Bayesuvius,
a small visual dictionary of Bayesian Networks

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Figure 1: View of Mount Vesuvius from Pompeii

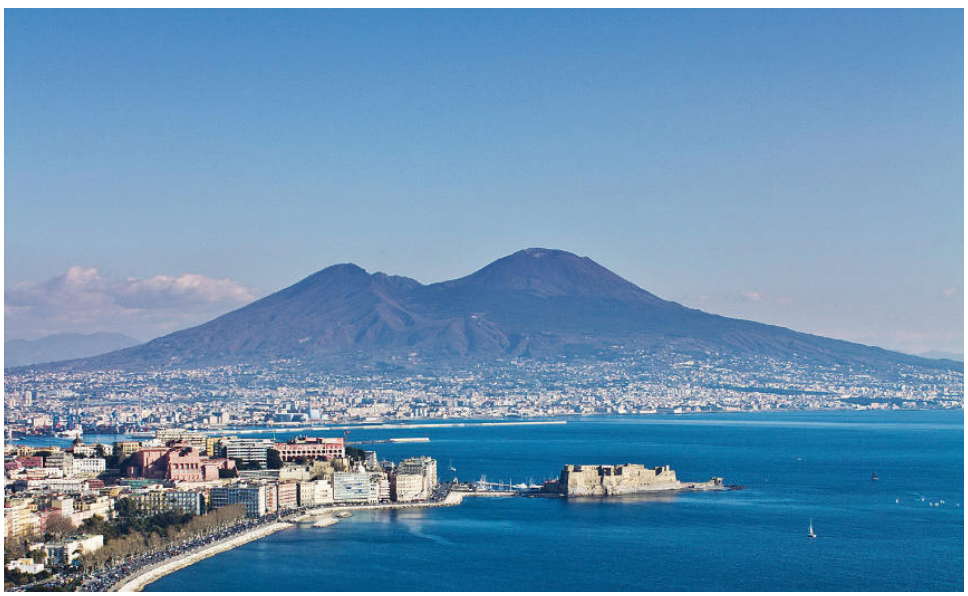


Figure 2: Mount Vesuvius and Bay of Naples

Contents

0.2 Navigating Judea Pearl's Books

Many of the greatest ideas in the bnet field were invented by Judea Pearl and his collaborators. Thus, this book is heavily indebted to those great scientists.

Those ideas have had no clearer and more generous expositor than Judea Pearl himself.

Pearl has 4 books that I have used in writing Bayesuvius.

His first 1988 book Ref.[1] was in his pre-causal days. That book deals with topics such as d-separation, belief propagation, Markov-blankets, and noisy-ors.

The other 3 books that came later are more fully devoted to causality. They are:

1. In 2000 (1st ed.), and 2013 (2nd ed.), Pearl published what is so far his most technical and exhaustive book on the subject of causality, Ref[2].

2. In 2016, he released together with Glymour and Jewell, a less advanced "primer" on causality, Ref.[3].

3. In 2018, he released together with Mackenzie his lovely "The Book of Why", Ref.[4].

Those 3 books deal with causality topics such as do-calculus, backdoor and front-door adjustments, linear systems with exogenous noise, and counterfactuals.

0.3 Notational Conventions and Preliminaries

Some abbreviations frequently used throughout this book.

- bnet= B net= Bayesian Network
- TPM= Transition Probability Matrix
- CPT = Conditional Probabilities Table, same as TPM
- $ch(\underline{a})$ = children of node $\underline{a}$.
- $pa(\underline{a})$ = parents of node $\underline{a}$.
- $nb(\underline{a}) = pa(\underline{a}) \cup ch(\underline{a})$ = neighbors of node $\underline{a}$.
- i.i.d.= independent identically distributed.

Define $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ to be the integers, real numbers and complex numbers, respectively.

For $a < b$, define $\mathbb{Z}_I$ to be the integers in the interval I, where $I = [a,b], [a,b), (a,b], (a,b)$ (i.e, I can be closed or open on either side).

$A_{>0} = \{k \in A : k > 0\}$ for $A = \mathbb{Z}, \mathbb{R}$.

Random Variables will be indicated by underlined letters and their values by non-underlined letters. Each node of a bnet will be labelled by a random variable. Thus, $\underline{x} = x$ means that node $\underline{x}$ is in state x.

$P_{\underline{x}}(x) = P(\underline{x} = x) = P(x)$ is the probability that random variable $\underline{x}$ equals $x \in S_{\underline{x}}$. $S_{\underline{x}}$ is the set of states (i.e., values) that $\underline{x}$ can assume and $n_{\underline{x}} = |S_{\underline{x}}|$ is the size (aka cardinality) of that set. Hence,

$$\sum_{x \in S_{\underline{x}}} P_{\underline{x}}(x) = 1 \tag{1}$$

$$P_{\underline{x},\underline{y}}(x,y) = P(\underline{x} = x, \underline{y} = y) = P(x,y) \tag{2}$$

$$P_{\underline{x}|\underline{y}}(x|y) = P(\underline{x} = x|\underline{y} = y) = P(x|y) = \frac{P(x,y)}{P(y)} \tag{3}$$

Kronecker delta function: For x, y in discrete set S,

$$\delta(x,y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases} \tag{4}$$

Dirac delta function: For $x, y \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} dx\ \delta(x-y) f(x) = f(y) \tag{5}$$

The TPM of a node of a bnet can be either a discrete or a continuous probability distribution. To go from continuous to discrete, one replaces integrals over states of a node by sums over new states, and Dirac delta functions by Kronecker delta functions. More precisely, consider a function $f : [a, b] \to \mathbb{R}$. Express $[a, b]$ as a union of small, disjoint (except for one point) closed sub-intervals (bins) of length Δx. Name one point in each bin to be the representative of that bin, and let $S_{\underline{x}}$ be the set of all the bin representatives. This is called discretization or binning. Then

$$\frac{1}{(b-a)} \int_{[a,b]} dx\ f(x) \to \frac{\Delta x}{(b-a)} \sum_{x \in S_{\underline{x}}} f(x) = \frac{1}{n_{\underline{x}}} \sum_{x \in S_{\underline{x}}} f(x)\ . \tag{6}$$

Both sides of last equation are 1 when $f(x) = 1$. Furthermore, if $y \in S_{\underline{x}}$, then

$$\int_{[a,b]} dx\ \delta(x-y) f(x) = f(y) \to \sum_{x \in S_{\underline{x}}} \delta(x,y) f(x) = f(y)\ . \tag{7}$$

Indicator function (aka Truth function):

$$\mathbb{1}(\mathcal{S}) = \begin{cases} 1 \text{ if } \mathcal{S} \text{ is true} \\ 0 \text{ if } \mathcal{S} \text{ is false} \end{cases} \tag{8}$$

For example, $\delta(x, y) = \mathbb{1}(x = y)$.

$$\vec{x} = (x[0], x[1], x[2] \ldots, x[nsam(\vec{x}) - 1]) = x[:] \tag{9}$$

$nsam(\vec{x})$ is the number of samples of $\vec{x}$. $\underline{x}[i] \in S_{\underline{x}}$ are i.i.d. (independent identically distributed) samples with

$$x[i] \sim P_{\underline{x}} \ \text{ (i.e. } P_{\underline{x[i]}} = P_{\underline{x}}) \tag{10}$$

$$P(\underline{x} = x) = \frac{1}{nsam(\vec{x})} \sum_i \mathbb{1}(x[i] = x) \tag{11}$$

Hence, for any $f : S_{\underline{x}} \to \mathbb{R}$,

$$\sum_x P(\underline{x} = x) f(x) = \frac{1}{nsam(\vec{x})} \sum_i f(x[i]) \tag{12}$$

If we use two sampled variables, say $\vec{x}$ and $\vec{y}$, in a given bnet, their number of samples $nsam(\vec{x})$ and $nsam(\vec{y})$ need not be equal.

$$P(\vec{x}) = \prod_i P(x[i]) \tag{13}$$

$$\sum_{\vec{x}} = \prod_i \sum_{x[i]} \tag{14}$$

$$\partial_{\vec{x}} = [\partial_{x[0]}, \partial_{x[1]}, \partial_{x[2]}, \ldots, \partial_{x[nsam(\vec{x})-1]}] \tag{15}$$

$$
\begin{aligned}
P(\vec{x}) &\approx [\prod_x P(x)^{P(x)}]^{nsam(\vec{x})} && (16)\\
&= e^{nsam(\vec{x})\sum_x P(x)\ln P(x)} && (17)\\
&= e^{-nsam(\vec{x})H(P_{\underline{x}})} && (18)
\end{aligned}
$$

$$f^{[1,\partial_x,\partial_y]}(x,y) = [f, \partial_x f, \partial_y f] \tag{19}$$

$$f^{+} = f^{[1,\partial_x,\partial_y]} \tag{20}$$

For probabilty distributions $p(x), q(x)$ of $x \in S_{\underline{x}}$

- Entropy:

$$H(p) = -\sum_x p(x)\ln p(x) \geq 0 \tag{21}$$

- Kullback-Liebler divergence:

$$D_{KL}(p \parallel q) = \sum_x p(x)\ln\frac{p(x)}{q(x)} \geq 0 \tag{22}$$

- Cross entropy:

$$
\begin{aligned}
CE(p \to q) &= -\sum_x p(x)\ln q(x) && (23)\\
&= H(p) + D_{KL}(p \parallel q) && (24)
\end{aligned}
$$

Normal Distribution: $x, \mu, \sigma \in \mathbb{R}$, $\sigma > 0$

$$\mathcal{N}(\mu, \sigma^2)(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{25}$$

Uniform Distribution: $a < b$, $x \in [a, b]$

$$\mathcal{U}(a,b)(x) = \frac{1}{b-a} \tag{26}$$

Expected Value and Variance

Given a random variable $\underline{x}$ with states $S_{\underline{x}}$ and a function $f : S_{\underline{x}} \to \mathbb{R}$, define

$$E_{\underline{x}}[f(\underline{x})] = E_{x\sim P(x)}[f(x)] = \sum_x P(x)f(x) \tag{27}$$

$$
\begin{aligned}
Var_{\underline{x}}[f(\underline{x})] &= E_{\underline{x}}\left[(f(\underline{x}) - E_{\underline{x}}[f(\underline{x})])^2\right] && (28)\\
&= E_{\underline{x}}[f(\underline{x})^2] - (E_{\underline{x}}[f(\underline{x})])^2 && (29)
\end{aligned}
$$

Conditional Expected Value

Given a random variable $\underline{x}$ with states $S_{\underline{x}}$, a random variable $\underline{y}$ with states $S_{\underline{y}}$, and a function $f : S_{\underline{x}} \times S_{\underline{y}} \to \mathbb{R}$, define

$$E_{\underline{x}|\underline{y}}[f(\underline{x},\underline{y})] = \sum_x P(x|\underline{y}) f(x,\underline{y}) \ , \tag{30}$$

$$E_{\underline{x}|\underline{y}=y}[f(\underline{x},y)] = E_{\underline{x}|y}[f(\underline{x},y)] = \sum_x P(x|y) f(x,y) \ . \tag{31}$$

Note that

$$\begin{aligned}
E_{\underline{y}}[E_{\underline{x}|\underline{y}}[f(\underline{x},\underline{y})]] &= \sum_{x,y} P(x|y)P(y)f(x,y) & (32)\\
&= \sum_{x,y} P(x,y)f(x,y) & (33)\\
&= E_{\underline{x},\underline{y}}[f(\underline{x},\underline{y})] \ . & (34)
\end{aligned}$$

Law of Total Variance

Claim 1 *Suppose $P : S_{\underline{x}} \times S_{\underline{y}} \to [0,1]$ is a probability distribution. Suppose $f : S_{\underline{x}} \times S_{\underline{y}} \to \mathbb{R}$ and $f = f(x,y)$. Then*

$$Var_{\underline{x},\underline{y}}(f) = E_{\underline{y}}[Var_{\underline{x}|\underline{y}}(f)] + Var_{\underline{y}}(E_{\underline{x}|\underline{y}}[f]) \ . \tag{35}$$

In particular,

$$Var_{\underline{x}}(x) = E_{\underline{y}}[Var_{\underline{x}|\underline{y}}(x)] + Var_{\underline{y}}(E_{\underline{x}|\underline{y}}[x]) \ . \tag{36}$$

proof:

Let

$$A = \sum_y P(y) \left(\sum_x P(x|y) f \right)^2 \ . \tag{37}$$

Then

$$\begin{aligned}
Var_{\underline{x},\underline{y}}(f) &= \sum_{x,y} P(x,y)f^2 - \left(\sum_{x,y} P(x,y) f \right)^2 & (38)\\
&= \left\{ \begin{array}{l} \sum_{x,y} P(x,y)f^2 - A \\ + \left(A - \left(\sum_{x,y} P(x,y) f \right)^2 \right) \end{array} \right. & (39)
\end{aligned}$$

$$\begin{aligned}
E_{\underline{y}}[Var_{\underline{x}|\underline{y}}(f)] &= \sum_y P(y) \left(\sum_x P(x|y) f^2 - \left(\sum_x P(x|y) f \right)^2 \right) & (40)\\
&= \sum_{x,y} P(x,y) f^2 - A & (41)
\end{aligned}$$

$$Var_{\underline{y}}(E_{\underline{x}|\underline{y}}[f]) = \sum_y P(y)\left(\sum_x P(x|y)f\right)^2 - \left(\sum_y P(y)\sum_x P(x|y)f\right)^2 \quad (42)$$

$$= A - \left(\sum_{x,y} P(x,y)f\right)^2 \quad (43)$$

QED

$\langle \underline{x}, \underline{y}\rangle$ notation, for covariances of any two random variables $\underline{x}, \underline{y}$.

Mean value of $\underline{x}$

$$\langle \underline{x}\rangle = E_{\underline{x}}[\underline{x}] \quad (44)$$

Signed distance of $\underline{x}$ to its mean value

$$\Delta\underline{x} = \underline{x} - \langle \underline{x}\rangle \quad (45)$$

Covariance of $(\underline{x}, \underline{y})$

$$\langle \underline{x}, \underline{y}\rangle = \langle \Delta\underline{x}\Delta\underline{y}\rangle = Cov(\underline{x}, \underline{y}) \quad (46)$$

Variance of $\underline{x}$

$$Var(\underline{x}) = \langle \underline{x}, \underline{x}\rangle \quad (47)$$

Standard deviation or $\underline{x}$

$$\sigma_{\underline{x}} = \sqrt{\langle \underline{x}, \underline{x}\rangle} \quad (48)$$

Correlation of $(\underline{x}, \underline{y})$

$$\rho_{\underline{x},\underline{y}} = \frac{\langle \underline{x}, \underline{y}\rangle}{\sqrt{\langle \underline{x}, \underline{x}\rangle\langle \underline{y}, \underline{y}\rangle}} \quad (49)$$

Sigmoid function: For $x \in \mathbb{R}$,

$$\text{sig}(x) = \frac{1}{1+e^{-x}} \quad (50)$$

$\mathcal{N}(!a)$ will denote a normalization constant that does not depend on a. For example, $P(x) = \mathcal{N}(!x)e^{-x}$ where $\int_0^\infty dx\ P(x) = 1$.

A **one hot** vector of zeros and ones is a vector with all entries zero with the exception of a single entry which is one. A **one cold** vector has all entries equal to one with the exception of a single entry which is zero. For example, if $x^n = (x_0, x_1, \ldots, x_{n-1})$ and $x_i = \delta(i, 0)$ then x^n is one hot.

Short Summary of Boolean Algebra.

See Ref.[5] for more info about this topic.

Suppose $x, y, z \in \{0, 1\}$. Define

$$x \text{ or } y = x \vee y = x + y - xy\ , \quad (51)$$

$$x \text{ and } y = x \wedge y = xy\ , \quad (52)$$

and

$$\text{not } x = \overline{x} = 1 - x \ , \tag{53}$$

where we are using normal addition and multiplication on the right hand sides.[1]

Associativity	$x \vee (y \vee z) = (x \vee y) \vee z$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
Commutativity	$x \vee y = y \vee x$ $x \wedge y = y \wedge x$
Distributivity	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
Identity	$x \vee 0 = x$ $x \wedge 1 = x$
Annihilator	$x \wedge 0 = 0$ $x \vee 1 = 1$
Idempotence	$x \vee x = x$ $x \wedge x = x$
Absorption	$x \wedge (x \vee y) = x$ $x \vee (x \wedge y) = x$
Complementation	$x \wedge \overline{x} = 0$ $x \vee \overline{x} = 1$
Double negation	$\overline{(\overline{x})} = x$
De Morgan Laws	$\overline{x} \wedge \overline{y} = \overline{(x \vee y)}$ $\overline{x} \vee \overline{y} = \overline{(x \wedge y)}$

Table 1: Boolean Algebra Identities

Actually, since $x \wedge y = xy$, we can omit writing the symbol $\wedge$. The symbol $\wedge$ is useful to exhibit the symmetry of the identities, and to remark about the analogous identities for sets, where $\wedge$ becomes intersection $\cap$ and $\vee$ becomes union $\cup$. However, for practical calculations, $\wedge$ is an unnecessary nuisance.

Since $x \in \{0, 1\}$,

$$P(\overline{x}) = 1 - P(x) \ . \tag{54}$$

Clearly, from analyzing the simple event space $(x, y) \in \{0, 1\}^2$,

$$P(x \vee y) = P(x) + P(y) - P(x \wedge y) \ . \tag{55}$$

[1]Note the difference between $\vee$ and modulus 2 addition $\oplus$. For $\oplus$ (aka XOR): $x \oplus y = x + y - 2xy$.

Chapter 1

Backdoor Adjustment

The backdoor (BD) adjustment theorem is proven in Chapter 9 from the rules of do-calculus. The goal of this chapter is to give examples of the use of that theorem. We will restate the theorem in this chapter, sans proof. There is no need to understand the theorem's proof in order to use it. However, you will need to skim Chapter 9 in order to familiarize yourself with the notation used to state the theorem. This chapter also assumes that you are comfortable with the rules for checking for d-separation. Those rules are covered in Chapter 10.

Suppose we have access to data that allows us to estimate a probability distribution $P(x., y., z.)$. Hence, the variables $\underline{x}., \underline{y}., \underline{z}.$ are all observed (i.e, not hidden). Then we say that $\underline{z}.$ satisfies the **backdoor criterion** relative to $(\underline{x}., \underline{y}.)$ if

1. All paths from $\underline{x}.$ to $\underline{y}.$ that start with an arrow pointing into $\underline{x}.$ (i.e., "a backdoor"), are blocked by $\underline{z}.$.
2. $\underline{z}. \notin de(\underline{x}.)$.

Claim 2 Backdoor Adjustment Theorem
If $\underline{z}.$ satisfies the backdoor criterion relative to $(\underline{x}., \underline{y}.)$, then

$$P(y.|\rho\underline{x}. = x.) = \sum_{z.} P(y.|x., z.)P(z.) \tag{1.1}$$

$$= \sum_{z.} \left\{ \begin{tikzcd} \underline{z}. = z. \arrow[dr] & \\ \rho\underline{x}. = x. \arrow[r] & \underline{y}. \end{tikzcd} \right\} \tag{1.2}$$

proof: See Chapter 9
QED

Examples:

1.

$$\begin{tikzcd} & \underline{z} \arrow[dr] & \\ \underline{x} \arrow[ur] \arrow[rr] & & \underline{y} \end{tikzcd} \tag{1.3}$$

BD criterion satisfied if $\underline{x}. = \underline{x}, \underline{y}. = \underline{y}, \underline{z}. = \emptyset$. No adjustment necessary.

$$P(y|\rho\underline{x} = x) = P(y|x) \tag{1.4}$$

2.

$$\begin{tikzcd} & \underline{z} \arrow[dl] \arrow[dr] & \\ \underline{x} \arrow[rr] & & \underline{y} \end{tikzcd} \tag{1.5}$$

BD criterion satisfied if $\underline{x}. = \underline{x}, \underline{y}. = \underline{y}, \underline{z}. = \underline{z}$.

Note that here the backdoor formula adjusts the parents of $\underline{x}$..

3.

$$\begin{tikzcd} & \underline{z} \arrow[dl] \arrow[dr] & \\ \underline{x} \arrow[r] & \underline{m} \arrow[r] & \underline{y} \end{tikzcd} \tag{1.6}$$

BD criterion satisfied if $\underline{x}. = \underline{x}, \underline{y}. = \underline{y}, \underline{z}. = \underline{z}$.

This bnet is also used to demonstrate the front-door criterion.

4.

$$\begin{tikzcd} \boxed{\underline{w}} \arrow[r] \arrow[d] & \underline{z} \arrow[d] \\ \underline{x} \arrow[r] & \underline{y} \end{tikzcd} \tag{1.7}$$

BD criterion satisfied if $\underline{x}. = \underline{x}, \underline{y}. = \underline{y}, \underline{z}. = \underline{z}$. Note that here the backdoor formula cannot adjust the single parent $\underline{w}$ of $\underline{x}$ because it is hidden, but we are able to block the backdoor path by conditioning on $\underline{z}$ instead.

5.

$$\begin{tikzcd} \boxed{\underline{e}} \arrow[r] \arrow[d] & \underline{z} \arrow[dl] \arrow[dr] & \underline{a} \arrow[l] \arrow[d] \\ \underline{x} \arrow[rr] & & \underline{y} \end{tikzcd} \tag{1.8}$$

Conditioning on $\underline{z}$ blocks backdoor path $\underline{x} - \underline{z} - \underline{y}$, but opens path $\underline{x} - \underline{e} - \underline{z} - \underline{a} - \underline{y}$ because $\underline{z}$ is a collider for that path. That path is blocked if we also condition on $\underline{a}$, which is possible because $\underline{a}$ is observed. In conclusion, the BD criterion is satisfied if $\underline{x}. = \underline{x}$, $\underline{y}. = \underline{y}$ and $\underline{z}. = (\underline{z}, \underline{a})$.

Conditioning on the parents of $\underline{x}.$ is often enough to block all backdoor paths. However, sometimes some of the parents are unobserved and one most condition on other nodes that are not parents of $\underline{x}.$ in order to satisfy the BD criterion.

6.
$$\begin{tikzcd}
\underline{z} \arrow[d] & & \underline{t} \arrow[ll] \arrow[d] \\
\underline{w} & \underline{x} \arrow[l] \arrow[r] & \underline{y}
\end{tikzcd} \tag{1.9}$$

No need to control anything because only possible backdoor path is blocked by collider $\underline{w}$. Hence,

$$P(y|\rho\underline{x} = x) = P(y|x) \ . \tag{1.10}$$

However, if for some reason we want to control $\underline{w}$, we can block the path by controlling $\underline{t}$ too. Thus, the BD criterion is satisfied if $\underline{x}. = \underline{x}$, $\underline{y}. = \underline{y}$ and $\underline{z}. = (\underline{w}, \underline{t})$. Therefore,

$$P(y|\rho\underline{x} = x) = \sum_{t,w} P(y|x,t,w)P(t,w) \ . \tag{1.11}$$

Alternatively, can condition on $\underline{w}$ a priori, and satisfy the BD criterion with $\underline{x}. = \underline{x}$, $\underline{y}. = \underline{y}$ and $\underline{z}. = \underline{t}$; thus,

$$P(y|\rho\underline{x} = x, w) = \sum_{t} P(y|x,t,w)P(t|w) \ . \tag{1.12}$$

Multiplying Eq.(1.12) by $P(w)$ and summing over w gives Eq.(1.11).

7. Discuss reasons why multiple possible sets $\underline{z}.$ that satisfy the BD criterion can be advantageous.

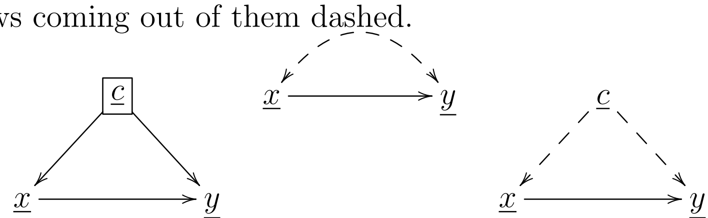
   - Can evaluate $P(y.|\rho\underline{x}. = x.)$ multiple ways and compare the results. This is a test that the causal bnet is correct.
   - Some $\underline{z}.$ might be easier or less expensive to get data on.

Chapter 9

Do-Calculus

The Do-calculus and associated ideas were invented by Judea Pearl and collaborators. This chapter is heavily based on Judea Pearl's books. (See 0.2).

$\underline{X}. = \underline{V}. \cup \underline{H}.$, $\underline{V}. \cap \underline{H}. = \emptyset$. $\underline{V}.$= visible, observed. $\underline{H}.$= hidden, uobserved. Hidden nodes will be indicated either by enclosing them their random variable in a box (as if it were a black box) or by making the arrows coming out of them dashed.



(9.1)

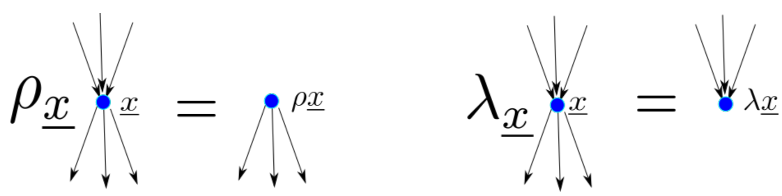


Figure 9.1: The operator $\rho_{\underline{x}}$ converts node $\underline{x}$ into a root node $\rho\underline{x}$. The operator $\lambda_{\underline{x}}$ converts node $\underline{x}$ into a leaf node $\lambda\underline{x}$.

$$\rho_{\underline{a}.}G = \prod_j \rho_{\underline{a}_j} G \;, \quad \lambda_{\underline{a}.}G = \prod_j \lambda_{\underline{a}_j} G \tag{9.2}$$

$$P(X. - a. | \rho\underline{a}. = a.) = \mathcal{N}(!(X. - a.)) \prod_{j:\underline{X}_j \notin \underline{a}.} P(X_j | pa(X_j)) \tag{9.3}$$

$\underline{b}. \subset \underline{X}. - \underline{a}.$

$$P(b. | \rho\underline{a}. = a.) = \sum_{X.-a.-b.} P(X. - a. | \rho\underline{a}. = a.) \tag{9.4}$$

$\underline{r}. \subset \underline{X}. - \underline{a}. - \underline{b}.$

$$P(b. | \rho\underline{a}. = a., r.) = \frac{P(b., r. | \rho\underline{a}. = a.)}{P(r. | \rho\underline{a}. = a.)} \tag{9.5}$$

Usually $P(b.|\rho\underline{a}. = a., s.)$ is denoted by $P(b.|do(\underline{a}. = a.), s.)$.

$P(b.|\rho\underline{a}. = a., s.)$ is said to be **identifiable** if it can be expressed as a product of conditional prob distributions that only depend on observed variables and that have no $do()$ conditions in them.

For $\underline{x}, rvy \in \{0, 1\}$, "causal effect difference , or 'average causal effect (ACE)

$$ACE = P(y = 1|\rho\underline{x} = 1) - P(y = 1|\rho\underline{x} = 0) \tag{9.6}$$

Risk Difference

$$RD = P(y = 1|\underline{x} = 1) - P(y = 1|\underline{x} = 0) \tag{9.7}$$

3 Rules of do-calculus

If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in G, then $P(b.|a., r., s.) = P(b.|r., s.)$

- **Rule 1:** Insertion or deletion of observations ($\underline{a}. = a. \leftrightarrow 1$)

  If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\rho_{\underline{r}.}G$, then $P(b.|a., \rho\underline{r}. = r., s.) = P(b.|\rho\underline{r}. = r., s.)$.

- **Rule 2:** Action or observation exchange ($\rho\underline{a}. = a. \leftrightarrow \underline{a}. = a.$)

  If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\lambda_{\underline{a}.}\rho_{\underline{r}.}G$, then $P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|a., \rho\underline{r}. = r., s.)$.

- **Rule 3:** Insertion and deletion of actions ($\rho\underline{a}. = a. \leftrightarrow 1$)

  If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\rho_{\underline{a}.-an(\underline{s}.)}\rho_{\underline{r}.}G$, then $P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|\rho\underline{r}. = r., s.)$.

Backdoor Adjustment

Back and front door **adjustment formulas**. Adjust a variable , average over it, control it.

See Chapter 1 for examples of the use of the backdoor adjustment formula. In this section, we are mainly concerned with proving that formula using do-calculus.

Suppose we have access to data that allows us to estimate a probability distribution $P(x., y., z.)$. Hence, the variables $\underline{x}., \underline{y}., \underline{z}.$ are all observed (i.e, not hidden). Then we say that $\underline{z}.$ satisfies the **backdoor criterion** relative to $(\underline{x}., \underline{y}.)$ if

1. All paths from $\underline{x}.$ to $\underline{y}.$ that start with an arrow pointing into $\underline{x}.$ (i.e., "a backdoor"), are blocked by $\underline{z}.$.
2. $\underline{z}. \notin de(\underline{x}.)$.

Claim 3 *Backdoor Adjustment*

If $\underline{z}.$ satisfies the backdoor criterion relative to $(\underline{x}., \underline{y}.)$, then

$$P(y.|\rho\underline{x}. = x.) = \sum_{z.} P(y.|x., z.)P(z.) \tag{9.8}$$

$$= \sum_{z.} \left\{ \begin{tikzcd} \underline{z}. = z. \arrow[dr] & \\ \rho\underline{x}. = x. \arrow[r] & \underline{y}. \end{tikzcd} \right\} \tag{9.9}$$

proof:

$$\begin{tikzcd} \underline{z} \arrow[d] \arrow[dr] & \\ \underline{x} \arrow[r] & \underline{y} \end{tikzcd} \tag{9.10}$$

$P(y|\rho\underline{x} = x) =$

$= \sum_m P(y|\rho\underline{x} = x, z)P(z|\rho\underline{x} = x)$

by Probability Axioms

$= \sum_P (y|x, z)P(z|\rho\underline{x} = x)$

$P(y|\rho\underline{x} = x, z) \to P(y|x, z)$

by Rule 2: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\lambda_{\underline{a}.}\rho_{\underline{r}.}G$, then $P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|a., \rho\underline{r}. = r., s.)$.

$\underline{y} \perp \underline{x}|\underline{z}$ in $\lambda_{\underline{x}}G$ $\begin{tikzcd} \underline{z} \arrow[d] \arrow[dr] & \\ \underline{x} & \underline{y} \end{tikzcd}$

$= \sum_z P(y|x, z)P(z)$

$P(z|\rho\underline{x} = x) \to P(z)$

by Rule 3: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\rho_{\underline{a}.-an(\underline{s}.)}\rho_{\underline{r}.}G$, then $P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|\rho\underline{r}. = r., s.)$.

$\underline{z} \perp \underline{x}$ in $\rho_{\underline{x}}G$ $\begin{tikzcd} \underline{z} \arrow[dr] & \\ \underline{x} \arrow[r] & \underline{y} \end{tikzcd}$

(9.11)

QED

$$P(y.|\rho\underline{x}. = x.) = \sum_{z.} P(y.|x., z.)P(z.) \tag{9.12}$$

$$= \sum_{z.} \frac{P(y., x., z.)}{P(x.|z.)} \tag{9.13}$$

$P(x.|z.)$ called propensity score, can be approximated.

Front Door Adjustment

See Chapter 13 for examples of the use of the backdoor adjustment formula. In this section, we are mainly concerned with proving that formula using do-calculus.

If $\underline{z}.$ satisfies the Suppose we have access to data that allows us to estimate a probability distribution $P(x., m., y.)$. Hence, the variables $\underline{x}., \underline{m}., \underline{y}.$ are all observed (i.e, not hidden). Then we say that $\underline{m}.$ satisfies the **front-door criterion** relative to $(\underline{x}., \underline{y}.)$ if

1. All directed paths from $\underline{x}.$ to $\underline{y}.$ are intercepted by (i.e., have a node in) $\underline{m}.$.
2. All backdoor paths from $\underline{x}.$ to $\underline{m}.$ are blocked.
3. All backdoor paths from on $\underline{m}.$ to $\underline{y}.$ are blocked by $\underline{x}.$.

Claim 4 *Front-Door Adjustment*
If $\underline{m}.$ satisfies the front-door criterion relative to $(\underline{x}., \underline{y}.)$, and $P(x., m.) > 0$, then

$$P(y.|\rho\underline{x}. = x.) = \sum_{m.} \underbrace{\left[\sum_{x.'} P(y.|x'., m.)P(x'.)\right]}_{P(y.|\rho\underline{m}.=m.)} \underbrace{P(m.|x.)}_{P(m.|\rho\underline{x}.=x.)} \tag{9.14}$$

$$= \sum_{m.,x.'} \left\{ \begin{tikzcd} & \underline{x}. = x.' \arrow[dr] & \\ \rho\underline{x}. = x. \arrow[r] & \underline{m}. = m. \arrow[r] & \underline{y}. \end{tikzcd} \right\} \tag{9.15}$$

proof: $\underline{c}$ confounder, hidden

$$\begin{tikzcd} & \boxed{\underline{c}} \arrow[dl] \arrow[dr] & \\ \underline{x} \arrow[r] & \underline{m} \arrow[r] & \underline{y} \end{tikzcd} \tag{9.16}$$

$$
\begin{aligned}
& Pcc(y|\rho\underline{x} = x) = \\
= \; & \textstyle\sum_m P(y|\rho\underline{x} = x, m) P(m|\rho\underline{x} = x) \\
& \text{by Probability Axioms} \\
= \; & \textstyle\sum_m P(y|\rho\underline{x} = x, \rho\underline{m} = m) P(m|\rho\underline{x} = x) \\
& P(y|\rho\underline{x} = x, m) \to P(y|\rho\underline{x} = x, \rho m = m) \\
& \text{by Rule 2: If } (\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.) \text{ in } \lambda_{\underline{a}.}\rho_{\underline{r}.}G, \text{ then } P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|a., \rho\underline{r}. = r., s.). \\
& \underline{y} \perp \underline{m}|\underline{x} \text{ in } \lambda_{\underline{m}}\rho_{\underline{x}}G
\end{aligned}
$$

$$
\begin{tikzcd}
 & \boxed{\underline{c}} \arrow[dr] & \\
\underline{x} \arrow[r] & \underline{m} & \underline{y}
\end{tikzcd}
$$

$$
\begin{aligned}
= \; & \textstyle\sum_m P(y|\rho\underline{x} = x, \rho\underline{m} = m) P(m|x) \\
& P(m|\rho\underline{x} = x) \to P(m|x) \\
& \text{by Rule 2: If } (\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.) \text{ in } \lambda_{\underline{a}.}\rho_{\underline{r}.}G, \text{ then } P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|a., \rho\underline{r}. = r., s.). \\
& \underline{m} \perp \underline{x} \text{ in } \lambda_{\underline{x}}G
\end{aligned}
$$

$$
\begin{tikzcd}
 & \boxed{\underline{c}} \arrow[dl] \arrow[dr] & \\
\underline{x} & \underline{m} \arrow[r] & \underline{y}
\end{tikzcd}
$$

$$
\begin{aligned}
= \; & \textstyle\sum_m P(y|\rho\underline{m} = m) P(m|x) \\
& P(y|\rho\underline{x} = x, \rho\underline{m} = m) \to P(y|\rho\underline{m} = m) \\
& \text{by Rule 3: If } (\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.) \text{ in } \rho_{\underline{a}.-an(\underline{s}.)}\rho_{\underline{r}.}G, \text{ then } P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|\rho\underline{r}. = r., s.). \\
& \underline{y} \perp \underline{x}|\underline{m} \text{ in } \rho_{\underline{x}}\rho_{\underline{m}}G
\end{aligned}
$$

$$
\begin{tikzcd}
 & \boxed{\underline{c}} \arrow[dr] & \\
\underline{x} & \underline{m} \arrow[r] & \underline{y}
\end{tikzcd}
$$

$$
\begin{aligned}
= \; & \textstyle\sum_{x'} \sum_m P(y|\rho\underline{m} = m, x') P(x'|\rho\underline{m} = m) P(m|x) \\
& \text{by Probability Axioms} \\
= \; & \textstyle\sum_{x'} \sum_m P(y|m, x') P(x'|\rho\underline{m} = m) P(m|x) \\
& P(y|\rho\underline{m} = m, x') \to P(y|m, x') \\
& \text{by Rule 2: If } (\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.) \text{ in } \lambda_{\underline{a}.}\rho_{\underline{r}.}G, \text{ then } P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|a., \rho\underline{r}. = r., s.). \\
& \underline{y} \perp \underline{m}|\underline{x} \text{ in } \lambda_{\underline{m}}G
\end{aligned}
$$

$$
\begin{tikzcd}
 & \boxed{\underline{c}} \arrow[dl] \arrow[dr] & \\
\underline{x} \arrow[r] & \underline{m} & \underline{y}
\end{tikzcd}
$$

$$
\begin{aligned}
= \; & \textstyle\sum_{x'} \sum_m P(y|m, x') P(x') P(m|x) \\
& P(x'|\rho\underline{m} = m) \to P(x') \\
& \text{by Rule 3: If } (\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.) \text{ in } \rho_{\underline{a}.-an(\underline{s}.)}\rho_{\underline{r}.}G, \text{ then } P(b.|\rho\underline{a}. = a., \rho\underline{r}. = r., s.) = P(b.|\rho\underline{r}. = r., s.). \\
& \underline{x} \perp \underline{m} \text{ in } \rho_{\underline{m}}G
\end{aligned}
$$

$$
\begin{tikzcd}
 & \boxed{\underline{c}} \arrow[dl] \arrow[dr] & \\
\underline{x} & \underline{m} \arrow[r] & \underline{y}
\end{tikzcd}
\tag{9.17}
$$

QED

Chapter 13

Front-door Adjustment

The front-door (FD) adjustment theorem is proven in Chapter 9 from the rules of do-calculus. The goal of this chapter is to give examples of the use of that theorem. We will restate the theorem in this chapter, sans proof. There is no need to understand the theorem's proof in order to use it. However, you will need to skim Chapter 9 in order to familiarize yourself with the notation used to state the theorem. This chapter also assumes that you are comfortable with the rules for checking for d-separation. Those rules are covered in Chapter 10.

If $\underline{z}.$ satisfies the Suppose we have access to data that allows us to estimate a probability distribution $P(x., m., y.)$. Hence, the variables $\underline{x}., \underline{m}., \underline{y}.$ are all observed (i.e, not hidden). Then we say that $\underline{m}.$ satisfies the **front-door criterion** relative to $(\underline{x}., \underline{y}.)$ if

1. All directed paths from $\underline{x}.$ to $\underline{y}.$ are intercepted by (i.e., have a node in) $\underline{m}.$.
2. All backdoor paths from $\underline{x}.$ to $\underline{m}.$ are blocked.
3. All backdoor paths from on $\underline{m}.$ to $\underline{y}.$ are blocked by $\underline{x}.$.

Claim 5 *Front-Door Adjustment If $\underline{m}.$ satisfies the front-door criterion relative to $(\underline{x}., \underline{y}.)$, and $P(x., m.) > 0$, then*

$$P(y.|\rho\underline{x}. = x.) = \sum_{m.} \underbrace{\left[\sum_{x.'} P(y.|x'., m.)P(x'.)\right]}_{P(y.|\rho\underline{m}.=m.)} \underbrace{P(m.|x.)}_{P(m.|\rho\underline{x}.=x.)} \tag{13.1}$$

$$= \sum_{m.,x.'} \left\{ \begin{array}{ccccc} & & \underline{x}. = x.' & & \\ & & & \searrow & \\ \rho\underline{x}. = x. & \longrightarrow & \underline{m}. = m. & \longrightarrow & \underline{y}. \end{array} \right\} \tag{13.2}$$

proof: See Chapter 9
QED

Chapter 21

Linear Systems with Exogenous Noise

In this chapter, we will consider bnets which were referred to, prior to the invention of bnets, as: Sewall Wright's Path Analysis (PA) and linear Structural Equations Model (SEM). Judea Pearl in his books calls them linear Structural Causal Models (SCM), because they are very convenient for doing causal analysis. We will follow Judea's convention and refer to them as scum.

To build a SCM, start with a deterministic bnet G. Now add to each node $\underline{a}$ of G a root node $\underline{U}_{\underline{a}}$ pointing into $\underline{a}$ only. The nodes $\underline{U}_{\underline{a}}$ are called the **exogenous (external) variables**. The exogenous variables make their children noisy. Their TPMs are priors and are assumed to be unobserved. Of course, by the fact that they are root nodes, they are assumed to be mutually independent.

Examples:

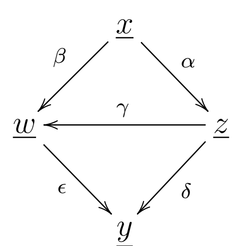


Figure 21.1:

$$P(y|w, z, U_{\underline{y}}) = \mathbb{1}(y = \epsilon w + \delta z + U_{\underline{y}}) \tag{21.1}$$

$$P(w|x, z, U_{\underline{w}}) = \mathbb{1}(w = \beta x + \gamma z + U_{\underline{w}}) \tag{21.2}$$

$$P(z|x, U_{\underline{z}}) = \mathbb{1}(z = \alpha x + U_{\underline{z}}) \tag{21.3}$$

$$P(x|U_{\underline{x}}) = \mathbb{1}(x = U_{\underline{x}}) \tag{21.4}$$

$$\begin{aligned}
y &= \epsilon w + \delta z + U_{\underline{y}} && (21.5)\\
&= \epsilon(\beta x + \gamma z + U_{\underline{w}}) + \delta z + U_{\underline{y}} && (21.6)\\
&= (\epsilon\gamma + \delta)z + \epsilon\beta x + \epsilon U_{\underline{w}} U_{\underline{y}} && (21.7)\\
&= (\epsilon\gamma + \delta)z + \epsilon\beta U_{\underline{x}} + \epsilon U_{\underline{w}} U_{\underline{y}} && (21.8)
\end{aligned}$$

$$\left(\frac{\partial y}{\partial z}\right)_{U.} = \epsilon\gamma + \delta \tag{21.9}$$

sum of terms, each of those terms represents a different causal (not blocked) path from $\underline{z}$ to $\underline{y}(\underline{z})$.

bnet with deterministic nodes $\underline{x}. = (\underline{x}_k)_{k=0,1,\ldots nx-1}$ and corresponding exogenous nodes $\underline{U}. = (\underline{U}_k)_{k=0,1,\ldots nx-1}$. Assume $\langle \underline{U}_i, \underline{U}_j \rangle = 0$ if $i \neq j$. The **structural coefficient** $\alpha_{j|i} > 0$ measures the strength of the connection $\underline{x}_i \rightarrow \underline{x}_j$. $\langle \underline{x}, \underline{y} \rangle$ notation, for covariances of any two random variables $\underline{x}, \underline{y}$ is explained in the Notational Conventions chapter 0.3.

- **Fully connected graph with** $nx = 2$

$$\begin{tikzcd}
\underline{x}_0 \arrow[d, "\alpha 1|0"] \\
\underline{x}_1
\end{tikzcd}$$

Figure 21.2: Fully connected graph with 2 $\underline{x}_j$ (exogenous nodes $\underline{U}_j$ not shown).

$$\begin{aligned}
\underline{x}_0 &= \underline{U}_0 && (21.10a)\\
\underline{x}_1 &= \alpha_{1|0}\underline{x}_0 + \underline{U}_1 && (21.10b)
\end{aligned}$$

Eqs.21.10 constitute a system of 2 linear equations in 2 unknowns (the $\underline{x}$'s) so can solve for the $\underline{x}$'s in terms of the α's and $\underline{U}$'s.

$$\langle \underline{x}_1, \underline{x}_0 \rangle = \alpha_{1|0} \langle \underline{x}_0, \underline{x}_0 \rangle \tag{21.11}$$

Thus, $\alpha_{1|0}$ can be estimated from the covariances $\langle \underline{x}_i, \underline{x}_j \rangle$.

$$
\begin{tikzcd}
\underline{x}_0 \arrow[d, "\alpha_{1|0}"'] \arrow[dr, "\alpha_{2|0}"] & \\
\underline{x}_1 \arrow[r, "\alpha_{2|1}"] & \underline{x}_2
\end{tikzcd}
$$

Figure 21.3: Fully connected graph with 3 $\underline{x}_j$ (exogenous nodes $\underline{U}_j$ not shown).

- **Fully connected graph with** $nx = 3$

$$\underline{x}_0 = \underline{U}_0 \tag{21.12a}$$
$$\underline{x}_1 = \alpha_{1|0}\underline{x}_0 + \underline{U}_1 \tag{21.12b}$$
$$\underline{x}_2 = \alpha_{2|1}\underline{x}_1 + \alpha_{2|0}\underline{x}_0 + \underline{U}_2 \tag{21.12c}$$

Eqs.21.12 constitute a system of 3 linear equations in 3 unknowns (the $\underline{x}$'s) so can solve for the $\underline{x}$'s in terms of the α's and $\underline{U}$'s.

$$\langle \underline{x}_1, \underline{x}_0 \rangle = \alpha_{1|0}\langle \underline{x}_0, \underline{x}_0 \rangle \tag{21.13a}$$
$$\langle \underline{x}_2, \underline{x}_0 \rangle = \alpha_{2|1}\langle \underline{x}_1, \underline{x}_0 \rangle + \alpha_{2|0}\langle \underline{x}_0, \underline{x}_0 \rangle \tag{21.13b}$$
$$\langle \underline{x}_2, \underline{x}_1 \rangle = \alpha_{2|1}\langle \underline{x}_1, \underline{x}_1 \rangle + \alpha_{2|0}\langle \underline{x}_0, \underline{x}_1 \rangle \tag{21.13c}$$

Eqs.21.13 constitute a system of 3 linear equations in 3 unknowns (the α's) so can solve solve for the α's in terms of covariances $\langle \underline{x}_i, \underline{x}_j \rangle$. This gives an estimate for the α's.

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