Bayesuvius,
a small visual dictionary of Bayesian Networks

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Figure 1: View of Mount Vesuvius from Pompeii

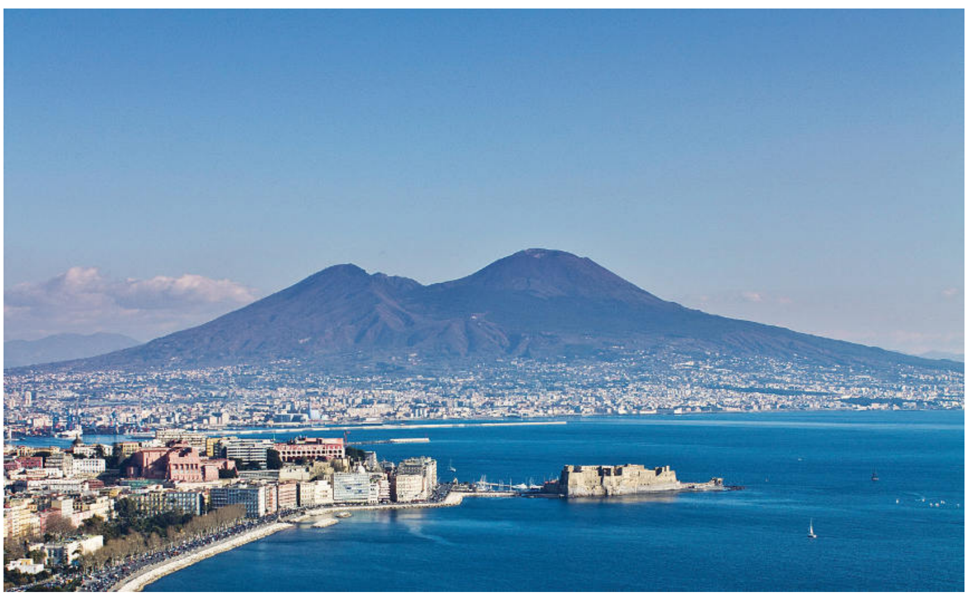


Figure 2: Mount Vesuvius and Bay of Naples

Contents

0.2 Notational Conventions

bnet=B net=Bayesian Network

Define $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ to be the integers, real numbers and complex numbers, respectively.

For $a < b$, define $\mathbb{Z}_I$ to be the integers in the interval I, where $I = [a, b], [a, b), (a, b], (a, b)$ (i.e, I can be closed or open on either side).

$A_{>0} = \{k \in A : k > 0\}$ for $A = \mathbb{Z}, \mathbb{R}$.

Random Variables will be indicated by underlined letters and their values by non-underlined letters. Each node of a bnet will be labelled by a random variable. Thus, $\underline{x} = x$ means that node $\underline{x}$ is in state x.

$P_{\underline{x}}(x) = P(\underline{x} = x) = P(x)$ is the probability that random variable $\underline{x}$ equals $x \in S_{\underline{x}}$. $S_{\underline{x}}$ is the set of states (i.e., values) that $\underline{x}$ can assume and $n_{\underline{x}} = |S_{\underline{x}}|$ is the size (aka cardinality) of that set. Hence,

$$\sum_{x \in S_{\underline{x}}} P_{\underline{x}}(x) = 1 \tag{1}$$

$$P_{\underline{x},\underline{y}}(x, y) = P(\underline{x} = x, \underline{y} = y) = P(x, y) \tag{2}$$

$$P_{\underline{x}|\underline{y}}(x|y) = P(\underline{x} = x|\underline{y} = y) = P(x|y) = \frac{P(x, y)}{P(y)} \tag{3}$$

Kronecker delta function: For x, y in discrete set S,

$$\delta(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases} \tag{4}$$

Dirac delta function: For $x, y \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} dx\ \delta(x - y) f(x) = f(y) \tag{5}$$

Transition probability matrix of a node of a bnet can be either a discrete or a continuous probability distribution. To go from continuous to discrete, one replaces integrals over states of node by sums over new states, and Dirac delta functions by Kronecker delta functions. More precisely, consider a function $f : S \to \mathbb{R}$. Let $S_{\underline{x}} \subset S$ and $S \to S_{\underline{x}}$ upon discretization (binning). Then

$$\int_S dx\ P_{\underline{x}}(x) f(x) \to \frac{1}{n_{\underline{x}}} \sum_{x \in S_{\underline{x}}} f(x) \ . \tag{6}$$

Both sides of last equation are 1 when $f(x) = 1$. Furthermore, if $y \in S_{\underline{x}}$, then

$$\int_S dx\ \delta(x - y) f(x) = f(y) \to \sum_{x \in S_{\underline{x}}} \delta(x, y) f(x) = f(y) \ . \tag{7}$$

Indicator function (aka Truth function):

$$\mathbb{1}(\mathcal{S}) = \begin{cases} 1 \text{ if } \mathcal{S} \text{ is true} \\ 0 \text{ if } \mathcal{S} \text{ is false} \end{cases} \tag{8}$$

For example, $\delta(x,y) = \mathbb{1}(x=y)$.

$$\vec{x} = (x[0], x[1], x[2] \ldots, x[nsam(\vec{x})-1]) = x[:] \tag{9}$$

$nsam(\vec{x})$ is the number of samples of $\vec{x}$. $\underline{x}[i]$ are i.d.d. (independent identically distributed) samples with

$$x[i] \sim P_{\underline{x}} \text{ (i.e. } P_{\underline{x[i]}} = P_{\underline{x}}) \tag{10}$$

$$P(\underline{x} = x) = \frac{1}{nsam(\vec{x})} \sum_i \mathbb{1}(x[i] = x) \tag{11}$$

If we use two sampled variables, say $\vec{x}$ and $\vec{y}$, in a given bnet, their number of samples $nsam(\vec{x})$ and $nsam(\vec{y})$ need not be equal.

$$P(\vec{x}) = \prod_i P(x[i]) \tag{12}$$

$$\sum_{\vec{x}} = \prod_i \sum_{x[i]} \tag{13}$$

$$\partial_{\vec{x}} = [\partial_{x[0]}, \partial_{x[1]}, \partial_{x[2]}, \ldots, \partial_{x[nsam(\vec{x})-1]}] \tag{14}$$

$$\begin{aligned} P(\vec{x}) &\approx [\prod_x P(x)^{P(x)}]^{nsam(\vec{x})} && (15)\\ &= e^{nsam(\vec{x})\sum_x P(x)\ln P(x)} && (16)\\ &= e^{-nsam(\vec{x})H(P_{\underline{x}})} && (17) \end{aligned}$$

$$f^{[1,\partial_x,\partial_y]}(x,y) = [f, \partial_x f, \partial_y f] \tag{18}$$

$$f^+ = f^{[1,\partial_x,\partial_y]} \tag{19}$$

For probabilty distributions $p(x), q(x)$ of $x \in S_{\underline{x}}$

- Entropy:

$$H(p) = -\sum_x p(x) \ln p(x) \geq 0 \tag{20}$$

- Kullback-Liebler divergence:

$$D_{KL}(p \parallel q) = \sum_x p(x) \ln \frac{p(x)}{q(x)} \geq 0 \tag{21}$$

- Cross entropy:

$$\begin{aligned} CE(p \to q) &= -\sum_x p(x) \ln q(x) && (22)\\ &= H(p) + D_{KL}(p \parallel q) && (23) \end{aligned}$$

Normal Distribution: $x, \mu, \sigma \in \mathbb{R}$, $\sigma > 0$

$$\mathcal{N}(\mu, \sigma^2)(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{24}$$

Uniform Distribution: $a < b$, $x \in [a, b]$

$$\mathcal{U}(a, b)(x) = \frac{1}{b-a} \tag{25}$$

Expected Value

Given a random variable $\underline{x}$ with states $S_{\underline{x}}$ and a function $f : S_{\underline{x}} \to \mathbb{R}$, define

$$E_{\underline{x}}[f(\underline{x})] = E_{x\sim P(x)}[f(x)] = \sum_x P(x) f(x) \tag{26}$$

Conditional Expected Value

Given a random variable $\underline{x}$ with states $S_{\underline{x}}$, a random variable $\underline{y}$ with states $S_{\underline{y}}$, and a function $f : S_{\underline{x}} \times S_{\underline{y}} \to \mathbb{R}$, define

$$E_{\underline{x}|\underline{y}}[f(\underline{x}, \underline{y})] = \sum_x P(x|\underline{y}) f(x, \underline{y}) \ , \tag{27}$$

$$E_{\underline{x}|\underline{y}=y}[f(\underline{x}, y)] = E_{\underline{x}|y}[f(\underline{x}, y)] = \sum_x P(x|y) f(x, y) \ . \tag{28}$$

Note that

$$\begin{aligned} E_{\underline{y}}[E_{\underline{x}|\underline{y}}[f(\underline{x}, \underline{y})]] &= \sum_{x,y} P(x|y)P(y) f(x, y) && (29) \\ &= \sum_{x,y} P(x, y) f(x, y) && (30) \\ &= E_{\underline{x},\underline{y}}[f(\underline{x}, \underline{y})] \ . && (31) \end{aligned}$$

Sigmoid function: For $x \in \mathbb{R}$,

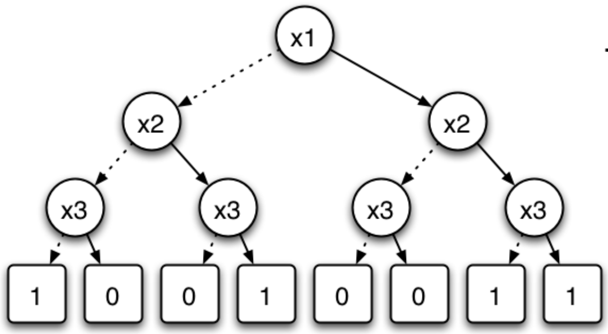
$$\mathrm{sig}(x) = \frac{1}{1 + e^{-x}} \tag{32}$$

$\mathcal{N}(!a)$ will denote a normalization constant that does not depend on a. For example, $P(x) = \mathcal{N}(!x)e^{-x}$ where $\int_0^\infty dx\ P(x) = 1$.

A **one hot** vector of zeros and ones is a vector with all entries zero with the exception of a single entry which is one. A **one cold** vector has all entries equal to one with the exception of a single entry which is zero. For example, if $x^n = (x_0, x_1, \dots, x_{n-1})$ and $x_i = \delta(i, 0)$ then x^n is one hot.

Chapter 4

Binary Decision Diagrams



x1	x2	x3	*f*
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Figure 4.1: Binary decision tree and truth table for the function $f(x_1, x_2, x_3) = \bar{x}_1(x_2 + \bar{x}_3) + x_1 x_2$

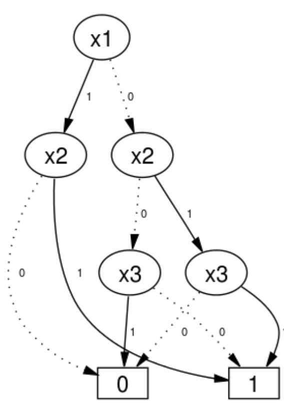


Figure 4.2: BDD for the function f of Fig.4.1.

This chapter is based on Wikipedia article Ref.[2].

Binary Decision Diagrams (BDDs) can be understood as a special case of Decision Trees (dtrees). We will assume that the reader has read Chapter 5 on dtrees before reading this chapter.

Both Figs.4.1 and 4.2 were taken from the aforementioned Wikipedia article. They give a simple example of a function $f : \{0,1\}^3 \rightarrow \{0,1\}$ represented in Fig.4.1 as a **binary decision tree** and in Fig.4.2 as a **binary decision diagram (BDD)**. The goal of this chapter is to find for each of those figures a bnet with the same graph structure.

We begin by noting that the function $f : \{0,1\}^3 \rightarrow \{0,1\}$ is a special case of a probability distribution $P : \{0,1\}^3 \rightarrow [0,1]$. In fact, if we restrict P to be deterministic, then $P_{det} : \{0,1\}^3 \rightarrow \{0,1\}$ has the same domain and range as f. Henceforth, we will refer to $f(x_1,x_2,x_3)$ as $P(x_1,x_2,x_3)$, keeping in mind that we are restricting our attention to deterministic probability distributions.

If we apply the chain rule for conditional probabilities to $P(x_1,x_2,x_3)$, we get

$$P(x_1,x_2,x_3) = P(x_3|x_1,x_2)P(x_2|x_1)P(x_1) \ , \tag{4.1}$$

which can be represented by the bnet:

$$\begin{tikzcd}
\underline{x}_1 \arrow[d] \arrow[dd, bend left] \\
\underline{x}_2 \arrow[d] \\
\underline{x}_3
\end{tikzcd} \tag{4.2}$$

But in Chapter 5, we learned how to represent the bnet of Eq.(4.2) as the bnet tree Eq.(4.3). In that tree, the nodes pose questions with 3 possible answers $0, 1, null$. In Eq.(4.3), $x_2|a?$ stands for "what is x_2 if $x_1 = a$?" and $x_3|a,b?$ stands for "what is x_3 if $x_1 = a, x_2 = b$?".

$$\begin{tikzcd}
\underline{x_1?} \arrow[d] \arrow[dr] & & & \\
\underline{x_2|0?} \arrow[d] \arrow[dr] & \underline{x_2|1?} \arrow[dr] \arrow[drr] & & \\
\underline{x_3|00?} & \underline{x_3|01?} & \underline{x_3|10?} & \underline{x_3|11?}
\end{tikzcd} \tag{4.3}$$

The node transition probability matrices, printed in blue, for the bnet of Eq.(4.3) are as follows. If $x_1, x_2, x_3 \in \{0, 1, null\}$ and $a, b \in \{0, 1\}$, then

$$P(\underline{x_1?} = x_1) = \left\{ \begin{array}{ll} P_{\underline{x}_1}(x_1) & \text{if } x_1 \in \{0,1\} \\ 0 & \text{if } x_1 = null \end{array} \right. \tag{4.4}$$

$$P(\underline{x_2|a?} = x_2 \mid \underline{x_1?} = x_1) = \left\{ \begin{array}{ll} P_{\underline{x}_2|\underline{x}_1}(x_2|a) & \text{if } x_1 = a \\ \mathbb{1}(x_2 = null) & \text{otherwise} \end{array} \right. \tag{4.5}$$

$$P(\underline{x_3|a,b?} = x_3 \mid \underline{x_2|b?} = x_2) = \left\{ \begin{array}{ll} P_{\underline{x}_3|\underline{x}_1,\underline{x}_2}(x_3|a,b) & \text{if } (x_1,x_2)=(a,b) \\ \mathbb{1}(x_3 = null) & \text{otherwise} \end{array} \right. \tag{4.6}$$

The bnet shown in Eq.(4.3) contains the same info and has the same graph structure as the binary decision tree Fig.4.1. As when we were converting dtrees to their image bnets, the info in the endpoint nodes of Fig.4.1 is implicit in the transition matrices of the image bnet Eq.(4.3). If one wants to make the endpoint node info more explicit in the image bnet, one can add it to the descriptors of the state names of the leaf nodes of the image bnet. For example, one can add descriptors "gives $f=0$" or "gives $f=1$" to the "0" or "1" states of those leaf nodes.

The BDD shown in Fig.4.2 emphasizes the fact that

$$P(x_1, x_2, x_3|x_1 = 1) = P(x_2|x_1 = 1) = x_2 \; . \tag{4.7}$$

The BDD of Fig.4.2 corresponds to the bnet of Eq.(4.8).

$$\begin{tikzcd} \underline{x_1?} \arrow[d] \arrow[dr] & & & \\ \underline{x_2|0?} \arrow[d] \arrow[dr] & \underline{x_2|1?} & & \\ \underline{x_3|00?} & \underline{x_3|01?} & \bullet & \bullet \end{tikzcd} \tag{4.8}$$

What happens if we consider an f for which $P(x_3|x_1,x_2) = P(x_3|x_2)$ so that one of the arcs of the fully connected bnet Eq.(4.2) is unnecessary? In that case,

$$P(x_1, x_2, x_3) = P(x_3|x_2)P(x_2|x_1)P(x_1) \; , \tag{4.9}$$

which can be represented by the Markov chain bnet:

$$\begin{tikzcd} \underline{x}_1 \arrow[d] \\ \underline{x}_2 \arrow[d] \\ \underline{x}_3 \end{tikzcd} \; . \tag{4.10}$$

Following the prescriptions of Chapter 5, we can represent the bnet of Eq.(4.10) as the bnet tree Eq.(4.11). In that tree, the nodes pose questions with 3 possible answers $0, 1, null$.

$$
\begin{tikzcd}
\underline{x_1?} \arrow[d] \arrow[dr] & & & \\
\underline{x_2|0?} \arrow[d] \arrow[dr] & \underline{x_2|1?} \arrow[dr] \arrow[drr] & & \\
\underline{x_3|_{\_},0?} & \underline{x_3|_{\_},1?} & \underline{x_3|_{\_},0?} & \underline{x_3|_{\_},1?}
\end{tikzcd}
\tag{4.11}
$$

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