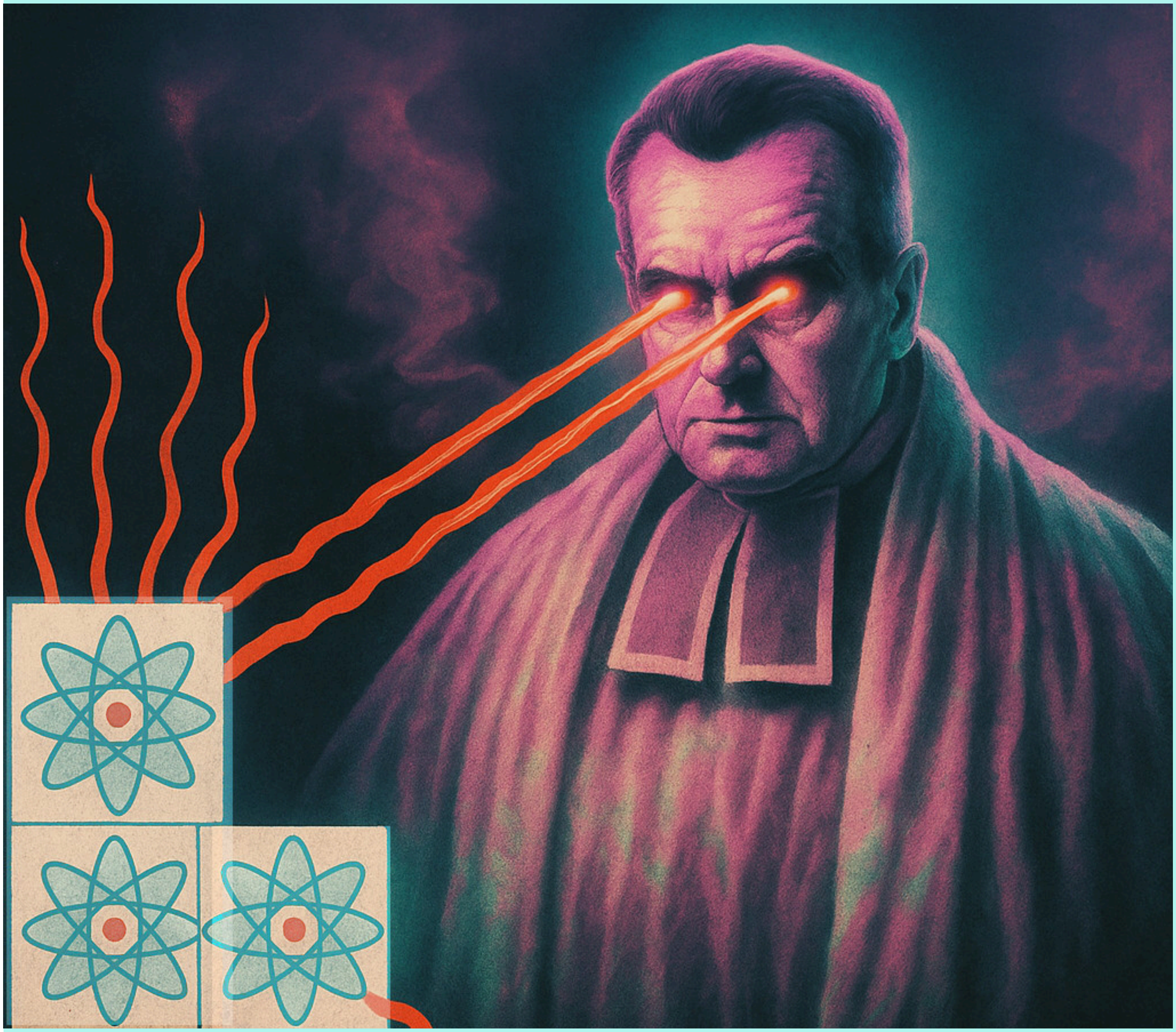
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayes Quantico
by Robert R. Tucci

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Appendix A

Notational Conventions and Preliminaries

A.1 Group

A **group** $\mathcal{G}$ is a set of elements with a multiplication map $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ such that

1. the multiplication is **associative** ; i.e.,

$$(ab)c = a(bc) \tag{A.1}$$

   for $a, b, c \in \mathcal{G}$.

2. there exists an **identity element** $e \in \mathcal{G}$ such that

$$ea = ae = a \tag{A.2}$$

   for all $a \in \mathcal{G}$

3. for any $g \in \mathcal{G}$, there exists an **inverse** $a^{-1} \in \mathcal{G}$ such that

$$aa^{-1} = a^{-1}a = e \tag{A.3}$$

The number of elements in any set S is denoted by $|S|$. $|\mathcal{G}|$ is called the **order** of the group.

If multiplication is **commutative** (i.e., $ab = ba$ for all $a, b \in \mathcal{G}$, the group is said to be **abelian**.

A **subgroup** $\mathcal{H}$ of $\mathcal{G}$ is a subset of $\mathcal{G}$ ($\mathcal{H} \subset \mathcal{G}$) which is also a group. It's easy to show that any $\mathcal{H} \subset \mathcal{G}$ is a group if it contains the identity and is **closed under multiplication** (i.e., $ab \in \mathcal{H}$ for all $a, b \in \mathcal{H}$)

A.2 Group Representation

A **group representation** of a group $\mathcal{G}$ is a map $\phi : \mathcal{G} \to \mathbb{C}^{n\times n}$[1] such that

$$\phi(a)\phi(b) = \phi(ab) \tag{A.4}$$

Such a map is called a **homomorphism**. When a group is defined using matrices, those matrices are called the **defining representation**. The map ϕ partitions $\mathcal{G}$ into disjoints subsets (equivalence classes), such that all elements of $\mathcal{G}$ in a disjoint set are represented by the same matrix.

For example, the group of **General Linear Transformations** is defined by

$$GL(n, \mathbb{C}) = \{M \in \mathbb{C}^{n\times n} : \det M \neq 0\} \tag{A.5}$$

A.3 Vector Notation

$(x_1, x_2, \ldots, x_n) = x^{:n} \in V^n = \mathbb{C}^{n\times 1}$

$y^b = \sum_b g^{ba} x^{:n}$

$(y^1, y^2, \ldots, y^n) = \bar{y}^{:n} \in \bar{V}^n = \mathbb{C}^{n\times 1}$. V^n and $\bar{V}^n$ are **dual vector spaces**.

Reverse of vector $rev(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_1)$

Implicit Summation Convention

$$G^b_a x_b = \sum_{b=1}^{n} G^b_a x_b \tag{A.6}$$

Suppose $G \in \mathcal{G} \subset GL(n, \mathbb{C})$ and $a_i, b_i \in \mathbb{Z}_{[1,n]}$. For $x \in V^n$,

$$(x')_a = G_a{}^b x_b \tag{A.7}$$

For $x \in \bar{V}^n$,

$$(x')^a = x^b (G^\dagger)_b{}^a = G^a{}_b x^b \tag{A.8}$$

so

$$\begin{aligned} (G^\dagger)_b{}^a &= G^a{}_b && \text{(A.9)} \\ &= (G_a{}^b)^* \quad \text{(only if } G \text{ is a unitary matrix)} && \text{(A.10)} \end{aligned}$$

[1] More generally, the $\mathbb{C}^{n\times n}$ can be replaced by $\mathbb{R}^{n\times n}$ or by $\mathbb{F}^{n\times n}$ for any field $\mathbb{F}$

A.4 Tensors

Suppose $a_i, b_i, c_i, d_i \in \mathbb{Z}_{[1,n]}$.

Note that for $x \in V^n$, $y \in \bar{V}^n$, and $G \in \mathcal{G} \subset GL(n, \mathbb{C})$,

$$(x')_b (y')^a = G^a{}_c G_b{}^d x_d y^c \tag{A.11}$$

For $x \in V^{n^p} \otimes \bar{V}^{n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{b^{\cdot p}}{}^{a^{\cdot q}} = \mathbb{G}_{b^{\cdot p}}{}^{a^{\cdot q}}{}_{rev(c^{\cdot q})}{}^{rev(d^{\cdot p})} x_{d^{\cdot p}}{}^{c^{\cdot q}} \tag{A.12}$$

where we define

$$\mathbb{G}_{b^{\cdot p}}{}^{a^{\cdot q}}{}_{rev(c^{\cdot q})}{}^{rev(d^{\cdot p})} \stackrel{\text{def}}{=} \prod_{i=1}^{q} G^{a_i}{}_{c_i} \prod_{i=1}^{p} G_{b_i}{}^{d_i} \tag{A.13}$$

Hermitian conjugation

$$(M^\dagger)_a{}^b = (M_b{}^a)^* \tag{A.14}$$

Hermitian matrix

$$M^\dagger = M, \quad (M^\dagger)_a{}^b = M_a{}^b \tag{A.15}$$

$$(M^\dagger)_{ab}{}^{cde} = (M_{ed}{}^{cba})^* \tag{A.16}$$

An issue that arises with tensors is this: When is it permissible to represent a tensor by T^{cd}_{ab}? If we define T^{cd}_{ab} by

$$T^{cd}_{ab} = T_{ab}{}^{cd} \tag{A.17}$$

then it's always permissible. Then one can define tensors like $T_a{}^{bcd}$ as

$$T_a{}^{bcd} = g^{bb'} T_{ab'}{}^{cd} = g^{bb'} T^{cd}_{ab'} \tag{A.18}$$

Hence, one drawback of using the notation T^{cd}_{ab} is that if one is interested in using versions of T^{cd}_{ab} with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing $T_a{}^{bcd}$, you'll have to write $g^{bb'} T^{cd}_{ab'}$. This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too elegantly concise.

A.5 Invariance

Given a **bilinear form**

$$m(\bar{x}^{:n}, y^{:n}) = x^a M_a{}^b y_b \tag{A.19}$$

is invariant if

$$m(\bar{x}^{:n}, y^{:n}) = m(\bar{x}^{:n} G^\dagger, G y^{:n}) \tag{A.20}$$

invariant matrix

$$M_a{}^b = (G^\dagger)_a{}^{a'} G_{b'}{}^b M_{a'}{}^{b'} \tag{A.21}$$

$$M = G^\dagger M G \tag{A.22}$$

$$GM = MG, \quad [G, M] = 0 \tag{A.23}$$

multilinear form

$$h(\bar{w}, \bar{x}, y, z) = h_{ab}{}^{cd} w^a x^b y_c z_d \tag{A.24}$$

is invariant if

$$h(\bar{w}, \bar{x}, y, z) = h(\bar{w} G^\dagger, \bar{x} G^\dagger, Gy, Gz) \tag{A.25}$$

invariant tensor

$$h_{ab}{}^{cd} = (G^\dagger)_a{}^{a'} (G^\dagger)_b{}^{b'} h_{a'b'}{}^{c'd'} G_{c'}{}^c G_{d'}{}^d \tag{A.26}$$

A.6 Algebras

A.7 Spectral Decomposition and Eigenvalue Projection Operators

$M \in \mathbb{C}^{d \times d}$

$$M|v\rangle = \lambda |v\rangle \tag{A.27}$$

If M is Hermitian ($H^\dagger = H$), its eigenvalues are real. ($\lambda = \langle \lambda | M \lambda \rangle \in \mathbb{R}$)

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M - \lambda) = 0 \tag{A.28}$$

If M is a Hermitain matrix, then there exists a unitary matric ($CC^\dagger = C^\dagger C = 1$) such that

$$CMC^\dagger = \begin{bmatrix} D_{\lambda_1} & 0 & 0 & 0 \\ 0 & D_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{\lambda_r} \end{bmatrix} \tag{A.29}$$

where

$$D_{\lambda_i} = \mathrm{diag}\,\underbrace{(\lambda_i, \lambda_i, \ldots, \lambda_i)}_{d_i \text{ times}} \tag{A.30}$$

$$d = \sum_{i=1}^{r} d_i \tag{A.31}$$

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{A.32}$$

$$CP_1C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2} \tag{A.33}$$

$$CP_2C^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1} \tag{A.34}$$

If $I^{d_i \times d_i}$ is the d_i dimensional unit matrix,

$$P_i = C^\dagger diag(0, \ldots, 0, I^{d_i \times d_i}, 0, \ldots, 0)C \tag{A.35}$$

$$= \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \tag{A.36}$$

Note that P_i are Hermitian ($P_i^\dagger = P_i$) because M is Hermitian and its eigenvalues are real.)

Note that P_i and M commute

$$[P_i, M] = P_iM - MP_i = 0 \tag{A.37}$$

orthogonal

$$P_iP_j = \delta(i, j)P_j \tag{A.38}$$

complete

$$\sum_i P_i = 1 \tag{A.39}$$

$$M = \sum_{i=1}^{r} P_iMP_i \tag{A.40}$$

$$d_i = \mathrm{tr} P_i \tag{A.41}$$

$$CMP_1C^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{A.42}$$

$$= \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{A.43}$$

$$MP_i = \lambda_i P_i \text{ (no } i \text{ sum)} \tag{A.44}$$

$$f(M)P_i = f(\lambda_i)P_i \text{ (no } i \text{ sum)} \tag{A.45}$$

$M^{(1)}, M^{(2)}$

$$[M^{(1)}, M^{(2)}] = 0 \tag{A.46}$$

Use $M^{(1)}$ to decompose V into $\bigoplus_i V_i$. Use $M^{(2)}$ to decompose V_i into $\bigoplus_j V_{i,j}$. If $M^{(1)}$ and $M^{(2)}$ don't commute, let $P_i^{(1)}$ be the eigenvalue projection operators of $M^{(1)}$. The replace $M^{(2)}$ by $P_i^{(1)} M^{(2)} P_i^{(1)}$

$$[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] = 0 \tag{A.47}$$

Appendix B

Birdtracks: COMING SOON

Cvitanovic Birdtracks book [1]
Elliott-Dawber book [2]
My paper “Quantum Bayesian Nets” [3]

B.1 Classical Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $P(y|x) \in [0,1]$ where $x \in val(\underline{x})$ and $y \in val(\underline{y})$

$$\sum_{y \in val(\underline{y})} P(y|x) = 1 \tag{B.1}$$

$$\mathcal{C} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ul] \arrow[ll] \end{tikzcd} \tag{B.2}$$

$$\mathcal{C}(a,b,c) = P(c|b,a)P(b|a)P(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ul] \arrow[ll] \end{tikzcd} \quad P(a) \tag{B.3}$$

$a^{:2} = (a_1, a_2)$

$$\mathcal{C}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^{:2} \arrow[ul, "\underline{a}_1"'] \arrow[ll, "\underline{a}_2"'] \end{tikzcd} \tag{B.4}$$

$$\mathcal{C}'(a^{:2},b,c) = P(c|b,a_2)P(a_2|a^{:2})P(b|a_1)P(a_1|a^{:2})P(a^{:2}) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^{:2} \arrow[ul, "a_1"'] \arrow[ll, "a_2"'] \end{tikzcd} \quad P(a^{:2}) \tag{B.5}$$

Marginalizer nodes $\underline{a}_1$ and $\underline{a}_2$ have the TPMs

$$P(a'_i|\underline{a}^{:2} = (a_1, a_2)) = \delta(a'_i, a_i) \tag{B.6}$$

for $i = 1, 2$

B.2 Quantum Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $A(y|x) \in \mathbb{C}$ where $x \in val(\underline{x})$ and $y \in val(\underline{y})$

$$\sum_{y \in val(\underline{y})} |A(y|x)|^2 = 1 \tag{B.7}$$

$$\mathcal{Q} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ul] \arrow[ll] \end{tikzcd} \tag{B.8}$$

$$\mathcal{Q}(a, b, c) = A(c|b, a)A(b|a)A(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ul] \arrow[ll] \end{tikzcd} \quad A(a) \tag{B.9}$$

$a^{:2} = (a_1, a_2)$

$$\mathcal{Q}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^{:2} \arrow[ul, "\underline{a}_1"'] \arrow[ll, "\underline{a}_2"] \end{tikzcd} \tag{B.10}$$

$$\mathcal{Q}'(a^{:2}, b, c) = A(c|b, a_2)A(a_2|a^{:2})A(b|a_1)A(a_1|a^{:2})A(a^{:2}) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^{:2} \arrow[ul, "a_1"'] \arrow[ll, "a_2"] \end{tikzcd} \quad A(a^{:2}) \tag{B.11}$$

Marginalizer nodes $\underline{a}_1$ and $\underline{a}_2$ have the TAMs

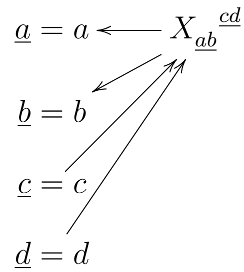
$$A(a'_i|\underline{a}^{:2} = (a_1, a_2)) = \delta(a'_i, a_i) \tag{B.12}$$

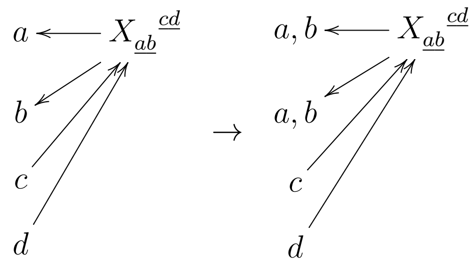
for $i = 1, 2$

B.3 Birdtracks

$$\delta(b,a) = \mathbb{1}(a=b) = \delta_a^b = \; a \leftarrow\!\bullet\!\!- \, b \tag{B.13}$$

$$\langle a, b | X_{\underline{ab}}{}^{\underline{cd}} | c, d\rangle = X_{ab}{}^{cd} = \tag{B.14}$$

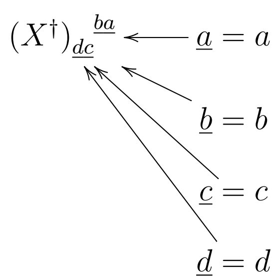




(B.15)

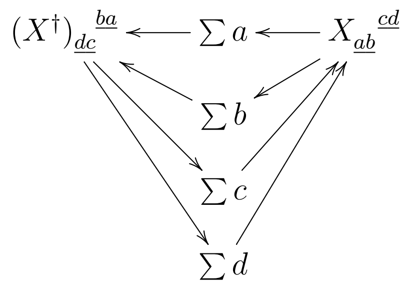
$X_{\underline{ab}}{}^{\underline{cd}} \in V^2 \otimes V_2$. Sometimes, we will omit denote this node simply by X. This if okay as long as we are not using, X to also denote a different version of $X_{\underline{ab}}{}^{\underline{cd}}$ with some of the indices raised or lowered or their order has been changed. [1]

$$(X^\dagger)_{dc}{}^{ba} = \tag{B.16}$$

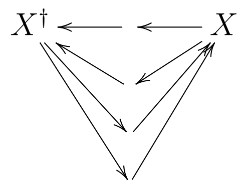


[1]For matrices, $(A^\dagger)_{i,j} = (A_{j,i})^*$ so taking a Hermitian conjugate involves both taking the complex conjugate of the matrix element and reversing the left-to-right (L2R) order of its indices. This generalizes to $(X^\dagger)_{dc}{}^{ba} = (X_{ab}{}^{cd})^*$. Besides raising and lowering indices, we reverse their L2R order.

$$(X^\dagger)_{dc}{}^{ba} X_{ab}{}^{cd} \quad = \tag{B.17}$$



$$= \tag{B.18}$$



Birdtracks originated as a graphical way to represent the tensors in General Relativity (Gravitation). In General Relativity, one deals with tensors such as $T_a{}^b{}_c$ which have some indices raised and some lowered. One can use the metric $g^{a,b}$ to raise all the lowered indices to get T^{abc}. If we represent this graphically as a node with incoming arrows a, b, c, we need to follow one of the following 2 conventions: either

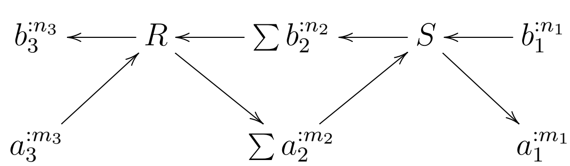
1. label the arrows as $\underline{a}$, $\underline{b}, \underline{c}$, and define the node as $T^{\underline{abc}}$, or

2. instead of labelling the arrows explicitly $\underline{a}, \underline{b}, \underline{c}$, indicate in the node where is the first arrow $\underline{a}$, and draw the arrows $\underline{a}, \underline{b}, \underline{c}$ so that they enter the node in **counterclockwise** (CC) order. The **left-to-right** (L2R) order of the indices on T corresponds the CC order of the arrows.

If we don't do either 1 or 2, we won't be able to distinguish between the graphical representations of $T^{1,2,3}$ and $T^{2,1,3}$, for example. Cvitanovic's Birdtracks book Ref.[1] follows Convention 2, but most of the time, in this book, we will follow Convention 1 [2] The reason I chose to do so is for the sake of consistency: Convention 2 is closer to the quantum bnet conventions.

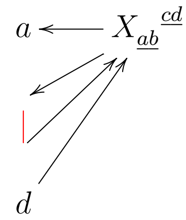
$a^{:m} \in \mathbb{Z}_+^m$

$$R^{a_3^{:m_3}, b_2^{:n_2}}_{b_3^{:n_3}, a_2^{:m_2}} S^{a_2^{:m_2}, b_1^{:n_1}}_{b_2^{:n_2}, a_1^{:m_1}} = \tag{B.19}$$



[2]If we follow Convention 1, we don't need to reverse the L2R order of the indices when taking a Hermitian conjugate. Thus, $(X^\dagger)^{\underline{ab}}{}_{\underline{cd}} = X_{\underline{ab}}{}^{\underline{cd}} = X^{\underline{dc}}{}_{\underline{ba}}$. As long as $\underline{a}, \underline{b}$ are lower indices and $\underline{c}, \underline{d}$ are upper indices of X, any L2R order of $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ is equivalent under Convention 1.

$$\mathrm{tr}_{\underline{b}} X_{a\underline{b}}{}^{\underline{b}d} = \sum_b X_{ab}{}^{bd} = \tag{B.20}$$



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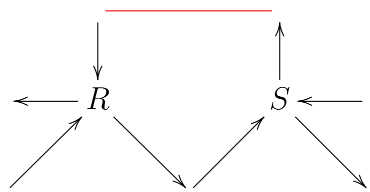


Chapter 1

Casimir Operators: COMING SOON

Chapter 2

Clebsch-Gordan Coefficients: COMING SOON

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d \times d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} C^{d \times d} \tag{2.1}$$

Let $b^{:nb} = (b_1, b_2, \ldots, b_{nb})$ where $b_i \in Z_{[0, db_i]}$ and $a \in Z_{[1, d_\lambda]}$. Hence,

$$d_\lambda = \prod_{i=1}^{:nb} db_i \tag{2.2}$$

$$(C_\lambda)_a^{b^{:nb}} = \begin{tikzcd} & & b_1 \arrow[dl] \\ a & C_\lambda \arrow[l] & b_2 \arrow[l] \\ & & b_{nb} \arrow[ul] \end{tikzcd} \tag{2.3}$$

$$\begin{bmatrix} 0 & (C^\dagger)_\lambda^{d \times d_\lambda} & 0 \end{bmatrix}^{d \times d} = (C^\dagger)^{d \times d} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} \tag{2.4}$$

$$(C_\lambda^\dagger)^a_{b^{:nb}} = \begin{tikzcd} b_1 & & \\ b_2 & (C_\lambda^\dagger) \arrow[ul] \arrow[l] \arrow[dl] & a \arrow[l] \\ b_{nb} & & \end{tikzcd} \tag{2.5}$$

More generally, some of the b_i indices may lowered and their arrows changed to outgoing instead of ingoing. Each b_i represents a different rep (or irrep)

$$\boxed{(C^\dagger_\lambda)^{b^{:nb}}_a (C_\lambda)^a_{(b')^{:nb}} = (P_\lambda)^{b^{:nb}}_{(b')^{:nb}}}$$

$$\begin{tikzcd}
b_1 & & & & b'_1 \arrow[dl] \\
b_2 & (C^\dagger_\lambda) \arrow[l] \arrow[ul] \arrow[dl] & \sum a \arrow[l] & C_\lambda \arrow[l] & b'_2 \arrow[l] \\
b_{nb} & & & & b'_{nb} \arrow[ul]
\end{tikzcd}
\;=\; b^{:nb} \longleftarrow P_\lambda \longleftarrow (b')^{:nb} \tag{2.6}$$

$$\boxed{(C_\lambda)^{a'}_{b^{:nb}} (C^\dagger_\mu)^{b^{:nb}}_a = \delta(\lambda,\mu)\delta^{a'}_a}$$

$$\begin{tikzcd}
 & & \sum b_1 \arrow[dl] & & \\
a & C_\lambda \arrow[l] & \sum b_2 \arrow[l] & (C^\dagger_\mu) \arrow[l] \arrow[ul] \arrow[dl] & a' \arrow[l] \\
 & & \sum b_{nb} \arrow[ul] & &
\end{tikzcd}
\;=\; \delta(\mu,\lambda)\; a \longleftarrow\!\bullet\!\!-\!\!-\; a' \tag{2.7}$$

Chapter 3

Determinants: COMING SOON

Chapter 4

General Relativity Nets: COMING SOON

Chapter 5

Group Integrals: COMING SOON

Chapter 6

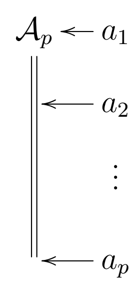
Levi-Civita Tensor

$$\epsilon^{123\ldots p} = \epsilon_{123\ldots p} = 1 \tag{6.1}$$

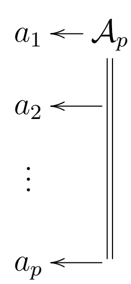
$$\epsilon_{rev(a^{:p})} = (-1)^{\binom{p}{2}}\epsilon_{a^{:p}} \tag{6.2}$$

where $rev(a^{:p})$ is the reverse of $a^{:p}$. $rev(a_1, a_2, \ldots, a_p) = (a_p, a_{p-1}, \ldots, a_1)$

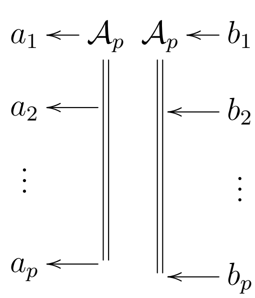
$$(C_{\mathcal{A}_p})_1^{a^{:p}} = e^{i\phi}\frac{\epsilon^{a^{:p}}}{\sqrt{p!}} = \tag{6.3}$$



$$(C^{\dagger}_{\mathcal{A}_p})^1_{a^{:p}} = e^{-i\phi}\frac{\epsilon_{a^{:p}}}{\sqrt{p!}} = \tag{6.4}$$



$$\boxed{\frac{1}{p!}\epsilon_{a^{:p}}\epsilon^{b^{:p}} = (\mathcal{A}_p)^{b^{:p}}_{a^{:p}}} \tag{6.5}$$



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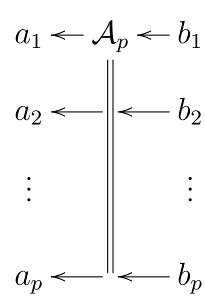
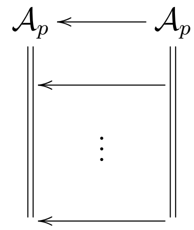


$$\boxed{e^{i2\phi}\frac{1}{p!}\epsilon^{a:n}\epsilon_{a:n} = \delta^1_1 = 1}$$



$$= 1 \tag{6.6}$$

For Convention 1, we will use $\phi = 0$.
For Convention 2, we must choose

$$e^{i2\phi} = (-1)^{\binom{p}{2}} = e^{i\pi\frac{p(p-1)}{2}} \tag{6.7}$$

so

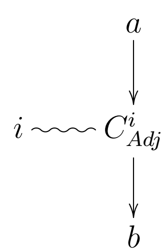
$$\phi = \frac{\pi}{4}p(p-1) \tag{6.8}$$

Chapter 7

Lie Algebra Definition: COMING SOON

$i \in \mathbb{Z}_{[1,N]},\ a, b \in \mathbb{Z}_{[1,n]}$

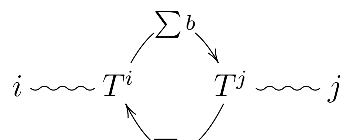
$$(C^i_{Adj})^a_b = \frac{1}{\sqrt{K}}(T^i)^a_b = $$



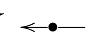
(7.1)

The matrices T^i are called the generators. It's customary to choose them so that they are Hermitian and $K = \frac{1}{2}$

$$\boxed{(T^i)^a_b (T^j)^b_a = \mathrm{tr}(T^i T^j) = K\delta(i,j)}$$

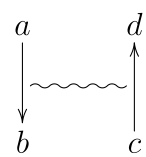


$= K$



(7.2)

$$(P_{Adj})^{a,c}_{b,d} = \sum_i \frac{1}{K}(T^i)^a_b (T^i)^c_d = \frac{1}{K}$$



(7.3)

$H \in V^{\underline{a}} \otimes V_{\underline{a}}$

$$(P_{Adj})^{ac}_{bd} H^d_c = \sum_i (T^i)^a_b \underbrace{\left[\frac{1}{K}(T^i)^c_d H^d_c\right]}_{h_i \in \mathbb{R}} \tag{7.4}$$

$G = 1 + iD \in \mathcal{G}$
$\epsilon_i \in \mathbb{R},\ |\epsilon_i| << 1$

$D = \sum_i \epsilon_i T^i = V^{\underline{a}} \otimes V_{\underline{a}}$

$$\mathcal{T}^i q = 0 \tag{7.5}$$

Chapter 8

Lie Algebra Classification, Dynkin Diagrams: COMING SOON

Chapter 9

Orthogonal Groups: COMING SOON

Chapter 10

Quantum Shannon Information Theory: COMING SOON

Chapter 11

Recoupling Equations: COMING SOON

Chapter 12

Reducibility: COMING SOON

Chapter 13

Spinors: COMING SOON

Chapter 14

Squashed Entanglement: COMING SOON

Chapter 15

Symplectic Groups: COMING SOON

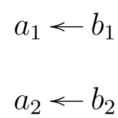
Chapter 16

Symmetrization and Antisymmetrization: COMING SOON

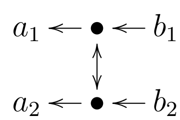
$(1,2)$ transposition, swaps 1 and 2, $1 \to 2 \to 1$. $(3,2,1)$ means $3 \to 2 \to 1 \to 3$. A reordering of $(1,2,3,\ldots,p)$ is a permutation on p letters. A permutation can be expressed as a product of transpositions $(3,2,1) = (3,2)(2,1)$ is an even permutation because it can be expressed as a product of an even number of transpositions. An odd permutation can be expressed as a product of an odd number of permutations.

16.1 Symmetrization

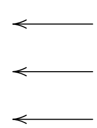
$$\mathbb{1}^{b_2,b_1}_{a_1,a_2} = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2} = \tag{16.1}$$



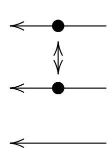
$$(\sigma_{(1,2)})^{b_2,b_1}_{a_1,a_2} = \delta^{b_2}_{a_1}\delta^{b_1}_{a_2} = \tag{16.2}$$

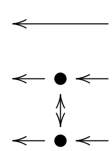


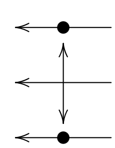
$$\mathbb{1} = \tag{16.3}$$



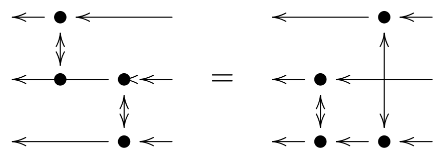
$$\sigma_{(1,2)} = \quad \sigma_{(2,3)} = \quad \sigma_{(1,3)} = \tag{16.4}$$





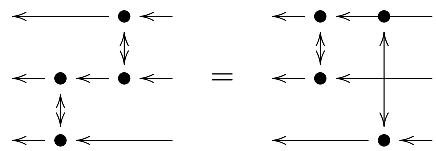


$\sigma_{(1,2,3)} =$

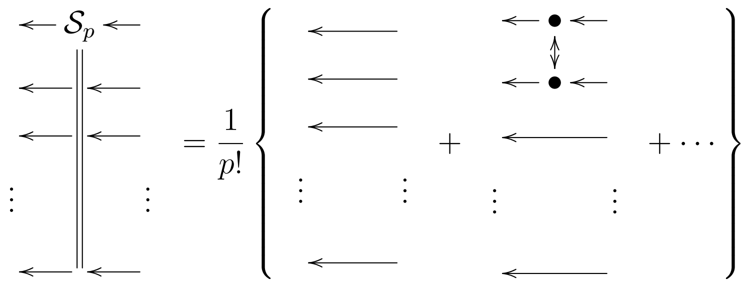


(16.5)

$\sigma_{(1,3,2)} =$

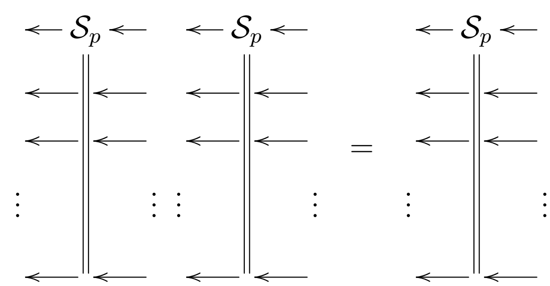


(16.6)



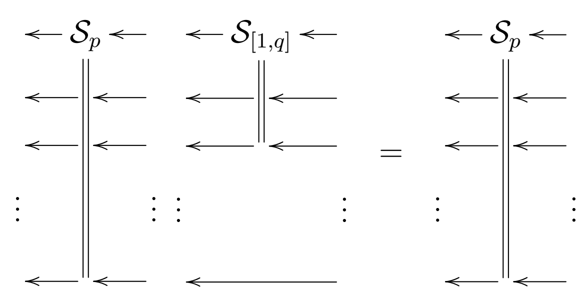
(16.7)

$\boxed{\mathcal{S}_p^2 = \mathcal{S}_p}$



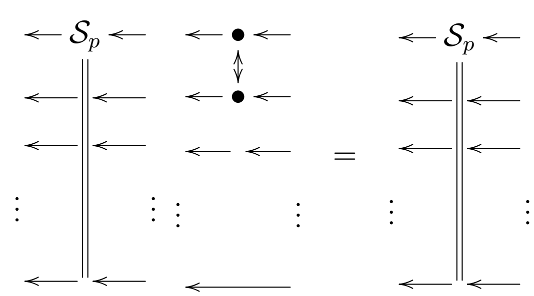
(16.8)

$\boxed{\mathcal{S}_p \mathcal{S}_{[1,q]} = \mathcal{S}_p}$



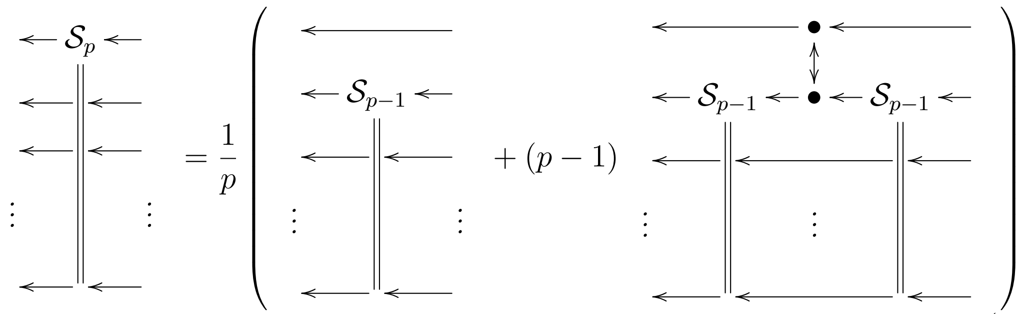
(16.9)

$\boxed{\mathcal{S}_p \sigma_{(1,2)} = \mathcal{S}_p}$



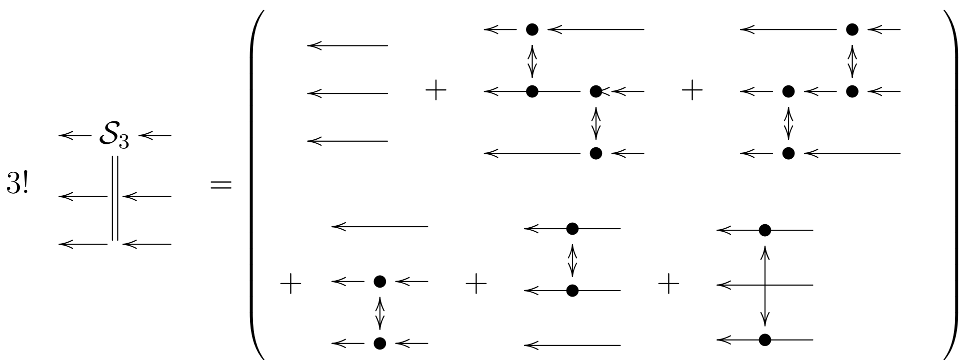
(16.10)

Claim 1

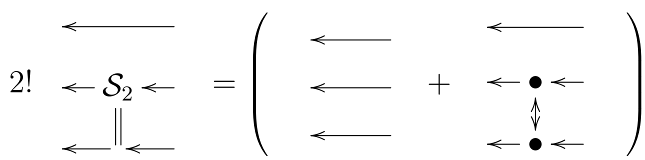


(16.11)

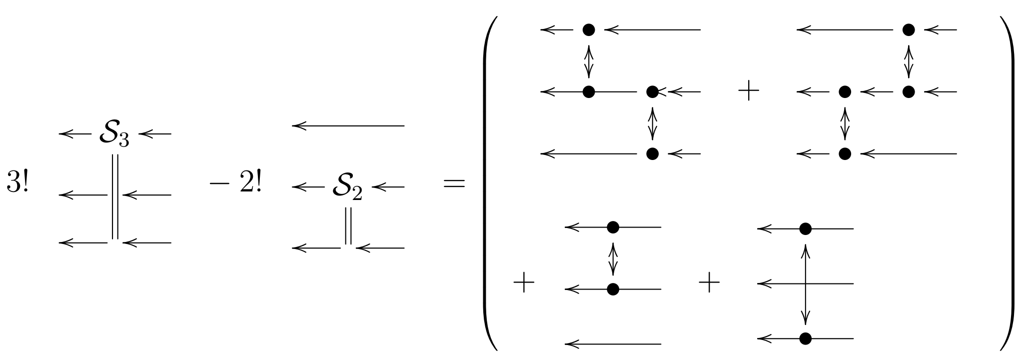
proof: We only prove it for $p = 3$.



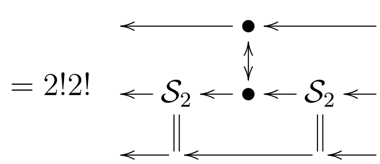
(16.12)



(16.13)

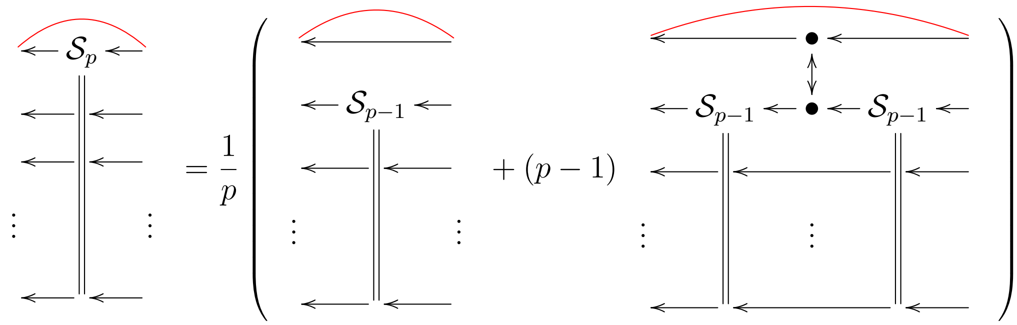


(16.14)

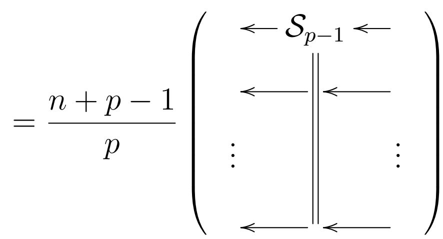


(16.15)

QED



(16.16)



(16.17)

$$\mathrm{tr}_{\underline{a}_1}\mathcal{S}_p = \frac{n+p-1}{p}\mathcal{S}_{p-1} \tag{16.18}$$

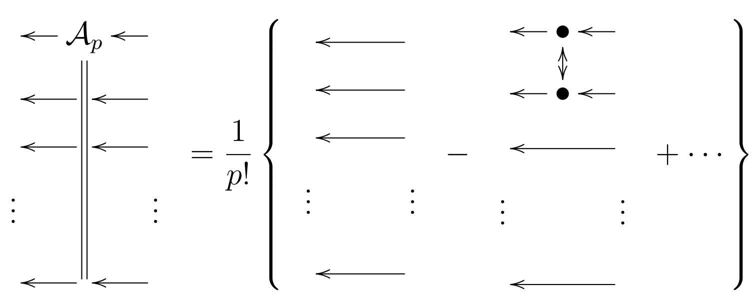
$$\mathrm{tr}_{\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_k}\mathcal{S}_p = \frac{(n+p-1)(n+p-2)\ldots(n=p-k)}{p(p-1)\ldots(p-k+1)}\mathcal{S}_{p-k} \tag{16.19}$$

$$d_{\mathcal{S}_p} = \mathrm{tr}_{\underline{a}^p}\mathcal{S}_p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \tag{16.20}$$

For $p = 2$,

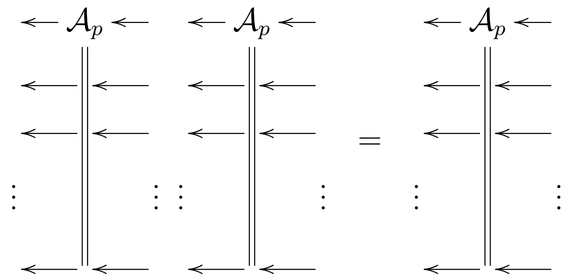
$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \tag{16.21}$$

16.2 Antisymmetrization

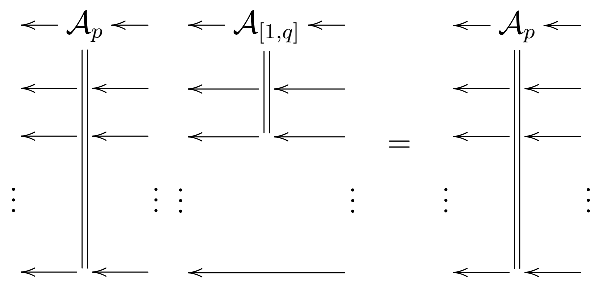


(16.22)

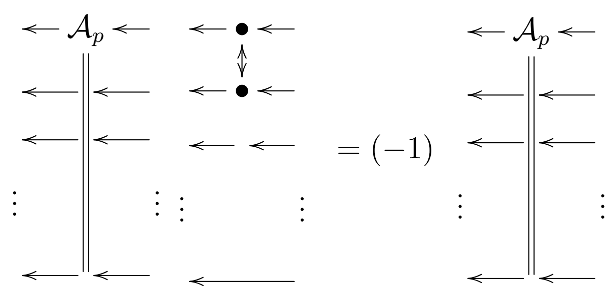
$$\boxed{\mathcal{A}_p^2 = \mathcal{A}_p} \tag{16.23}$$



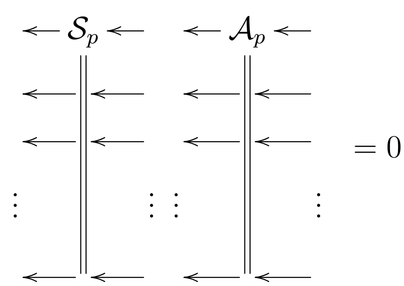
$$\boxed{\mathcal{A}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p} \tag{16.24}$$



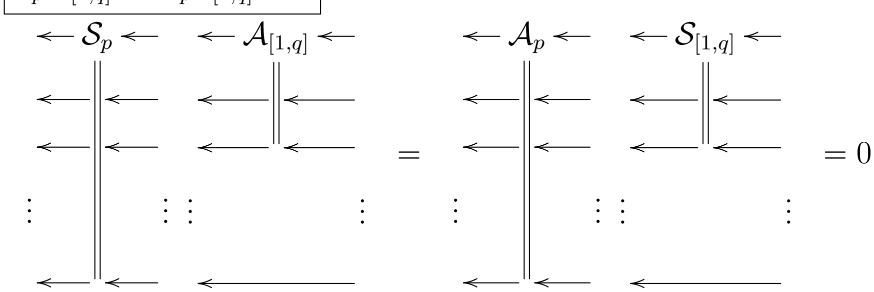
$$\boxed{\mathcal{A}_p \sigma_{(1,2)} = -\mathcal{A}_p} \tag{16.25}$$



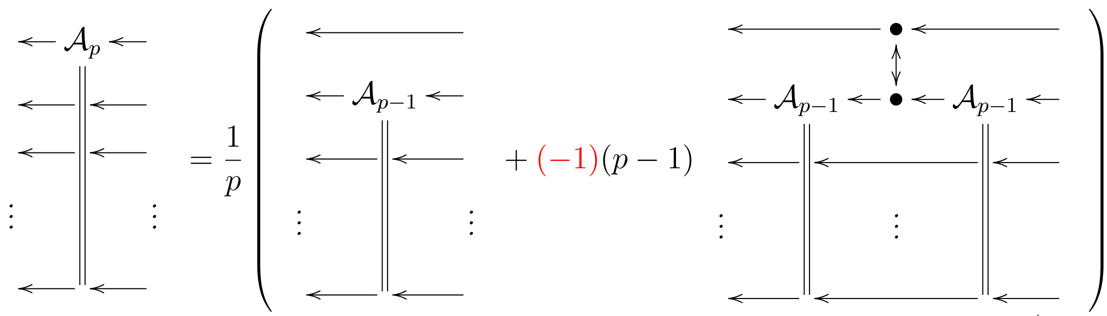
$$\boxed{\mathcal{S}_p \mathcal{A}_q = \mathcal{A}_p \mathcal{S}_q = 0} \tag{16.26}$$



$$\boxed{\mathcal{S}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p \mathcal{S}_{[1,q]} = 0} \tag{16.27}$$

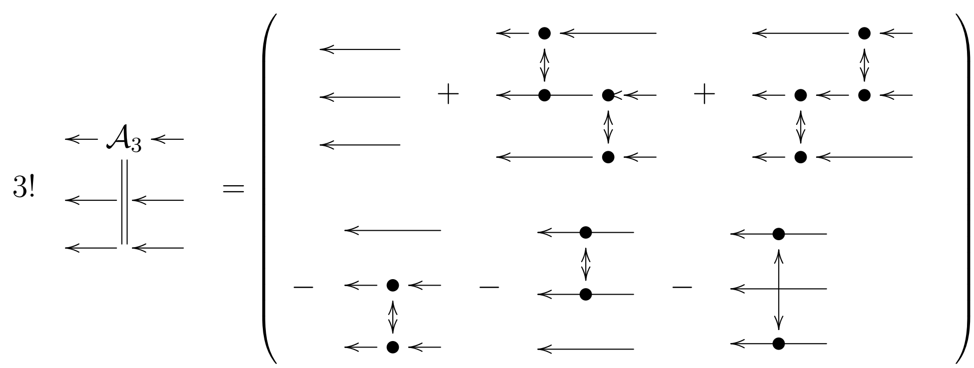


Claim 2

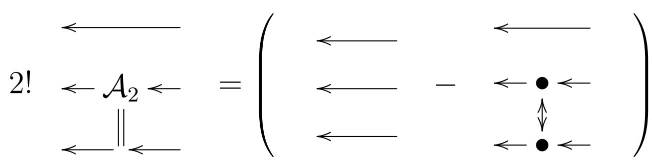


(16.28)

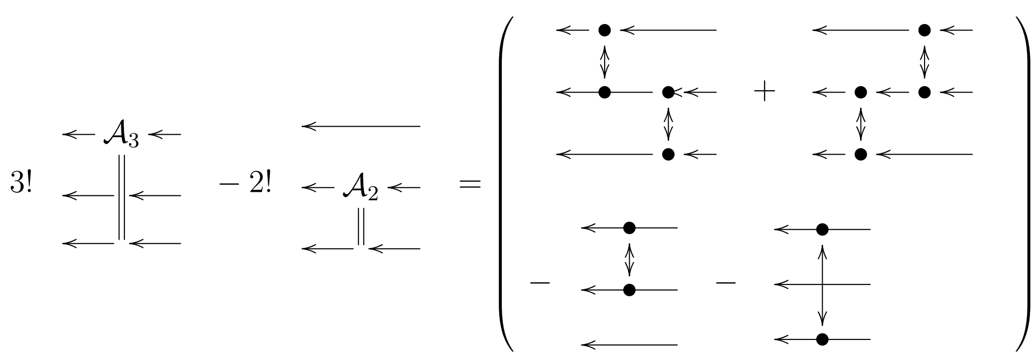
proof: We only prove it for $p = 3$.



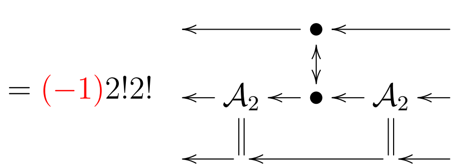
(16.29)



(16.30)

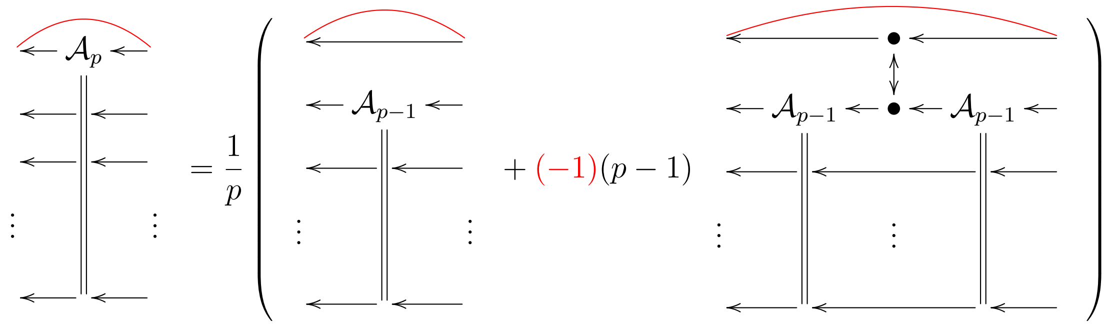


(16.31)



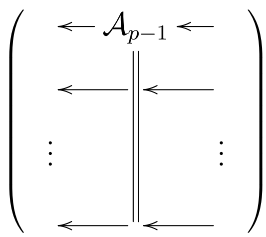
(16.32)

QED



(16.33)

$$= \frac{n + (-1)(p-1)}{p} \tag{16.34}$$



$$\mathrm{tr}_{\underline{a}_1} \mathcal{A}_p = \frac{n-p+1}{p} \mathcal{A}_{p-1} \tag{16.35}$$

$$\mathrm{tr}_{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k} \mathcal{A}_p = \frac{(n-p+1)(n-p+2)\ldots(n-p+k)}{p(p-1)\ldots(p-k+1)} \mathcal{A}_{p-k} \tag{16.36}$$

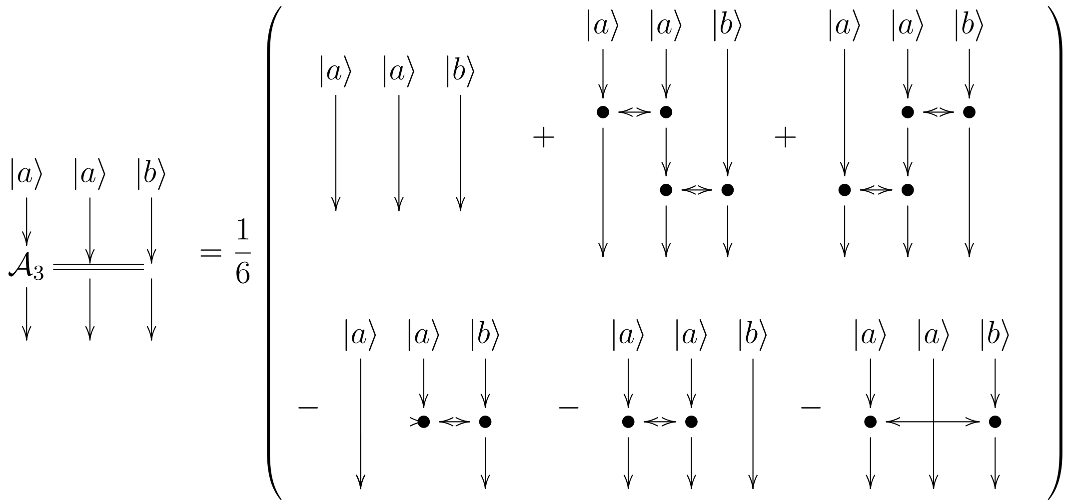
$$\begin{aligned} d_{\mathcal{A}_p} = \mathrm{tr}_{\underline{a}^p} \mathcal{A}_p &= \frac{\prod_{i=n-p+1}^{n} i}{p!} && (16.37) \\ &= \frac{\prod_{i=n}^{n-p+1} i}{p!} && (16.38) \\ &= \begin{cases} \frac{n!}{p!(n-p)!} = \binom{n}{p} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} && (16.39) \end{aligned}$$

For $p = 2 \leq n$,

$$d_{\mathcal{A}_2} = \binom{n}{2} \tag{16.40}$$

$$\mathcal{A}_p = 0 \text{ if } n < p \tag{16.41}$$

For example, for $n = 2$ and $p = 3$



(16.42)

$$
\begin{aligned}
\mathcal{A}_3|a,a,b\rangle &= \frac{1}{6}\begin{pmatrix} |a,a,b\rangle + |a,b,a\rangle + |b,a,a\rangle \\ -|a,b,a\rangle - |a,a,b\rangle - |b,a,a\rangle \end{pmatrix} && (16.43)\\
&= 0 && (16.44)
\end{aligned}
$$

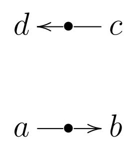
Chapter 17

Unitary Groups: COMING SOON

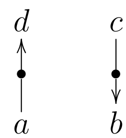
17.1 SU(n)

$$m(p,q) = \delta^a_b \sum_{a=1}^{n} (p_a)^* q_a \tag{17.1}$$

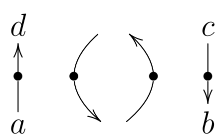
$$\mathbb{1}^{a,c}_{d,b} = \delta^a_b \delta^c_d = \tag{17.2}$$



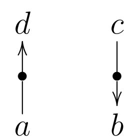
$$\uparrow\downarrow^{a,c}_{d,b} = \delta^a_d \delta^c_b = \tag{17.3}$$



$$\boxed{\uparrow\downarrow^2 = n \uparrow\downarrow} \tag{17.4}$$



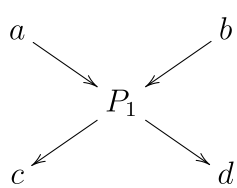
$= n$



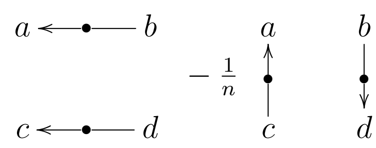
$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \tag{17.5}$$

$\lambda_1 = n$

$$\boxed{P_1 = \frac{\uparrow\downarrow - n}{0 - n} = 1 - \frac{1}{n} \uparrow\downarrow} \tag{17.6}$$

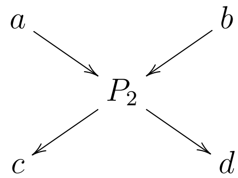


$=$

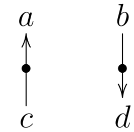


$\lambda_2 = 0$

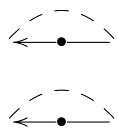
$$\boxed{P_2 = \frac{\uparrow\downarrow - 0}{n-0} = \frac{1}{n}\uparrow\downarrow} \qquad \text{(17.7)}$$



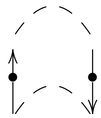
$= \frac{1}{n}$



$$\mathrm{tr}P_1 \ = \quad \text{(17.8)}$$

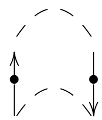


$- \frac{1}{n}$



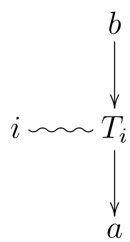
$$= \ n^2 - 1 \qquad \text{(17.9)}$$

$$\mathrm{tr}P_2 \ = \ \frac{1}{n} \quad \text{(17.10)}$$



$$= \ 1 \qquad \text{(17.11)}$$

$$(T_i)_a^b = \quad \text{(17.12)}$$



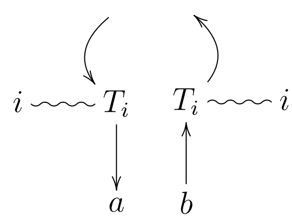
$$T_i^\dagger = T_i \qquad \text{(17.13)}$$

Claim 3

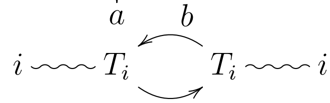
$$C_F \delta_a^b = (T_i T_i)_a^b = \frac{n^2-1}{n}\delta_a^b \qquad \text{(17.14)}$$

proof:

$$(T_i T_i)_a^b \ = \ \sum_i \quad \text{(17.15)}$$



$$= \ \sum_i \quad \text{(17.16)}$$



QED

Chapter 18

Wigner Coefficients: COMING SOON

Chapter 19

Wigner-Ekart Theorem: COMING SOON

Chapter 20

Young Tableau: COMING SOON

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