

BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF
QUANTUM BAYESIAN NETWORKS



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Bayesuvious Quantico, a visual dictionary of Quantum Bayesian Networks

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July 31, 2025

This book is constantly being expanded and improved. To download
the latest version, go to

<https://github.com/rrtucci/bayes-quantico>

Bayes Quantico

by Robert R. Tucci

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Appendices

Appendix A

Notational Conventions and Preliminaries

A.1 Group

A **group** \mathcal{G} is a set of elements with a multiplication map $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ such that

1. the multiplication is **associative** ; i.e.,

$$(ab)c = a(bc) \tag{A.1}$$

for $a, b, c \in \mathcal{G}$.

2. there exists an **identity element** $e \in \mathcal{G}$ such that

$$ea = ae = a \tag{A.2}$$

for all $a \in \mathcal{G}$

3. for any $g \in \mathcal{G}$, there exists an **inverse** $a^{-1} \in \mathcal{G}$ such that

$$aa^{-1} = a^{-1}a = e \tag{A.3}$$

The number of elements in any set S is denoted by $|S|$. $|\mathcal{G}|$ is called the **order** of the group.

If multiplication is **commutative** (i.e., $ab = ba$ for all $a, b \in \mathcal{G}$, the group is said to be **abelian**.

A **subgroup** \mathcal{H} of \mathcal{G} is a subset of \mathcal{G} ($\mathcal{H} \subset \mathcal{G}$) which is also a group. It's easy to show that any $\mathcal{H} \subset \mathcal{G}$ is a group if it contains the identity and is **closed under multiplication** (i.e., $ab \in \mathcal{H}$ for all $a, b \in \mathcal{H}$)

A.2 Group Representation

A **group representation** of a group \mathcal{G} is a map $\phi : \mathcal{G} \rightarrow \mathbb{C}^{n \times n}$ ¹ such that

$$\phi(a)\phi(b) = \phi(ab) \quad (\text{A.4})$$

Such a map is called a **homomorphism**. When a group is defined using matrices, those matrices are called the **defining representation**. The map ϕ partitions \mathcal{G} into disjoint subsets (equivalence classes), such that all elements of \mathcal{G} in a disjoint set are represented by the same matrix.

For example, the group of **General Linear Transformations** is defined by

$$GL(n, \mathbb{C}) = \{M \in \mathbb{C}^{n \times n} : \det M \neq 0\} \quad (\text{A.5})$$

A.3 Vector Space and Algebra over a field \mathbb{F}

A vector (or linear) space \mathcal{V} is defined as a set endowed with two operations: vector addition $+: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$, and scalar multiplication $\mathbb{F} \times \mathcal{V} \rightarrow \mathcal{V}$, such that

- \mathcal{V} is an abelian group under $+$ with identity 0 and inverse of $x \in \mathcal{V}$ equal to $-x \in \mathcal{V}$
- For $\alpha, \beta \in \mathbb{F}$ and $x, y \in \mathcal{V}$

$$\alpha(x + y) = \alpha x + \alpha y \quad (\text{A.6})$$

$$(\alpha + \beta)x = \alpha x + \beta x \quad (\text{A.7})$$

$$\alpha(\beta x) = (\alpha\beta)x \quad (\text{A.8})$$

$$1x = x \quad (\text{A.9})$$

$$0x = 0 \quad (\text{A.10})$$

In this book, we will always use either \mathbb{C} or \mathbb{R} for \mathbb{F} . Both of these fields are infinite but some fields are finite.

An **algebra** \mathcal{A} is a vector space which, besides being endowed with vector addition and scalar multiplication with which all vector spaces are, it has a **bilinear vector product**. A bilinear vector product is a product that is linear on both sides; i.e.,

$$(\alpha x + \beta y) \cdot z = \alpha x \cdot z + \beta y \cdot z \quad (\text{A.11})$$

and

$$z \cdot (\alpha x + \beta y) = \alpha z \cdot x + \beta z \cdot y \quad (\text{A.12})$$

¹More generally, the $\mathbb{C}^{n \times n}$ can be replaced by $\mathbb{R}^{n \times n}$ or by $\mathbb{F}^{n \times n}$ for any field \mathbb{F}

for $x, y, z \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. The cross product (but not the dot product) for vectors in \mathbb{R}^3 , the multiplication of 2 complex numbers, and the commutator for square matrices, are all good examples of bilinear vector products.

Let $B = \{\tau_i : i = 1, 2, \dots, r\}$ be a basis for the vector space \mathcal{A} . Then note that B is closed under vector multiplication.

$$\tau_i \cdot \tau_j = \sum_k c_{ij}^k \tau_k \quad (\text{A.13})$$

where $c_{ij}^k \in \mathbb{C}$. The c_{ij}^k are called **structure constants** of B .

An **associative algebra** satisfies $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for $x, y, z \in \mathcal{A}$.

- Not associative: cross product for vectors in \mathbb{R}^3 .
- Associative: the commutator for square matrices and product of complex numbers

A.4 Tensors

$$(x_1, x_2, \dots, x_n) = x^{;n} \in V^n = \mathbb{C}^{n \times 1}$$

Reverse of vector $rev(x_1, x_2, \dots, x_n) = (x_n, x_{n-1}, \dots, x_1)$

$$y^b = \sum_b g^{ba} x_{;n}$$

$(y^1, y^2, \dots, y^n) = \bar{y}^{;n} \in \bar{V}^n = \mathbb{C}^{n \times 1}$. V^n and \bar{V}^n are **dual vector spaces**.

$$M_a^b \in \mathbb{C}^{n \times n}, a, b \in \mathbb{Z}_{[1,n]}$$

Implicit Summation Convention

$$M_a^b x_b = \sum_{b=1}^n M_a^b x_b \quad (\text{A.14})$$

$$(M^\dagger)_b^a = (M^*)_a^b \quad (\text{A.15})$$

$$= M_b^a \quad (\text{only if } M \text{ is a unitary matrix}) \quad (\text{A.16})$$

For $x_a \in V^n$,

$$(x')_a = M_a^b x_b \quad (\text{A.17})$$

For $x^a \in \bar{V}^n$,

$$(x'^*)^a = x^{*b} (M^*)_b^a \quad (\text{A.18})$$

$$= x^{*b} (M^\dagger)_b^a \quad (\text{A.19})$$

so

Suppose $a_i, b_i, c_i, d_i \in \mathbb{Z}_{[1,n]}$. From Fig.A.1

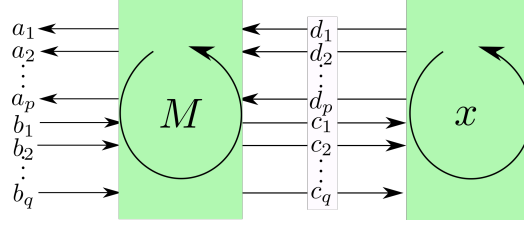


Figure A.1: Index labels for Mx where $M \in \mathbb{C}^{n^{p+q} \times n^{p+q}}$ and $x \in V^{n^p} \otimes \bar{V}^{n^q}$

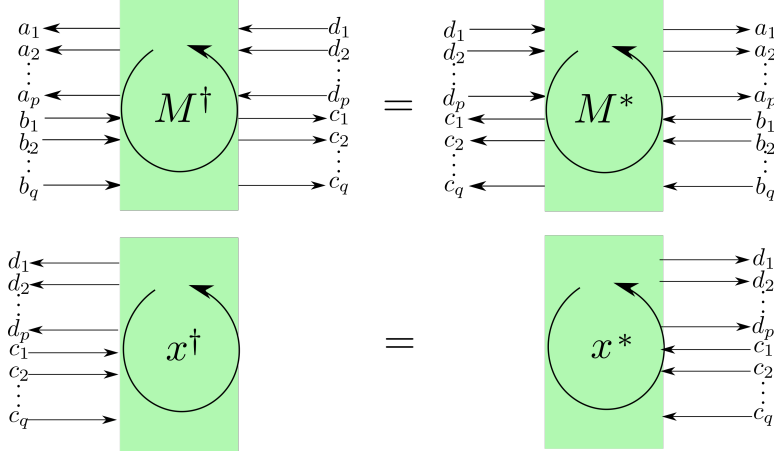


Figure A.2: Index labels for M^\dagger where $M \in \mathbb{C}^{n^{p+q} \times n^{p+q}}$

$$y_{a:p}^{b:q} = M_{a:p}^{b:q}{}_{rev(c:q)}^{rev(d:p)} x_{d:p}^{c:q} \quad (\text{A.20})$$

$$X_\alpha = X_{a:p}^{b:q}, \quad X^\alpha = X_{rev(b:q)}^{rev(a:p)} \quad (\text{A.21})$$

$$x_\alpha = M_\alpha^\beta x_\beta \quad (\text{A.22})$$

Hermitian conjugation (see Fig.A.2)

$$\begin{cases} (M^\dagger)_a^d = (M_a^d)^* \\ (M^\dagger)_\alpha^\delta = (M^{rev(\delta)}_{rev(\alpha)})^* \end{cases} \quad (\text{A.23})$$

Hermitian matrix

$$M^\dagger = M, \quad \begin{cases} (M_a^d)^* = M_a^d \\ (M^{rev(\delta)}_{rev(\alpha)})^* = M_\alpha^\delta \end{cases} \quad (\text{A.24})$$

Note that for $x \in V^n$, $y \in \bar{V}^n$, and $G \in \mathcal{G} \subset GL(n, \mathbb{C})$,

$$(x')_a (y')^b = G^b_c G_a^d x_d y^c \quad (\text{A.25})$$

If $x \in V^{n^p} \otimes \bar{V}^{n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a:p}{}^{b:q} = \mathbb{G}_{a:p}{}^{b:q}{}_{rev(c:q)}{}^{rev(d:p)} x_{d:p}{}^{c:q} \quad (\text{A.26})$$

where we define

$$\mathbb{G}_{a:p}{}^{b:q}{}_{rev(c:q)}{}^{rev(d:p)} \stackrel{\text{def}}{=} \prod_{i=1}^q G_{c_i}^{b_i} \prod_{i=1}^p G_{a_i}^{d_i} \quad (\text{A.27})$$

An issue that arises with tensors is this: When is it permissible to represent a tensor by T_{ab}^{cd} ? If we define T_{ab}^{cd} by

$$T_{ab}^{cd} = T_{ab}{}^{cd} \quad (\text{A.28})$$

then it's always permissible. Then one can define tensors like $T_a{}^{bcd}$ as

$$T_a{}^{bcd} = g^{bb'} T_{ab'}{}^{cd} = g^{bb'} T_{ab'}^{cd} \quad (\text{A.29})$$

Hence, one drawback of using the notation T_{ab}^{cd} is that if one is interested in using versions of T_{ab}^{cd} with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing $T_a{}^{bcd}$, you'll have to write $g^{bb'} T_{ab'}^{cd}$. This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too elegantly concise.

A.5 Spectral Decomposition and Eigenvalue Projection Operators

$$M \in \mathbb{C}^{d \times d}$$

$$M|v\rangle = \lambda|v\rangle \quad (\text{A.30})$$

If M is Hermitian ($H^\dagger = H$), its eigenvalues are real. ($\lambda = \langle \lambda | M | \lambda \rangle \in \mathbb{R}$)

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M - \lambda) = 0 \quad (\text{A.31})$$

If M is a Hermitian matrix, then there exists a unitary matrix ($CC^\dagger = C^\dagger C = 1$) such that

$$CMC^\dagger = \begin{bmatrix} D_{\lambda_1} & 0 & 0 & 0 \\ 0 & D_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{\lambda_r} \end{bmatrix} \quad (\text{A.32})$$

where

$$D_{\lambda_i} = \text{diag}(\underbrace{\lambda_i, \lambda_i, \dots, \lambda_i}_{d_i \text{ times}}) \quad (\text{A.33})$$

$$d = \sum_{i=1}^r d_i \quad (\text{A.34})$$

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (\text{A.35})$$

$$CP_1C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2} \quad (\text{A.36})$$

$$CP_2C^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1} \quad (\text{A.37})$$

If $I^{d_i \times d_i}$ is the d_i dimensional unit matrix,

$$P_i = C^\dagger \text{diag}(0, \dots, 0, I^{d_i \times d_i}, 0, \dots, 0)C \quad (\text{A.38})$$

$$= \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (\text{A.39})$$

Note that P_i are Hermitian ($P_i^\dagger = P_i$) because M is Hermitian and its eigenvalues are real.)

Note that P_i and M commute

$$[P_i, M] = P_i M - M P_i = 0 \quad (\text{A.40})$$

orthogonal

$$P_i P_j = \delta(i, j) P_j \quad (\text{A.41})$$

complete

$$\sum_i P_i = 1 \quad (\text{A.42})$$

$$M = \sum_{i=1}^r P_i M P_i \quad (\text{A.43})$$

$$d_i = \text{tr} P_i \quad (\text{A.44})$$

$$CMP_1C^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.45})$$

$$= \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.46})$$

$$MP_i = \lambda_i P_i \text{ (no } i \text{ sum)} \quad (\text{A.47})$$

$$f(M)P_i = f(\lambda_i)P_i \text{ (no } i \text{ sum)} \quad (\text{A.48})$$

$$M^{(1)}, M^{(2)}$$

$$[M^{(1)}, M^{(2)}] = 0 \quad (\text{A.49})$$

Use $M^{(1)}$ to decompose V into $\bigoplus_i V_i$. Use $M^{(2)}$ to decompose V_i into $\bigoplus_j V_{i,j}$. If $M^{(1)}$ and $M^{(2)}$ don't commute, let $P_i^{(1)}$ be the eigenvalue projection operators of $M^{(1)}$. The replace $M^{(2)}$ by $P_i^{(1)}M^{(2)}P_i^{(1)}$

$$[M^{(1)}, P_i^{(1)}M^{(2)}P_i^{(1)}] = 0 \quad (\text{A.50})$$

Appendix B

Birdtracks: COMING SOON

Cvitanovic Birdtracks book [1]

Elliott-Dawber book [2]

My paper “Quantum Bayesian Nets” [3]

B.1 Classical Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $P(y|x) \in [0, 1]$ where $x \in \text{val}(\underline{x})$ and $y \in \text{val}(\underline{y})$

$$\sum_{y \in \text{val}(\underline{y})} P(y|x) = 1 \quad (\text{B.1})$$

$$\mathcal{C} = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow \\ \underline{c} & \longleftarrow & \underline{a} \end{array} \quad (\text{B.2})$$

$$\mathcal{C}(a, b, c) = P(c|b, a)P(b|a)P(a) = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow \\ c & \longleftarrow & a \end{array} P(a) \quad (\text{B.3})$$

$$a^{:2} = (a_1, a_2)$$

$$\mathcal{C}' = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow_{a_1} \\ \underline{c} & \longleftarrow_{a_2} & \underline{a}^{:2} \end{array} \quad (\text{B.4})$$

$$\mathcal{C}'(a^{:2}, b, c) = P(c|b, a_2)P(a_2|a^{:2})P(b|a_1)P(a_1|a^{:2})P(a^{:2}) = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow_{a_1} \\ c & \longleftarrow_{a_2} & a^{:2} \end{array} P(a^{:2}) \quad (\text{B.5})$$

Marginalizer nodes \underline{a}_1 and \underline{a}_2 have the TPMs

$$P(a'_i|\underline{a}^{i2} = (a_1, a_2)) = \delta(a'_i, a_i) \quad (\text{B.6})$$

for $i = 1, 2$

B.2 Quantum Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $A(y|x) \in \mathbb{C}$ where $x \in \text{val}(\underline{x})$ and $y \in \text{val}(\underline{y})$

$$\sum_{y \in \text{val}(\underline{y})} |A(y|x)|^2 = 1 \quad (\text{B.7})$$

$$\mathcal{Q} = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow & \underline{a} \end{array} \quad (\text{B.8})$$

$$\mathcal{Q}(a, b, c) = A(c|b, a)A(b|a)A(a) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow & a \end{array} A(a) \quad (\text{B.9})$$

$$a^{i2} = (a_1, a_2)$$

$$\mathcal{Q}' = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow \underline{a}_2 & \underline{a}^{i2} \end{array} \quad (\text{B.10})$$

$$\mathcal{Q}'(a^{i2}, b, c) = A(c|b, a_2)A(a_2|a^{i2})A(b|a_1)A(a_1|a^{i2})A(a^{i2}) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow a_2 & a^{i2} \end{array} A(a^{i2}) \quad (\text{B.11})$$

Marginalizer nodes \underline{a}_1 and \underline{a}_2 have the TAMs

$$A(a'_i|\underline{a}^{i2} = (a_1, a_2)) = \delta(a'_i, a_i) \quad (\text{B.12})$$

for $i = 1, 2$

B.3 Birdtracks

$$\delta(b, a) = \mathbb{1}(a = b) = \delta_a^b = a \leftarrow \bullet \rightarrow b \quad (\text{B.13})$$

$$\langle a, b | X_{\underline{ab}}^{\underline{cd}} | c, d \rangle = X_{ab}^{cd} = \begin{array}{c} \underline{a} = a \leftarrow X_{\underline{ab}}^{\underline{cd}} \\ \swarrow \quad \nearrow \\ \underline{b} = b \\ \swarrow \quad \nearrow \\ \underline{c} = c \\ \swarrow \quad \nearrow \\ \underline{d} = d \end{array} \quad (\text{B.14})$$

$$\begin{array}{c} a \leftarrow X_{\underline{ab}}^{\underline{cd}} \\ \swarrow \quad \nearrow \\ b \\ \swarrow \quad \nearrow \\ c \\ \swarrow \quad \nearrow \\ d \end{array} \rightarrow \begin{array}{c} a, b \leftarrow X_{\underline{ab}}^{\underline{cd}} \\ \swarrow \quad \nearrow \\ a, b \\ \swarrow \quad \nearrow \\ c \\ \swarrow \quad \nearrow \\ d \end{array} \quad (\text{B.15})$$

$X_{\underline{ab}}^{\underline{cd}} \in V^2 \otimes V_2$. Sometimes, we will omit denote this node simply by X . This is okay as long as we are not using, X to also denote a different version of $X_{\underline{ab}}^{\underline{cd}}$ with some of the indices raised or lowered or their order has been changed. ¹

$$(X^\dagger)_{dc}^{ba} = \begin{array}{c} (X^\dagger)_{dc}^{ba} \leftarrow \underline{a} = a \\ \swarrow \quad \nearrow \\ \underline{b} = b \\ \swarrow \quad \nearrow \\ \underline{c} = c \\ \swarrow \quad \nearrow \\ \underline{d} = d \end{array} \quad (\text{B.16})$$

¹For matrices, $(A^\dagger)_{i,j} = (A_{j,i})^*$ so taking a Hermitian conjugate involves both taking the complex conjugate of the matrix element and reversing the left-to-right (L2R) order of its indices. This generalizes to $(X^\dagger)_{dc}^{ba} = (X_{ab}^{cd})^*$. Besides raising and lowering indices, we reverse their L2R order.

$$\begin{array}{c}
(X^\dagger)_{dc}^{ba} \longleftarrow \sum a \longleftarrow X_{ab}^{cd} \\
\swarrow \quad \searrow \quad \swarrow \quad \searrow \\
\sum b \quad \sum c \\
\searrow \quad \swarrow \\
\sum d
\end{array}
\quad (B.17)$$

$$\begin{array}{c}
X^\dagger \longleftarrow X \\
\swarrow \quad \searrow \quad \swarrow \quad \searrow \\
\text{diagonal lines}
\end{array}
\quad (B.18)$$

Birdtracks originated as a graphical way to represent the tensors in General Relativity (Gravitation). In General Relativity, one deals with tensors such as $T_a^b{}_c$ which have some indices raised and some lowered. One can use the metric $g^{a,b}$ to raise all the lowered indices to get T^{abc} . If we represent this graphically as a node with incoming arrows a, b, c , we need to follow one of the following 2 conventions: either

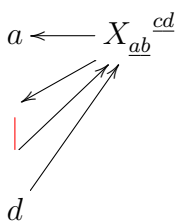
1. label the arrows as $\underline{a}, \underline{b}, \underline{c}$, and define the node as $T^{\underline{abc}}$, or
2. instead of labelling the arrows explicitly $\underline{a}, \underline{b}, \underline{c}$, indicate in the node where is the first arrow \underline{a} , and draw the arrows $\underline{a}, \underline{b}, \underline{c}$ so that they enter the node in **counterclockwise** (CC) order. The **left-to-right** (L2R) order of the indices on T corresponds the CC order of the arrows.

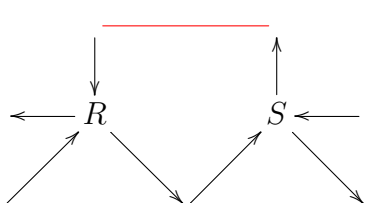
If we don't do either 1 or 2, we won't be able to distinguish between the graphical representations of $T^{1,2,3}$ and $T^{2,1,3}$, for example. Cvitanovic's Birdtracks book Ref.[1] follows Convention 2, but most of the time, in this book, we will follow Convention 1² The reason I chose to do so is for the sake of consistency: Convention 2 is closer to the quantum bnet conventions.

$$a^{:m} \in \mathbb{Z}_+^m$$

$$\begin{array}{c}
b_3^{:n_3} \longleftarrow R \longleftarrow \sum b_2^{:n_2} \longleftarrow S \longleftarrow b_1^{:n_1} \\
\swarrow \quad \searrow \quad \swarrow \quad \searrow \\
a_3^{:m_3} \quad \sum a_2^{:m_2} \quad a_1^{:m_1}
\end{array}
\quad (B.19)$$

²If we follow Convention 1, we don't need to reverse the L2R order of the indices when taking a Hermitian conjugate. Thus, $(X^\dagger)_{cd}^{ab} = X_{ab}^{cd} = X_{ba}^{dc}$. As long as $\underline{a}, \underline{b}$ are lower indices and $\underline{c}, \underline{d}$ are upper indices of X , any L2R order of $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ is equivalent under Convention 1.

$$\text{tr}_{\underline{b}} X_{\underline{a}\underline{b}}{}^{\underline{b}d} = \sum_b X_{ab}{}^{bd} =$$

(B.20)


(B.21)

Chapter 1

**Casimir Operators: COMING
SOON**

Chapter 2

Clebsch-Gordan Coefficients: COMING SOON

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d \times d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} C^{d \times d} \quad (2.1)$$

Let $b^{nb} = (b_1, b_2, \dots, b_{nb})$ where $b_i \in Z_{[0, db_i]}$ and $a \in Z_{[1, d_\lambda]}$. Hence,

$$d_\lambda = \prod_{i=1}^{nb} db_i \quad (2.2)$$

$$(C_\lambda)_{a^{b^{nb}}} = a \longleftarrow C_\lambda \begin{matrix} \swarrow b_1 \\ \longleftarrow b_2 \\ \searrow b_{nb} \end{matrix} \quad (2.3)$$

$$\begin{bmatrix} 0 & (C^\dagger)_\lambda^{d \times d_\lambda} & 0 \end{bmatrix}^{d \times d} = (C^\dagger)^{d \times d} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} \quad (2.4)$$

$$(C^\dagger_\lambda)_{b^{nb}^a} = \begin{matrix} \swarrow b_1 \\ b_2 \longleftarrow \\ \searrow b_{nb} \end{matrix} (C^\dagger_\lambda) \longleftarrow a \quad (2.5)$$

More generally, some of the b_i indices may be lowered and their arrows changed to outgoing instead of ingoing. Each b_i represents a different rep (or irrep)

$$\boxed{(C_\lambda^\dagger)_a^{b:nb} (C_\lambda)_a^{(b') :nb} = (P_\lambda)_{(b') :nb}^{b:nb}}$$

$$\begin{array}{c}
b_1 \swarrow \\
b_2 \leftarrow (C_\lambda^\dagger) \leftarrow \sum a \leftarrow C_\lambda \leftarrow b'_2 \\
b_{nb} \searrow
\end{array}
\begin{array}{c}
b'_1 \swarrow \\
b'_2 \leftarrow C_\lambda \leftarrow b'_{nb}
\end{array}
= b:nb \leftarrow P_\lambda \leftarrow (b') :nb
\quad (2.6)$$

$$\boxed{(C_\lambda)_{b:nb}^{a'} (C_\mu^\dagger)_a^{b:nb} = \delta(\lambda, \mu) \delta_a^{a'}}$$

$$\begin{array}{c}
\sum b_1 \swarrow \\
a \leftarrow C_\lambda \leftarrow \sum b_2 \leftarrow (C_\mu^\dagger) \leftarrow a' \\
\sum b_{nb} \searrow
\end{array}
= \delta(\mu, \lambda) a \leftarrow \bullet a'
\quad (2.7)$$

Chapter 3

Determinants: COMING SOON

Chapter 4

**General Relativity Nets: COMING
SOON**

Chapter 5

Group Integrals: COMING SOON

Chapter 6

Invariants

Given a **bilinear form**

$$m(\bar{x}^{\cdot n}, y^{\cdot n}) = x^a M_a^{\cdot b} y_b \quad \begin{array}{c} M \\ \swarrow \quad \searrow \\ a \quad b \end{array} \quad (6.1)$$

is invariant if

$$m(\bar{x}^{\cdot n}, y^{\cdot n}) = m(\bar{x}^{\cdot n} G^\dagger, G y^{\cdot n}) \quad (6.2)$$

matrix invariant

$$\boxed{M_a^{\cdot b} = (G^\dagger)_a^{\cdot a'} G_{b'}^{\cdot b} M_{a'}^{\cdot b'}} \quad \begin{array}{c} M \\ \swarrow \quad \searrow \\ a \quad b \end{array} = \begin{array}{c} M \\ \swarrow \quad \searrow \\ a \quad b \end{array} \quad (6.3)$$

$$M = G^\dagger M G \quad (6.4)$$

$$GM = MG, \quad [G, M] = 0 \quad (6.5)$$

multilinear form

$$h(\bar{w}, \bar{x}, y, z) = h_{ab}^{\cdot cd} w^a x^b y_c z_d \quad \begin{array}{c} h \\ \downarrow \quad \swarrow \quad \searrow \quad \searrow \\ a \quad b \quad c \quad d \end{array} \quad (6.6)$$

is invariant if

$$h(\bar{w}, \bar{x}, y, z) = h(\bar{w} G^\dagger, \bar{x} G^\dagger, G y, G z) \quad (6.7)$$

tensor invariant

$$\boxed{h_{ab}{}^{cd} = (G^\dagger)_a{}^{a'} (G^\dagger)_b{}^{b'} h_{a'b'}{}^{c'd'} G_{c'}{}^c G_{d'}{}^d} \quad \begin{array}{c} h \\ \swarrow \downarrow \searrow \\ a \quad b \quad c \quad d \end{array} = \begin{array}{c} h \\ \swarrow \downarrow \searrow \\ G^\dagger \downarrow G^\dagger \downarrow G \downarrow G \\ a \quad b \quad c \quad d \end{array} \quad (6.8)$$

Let $\mathcal{P} = (p_1, p_2, \dots, p_k)$ be a full set of invariant multilinear forms (“primitives”). By “full”, we mean no others exist.

An **invariance group** \mathcal{G} is the set of all linear transformation $G \in \mathcal{G}$ such that

$$p_1(x, \bar{y}) = p_1(Gx, \bar{y}G^\dagger) \quad (6.9)$$

$$p_2(w, x, \bar{y}, \bar{z}) = p_2(Gw, Gx, \bar{y}G^\dagger, \bar{z}G^\dagger) \quad (6.10)$$

$$\text{etc.} \quad (6.11)$$

Example

$$p(\bar{x}, y) = \delta_a^b x^a y_b = x^b y_b \quad (6.12)$$

$$(x')^a (y')_a = x^b (G^\dagger G)_b^c y_c = x^b y_b \quad (6.13)$$

So G must be unitary

$$G^\dagger G = 1 \quad (6.14)$$

The group of n dimensional unitary matrices is called $U(n)$

Chapter 7

Levi-Civita Tensor

$$\epsilon^{123\dots p} = \epsilon_{123\dots p} = 1 \quad (7.1)$$

$$\epsilon_{rev(a^{\cdot p})} = (-1)^{\binom{p}{2}} \epsilon_{a^{\cdot p}} \quad (7.2)$$

where $rev(a^{\cdot p})$ is the reverse of $a^{\cdot p}$. $rev(a_1, a_2, \dots, a_p) = (a_p, a_{p-1}, \dots, a_1)$

$$(C_{\mathcal{A}_p})_1^{a^{\cdot p}} = e^{i\phi} \frac{\epsilon_{a^{\cdot p}}}{\sqrt{p!}} = \begin{array}{c} \mathcal{A}_p \leftarrow a_1 \\ \parallel \\ \leftarrow a_2 \\ \vdots \\ \leftarrow a_p \end{array} \quad (7.3)$$

$$(C_{\mathcal{A}_p}^\dagger)_{a^{\cdot p}}^1 = e^{-i\phi} \frac{\epsilon_{a^{\cdot p}}}{\sqrt{p!}} = \begin{array}{c} a_1 \leftarrow \mathcal{A}_p \\ \parallel \\ a_2 \leftarrow \\ \vdots \\ a_p \leftarrow \parallel \end{array} \quad (7.4)$$

$$\boxed{\frac{1}{p!} \epsilon_{a^{\cdot p}} \epsilon^{b^{\cdot p}} = (\mathcal{A}_p)_{a^{\cdot p}}^{b^{\cdot p}}} \quad \begin{array}{c} a_1 \leftarrow \mathcal{A}_p \quad \mathcal{A}_p \leftarrow b_1 \\ \parallel \quad \parallel \\ a_2 \leftarrow \quad \leftarrow b_2 \\ \vdots \quad \vdots \\ a_p \leftarrow \quad \leftarrow b_p \end{array} = \begin{array}{c} a_1 \leftarrow \mathcal{A}_p \leftarrow b_1 \\ \parallel \quad \parallel \\ a_2 \leftarrow \quad \leftarrow b_2 \\ \vdots \quad \vdots \\ a_p \leftarrow \quad \leftarrow b_p \end{array} \quad (7.5)$$

$$\boxed{e^{i2\phi} \frac{1}{p!} \epsilon^{a:n} \epsilon_{a:n} = \delta_1^1 = 1} \quad \begin{array}{c} \mathcal{A}_p \longleftarrow \mathcal{A}_p \\ \parallel \\ \longleftarrow \\ \vdots \\ \longleftarrow \\ \parallel \end{array} = 1 \quad (7.6)$$

For Convention 1, we will use $\phi = 0$.
For Convention 2, we must choose

$$e^{i2\phi} = (-1)^{\binom{p}{2}} = e^{i\pi \frac{p(p-1)}{2}} \quad (7.7)$$

so

$$\phi = \frac{\pi}{4} p(p-1) \quad (7.8)$$

Chapter 8

Lie Algebra Definition: COMING SOON

$$i \in \mathbb{Z}_{[1,N]}, \quad a, b \in \mathbb{Z}_{[1,n]}$$

$$(C_{Adj}^i)_b^a = \frac{1}{\sqrt{K}} (T^i)_b^a = \quad i \text{ --- } C_{Adj}^i \quad \begin{array}{c} a \\ \downarrow \\ b \end{array} \quad (8.1)$$

The matrices T^i are called the generators. It's customary to choose them so that they are Hermitian and $K = \frac{1}{2}$

$$\boxed{(T^i)_b^a (T^j)_a^b = \text{tr}(T^i T^j) = K \delta(i, j)} \quad i \text{ --- } T^i \quad \begin{array}{c} \xrightarrow{\sum b} \\ T^j \text{ --- } j \\ \xleftarrow{\sum a} \end{array} = K \quad \leftarrow \bullet \text{ ---} \quad (8.2)$$

$$(P_{Adj})_{b,d}^{a,c} = \sum_i \frac{1}{K} (T^i)_b^a (T^i)_d^c = \frac{1}{K} \quad \begin{array}{c} a \\ \downarrow \\ b \end{array} \text{ --- } \begin{array}{c} d \\ \uparrow \\ c \end{array} \quad (8.3)$$

$$H \in V^a \otimes V_{\underline{a}}$$

$$(P_{Adj})_{bd}^{ac} H_c^d = \sum_i (T^i)_b^a \underbrace{\left[\frac{1}{K} (T^i)_d^c H_c^d \right]}_{h_i \in \mathbb{R}} \quad (8.4)$$

$$G = 1 + iD \in \mathcal{G}$$

$$\epsilon_i \in \mathbb{R}, \quad |\epsilon_i| \ll 1$$

$$D = \sum_i \epsilon_i T^i = V^{\underline{a}} \otimes V_{\underline{a}}$$

$$\mathcal{T}^i q = 0 \tag{8.5}$$

Chapter 9

Lie Algebra Classification, Dynkin Diagrams: COMING SOON

Chapter 10

**Orthogonal Groups: COMING
SOON**

Chapter 11

Quantum Shannon Information Theory: COMING SOON

Chapter 12

**Recoupling Equations: COMING
SOON**

Chapter 13

Reducibility: COMING SOON

Chapter 14

Spinors: COMING SOON

Chapter 15

Squashed Entanglement: COMING SOON

Chapter 16

**Symplectic Groups: COMING
SOON**

Chapter 17

Symmetrization and Antisymmetrization: COMING SOON

$(1, 2)$ transposition, swaps 1 and 2, $1 \rightarrow 2 \rightarrow 1$. $(3, 2, 1)$ means $3 \rightarrow 2 \rightarrow 1 \rightarrow 3$. A reordering of $(1, 2, 3, \dots, p)$ is a permutation on p letters. A permutation can be expressed as a product of transpositions $(3, 2, 1) = (3, 2)(2, 1)$ is an even permutation because it can be expressed as a product of an even number of transpositions. An odd permutation can be expressed as a product of an odd number of permutations.

17.1 Symmetrization

$$\mathbb{1}_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} = \begin{array}{c} a_1 \leftarrow b_1 \\ a_2 \leftarrow b_2 \end{array} \quad (17.1)$$

$$(\sigma_{(1,2)})_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} = \begin{array}{c} a_1 \leftarrow \bullet \leftarrow b_1 \\ \updownarrow \\ a_2 \leftarrow \bullet \leftarrow b_2 \end{array} \quad (17.2)$$

$$\mathbb{1} = \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \quad (17.3)$$

$$\sigma_{(1,2)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \end{array} \quad \sigma_{(2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad \sigma_{(1,3)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \end{array} \quad (17.4)$$

$$\sigma_{(1,2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \end{array} \quad (17.5)$$

$$\sigma_{(1,3,2)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad (17.6)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p!} \left\{ \begin{array}{cc} \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \\ \vdots & \vdots \\ \leftarrow & \leftarrow \end{array} + \begin{array}{cc} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \\ \vdots \end{array} + \dots \right\} \quad (17.7)$$

$$\boxed{\mathcal{S}_p^2 = \mathcal{S}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{S}_p \leftarrow \\ \parallel & \parallel & \parallel \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (17.8)$$

$$\boxed{\mathcal{S}_p \mathcal{S}_{[1,q]} = \mathcal{S}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{S}_{[1,q]} \leftarrow & \leftarrow \mathcal{S}_p \leftarrow \\ \parallel & \parallel & \parallel \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (17.9)$$

$$\boxed{\mathcal{S}_p \sigma_{(1,2)} = \mathcal{S}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \bullet \leftarrow \\ \parallel & \updownarrow \\ \leftarrow \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (17.10)$$

Claim 1

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} = \frac{1}{p} \left(\begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} + (p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow \leftarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \end{array} \right) \quad (17.11)$$

proof: We only prove it for $p = 3$.

$$\begin{array}{c} 3! \leftarrow \mathcal{S}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \bullet \leftarrow \bullet \leftarrow \leftarrow \bullet \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \bullet \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ + \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (17.12)$$

$$\begin{array}{c} 2! \leftarrow \mathcal{S}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (17.13)$$

$$\begin{array}{c} 3! \leftarrow \mathcal{S}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} - 2! \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \bullet \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ + \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (17.14)$$

$$\begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \mathcal{S}_2 \leftarrow \bullet \leftarrow \mathcal{S}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \end{array} = 2!2! \quad (17.15)$$

QED

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p} \left(\begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} + (p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \quad \parallel \\ \leftarrow \leftarrow \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \leftarrow \leftarrow \end{array} \right) \quad (17.16)$$

$$= \frac{n+p-1}{p} \left(\begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} \right) \quad (17.17)$$

$$\text{tr}_{\underline{a}_1} \mathcal{S}_p = \frac{n+p-1}{p} \mathcal{S}_{p-1} \quad (17.18)$$

$$\text{tr}_{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k} \mathcal{S}_p = \frac{(n+p-1)(n+p-2) \dots (n=p-k)}{p(p-1) \dots (p-k+1)} \mathcal{S}_{p-k} \quad (17.19)$$

$$d_{\mathcal{S}_p} = \text{tr}_{\underline{a}^p} \mathcal{S}_p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \quad (17.20)$$

For $p = 2$,

$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \quad (17.21)$$

17.2 Antisymmetrization

$$\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p!} \left\{ \begin{array}{c} \leftarrow \leftarrow \quad \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \quad \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \quad \leftarrow \leftarrow \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \leftarrow \leftarrow \quad \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} + \dots \right\} \quad (17.22)$$

$$\boxed{\mathcal{A}_p^2 = \mathcal{A}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \end{array} = \quad (17.23)$$

$$\boxed{\mathcal{A}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_{[1,q]} \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow & \leftarrow \parallel \leftarrow \end{array} = \quad (17.24)$$

$$\boxed{\mathcal{A}_p \sigma_{(1,2)} = -\mathcal{A}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \updownarrow \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow & \leftarrow \parallel \leftarrow \end{array} = (-1) \quad (17.25)$$

$$\boxed{\mathcal{S}_p \mathcal{A}_q = \mathcal{A}_p \mathcal{S}_q = 0} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow & \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \\ \leftarrow \leftarrow & \leftarrow \leftarrow & = 0 \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \end{array} \quad (17.26)$$

$$\boxed{\mathcal{S}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p \mathcal{S}_{[1,q]} = 0} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{A}_{[1,q]} \leftarrow & \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{S}_{[1,q]} \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \vdots & \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \end{array} = \quad = 0 \quad (17.27)$$

Claim 2

$$\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} = \frac{1}{p} \left(\begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} + (-1)(p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \end{array} \right) \quad (17.28)$$

proof: We only prove it for $p = 3$.

$$\begin{array}{c} 3! \\ \leftarrow \mathcal{A}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \bullet \leftarrow + \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \leftarrow \\ - \leftarrow \bullet \leftarrow \leftarrow - \leftarrow \bullet \leftarrow \leftarrow - \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (17.29)$$

$$\begin{array}{c} 2! \\ \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow - \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (17.30)$$

$$\begin{array}{c} 3! \\ \leftarrow \mathcal{A}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \end{array} - \begin{array}{c} 2! \\ \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow + \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \\ - \leftarrow \bullet \leftarrow \leftarrow - \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (17.31)$$

$$= (-1)2!2! \begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \mathcal{A}_2 \leftarrow \bullet \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \end{array} \quad (17.32)$$

QED

$$\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} = \frac{1}{p} \left(\begin{array}{c} \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} + (-1)(p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} \right) \quad (17.33)$$

$$= \frac{n + (-1)(p-1)}{p} \left(\begin{array}{c} \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} \right) \quad (17.34)$$

$$\text{tr}_{\underline{a}_1} \mathcal{A}_p = \frac{n-p+1}{p} \mathcal{A}_{p-1} \quad (17.35)$$

$$\text{tr}_{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k} \mathcal{A}_p = \frac{(n-p+1)(n-p+2) \dots (n-p+k)}{p(p-1) \dots (p-k+1)} \mathcal{A}_{p-k} \quad (17.36)$$

$$d_{\mathcal{A}_p} = \text{tr}_{\underline{a}^p} \mathcal{A}_p = \frac{\prod_{i=n-p+1}^n i}{p!} \quad (17.37)$$

$$= \frac{\prod_{i=n}^{n-p+1} i}{p!} \quad (17.38)$$

$$= \begin{cases} \frac{n!}{p!(n-p)!} = \binom{n}{p} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} \quad (17.39)$$

For $p = 2 \leq n$,

$$d_{\mathcal{A}_2} = \binom{n}{2} \quad (17.40)$$

$$\mathcal{A}_p = 0 \text{ if } n < p \quad (17.41)$$

For example, for $n = 2$ and $p = 3$

$$\begin{array}{c} |a\rangle \\ \downarrow \\ \mathcal{A}_3 \\ \downarrow \\ |a\rangle \end{array} \quad \begin{array}{c} |a\rangle \\ \downarrow \\ \mathcal{A}_3 \\ \downarrow \\ |a\rangle \end{array} \quad \begin{array}{c} |b\rangle \\ \downarrow \\ \mathcal{A}_3 \\ \downarrow \\ |b\rangle \end{array} = \frac{1}{6} \left(\begin{array}{c} \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \right. \\ \left. - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \leftarrow \rightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \right) \quad (17.42)
\end{array}$$

$$\mathcal{A}_3|a, a, b\rangle = \frac{1}{6} \left(\begin{array}{c} |a, a, b\rangle + |a, b, a\rangle + |b, a, a\rangle \\ -|a, b, a\rangle - |a, a, b\rangle - |b, a, a\rangle \end{array} \right) \quad (17.43)$$

$$= 0 \quad (17.44)$$

Chapter 18

Unitary Groups: COMING SOON

18.1 SU(n)

$$m(p, q) = \delta_b^a \sum_{a=1}^n (p_a)^* q_a \quad (18.1)$$

$$\mathbb{1}_{d,b}^{a,c} = \delta_b^a \delta_d^c = \begin{array}{c} d \leftarrow \bullet \longrightarrow c \\ a \longrightarrow \bullet \longrightarrow b \end{array} \quad (18.2)$$

$$\uparrow \downarrow_{d,b}^{a,c} = \delta_d^a \delta_b^c = \begin{array}{cc} d & c \\ \uparrow & \downarrow \\ \bullet & \bullet \\ | & | \\ a & b \end{array} \quad (18.3)$$

$$\boxed{\uparrow \downarrow^2 = n \uparrow \downarrow} \quad \begin{array}{c} d \\ \uparrow \\ \bullet \\ | \\ a \end{array} \quad \begin{array}{c} \curvearrowright \\ \bullet \\ \curvearrowleft \end{array} \quad \begin{array}{c} c \\ \downarrow \\ \bullet \\ | \\ b \end{array} = n \quad \begin{array}{c} d \\ \uparrow \\ \bullet \\ | \\ a \end{array} \quad \begin{array}{c} c \\ \downarrow \\ \bullet \\ | \\ b \end{array} \quad (18.4)$$

$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (18.5)$$

$$\lambda_1 = n$$

$$\boxed{P_1 = \frac{\uparrow \downarrow - n}{0 - n} = 1 - \frac{1}{n} \uparrow \downarrow} \quad \begin{array}{ccc} a & & b \\ & \searrow \quad \swarrow & \\ & P_1 & \\ & \swarrow \quad \searrow & \\ c & & d \end{array} = \begin{array}{c} a \leftarrow \bullet \longrightarrow b \\ c \leftarrow \bullet \longrightarrow d \end{array} - \frac{1}{n} \quad \begin{array}{c} a \\ \uparrow \\ \bullet \\ | \\ c \end{array} \quad \begin{array}{c} b \\ \downarrow \\ \bullet \\ | \\ d \end{array} \quad (18.6)$$

$$\lambda_2 = 0$$

$$\boxed{P_2 = \frac{\uparrow\downarrow - 0}{n - 0} = \frac{1}{n} \uparrow\downarrow} \quad
\begin{array}{c} a \\ \searrow \\ P_2 \\ \swarrow \\ c \end{array}
\begin{array}{c} b \\ \swarrow \\ P_2 \\ \searrow \\ d \end{array}
= \frac{1}{n}
\begin{array}{c} a \\ \bullet \\ \uparrow \\ c \end{array}
\begin{array}{c} b \\ \bullet \\ \downarrow \\ d \end{array} \quad (18.7)$$

$$\text{tr} P_1 = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} - \frac{1}{n} \begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \bullet \downarrow \end{array} \quad (18.8)$$

$$= n^2 - 1 \quad (18.9)$$

$$\text{tr} P_2 = \frac{1}{n} \begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \bullet \downarrow \end{array} \quad (18.10)$$

$$= 1 \quad (18.11)$$

$$\begin{array}{c} b \\ \downarrow \\ (T_i)_a^b = i \text{---} T_i \\ \downarrow \\ a \end{array} \quad (18.12)$$

$$T_i^\dagger = T_i \quad (18.13)$$

Claim 3

$$C_F \delta_a^b = (T_i T_i)_a^b = \frac{n^2 - 1}{n} \delta_a^b \quad (18.14)$$

proof:

$$(T_i T_i)_a^b = \sum_i \begin{array}{c} \text{---} \bullet \text{---} \\ \downarrow \\ a \end{array} \begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \\ b \end{array} \quad (18.15)$$

$$= \sum_i \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \quad (18.16)$$

QED

Chapter 19

**Wigner Coefficients: COMING
SOON**

Chapter 20

**Wigner-Ekart Theorem: COMING
SOON**

Chapter 21

Young Tableau: COMING SOON

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