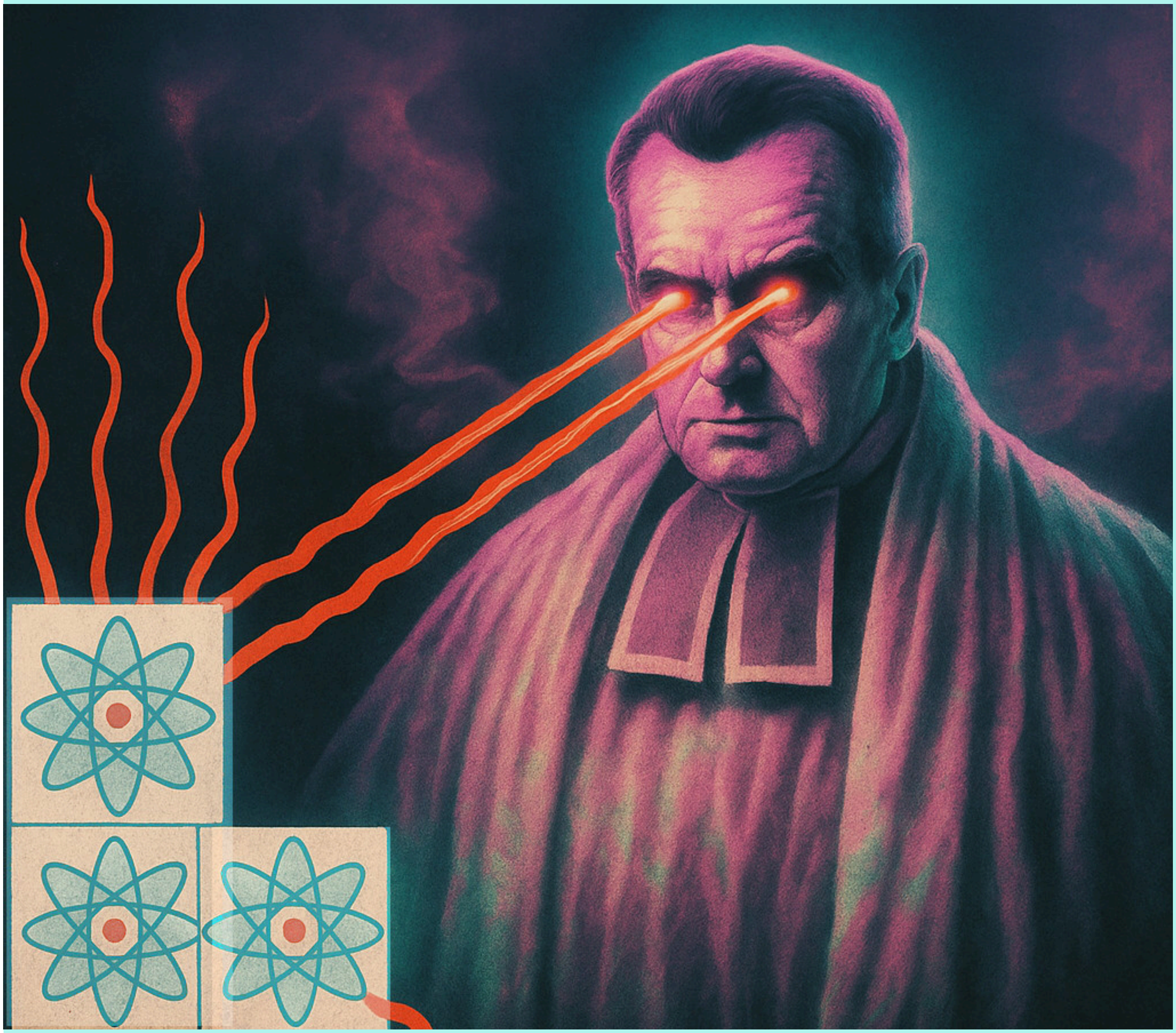
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

Robert R. Tucci
www.ar-tiste.xyz

August 14, 2025

This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayesuvius Quantico
by Robert R. Tucci

Contents

Appendices

Chapter 8

Lie Algebras

This chapter is based on Ref.[1].

8.1 Generators (infinitesimal transformations)

For some group $\mathcal{G}$, assume that any group element $G \in \mathcal{G}$ that is infinitesimal close to the identity 1 can be parametrized by

$$G = 1 + i \sum_i \epsilon_i T^i \tag{8.1}$$

where $T^i \in \mathbb{C}^{n \times n}$ for $i = 1, 2, \ldots, N$, $\epsilon_i \in \mathbb{R}$ and $|\epsilon_i| << 1$.

Assume that the T^i matrices are Hermitian and that they satisfy

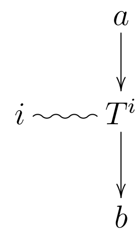
$$\mathrm{tr}(T^i T^j) = K\delta(i, j) \tag{8.2}$$

We will call these matrices the **generators** of infinitesimal transformations for group $\mathcal{G}$.

It's customary to choose generators so that $K = \frac{1}{2}$.[1] However, we will often set $K = 1$ for intermediate calculations and restore $K \neq 1$ at the end by dimensional analysis. Just remember that each T^j scales as $\sqrt{K}$. For example, given the equation $\mathrm{tr}(T^i T^j) = \delta(i, j)$, we know that when $K \neq 1$, $\mathrm{tr}(T^i T^j) = K\delta(i, j)$ so both sides of the equation scale as K.

We will use the following scaled version of T^j as a birdtrack. Define

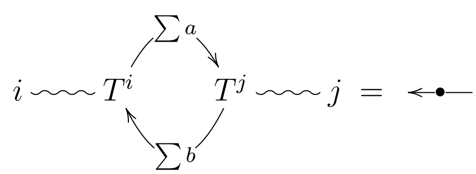
[1]For $SU(2)$, it is customary to choose $T^i = \frac{1}{2}\sigma_i$, where σ_i for $i = 1, 2, 3$ are the Pauli matrices. For $SU(3)$, it is customary to choose $T^i = \frac{1}{2}\lambda_i$ where λ_i for $i = 1, 2, \ldots, 8$ are the Gell-Mann matrices. For both of these choices, $K = \frac{1}{2}$.

$$(C^i_{Adj})_b{}^a = \frac{1}{\sqrt{K}}(T^i)_b{}^a = \frac{1}{\sqrt{K}}$$  (8.3)

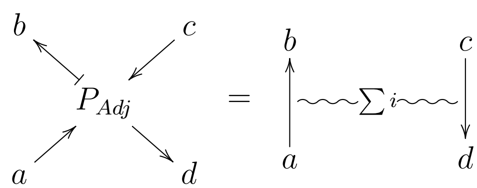
In the CC convention, we will always start reading the indices of this node at the wavy undirected leg.

Adj stands the Adjoint. In this node (vertex), an adjoint representation (adrep) particle (wavy line, gluon) is generated (released) by a fundamental representation (funrep) particle (straight solid line, arrow).

In terms of birdtracks, Eq.(8.2) becomes

$$\boxed{(T^i)^b{}_a(T^j)^a{}_b = \mathrm{tr}(T^iT^j) = \delta(i,j)}$$  (8.4)

We can now define the projection operator for the adrep (gluon exchange between 2 funrep particles)

$$\boxed{(P_{Adj})_b{}^a{}_d{}^c = \sum_i (T^i)_b{}^a(T^i)_d{}^c}$$  (8.5)

The arrow that starts with a bar as in $\longleftarrow\!|$ indicates this is the first index in the CC convention.

Note that if $x \in V^n \otimes V^{\dagger n}$, then

$$(P_{Adj})_b{}^a{}_d{}^c x_c{}^d = \sum_i (T^i)_b{}^a \underbrace{\left[(T^i)_d{}^c x_c{}^d\right]}_{\epsilon_i \in \mathbb{R}} \tag{8.6}$$

Recall Eq.(A.24). If $x \in V^{n^p} \otimes V^{\dagger n^q}$, and $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a^{:p}}{}^{b^{:q}} = \mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}}, \quad x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta \tag{8.7}$$

where we define

$$\mathbb{G}_\alpha{}^\beta \stackrel{\text{def}}{=} \prod_{i=1}^p G_{a_i}{}^{d_i} \prod_{i=1}^q G^{\dagger b_i}{}_{c_i} \tag{8.8}$$

If $\mathbb{G}$ is infinitesimally close to the identity, then we can parametrize it as

$$
\begin{aligned}
\mathbb{G}_\alpha{}^\beta &= 1 + i\sum_j \epsilon_j (\mathbb{T}^j)_\alpha{}^\beta && (8.9)\\
G_{a_i}{}^{d_i} &= 1 + i\sum_j \epsilon_j (T^j)_{a_i}{}^{d_i} && (8.10)\\
G^{\dagger b_i}{}_{c_i} &= 1 - i\sum_j \epsilon_j (T^j)^{b_i}{}_{c_i} && (8.11)
\end{aligned}
$$

Define

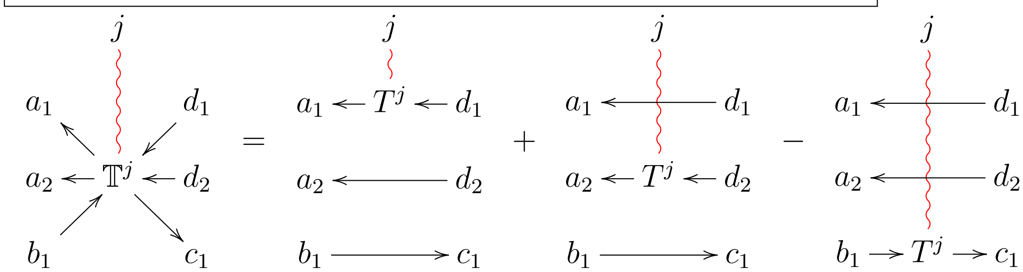
$$
(\mathbb{T}^j)_\alpha{}^\beta = \left[(T^j)_{a_i}{}^{d_i}\frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}{}_{c_i}\frac{1}{\delta_{c_i}^{b_i}}\right]\delta_{a^{:p}}^{d^{:p}}\delta_{c^{:q}}^{b^{:q}} \tag{8.12}
$$

When $x'_\alpha = x_\alpha$, to first order in ϵ_i,

$$
0 = (\mathbb{T}^j)_\alpha{}^\beta x_\beta = \left[(T^j)_{a_i}{}^{d_i}\frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}{}_{c_i}\frac{1}{\delta_{c_i}^{b_i}}\right]\delta_{a^{:p}}^{d^{:p}}\delta_{c^{:q}}^{b^{:q}}x_{d^{:p}}{}^{c^{:q}} \tag{8.13}
$$

For example, if we define

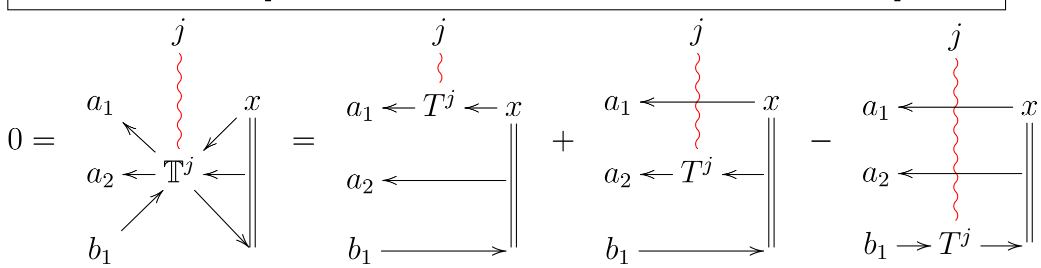
$$
\boxed{(\mathbb{T}^j)_{a_1a_2}{}^{b_1}{}_{c_1}{}^{d_2d_1} = (T^j)_{a_1}{}^{d_1}\delta_{a_2}^{d_2}\delta_{c_1}^{b_1} + \delta_{a_1}^{d_1}(T^j)_{a_2}{}^{d_2}\delta_{c_1}^{b_1} - \delta_{a_1}^{d_1}\delta_{a_2}^{d_2}(T^j)^{b_1}{}_{c_1}}
$$



(8.14)

then

$$
\boxed{0 = (\mathbb{T}^j x)_{a_1a_2}{}^{b_1} = \left[(T^j)_{a_1}{}^{d_1}\delta_{a_2}^{d_2}\delta_{c_1}^{b_1} + \delta_{a_1}^{d_1}(T^j)_{a_2}{}^{d_2}\delta_{c_1}^{b_1} - \delta_{a_1}^{d_1}\delta_{a_2}^{d_2}(T^j)^{b_1}{}_{c_1}\right]x_{d_1d_2}{}^{c_1}}
$$



(8.15)

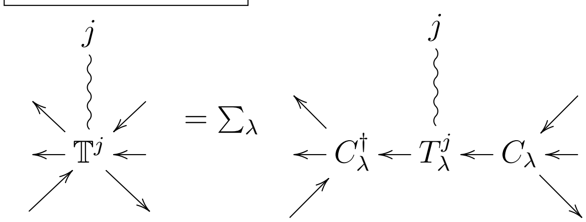
8.2 Clebsch-Gordan matrices

The Clebsch Gordan (CG) matrices are introduced in Ch.2. Note that the generators $(T^i)_a{}^b$ are a simple kind of CG matrix, one with

- a gluon (adrep) particle instead of a general λ rep particle emanating from the i index,
- a particle of the funrep entering and another leaving the node, instead of any number of funrep particles entering and leaving.

Since $\mathbb{G} = 1 + i\sum_j \epsilon_j \mathbb{T}^j$, generators decompose in the same way as the group elements

$$\boxed{\mathbb{T}^j = \sum_\lambda C_\lambda^\dagger T_\lambda^j C_\lambda}$$



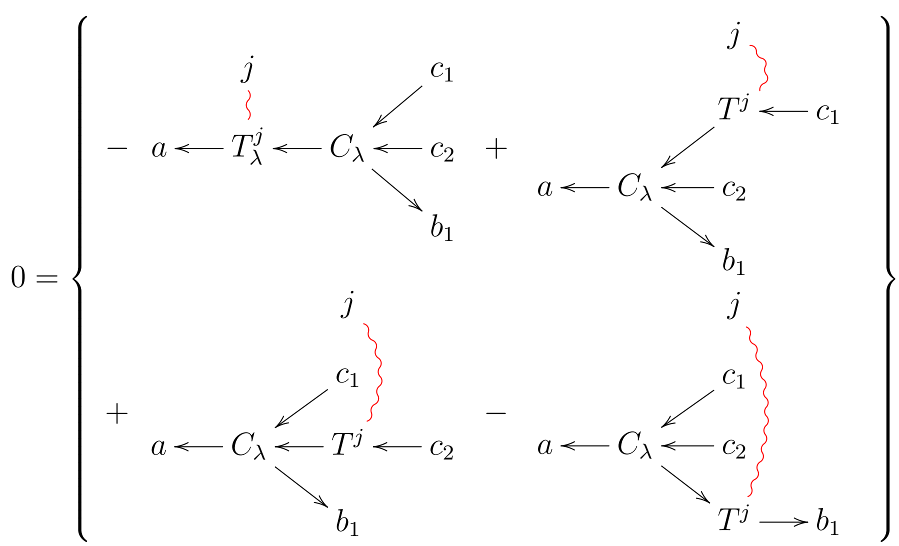
(8.16)

The CG matrices are matrix invariants.

$$C_\lambda = G_\lambda^\dagger C_\lambda G \tag{8.17}$$

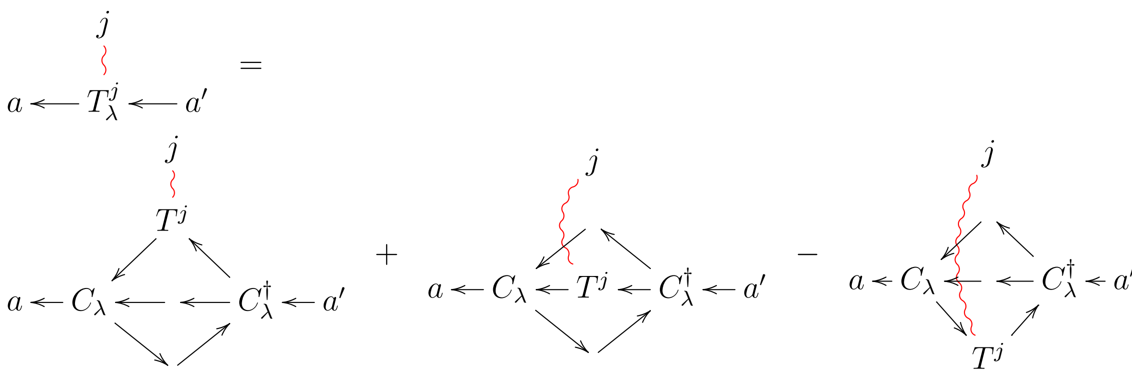
Hence,

$$0 = -T_\lambda^j C_\lambda + C_\lambda T^j \tag{8.18}$$



(8.19)

Multiplying on the left by $C_\lambda^\dagger$, we obtain an expression for the generator T_λ^i in term the generators T^j (and C_λ CG matrices).

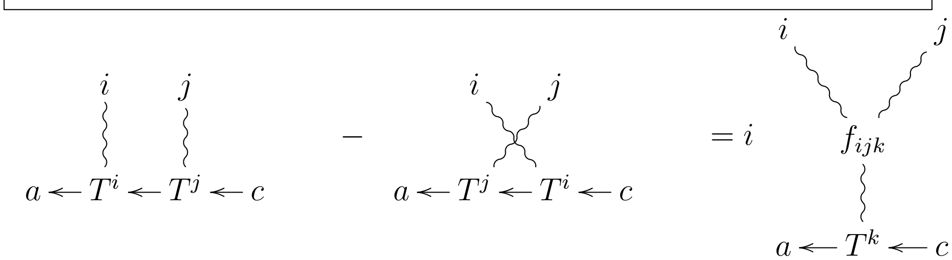


(8.20)

8.3 Structure Constants (3 gluon vertex)

See Sec.A.5 for the definition of an algebra over a field.

$$\boxed{\underbrace{T^iT^j - T^jT^i}_{[T^i,T^j]} = if_{ijk}T^k \quad \textbf{(Lie Algebra commutation relations)}}$$



(8.21)

The f_{ijk} tensors are called the **structure constants** of the Lie Algebra. They define a 3 gluon vertex in term of the generators T^i.[2]

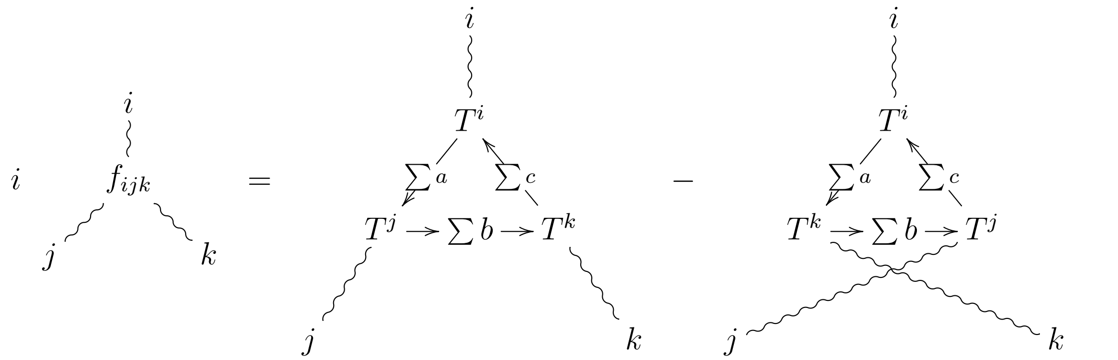
If $(T^j)_a{}^b$ are the matrix rep (the funrep) of the generators of a group $\mathcal{G}$, then Eq.(8.21) shows that the matrices $(M^k)_{ij} = -iC_{ijk}$ are also a matrix rep (the adrep) of the generators of $\mathcal{G}$.

Since $\mathrm{tr}(T^kT^{k'}) = \delta(k, k')$, Eq.(8.21) implies

$$\mathrm{tr}([T^i, T^j]T^k) = if_{ijk} \tag{8.22}$$

[2]It's possible to distinguish between upper and lower gluon indices (i.e., to give the gluon arrows a direction. In that case, the Lie Algebra commutation relations would be $[T^i, T^j] = f^{ij}{}_k T^k$ and the gluon indices could be lowered and raised using the metric $g_{ij} = \mathrm{tr}(T^iT^j)$. But since we are assuming $g_{ij} = K\delta_i^j$, there is no need to do this.

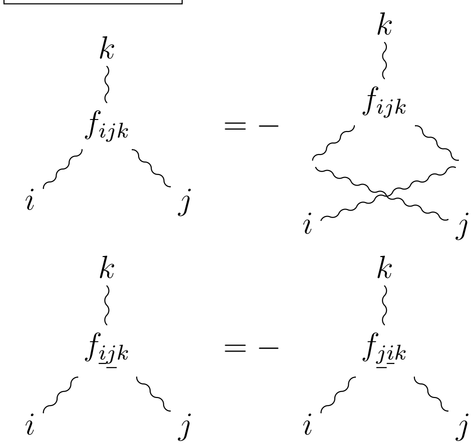
$$\boxed{if_{ijk} = \mathrm{tr}([T^i, T^j]T^k) = (T^i)_a{}^c (T^j)_b{}^a (T^k)_c{}^b - (T^j)_c{}^b (T^i)_a{}^c (T^k)_b{}^a}$$



(8.23)

Note that

$$\boxed{f_{ijk} = -f_{jik}}$$



(Convention CC)

(8.24)

(Convention L)

In fact, the tensor f_{ijk} is **totally antisymmetric** (i.e., it changes sign under a transposition of any two indices).

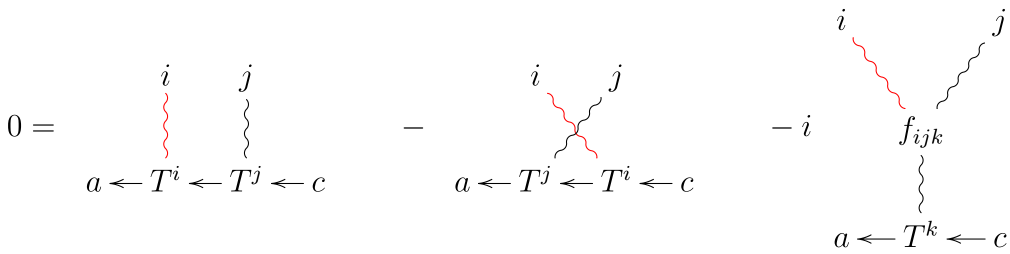
Claim 1 *f_{ijk} is a real number.*

proof:

$$\begin{aligned}
\left[i\mathrm{tr}([T^i, T^j]T^k)\right]^\dagger &= (-i)\mathrm{tr}(T^k[T^j, T^i]) && (8.25)\\
&= (-i)\mathrm{tr}([T^j, T^i]T^k) && (8.26)\\
&= i\mathrm{tr}([T^j, T^k]T^k) && (8.27)
\end{aligned}$$

QED

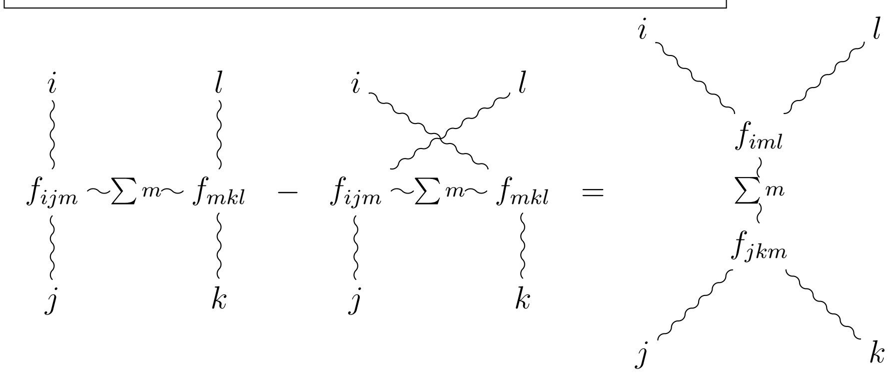
Note that the birdtrack for the Lie Algebra commutation relations Eq.(8.21) can be understood as the statement that the generators T^j are matrix invariants. Below we restate Eq.(8.21) to make that obvious



(8.28)

Claim 2

$$\boxed{f_{ijm}f_{mkl} - f_{ljm}f_{mki} = f_{iml}f_{jkm} \quad \textbf{(Jacobi identity)}}$$



(8.29)

proof:

Note that

$$\begin{aligned}
\mathrm{tr}\left([[T^i, T^j], T^k]T^l\right) &= \mathrm{tr}\left(f_{ijm}[T^m, T^k]\right) && (8.30)\\
&= \mathrm{tr}\left(f_{ijm}f_{mkl'}T^{l'}T^l\right) && (8.31)\\
&= f_{ijm}f_{mkl} && (8.32)
\end{aligned}$$

so the Jacobi identity can be restated as

$$\mathrm{tr}\left(\left\{[[T^i, T^j], T^k] + [[T^j, T^k], T^i] + [[T^k, T^i], T^j]\right\} T^l\right) = 0 \tag{8.33}$$

Hence, the claim follows if we can prove that

$$\underbrace{[[T^i, T^j], T^k] + [[T^j, T^k], T^i] + [[T^k, T^i], T^j]}_{\text{cyclic permutations of } ijk} = 0 \tag{8.34}$$

If we expand the left hand side on Eq.(8.34), we find 6 terms that cancel in pairs.
QED

Note Claim 2 can be undertood as the Lie Algebra commutation relations Eq.(8.21), but stated in the adrep instead of the funrep. Indeed, if

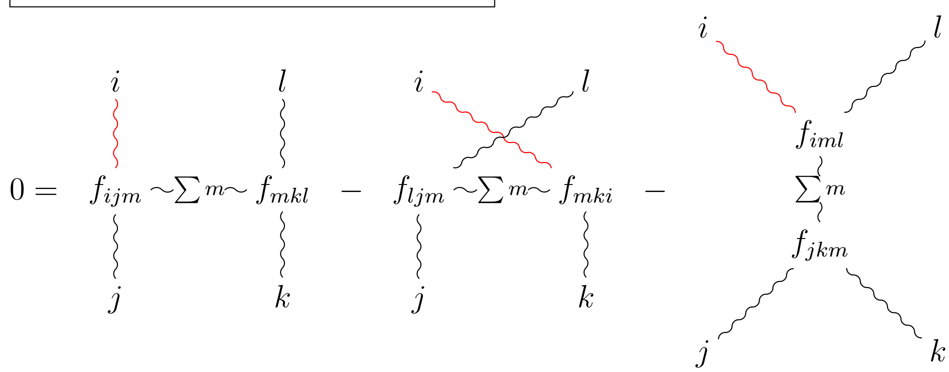
$$\mathbb{T}^i_{jk} = -iC_{ijk} \tag{8.35}$$

then Claim 2 becomes

$$(\mathbb{T}^i\mathbb{T}^l - \mathbb{T}^l\mathbb{T}^i)_{jk} = iC_{ilm}(\mathbb{T}^m)_{jk} \tag{8.36}$$

Note that Claim 2 can be understood as a statement of the fact that f_{ijk} is a tensor invariant.

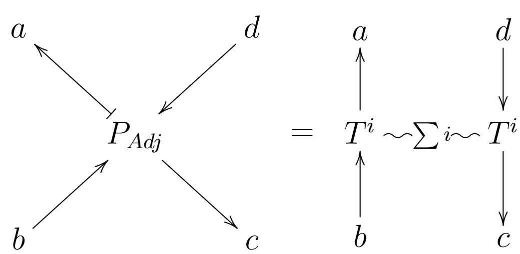
$$\boxed{0 = f_{ijm}f_{mkl} - f_{ljm}f_{mki} - f_{iml}f_{jkm}}$$



(8.37)

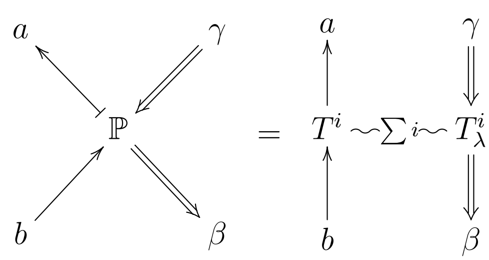
8.4 Two types of gluon exchanges

$$\boxed{(P_{Adj})_{a\ c}^{\ b\ d} = \sum_i (T^i)_a^{\ b}(T^i)_c^{\ d}}$$

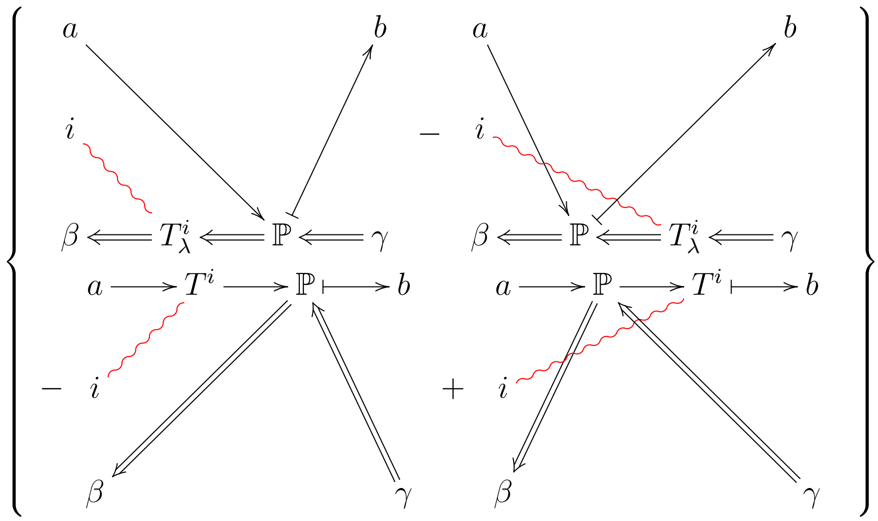


(8.38)

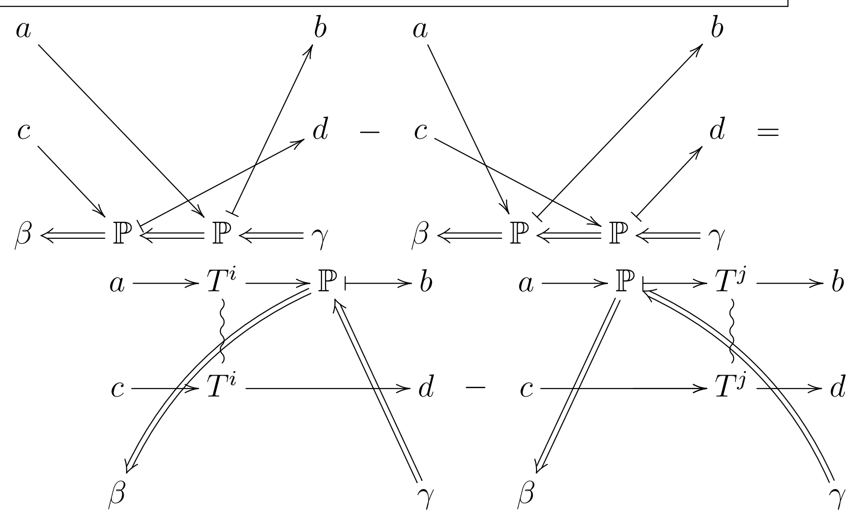
$$\boxed{\mathbb{P}_{a\ \beta}^{\ b\ \gamma} = \sum_i (T^i)_a^{\ b}(T^i_\lambda)_\beta^{\ \gamma}}$$



(8.39)

$0 =$  (8.40)

$$\boxed{\mathbb{P}_d{}^c{}_\beta{}^\nu \mathbb{P}_b{}^a{}_\nu{}^\gamma - \mathbb{P}_b{}^a{}_\beta{}^\nu \mathbb{P}_d{}^c{}_\nu{}^\gamma = (P_{Adj})_x{}^a{}_d{}^c \mathbb{P}_b{}^x{}_\beta{}^\gamma - \mathbb{P}_x{}^a{}_\beta{}^\gamma (P_{Adj})_b{}^x{}_d{}^c}$$

 (8.41)

$$\sum_{i,j} (T^i_\lambda)_\beta{}^\nu (T^j_\lambda)_\nu{}^\gamma \left[(T^i)_d{}^c (T^j)_b{}^a - (T^i)_b{}^a (T^j)_d{}^c \right] = \sum_k (T^k_\lambda)_\beta{}^\gamma \left[\begin{array}{l} (T^k)_b{}^x (P_{Adj})_x{}^a{}_d{}^c \\ -(P_{Adj})_b{}^x{}_d{}^c (T^k)_x{}^a \end{array} \right] \tag{8.42}$$

$$= \sum_{k,i} (T^i)_d{}^c (T^k_\lambda)_\beta{}^\gamma \left[\begin{array}{l} (T^k)_b{}^x (T^i)_x{}^a \\ -(T^i)_b{}^x (T^k)_x{}^a \end{array} \right] \tag{8.43}$$

$$\sum_{ijk} \frac{1}{2} i C_{ijk} (T^k_\lambda)_\beta{}^\gamma \left[(T^i)_d{}^c (T^j)_b{}^a - (T^i)_b{}^a (T^j)_d{}^c \right] = \sum_k (T^k_\lambda)_\beta{}^\gamma \left[\begin{array}{l} (T^k)_b{}^x (P_{Adj})_x{}^a{}_d{}^c \\ -(P_{Adj})_b{}^x{}_d{}^c (T^k)_x{}^a \end{array} \right] \tag{8.44}$$

$$\sum_{ij} \frac{1}{2} i C_{ijk} \left[(T^i)_d{}^c (T^j)_b{}^a - (T^i)_b{}^a (T^j)_d{}^c \right] = \left[\begin{array}{c} (T^k)_b{}^x (P_{Adj})_x{}^a{}_d{}^c \\ -(P_{Adj})_b{}^x{}_d{}^c (T^k)_x{}^a \end{array} \right] \tag{8.45}$$

$$T_d{}^c T_b{}^a - T_b{}^a T_d{}^c = (P_{Adj})^a{}_b{}^c{}_x T^x{}_d - T_x{}^c (P_{Adj})_b{}^{ax}{}_d \tag{8.46}$$

$$T^c{}_d T^a{}_b - T^a{}_b T^c{}_d = T^a{}_d \delta^c_b - T^c{}_b \delta^a_d \tag{8.47}$$

For $SU(N)$,

$$(P_{Adj})_b{}^x{}_d{}^c = \delta^{xc}_{bd} - \delta^{xc}_{db} \tag{8.48}$$

$$T^c{}_d T^a{}_b - T^a{}_b T^c{}_d = T^a{}_d \delta^c_b - T^c{}_b \delta^a_d \tag{8.49}$$

Bibliography

[1] Predrag Cvitanovic. *Group theory: birdtracks, Lie's, and exceptional groups.* Princeton University Press, 2008. `https://birdtracks.eu/course3/notes.pdf`.

[2] JP Elliott and PG Dawber. *Symmetry in Physics, vols. 1, 2.* Springer, 1979.

[3] Robert R. Tucci. Bayesuvius (free book). `https://github.com/rrtucci/Bayesuvius`.

[4] Robert R. Tucci. Quantum Bayesian nets. *International Journal of Modern Physics B*, 09(03):295–337, January 1995.