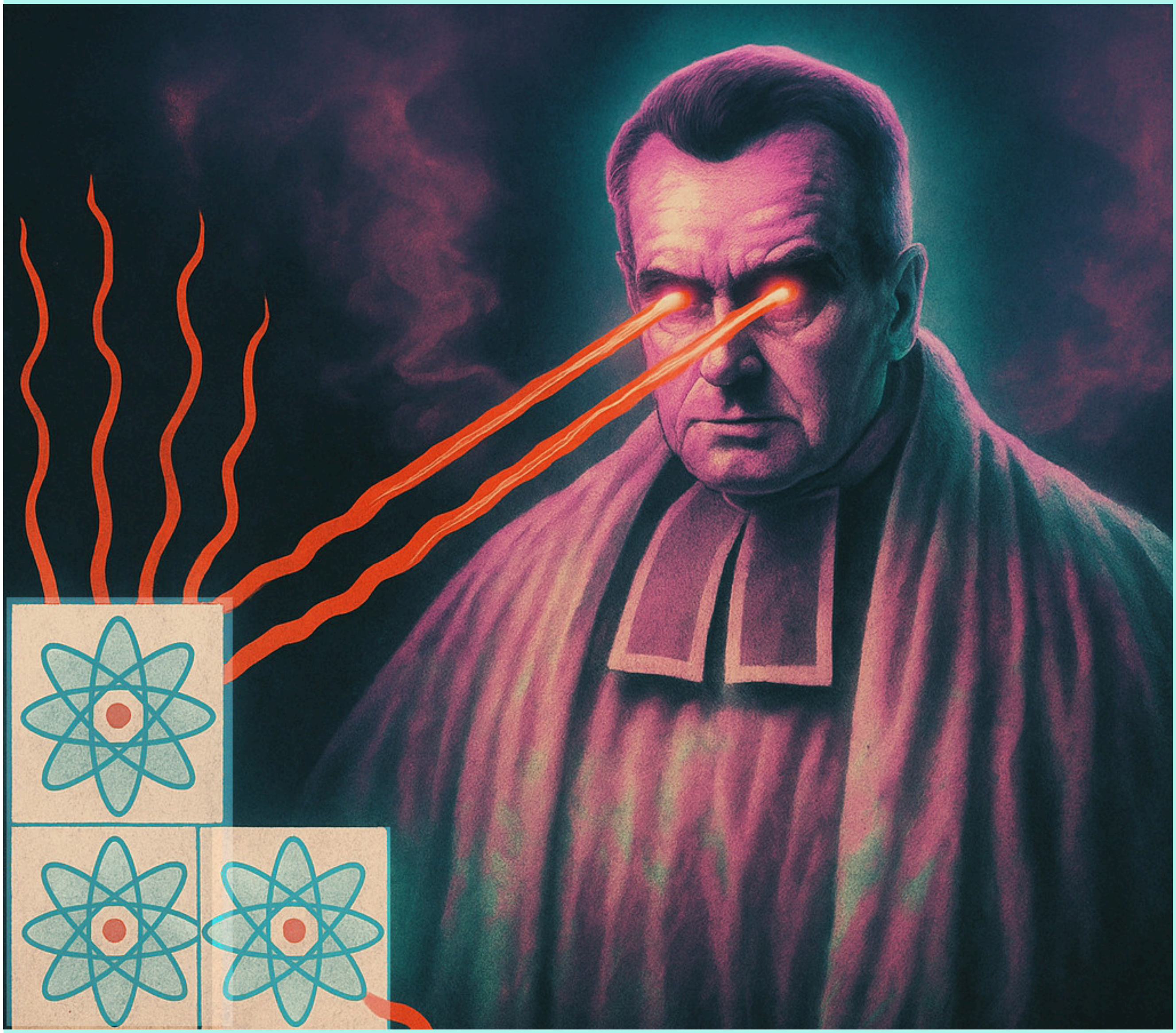
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayes Quantico
by Robert R. Tucci

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Appendix A

Spectral Decomposition and Eigenvalue Projection Operators: COMING SOON

$M \in \mathbb{C}^{d\times d}$

$$M|v\rangle = \lambda|v\rangle \tag{A.1}$$

If M is Hermitian ($H^\dagger = H$), its eigenvalues are real. ($\lambda = \langle\lambda|M\lambda\rangle \in \mathbb{R}$)

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M-\lambda) = 0 \tag{A.2}$$

If M is a Hermitain matrix, then there exists a unitary matric ($CC^\dagger = C^\dagger C = 1$) such that

$$CMC^\dagger = \begin{bmatrix} D_{\lambda_1} & 0 & 0 & 0 \\ 0 & D_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{\lambda_r} \end{bmatrix} \tag{A.3}$$

where

$$D_{\lambda_i} = \mathrm{diag}\underbrace{(\lambda_i, \lambda_i, \ldots, \lambda_i)}_{d_i \text{ times}} \tag{A.4}$$

$$d = \sum_{i=1}^{r} d_i \tag{A.5}$$

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{A.6}$$

$$CP_1C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2} \tag{A.7}$$

$$CP_2C^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1} \tag{A.8}$$

If $I^{d_i \times d_i}$ is the d_i dimensional unit matrix,

$$\begin{aligned} P_i &= C^\dagger diag(0, \ldots, 0, I^{d_i \times d_i}, 0, \ldots, 0)C & \text{(A.9)} \\ &= \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} & \text{(A.10)} \end{aligned}$$

Note that P_i are Hermitian ($P_i^\dagger = P_i$) because M is Hermitian and its eigenvalues are real.)

Note that P_i and M commute

$$[P_i, M] = P_i M - M P_i = 0 \tag{A.11}$$

orthogonal

$$P_i P_j = \delta(i,j) P_j \tag{A.12}$$

complete

$$\sum_i P_i = 1 \tag{A.13}$$

$$M = \sum_{i=1}^{r} P_i M P_i \tag{A.14}$$

$$d_i = \mathrm{tr} P_i \tag{A.15}$$

$$\begin{aligned} CMP_1C^\dagger &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \text{(A.16)} \\ &= \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \text{(A.17)} \end{aligned}$$

$$MP_i = \lambda_i P_i \text{ (no } i \text{ sum)} \tag{A.18}$$

$$f(M)P_i = f(\lambda_i) P_i \text{ (no } i \text{ sum)} \tag{A.19}$$

$M^{(1)}, M^{(2)}$

$$[M^{(1)}, M^{(2)}] = 0 \tag{A.20}$$

Use $M^{(1)}$ to decompose V into $\bigoplus_i V_i$. Use $M^{(2)}$ to decompose V_i into $\bigoplus_j V_{i,j}$. If $M^{(1)}$ and $M^{(2)}$ don't commute, let $P_i^{(1)}$ be the eigenvalue projection operators of $M^{(1)}$. The replace $M^{(2)}$ by $P_i^{(1)} M^{(2)} P_i^{(1)}$

$$[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] = 0 \tag{A.21}$$

Appendix B

Birdtracks: COMING SOON

B.1 Classical Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $P(y|x) \in [0,1]$ where $x \in val(\underline{x})$ and $y \in val(\underline{y})$

$$\sum_{y \in val(\underline{y})} P(y|x) = 1 \tag{B.1}$$

$$\mathcal{C} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ul] \arrow[ll] \end{tikzcd} \tag{B.2}$$

$$\mathcal{C}(a,b,c) = P(c|b,a)P(b|a)P(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ul] \arrow[ll] \end{tikzcd} \quad P(a) \tag{B.3}$$

$a^2 = (a_1, a_2)$

$$\mathcal{C}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^2 \arrow[ul, "\underline{a}_1"'] \arrow[ll, "\underline{a}_2"] \end{tikzcd} \tag{B.4}$$

$$\mathcal{C}'(a^2,b,c) = P(c|b,a_2)P(a_2|a^2)P(b|a_1)P(a_1|a^2)P(a^2) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^2 \arrow[ul, "a_1"'] \arrow[ll, "a_2"] \end{tikzcd} \quad P(a^2) \tag{B.5}$$

Marginalizer nodes $\underline{a}_1$ and $\underline{a}_2$ have the TPMs

$$P(a_i' | \underline{a}^2 = (a_1, a_2)) = \delta(a_i', a_i) \tag{B.6}$$

for $i = 1, 2$

B.2 Quantum Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $A(y|x) \in \mathbb{C}$ where $x \in val(\underline{x})$ and $y \in val(\underline{y})$

$$\sum_{y \in val(\underline{y})} |A(y|x)|^2 = 1 \tag{B.7}$$

$$\mathcal{Q} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ll] \arrow[ul] \end{tikzcd} \tag{B.8}$$

$$\mathcal{Q}(a,b,c) = A(c|b,a)A(b|a)A(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ll] \arrow[ul] \end{tikzcd} \quad A(a) \tag{B.9}$$

$a^2 = (a_1, a_2)$

$$\mathcal{Q}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^2 \arrow[ll, "\underline{a}_2"] \arrow[ul, "\underline{a}_1"'] \end{tikzcd} \tag{B.10}$$

$$\mathcal{Q}'(a^2,b,c) = A(c|b,a_2)A(a_2|a^2)A(b|a_1)A(a_1|a^2)A(a^2) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^2 \arrow[ll, "a_2"] \arrow[ul, "a_1"'] \end{tikzcd} \quad A(a^2) \tag{B.11}$$

Marginalizer nodes $\underline{a}_1$ and $\underline{a}_2$ have the TAMs

$$A(a_i'|\underline{a}^2 = (a_1,a_2)) = \delta(a_i', a_i) \tag{B.12}$$

for $i = 1, 2$

B.3 Birdtracks

$$\delta(b,a) = \mathbb{1}(a=b) = \delta_a^b = \; a \leftarrow\!\bullet\!\!-\!\!-\, b \tag{B.13}$$

$$\langle a,b|X^{\bullet,\underline{c},\bullet,\underline{d}}_{\underline{a},\bullet,\underline{b},\bullet}|c,d\rangle = X^{\bullet,c,\bullet,d}_{a,\bullet,b,\bullet} = \begin{tikzcd} a & X^{\bullet,\underline{c},\bullet,\underline{d}}_{\underline{a},\bullet,\underline{b},\bullet} \arrow[l] \arrow[dl] \\ b & \\ c \arrow[uur] & \\ d \arrow[uuur] & \end{tikzcd} \tag{B.14}$$

$$\begin{tikzcd} a & X^{\bullet,\underline{c},\bullet,\underline{d}}_{\underline{a},\bullet,\underline{b},\bullet} \arrow[l] \arrow[dl] \\ b & \\ c \arrow[uur] & \\ d \arrow[uuur] & \end{tikzcd} \quad \to \quad \begin{tikzcd} a,b & X^{\bullet,\underline{c},\bullet,\underline{d}}_{\underline{a},\bullet,\underline{b},\bullet} \arrow[l] \arrow[dl] \\ a,b & \\ c \arrow[uur] & \\ d \arrow[uuur] & \end{tikzcd} \tag{B.15}$$

$X^{\bullet,\underline{c},\bullet,\underline{d}}_{\underline{a},\bullet,\underline{b},\bullet} \in V^2 \otimes V_2$. Sometimes, we will omit denote this node simply b y X. This if okay as long as we are not using, X to also denote a different version of $X^{\underline{a},\bullet,\underline{b},\bullet}_{\bullet,\underline{c},\bullet,\underline{d}}$ with some of the indices raised or lowered or their order has been changed.

$$(X^\dagger)^{a,\bullet,b,\bullet}_{\bullet,c,\bullet,d} = \begin{tikzcd} a \arrow[r] & (X^\dagger)^{\underline{a},\bullet,\underline{b},\bullet}_{\bullet,\underline{c},\bullet,\underline{d}} \arrow[ddl] \arrow[dddl] \\ b \arrow[ur] & \\ c & \\ d & \end{tikzcd} \tag{B.16}$$

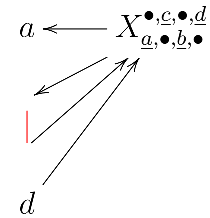
$$(X^\dagger)^{a,\bullet,b,\bullet}_{\bullet,c,\bullet,d} X^{\bullet,c,\bullet,d}_{a,\bullet,b,\bullet} = \begin{tikzcd} (X^\dagger)^{\underline{a},\bullet,\underline{b},\bullet}_{\bullet,\underline{c},\bullet,\underline{d}} \arrow[ddr] \arrow[dddr] & \sum a \arrow[l] & X^{\bullet,\underline{c},\bullet,\underline{d}}_{\underline{a},\bullet,\underline{b},\bullet} \arrow[l] \arrow[dl] \\ & \sum b \arrow[ul] & \\ & \sum c \arrow[uur] & \\ & \sum d \arrow[uuur] & \end{tikzcd} \tag{B.17}$$

$$= \begin{tikzcd} X^\dagger \arrow[ddr] \arrow[dddr] & \arrow[l] & X \arrow[l] \arrow[dl] \\ & \arrow[ul] & \\ & \arrow[uur] & \\ & \arrow[uuur] & \end{tikzcd} \tag{B.18}$$

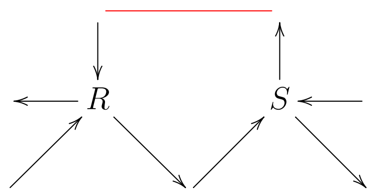
$a^m \in \mathbb{Z}_+^m$

$$R^{a_3^{m_3},b_2^{n_2}}_{b_3^{n_3},a_2^{m_2}} S^{a_2^{m_2},b_1^{n_1}}_{b_2^{n_2},a_1^{m_1}} = \begin{tikzcd} b_3^{n_3} & R \arrow[l] \arrow[dr] & \sum b_2^{n_2} \arrow[l] & S \arrow[l] \arrow[dr] & b_1^{n_1} \arrow[l] \\ a_3^{m_3} \arrow[ur] & & \sum a_2^{m_2} \arrow[ur] & & a_1^{m_1} \end{tikzcd} \tag{B.19}$$

$$\mathrm{tr}_{\underline{b}} X^{\bullet,\underline{b},\bullet,d}_{a,\bullet,\underline{b},\bullet} = \sum_b X^{\bullet,b,\bullet,d}_{a,\bullet,b,\bullet} = \tag{B.20}$$



(B.21)



Appendix C

Clebsch-Gordan Coefficients: COMING SOON

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d\times d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d\times d} C^{d\times d} \tag{C.1}$$

Let $b^{nb} = (b_1, b_2, \ldots, b_{nb})$ where $b_i \in Z_{[0,nb_i]}$ and $a \in Z_{[1,d_\lambda]}$. Hence,

$$d_\lambda = \prod_{i=1}^{nb} nb_i \tag{C.2}$$

$$(C_\lambda)_a^{b^{nb}} = \begin{tikzcd} & & b_1 \arrow[dl] \\ a & C_\lambda \arrow[l] & b_2 \arrow[l] \\ & & b_{nb} \arrow[ul] \end{tikzcd} \tag{C.3}$$

$$\begin{bmatrix} 0 & (C^\dagger)_\lambda^{d\times d_\lambda} & 0 \end{bmatrix}^{d\times d} = (C^\dagger)^{d\times d} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d\times d} \tag{C.4}$$

$$(C_\lambda^\dagger)^a_{b^{nb}} = \begin{tikzcd} b_1 & & \\ b_2 & (C_\lambda^\dagger) \arrow[l] \arrow[ul] \arrow[dl] & \lambda, a \arrow[l] \\ b_{nb} & & \end{tikzcd} \tag{C.5}$$

More generally, some of the b_i indices may lowered and their arrows changed to outgoing instead of ingoing. Each b_i represents a different rep (or irrep)

$$(C^\dagger_\lambda)^{b^{nb}}_a (C_\lambda)^a_{(b')^{nb}} = (P_\lambda)^{b^{nb}}_{(b')^{nb}} \tag{C.6}$$

$$\begin{tikzcd}
b_1 & & & & b'_1 \arrow[dl] & & & \\
b_2 & (C^\dagger_\lambda) \arrow[l] \arrow[ul] \arrow[dl] & \sum a \arrow[l] & C_\lambda \arrow[l] & b'_2 \arrow[l] & = b^{nb} & P_\lambda \arrow[l] & (b')^{nb} \arrow[l] \\
b_{nb} & & & & b'_{nb} \arrow[ul] & & &
\end{tikzcd} \tag{C.7}$$

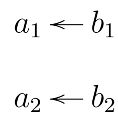
$$(C_\lambda)^{(a')^{na}}_{b^{nb}} (C^\dagger_\mu)^{b^{nb}}_a = \delta(\lambda, \mu) \delta^{(a')^{na}}_a \tag{C.8}$$

$$\begin{tikzcd}
 & & \sum b_1 \arrow[dl] & & & & & \\
a & C_\lambda \arrow[l] & \sum b_2 \arrow[l] & (C^\dagger_\mu) \arrow[l] \arrow[ul] \arrow[dl] & a' \arrow[l] & = \delta(\mu, \lambda)\ a & \bullet \arrow[l] & a' \arrow[l, dash] \\
 & & \sum b_{nb} \arrow[ul] & & & & &
\end{tikzcd} \tag{C.9}$$

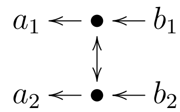
Chapter 18

Symmetrization: COMING SOON

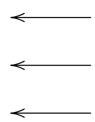
$$\mathbb{1}^{b_2,b_1}_{a_1,a_2} = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2} = \tag{18.1}$$



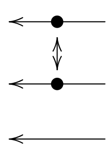
$$(\sigma_{(1,2)})^{b_2,b_1}_{a_1,a_2} = \delta^{b_2}_{a_1}\delta^{b_1}_{a_2} = \tag{18.2}$$



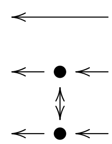
$$\mathbb{1} = \tag{18.3}$$



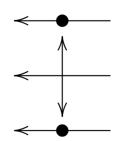
$$\sigma_{(1,2)} =$$



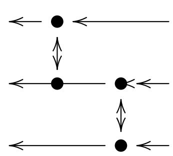
$$\sigma_{(2,3)} =$$



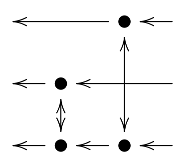
$$\sigma_{(1,3)} = \tag{18.4}$$



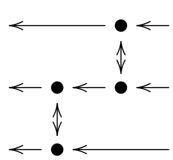
$$\sigma_{(1,2,3)} =$$



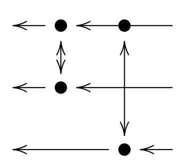
$$= \tag{18.5}$$

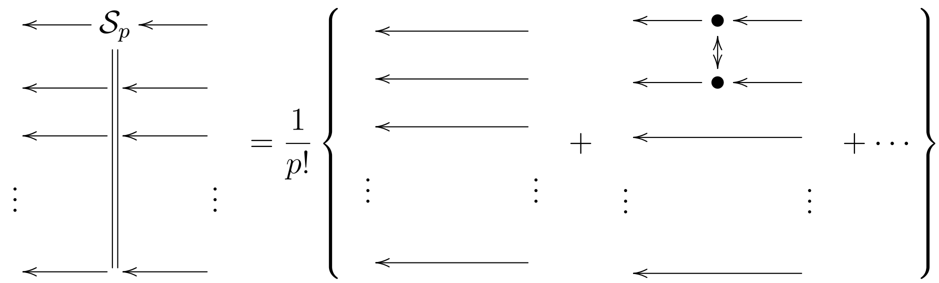


$$\sigma_{(1,3,2)} =$$



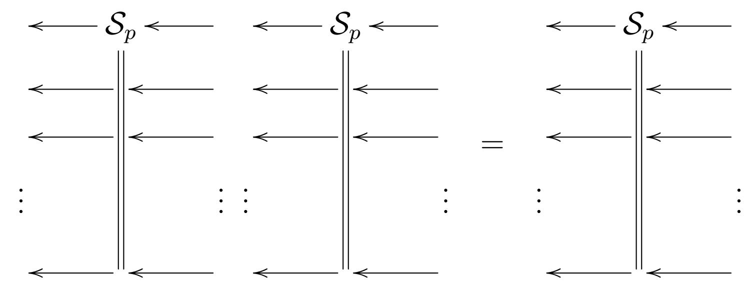
$$= \tag{18.6}$$





(18.7)

$$\mathcal{S}_p^2 = \mathcal{S}_p \tag{18.8}$$



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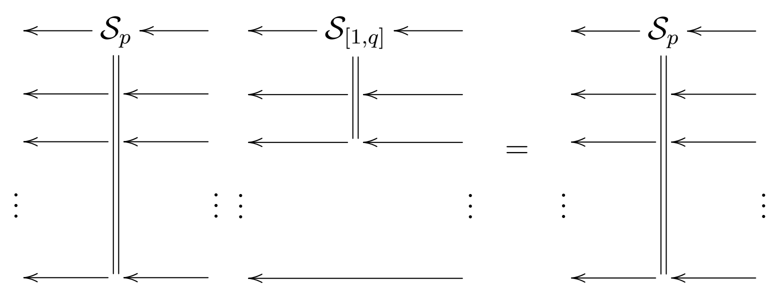

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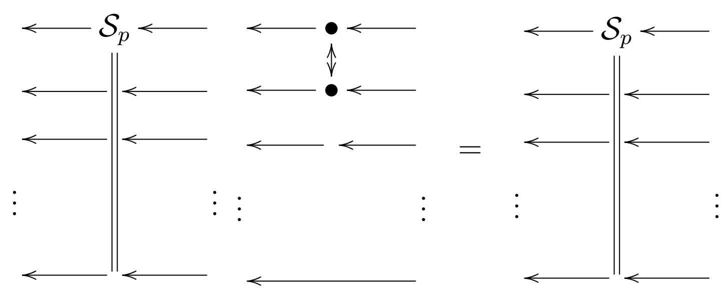
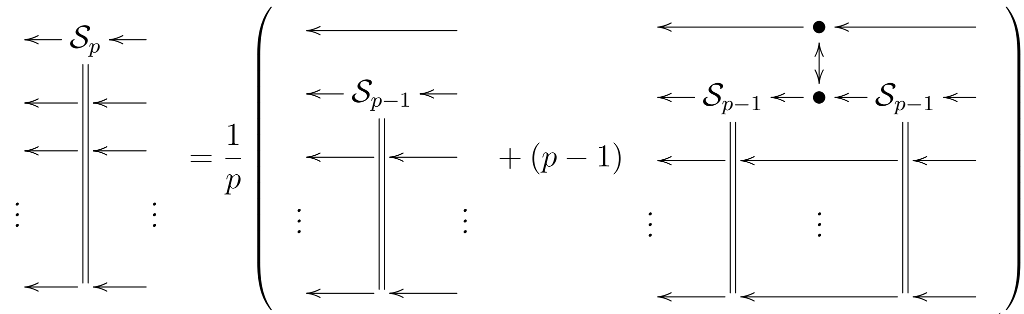


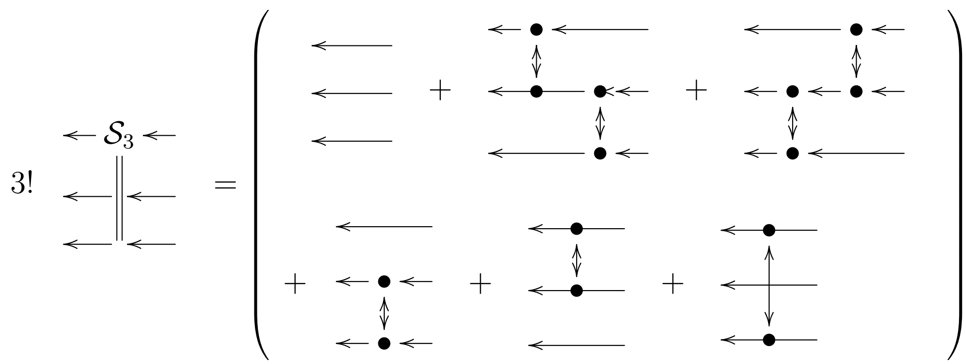
(18.11)

Claim 1



(18.12)

proof: We only prove it for $p = 3$.



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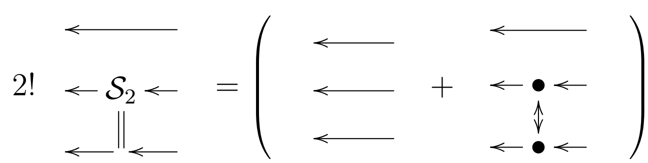

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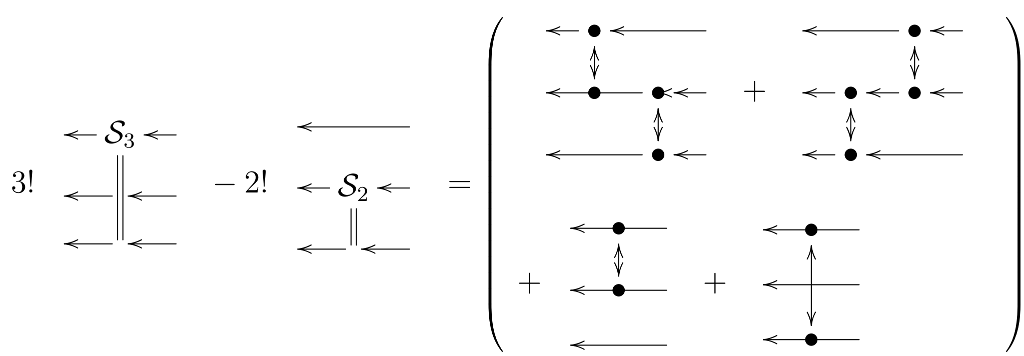

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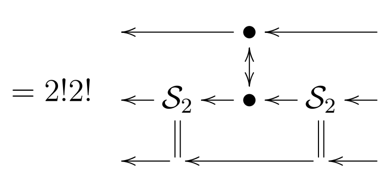
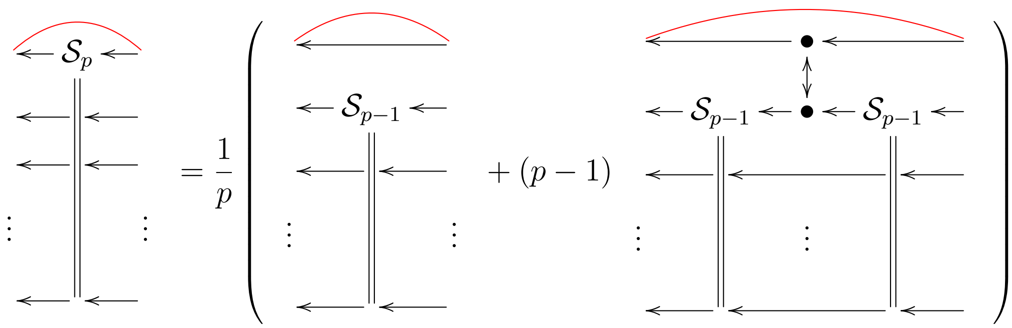


(18.16)

QED



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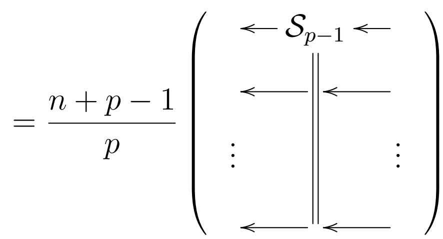

(18.18)

$$\mathrm{tr}_{\underline{a}_1} \mathcal{S}_p = \frac{n+p-1}{p} \mathcal{S}_{p-1} \tag{18.19}$$

$$\mathrm{tr}_{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k} \mathcal{S}_p = \frac{(n+p-1)(n-p-2)\ldots(n-p-k)}{p(p-1)\ldots(p-k+1)} \mathcal{S}_{p-k} \tag{18.20}$$

$$d_{\mathcal{S}_p} = \mathrm{tr}_{\underline{a}^p} \mathcal{S} = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \tag{18.21}$$

For $p = 2$,

$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \tag{18.22}$$

Bibliography