

# BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF  
QUANTUM BAYESIAN NETWORKS



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# **Bayesuvious Quantico,** a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download  
the latest version, go to

<https://github.com/rrtucci/bayes-quantico>

## **Bayes Quantico**

by Robert R. Tucci

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# Appendix A

## Spectral Decomposition and Eigenvalue Projection Operators: COMING SOON

$$M \in \mathbb{C}^{d \times d}$$

$$M|v\rangle = \lambda|v\rangle \quad (\text{A.1})$$

If  $M$  is Hermitian ( $H^\dagger = H$ ), its eigenvalues are real. ( $\lambda = \langle \lambda | M | \lambda \rangle \in \mathbb{R}$ )

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M - \lambda) = 0 \quad (\text{A.2})$$

If  $M$  is a Hermitain matrix, then there exists a unitary matrix ( $CC^\dagger = C^\dagger C = 1$ ) such that

$$CMC^\dagger = \begin{bmatrix} D_{\lambda_1} & 0 & 0 & 0 \\ 0 & D_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{\lambda_r} \end{bmatrix} \quad (\text{A.3})$$

where

$$D_{\lambda_i} = \text{diag}(\underbrace{\lambda_i, \lambda_i, \dots, \lambda_i}_{d_i \text{ times}}) \quad (\text{A.4})$$

$$d = \sum_{i=1}^r d_i \quad (\text{A.5})$$

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (\text{A.6})$$

$$CP_1C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2} \quad (\text{A.7})$$

$$CP_2C^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1} \quad (\text{A.8})$$

If  $I^{d_i \times d_i}$  is the  $d_i$  dimensional unit matrix,

$$P_i = C^\dagger \text{diag}(0, \dots, 0, I^{d_i \times d_i}, 0, \dots, 0)C \quad (\text{A.9})$$

$$= \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (\text{A.10})$$

Note that  $P_i$  are Hermitian ( $P_i^\dagger = P_i$ ) because  $M$  is Hermitian and its eigenvalues are real.)

Note that  $P_i$  and  $M$  commute

$$[P_i, M] = P_i M - M P_i = 0 \quad (\text{A.11})$$

orthogonal

$$P_i P_j = \delta(i, j) P_j \quad (\text{A.12})$$

complete

$$\sum_i P_i = 1 \quad (\text{A.13})$$

$$M = \sum_{i=1}^r P_i M P_i \quad (\text{A.14})$$

$$d_i = \text{tr} P_i \quad (\text{A.15})$$

$$C M P_1 C^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.16})$$

$$= \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.17})$$

$$M P_i = \lambda_i P_i \text{ (no } i \text{ sum)} \quad (\text{A.18})$$

$$f(M) P_i = f(\lambda_i) P_i \text{ (no } i \text{ sum)} \quad (\text{A.19})$$

$M^{(1)}, M^{(2)}$

$$[M^{(1)}, M^{(2)}] = 0 \quad (\text{A.20})$$

Use  $M^{(1)}$  to decompose  $V$  into  $\bigoplus_i V_i$ . Use  $M^{(2)}$  to decompose  $V_i$  into  $\bigoplus_j V_{i,j}$ . If  $M^{(1)}$  and  $M^{(2)}$  don't commute, let  $P_i^{(1)}$  be the eigenvalue projection operators of  $M^{(1)}$ . The replace  $M^{(2)}$  by  $P_i^{(1)} M^{(2)} P_i^{(1)}$

$$[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] = 0 \quad (\text{A.21})$$



# Appendix B

## Birdtracks: COMING SOON

### B.1 Classical Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix)  $P(y|x) \in [0, 1]$  where  $x \in \text{val}(\underline{x})$  and  $y \in \text{val}(\underline{y})$

$$\sum_{y \in \text{val}(\underline{y})} P(y|x) = 1 \quad (\text{B.1})$$

$$\mathcal{C} = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow \\ \underline{c} & \longleftarrow & \underline{a} \end{array} \quad (\text{B.2})$$

$$\mathcal{C}(a, b, c) = P(c|b, a)P(b|a)P(a) = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow \\ c & \longleftarrow & a \end{array} P(a) \quad (\text{B.3})$$

$$a^2 = (a_1, a_2)$$

$$\mathcal{C}' = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow \\ \underline{c} & \longleftarrow \underline{a_2} & \underline{a^2} \end{array} \quad (\text{B.4})$$

$$\mathcal{C}'(a^2, b, c) = P(c|b, a_2)P(a_2|a^2)P(b|a_1)P(a_1|a^2)P(a^2) = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow \\ c & \longleftarrow a_2 & a^2 \end{array} P(a^2) \quad (\text{B.5})$$

Marginalizer nodes  $\underline{a_1}$  and  $\underline{a_2}$  have the TPMs

$$P(a'_i|\underline{a^2} = (a_1, a_2)) = \delta(a'_i, a_i) \quad (\text{B.6})$$

for  $i = 1, 2$

## B.2 Quantum Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix)  $A(y|x) \in \mathbb{C}$  where  $x \in \text{val}(\underline{x})$  and  $y \in \text{val}(\underline{y})$

$$\sum_{y \in \text{val}(\underline{y})} |A(y|x)|^2 = 1 \quad (\text{B.7})$$

$$\mathcal{Q} = \begin{array}{ccc} & \underline{b} & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow & \underline{a} \end{array} \quad (\text{B.8})$$

$$\mathcal{Q}(a, b, c) = A(c|b, a)A(b|a)A(a) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow & a \end{array} A(a) \quad (\text{B.9})$$

$$a^2 = (a_1, a_2)$$

$$\mathcal{Q}' = \begin{array}{ccc} & \underline{b} & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow \underline{a_2} & \underline{a^2} \end{array} \quad (\text{B.10})$$

$$\mathcal{Q}'(a^2, b, c) = A(c|b, a_2)A(a_2|a^2)A(b|a_1)A(a_1|a^2)A(a^2) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow a_2 & a^2 \end{array} A(a^2) \quad (\text{B.11})$$

Marginalizer nodes  $\underline{a_1}$  and  $\underline{a_2}$  have the TAMs

$$A(a'_i|\underline{a^2} = (a_1, a_2)) = \delta(a'_i, a_i) \quad (\text{B.12})$$

for  $i = 1, 2$

## B.3 Birdtracks

$$\delta(b, a) = \mathbb{1}(a = b) = \delta_a^b = a \leftarrow \bullet \longrightarrow b \quad (\text{B.13})$$

$$\langle a, b | X_{\underline{a}, \bullet, \underline{b}, \bullet}^{\bullet, \underline{c}, \bullet, \underline{d}} | c, d \rangle = X_{\underline{a}, \bullet, \underline{b}, \bullet}^{\bullet, \underline{c}, \bullet, \underline{d}} = \begin{array}{c} a \longleftarrow X_{\underline{a}, \bullet, \underline{b}, \bullet}^{\bullet, \underline{c}, \bullet, \underline{d}} \\ \swarrow \quad \nearrow \\ b \quad c \\ \swarrow \quad \nearrow \\ d \end{array} \quad (\text{B.14})$$

$$\begin{array}{ccc}
a \longleftarrow X_{\underline{a}, \bullet, \underline{b}, \bullet}^{\bullet, \underline{c}, \bullet, \underline{d}} & & a, b \longleftarrow X_{\underline{a}, \bullet, \underline{b}, \bullet}^{\bullet, \underline{c}, \bullet, \underline{d}} \\
b \swarrow & \rightarrow & a, b \swarrow \\
c \nearrow & & c \nearrow \\
d \nearrow & & d \nearrow
\end{array} \quad (\text{B.15})$$

$X_{\underline{a}, \bullet, \underline{b}, \bullet}^{\bullet, \underline{c}, \bullet, \underline{d}} \in V^2 \otimes V_2$ . Sometimes, we will omit denote this node simply by  $X$ . This is okay as long as we are not using,  $X$  to also denote a different version of  $X_{\bullet, \underline{c}, \bullet, \underline{d}}^{\underline{a}, \bullet, \underline{b}, \bullet}$  with some of the indices raised or lowered or their order has been changed.

$$\begin{array}{ccc}
a \longrightarrow (X^\dagger)_{\bullet, \underline{c}, \bullet, \underline{d}}^{\underline{a}, \bullet, \underline{b}, \bullet} & & \\
b \swarrow & & \\
c \nearrow & & \\
d \nearrow & &
\end{array} \quad (\text{B.16})$$

$$\begin{array}{ccc}
(X^\dagger)_{\bullet, \underline{c}, \bullet, \underline{d}}^{\underline{a}, \bullet, \underline{b}, \bullet} \longleftarrow \sum a \longleftarrow X_{\underline{a}, \bullet, \underline{b}, \bullet}^{\bullet, \underline{c}, \bullet, \underline{d}} & & \\
\swarrow \quad \searrow & \swarrow \quad \searrow & \\
\sum b & & \\
\swarrow \quad \searrow & & \\
\sum c & & \\
\swarrow \quad \searrow & & \\
\sum d & &
\end{array} \quad (\text{B.17})$$

$$\begin{array}{ccc}
X^\dagger \longleftarrow \longleftarrow X & & \\
\swarrow \quad \searrow & \swarrow \quad \searrow & \\
& &
\end{array} \quad (\text{B.18})$$

$$a^m \in \mathbb{Z}_+^m$$

$$\begin{array}{ccc}
b_3^{n_3} \longleftarrow R \longleftarrow \sum b_2^{n_2} \longleftarrow S \longleftarrow b_1^{n_1} & & \\
a_3^{m_3} \nearrow & \nearrow \quad \searrow & \nearrow \\
& \sum a_2^{m_2} & \\
& \searrow & \\
& a_1^{m_1} &
\end{array} \quad (\text{B.19})$$

$$\text{tr}_{\underline{b}} X_{a, \bullet, \underline{b}, \bullet}^{\bullet, b, \bullet, d} = \sum_b X_{a, \bullet, b, \bullet}^{\bullet, b, \bullet, d} = \begin{array}{c} \begin{array}{c} a \longleftarrow X_{\underline{a}, \bullet, \underline{b}, \bullet}^{\bullet, \underline{c}, \bullet, \underline{d}} \\ \swarrow \nearrow \\ d \end{array} \end{array} \quad (\text{B.20})$$

$$\begin{array}{c} \text{---} \\ \downarrow \quad \uparrow \\ \leftarrow R \quad S \rightarrow \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \end{array} \quad (\text{B.21})$$

# Appendix C

## Clebsch-Gordan Coefficients: COMING SOON

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d \times d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} C^{d \times d} \quad (\text{C.1})$$

Let  $b^{nb} = (b_1, b_2, \dots, b_{nb})$  where  $b_i \in Z_{[0, nb_i]}$  and  $a \in Z_{[1, d_\lambda]}$ . Hence,

$$d_\lambda = \prod_{i=1}^{nb} nb_i \quad (\text{C.2})$$

$$(C_\lambda)_{a^{b^{nb}}} = a \longleftarrow C_\lambda \begin{matrix} \swarrow b_1 \\ \longleftarrow b_2 \\ \searrow b_{nb} \end{matrix} \quad (\text{C.3})$$

$$\begin{bmatrix} 0 & (C^\dagger)_\lambda^{d \times d_\lambda} & 0 \end{bmatrix}^{d \times d} = (C^\dagger)^{d \times d} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} \quad (\text{C.4})$$

$$(C_\lambda^\dagger)_{b^{nb}}^a = \begin{matrix} \swarrow b_1 \\ \longleftarrow b_2 \\ \searrow b_{nb} \end{matrix} (C_\lambda^\dagger) \longleftarrow \lambda, a \quad (\text{C.5})$$

More generally, some of the  $b_i$  indices may be lowered and their arrows changed to outgoing instead of ingoing. Each  $b_i$  represents a different rep (or irrep)

$$(C_\lambda^\dagger)_a^{b^{nb}} (C_\lambda)_a^b = (P_\lambda)_{(b')^{nb}}^{b^{nb}} \quad (\text{C.6})$$

$$\begin{array}{c}
b_1 \swarrow \\
b_2 \longleftarrow (C_\lambda^\dagger) \longleftarrow \sum a \longleftarrow C_\lambda \longleftarrow b'_2 \quad = \quad b^{nb} \longleftarrow P_\lambda \longleftarrow (b')^{nb} \\
b_{nb} \searrow \qquad \qquad \qquad \swarrow b'_{nb}
\end{array} \quad (\text{C.7})$$

$$(C_\lambda)_{b^{nb}}^{(a')^{na}} (C_\mu^\dagger)_a^{b^{nb}} = \delta(\lambda, \mu) \delta_a^{(a')^{na}} \quad (\text{C.8})$$

$$\begin{array}{c}
\qquad \qquad \qquad \sum b_1 \\
\qquad \qquad \swarrow \qquad \nwarrow \\
a \longleftarrow C_\lambda \longleftarrow \sum b_2 \longleftarrow (C_\mu^\dagger) \longleftarrow a' \quad = \quad \delta(\mu, \lambda) \quad a \longleftarrow \bullet \longleftarrow a' \\
\qquad \qquad \nwarrow \qquad \swarrow \\
\qquad \qquad \qquad \sum b_{nb}
\end{array} \quad (\text{C.9})$$

# Chapter 18

## Symmetrization: COMING SOON

$$\mathbb{1}_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} = \begin{array}{c} a_1 \leftarrow b_1 \\ a_2 \leftarrow b_2 \end{array} \quad (18.1)$$

$$(\sigma_{(1,2)})_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} = \begin{array}{c} a_1 \leftarrow \bullet \leftarrow b_1 \\ \updownarrow \\ a_2 \leftarrow \bullet \leftarrow b_2 \end{array} \quad (18.2)$$

$$\mathbb{1} = \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \quad (18.3)$$

$$\sigma_{(1,2)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \end{array} \quad \sigma_{(2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad \sigma_{(1,3)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \bullet \end{array} \quad (18.4)$$

$$\sigma_{(1,2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \end{array} \quad (18.5)$$

$$\sigma_{(1,3,2)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad (18.6)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \left\| \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \right. = \frac{1}{p!} \left\{ \begin{array}{cc} \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \\ \vdots & \vdots \\ \leftarrow & \leftarrow \end{array} + \begin{array}{cc} \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \\ \vdots & \vdots \\ \leftarrow & \leftarrow \end{array} + \dots \right\} \quad (18.7)$$

$$\mathcal{S}_p^2 = \mathcal{S}_p \quad (18.8)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \left\| \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \right. = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \left\| \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \right. \quad (18.9)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \left\| \begin{array}{c} \leftarrow \mathcal{S}_{[1,q]} \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \right. = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \left\| \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \right. \quad (18.10)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \left\| \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \right. = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \left\| \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{array} \right. \quad (18.11)$$



### Claim 1

$$= \frac{1}{p} \left( \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \qquad \qquad \vdots \\ \leftarrow \parallel \leftarrow \end{array} \right. + (p-1) \left. \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \vdots \qquad \qquad \vdots \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} \right) \quad (18.12)$$

**proof:** We only prove it for  $p = 3$ .

$$3! \left( \begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \\ + \\ \text{Diagram 3} \\ + \\ \text{Diagram 4} \\ + \\ \text{Diagram 5} \\ + \\ \text{Diagram 6} \\ + \\ \text{Diagram 7} \\ + \\ \text{Diagram 8} \end{array} \right) = \dots \quad (18.13)$$

$$2! \begin{array}{c} \leftarrow \\ \leftarrow \mathcal{S}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left( \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} + \begin{array}{c} \leftarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \updownarrow \bullet \leftarrow \end{array} \right) \quad (18.14)$$

$$3! \begin{array}{c} \leftarrow \mathcal{S}_3 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} - 2! \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} = \left( \begin{array}{cc} \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} & \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \\ + & + \\ \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \end{array} & \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \end{array} \end{array} \right) \quad (18.15)$$

$$= 2!2! \quad (18.16)$$

QED

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p} \left( \begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} + (p-1) \begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \leftarrow \leftarrow \\ \parallel \quad \parallel \\ \vdots \\ \leftarrow \leftarrow \leftarrow \leftarrow \end{array} \right) \quad (18.17)$$

$$= \frac{n+p-1}{p} \left( \begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \right) \quad (18.18)$$

$$\text{tr}_{\underline{a}_1} \mathcal{S}_p = \frac{n+p-1}{p} \mathcal{S}_{p-1} \quad (18.19)$$

$$\text{tr}_{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k} \mathcal{S}_p = \frac{(n+p-1)(n-p-2) \dots (n-p-k)}{p(p-1) \dots (p-k+1)} \mathcal{S}_{p-k} \quad (18.20)$$

$$d_{\mathcal{S}_p} = \text{tr}_{\underline{a}^p} \mathcal{S} = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \quad (18.21)$$

For  $p = 2$ ,

$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \quad (18.22)$$

# Bibliography