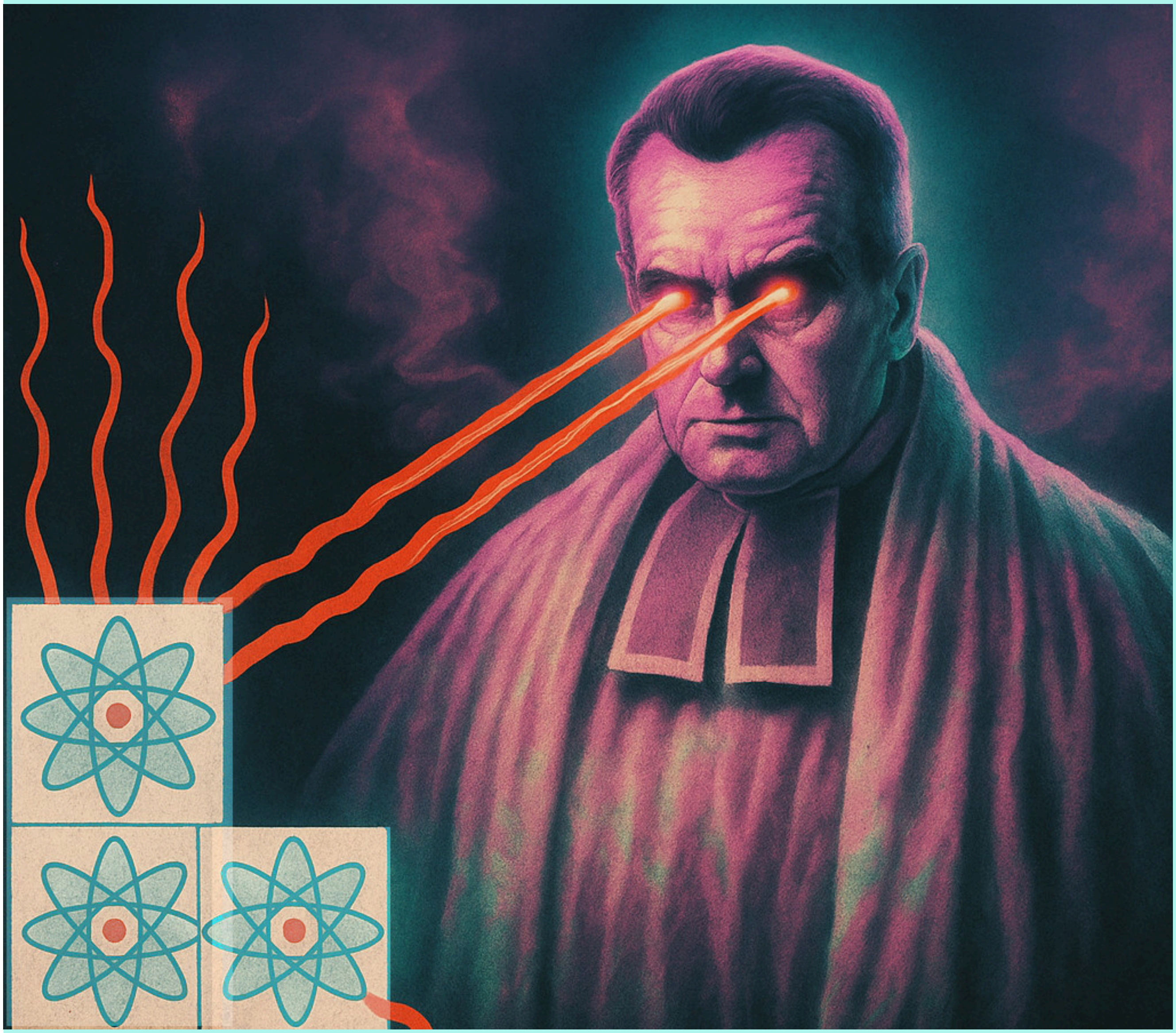
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

Robert R. Tucci
www.ar-tiste.xyz

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayesuvius Quantico
by Robert R. Tucci

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Appendices

Chapter 17

Symplectic Groups: COMING SOON

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

n even

$$f = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \tag{17.1}$$

$$f^T f = I_n, \quad f^2 = -I_n \tag{17.2}$$

f^2 is an invariant matrix so, by Schur's Lemma, it must be proportional to the identity.

$$Sp(n) = \{G \in GL(n, \mathbb{C}) : G^T f G = f\} \tag{17.3}$$

Claim 21 *If $G \in Sp(n)$, then $det(G) = 1$.*

proof:

$$det(G^T f G) = det(f) \tag{17.4}$$

$$det() = det(I_{n/2})det(-I_{n/2}) = (-1)^{n/2} \tag{17.5}$$

$$det(G^T f G) = det^2(G)(-1)^{n/2} \tag{17.6}$$

$$det^2(G) = 1 \tag{17.7}$$

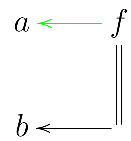
$det(G) = \pm 1$. $Sp(n)$ is connected, $I_n \in Sp(n)$ and $det(I_n) = 1$. Hence $det(G) = 1$.

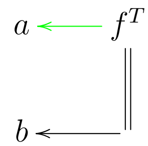
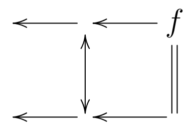
QED

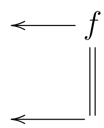
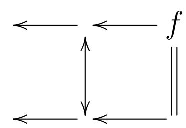
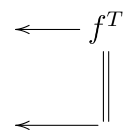
$a, , b \in \{1, 2, \ldots, n\}$. n even

Indices may be raised or lowered without changing tensors. In particular

$$f_{ab} = f^a{}_b = f_a{}^b = f^{ab} \tag{17.8}$$

$f_{ab} =$  $= a \leftarrow f \longrightarrow b$ (17.9)

$f_{ba} = f^T_{ab} =$  $=$  (17.10)

$\boxed{f_{ab} = -f_{ba}}$,  $= -$  $= -$  (17.11)

$Sp(n)$ leaves invariant the skew symmetric form

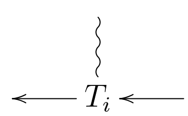
$$h(p,q) = f_{ab} p^a q^b \tag{17.12}$$

$$h(Gp, Gq) = h(p,q) \implies G^{b'}{}_b G^{a'}{}_a f_{a'b'} = f_{ab} \implies G^T f G = f \tag{17.13}$$

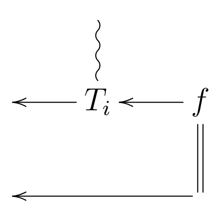
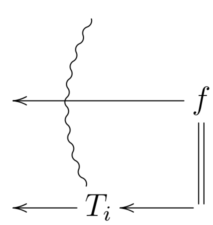
$$\boxed{f^T_{ac} f^{cb} = \delta^b_a}, \qquad \longleftarrow f^T \longrightarrow f \longleftarrow \;=\; \longleftarrow \tag{17.14}$$

$$\boxed{f_{ac} f^{cb} = -\delta^b_a}, \qquad \longleftarrow f \longrightarrow f \longleftarrow \;=\; - \longleftarrow \tag{17.15}$$

Generator $(T_i)_{ab}$:

$(T_i)_a{}^b =$  (17.16)

$$\boxed{(T_i)_a{}^c f_{cb} + \underbrace{(T_i)_b{}^c f_{ac}}_{f_{ac}(T_i^T)^c{}_b} = 0}$$

 $+$  $= 0$ (17.17)

Bibliography

[1] Predrag Cvitanovic. *Group theory: birdtracks, Lie's, and exceptional groups*. Princeton University Press, 2008. `https://birdtracks.eu/course3/notes.pdf`.

[2] JP Elliott and PG Dawber. *Symmetry in Physics, vols. 1, 2*. Springer, 1979.

[3] Robert R. Tucci. Bayesuvius (free book). `https://github.com/rrtucci/Bayesuvius`.

[4] Robert R. Tucci. Quantum Bayesian nets. *International Journal of Modern Physics B*, 09(03):295–337, January 1995.