

BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF
QUANTUM BAYESIAN NETWORKS



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Bayesuvious Quantico, a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download
the latest version, go to

<https://github.com/rrtucci/bayes-quantico>

Bayes Quantico

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Appendices

Appendix A

Spectral Decomposition and Eigenvalue Projection Operators: COMING SOON

$$M \in \mathbb{C}^{d \times d}$$

$$M|v\rangle = \lambda|v\rangle \quad (\text{A.1})$$

If M is Hermitian ($H^\dagger = H$), its eigenvalues are real. ($\lambda = \langle \lambda | M | \lambda \rangle \in \mathbb{R}$)

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M - \lambda) = 0 \quad (\text{A.2})$$

If M is a Hermitain matrix, then there exists a unitary matrix ($CC^\dagger = C^\dagger C = 1$) such that

$$CMC^\dagger = \begin{bmatrix} D_{\lambda_1} & 0 & 0 & 0 \\ 0 & D_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{\lambda_r} \end{bmatrix} \quad (\text{A.3})$$

where

$$D_{\lambda_i} = \text{diag}(\underbrace{\lambda_i, \lambda_i, \dots, \lambda_i}_{d_i \text{ times}}) \quad (\text{A.4})$$

$$d = \sum_{i=1}^r d_i \quad (\text{A.5})$$

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (\text{A.6})$$

$$CP_1C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2} \quad (\text{A.7})$$

$$CP_2C^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1} \quad (\text{A.8})$$

If $I^{d_i \times d_i}$ is the d_i dimensional unit matrix,

$$P_i = C^\dagger \text{diag}(0, \dots, 0, I^{d_i \times d_i}, 0, \dots, 0)C \quad (\text{A.9})$$

$$= \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (\text{A.10})$$

Note that P_i are Hermitian ($P_i^\dagger = P_i$) because M is Hermitian and its eigenvalues are real.)

Note that P_i and M commute

$$[P_i, M] = P_i M - M P_i = 0 \quad (\text{A.11})$$

orthogonal

$$P_i P_j = \delta(i, j) P_j \quad (\text{A.12})$$

complete

$$\sum_i P_i = 1 \quad (\text{A.13})$$

$$M = \sum_{i=1}^r P_i M P_i \quad (\text{A.14})$$

$$d_i = \text{tr} P_i \quad (\text{A.15})$$

$$CMP_1C^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.16})$$

$$= \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.17})$$

$$M P_i = \lambda_i P_i \text{ (no } i \text{ sum)} \quad (\text{A.18})$$

$$f(M) P_i = f(\lambda_i) P_i \text{ (no } i \text{ sum)} \quad (\text{A.19})$$

$M^{(1)}, M^{(2)}$

$$[M^{(1)}, M^{(2)}] = 0 \quad (\text{A.20})$$

Use $M^{(1)}$ to decompose V into $\bigoplus_i V_i$. Use $M^{(2)}$ to decompose V_i into $\bigoplus_j V_{i,j}$. If $M^{(1)}$ and $M^{(2)}$ don't commute, let $P_i^{(1)}$ be the eigenvalue projection operators of $M^{(1)}$. The replace $M^{(2)}$ by $P_i^{(1)} M^{(2)} P_i^{(1)}$

$$[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] = 0 \quad (\text{A.21})$$

Appendix B

Birdtracks: COMING SOON

B.1 Classical Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $P(y|x) \in [0, 1]$ where $x \in \text{val}(\underline{x})$ and $y \in \text{val}(\underline{y})$

$$\sum_{y \in \text{val}(\underline{y})} P(y|x) = 1 \quad (\text{B.1})$$

$$\mathcal{C} = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow & \underline{a} \end{array} \quad (\text{B.2})$$

$$\mathcal{C}(a, b, c) = P(c|b, a)P(b|a)P(a) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow & a \end{array} P(a) \quad (\text{B.3})$$

$$a^2 = (a_1, a_2)$$

$$\mathcal{C}' = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow \underline{a_2} & \underline{a^2} \end{array} \quad (\text{B.4})$$

$$\mathcal{C}'(a^2, b, c) = P(c|b, a_2)P(a_2|a^2)P(b|a_1)P(a_1|a^2)P(a^2) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow a_2 & a^2 \end{array} P(a^2) \quad (\text{B.5})$$

Marginalizer nodes $\underline{a_1}$ and $\underline{a_2}$ have the TPMs

$$P(a'_i | \underline{a^2} = (a_1, a_2)) = \delta(a'_i, a_i) \quad (\text{B.6})$$

for $i = 1, 2$

B.2 Quantum Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $A(y|x) \in \mathbb{C}$ where $x \in \text{val}(\underline{x})$ and $y \in \text{val}(\underline{y})$

$$\sum_{y \in \text{val}(\underline{y})} |A(y|x)|^2 = 1 \quad (\text{B.7})$$

$$\mathcal{Q} = \begin{array}{c} \underline{b} \\ \swarrow \quad \searrow \\ \underline{c} \longleftarrow \underline{a} \end{array} \quad (\text{B.8})$$

$$\mathcal{Q}(a, b, c) = A(c|b, a)A(b|a)A(a) = \begin{array}{c} b \\ \swarrow \quad \searrow \\ c \longleftarrow a \end{array} A(a) \quad (\text{B.9})$$

$$a^2 = (a_1, a_2)$$

$$\mathcal{Q}' = \begin{array}{c} \underline{b} \\ \swarrow \quad \searrow \underline{a_1} \\ \underline{c} \longleftarrow \underline{a_2} \longrightarrow \underline{a^2} \end{array} \quad (\text{B.10})$$

$$\mathcal{Q}'(a^2, b, c) = A(c|b, a_2)A(a_2|a^2)A(b|a_1)A(a_1|a^2)A(a^2) = \begin{array}{c} b \\ \swarrow \quad \searrow \underline{a_1} \\ c \longleftarrow \underline{a_2} \longrightarrow a^2 \end{array} A(a^2) \quad (\text{B.11})$$

Marginalizer nodes $\underline{a_1}$ and $\underline{a_2}$ have the TAMs

$$A(a'_i|\underline{a^2} = (a_1, a_2)) = \delta(a'_i, a_i) \quad (\text{B.12})$$

for $i = 1, 2$

B.3 Birdtracks

$$\delta(b, a) = \mathbb{1}(a = b) = \delta_a^b = a \leftarrow \bullet \longrightarrow b \quad (\text{B.13})$$

$$\langle a, b | X_{\underline{ab}}^{\underline{cd}} | c, d \rangle = X_{ab}^{cd} = \begin{array}{c} a \longleftarrow X_{\underline{ab}}^{\underline{cd}} \\ \swarrow \quad \nearrow \\ b \longleftarrow \nearrow \\ \swarrow \quad \nearrow \\ c \longleftarrow \nearrow \\ \swarrow \quad \nearrow \\ d \longleftarrow \nearrow \end{array} \quad (\text{B.14})$$

$$\begin{array}{ccc}
\begin{array}{c} a \longleftarrow X_{ab}^{\underline{cd}} \\ b \nearrow \\ c \nearrow \\ d \nearrow \end{array} & \rightarrow & \begin{array}{c} a, b \longleftarrow X_{ab}^{\underline{cd}} \\ a, b \nearrow \\ c \nearrow \\ d \nearrow \end{array}
\end{array} \tag{B.15}$$

$X_{ab}^{\underline{cd}} \in V^2 \otimes V_2$. Sometimes, we will omit denote this node simply by X . This is okay as long as we are not using, X to also denote a different version of $X_{ab}^{\underline{cd}}$ with some of the indices raised or lowered or their order has been changed.

$$\begin{array}{ccc}
& a \longrightarrow (X^\dagger)^{\underline{ab}}_{\underline{cd}} & \\
(X^\dagger)^{\underline{ab}}_{\underline{cd}} = & \begin{array}{c} b \nearrow \\ c \nearrow \\ d \nearrow \end{array} & \tag{B.16}
\end{array}$$

$$\begin{array}{ccc}
& (X^\dagger)^{\underline{ab}}_{\underline{cd}} \longleftarrow \sum a \longleftarrow X_{ab}^{\underline{cd}} & \\
(X^\dagger)^{\underline{ab}}_{\underline{cd}} X_{ab}^{\underline{cd}} = & \begin{array}{c} \swarrow \quad \searrow \\ \sum b \quad \swarrow \\ \sum c \quad \searrow \\ \sum d \end{array} & \tag{B.17}
\end{array}$$

$$\begin{array}{ccc}
& X^\dagger \longleftarrow \longleftarrow X & \\
= & \begin{array}{c} \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \end{array} & \tag{B.18}
\end{array}$$

Birdtracks originated as a graphical way to represent the tensors in General Relativity (Gravitation). In General Relativity, one deals with tensors such as $T_a^b{}_c$ which have some indices raised and some lowered. One can use the metric $g^{a,b}$ to raise all the lowered indices to get T^{abc} . If we represent this graphically as a node with incoming arrows a, b, c , we need to either

1. label the arrows as $\underline{a}, \underline{b}, \underline{c}$, and define the node as $T^{\underline{abc}}$, or
2. instead of labelling the arrows explicitly $\underline{a}, \underline{b}, \underline{c}$, indicate in the node where is the first arrow \underline{a} , and draw the arrows $\underline{a}, \underline{b}, \underline{c}$ so that they enter the node in counterclockwise order.

If we don't do either 1 or 2, we won't be able to distinguish between the graphical representations of $T^{1,2,3}$ and $T^{2,1,3}$, for example. Most of the time, we will do the explicit labeling (alternative 1)

Another issue that arises in using birdtracks is this. When is it permissible to represent a tensor by T_{ab}^{cd} ? If we define T_{ab}^{cd} by

$$T_{ab}^{cd} = T_{ab}{}^{cd} \quad (\text{B.19})$$

then it's always permissible. Then one can define tensors like $T_a{}^{bcd}$ as

$$T_a{}^{bcd} = g^{bb'} T_{ab'}{}^{cd} = g^{bb'} T_{ab'}^{cd} \quad (\text{B.20})$$

Hence, one drawback of using the notation T_{ab}^{cd} is that if one is interested in using versions of T_{ab}^{cd} with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing $T_a{}^{bcd}$, you'll have to write $g^{bb'} T_{ab'}^{cd}$. This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too elegantly concise.

$$a^m \in \mathbb{Z}_+^m$$

$$R_{b_3^{n_3}, a_2^{m_2}}^{a_3^{m_3}, b_2^{n_2}} S_{b_2^{n_2}, a_1^{m_1}}^{a_2^{m_2}, b_1^{n_1}} = \begin{array}{ccccc} & b_3^{n_3} & \leftarrow R & \leftarrow \sum b_2^{n_2} & \leftarrow S & \leftarrow b_1^{n_1} \\ & \nearrow & & \searrow & \nearrow & \\ a_3^{m_3} & & & \sum a_2^{m_2} & & a_1^{m_1} \end{array} \quad (\text{B.21})$$

$$\text{tr}_{\underline{b}} X_{\underline{a}\underline{b}}{}^{\underline{b}\underline{d}} = \sum_b X_{ab}{}^{bd} = \begin{array}{c} a \leftarrow X_{ab}{}^{\underline{cd}} \\ \nearrow \nearrow \nearrow \\ \text{red line} \\ \searrow \\ d \end{array} \quad (\text{B.22})$$

$$\begin{array}{ccc} & \text{red line} & \\ \downarrow & & \uparrow \\ \leftarrow R & & S \leftarrow \\ \nearrow & & \nearrow \\ & & \searrow \end{array} \quad (\text{B.23})$$

Appendix C

Clebsch-Gordan Coefficients: COMING SOON

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d \times d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} C^{d \times d} \quad (\text{C.1})$$

Let $b^{nb} = (b_1, b_2, \dots, b_{nb})$ where $b_i \in Z_{[0, nb_i]}$ and $a \in Z_{[1, d_\lambda]}$. Hence,

$$d_\lambda = \prod_{i=1}^{nb} nb_i \quad (\text{C.2})$$

$$(C_\lambda)_{a^{b^{nb}}} = a \longleftarrow C_\lambda \begin{matrix} \swarrow b_1 \\ \longleftarrow b_2 \\ \searrow b_{nb} \end{matrix} \quad (\text{C.3})$$

$$\begin{bmatrix} 0 & (C^\dagger)_\lambda^{d \times d_\lambda} & 0 \end{bmatrix}^{d \times d} = (C^\dagger)^{d \times d} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} \quad (\text{C.4})$$

$$(C_\lambda^\dagger)_{b^{nb}}^a = \begin{matrix} \swarrow b_1 \\ \longleftarrow b_2 \\ \searrow b_{nb} \end{matrix} (C_\lambda^\dagger) \longleftarrow \lambda, a \quad (\text{C.5})$$

More generally, some of the b_i indices may be lowered and their arrows changed to outgoing instead of ingoing. Each b_i represents a different rep (or irrep)

$$(C_\lambda^\dagger)_a^{b^{nb}} (C_\lambda)_a^b = (P_\lambda)_{(b')^{nb}}^{b^{nb}} \quad (\text{C.6})$$

$$\begin{array}{c}
b_1 \swarrow \\
b_2 \longleftarrow (C_\lambda^\dagger) \longleftarrow \sum a \longleftarrow C_\lambda \longleftarrow b'_2 \quad = \quad b^{nb} \longleftarrow P_\lambda \longleftarrow (b')^{nb} \\
b_{nb} \searrow \qquad \qquad \qquad \swarrow b'_{nb}
\end{array} \quad (\text{C.7})$$

$$(C_\lambda)_{b^{nb}}^{(a')^{na}} (C_\mu^\dagger)_a^{b^{nb}} = \delta(\lambda, \mu) \delta_a^{(a')^{na}} \quad (\text{C.8})$$

$$\begin{array}{c}
\qquad \qquad \qquad \sum b_1 \\
\qquad \qquad \swarrow \qquad \nwarrow \\
a \longleftarrow C_\lambda \longleftarrow \sum b_2 \longleftarrow (C_\mu^\dagger) \longleftarrow a' \quad = \quad \delta(\mu, \lambda) \quad a \longleftarrow \bullet \longleftarrow a' \\
\qquad \qquad \nwarrow \qquad \swarrow \\
\qquad \qquad \qquad \sum b_{nb}
\end{array} \quad (\text{C.9})$$

Bibliography