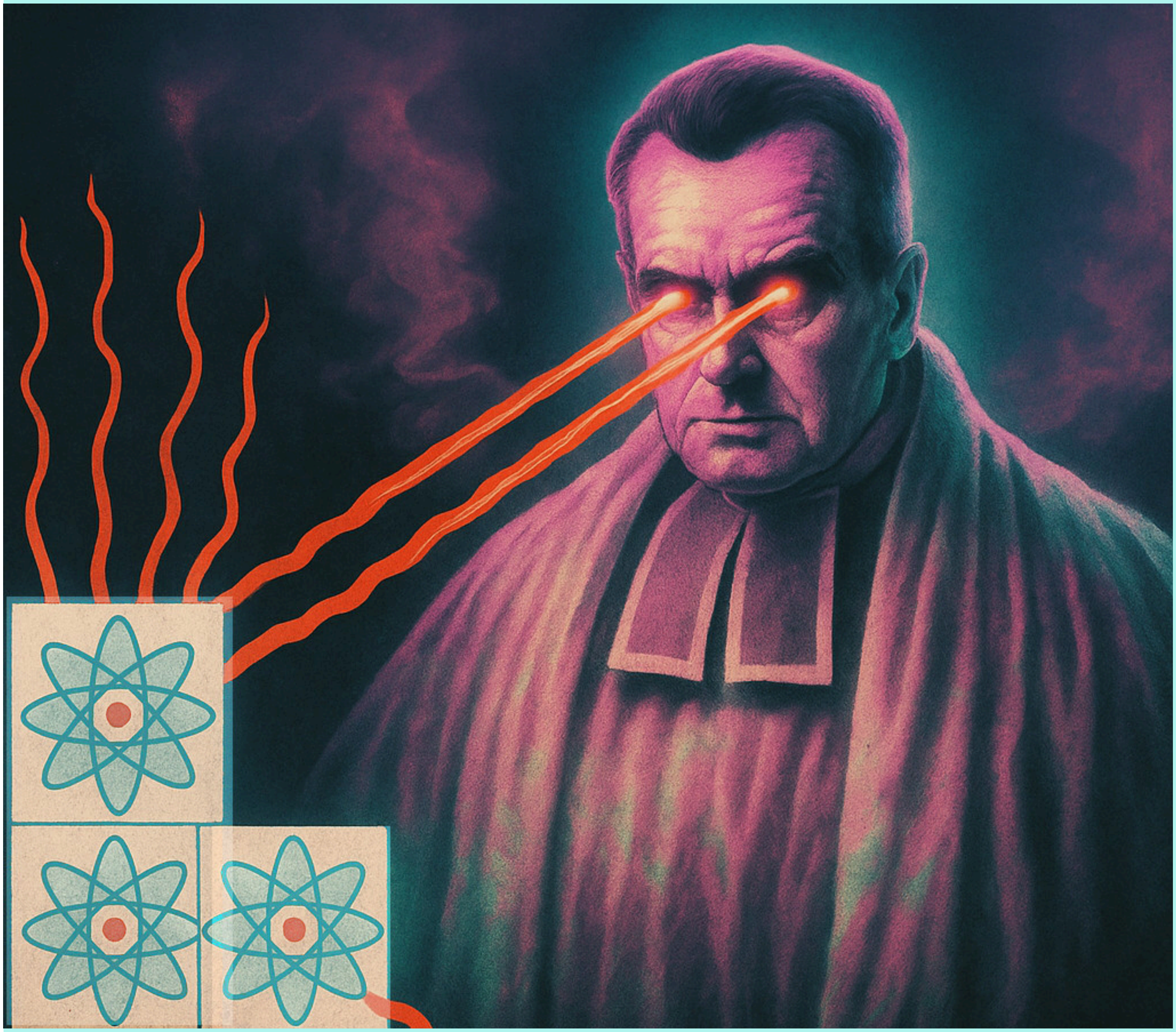
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



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Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayesuvius Quantico
by Robert R. Tucci

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Appendix A

Notational Conventions and Preliminaries

This book is a sequel to my book entitled "Bayesuvius" (see Ref.[3]). For consistency, I have tried to follow in this book the same notational conventions used in the prior book. If any notation is not defined in this book, check in the prior book. It might be defined there.

A.1 Set notation

Definitions

$|S| ==$ the number of elements in a set S. (known as its **order, size, length, cardinality**)

$\mathbb{Z}$ = integers

$\mathbb{Z}_{>0}$ = positive integers

$\mathbb{Z}_{[a,b]} = a, a+1, \ldots, b$ for some integers a, b such that $a \leq b$

$\mathbb{R}$ = reals

$\mathbb{C}$= complex numbers

$\mathbb{C}^{n\times m} = n \times m$ matrices of complex numbers

A.2 Commutator and Anti-commutator

Let

commutator of A and B

$$[A, B] = AB - BA \tag{A.1}$$

Anti-commutator of A and B

$$[A, B]_+ = AB + BA \tag{A.2}$$

A.3 Group Theory References

Much of this book deals with Group Theory (GT).

GT is a vast subject. Who would have thought that the simple definition of a group would generate so many elegant and useful results.

GT books by mathematicians are very different from GT books by physicists, even though, of course, they agree on the definitions. Mathematicians are, as to be expected, more rigorous and abstract. But it goes much further than that. Physicists are much more interested in applications to physical systems, especially Quantum Mechanics (QM). Soon after QM was invented, it was realized that Linear Algebra (LA) and GT (especially Group Representation Theory, which combines GT and LA) are extremely relevant and useful in QM. Hermann Weyl, Eugene Wigner, Hans Bethe, Linus Pauling, etc. combined QM and GT to understand the spectra and chemistry of atoms and molecules, and later GT was heavily used in Quantum Field Theory and Particle Physics to devise the Standard Model. Condensed Matter physicists have also used it to understand crystalline solids and to predict quasi particles that can be detected in the lab.

My PhD is in physics so in this book I cover GT topics that are mainly of interests to physicists and engineers. Furthermore, I am nowhere as abstract and rigorous as mathematicians usually are.

My favorite books about GT for physicists are the Elliott & Dawber's (ED) 2 volume series Ref. [2] and Predrag Cvitanovic's Birdtracks book Ref.[1]. I highly recommend both of these references. I think both of them are excellent.

The Birdtracks book explains key concepts in GT representation theory using network diagrams (Cvitanovic calls such diagrams birdtracks) The ED books, on the other hand, do not use birdtracks. They use algebra instead. In fact, most GT books don't use birdtracks either. But since this is a book about visualization using network diagrams (quantum bnets), we use birdtracks. In fact, many of the chapters in this book were heavily influenced by Ref.[1] by Cvitanovic. I hope he doesn't mind. I really love his book.

A.4 Group

A **group** $\mathcal{G}$ is a set of elements with a multiplication map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ such that

1. the multiplication is **associative** ; i.e.,

$$(ab)c = a(bc) \tag{A.3}$$

   for $a, b, c \in \mathcal{G}$.

2. there exists an **identity element** $e \in \mathcal{G}$ such that

$$ea = ae = a \tag{A.4}$$

for all $a \in \mathcal{G}$

3. for any $a \in \mathcal{G}$, there exists an **inverse** $a^{-1} \in \mathcal{G}$ such that

$$aa^{-1} = a^{-1}a = e \tag{A.5}$$

$|\mathcal{G}|$ (i.e., number of elements in $\mathcal{G}$) is called the **order** of the group.

If multiplication is **commutative** (i.e., $ab = ba$ for all $a, b \in \mathcal{G}$), the group is said to be **abelian**.

A **subgroup** $\mathcal{H}$ of $\mathcal{G}$ is a subset of $\mathcal{G}$ ($\mathcal{H} \subset \mathcal{G}$) which is also a group. It's easy to show that any $\mathcal{H} \subset \mathcal{G}$ is a group if it contains the identity and is **closed under multiplication** (i.e., $ab \in \mathcal{H}$ for all $a, b \in \mathcal{H}$)

A.5 Group Representation

A **group representation** (rep) of a group $\mathcal{G}$ is a map $\phi : \mathcal{G} \to \mathbb{C}^{n\times n}$[1] such that

$$\phi(a)\phi(b) = \phi(ab), \quad \phi(e) = I \tag{A.6}$$

where e is the identity of the group and I is the identity matrix. Such a map is called a **homomorphism** (because it preserves an operation). The map ϕ partitions $\mathcal{G}$ into disjoints subsets (equivalence classes), such that all elements of $\mathcal{G}$ in each disjoint subset are represented by the same matrix.

In this book, we will usually label reps by a Greek letter such as λ, and we will refer to $\phi(g) = G_\lambda(g) = G_\lambda$ as the **representation matrix** (rep-matrix) of $g \in \mathcal{G}$.

One way to specify a representation is to give the effect of each group element $a \in \mathcal{G}$ on a basis of vectors $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$.

$$\phi(a)|i\rangle = \sum_j M_{ij}|j\rangle \implies \langle i|\phi(a)|j\rangle = M_{ij} \tag{A.7}$$

If the map ϕ is 1-1, onto, we call it a **faithful representation**

A **singlet (or invariant or conserved) quantity of group** $\mathcal{G}$ is a quantity that is invariant under the group transformations $g \in \mathcal{G}$. The **trivial or singlet representation** is the rep with $\phi(g) = 1 = [1]^{1\times 1}$ for all $g \in \mathcal{G}$. The dimension of this rep is $d_\lambda = 1$. If $\phi(g) = diag(1, 1)$, this is referred to as two identical copies of a singlet rep. The **singlet projection operator** $\delta^b_a \delta^d_c$ when acting on ${z_c}^d$ gives a $\mathrm{tr}(z) diag(1, 1, \ldots, 1)$ where $\mathrm{tr}(z) = {z_c}^c$, so it projects to out a singlet quantity. A singlet projection operator P_λ is associated with a singlet rep λ with rep-matrices

[1] More generally, the $\mathbb{C}^{n\times n}$ can be replaced by $\mathbb{R}^{n\times n}$ or by $\mathbb{F}^{n\times n}$ for any field $\mathbb{F}$

$G_\lambda(g) = 1$. For example, $P_\lambda = \delta_a^b \delta_c^d$ is associated with a rep λ with rep-matrices $G_\lambda \otimes G_\lambda^\dagger = 1$

A **1-dimensional (1-D or 1dim) representation** assigns a complex number to each $g \in \mathcal{G}$.[2] For example, the rep with $\phi(g) = e^{i\beta(g)} = [e^{i\beta(g)}]^{1\times 1}$ for all $g \in \mathcal{G}$, where $\beta(g) \in \mathbb{R}$. The the trivial/singlet rep is a special 1-dim rep. The dimension of this rep is $d_\lambda = 1$.

When a group is defined using matrices, those matrices are called the **defining representation** (def-rep). For example, the group of **General Linear Transformations** is defined by

$$GL(n;\mathbb{C}) = \{M \in \mathbb{C}^{n\times n} : \det M \neq 0\} \tag{A.8}$$

The **adjoint representation** (adj-rep) is defined in terms of the structure constants of the Lie Algebra. If the Lie Algebra satisfies $[T^i, T^j] = i f_{ijk} T^k$, then the adj-rep is given by the matrices with i, j entries $M_{ij}^k = -i f^k{}_{ij}$. Let $|x\rangle = x_i |T^i\rangle$. Then

$$[|x\rangle, \cdot]|T^j\rangle = |[x, T^j]\rangle = i x_i f_{ijk} |T^k\rangle \implies \langle T^k|[|x\rangle, \cdot]|T^j\rangle = i x_i f_{ijk} \tag{A.9}$$

Irreducible representations (irreps) are defined in Ch. 15

The **fundamental representation** (fun-rep) is defined as the smallest irrep. The def-rep equals the fun-rep for $SU(n), SO(n), SP(n)$, but not for E_8.

The **regular representation** is defined in Chapter 23 for the symmetric group on n_b letters (or n_b boxes) S_{n_b}.

A.6 Dimensions

In Physics/Math, the term “dimension” can mean various things. For example, it might mean

1. (vector space dimension) the number of vectors in a basis of a vector space
2. (matrix row or column dimensions) the number of rows or columns in a matrix $M_{a,b}$.
3. (vector dimension) the number of components of a vector x_a

These 3 uses of the term “dimension” are all closely related but not the same. Sometimes, there are several dimensions at play in the same conversation.

Let MD stand for matrix dimension. A rep λ with rep-matrices G_λ has 2 MDs associated with it that we must distinguish:

[2] Note that a 1-dim rep and a tensor with one index x_a, where $a = 1, 2, \ldots, n$ are not the same thing. x_a is not even a rep. x_a is often referred to as an n-dim vector. x_a might transform as the n-dim rep with rep-matrices $G_b{}^a$ where $b = 1, 2, \ldots, n$. Always associate the dim of a rep with the matrix dimension of a square matrix.

1. (adjoint rep MD) the number N of generators T_i, where $i = 1, 2, \ldots, N$.

2. (def rep MD) the number of rows and columns of the square rep-matrix G_λ

For example, the Pauli matrices are 3 2×2 matrices.

For $SU(n)$ and $U(n)$

$n = d_{def}$ = MD of rep-matrices G in defining rep of $U(n)$ or $SU(n)$. This MD equals n for both $U(n)$ and $SU(n)$.

$N = d_{adj}$ = MD of rep-matrices G in adjoint rep of $U(n)$ or $SU(n)$. As we shall prove in Chapter 21, $N = n^2$ for $U(n)$ but $N = n^2 - 1$ for $SU(n)$.

A.7 Vector Space and Algebra Over a Field $\mathbb{F}$

A **vector space (a.k.a. linear space) $\mathcal{V}$ over a field** $\mathbb{F}$ is defined as a set $\mathcal{V}$ endowed with two operations: vector addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and scalar multiplication $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$, such that

- $\mathcal{V}$ is an abelian group under $+$ with identity 0 and inverse of $x \in \mathcal{V}$ equal to $-x \in \mathcal{V}$

- For $\alpha, \beta \in \mathbb{F}$ and $x, y \in \mathcal{V}$

$$\alpha(x+y) = \alpha x + \alpha y \tag{A.10}$$
$$(\alpha+\beta)x = \alpha x + \beta x \tag{A.11}$$
$$\alpha(\beta x) = (\alpha\beta)x \tag{A.12}$$
$$1x = x \tag{A.13}$$
$$0x = 0 \tag{A.14}$$

In this book, we will always use either $\mathbb{C}$ or $\mathbb{R}$ for $\mathbb{F}$. Both of these fields are infinite but some fields are finite.

An **algebra** $\mathcal{A}$ is a vector space which, besides being endowed with vector addition and scalar multiplication as all vector spaces are, it has a **bilinear vector product**. A bilinear vector product is a product that is linear on both sides; i.e.,

$$(\alpha x + \beta y) \cdot z = \alpha x \cdot z + \beta y \cdot z \tag{A.15}$$

and

$$z \cdot (\alpha x + \beta y) = \alpha z \cdot x + \beta z \cdot y \tag{A.16}$$

for $x, y, z \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. The cross product (but not the dot product) for vectors in $\mathbb{R}^3$, the multiplication of 2 complex numbers, the matrix product or matrix commutator of 2 square matrices, are all good examples of bilinear vector products.

Let $B = \{\tau_i : i = 1, 2, \ldots, r\}$ be a basis for the vector space $\mathcal{A}$. Then note that $\mathcal{A}$ is closed under vector multiplication.

$$\tau_i \cdot \tau_j = \sum_k c_{ij}{}^k \tau_k \tag{A.17}$$

where $c_{ij}{}^k \in \mathbb{C}$. The $c_{ij}{}^k$ are called **structure constants** of $\mathcal{A}$. In Dirac notation

$$\tau_i|\tau_j\rangle = |\tau_i \cdot \tau_j\rangle = \sum_k c_{ij}{}^k |\tau_k\rangle \tag{A.18}$$

$$\langle\tau_k|\tau_i|\tau_j\rangle = c_{ij}{}^k \tag{A.19}$$

An **associative algebra** satisfies $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for $x, y, z \in \mathcal{A}$.

- Not associative: cross product for vectors in $\mathbb{R}^3$.
- Associative: the matrix product or matrix commutator of 2 square matrices and the product of complex numbers

A.8 Tensors

Let

$(x_a) = (x_1, x_2, \ldots, x_n) = x^{:n} \in V^n = \mathbb{C}^{n\times 1}$

Reverse of vector $rev(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_1)$

$x^b = \sum_a g^{ba} x_a$, g^{ab} is the **metric tensor**

$(y^b) = (y^1, y^2, \ldots, y^n) = y^{\dagger:n} \in V^{\dagger n} = \mathbb{C}^{n\times 1}$. V^n is the lower indices vector space and $V^{\dagger n}$ is its **dual vector space** (i.e., with upper indices).

$M_a{}^b \in \mathbb{C}^{n\times n}$, $a, b \in \mathbb{Z}_{[1,n]}$

Implicit Summation Convention

$$M_a{}^b x_b = \sum_{b=1}^{n} M_a{}^b x_b \tag{A.20}$$

The **Hermitian conjugate** $\dagger$ equals $*T$ where $*$ is complex conjugation and T is transpose. Hence

$$\boxed{(M^T)_a{}^b = M_b{}^a} \qquad a \longleftarrow M^T \longleftarrow b \;=\; b \longleftarrow M \longleftarrow a \tag{A.21}$$

$$\boxed{(M^\dagger)_a{}^b = (M_b{}^a)^*} \qquad a \longleftarrow M^\dagger \longleftarrow b \;=\; b \longleftarrow M^* \longleftarrow a \tag{A.22}$$

To avoid confusion, follow the golden rule: write $\dagger$ and T only before declaring the indices; and write the $*$ only after declaring the indices. Note that $\dagger$ does 3 things:

1. reverse the horizontal order of the indices

2. reverse vertical positions of the indices; i.e., lower upper indices and raise lower indices.

3. replace the tensor components by their complex conjugates

Transposing only does items 1 and 2.

If M is a Hermitian matrix (i.e., $M^\dagger = M$), then

$$\boxed{M_a{}^b = (M_b{}^a)^*} \qquad a \longleftarrow M \longleftarrow b \;=\; b \longleftarrow M^* \longleftarrow a \tag{A.23}$$

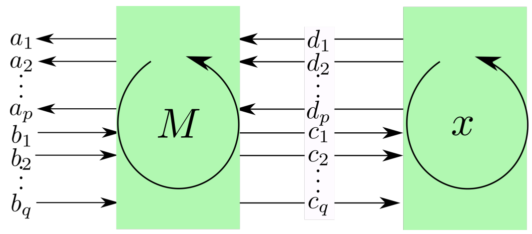


Figure A.1: Index labels for Mx where $M \in \mathbb{C}^{n^{p+q}\times n^{p+q}}$ and $x \in V^{n^p} \otimes V^{\dagger n^q}$. Note that we list indices in counterclockwise (CC) direction, starting at the top.

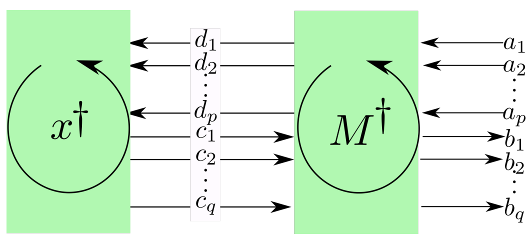


Figure A.2: Index labels for $x^\dagger M^\dagger$ corresponding to Fig.A.2. Note that we list indices in counterclockwise (CC) direction, starting at the top.

Suppose $a_i, b_i, c_i, d_i \in \mathbb{Z}_{[1,n]}$. From Fig.A.1

$$y_{a^{\cdot p}}{}^{b^{\cdot q}} = M_{a^{\cdot p}}{}^{b^{\cdot q}}{}_{rev(c^{\cdot q})}{}^{rev(d^{\cdot p})} x_{d^{\cdot p}}{}^{c^{\cdot q}} \tag{A.24}$$

If we define x_α and x^α by

$$x_\alpha = x_{a^{\cdot p}}{}^{b^{\cdot q}}, \quad x^\alpha = x_{rev(b^{\cdot q})}{}^{rev(a^{\cdot p})} \tag{A.25}$$

then

$$x_\alpha = M_\alpha{}^\beta x_\beta \tag{A.26}$$

Hermitian conjugation (see Fig.A.2)

$$\left\{\begin{array}{l}(M^\dagger)_a{}^d = (M_d{}^a)^* \\ (M^\dagger)_\alpha{}^\delta = (M_{rev(\delta)}{}^{rev(\alpha)})^*\end{array}\right. \qquad \alpha \longleftarrow M^\dagger \longleftarrow \delta \;=\; rev(\delta) \longleftarrow M^* \longleftarrow rev(\alpha) \tag{A.27}$$

Note that † does 3 things to the birdtrack:

1. It flips the horizontal axis of the figure. (In the algebraic expression of the tensor, this corresponds to reversing the horizontal order of the indices.)

2. For each node, it changes incoming arrows to outgoing ones and vice versa. (In the algebraic expression of the tensor, this corresponds reversing the vertical positions of the indices; i.e., lowering upper indices and raising lower ones.)

3. It replaces the tensor component by its complex conjugate

Hermitian matrix

$$M^\dagger = M, \quad \left\{\begin{array}{l}M_a{}^d = (M_d{}^a)^* \\ M_\alpha{}^\delta = (M_{rev(\delta)}{}^{rev(\alpha)})^*\end{array}\right. \tag{A.28}$$

Unitary matrix

$$M^\dagger M = 1, \quad \left\{\begin{array}{l}(M_b{}^a)^* M_a{}^c = \delta_b^c \\ (M_{rev(\beta)}{}^{rev(\alpha)})^* M_\alpha{}^\gamma = \delta^\gamma_{rev(\beta)}\end{array}\right. \tag{A.29}$$

Note that for $x \in V^n$, $y \in {V^\dagger}^n$, and $G \in \mathcal{G} \subset GL(n;\mathbb{C})$,

$$(x')_a (y')^b = G^b{}_c G_a{}^d x_d y^c \tag{A.30}$$

If $x \in V^{n^p} \otimes {V^\dagger}^{n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q};\mathbb{C})$,

$$(x')_{a^{:p}}{}^{b^{:q}} = \mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}}, \quad (x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta) \tag{A.31}$$

where we define

$$\mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} \stackrel{\text{def}}{=} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \prod_{i=1}^{q} {G^\dagger}^{b_i}{}_{c_i} \tag{A.32}$$

An issue that arises with tensors is this: When is it permissible to represent a tensor by M^{cd}_{ab}? If we define M^{cd}_{ab} by

$$M^{cd}_{ab} = M_{ab}{}^{cd} \tag{A.33}$$

then it's always permissible. Then one can define tensors like $M_a{}^{bcd}$ as

$$M_a{}^{bcd} = g^{bb'} M_{ab'}{}^{cd} = g^{bb'} M^{cd}_{ab'} \tag{A.34}$$

One drawback of using the notation M_{ab}^{cd} is that if one is interested in using several versions of M_{ab}^{cd} with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing $M_a{}^{bcd}$, you'll have to write $g^{bb'} M_{ab'}^{cd}$. This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too succinct.

A.9 Permutations

Some well known notation and results about permutations are these.

$(1,2)$ stands for a **transposition**; i.e., a map that swaps 1 and 2:

$$\left(\begin{tikzcd} 1 \arrow[dr] & 2 \arrow[dl] & 3 \arrow[d] & \dots & p \arrow[d] \\ 1 & 2 & 3 & \dots & p \end{tikzcd}\right) \tag{A.35}$$

$(3,2,1)$ stands for a **permutation**; i.e., a map that maps $3 \to 2 \to 1 \to 3$.

$$\left(\begin{tikzcd} 1 \arrow[drr] & 2 \arrow[dl] & 3 \arrow[dl] & 4 \arrow[d] & \dots & p \arrow[d] \\ 1 & 2 & 3 & 4 & \dots & p \end{tikzcd}\right) \tag{A.36}$$

Any reordering of $(1,2,3,\dots,p)$ is a permutation of p letters (or numbers or elements).

The set S_p of all permutation of p letters is called the **symmetric group in p letters**. It has $p!$ elements (i.e., $|S_p| = p!$) and is a group, where the group's product is map composition and the group's identity element is the identity map.

Any permutation can be expressed as a product of transpositions, For example, $(3,2,1) = (3,2)(2,1)$.

An **even permutation** such as $(3,2,1)$ can be expressed as a product of an even number of transpositions. An **odd permutation** can be expressed as a product of an odd number of transpositions.

Appendix B

Birdtracks

This chapter is based on Cvitanovic's Birdtracks book Ref. [1] and my paper Ref. [4]

The tensor notation discussed in Sec.A.8 is succinct and straightforward, but it's not visually illuminating. The birdtrack notation that we shall discuss in this chapter, is not as succinct as the tensor notation, and can lead to sign errors if you are careless, but it is very visually illuminating. Thus, the tensor and birdtrack notations complement each other well. We will often display results using both, side by side.

B.1 Classical Bayesian Networks and their Instantiations

Classical Bayesian Networks (bnets) are discussed exhaustively in the first book of this series, Ref.[3]. This is a brief section to remind the reader of how they are defined.

Let PD stand for probability distribution.

We call $P_{\underline{y}|\underline{x}} : val(\underline{y}) \times val(\underline{x}) \rightarrow [0,1]$ a **Transition Probability Matrix** (TPM)[1] if

$$\sum_{y \in val(\underline{y})} P_{\underline{y}|\underline{x}}(y|x) = 1 \tag{B.1}$$

In other words, a TPM is a conditional PD. A TPM of the form

$$P(y|x) = \delta(y, f(x)) \tag{B.2}$$

for some function $f : val(\underline{x}) \rightarrow val(\underline{y})$ is said to be **deterministic**.

A bnet is a **Directed Acyclic Graph** (DAG) with the nodes labelled by random variables[2]. Each bnet stands for a full PD of the node random variables expressed as a product of a TPM for each node. For example, the bnet

[1] A TPM is also known as a Conditional Probability Table (CPT).

[2] As in the first volume of this series, we indicate random variables by underlined letters

$$
\mathcal{C} = \begin{tikzcd}[column sep=small]
 & \underline{b} \arrow[dl] & \\
\underline{c} & & \underline{a} \arrow[ul] \arrow[ll]
\end{tikzcd}
\tag{B.3}
$$

stands for the full PD

$$
P(a,b,c) = P(c|b,a)P(b|a)P(a) \tag{B.4}
$$

Bnets do not have free indices because their nodes are labelled by random variables. It is convenient to draw the DAG for a bnet but with the underlining removed from the random variables, and then to assign a numerical value to this new DAG. The resultant DAG now has free indices. We call it an **instantiation of the bnet**. For example, from the bnet $\mathcal{C}$ of Eq.(B.3), we get the instantiation[3]

$$
P(a,b,c) = P(c|b,a)P(b|a)P(a) = \begin{tikzcd}[column sep=small]
 & b \arrow[dl] & \\
c & & a \arrow[ul] \arrow[ll]
\end{tikzcd} \quad P(a)
\tag{B.5}
$$

Let $a^{:2} = (a_1, a_2)$. Based on the bnet $\mathcal{C}$ of Eq.(B.3), define a new bnet $\mathcal{C}'$ as follows

$$
\mathcal{C}' = \begin{tikzcd}[column sep=small]
 & \underline{b} \arrow[dl] & \\
\underline{c} & & \underline{a}^{:2} \arrow[ul, "\underline{a}_1"'] \arrow[ll, "\underline{a}_2"]
\end{tikzcd}
\tag{B.6}
$$

$\mathcal{C}'$ represents the the full PD

$$
P(a^{:2}, b, c) = P(c|b, a_2)P(a_2|a^{:2})P(b|a_1)P(a_1|a^{:2})P(a^{:2}) \tag{B.7}
$$

The 2 new nodes $\underline{a}_1$ and $\underline{a}_2$ of bnet $\mathcal{C}'$ are called **marginalizer nodes**. We assign to them the following TPMs (printed in blue):

$$
P[a'_i|\underline{a}^{:2} = (a_1, a_2)] = \delta(a'_i, a_i) \tag{B.8}
$$

for $i = 1, 2$. We can also define an instantiation of $\mathcal{C}'$ as follows:

$$
P'(a^{:2}, b, c) = \begin{tikzcd}[column sep=small]
 & b \arrow[dl] & \\
c & & a^{:2} \arrow[ul, "a_1"'] \arrow[ll, "a_2"]
\end{tikzcd} \quad P(a^{:2})
\tag{B.9}
$$

[3]Note that we don't include the root node probabilities as part of the graph value. Thus, $P(a,b) = \underbrace{b \leftarrow a}_{P(b|a)} P(a)$

B.2 Quantum Bayesian Networks and their Instantiations

As far as I know, Quantum Bayesian Networks (qbnets) were invented by me in Ref.[4].

qbnets are closely analogous to classical bnets, but the TPM are replaced by **Transition Amplitude Matrices (TAM)**.

Let PA stand for probability amplitude.

We call $A_{\underline{y}|\underline{x}} : val(\underline{y}) \times val(\underline{x}) \to \mathbb{C}$ a TAM if

$$\sum_{y \in val(\underline{y})} |A(y|x)|^2 = 1 \tag{B.10}$$

Note that if A is the matrix with entries $\langle y|A|x\rangle = A(y|x)$, then

$$\left\langle x|A^\dagger A|x\right\rangle = \sum_{y \in val(\underline{y})} |A(y|x)|^2 = 1 \tag{B.11}$$

If A is a unitary matrix, then $A^\dagger A = AA^\dagger = 1$ so "half" ($A^\dagger A = 1$) of the definition of unitary matrix is satisfied by a TAM. If both halves were satisfied, A would have to be a square matrix.

A qbnet is a DAG with the nodes labelled by random variables. Each qbnet stands for a full PA of the node random variables expressed as a product of a TAM for each node. For example, the qbnet

$$\mathcal{Q} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ll] \arrow[ul] \end{tikzcd} \tag{B.12}$$

stands for the full PA

$$A(a,b,c) = A(c|b,a)A(b|a)A(a) \tag{B.13}$$

Qbnets do not have free indices because their nodes are labelled by random variables. It is convenient to draw the DAG for a qbnet but with the underlining removed from the random variables, and then to assign a numerical value to this new DAG. The resultant DAG now has free indices. We call it an **instantiation of the qbnet**. For example, from the bnet $\mathcal{Q}$ of Eq.(B.12), we get the instantiation

$$A(a,b,c) = A(c|b,a)A(b|a)A(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ll] \arrow[ul] \end{tikzcd} \quad A(a) \tag{B.14}$$

Let $a^{:2} = (a_1, a_2)$. Based on the qbnet $\mathcal{Q}$ of Eq.(B.12), define a new qbnet $\mathcal{Q}'$ as follows

$$\mathcal{Q}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^{:2} \arrow[ul, "\underline{a}_1"'] \arrow[ll, "\underline{a}_2"] \end{tikzcd} \tag{B.15}$$

$\mathcal{Q}'$ represents the the full PA

$$A(a^{:2}, b, c) = A(c|b, a_2)A(a_2|a^{:2})A(b|a_1)A(a_1|a^{:2})A(a^{:2}) \tag{B.16}$$

The 2 new nodes $\underline{a}_1$ and $\underline{a}_2$ of qbnet $\mathcal{Q}'$ are called **marginalizer nodes**. We assign to them the following TAMs (printed in blue):

$$A[a'_i|\underline{a}^{:2} = (a_1, a_2)] = \delta(a'_i, a_i) \tag{B.17}$$

for $i = 1, 2$. We can also define an instantiation of $\mathcal{Q}'$ as follows:

$$A(a^{:2}, b, c) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^{:2} \arrow[ul, "a_1"'] \arrow[ll, "a_2"] \end{tikzcd} \quad A(a^{:2}) \tag{B.18}$$

B.3 Birdtracks

Tensors written in **algebraic notation** such as $T_a{}^{bc}$ were already discussed in Section A.8

Birdtracks are a DAG used to represent algebraic tensor equations. The nodes of the DAG are labelled by tensors and the arrows are labeled by the indices of the tensors: upper indices of a tensor are pictured as incoming arrows of the node, and lower indices as outgoing arrows.

We've already discussed in Section A.8 what we will call the **Counter Clockwise (CC) convention** of drawing birdtrack nodes. Now that we have discussed classical and quantum bnets, we would like to introduce an equivalent, more bnet like, convention that we will call the **Fully Label (FL) convention**. Cvitanovic's birdtracks book Ref.[1] uses the CC convention. We will use both. No confusion will arise, as long as it is clear from context which convention is being used.

Next we review the CC convention and then describe the FL convention for the first time.

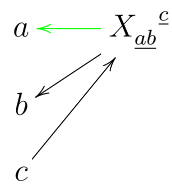
---

1. **CC convention**

   In the CC convention, we must specify for each the node, which arrow is first, and then the CC order in which the arrows enter or leave the node is drawn so that it reproduces the horizontal order of the indices in the algebraic notation for the tensor. We shall often indicate the first arrow by coloring it green.

   For example,

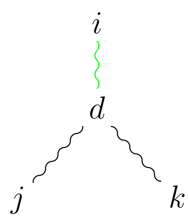
$$\delta(b, a) = \mathbb{1}(a = b) = \delta_a^b = \; a \leftarrow\!\bullet\!- b \tag{B.19}$$

$$X_{ab}{}^{c} = $$ 

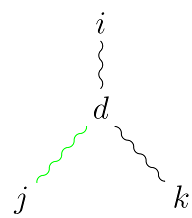
(B.20)

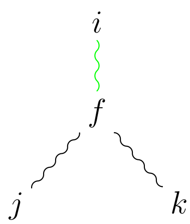
In this picture, the green arrow indicates which tensor index is first horizontally in the algebraic representation of the tensor.

Sometimes there is no need to indicate which arrow is first by drawing it in green, because all choices give the same number. For example, in the birdtracks for δ_a^b, starting with the incoming arrow or the outgoing arrow leads to the same number. Likewise, with the totally symmetric tensor d_{ijk} (doesn't change sign under swap of any two indices) and the totally antisymmetric tensor f_{ijk} (changes sign under swap of any two indices), it doesn't matter if one starts at i, j or k. This is shown below.

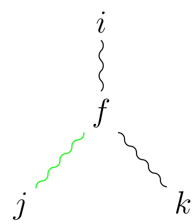


$$= d_{ijk} = d_{jki} = \tag{B.21}$$

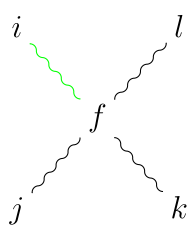




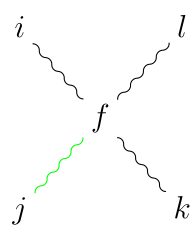
$$= f_{ijk} = f_{jki} = \tag{B.22}$$



Note that for a totally antisymmetic tensor with an even number of indices, the beginning arrow can change the sign. Indeed,



$$= f_{ijkl} = -f_{jkli} = (-1) \tag{B.23}$$



2. **FL convention**

   In the FL convention, the arrows must be labelled by random (underlined) variables, and the names of the nodes must also indicate by underlined variables what is the the order of the indices

For example,

$$\delta(b,a) = \mathbb{1}(a=b) = \delta^b_a = \; a \leftarrow\!\bullet\!\!-\!\!- \, b \tag{B.24}$$

$$\langle a, b|X_{\underline{ab}}{}^{\underline{c}}|c\rangle = X_{ab}{}^{c} = \begin{tikzcd} \underline{a}=a & X_{\underline{ab}}{}^{\underline{c}} \arrow[l] \arrow[dl] \\ \underline{b}=b & \\ \underline{c}=c \arrow[uur] & \end{tikzcd} \tag{B.25}$$

Sometimes, we will denote this node simply by X. This is okay as long as we state that $X = X_{\underline{ab}}{}^{\underline{cd}}$, and we don't start using X to represent a different version of $X_{\underline{ab}}{}^{\underline{cd}}$ with some of the indices raised or lowered or their horizontal order changed.

Often, we will write simply a instead of $\underline{a} = a$. This is similar to the shorthand $P(\underline{a} = a) = P(a)$.

Note that, unlike in the CC convention, in the FL convention, the CC order in which the arrows enter or leave the node, is meaningless. All orders are equivalent. This is akin to the notation for bnets and qbnets.

If we don't follow either convention CC or FL, we won't be able to distinguish between the graphical representations of $T^{1,2,3}$ and $T^{2,1,3}$, for example.

Two other features of the CC and FL conventions that we would like to discuss before ending this section are how to indicate

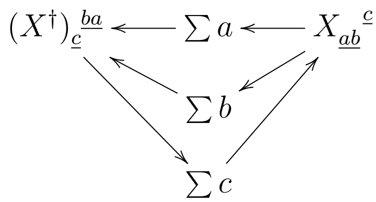
- **noncyclic index contractions**; i.e., index contractions (i.e., summations) that do not introduce cycles, and
- **traces**; i.e., index contractions that do introduce cycles.

Noncyclic index contractions will be indicated by an arrow connecting two nodes, with the symbol $\sum a$ midway in the arrow if the index a is being contracted. For simplicity, we often omit writing the $\sum a$ altogether.

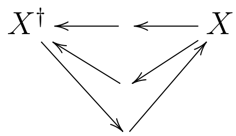
For example (in CC convention),

$$X_{ab}{}^{c} = \begin{tikzcd} a & X_{\underline{ab}}{}^{\underline{c}} \arrow[l, green] \arrow[dl] \\ b & \\ c \arrow[uur] & \end{tikzcd} \quad , \quad (X^\dagger)_c{}^{ba} = \begin{tikzcd} (X^\dagger)_{\underline{c}}{}^{\underline{ba}} & a \arrow[l] \\ & b \arrow[ul] \\ & c \arrow[uul, green] \end{tikzcd} \tag{B.26}$$

$$(X^{\dagger})_c{}^{ba} X_{ab}{}^c \quad = \qquad \text{(B.27)}$$



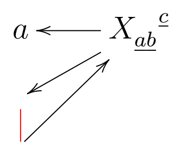
$$= \qquad \text{(B.28)}$$



Birdtracks are DAGs until we are asked to take a trace of one of their indices. Tracing ruins their acyclicity. The acyclicity of DAGs is mandated by causality. The acyclicity of tracing hints to its acausal (or feedback) nature.

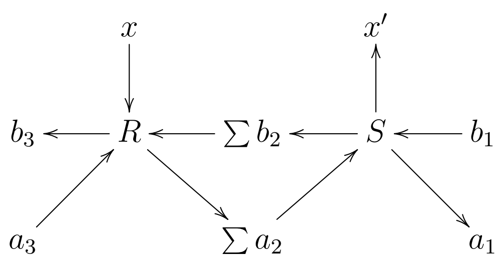
In this book, we will indicate tracing with a red undirected arrow. For example, in the CC convention,

$$\mathrm{tr}_{\underline{b}} X_{a\underline{b}}{}^{\underline{b}} = \sum_b X_{ab}{}^b = \qquad \text{(B.29)}$$



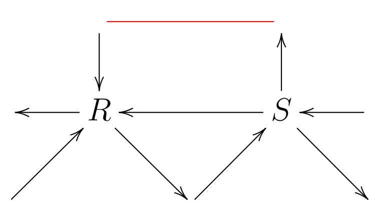
If

$$R^x{}_{b_3}{}^{a_3}{}_{a_2}{}^{b_2} S_{x'b_2}{}^{a_2}{}_{a_1}{}^{b_1} = \qquad \text{(B.30)}$$



then

$$\mathrm{tr}_{\underline{x}} R^{\underline{x}}{}_{b_3}{}^{a_3}{}_{a_2}{}^{b_2} S_{\underline{x}b_2}{}^{a_2}{}_{a_1}{}^{b_1} = \qquad \text{(B.31)}$$



When using the FL convention, it becomes clear that birdtracks can be understood as instantiations of qbnets, provided that we weaken slightly the definition of qbnets, by not requiring that the unitarity condition Eq.(B.10) be satisfied. Also, the outgoing arrows of the nodes of a birdtrack must be understood as the result of marginalizer nodes. For example, if the arrows leaving a node are labelled a_1 and a_2, then these two arrows must be understood as the result of marginalizing an arrow $a^{:2} = (a_1, a_2)$.

Appendix C

Clebsch-Gordan Series Tables

In this Appendix, we present jpgs of some interesting tables from the Birdtracks book Ref.[1] by Cvitanovic. The tables give Clebsch-Gordan series (tensor product decompositions) of $SU(n)$ and $SO(n)$

A black (white) filled rectangle with p incoming and p outgoing legs represents the anti-symmetrizer $\mathcal{A}_p$ (symmetrizer $\mathcal{S}_p$).

Thick lines represent particles in the defining representation. These lines range over $1, 2, \ldots, n$.

Thin lines represent particles in the adjoint rep. (In this book we represent such particles by a wavy line $\sim\sim$.)

UNITARY GROUPS 97

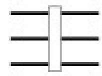
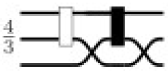
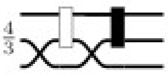
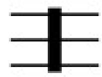
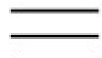
Y_a	$\mathbf{P}_{Y_a}$	d_{Y_a}
1 2 3		$\frac{n(n+1)(n+2)}{6}$
1 2 / 3	$\frac{4}{3}$ 	$\frac{n(n^2-1)}{3}$
1 3 / 2	$\frac{4}{3}$ 	
1 / 2 / 3		$\frac{(n-2)(n-1)n}{6}$
$1 \otimes 2 \otimes 3$		n^3

Table 9.1 Reduction of 3-index tensor. The last row shows the direct sum of the Young tableaux, the sum of the dimensions of the irreps adding up to n^3, and the sum of the projection operators adding up to the identity as verification of completeness (3.51).

ORTHOGONAL GROUPS 123

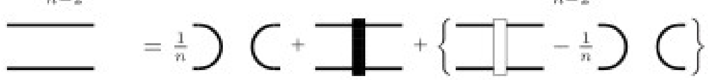
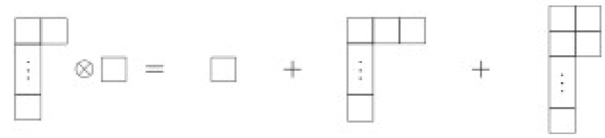
Young tableaux							
Dynkin labels	$(10\ldots)\times(10\ldots)$	=	$(00\ldots)$	+	$(010\ldots)$	+	$(20\ldots)$
Dimensions	n^2	=	1	+	$\frac{n(n-1)}{2}$	+	$\frac{(n+2)(n-1)}{2}$
Dynkin indices	$2n\frac{1}{n-2}$	=	0	+	1	+	$\frac{n+2}{n-2}$
Projectors							

Table 10.1 $SO(n)$ Clebsch-Gordan series for $V \otimes V$.

	$\underline{A} \otimes \underline{q}$	$=$	$\underline{V_1}$	$\oplus$	$\underline{V_2}$	$\oplus$	$\underline{V_3}$
Dynkin labels	$(10\ldots1)\otimes(10\ldots)$	$=$	$(10\ldots)$	$\oplus$	$(200\ldots01)$	$\oplus$	$(010\ldots01)$
							
Dimensions:	$(n^2-1)n$	$=$	n	$+$	$\frac{n(n-1)(n+2)}{2}$	$+$	$\frac{n(n+1)(n-2)}{2}$
Indices:	$n+\frac{n^2-1}{2n}$	$=$	$\frac{1}{2n}$	$+$	$\frac{(n+2)(3n-1)}{4n}$	$+$	$\frac{(n-2)(3n+1)}{4n}$
SU(3) example:							
Dimensions:	$8\cdot3$	$=$	3	$+$	15	$+$	6
Indices:	$13/3$	$=$	$1/6$	$+$	$10/3$	$+$	$5/6$
SU(4) example:							
Dimensions:	$15\cdot4$	$=$	4	$+$	36	$+$	20
Indices:	$47/8$	$=$	$1/8$	$+$	$33/8$	$+$	$13/8$

Projection operators:

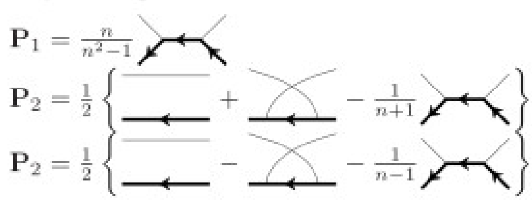


Table 9.3 $SU(n)$ $V \otimes A$ Clebsch-Gordan series.

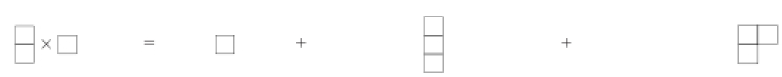
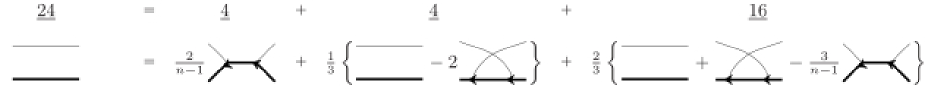
Young tableaux							
Dynkin labels	$(010\ldots)\times(100\ldots)$	$=$	$(100\ldots)$	$+$	$(0010\ldots)$	$+$	$(110\ldots)$
Dimensions	$\frac{n^2(n-1)}{2}$	$=$	n	$+$	$\frac{n(n-1)(n-2)}{6}$	$+$	$\frac{n(n^2-4)}{3}$
$SO(3)$	$\underline{9}$	$=$	$\underline{3}$	$+$	$\underline{1}$	$+$	$\underline{5}$
$SO(4)$	$\underline{24}$	$=$	$\underline{4}$	$+$	$\underline{4}$	$+$	$\underline{16}$
Projectors							

Table 10.2 $SO(n)$ $A\otimes V$ Clebsch-Gordan series.

Chapter 1

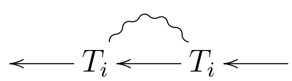
Casimir Operators

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

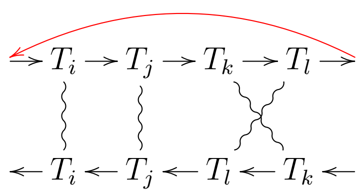
The term **Casimir operator** will be used to refer to 2 types of operators: a Casimir matrix or a Casimir sun

Casimir matrix

Examples:

$$M_2 =$$ 

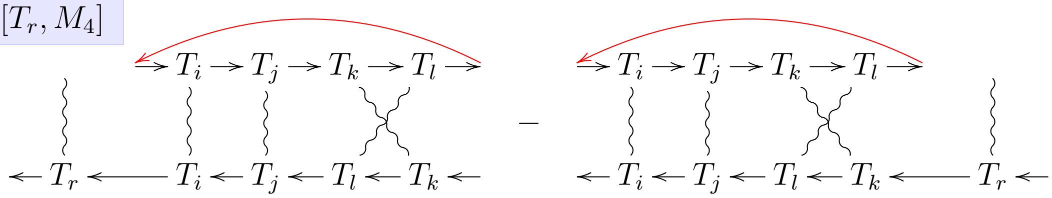
(1.1)

$$M_4 =$$ 

(1.2)

Casimir matrices are invariant matrices so they satisfy

$\boxed{0 = [T_r, M_4]}$

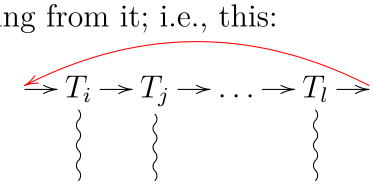
$$0 =$$ 

(1.3)

Because Casimir matrices are invariant matrices, they commute with each other. For example,

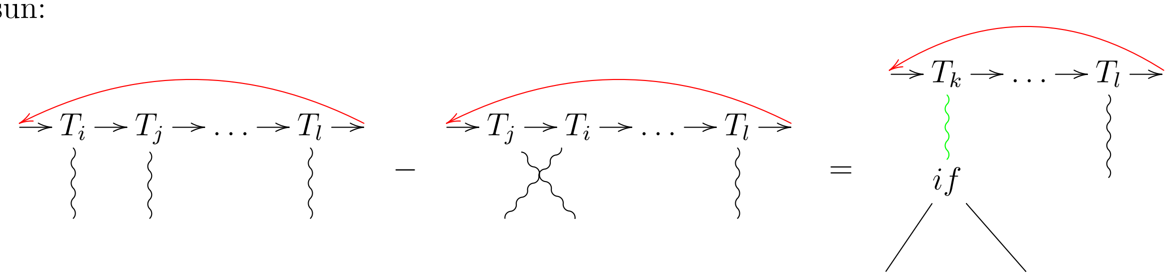
$$M_2 M_4 = M_4 M_2 \tag{1.4}$$

Casimir sun. By this we mean a tensor consisting of a loop of fundamental particles with gluons (rays) emanating from it; i.e., this:

$$\mathrm{tr}(T_i T_j \dots T_l) =$$ 

(1.5)

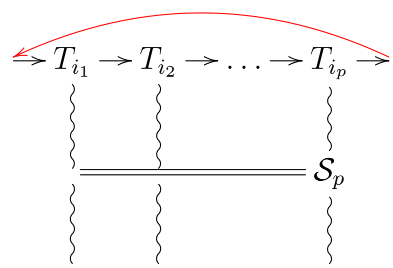
Note that the Lie Algebra commutation relations can be applies to a Casimir sun:



(1.6)

Note also that we can define a symmetrized version of a Casimir sun:

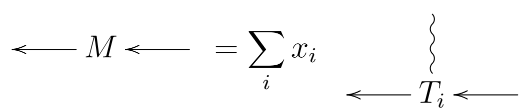
$$h_{i_1 i_2 \ldots i_p} = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} \mathrm{tr}(T_{\sigma(i_1)} T_{\sigma(i_2)} \ldots T_{\sigma(i_p)}) = \tag{1.7}$$



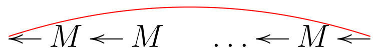
1.1 Independent Casimirs of Simple Lie Groups

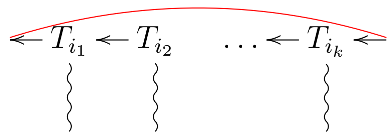
So as not to have any gluon free indices, it is convenient to contract with a matrix M, all the outgoing gluons of a Casimir sun . Let

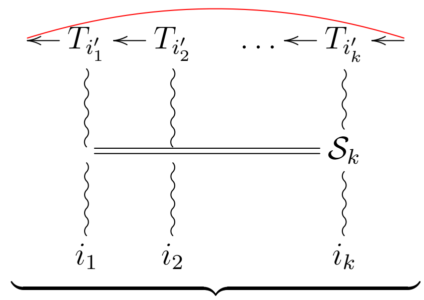
$$\boxed{M = \sum_i T_i x_i} \tag{1.8}$$



Then

$$\mathrm{tr}(M^k) = \quad$$  $$(1.9)$$

$$= \sum_{i_1 i_2 \ldots i_k}$$  $$x_{i_1} x_{i_2} \ldots x_{i_k} \qquad (1.10)$$

$$= \sum_{i_1 i_2 \ldots i_k}$$  $$x_{i_1} x_{i_2} \ldots x_{i_k} \qquad (1.11)$$

$h_{i_1 i_2 \ldots i_k}$

Recall Eq.(2.22) for the general characteristic equation of a matrix M

$$0 = \sum_{k=0}^{n} (-1)^k \left(\mathrm{tr}_{1\ldots n-k} \mathcal{A}_{n-k} M^{\otimes n-k}\right) M^k \qquad (1.12)$$

$$= \begin{cases} M^n \\ -M^{n-1}(\mathrm{tr}M) \\ +M^{n-2}(\mathrm{tr}_{1\ldots 2}\mathcal{A}_2 M^{\otimes 2}) \\ \ldots \\ (-1)^n det(M) \end{cases} \qquad (1.13)$$

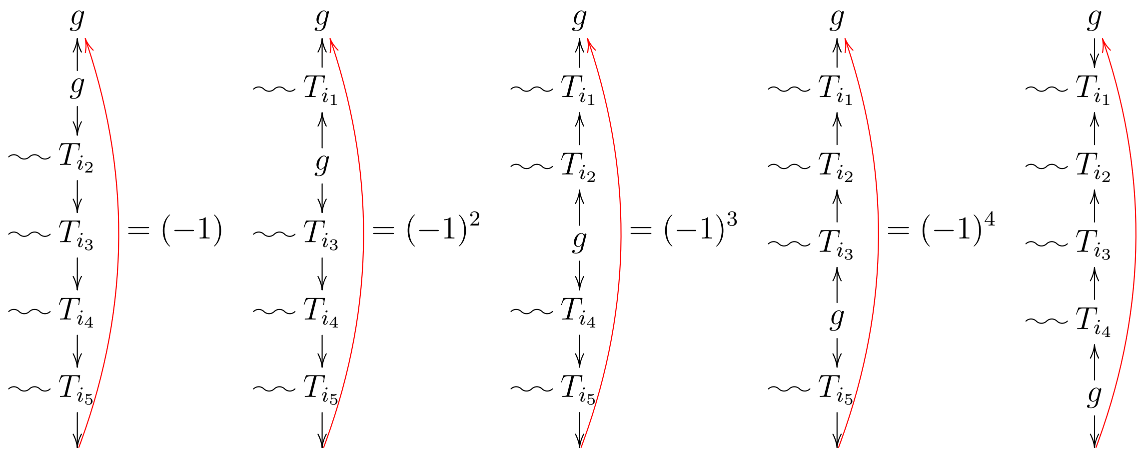
Note that that $\mathrm{tr}_{12}\mathcal{A}_2 M^{\otimes 2}$ can be expressed in terms of $\mathrm{tr}(M)$ and $\mathrm{tr}(M^2)$. Likewise, $\mathrm{tr}_{123}\mathcal{A}_3 M^{\otimes 3}$ can be expressed in terms of $\mathrm{tr}(M)$, $\mathrm{tr}(M^2)$ and $\mathrm{tr}(M^3)$. If we take the trace of the above equation, we get an equation constraining $\mathrm{tr}(M^k)$ for $k = 1, 2, \ldots n$.

The **Betti number** of the Casimir $\mathrm{tr}(M^k) \neq 0$ is the integer k. Table 1.1 gives all the Betti numbers for the simple Lie Algebras. Note that the Betti numbers in Table 1.1 are all even except for $SU(n)$.

For all simple Lie Groups except for $SU(n)$, there is a invertible symmetric or skew-symmetric bilinear invariant matrix g_{ab} satisfying $g_{ab}g^{bc} = \delta_a^c$. Hence

$A_r = \mathfrak{su}(r+1)$	$2, 3, \ldots, r+1$
$B_r = \mathfrak{so}(2r+1)$	$2, 4, 6, \ldots, 2r$
$C_r = \mathfrak{sp}(2r)$	$2, 4, 6, \ldots, 2r$
$D_r = \mathfrak{so}(2r)$	$2, 4, \ldots, 2r-2, 2r$
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$6, 8, 10, 12, 14, 18$
E_8	$8, 12, 14, 18, 20, 24, 30$

Table 1.1: Betti numbers for the simple Lie Algebras

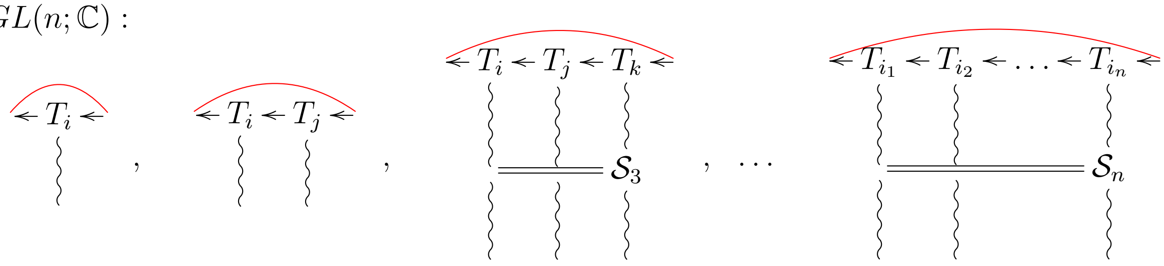


(1.14)

As illustrated in Eq.(1.14), if such a g^{ab} exists, a Casimir $\mathrm{tr}(M^k)$ equals itself times $(-1)^k$. Hence, only Casimirs with even k are non-zero.

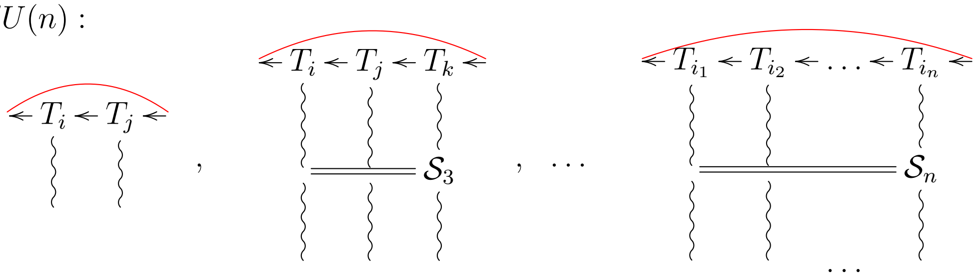
Claim 1 *The following are a complete set of Casimir operators for the given groups*

$GL(n;\mathbb{C})$:



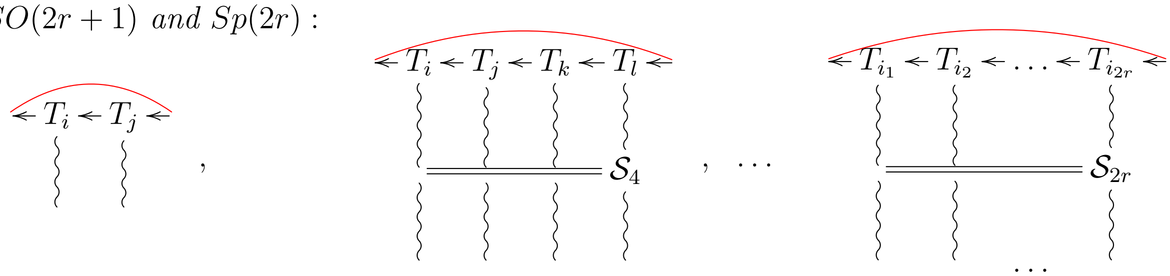
(1.15)

$SU(n)$:



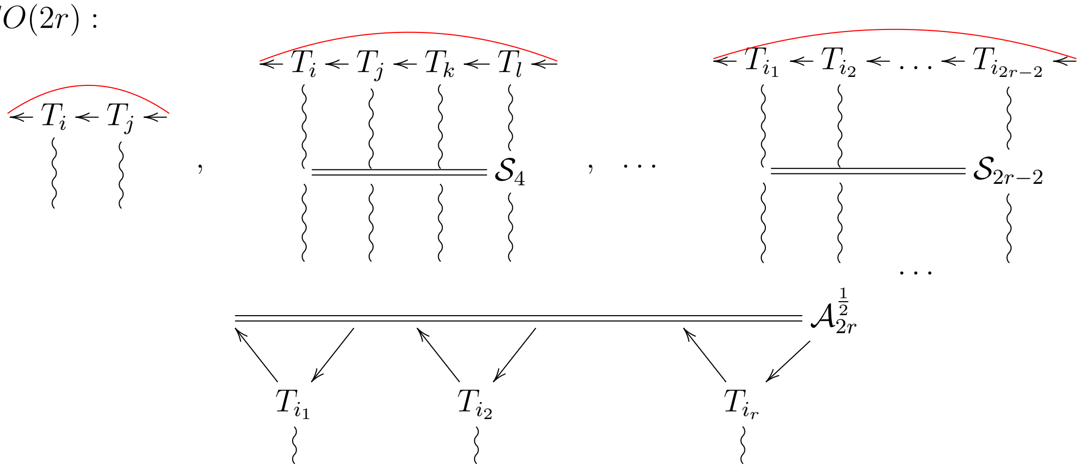
(1.16)

$SO(2r+1)$ *and* $Sp(2r)$:



(1.17)

$SO(2r)$:

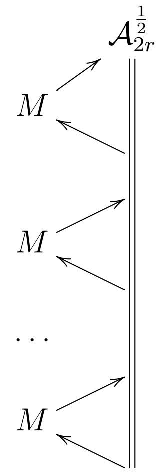


(1.18)

proof:

Define

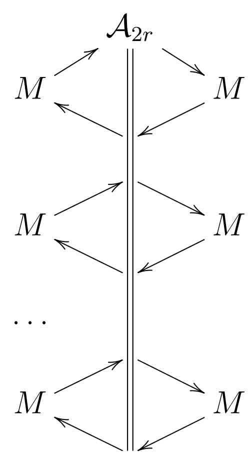
$$I_r(x) =$$



(1.19)

where $\mathcal{A}^{\frac{1}{2}}$ is the Levi Civita tensor.(see Chapter 20). Then an expansion of $I_r^2(x)$ contains $\mathrm{tr}(M^{2r})$ among its summands.

$$I_r^2(x) =$$



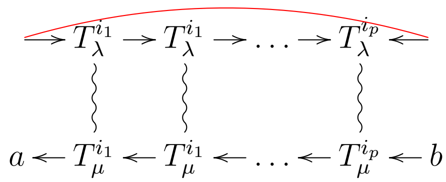
$$= \mathrm{tr}(M^{2r}) + \ldots \qquad (1.20)$$

QED

1.2 Casimir Matrix Expressed in Terms of $6j$ Coefficients

Define the Casimir matrix I_p as

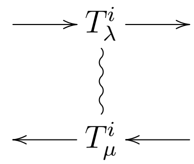
$$(I_p)_a{}^b = \mathrm{tr}(T_\lambda^{i_1} T_\lambda^{i_2} \dots T_\lambda^{i_p})(T_\mu^{i_1} T_\mu^{i_2} \dots T_\mu^{i_p})_a{}^b \tag{1.21}$$

$=$ 

(1.22)

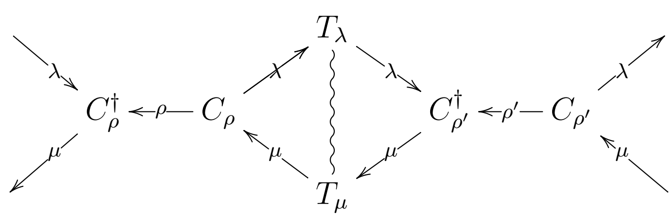
The goal of this section is to express I_p in terms of $6j$ coefficients.

Let

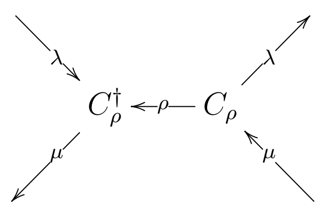
$M =$ 

(1.23)

We will first decompose M in terms of $6j$ coefficients, and then use that result to decompose I_p for $p = 1, 2, 3, \dots$. Note that

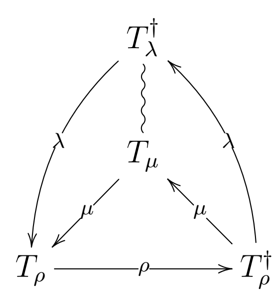
$M = \sum_{\rho,\rho'}$ 

(1.24)

$= \sum_\rho A(\lambda, \rho, \mu)$ 

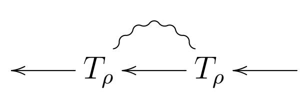
(1.25)

where

$A(\lambda, \rho, \mu) = \frac{1}{d_\rho}$ 

(1.26)

Claim 2 *If*

$\Gamma_2(\rho) =$ 

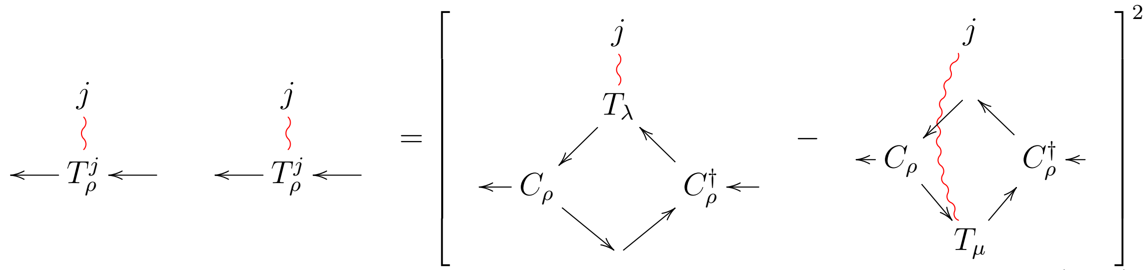
(1.27)

then

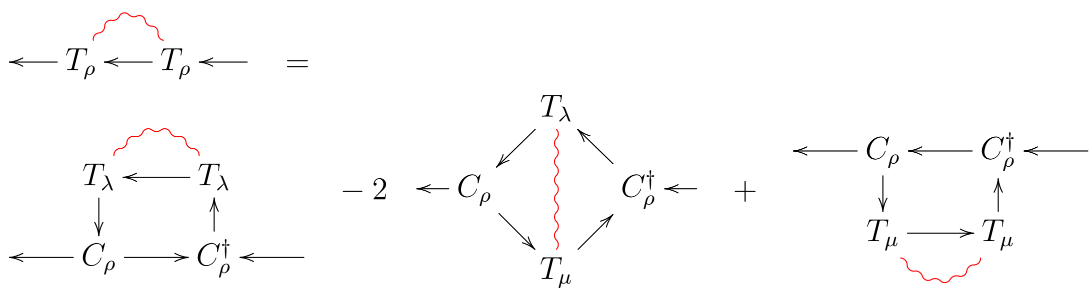
$$A(\lambda,\mu,\rho) = -\frac{1}{2}\left[\Gamma_2(\rho) - \Gamma_2(\lambda) - \Gamma_2(\mu)\right] \tag{1.28}$$

proof:

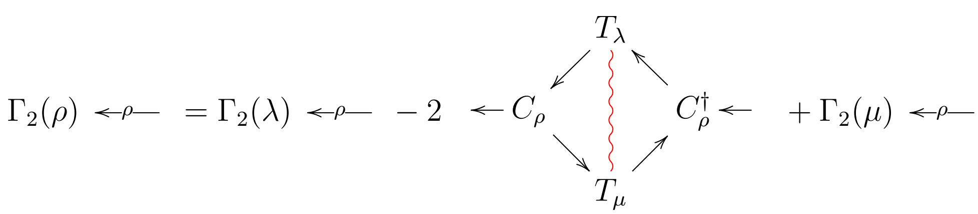
Recall Eq.(8.22). Square both sides of the equation.



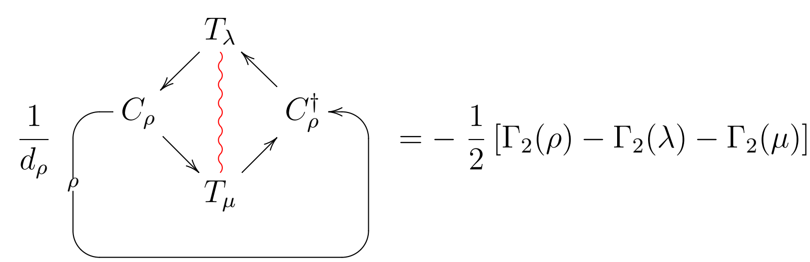
(1.29)



(1.30)



(1.31)



(1.32)

This is similar to assuming

$$\vec{J} = \vec{L} + \vec{S} \tag{1.33}$$

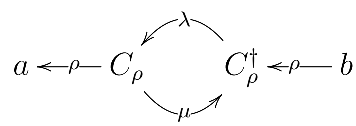
where $\vec{J}, \vec{L}, \vec{S}$ are the total, orbital and spin angular momentum. Then

$$\vec{L} \cdot \vec{S} = \frac{1}{2}\left[J^2 - L^2 - S^2\right] \tag{1.34}$$

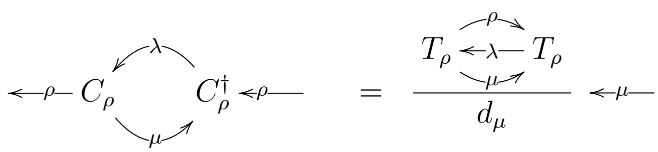
QED Next note that

$$(I_p)_a{}^b = (M^p)^c{}_a{}^b{}_c \tag{1.35}$$

$$= \sum_{\rho \in irreps} [A(\lambda, \mu, \rho)]^p \tag{1.36}$$



If μ is an irrep,

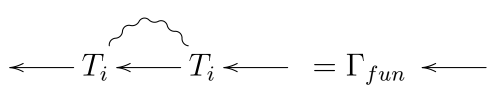


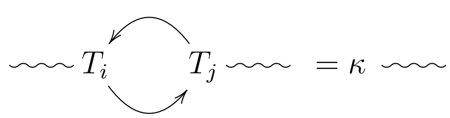
(1.37)

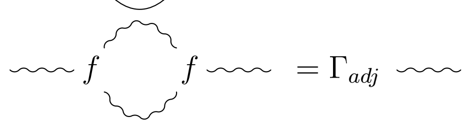
$$= \frac{d_\rho}{d_\lambda} \xleftarrow{\mu} \text{ (because } \rho \text{ is an irrep)} \tag{1.38}$$

1.3 $\mathrm{tr}(M^2)$ and $\mathrm{tr}(M^3)$

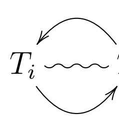
There are 3 quadratic Casimir ($\mathrm{tr}(M^2)$) matrices:

1. $\boxed{(T_iT_i)_a{}^b = \Gamma_{fun}\delta_a^b}$  (1.39)

2. $\boxed{\mathrm{tr}(T_iT_j) = \kappa\delta_i^j}$  (1.40)

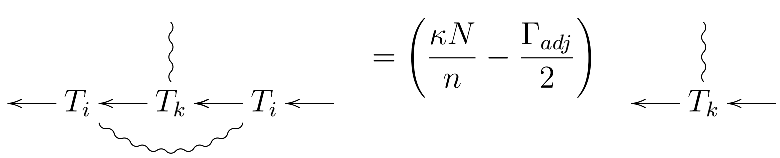
3. $\boxed{f_{ijk}f_{kji'} = \Gamma_{adj}\delta_i^{i'}}$  (1.41)

Note that

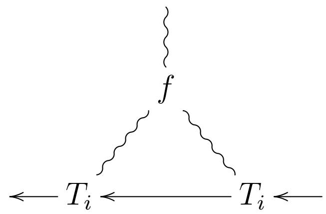


$$= n\Gamma_{fun} = N\kappa \tag{1.42}$$

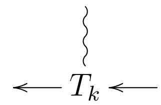
Claim 3



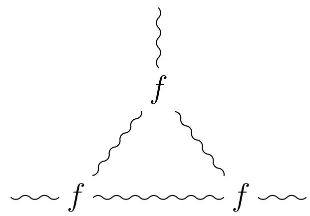
(1.43)



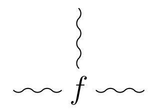
$$= \frac{\Gamma_{adj}}{2}$$



(1.44)



$$= \frac{\Gamma_{adj}}{2}$$

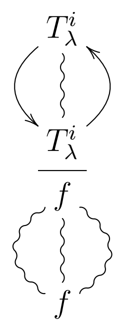


(1.45)

proof:
QED

1.4 Dynkin Index

$$DI_\lambda = \frac{\mathrm{tr}(T^i_\lambda T^i_\lambda)}{f^i_{jk} f^i_{kj}} = \tag{1.46}$$



Chapter 2

Characteristic Equations

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Let

$$M_a{}^b = \; a \longleftarrow M \longleftarrow b \tag{2.1}$$

for $a, b = 1, 2, \ldots, n$.

The goal of this chapter is to express the coefficients of the characteristic equation (i.e., $det(M - \lambda) = 0$) of M as traces.

For starters, note the difference between birdtracks for a matrix power and a tensor power of M.

$$M^2 = \; \longleftarrow M^2 \longleftarrow \; = \; \longleftarrow M \longleftarrow M \longleftarrow \tag{2.2}$$

$$M \otimes M = M^{\otimes 2} = \begin{array}{c} \longleftarrow M \longleftarrow \\ \\ \longleftarrow M \longleftarrow \end{array} \tag{2.3}$$

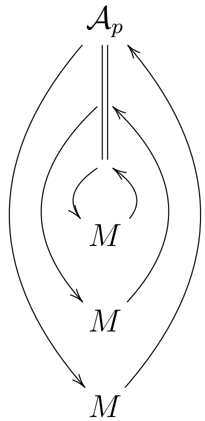
In general, $M^{\otimes p}$ is defined by

$$\boxed{(M^{\otimes p})_\alpha{}^\beta = (M^{\otimes p})_{a^{\cdot p}}{}^{rev(b^p)} = M_{a_1}{}^{b_1} M_{a_2}{}^{b_2} \ldots M_{a_p}{}^{b_p}}$$

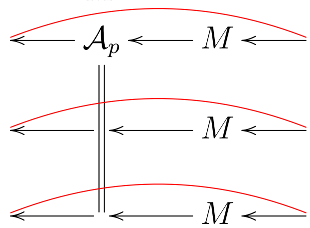
$$\begin{array}{ccc} \longleftarrow M^{\otimes p} \longleftarrow & & \longleftarrow M \longleftarrow \\ \longleftarrow\!\!\|\!\!\longleftarrow & & \longleftarrow M \longleftarrow \\ \vdots \quad\quad \vdots & = & \vdots \quad\quad \vdots \\ \longleftarrow\!\!\|\!\!\longleftarrow & & \longleftarrow M \longleftarrow \end{array} \tag{2.4}$$

where $a_i, b_i \in \mathbb{Z}_{[1,n]}$, and we define the anti-symmetrized trace of $M^{\otimes p}$ by

$$\mathrm{tr}_{1\ldots p}\mathcal{A}[M^{\otimes p}] \;=\; \mathcal{A}_{a^{:p}}{}^{rev(b^{:p})}\prod_{i=1}^{p} M_{b_i}{}^{a_i} \tag{2.5}$$

$=$ 

(Cvitanovic Drawing Style) (2.6)

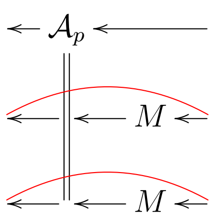
$=$ 

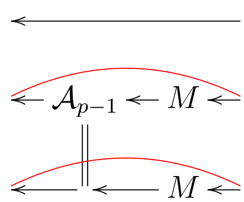
(This book's drawing style) (2.7)

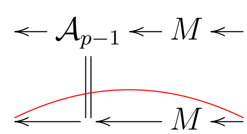
Note that the determinant of M is one of those traces

$$detM = \mathrm{tr}_{1\ldots n}\mathcal{A}[M^{\otimes n}] \tag{2.8}$$

Claim 4



$= \dfrac{1}{p}\Big[$ 

$-\,(p-1)$ 

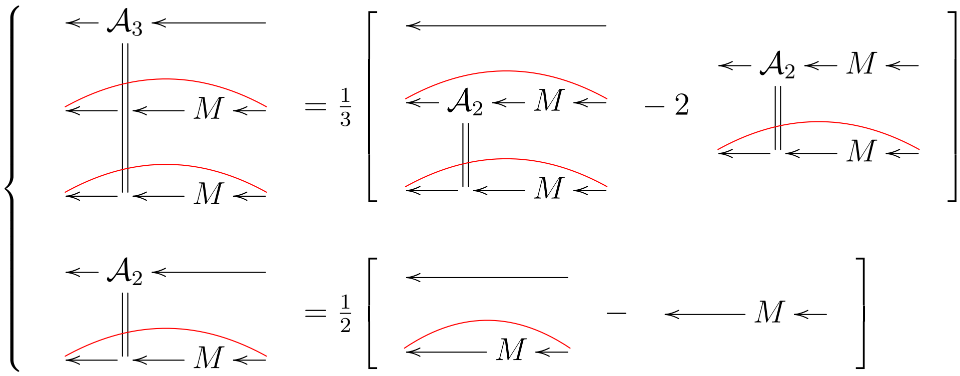
$\Big]$ (2.9)

proof:

See Chapter 20.

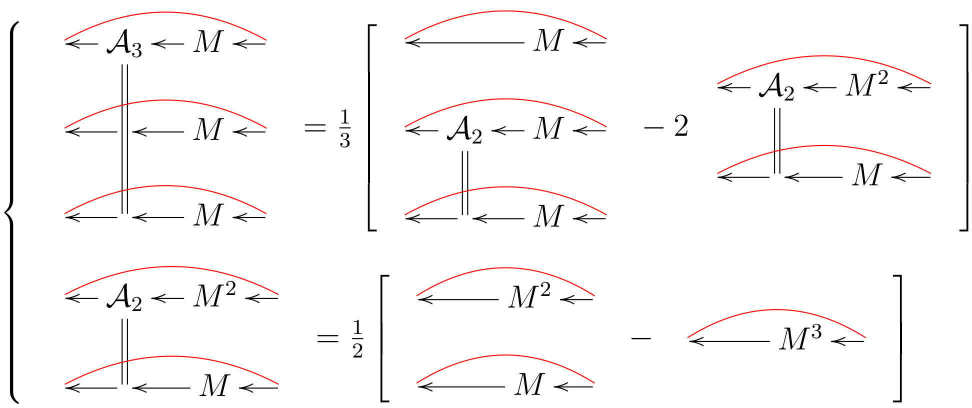
QED

Consider the above claim for $p = 2, 3$.



(2.10)

If we multiply from the right, by M^d for $d = 1, 2$, the first row of Eq.(2.10) and then take the trace of that row, we get



(2.11)

Let

$$\tau = \mathrm{tr}(M) \tag{2.12}$$

Then Eqs.(2.11) can be expressed algebraically by

$$\mathrm{tr}_{1,2,3}\mathcal{A}_3(M^{\otimes 3}) = \frac{1}{3}\left[\tau \mathrm{tr}_{1,2}\mathcal{A}_2(M^{\otimes 2}) - \mathrm{tr}(M^2)\tau + \mathrm{tr}M^3\right] \tag{2.13}$$

and

$$\mathrm{tr}_{1,2}\mathcal{A}_2 M^{\otimes 2} = \frac{1}{2}\left[\tau^2 - \mathrm{tr}(M^2)\right] \tag{2.14}$$

Therefore,

$$\begin{aligned} \mathrm{tr}_{1,2,3}\mathcal{A}_3(M^{\otimes 3}) &= \frac{1}{3}\left[\frac{1}{2}\tau^3 - \frac{3}{2}\mathrm{tr}(M^2)\tau + \mathrm{tr}M^3\right] && (2.15)\\ &= \frac{1}{3!}\left[\tau^3 - 3\mathrm{tr}(M^2)\tau + 2\mathrm{tr}M^3\right] && (2.16) \end{aligned}$$

In general,

$$\mathrm{tr}_{1\ldots p}\mathcal{A}_p M = \frac{1}{p}\sum_{k=1}^{p}(-1)^{k-1}\left(\mathrm{tr}_{1\ldots p-k}\mathcal{A}_{p-k}M^{\otimes p-k}\right)\mathrm{tr}(M^k) \tag{2.17}$$

Next note that

$$\mathcal{A}_p = 0 \quad \text{if } p > n \tag{2.18}$$

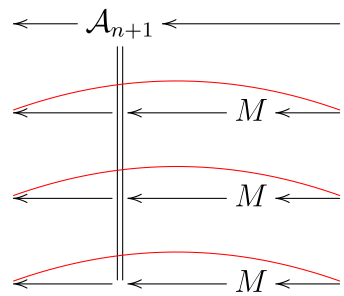
This follows because the Levi Civita tensor with more than n indices is zero.; i.e.,

$$\epsilon_{a_1,a_2,\ldots,a_{n+1}} = 0 \tag{2.19}$$

Indeed, two of the a_i must be equal, so that element of the ϵ tensor is zero

Let I be the $n \times n$ identity matrix. Then, since $\mathcal{A}_{n+1} = 0$, the following is true

$$\boxed{0 = \mathrm{tr}_{2\ldots n+1}\mathcal{A}_{n+1} I \otimes M^{\otimes n}} \qquad 0 = \tag{2.20}$$



We can now expand the right hand side of Eq.(2.20) using identity Eq.(2.17)

$$0 = \sum_{k=0}^{n}(-1)^k\left(\mathrm{tr}_{1\ldots n-k}\mathcal{A}_{n-k}M^{\otimes n-k}\right)M^k \tag{2.21}$$

$$= \left\{\begin{array}{l} M^n \\ -M^{n-1}(\mathrm{tr}M) \\ +M^{n-2}(\mathrm{tr}_{1\ldots 2}\mathcal{A}_2 M^{\otimes 2}) \\ \ldots \\ (-1)^n det(M) \end{array}\right. \tag{2.22}$$

Viola. The last equation is none other than the characteristic equation of M. As promised, the coefficients of this polynomial in M, are expressed as traces.

Chapter 3

Clebsch-Gordan Coefficients

This chapter is based on Cvitanovic's Birdtracks book Ref.[1].

Recall that if $|x\rangle$ for $x \in val(\underline{x})$ is a complete, orthonormal basis in Quantum Mechanics, then

$$\langle x|y\rangle = \delta(x, y) \quad \text{(orthonormality)} \tag{3.1}$$

and

$$\sum_x |x\rangle\langle x| = 1 \quad \text{(completeness)} \tag{3.2}$$

Furthermore, if we define

$$\pi_x = |x\rangle\langle x| \tag{3.3}$$

then π_x is a is a projection operator so

$$\pi_x \pi_x = \pi_x \tag{3.4}$$

and

$$\pi_x|y\rangle = |y\rangle\delta(x, y), \quad \langle y|\pi_x = \langle y|\delta(x, y) \tag{3.5}$$

Below, we will define matrices $C_\lambda = \langle\lambda|$ and $C^\dagger_\lambda = |\lambda\rangle$. If we identify $\langle\lambda|$ with $\langle x|$, and $|\lambda\rangle$ with $|x\rangle$, then $\langle\lambda|$ and $|\lambda\rangle$ satisfy identities similar to those satisfied by $\langle x|$ and $|x\rangle$. We will show this in this chapter.

3.1 CB Coefficients as Matrices

Suppose that $M \in \mathbb{C}^{d\times d}$ is a Hermitian matrix. Then we have

$$M = C^\dagger \Lambda C \tag{3.6}$$

where $C \in \mathbb{C}^{d\times d}$ is a unitary matrix, and Λ is a diagonal matrix.

One can partition C into rectangular submatrices $\langle\lambda|$ that have d_λ rows with $d_\lambda < d$, such that we have one $\langle\lambda|$ for each eigenvalue λ of C. Likewise, we can partition $C^\dagger$ into rectangular submatrices $C_\lambda^\dagger$ that have d_λ columns with $d_\lambda < d$, such that we have one $|\lambda\rangle$ for each eigenvalue λ of C. Thus, if $I^{d_\lambda \times d_\lambda}$ is the $d_\lambda \times d_\lambda$ identity matrix,

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d\times d} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d\times d}}_{\pi_\lambda} C^{d\times d} \tag{3.7}$$

$$\begin{bmatrix} 0 & (C_\lambda^\dagger)^{d\times d_\lambda} & 0 \end{bmatrix}^{d\times d} = (C^\dagger)^{d\times d} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d\times d}}_{\pi_\lambda} \tag{3.8}$$

Henceforth in this chapter, we will use C_λ and $\langle\lambda|$ interchangeably. Likewise, we will use $C_\lambda^\dagger$ and $|\lambda\rangle$ interchangeably The matrices $C_\lambda = \langle\lambda|$ are called the **Clebsch-Gordan (CG) coefficients** for M.

The matrices π_λ obviously form a complete orthogonal set of projection operators:

$$\sum_\lambda \pi_\lambda = 1, \quad \pi_\lambda \pi_\mu = \pi_\lambda \delta(\lambda, \mu) \tag{3.9}$$

We now have

$$\pi_\lambda C = \langle\lambda|, \quad C^\dagger \pi_\lambda = |\lambda\rangle \tag{3.10}$$

$$\begin{aligned} \langle\lambda||\lambda\rangle &= \pi_\lambda C C^\dagger \pi_\lambda &(3.11)\\ &= \pi_\lambda &(3.12) \end{aligned}$$

$$\begin{aligned} M &= C^\dagger \Lambda C &(3.13)\\ &= C^\dagger \left(\sum_\lambda \lambda \pi_\lambda \right) C &(3.14)\\ &= \sum_\lambda \lambda |\lambda\rangle\langle\lambda| &(3.15) \end{aligned}$$

$$\begin{aligned} I^{d\times d} &= C^\dagger C &(3.16)\\ &= \sum_\lambda C^\dagger \pi_\lambda C &(3.17)\\ &= \sum_\lambda \underbrace{|\lambda\rangle\langle\lambda|}_{P_\lambda} &(3.18) \end{aligned}$$

We will call Eq.(3.18) a **Clebsch-Gordan (CG) series or decomposition** either in that form, or after multiplying by a vector space V, as in

$$V = \sum_\lambda P_\lambda V \tag{3.19}$$

A simple example of a CG series is

$$\vec{r} = \widehat{x} \oplus \widehat{y} \oplus \widehat{z} \tag{3.20}$$

for a vector $\vec{r} \in \mathbb{R}^3$. In this expression, the vectors $\widehat{x}, \widehat{y}, \widehat{z}$ constitute a complete (i.e., basis) orthonormal set for the vectors acted upon by $SO(3)$. Any generic vector $\vec{r} \in \mathbb{R}^3$ can be expressed as[1]

$$\vec{r} = a\widehat{x} + b\widehat{y} + c\widehat{z} \tag{3.21}$$

for some $a, b, c \in \mathbb{R}$.

So far, we have established that

$$P_\lambda = |\lambda\rangle\langle\lambda| = C^\dagger \pi_\lambda C, \tag{3.22}$$
$$\pi_\lambda = \langle\lambda||\lambda\rangle \tag{3.23}$$
$$1 = \sum_\lambda \underbrace{|\lambda\rangle\langle\lambda|}_{P_\lambda} = \sum_\lambda \underbrace{\langle\lambda||\lambda\rangle}_{\pi_\lambda} \tag{3.24}$$

In fact, the P_λ form a complete orthogonal set of projection operators, just like the π_λ.

$$\sum_\lambda P_\lambda = 1, \quad P_\lambda P_\mu = P_\lambda \delta(\mu, \nu) \tag{3.25}$$

Whereas the π_λ satisfy

$$\pi_\lambda C = \langle\lambda|, \quad C^\dagger \pi_\lambda = |\lambda\rangle \tag{3.26}$$

the P_λ satisfy

$$C P_\lambda = \langle\lambda|, \quad P_\lambda C^\dagger = |\lambda\rangle \tag{3.27}$$

Since we are assuming M is Hermitian, its eigenvalues are real. Thus, we can absorb the eigenvalue λ into the CG coefficients by defining

$$\mathcal{C}_\lambda = \sqrt{\lambda}\langle\lambda| \tag{3.28}$$

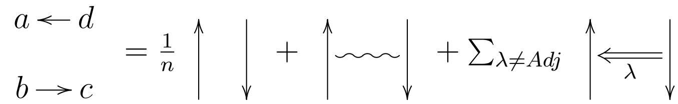
and writing

$$M = \sum_\lambda \mathcal{C}_\lambda^\dagger \mathcal{C}_\lambda \tag{3.29}$$

[1] Recall that a direct sum of two vector spaces $V = V_1 \oplus V_2$ means $V_1 \cap V_2 = \{0\}$

Here is an example of a CG series. One can decompose $V^n \otimes V^{\dagger n} = \sum_\lambda V_\lambda$ as follows

$$\boxed{1 = \frac{1}{n} P_S + P_{adj} + \sum_{\lambda \neq Adj} P_\lambda,}$$
$$\boxed{\delta^a_d \delta^c_d = \frac{1}{n} \delta^a_b \delta^c_d + (P_{adj})_a{}^b{}_c{}^d + \sum_{\lambda \neq Adj} (P_\lambda)_a{}^b{}_c{}^d} \tag{3.30}$$



3.2 Generalization From Matrices to Tensors

Let $b^{:nb} = (b_1, b_2, \ldots, b_{nb})$ where $b_i \in Z_{[0,d_{\mu_i}]}$ and $a \in Z_{[1,d_\lambda]}$. Assume that

$$d_\lambda = \prod_{i=1}^{:nb} d_{\mu_i} \tag{3.31}$$

Now define the birdtracks

$$(\langle\lambda|)_a{}^{rev(b^{:nb})} = \begin{tikzcd} & & \mu_1 b_1 \arrow[dl] \\ \lambda a & \langle\lambda| \arrow[l] & \mu_2 b_2 \arrow[l] \\ & & \mu_{nb} b_{nb} \arrow[ul] \end{tikzcd} \tag{3.32}$$

and

$$(|\lambda\rangle)^a{}_{b^{:nb}} = \begin{tikzcd} \mu_1 b_1 & & \\ \mu_2 b_2 & |\lambda\rangle \arrow[l] \arrow[ul] \arrow[dl] & \lambda a \arrow[l] \\ \mu_{nb} b_{nb} & & \end{tikzcd} \tag{3.33}$$

We will assume there is no difference between when a Greek letter is lowered and when it is raised. Also, all summations over a Greek letter will be stated explicitly; i.e., no implicit summations over repeated Greek letters.

On the other hand, the Latin letter indices b_i, a of $\langle\lambda|$ and $|\lambda\rangle$ may be lowered or raised and their arrows changed from outgoing to incoming or vice versa. Furthermore, we will use implicit summation over repeated Latin letters.

The Greek letters label representation of the group (not necessarily irreps). Each b_i labels a member of μ_i, and each a labels a member of λ.

$$\boxed{(\langle\lambda|)_a{}^{rev((b')^{:nb})}(P_\mu)_{(b')^{:nb}}{}^{rev(b^{:nb})} = \delta(\mu,\lambda)(\langle\mu|)_a{}^{rev(b^{:nb})}, \quad \langle\lambda|P_\mu = \delta(\mu,\lambda)\langle\mu|}$$

$$\begin{tikzcd}
 & & \sum b'_1 \arrow[dl] & {} & & b_1 \arrow[dl] & & & & b_1 \arrow[dl] \\
a & \langle\lambda| \arrow[l] & \sum b'_2 \arrow[l] & {} & P_\mu \arrow[l] \arrow[ul] \arrow[dl] & b_2 \arrow[l] & = \delta(\mu,\lambda) & a & \langle\lambda| \arrow[l] & b_2 \arrow[l] \\
 & & \sum b'_{nb} \arrow[ul] & {} & & b_{nb} \arrow[ul] & & & & b_{nb} \arrow[ul]
\end{tikzcd} \tag{3.34}$$

$$\boxed{(P_\mu)_{b^{:nb}}{}^{rev((b')^{:nb})}(|\lambda\rangle)^a{}_{(b')^{:nb}} = \delta(\mu,\lambda)(|\mu\rangle)^a{}_{b^{:nb}}, \quad P_\mu|\lambda\rangle = \delta(\mu,\lambda)|\mu\rangle}$$

$$\begin{tikzcd}
b_1 & & \sum b'_1 \arrow[dl] & {} & & & b_1 & & \\
b_2 & P_\mu \arrow[l] \arrow[ul] \arrow[dl] & \sum b'_2 \arrow[l] & {} & |\lambda\rangle \arrow[l] \arrow[ul] \arrow[dl] & a \arrow[l] & = \delta(\mu,\lambda) \; b_2 & |\lambda\rangle \arrow[l] \arrow[ul] \arrow[dl] & a \arrow[l] \\
b_{nb} & & \sum b'_{nb} \arrow[ul] & {} & & & b_{nb} & &
\end{tikzcd} \tag{3.35}$$

$$\boxed{(\langle\lambda|)_a{}^{rev(b^{:nb})}(|\mu\rangle)^{a'}{}_{b^{:nb}} = \delta(\lambda,\mu)\delta_a^{a'}, \quad \langle\lambda||\mu\rangle = \delta(\mu,\lambda)}$$

$$\begin{tikzcd}
 & & \sum b_1 \arrow[dl] & {} & & & \\
a & \langle\lambda| \arrow[l] & \sum b_2 \arrow[l] & {} & |\mu\rangle \arrow[l] \arrow[ul] \arrow[dl] & a' \arrow[l] & = \delta(\mu,\lambda) \; a \longleftarrow\!\bullet\!\!\longrightarrow a' \\
 & & \sum b_{nb} \arrow[ul] & {} & & &
\end{tikzcd} \tag{3.36}$$

$$\boxed{\sum_\lambda (|\lambda\rangle)^a{}_{b^{:nb}}(\langle\lambda|)_a{}^{rev((b')^{:nb})} = \delta_{b^{:nb}}^{rev((b')^{:nb})}, \quad \sum_\lambda |\lambda\rangle\langle\lambda| = 1}$$

$$\begin{tikzcd}
 & b_1 & & & & & b'_1 \arrow[dl] & & b_1 & \bullet \arrow[l] & b'_1 \arrow[l, no head] \\
\sum_\lambda & b_2 & |\lambda\rangle \arrow[l] \arrow[ul] \arrow[dl] & \sum a \arrow[l] & {} & \langle\lambda| \arrow[l] & b'_2 \arrow[l] & = & b_2 & \bullet \arrow[l] & b'_2 \arrow[l, no head] \\
 & b_{nb} & & & & & b'_{nb} \arrow[ul] & & b_{nb} & \bullet \arrow[l] & b'_{nb} \arrow[l, no head]
\end{tikzcd} \tag{3.37}$$

Chapter 4

Dynkin Diagrams

This chapter is based on Ref.[2], section 20.4.

This chapter is an overview of the classification of simple Lie algebras, a classification that was invented mainly by Killing, Cartan and Dynkin, in that historical order. The classification is valid for Lie algebras over $\mathbb{C}$[1] This caveat is important because there are more simple Lie algebras over $\mathbb{R}$ than over $\mathbb{C}$. When defining the generators of Lie algebras in other chapters, we defined a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ (a vector space over $\mathbb{R}$) which exponentiates to a group $\mathcal{G}_{\mathbb{R}} = \exp(\mathfrak{g}_{\mathbb{R}})$. This chapter refers to the complexification $\mathfrak{g}_{\mathbb{C}}$ (a vector space over $\mathbb{C}$) of $\mathfrak{g}_{\mathbb{R}}$ and to the group $\mathcal{G}_{\mathbb{C}} = \exp(\mathfrak{g}_{\mathbb{C}})$

Suppose $\mathfrak{g}_{\mathbb{C}}$ has generators X_s for $s \in S = \{1, 2, \ldots \mathcal{D}\}$ where

$\mathcal{D} = \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{C}} = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} =$ half the number of real degrees of freedom of the Lie algebra.

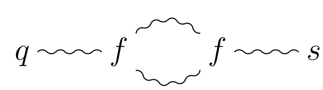
By definition, the generators X_s are closed under commutation so

$$[X_q, X_p] = \sum_t f_{qp}{}^t X_t \tag{4.1}$$

for some $f_{qp}{}^t \in \mathbb{C}$.

Define

$$g_{qs} = \sum_{p,t} f_{qp}{}^t f_{st}{}^p =$$



(4.2)

If $\det g = 0$, we can find disjoint sets S_1, S_2 so that $S = S_1 \cup S_2$ and

$$[X_a, X_b] = 0 \quad \text{for } a, b \in S_1 \tag{4.3}$$

and

$$[X_q, X_p] = \sum_t f_{qp}{}^t X_t \quad \text{for } p, q, t \in S_2 \tag{4.4}$$

Cartan Criterion: $\det g \neq 0$

[1] More generally, Cartan's classification is valid for Lie algebras over $\mathbb{F}$, where $\mathbb{F}$ is an algebraically closed field of characteristic zero. $\mathbb{C}$ satisfies both of these constraints. $\mathbb{R}$ has characteristic zero but is not algebraically closed.

We will assume that the CC is satisfied. This implies that the Lie algebra is semi-simple and that we can and will assume that g_{st} is diagonal.

$$g_{st} = \delta(s,t) = \quad \sim\sim\sim \tag{4.5}$$

$$f_{qp}{}^{t} = f_{qpt} \tag{4.6}$$

Will not, however, assume that f_{qpt} is totally antisymmetric, as is often assumed.

For any lower case latin letter q and Greek letter α, let
$q_{-} = 1, 2, \ldots, \mathcal{R}$
$\vec{\alpha} = 1, 2, \ldots, \mathcal{D} - \mathcal{R}$
$q =$ either q_{-} or $\vec{\alpha}$ but not both.

Let $\{H_{i_-}\}_{i_-=1}^{\mathcal{R}}$ be the largest possible set of mutually commuting X_p. $\mathcal{R}$ is called the **rank** of the group.

$$\boxed{[H_{i_-}, H_{j_-}] = 0} \tag{4.7}$$

Let $E_{\vec{\alpha}}$ be eigenvectors of H_{i_-} for a commutator "product" instead of a matrix multiplication product

$$\boxed{[H_{i_-}, E_{\vec{\alpha}}] = \underbrace{\alpha_{i_-}}_{f_{i_-,\vec{\alpha},\vec{\alpha}}} E_{\vec{\alpha}}} \tag{4.8}$$

The vectors $\vec{\alpha} \in \mathbb{R}^{\mathcal{R}}$ of eigenvalues are called **root vectors**.

From the properties of a commutator bracket[2]

$$\begin{aligned} [H_i, [E_{\vec{\alpha}}, E_{\vec{\beta}}]] &= [[H_i, E_{\vec{\alpha}}], E_{\vec{\beta}}] + [E_{\vec{\alpha}}, [H_i, E_{\vec{\beta}}]] && (4.9) \\ &= (\alpha_i + \beta_i)[E_{\vec{\alpha}}, E_{\vec{\beta}}] && (4.10) \end{aligned}$$

If $\vec{\alpha} + \vec{\beta} = 0$, $[H_i, [E_{\vec{\alpha}}, E_{\vec{\beta}}]] = 0$ so

$$\boxed{[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \sum_{i_-} f_{\vec{\alpha},-\vec{\alpha},i_-} H_{i_-}} \tag{4.11}$$

If $\vec{\alpha} + \vec{\beta} \neq 0$,

$$\boxed{[E_{\vec{\alpha}}, E_{\vec{\beta}}] = N_{\vec{\alpha},\vec{\beta}} E_{\vec{\alpha}+\vec{\beta}} \quad \text{if } \vec{\alpha} + \vec{\beta} \neq 0} \tag{4.12}$$

for some $N_{\vec{\alpha},\vec{\beta}} \in \mathbb{C}$.

The roots (**root system**) of a semisimple Lie algebra form a real $\mathcal{R}$ dimensional vector space.

[2]The commutator $[x, y] = xy - yx$ acts like a derivative operator: $[x[a, b]] = [[x, a], b] + [a, [x, b]]$

A **positive/negative (P/N) root** $\vec{\alpha}$ is a root for which the first component α_1 is positive/negative. If the first component of $\vec{\alpha}$ is zero, then decide the root's sign from its second component α_2. And so on. Note that the definition of P/N root is basis dependent.

A P root $\vec{\alpha}$ is a **simple positive (SP) root** if there are no P roots $\vec{\rho}$ and $\vec{\sigma}$ such that $\vec{\alpha} = \vec{\rho} + \vec{\sigma}$. Hence, SP roots are like the atoms or indivisible constituents of the root system.

Properties of root vectors $\vec{\alpha}, \vec{\beta}, \ldots \in \mathbb{R}^{\mathcal{R}}$

1. If $\vec{\alpha}$ is a root, then $-\vec{\alpha}$ is too.

2. We can find a basis of SP roots for the root space.

3. **Claim 5** *If $\vec{\alpha}$ and $\vec{\rho}$ are SP roots, then $\vec{\alpha} - \vec{\rho}$ is not a root of any kind.*

   proof:

   Assume $\vec{\alpha} - \vec{\rho}$ is either a P root or an N root.

   If $\vec{\alpha} - \vec{\rho}$ is a P root $\vec{\sigma}$, then $\vec{\alpha} = \vec{\rho} + \vec{\sigma}$ so $\vec{\alpha}$ is not a SP root.

   Likewise, if $\vec{\alpha} - \vec{\rho}$ is an N root $\vec{\sigma}$, then $-\vec{\sigma}$ is a P root and $\vec{\alpha} = \vec{\rho} + (-\vec{\sigma})$.
   QED

4. If $\vec{\alpha}$ and $\vec{\beta}$ are SP, then there can be roots

$$\{\vec{\beta} + n\vec{\alpha} | i = 0, 1, 2, \ldots, n\} \tag{4.13}$$

   for some terminal integer $n \geq 0$ defined by (see Fig.4.1)

$$n = \frac{-2\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}} \tag{4.14}$$

   The following roots are also possible

$$\{\vec{\alpha} + p\vec{\beta} | i = 0, 1, 2, \ldots, p\} \tag{4.15}$$

   for some terminal integer $p \geq 0$ defined by (see Fig.4.1)

$$p = \frac{-2\vec{\alpha} \cdot \vec{\beta}}{\vec{\beta} \cdot \vec{\beta}} \tag{4.16}$$

5. angle constraint

   Multiplying Eqs.(4.14) and (4.16), we get

$$-\sqrt{\frac{np}{4}} = \hat{\alpha} \cdot \hat{\beta} \in [-1, 0] \tag{4.17}$$

   This angle constraint implies that the angle between the two SP roots $\vec{\alpha}$ and $\vec{\beta}$ can only have one of the 4 possible values listed in Table 4.1.

6. length ratio constraint

Dividing Eqs.(4.14) and (4.16), we get

$$\sqrt{\frac{n}{p}} = \frac{|\vec{\beta}|}{|\vec{\alpha}|} \tag{4.18}$$

Table 4.2 gives the possible length ratios implied by Eq.(4.18).

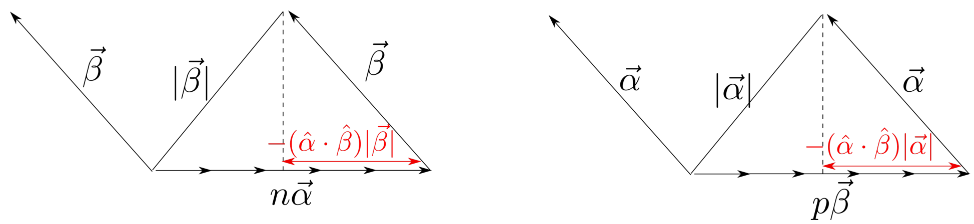


Figure 4.1: Pictorial representation of Eqs.(4.14) and (4.16).

np	$\sqrt{np/4}$	$\angle(\vec{\alpha},\vec{\beta}) = \arccos\left(-\sqrt{np/4}\right)$
0	0	$\frac{\pi}{2} = 90^o$
1	$\frac{1}{2}$	$\frac{2\pi}{3} = 120^o$
2	$\frac{1}{\sqrt{2}}$	$\frac{3\pi}{4} = 135^o$
3	$\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6} = 150^o$

Table 4.1: The 4 possible angles between two PS roots, as dictated by the angle constraint Eq.(4.17).

np	(n,p)	$\lvert\vec{\beta}\rvert/\lvert\vec{\alpha}\rvert$
0	(0, 1) or (1,0)	0 or ∞
1	(1,1)	1
2	(1,2) or (2,1)	$1/\sqrt{2}$ or $\sqrt{2}$
3	(1,3) or (3,1)	$1/\sqrt{3}$ or $\sqrt{3}$

Table 4.2: Possible length ratios between two PS roots, as dictated by the length ratio constraint Eq.(4.18).

Using the definition of SP roots, and the angle and length constraints on SP roots, one can show that all possible simple Lie algebras have one of the root systems given by Fig.4.2. In that figure, the parameters of a root system are specified via Dynkin diagrams.

Rules for drawing **Dynkin Diagrams (DD)**

1. One dot for each SP root. k subscript in A_k, B_k, C_k, D_k is number of dots
2. Dots connected by np number of lines
3. If $np > 1$, draw arrowhead (i.e., greater-than sign $>$) pointing from bigger to smaller root.
4. Draw one connected diagram (CD) for a simple Lie algebra. Draw multiple disconnected CDs for a semisimple Lie algebra.

   This follows because the Lie algebra of a semisimple Lie algebra $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \mathfrak{g}_t$ of simple Lie algebras $\mathfrak{g}_i$ and the root vectors of any two of those Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal so $np = 0$ and there is no line connecting the roots of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

4.1 Examples

- DD for $SO(3)$ and its double cover $SU(2)$ is a single dot
- $SO(4) \cong SO(3) \times SO(3)$ is not a simple Lie algebra. Its DD is two disconnected dots
- For $SU(3)$, the DD is $\bullet$ —— $\bullet$

$$H_1 = T_z, \quad H_2 = \frac{\sqrt{3}}{2} Y \tag{4.19}$$

$$E_{\vec{\alpha}} = \frac{1}{\sqrt{2}} T_+, \quad E_{\vec{\beta}} = \frac{1}{\sqrt{2}} U_+, \quad E_{\vec{\alpha}+\vec{\beta}} = \frac{1}{\sqrt{2}} V_- \tag{4.20}$$

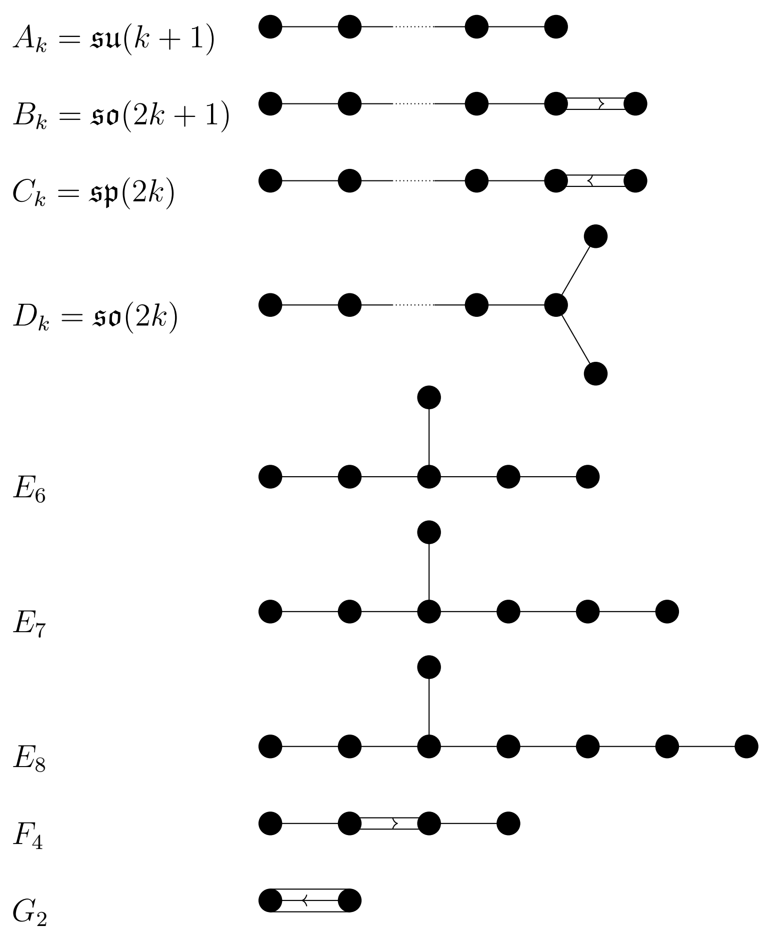


Figure 4.2: Dynkin diagrams for the simple Lie groups. n subscript in A_k, B_k, C_k, D_k is the number of dots, which equals the number of SP roots.

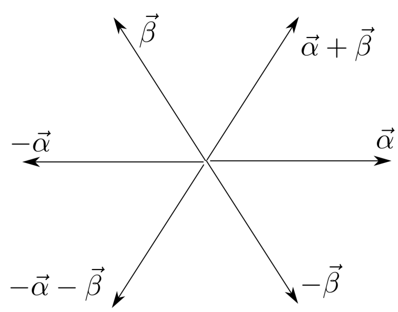


Figure 4.3: Root system for $SU(3)$

Chapter 5

General Relativity Nets: COMING SOON

T.his chapter is based on Cvitanovic's Birdtracks book Ref. [1]

Chapter 6

Integrals over a Group

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

For a group $\mathcal{G}$, let

$SR =$ set of singlet reps, and $SR^c =$ set of nonsinglet reps,

Let g be an element of $\mathcal{G}$ with a rep-matrix G. The goal of this chapter is to show how to evaluate integrals over a group $\mathcal{G}$, of the form:

$$\int dg \; G_a{}^b G_c{}^d \ldots (G^\dagger)_e{}^f (G^\dagger)_g{}^h \tag{6.1}$$

subject to the constraints that:

$$\int dg = 1 \tag{6.2}$$

and

$$\int dg \; G_\lambda = 0 \quad \text{if } \lambda \in SR^c \tag{6.3}$$

We will represent the rep-matrix G by

$$G_a{}^b = \; a \longleftarrow G \longleftarrow b \,, \quad (G^\dagger)_b{}^a = \; b \longleftarrow G^\dagger \longleftarrow a \tag{6.4}$$

Note that we will always take the out arrow (green) as the first one.

We will assume that G is a unitary matrix

$$\boxed{G^\dagger G = G G^\dagger = 1} \qquad \longleftarrow G^\dagger \longleftarrow G \longleftarrow \;=\; \longleftarrow G \longleftarrow G^\dagger \longleftarrow \;=\; \longleftarrow \bullet \longleftarrow \tag{6.5}$$

Tensor products of G's will be represented thus

$$G \otimes G \otimes G^\dagger = \begin{array}{c} \longleftarrow G \longleftarrow \\[1em] \longleftarrow G \longleftarrow \\[1em] \longleftarrow G^\dagger \longleftarrow \end{array} \tag{6.6}$$

6.1 $\int dg\ G$

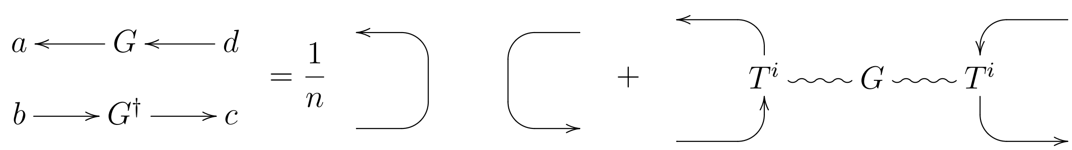
To evaluate $\int dg\ G$, we expand G in its Clebsch-Gordan series. Such series and the Clebsch-Gordan coefficients C_λ are discussed in Chapter 3.

$$\begin{aligned} \int dg\ G &= \sum_\lambda C_\lambda^\dagger \left[\int dg\ G_\lambda\right] C_\lambda && (6.7)\\ &= \sum_{\lambda \in SR} C_\lambda^\dagger C_\lambda && (6.8)\\ &= \sum_{\lambda \in SR} P_\lambda && (6.9)\end{aligned}$$

This result is valid for any group $\mathcal{G}$ and any rep-matrix G of that group.

6.2 $\int dg\ G \otimes G^\dagger$

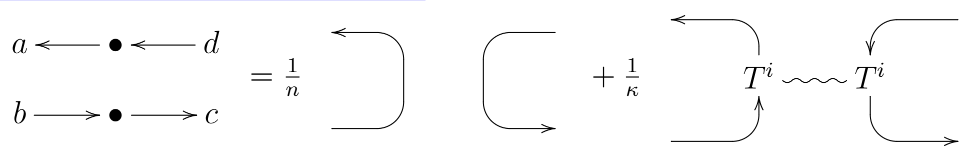
Claim 6 *For $G \in SU(n) \subset \mathbb{C}^{n\times n}$,*



(6.10)

proof: Recall that

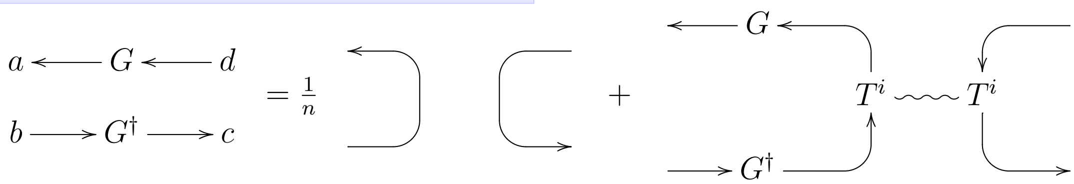
$$\boxed{\delta_a^d \delta_c^b = \frac{1}{n}\delta_a^b \delta_c^d + \frac{1}{\kappa}(T^i)_a{}^b (T^i)_c{}^d}$$



(6.11)

Will set $\kappa = 1$ from here on. Multiplying both sides from the left by $G \otimes G^\dagger$, we get

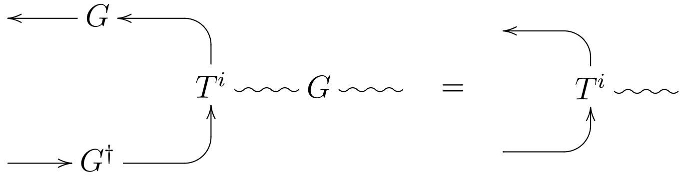
$$\boxed{G_a{}^d (G^\dagger)_c{}^b = \frac{1}{n}\delta_a^b \delta_c^d + (G^\dagger T^i G)_a{}^b (T^i)_c{}^d}$$



(6.12)

Since the generators T^i are invariant tensors,

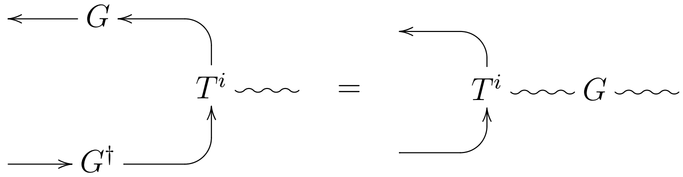
$$\boxed{G_a{}^{a'}(G^\dagger)_{b'}{}^{b}(T^{i'})_{a'}{}^{b'}G_{i'i} = (T^i)_a{}^b}$$



(6.13)

Hence,

$$\boxed{G_a{}^{a'}(G^\dagger)_{b'}{}^{b}(T^{i})_{a'}{}^{b'} = (T^i)_a{}^b G_{ii'}}$$

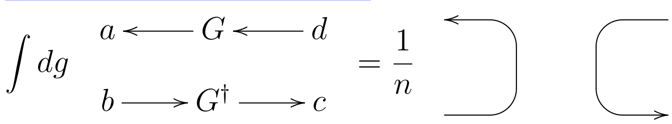


(6.14)

QED

Claim 7 *For $G \in SU(n) \subset \mathbb{C}^{n\times n}$,*

$$\boxed{\int dg\; G_a{}^{d}(G^\dagger)_{b}{}^{c} = \frac{1}{n}\delta_a^b\delta_c^d}$$



(6.15)

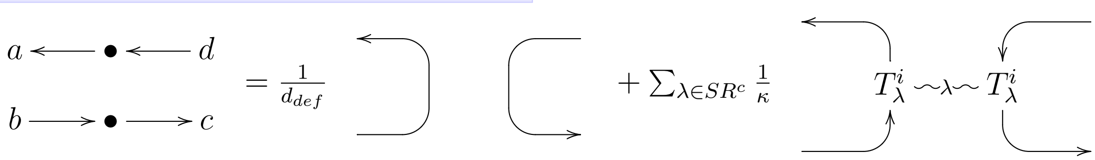
proof:

This claim follows immediately from the previous one.

QED

Claim 7 can be extended to any group $\mathcal{G}$ that has a single singlet rep. For such groups, we have, if $G \in \mathbb{C}^{d_{def}\times d_{def}}$ is the defining rep so that $a, b, c, d \in \{1, 2, \ldots d_{def}\}$,

$$\boxed{\delta_a^d\delta_c^b = \frac{1}{d_{def}}\delta_a^b\delta_c^d + \sum_{\lambda\in SR^c}\frac{1}{\kappa}(T^i_\lambda)_a{}^b(T^i_\lambda)_c{}^d}$$



(6.16)

so Eq.(6.15) is valid with n replaced by d_{def}.

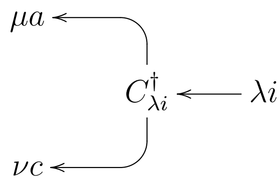
Claim 8 *For any group $\mathcal{G}$ with rep-matrices G_μ and G_ν (μ, ν are not necessarily irreps)*

$$\int dg \; (G_\mu)_{ab}(G_\nu)^{cd} = \sum_{\lambda \in SR} (P_\lambda)_{ab}{}^{cd} \tag{6.17}$$

proof:

Let

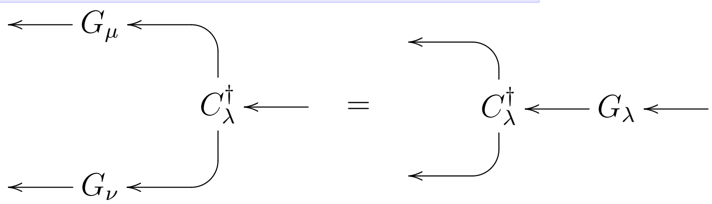
$$(C^\dagger_{\lambda i})_{ac} = \tag{6.18}$$



represent the Clebsch-Gordan coefficients for the Clebsch-Gordan series $V_\mu \otimes V_\nu = \sum_\lambda V_\lambda$.

Since the C_λ are invariant tensors:

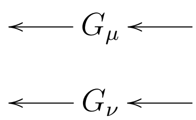
$$(G_\mu)_a{}^{a'}(G_\nu)_{b'}{}^{b}(C^\dagger_{\lambda i})_{a'}{}^{b'} = (C^\dagger_{\lambda i'})_a{}^{b}(G_\lambda)_{i'i}$$



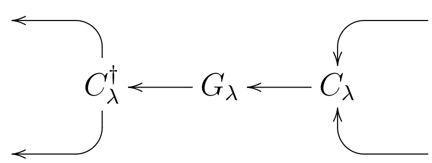
(6.19)

Therefore,

$$\int dg$$



$$= \int dg \sum_\lambda \tag{6.20}$$



$$= \sum_{ij} \sum_\lambda (C^\dagger_{\lambda i})_{ab}(C_{\lambda j})^{cd} \int dg (G_\lambda)_{ij} \tag{6.21}$$

$$= \sum_i \sum_{\lambda \in SR} (C^\dagger_{\lambda i})_{ab}(C_{\lambda i})^{cd} \tag{6.22}$$

$$= \sum_{\lambda \in SR} (P_\lambda)_{ab}{}^{cd} \tag{6.23}$$

QED

6.3 Character Orthonormality Relation

For any rep-matrix G_λ in rep λ such that G_λ represents the group element g in the Group $\mathcal{G}$, define the **character of** g **in rep** λ by

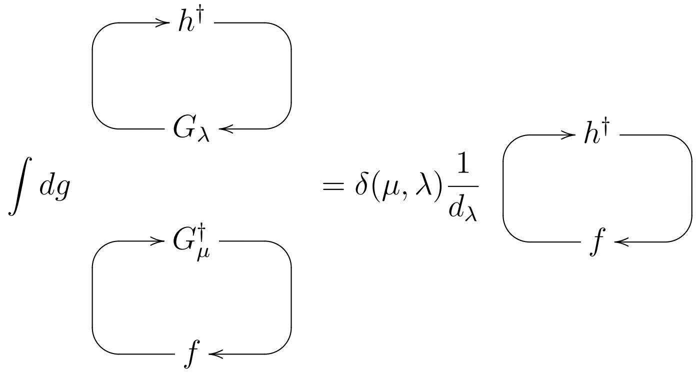
$$\chi_\lambda(g) = \chi_\lambda(G_\lambda) \stackrel{\text{def}}{=} \text{tr}G_\lambda = (G_\lambda)_a{}^a \tag{6.24}$$

Note that

$$\text{tr}G_\lambda^\dagger = (G_a{}^a)^* = \chi_\lambda(g)^* \tag{6.25}$$

Claim 9 *Suppose G_λ and G_μ are rep-matrices in irreps λ and μ, respectively. Suppose $h, G_\lambda \in \mathbb{C}^{d_\lambda \times d_\lambda}$ and $f, G_\mu \in \mathbb{C}^{d_\mu \times d_\mu}$. Then*

$$\boxed{\int dg\ \chi_\lambda(h^\dagger G_\lambda)\chi_\mu^*(G_\mu^\dagger f) = \delta(\mu,\lambda)\frac{1}{d_\mu}\chi_\lambda(h^\dagger f)}$$



(6.26)

proof:

This claim follows from Eq.(6.16 once we prove that the left hand side of Eq.(6.26) is zero if $\lambda \neq \mu$. Because λ and μ are both irreps, there can be no matrix connecting G_μ and G_λ when $\lambda \neq \mu$, so the left hand side of Eq.(6.26) is indeed zero. Even when $\lambda = \mu$, there can only be one matrix, namely a Kronecker delta, connecting G_λ and G_μ, so the group must have only one singlet rep.
QED

Note that since the matrices $h, f \in \mathbb{C}^{d_\mu \times d_\mu}$ are arbitrary, differentiation can be used to retrieve G_μ from its character with various h:

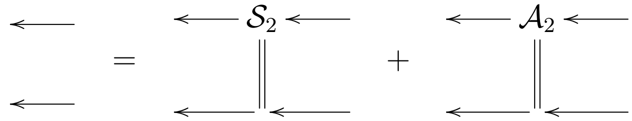
$$G_a{}^b = \frac{d}{d(h^\dagger)_b{}^a}\underbrace{\chi_\mu(h^\dagger G)}_{(h^\dagger)_b{}^a G_a{}^b} \tag{6.27}$$

6.4 $SU(n)$ Examples

In $SU(n)$, $n = d_{def}$, where d_{def} is the dimension of the defining rep. ($\mathcal{G} \subset \mathbb{C}^{n\times n}$). In this section, all matrices G are elements of $\mathbb{C}^{n\times n}$.

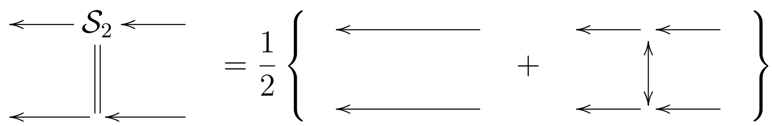
6.4.1 $\int dg\; G \otimes G$

Consider $V \otimes V$. We have



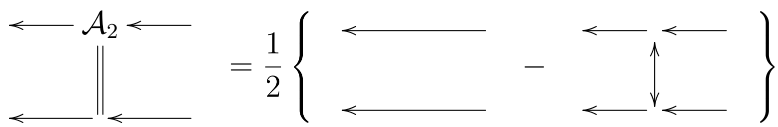
(6.28)

because



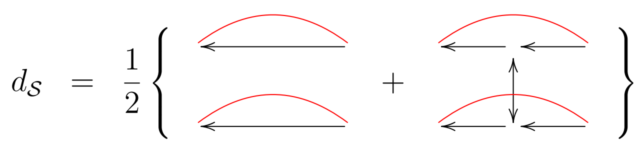
(6.29)

and



(6.30)

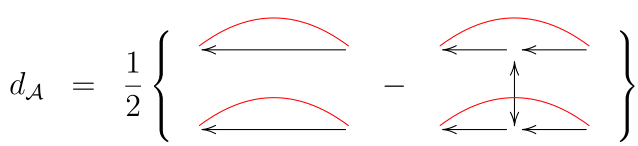
Thus



(6.31)

$$= \frac{n(n+1)}{2} \tag{6.32}$$

and



(6.33)

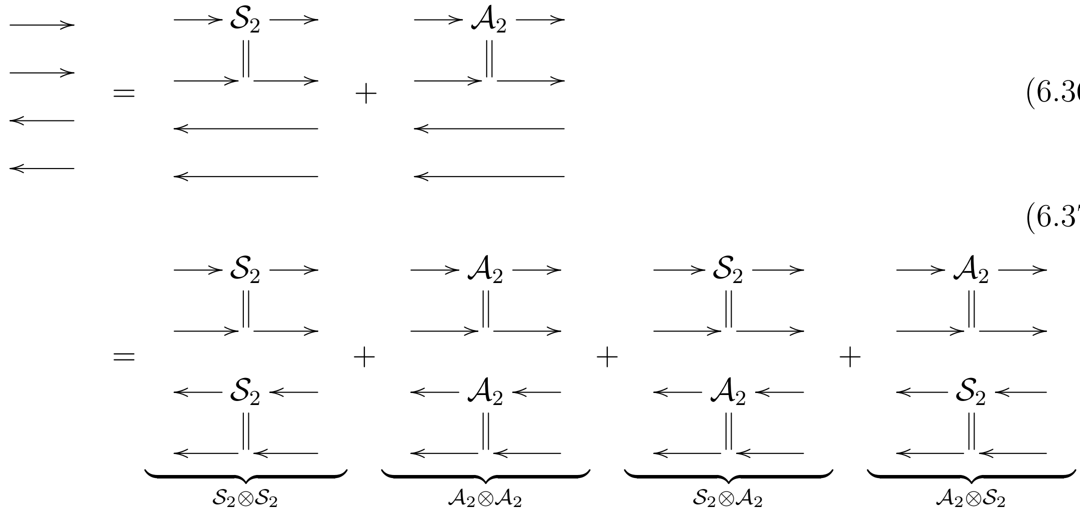
$$= \frac{n(n-1)}{2} \tag{6.34}$$

Note that $d_{\mathcal{S}} = 1$ iff $n = 1$, and $d_{\mathcal{A}} = 1$ iff $n = 2$. Therefore, for $SU(n)$

$$\int dg\; G \otimes G = 0 \quad \text{if } n > 2 \tag{6.35}$$

6.4.2 $\int dg\; G^\dagger \otimes G^\dagger \otimes G \otimes G$

Consider $V^\dagger \otimes V^\dagger \otimes V \otimes V$. We have



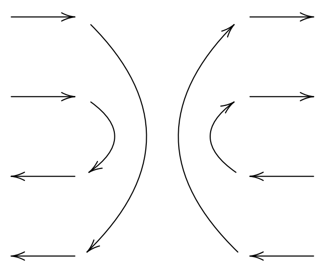
(6.36)

(6.37)

(6.38)

Let

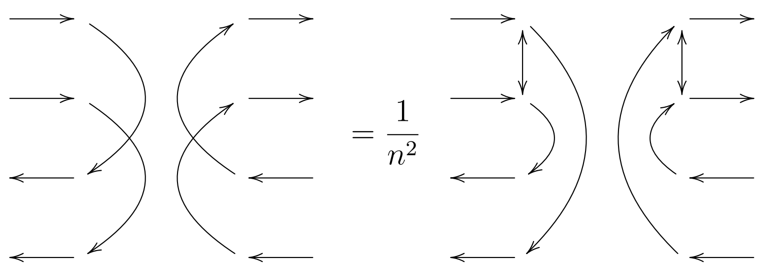
$$P_1 = \frac{1}{n^2}$$



(6.39)

and

$$P_2 = \frac{1}{n^2}$$



(6.40)

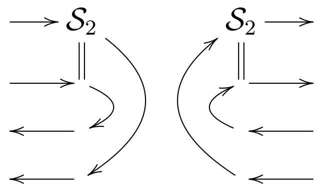
Then

$$P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1 P_2 \neq 0 \tag{6.41}$$

$$dim(P_1) = dim(P_2) = 1 \tag{6.42}$$

This hints to the possibility of two orthogonal projectors, if only we include terms where there is a single swap on either the right or the left side, but not on both sides as in Eq.(6.40). So define

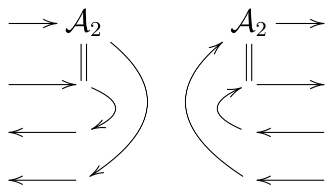
$$\pi_{\mathcal{S}} = \frac{1}{d_{\mathcal{S}}}$$



$$\text{where } d_{\mathcal{S}} = \frac{n(n+1)}{2} \tag{6.43}$$

and

$$\pi_{\mathcal{A}} = \frac{1}{d_{\mathcal{A}}}$$

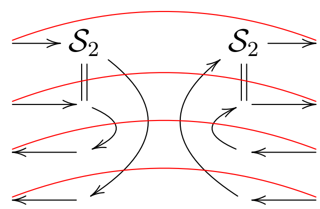


$$\text{where } d_{\mathcal{A}} = \frac{n(n-1)}{2} \tag{6.44}$$

Then

$$\pi_{\mathcal{A}}^2 = \pi_{\mathcal{A}}, \quad \pi_{\mathcal{S}}^2 = \pi_{\mathcal{S}}, \quad \pi_{\mathcal{A}}\pi_{\mathcal{S}} = 0 \tag{6.45}$$

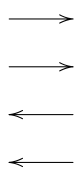
$$dim(\pi_{\mathcal{S}}) = \mathrm{tr}\pi_{\mathcal{S}} = \frac{1}{d_{\mathcal{S}}}$$



$$= 1 \tag{6.46}$$

$$dim(\pi_{\mathcal{A}}) = 1 \tag{6.47}$$

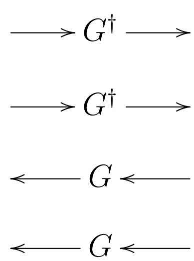
Thus



$$= \pi_{\mathcal{S}} + \pi_{\mathcal{A}} + \text{ non-singlet projectors} \tag{6.48}$$

Hence

$$\int dg$$



$$= \pi_{\mathcal{S}} + \pi_{\mathcal{A}} \tag{6.49}$$

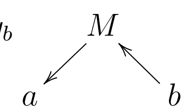
Chapter 7

Invariant Tensors

This chapter is based on Cvitanovic's Birdtracks book Ref.[1].

A **bilinear form** is a linear function $m : V^{\dagger n} \times V^n \to \mathbb{C}$ usually with $V^{\dagger n}, V^n = \mathbb{C}^n$. For example,

$$m(x^{\dagger :n}, y^{:n}) = x^{\dagger a} M_a{}^b y_b \tag{7.1}$$

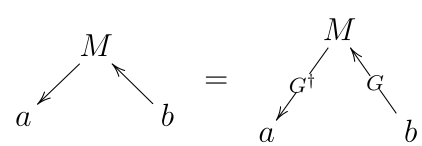


$m()$ is said to be invariant if

$$m(x^{\dagger :n}, y^{:n}) = m(x^{\dagger :n} G^\dagger, G y^{:n}) \tag{7.2}$$

$m()$ is invariant iff matrix M is an **invariant matrix**; i.e., iff

$$M_a{}^b = (G^\dagger)_a{}^{a'} G_{b'}{}^b M_{a'}{}^{b'} \tag{7.3}$$



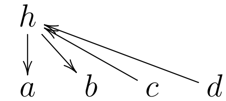
$$M = G^\dagger M G \tag{7.4}$$

If G is unitary,

$$GM = MG, \quad [G, M] = 0 \tag{7.5}$$

A **multilinear form** is a linear function $h : V^{\dagger n^p} \times V^{n^q} \to \mathbb{C}$, usually with $V^\dagger, V = \mathbb{C}$. For example,

$$h(w^\dagger, x^\dagger, y, z) = h_{ab}{}^{cd} w^{\dagger a} x^{\dagger b} y_c z_d \tag{7.6}$$

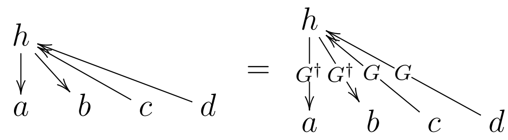


$h()$ is said to be invariant if

$$h(w^\dagger, x^\dagger, y, z) = h(w^\dagger G^\dagger, x^\dagger G^\dagger, Gy, Gz) \tag{7.7}$$

$h()$ is invariant iff tensor $h_{ab}{}^{cd}$ is a **invariant tensor** (IT); i.e., iff

$$\boxed{h_{ab}{}^{cd} = (G^{\dagger})_a{}^{a'}(G^{\dagger})_b{}^{b'}h_{a'b'}{}^{c'd'}G_{c'}{}^{c}G_{d'}{}^{d}} \tag{7.8}$$



A **composed IT** is an IT that can be written as a product or contraction of ITs.

A **tree IT** is a composed IT without any loops.

A **primitive IT** is an IT that can be expressed as a linear combination of a finite number of tree ITs.

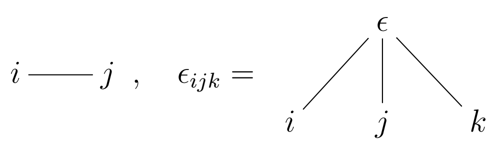
The **primitiveness assumption**: All IT are primitive.

Examples. Suppose $x, y, z \in \mathbb{R}^3$ and $i, j, k \in \{1, 2, 3\}$.

- Primitive ITs

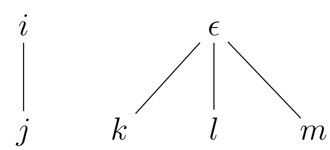
$$length(x) = \delta_{ij}x_i x_i \quad volume(x, y, z) = \epsilon_{ijk}x_i y_j z_k \tag{7.9}$$

$$\delta_{ij} = \quad , \quad \epsilon_{ijk} = \tag{7.10}$$

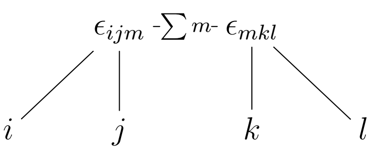


- Tree ITs

$$\delta_{ij}\epsilon_{klm} = \tag{7.11}$$

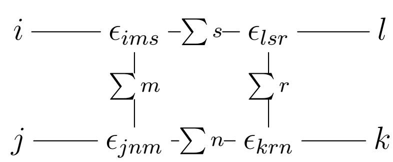


$$\epsilon_{ijm}\delta_{mn}\epsilon_{nkl} = \tag{7.12}$$

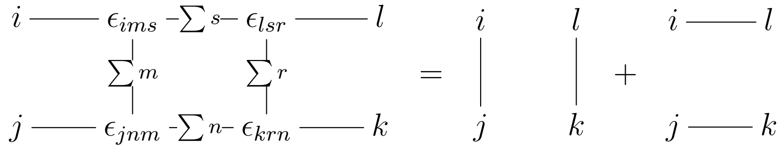


- Non-tree IT

$$\epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{lsr} = \tag{7.13}$$



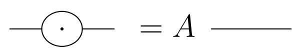
$$\epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{lsr} = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}$$



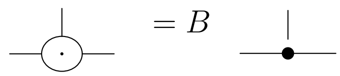
(7.14)

- Primitiveness Assumption

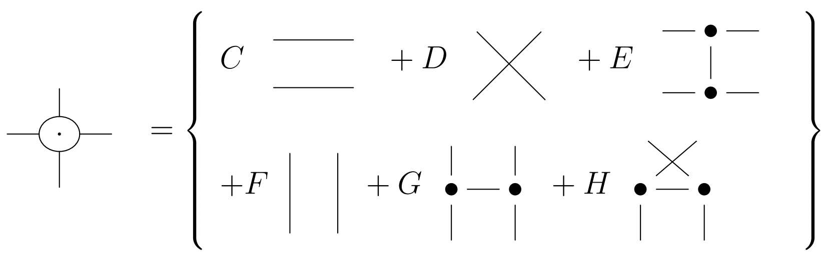
  Suppose $\mathcal{P} = \{\delta_{ij}, f_{ijk}\}$ where f_{ijk} is not ϵ_{ijk}. For some $A, B, C, \ldots H \in \mathbb{C}$, one has



(7.15)



(7.16)



(7.17)

Let $\mathcal{P} = (p_1, p_2, \ldots, p_k)$ be a **full set of primitive ITs**. By "full", we mean no others exist. $\mathcal{P}$ is a basis for an **algebra of invariants**.[1]

An **invariance group** $\mathcal{G}$ with a full set of primitive ITs $\mathcal{P} = \{p_1, p_2, \ldots, p_k\}$ is the set of all linear transformation $G \in \mathcal{G}$ such that

$$\begin{aligned}
p_1(x^\dagger, y) &= p_1(x^\dagger G^\dagger, Gy) && (7.18)\\
p_2(w^\dagger, x^\dagger, y, z) &= p_2(w^\dagger G^\dagger, x^\dagger G^\dagger, Gy, Gz) && (7.19)\\
&\text{etc.} && (7.20)
\end{aligned}$$

Example. Consider an invariance group with a single primitive IT $p()$ defined by

$$p(x^\dagger, y) = \delta^b_a {x^\dagger}^a y_b = {x^\dagger}^b y_b \tag{7.21}$$

Then

$$(x')^{\dagger a}(y')_a = {x^\dagger}^b (G^\dagger G)_b{}^c y_c = {x^\dagger}^b y_b \tag{7.22}$$

[1] An algebra over a field is defined in Sec.A.7

so G must be unitary

$$G^{\dagger}G = 1 \tag{7.23}$$

The group of n dimensional unitary matrices is called $U(n)$

Chapter 8

Lie Algebras

This chapter is based on Ref.[1].

8.1 Generators of Infinitesimal Transformations

For some group $\mathcal{G}$, assume that any group element $G \in \mathcal{G}$ that is infinitesimal close to the identity 1 can be parametrized by

$$G = 1 + i \sum_i \epsilon_i T_i \tag{8.1}$$

where $T_i \in \mathbb{C}^{n\times n}$ for $i = 1, 2, \ldots, N$, $\epsilon_i \in \mathbb{R}$ and $|\epsilon_i| << 1$ [1].

The T_i matrices are called the **generators of infinitesimal transformations** for group $\mathcal{G}$. The generators of a group $\mathcal{G}$ span a vector space called a Lie algebra $\mathfrak{g}$.[2] For example, the generators of the group SU(2) span the **Lie algebra** $\mathfrak{su}(2)$.

The tensor

$$g_{ij} = \mathrm{tr}(T_i^\dagger T_j) \tag{8.2}$$

is called the **Cartan-Killing form**. This tensor can be used to raise and lower the the adjoint rep indices i, j, k in a tensor such as M_{ijk}:

$$M^i{}_{jk} = g^{ii'} M_{i'jk} \tag{8.3}$$

Assume that the T_i matrices are Hermitian and that they satisfy

$$g_{ij} = \mathrm{tr}(T_i T_j) = \kappa\delta(i, j) \tag{8.4}$$

A Lie algebra that satisfies Eq.(8.4) is called a **simple Lie algebra**.

A **semi-simple Lie algebra** is a direct sum of simple Lie algebras.

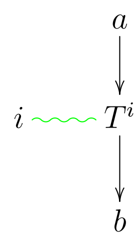
[1]Note that the ϵ_i are real, not complex.

[2]See Sec.A.7 for the definition of an algebra over a field.

It's customary to choose generators so that $\kappa = \frac{1}{2}$.[3] However, we will often set $\kappa = 1$ for intermediate calculations and restore $\kappa \neq 1$ at the end by dimensional analysis. Just remember that each T^j scales as $\sqrt{\kappa}$. For example, given the equation $\text{tr}(T^iT^j) = \delta(i,j)$, we know that when $\kappa \neq 1$, $\text{tr}(T^iT^j) = \kappa\delta(i,j)$ so both sides of the equation scale as κ.

We will use the following scaled version of T^j as a birdtrack. Define

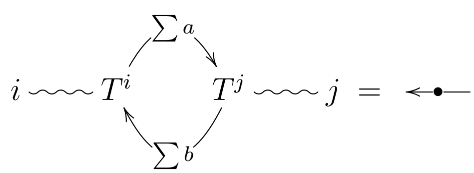
$$(C^i_{adj})_b{}^a = \frac{1}{\sqrt{\kappa}}(T^i)_b{}^a = \frac{1}{\sqrt{\kappa}} \tag{8.5}$$



In the CC convention, we will always start reading the indices of this node at the wavy undirected green leg. *adj* stands the adjoint. In this node (vertex), an adj-rep particle (wavy line, gluon) is generated (released) by a def-rep particle (straight solid line, arrow).

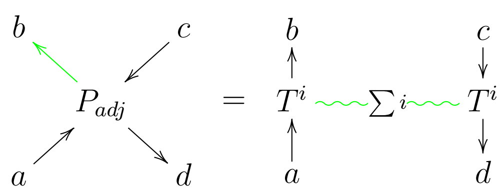
In terms of birdtracks, Eq.(8.4) becomes

$$\boxed{(T^i)^b{}_a(T^j)^a{}_b = \text{tr}(T^iT^j) = \delta(i,j)} \tag{8.6}$$



We can now define the projection operator for the adj-rep. This projection operator represent a gluon exchange between 2 def-rep particles.

$$\boxed{(P_{adj})_b{}^a{}_d{}^c = \sum_i (T^i)_b{}^a(T^i)_d{}^c} \tag{8.7}$$



The green arrow is the first index in the CC convention.

Note that if $x \in V^n \otimes V^{\dagger n}$, then

$$(P_{adj})_b{}^a{}_d{}^c x_c{}^d = \sum_i (T^i)_b{}^a \underbrace{\left[(T^i)_d{}^c x_c{}^d\right]}_{\epsilon_i \in \mathbb{R}} \tag{8.8}$$

---

[3]For $SU(2)$, it is customary to choose $T^i = \frac{1}{2}\sigma_i$, where σ_i for $i = 1,2,3$ are the Pauli matrices. For $SU(3)$, it is customary to choose $T^i = \frac{1}{2}\lambda_i$ where λ_i for $i = 1,2,\ldots,8$ are the Gell-Mann matrices. For both of these choices, $\kappa = \frac{1}{2}$.

8.2 Tensor Invariance Conditions

Recall Eq.(A.31). If $x \in V^{n^p} \otimes V^{\dagger n^q}$, and $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}; \mathbb{C})$,

$$(x')_{a^{:p}}{}^{b^{:q}} = \mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}}, \quad x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta \tag{8.9}$$

where we define

$$\mathbb{G}_\alpha{}^\beta \stackrel{\text{def}}{=} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \prod_{i=1}^{q} G^{\dagger}{}^{b_i}{}_{c_i} \tag{8.10}$$

If $\mathbb{G}$ is infinitesimally close to the identity, then we can parametrize it as

$$\mathbb{G}_\alpha{}^\beta = 1 + i \sum_j \epsilon_j (M^j)_\alpha{}^\beta \tag{8.11}$$

$$G_{a_i}{}^{d_i} = 1 + i \sum_j \epsilon_j (T^j)_{a_i}{}^{d_i} \tag{8.12}$$

$$G^{\dagger}{}^{b_i}{}_{c_i} = 1 - i \sum_j \epsilon_j (T^j)^{b_i}{}_{c_i} \tag{8.13}$$

Define

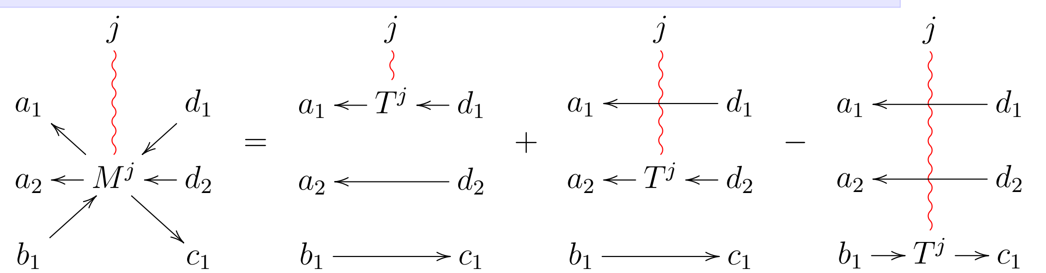
$$(M^j)_\alpha{}^\beta = \left[(T^j)_{a_i}{}^{d_i} \frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}{}_{c_i} \frac{1}{\delta_{c_i}^{b_i}} \right] \delta_{a^{:p}}^{d^{:p}} \delta_{c^{:q}}^{b^{:q}} \tag{8.14}$$

When $x'_\alpha = x_\alpha$, to first order in ϵ_i,

$$0 = (M^j)_\alpha{}^\beta x_\beta = \left[(T^j)_{a_i}{}^{d_i} \frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}{}_{c_i} \frac{1}{\delta_{c_i}^{b_i}} \right] \delta_{a^{:p}}^{d^{:p}} \delta_{c^{:q}}^{b^{:q}} x_{d^{:p}}{}^{c^{:q}} \tag{8.15}$$

For example, if we define

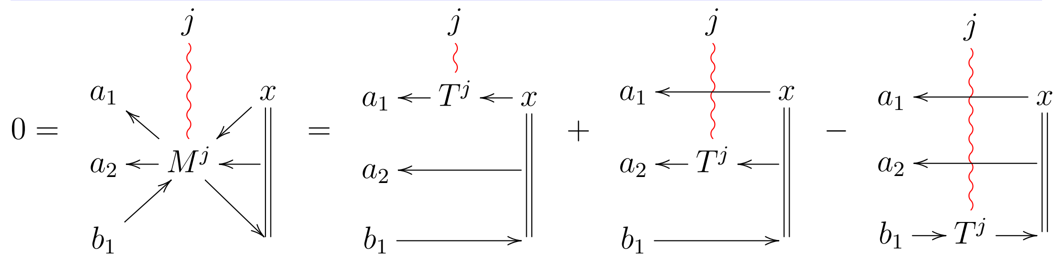
$$\boxed{(M^j)_{a_1 a_2}{}^{b_1}{}_{c_1}{}^{d_2 d_1} = (T^j)_{a_1}{}^{d_1} \delta_{a_2}^{d_2} \delta_{c_1}^{b_1} + \delta_{a_1}^{d_1} (T^j)_{a_2}{}^{d_2} \delta_{c_1}^{b_1} - \delta_{a_1}^{d_1} \delta_{a_2}^{d_2} (T^j)^{b_1}{}_{c_1}}$$



(8.16)

then

$$\boxed{0 = (M^j x)_{a_1 a_2}{}^{b_1} = \left[(T^j)_{a_1}{}^{d_1} \delta^{d_2}_{a_2} \delta^{b_1}_{c_1} + \delta^{d_1}_{a_1} (T^j)_{a_2}{}^{d_2} \delta^{b_1}_{c_1} - \delta^{d_1}_{a_1} \delta^{d_2}_{a_2} (T^j)^{b_1}{}_{c_1}\right] x_{d_1 d_2}{}^{c_1}}$$



(8.17)

We will refer to identities such as Eq.(8.16) and (8.17) as **tensor invariance conditions**.

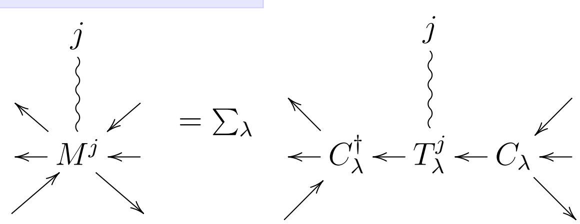
8.3 Clebsch-Gordan Coefficients

The Clebsch Gordan (CG) coefficients are introduced in Ch.3. Note that the generators $(T^i)_a{}^b$ are a simple kind of CG coefficient, one with

- a gluon (adj-rep) particle instead of a general λ rep particle emanating from the i index,
- a particle of the def-rep entering and another leaving the node, instead of any number of def-rep particles entering and leaving.

Since $\mathbb{G} = 1 + i \sum_j \epsilon_j M^j$, generators decompose in the same way as the group elements

$$\boxed{M^j = \sum_\lambda C^\dagger_\lambda T^j_\lambda C_\lambda}$$



(8.18)

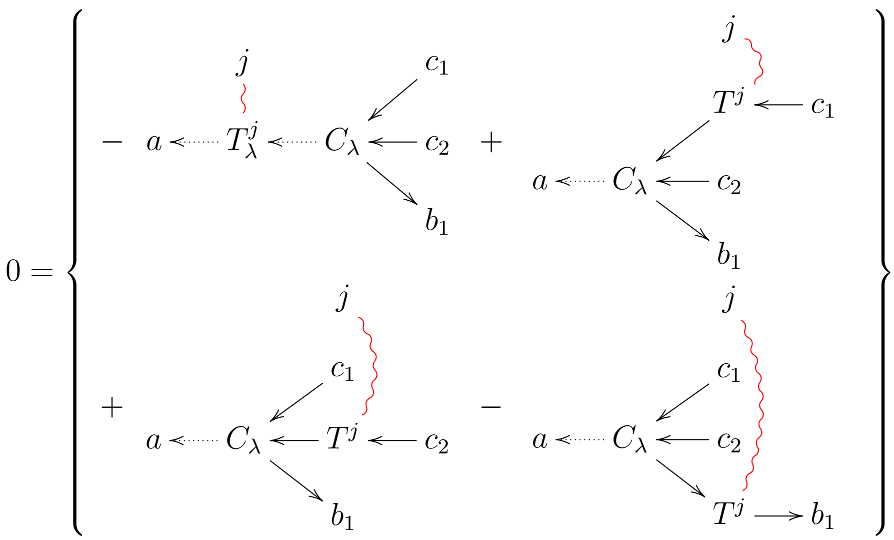
The CG coefficients are invariant tensors.

$$C_\lambda = G^\dagger_\lambda C_\lambda G \tag{8.19}$$

Hence,

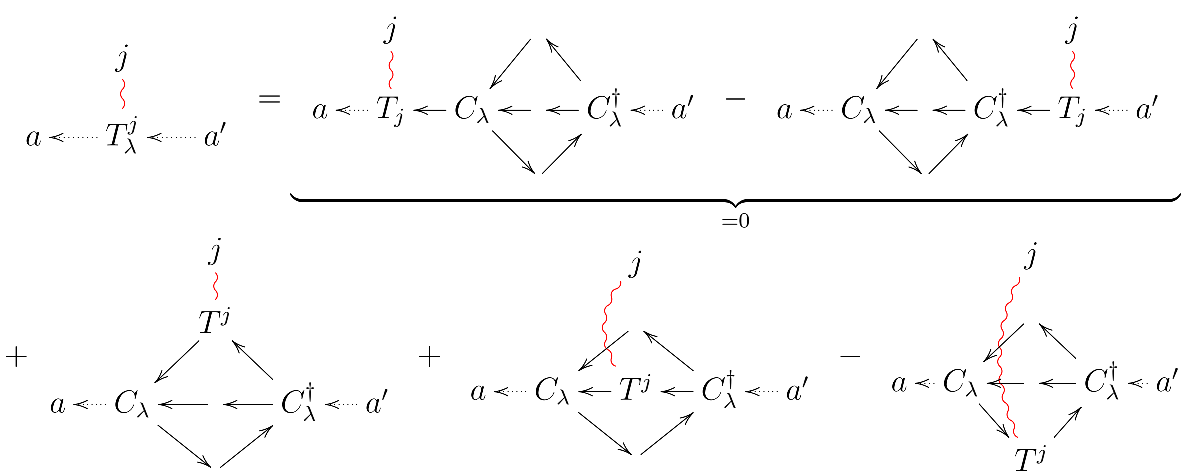
$$0 = -T^j_\lambda C_\lambda + C_\lambda T^j \tag{8.20}$$

Note that in the last equation, T^j_λ and T^j are different. In terms of birdtracks, we might have, for example,



(8.21)

Multiplying Eq.(8.21) on the left by $C^\dagger_\lambda$, and moving the first tem to the right side, we obtain an expression for the generator T^i_λ in term the generators T^j (and C_λ CG coefficients).



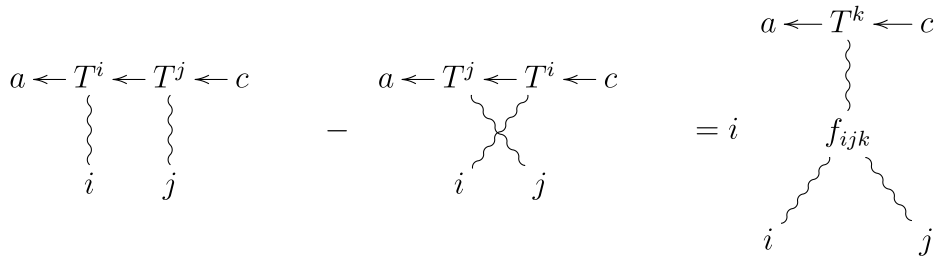
(8.22)

The term with the underbrace in Eq.(8.22) does not come from Eq.(8.21). I included it to demonstrate to the reader that Eq.(8.22) is just another tensor invariance condition that touches all the incoming and outgoing arrows.

8.4 Structure Constants (3 gluon vertex)

A **Lie Algebra** is an algebra over the field $\mathbb{C}$ such that its vector product is the matrix commutator (see Section A.7). Simply put, a Lie Algebra is a set of square Hermitian matrices $\{T^i\}_{i=1}^N$ that satisfy

$$\boxed{\underbrace{T^iT^j - T^jT^i}_{[T^i,T^j]} = if_{ijk}T^k \quad \textbf{(Lie Algebra commutation relations)}}$$



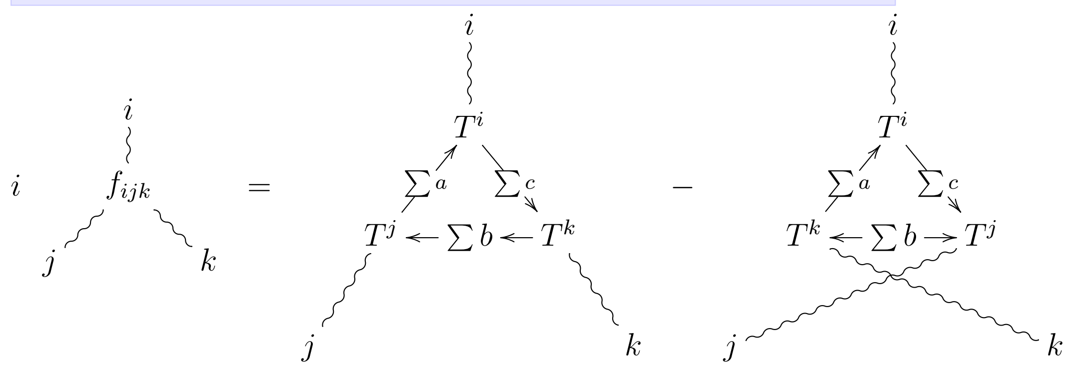
(8.23)

The f_{ijk} tensors are called the **structure constants** of the Lie Algebra. They define a 3 gluon vertex in term of the generators T^i.[4]

If $(T^j)_a{}^b$ are the rep-matrices (in the def-rep) of the generators of a group $\mathcal{G}$, then Eq.(8.23) shows that the matrices $(M^k)_{ij} = if_{ijk}$ are also a rep-matrix (in the adj-rep) of the generators of $\mathcal{G}$.

Since $\mathrm{tr}(T^kT^{k'}) = \delta(k,k')$, Eq.(8.23) implies

$$\boxed{if_{ijk} = \mathrm{tr}([T^i,T^j]T^k) = (T^i)_a{}^c(T^k)_c{}^b(T^j)_b{}^a - (T^i)_a{}^c(T^j)_c{}^b(T^k)_b{}^a}$$

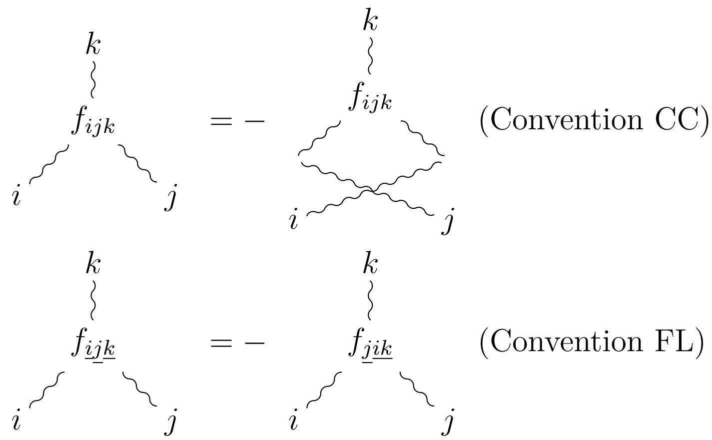


(8.24)

Note that

[4]It's possible to distinguish between upper and lower gluon indices (i.e., to give the gluon arrows a direction. In that case, the Lie Algebra commutation relations would be $[T^i,T^j] = f^{ij}{}_kT^k$ and the gluon indices could be lowered and raised using the metric (called the **Cartan-Killing form**) $g_{ij} = \mathrm{tr}((T^i)^\dagger T^j)$. But since we are assuming $g_{ij} = \kappa\delta^j_i$, there is no need to do this.

$$\boxed{f_{ijk} = -f_{jik}}$$



(8.25)

In fact, the tensor f_{ijk} is **totally antisymmetric** (i.e., it changes sign under a transposition of any two indices).

Claim 10 *f_{ijk} is a real number.*

proof:

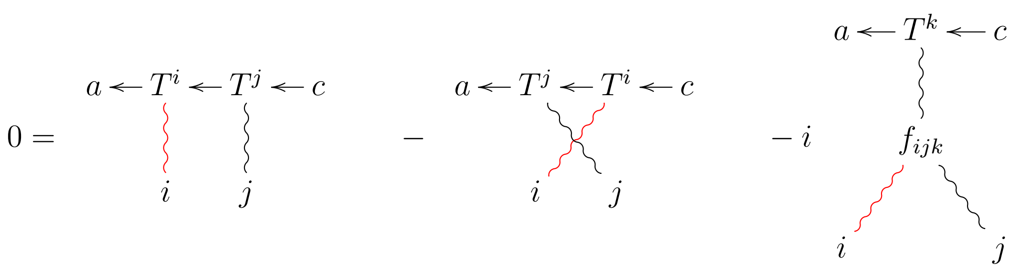
$$\left[i\mathrm{tr}([T^i, T^j]T^k)\right]^\dagger = (-i)\mathrm{tr}(T^k[T^j, T^i]) \tag{8.26}$$

$$= (-i)\mathrm{tr}([T^j, T^i]T^k) \tag{8.27}$$

$$= i\mathrm{tr}([T^i, T^j]T^k) \tag{8.28}$$

QED

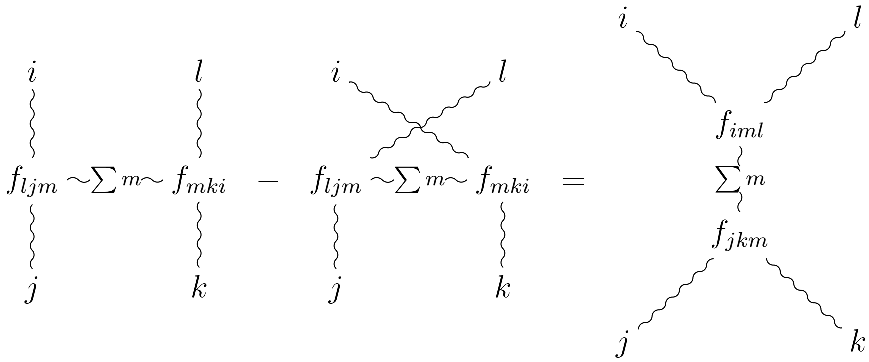
Note that the birdtrack for the Lie Algebra commutation relations Eq.(8.23) can be understood as the statement that the generators T^j are invariant matrices. Below we restate Eq.(8.23) to make that obvious



(8.29)

Claim 11

$$\boxed{f_{ijm}f_{mkl} - f_{ljm}f_{mki} = f_{iml}f_{jkm} \quad \textbf{(Jacobi identity)}}$$



(8.30)

proof:

Note that

$$\begin{aligned}
\mathrm{tr}\left([[T^i, T^j], T^k]T^l\right) &= \mathrm{tr}\left(f_{ijm}[T^m, T^k]T^l\right) &(8.31)\\
&= \mathrm{tr}\left(f_{ijm}f_{mkl'}T^{l'}T^l\right) &(8.32)\\
&= f_{ijm}f_{mkl} &(8.33)
\end{aligned}$$

so the Jacobi identity can be restated as

$$\mathrm{tr}\left(\left\{[[T^i, T^j], T^k] + [[T^j, T^k], T^i] + [[T^k, T^i], T^j]\right\} T^l\right) = 0 \tag{8.34}$$

Hence, the claim follows if we can prove that

$$\underbrace{[[T^i, T^j], T^k] + [[T^j, T^k], T^i] + [[T^k, T^i], T^j]}_{\text{cyclic permutations of } ijk} = 0 \tag{8.35}$$

If we expand the left hand side on Eq.(8.35), we find 6 terms that cancel in pairs.
QED

Note Claim 11 can be undertood as the Lie Algebra commutation relations Eq.(8.23), but stated in the adj-rep instead of the def-rep. Indeed, if

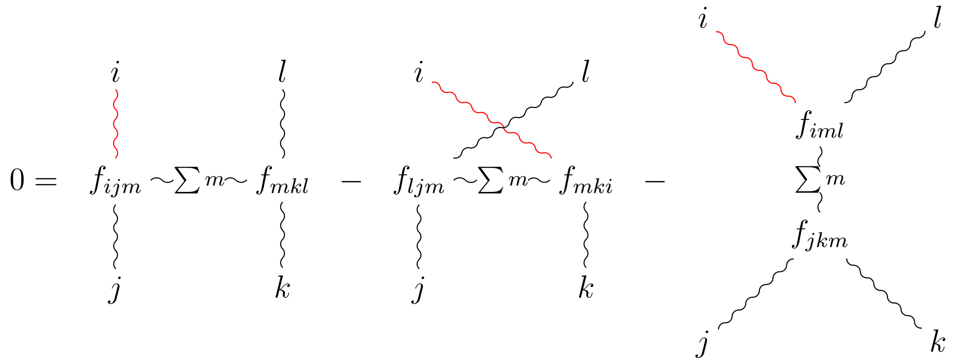
$$M^i_{jk} = if_{ijk} \tag{8.36}$$

then Claim 11 becomes

$$(M^iM^l - M^lM^i)_{jk} = if_{ilm}(M^m)_{jk} \tag{8.37}$$

Note that Claim 11 can be understood as a statement of the fact that f_{ijk} is an invariant tensor.

$$\boxed{0 = f_{ijm}f_{mkl} - f_{ljm}f_{mki} - f_{iml}f_{jkm}}$$

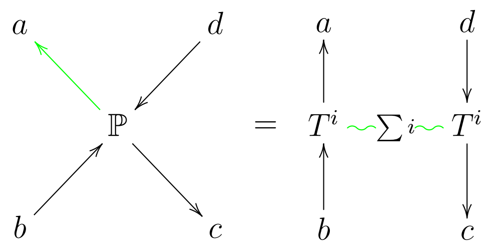


(8.38)

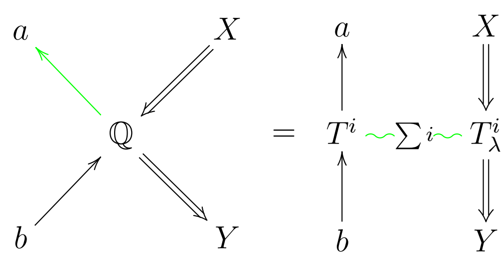
8.5 Other Forms of Lie Algebra Commutators

Consider the following two gluon exchange operators. Note that $\mathbb{P}^2 = \mathbb{P}$, but $\mathbb{Q}^2 \neq \mathbb{Q}$, so $\mathbb{P}$ is a bonafide projection operator but $\mathbb{Q}$ isn't. $\mathbb{Q}\mathbb{Q}^\dagger = \mathbb{P}$ so $\mathbb{Q}$ behaves like half of a projection operator.

$$\boxed{\mathbb{P}_a{}^b{}_c{}^d = \sum_i (T^i)_a{}^b (T^i)_c{}^d} \tag{8.39}$$



$$\boxed{\mathbb{Q}_a{}^b{}_Y{}^X = \sum_i (T^i)_a{}^b (T^i_\lambda)_Y{}^X} \tag{8.40}$$

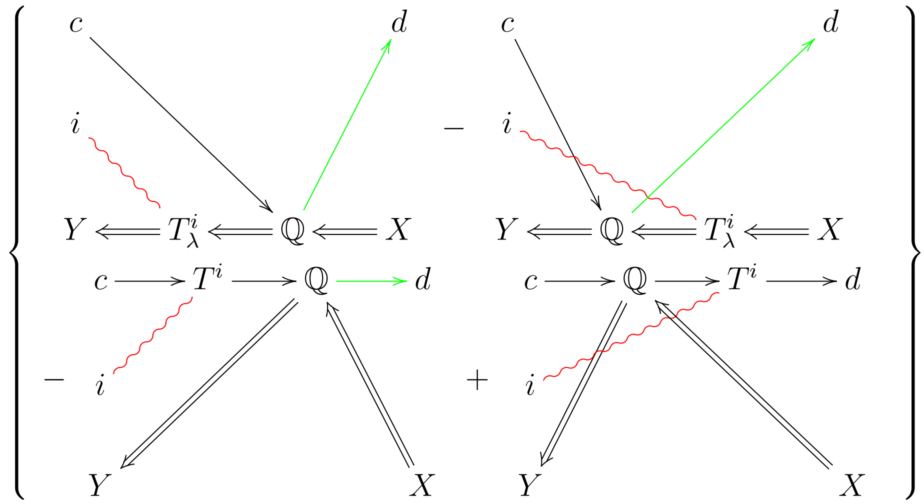


Claim 12 *If $\mathbb{Q}_b{}^a$ is the matrix with (Z, X) entries $\mathbb{Q}_b{}^a{}_Z{}^X$, then*

$$[\mathbb{Q}_b{}^a, \mathbb{Q}_d{}^c] = \mathbb{P}_{c'}{}^{ca}{}_b \mathbb{Q}_d{}^{c'} - \mathbb{Q}_{d'}{}^c \mathbb{P}_d{}^{d'a}{}_b \tag{8.41}$$

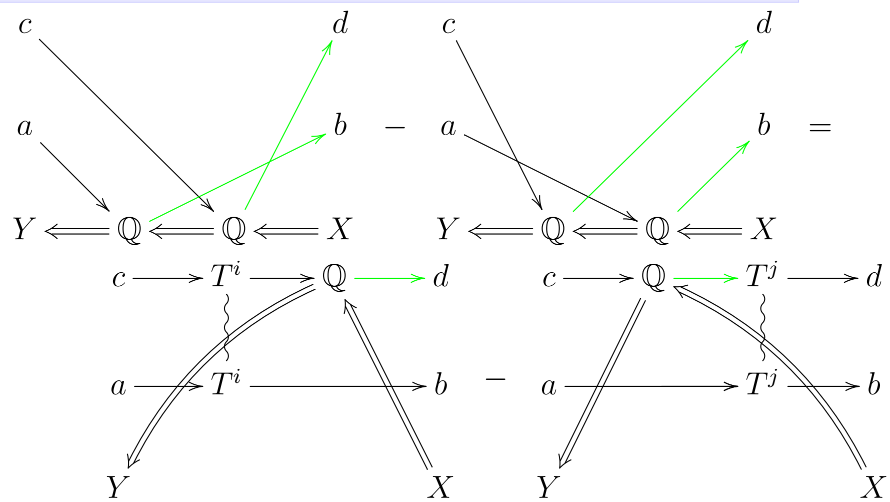
proof:

This claim can be visualized as follows. $\mathbb{Q}$ is an invariant tensor so

0 =  (8.42)

Now multiplying by $(T^i)_a{}^b$, we get

$$\boxed{\mathbb{Q}_b{}^a{}_Y{}^Z \mathbb{Q}_d{}^c{}_Z{}^X - \mathbb{Q}_d{}^c{}_Y{}^Z \mathbb{Q}_b{}^a{}_Z{}^X = \mathbb{P}_{c'}{}^{ca}{}_b \mathbb{Q}_d{}^{c'}{}_Y{}^X - \mathbb{Q}_{d'}{}^c{}_Y{}^X \mathbb{P}_d{}^{d'a}{}_b}$$

 (8.43)

Finally, if we hide the capital letter indices to obtain a statement about matrices with capital letter indices, we get

$$\mathbb{Q}_b{}^a \mathbb{Q}_d{}^c - \mathbb{Q}_d{}^c \mathbb{Q}_b{}^a = \mathbb{P}_{c'}{}^{ca}{}_b \mathbb{Q}_d{}^{c'} - \mathbb{Q}_{d'}{}^c \mathbb{P}_d{}^{d'a}{}_b \tag{8.44}$$

QED

Chapter 9

Lie Algebras of Classical Groups

In this chapter, we present an overview of the Lie algebras for the classical simple Lie groups.

Recall from Chapter 4 that the Lie algebras of the simple Lie groups can be divided into two classes: the Classical and the Exceptional. The Lie algebras of the classical simple Lie groups are

- $A_k = \mathfrak{su}(k+1)$
- $B_k = \mathfrak{so}(2k+1)$
- $C_k = \mathfrak{sp}(2k)$
- $D_k = \mathfrak{so}(2k)$

and the Lie algebras of Exceptional simple Lie groups are:

- E_6, E_7, E_8
- F_4
- G_2

9.1 $SU(n)$

$$SU(n) = \{U \in \mathbb{C}^{n\times n} : U^\dagger U = I,\ \det U = 1\} \tag{9.1}$$

Note that

$$1 = U^\dagger U = e^{X^\dagger} e^X \implies X^\dagger = -X \tag{9.2}$$

$$1 = \det U = \det e^X = e^{\mathrm{tr}X} \implies \mathrm{tr}X = 0 \tag{9.3}$$

Thus:

$$\mathfrak{su}(n) = \{X \in \mathbb{C}^{n\times n} : X^\dagger = -X, \quad \mathrm{tr}X = 0\} \tag{9.4}$$

Claim 13 *(Real dimension)*

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1 \tag{9.5}$$

proof:

Real parameter count:

- $+1$ for each diagonal entry
- $+1$ for the real part of each entry above the diagonal
- $+1$ for the imaginary part of each entry below the diagonal
- -1 for the zero trace constraint.

This adds up to $n^2 - 1$.
QED

Let X_j be the generators of the Lie algebra, and set

$$X_j = -iT_j. \tag{9.6}$$

By definition, the generators X_j are closed under commutation

$$[X_i, X_j] = f_{ijk}X_k \implies [T_i, T_j] = if_{ijk}T_k \tag{9.7}$$

where f_{ijk} are the real structure constants of $\mathfrak{su}(n)$

$$X^\dagger = -X, \quad \mathrm{tr}X = 0 \implies (T_i)^\dagger = T_i, \quad \mathrm{tr}(T_i) = 0 \tag{9.8}$$

It's also possible to assume

$$\mathrm{tr}(T_iT_j) = \frac{1}{2}\delta_{ij} \tag{9.9}$$

For $SU(2)$, $T^i = \frac{1}{2}\sigma_i$ (Pauli matrices)
For $SU(3)$, $T^i = \frac{1}{2}\lambda_i$ (Gel-Mann matrices)

9.2 $SO(n)$

$$SO(n) = \{G \in \mathbb{R}^{n\times n} : G^TG = 1, \quad \det G = 1\} \tag{9.10}$$

$$\mathfrak{so}(n) = \{X \in \mathbb{R}^{n\times n} | X^T = -X\} \tag{9.11}$$

Note that $X^T = -X$ implies that the diagonal of X is zero, so $\mathrm{tr}X = 0$ automatically.

Claim 14

$$\dim_{\mathbb{R}} \mathfrak{so}(n) = \frac{n(n-1)}{2} \tag{9.12}$$

proof:

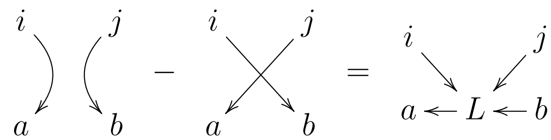
Real parameter count

- +1 for each entry above the diagonal

This adds up to $\frac{(n^2-n)}{2} = \frac{n(n-1)}{2}$.
QED

Define the generator basis $\{L_{ij}|i<j\}$, where

$$(L_{ij})_{ab} = \delta_{ia}\delta_{jb} - \delta_{ja}\delta_{ib} =$$

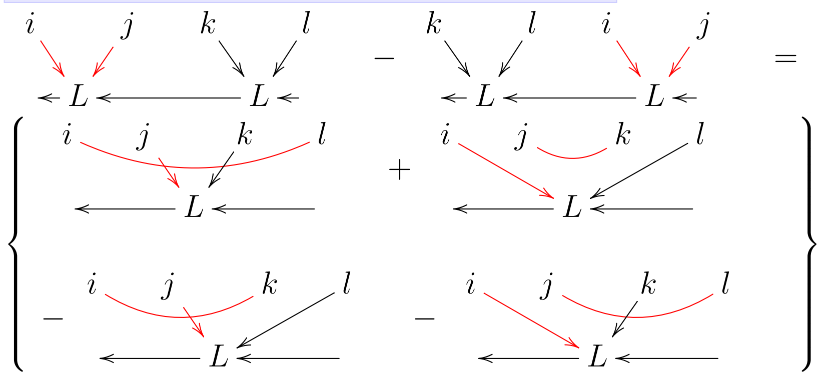


(9.13)

Note that $L_{ij}^T = L_{ji} = -L_{ij}$ and $L_{ii} = 0$. Hence these generators satisfy $X^T = -X$. Physicists often define generators $T_{ij} = iL_{ij}$ that are Hermitian.

Claim 15

$$\boxed{[L_{ij}, L_{kl}] = \delta_{jk}L_{il} - \delta_{ik}L_{jl} - \delta_{jl}L_{ik} + \delta_{il}L_{jk}}$$

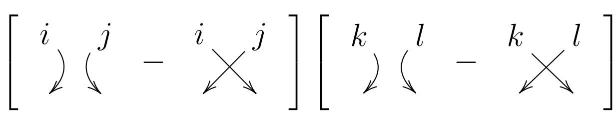


(9.14)

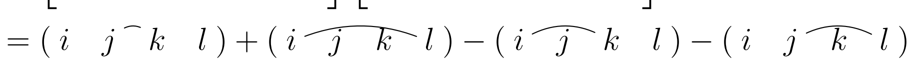
proof:

Just calculate the commutator $[L_{ij}, L_{kl}]$ using the definition Eq.(9.13) of L_{ij}

$L_{ij}L_{kl} =$

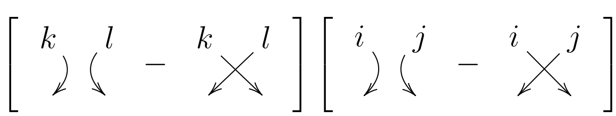


(9.15)

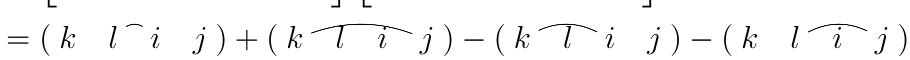


(9.16)

$L_{kl}L_{ij} =$



(9.17)



(9.18)

$$(i \quad j \frown k \quad l) - (k \quad l \quad i \quad j) = \{ i \quad j \frown k \quad l \} = \delta_{jk} L_{il} \tag{9.19}$$

$$[L_{ij}, L_{kl}] = \{ i \quad j \frown k \quad l \} + \{ i \quad j \quad k \quad l \} - \{ i \quad j \quad k \quad l \} - \{ i \quad j \quad k \quad l \} \tag{9.20}$$

QED

9.3 $Sp(n)$

Assume n is even. Then

$$Sp(n) = \{U \in \mathbb{C}^{n\times n} : U^\dagger U = I, \quad U^T J U = J\} \tag{9.21}$$

where

$$J = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \tag{9.22}$$

Note that J satisfies

$$J^T = -J, \quad J^T J = 1 \tag{9.23}$$

Note that

$$1 = U^\dagger U = e^{X^\dagger} e^X \implies X^\dagger = -X. \tag{9.24}$$

$$U^T J U = J \implies 1 = J^T U^T J U = e^{J^T X^T J} e^X \implies J^T X^T J = -X \tag{9.25}$$

Hence

$$\mathfrak{sp}(n) = \{X \in \mathbb{C}^{n\times n} : X^\dagger = -X, \quad J^T X^T J = -X\} \tag{9.26}$$

Claim 16 $X^\dagger = -X$ *and* $J^T X^T J = -X$ *iff* X *has the* **A-B form**

$$X = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \tag{9.27}$$

where

$$A^\dagger = -A, \quad B^T = B \tag{9.28}$$

proof:

Let

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{9.29}$$

Then

$$\begin{aligned} J^T X^T J &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (9.30) \\ &= \begin{pmatrix} -B^T & -D^T \\ A^T & C^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (9.31) \\ &= \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} & (9.32) \end{aligned}$$

$$J^T X^T J + X = \begin{pmatrix} D^T + A & -B^T + B \\ -C^T + C & A^T + D \end{pmatrix} = 0 \tag{9.33}$$

Hence

$$\boxed{B^T = B, \quad C^T = C, \quad A^T = -D} \tag{9.34}$$

Also

$$X^\dagger + X = \begin{pmatrix} A^\dagger + A & C^\dagger + B \\ B^\dagger + C & D^\dagger + D \end{pmatrix} = 0 \tag{9.35}$$

Hence

$$\boxed{A^\dagger = -A, \quad D^\dagger = -D, \quad C^\dagger = -B} \tag{9.36}$$

Combining Eqs.(9.34) and (9.36), we get

$$C = -B^*, \quad D = A^* \tag{9.37}$$

QED

Let

$$X = X_A + \epsilon_B \tag{9.38}$$

where

$$X_A = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}, \qquad X_A^\dagger = -X_A \tag{9.39}$$

and

$$\epsilon_B = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \quad \epsilon_B^\dagger = -\epsilon_B \tag{9.40}$$

Note that

$$J^T \epsilon_B^\dagger J = -\epsilon_B, \quad J^T X_A^\dagger J = -X_A \tag{9.41}$$

Claim 17 *(Lie Algebra is closed)*

$$[X_1, X_2] \in \mathfrak{sp}(n), \tag{9.42}$$

proof:

If $X, Y \in \mathfrak{sp}(n)$,

$$[X,Y]^\dagger = [Y^\dagger, X^\dagger] = [-Y,-X] = -[X,Y] \tag{9.43}$$

$$\begin{aligned} X_A \epsilon_B &= \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} && (9.44) \\ &= \begin{pmatrix} 0 & AB \\ -(AB)^* & 0 \end{pmatrix} && (9.45) \\ &= \epsilon_{AB} && (9.46) \end{aligned}$$

$$\begin{aligned} \epsilon_B X_A &= \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} && (9.47) \\ &= \begin{pmatrix} 0 & BA^* \\ -B^* A & 0 \end{pmatrix} && (9.48) \\ &= \epsilon_{BA^*} && (9.49) \end{aligned}$$

$$X_{A_1} X_{A_2} = X_{A_1 A_2} \tag{9.50}$$

$$\begin{aligned} \epsilon_{B_1} \epsilon_{B_2} &= \begin{pmatrix} 0 & B_1 \\ -B_1^* & 0 \end{pmatrix} \begin{pmatrix} 0 & B_2 \\ -B_2^* & 0 \end{pmatrix} && (9.51) \\ &= \begin{pmatrix} -B_1 B_2^* & 0 \\ 0 & -B_1^* B_2 \end{pmatrix} && (9.52) \\ &= X_{-B_1 B_2^*} && (9.53) \end{aligned}$$

$$[X_A, \epsilon_B] = \epsilon_{AB - BA^*} \tag{9.54}$$

$$[X_{A_1}, X_{A_2}] = X_{[A_1, A_2]} \tag{9.55}$$

$$[\epsilon_{B_1}, \epsilon_{B_2}] = X_{-B_1 B_2^* - B_2 B_1^*} \tag{9.56}$$

QED

Claim 18 *(Real dimension)*

$$\dim_{\mathbb{R}} \mathfrak{sp}(n) = n(2n+1) \tag{9.57}$$

proof:

$A^\dagger = -A$ so A contributes n^2 real parameters.
$B^T = B$ so B contributes

$$\frac{(n^2-n)}{2} + n = \frac{(n^2+n)}{2} = \frac{n(n+1)}{2} \tag{9.58}$$

complex parameters.

This adds up to

$$n^2 + n(n+1) = n(2n+1) \tag{9.59}$$

real parameters.
QED

Chapter 10

Orthogonal Groups

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

$O(n)$ and $SO(n)$ are defined as the following groups of real matrices:

$$O(n) = \{G \in \mathbb{R}^{n\times n} : G^T G = 1\} \tag{10.1}$$

$$SO(n) = \{G \in O(n) : \det G = 1\} \tag{10.2}$$

$O(n)$ contains orthogonal matrices G ($G^T G = GG^T = 1$) with $\det G \in Z_2 = \{-1, 1\}$. $SO(n)$ only contains those with $\det G = 1$. $O(n) \cong Z_2 \times SO(n)$ where $Z_2 = \{-1, 1\}$ corresponds to the sign of $\det G$. Hence, $O(n)$ is a **double cover** of $SO(n)$. $O(n)$ consists of 2 **connected components** (CC), whereas $SO(n)$ has only one CC.

An example of a $G \in O(n)$ that is not in $SO(n)$ is a reflection $Gx = -x$ for odd n. Another example for $O(2)$ is

$$\text{rotation:} \quad G = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \det G = 1 \tag{10.3}$$

$$\text{improper rotation:} \quad G = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad \det G = -1 \tag{10.4}$$

The irreps of $SO(n)$ depend on whether n is even or odd, and whether the rep is spinor or non-spinor. This chapter deals with **non-spinor reps** (with either even or odd n.). Chapter 16 deals with the **spinor reps**.

We assume that the **metric tensor** $g_{\mu\nu}$ is a primitive invariant that satisfies:

$$g_{\mu\nu} = g_{\nu\mu} = [g]_{\mu\nu}, \quad g^{\mu\nu} = g^{\nu\mu} = [g]_{\mu\nu}, \quad g_\mu{}^\nu = g_\nu{}^\mu = \delta_\mu^\nu \tag{10.5}$$

$$g_{\nu\mu} x^\mu = x_\nu, \quad g^{\rho\nu} x_\nu = x^\rho \quad (\text{so } g_{\nu\mu} g^{\rho\nu} = \delta_\mu^\rho) \tag{10.6}$$

where $\mu, \nu, \rho \in \{1, 2, \ldots, n\}$ and x_μ is any tensor.

In this section, we will call **orthogonal groups** the group of matrices under which the following symmetric quadratic form is invariant

$$h(x) = g_{\mu\nu} x^\mu x^\nu \tag{10.7}$$

where $\mu, \nu \in \{1, \dots, n\}$. Thus

$$h(Gx) = h(x) \tag{10.8}$$

$$g_{\mu\nu} G^\mu{}_\alpha G^\nu{}_\beta x^\alpha x^\beta = g_{\alpha\beta} x^\alpha x^\beta \implies g_{\mu\nu} G^\mu{}_\alpha G^\nu{}_\beta = g_{\alpha\beta} \implies G^T g G = g \tag{10.9}$$

This condition guarantees that $G \in O(n)$ is orthogonal for $g_{\mu\nu} = \delta^\nu_\mu$ but not that $\det G = 1$. When $g_{\mu\nu}$ is not the Kronecker delta function, we get a different group. For example, if $g_{\mu\nu} = diag(1, -1, -1, -1)$ and $\det G = 1$, we get the **Lorentz group** $SO(1, 3)$ used in Special Relativity.

In this chapter (and in this book), we will point the arrows in a birdtrack so that the birdtrack is a DAG. Cycles that make the birdtrack not acyclic will have a segment in red. Without that red segment, the birdtrack becomes acyclic. The reason we follow this arrow convention is that it promotes acyclic birdtracks which are more akin to bnets. We will eschew undirected birdtracks for the same reason: bnets are directed.

Let

$$g_\mu{}^\nu = \delta^\nu_\mu, \qquad \longleftarrow g \longleftarrow \;=\; \longleftarrow \tag{10.10}$$

$$g^\mu{}_\nu = \delta^\mu_\nu, \qquad \longrightarrow g \longrightarrow \;=\; \longrightarrow \tag{10.11}$$

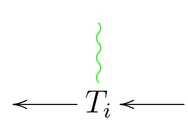
$$g_{\mu\sigma} g^{\sigma\nu} = \delta^\nu_\mu \qquad \longleftarrow \underline{g} \longrightarrow \overline{g} \longleftarrow \;=\; \longleftarrow \tag{10.12}$$

Note that we used

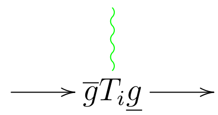
$$\underline{g} = [g_{\mu\nu}], \quad \overline{g} = [g^{\mu\nu}] \tag{10.13}$$

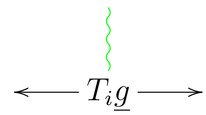
We could write Eq.(10.12) without the overline and underline on g. Those g-decorations are redundant as omitting them would not introduce any ambiguity. However, we will use them because they make spotting errors in the arrow directions easier.

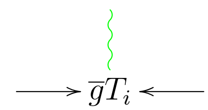
The generators of orthogonal groups will be represented by:

$(T_i)_\mu{}^\nu =$  (10.14)

We will also use

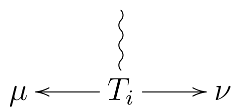
$(T_i)^{\mu}{}_{\nu} =$ 

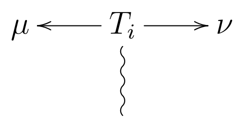
$(T_i)_{\mu\nu} =$ 

$(T_i)^{\mu\nu} =$ 

(10.15)

For $G \in O(n)$, $G^T G = 1$ with $G = e^{iT_i\epsilon_i}$ where $\epsilon_i \in \mathbb{R}$. Hence, the generators T_i must be anti-symmetric ($T_i^T = -T_i$).

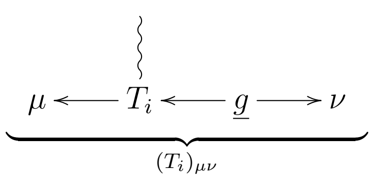
$\boxed{(T_i)_{\mu\nu} = -(T_i)_{\nu\mu}}$ 

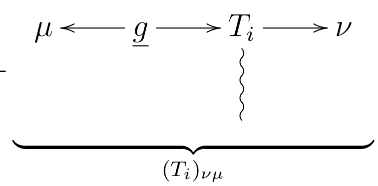
$= -$ 

(10.16)

$g_\mu^\nu = \delta_\mu^\nu$ is obviously an invariant matrix. $g_{\mu\nu}$ must be invariant too, so

$$\boxed{\underbrace{(T_i)_{\mu}{}^{\sigma} g_{\sigma\nu} + (T_i)_{\nu}{}^{\sigma} g_{\mu\sigma} = 0}_{(T_i)_{\mu\nu} = -(T_i)_{\nu\mu}}}$$

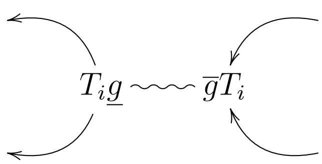


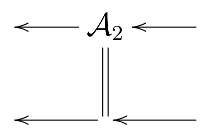
$+$ 

$= 0$ (10.17)

Hence, the invariance condition Eq.(10.17) reduces to to the statement that $(T_i)_{\mu\nu}$ is antisymmetric.

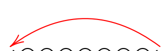
The anti-symmetrizer $\mathcal{A}_2$ is an invariant tensor (see Section 20.3). Other projectors of the $V \otimes V$ are not invariant tensors. Therefore, we must have



$=$ 

(10.18)

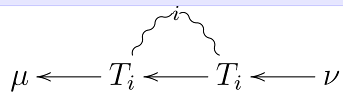
For $SO(n)$ and $O(n)$, the dimension N of the adjoint rep (= number of generators) is

$$N = \frac{n(n-1)}{2} =$$  (10.19)

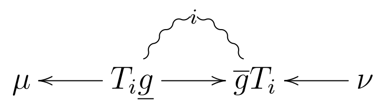
If you take an $n \times n$ matrix and remove its diagonal, this N is the number of entries in the upper (or lower) triangular sector of the matrix. Recall that for $U(n)$, $N = n^2$, and for $SU(n)$, $N = n^2 - 1$. So for $U(n)$ (or $SU(n)$), there is a generator for each entry (or each entry minus one) of an $n \times n$ matrix.

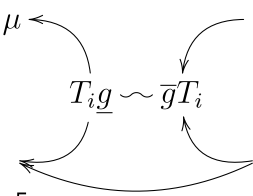
Claim 19

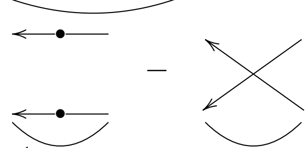
$$\boxed{\Gamma_{fun}\delta_\mu^\nu = \sum_i (T_iT_i)_\mu{}^\nu = \frac{n-1}{2}\delta_\mu^\nu} \tag{10.20}$$

$\sum_i$  $= \left(\frac{n-1}{2}\right)\ \mu \leftarrow\!\bullet\!- \nu$

proof:

$(T_iT_i)_\mu{}^\nu \quad =$  (10.21)

$=$  (10.22)

$= \frac{1}{2}\Big[$  $\Big]$ (10.23)

$$= \left(\frac{n-1}{2}\right)\ \mu \leftarrow\!\bullet\!- \nu \tag{10.24}$$

QED

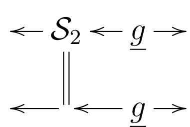
10.1 $V_{def} \otimes V_{def}$ Decomposition

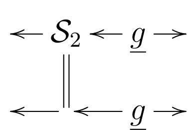
Let

$V_{def} = V =$ vector space in defining representation $\{|\mu\rangle\}_{\mu=1}^n$.

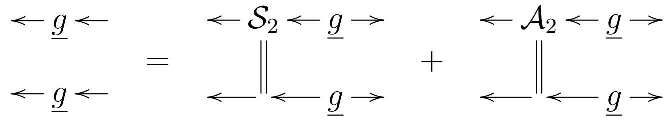
Note that the symmetrizer $\mathcal{S}_2$ originaly has two upper anf two lower indices. Its two upper indices can be lowered using the metric tensor:

$$\begin{aligned}(\mathcal{S}_2)_{\mu\nu,\rho\sigma} &= g_{\rho\rho'}g_{\sigma\sigma'}(\mathcal{S}_2)_{\mu\nu}{}^{\rho'\sigma'} && (10.25)\\ &= \frac{1}{2}\left(g_{\mu\nu}g_{\nu\rho} + g_{\mu,\rho}g_{\mu\sigma}\right) && (10.26)\end{aligned}$$

$=$  (10.27)

$=$  (10.28)

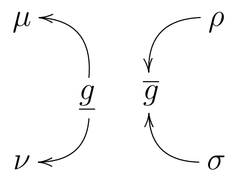
Likewise



$$(10.29)$$

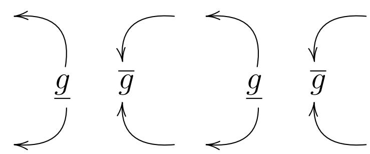
Define tensor M by

$$M_{\mu\nu}{}^{\rho\sigma} = g_{\mu\nu}g^{\rho\sigma} = \qquad (10.30)$$



Note that

$$M^2 = \qquad = nM \qquad (10.31)$$



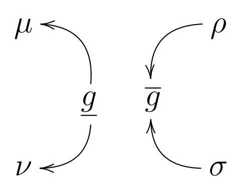
Hence, $(M-n)M = 0$ so M has two eigenvalues $\lambda = 0, n$.

Next we will use the following equation from Chapter 15 [1] to obtain a projection (PO) operator for each eigenvalue

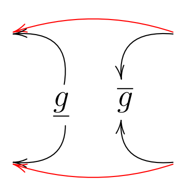
$$P_i = \sum_{j\neq i} \frac{M-\lambda_j}{\lambda_i - \lambda_j} \qquad (10.32)$$

1. Singlet PO

$$(P_S)_{\mu\nu}{}^{\rho\sigma} = \frac{1}{n} \qquad (10.33)$$



$$dim(P_S) = \frac{1}{n} \qquad (10.34)$$



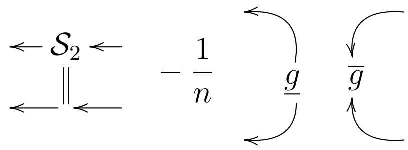
$$= 1 \qquad (10.35)$$

2. Traceless Symmetric PO[2]

[1] Note that this equation projects to zero all eigenvalues except one.

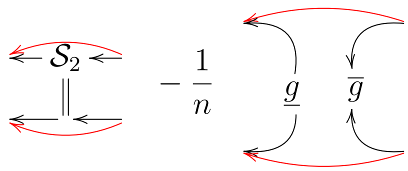
[2] Traceless here refers to $P_a{}^a{}_c{}^d V_d{}^c = (PV)_a{}^a = 0$ for any vector $V_d{}^c$. It does not refer to $P_a{}^b{}_b{}^a = 0$

$$(P_{TS})_{\mu\nu}{}^{\rho\sigma} =$$



(10.36)

$$dim(P_{TS}) =$$



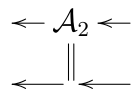
(10.37)

$$= \frac{1}{2}(n^2+n) - 1 \tag{10.38}$$

$$= \frac{1}{2}(n+2)(n-1) \tag{10.39}$$

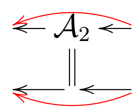
3. Anti-symmetric PO

$$(P_A)_{\mu\nu}{}^{\rho\sigma} =$$



(10.40)

$$dim(P_A) =$$



(10.41)

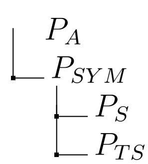
$$= \frac{1}{2}(n^2-n) \tag{10.42}$$

$$= \frac{1}{2}n(n-1) \tag{10.43}$$

Claim 20 *The Clebsch-Gordan series for $V \otimes V$ (i.e., decomposition of $V \otimes V$) is*

$$\begin{array}{ccccccc} \overbrace{V \otimes V}^{\mathcal{V}} & = & P_S\mathcal{V} & \oplus & P_{TS}\mathcal{V} & \oplus & P_A\mathcal{V} \\ \square \otimes \square & = & \bullet & \oplus & \square\square & \oplus & \begin{array}{c}\square\\ \square\end{array} \\ n^2 & = & 1 & + & \frac{1}{2}(n+2)(n-1) & + & \frac{1}{2}n(n-1) \end{array} \tag{10.44}$$

The projection operator tree is



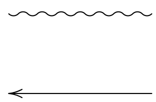
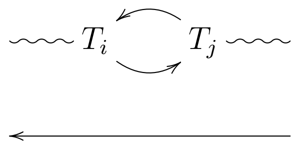
where $P_{SYM} = \mathcal{S}_2$.

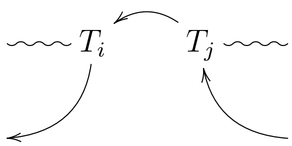
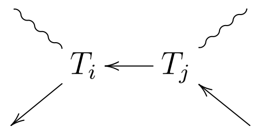
proof:
QED

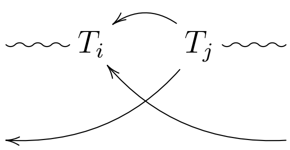
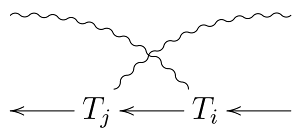
10.2 $V_{adj} \otimes V_{def}$ Decomposition

Let

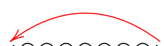
$V_{def} = V =$ vector space in defining representation $\{|\mu\rangle\}_{\mu=1}^{n}$.
$V_{adj} =$ vector space in adjoint representation $\{|i\rangle\}_{i=1}^{N}$.
$V_{adj} \otimes V \cong (V \otimes V^{\dagger}) \otimes V$

$e =$  $\cong$  (10.45)

$R =$  $=$  (10.46)

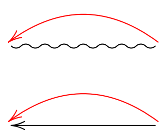
$Q =$  $=$  (10.47)

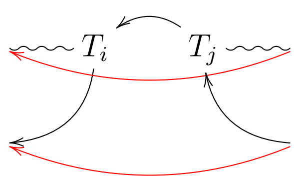
Recall that for $SO(n)$ and $O(n)$, the dimension N of the adjoint rep is

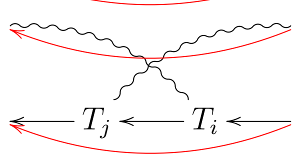
$N = \dfrac{n(n-1)}{2} =$  (10.48)

For example, for $SO(3)$, $N = 3$.

Note that

$\mathrm{tr}(e) =$  $= Nn$ (10.49)

$\mathrm{tr}(R) =$  $= N$ (10.50)

$\mathrm{tr}(Q) =$  $= N$ (10.51)

Claim 21

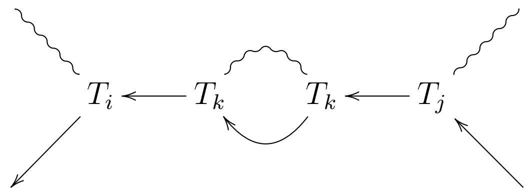
$$R^2 = \frac{n-1}{2} R \qquad (10.52)$$

$$QR = RQ = \frac{1}{2}R \tag{10.53}$$

$$Q^2 = \frac{e - Q}{2} \tag{10.54}$$

proof:

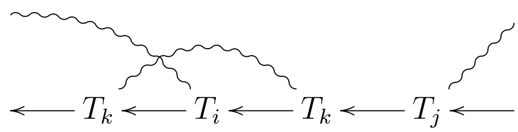
$R^2 \quad =$



(10.55)

$$= \quad \frac{n-1}{2}R \quad \text{(by Eq.(10.20))} \tag{10.56}$$

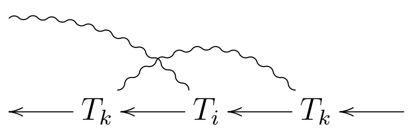
$QR \quad =$



(10.57)

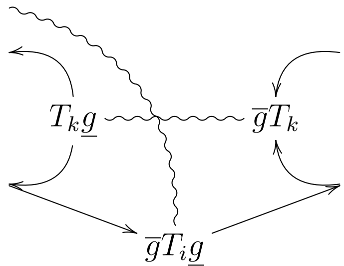
Define

$X =$



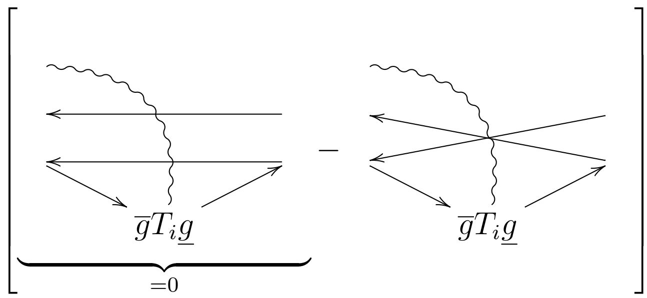
(10.58)

$X \quad =$



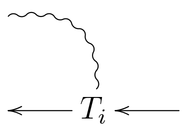
(10.59)

$= \quad \dfrac{1}{2}$



(10.60)

$= \quad \dfrac{1}{2}$

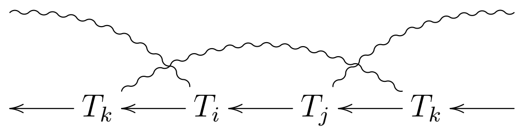


(10.61)

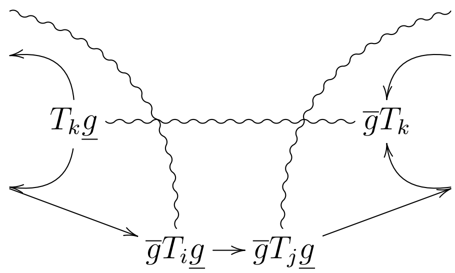
so

$$QR = RQ = \frac{1}{2}R \tag{10.62}$$

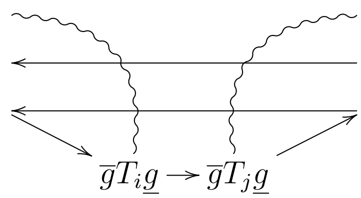
$$Q^2 = \tag{10.63}$$



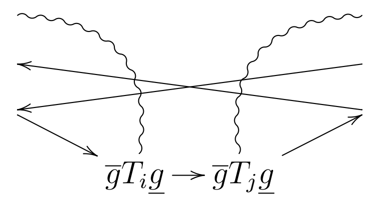
$$= \tag{10.64}$$



$$= \frac{1}{2}\Bigg[\tag{10.65}$$

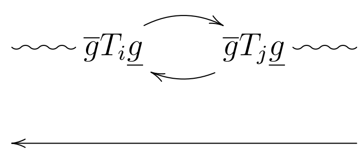


$$-$$

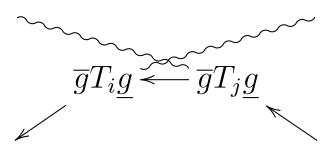


$$\Bigg]$$

$$= \frac{1}{2}\Bigg[\tag{10.66}$$



$$-$$



$$\Bigg]$$

$$= \frac{e-Q}{2} \tag{10.67}$$

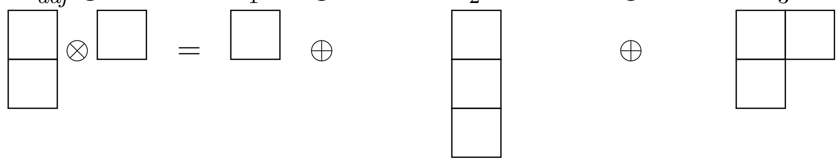
QED

Claim 22

$$\begin{aligned}
P_1 &= \frac{2}{n-1}R && (10.68)\\
P_2 &= \frac{1}{3}P_4(1-2Q) = \frac{1}{3}\left[e-2Q\right] && (10.69)\\
P_3 &= \frac{2}{3}P_4(1+Q) = \frac{2}{3}\left[e+Q-\frac{3}{n-1}R\right] && (10.70)\\
P_4 &= e-P_1 && (10.71)
\end{aligned}$$

are projectors for $O(n)$ and $SO(n)$. The $V_{adj} \otimes V = \sum_\lambda V_\lambda$ Clebsch-Gordan series is given by

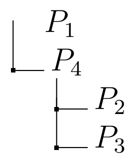
$$\overbrace{V_{adj} \otimes V}^{\mathcal{V}} = P_1\mathcal{V} \oplus P_2\mathcal{V} \oplus P_3\mathcal{V}$$



(10.72)

$$\begin{array}{rcccccc}
\frac{1}{2}n^2(n-1) & = & n & + & \frac{1}{6}n(n-1)(n-2) & + & \frac{1}{3}n(n+2)(n-2) \\
SO(3):9 & = & 3 & + & 1 & + & 5 \\
SO(4):24 & = & 4 & + & 4 & + & 16
\end{array}$$

The projection operator tree is



proof:

$$\mathrm{tr}(P_1) = \frac{2}{n-1}N \qquad (10.73)$$
$$= \frac{1}{n-1}n(n-1) \qquad (10.74)$$
$$= n \qquad (10.75)$$

$$\mathrm{tr}(P_2) = \frac{N}{3}(n-2) \qquad (10.76)$$
$$= \frac{n(n-1)}{6}(n-2) \qquad (10.77)$$

$$\mathrm{tr}(P_3) = \frac{2N}{3}\left(n+1-\frac{3}{n-1}\right) \qquad (10.78)$$
$$= \frac{n}{3}\left(n^2-4\right) \qquad (10.79)$$

From $R^2 = \frac{2}{n-1}R$,

$$P_1 = \frac{2}{n-1}R \qquad (10.80)$$

Define

$$P_4 = e - P_1 \tag{10.81}$$

From $Q^2 = \frac{1}{2}(1-Q)$, we get

$$2Q^2 + Q - 1 = (2Q-1)(Q+1) = 0 \tag{10.82}$$

Let

$$P_2 = \frac{1}{3}P_4(1-2Q) \quad P_3 = \frac{2}{3}P_4(1+Q) \tag{10.83}$$

and

$$a = \frac{2}{n-1} \tag{10.84}$$

Then

$$\begin{aligned}
P_3 &= \frac{2}{3}P_4(1+Q) && (10.85)\\
&= \frac{2}{3}(e-aR)(1+Q) && (10.86)\\
&= \frac{2}{3}(e-aR+Q-aRQ) && (10.87)\\
&= \frac{2}{3}\left(e-\frac{3}{2}aR+Q\right) \quad (\text{use } QR = \frac{1}{2}R) && (10.88)\\
&= \frac{2}{3}\left(e-\frac{3}{n-1}R+Q\right) && (10.89)
\end{aligned}$$

Furthermore

$$\begin{aligned}
P_2 &= \frac{1}{3}P_4(1-2Q) && (10.90)\\
&= \frac{1}{3}(e-aR)(1-2Q) && (10.91)\\
&= \frac{1}{3}(e-aR-2Q+2aRQ) && (10.92)\\
&= \frac{1}{3}(e-2Q) \quad (\text{use } QR = \frac{1}{2}R) && (10.93)
\end{aligned}$$

QED

Chapter 11

Paulions and Gammions

11.1 Paulions

Let

$\vec{a} \in \mathbb{R}^3$, $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$
$i = 1, 2, 3$, $\mu = 0, 1, 2, 3$
The Pauli matrices are defined by

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.1}$$

Note that they are Hermitian

$$\sigma_i^\dagger = \sigma_i \tag{11.2}$$

and their multiplication table is given by

$$\sigma_i \sigma_j = \delta_{ij} I_2 + i\epsilon_{ijk}\sigma_k \tag{11.3}$$

This multiplication table implies

- Square is 1 (unitary too because Hermitian)

$$\sigma_i^2 = 1 \tag{11.4}$$

- Different ones anticommute and proportional to third

$$\sigma_x \sigma_y = -\sigma_y \sigma_x \tag{11.5}$$

$$\sigma_x \sigma_y = i\sigma_z \tag{11.6}$$

It's common to write

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \tag{11.7}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_\mu = (\sigma_0, \vec{\sigma}) \tag{11.8}$$

Suppose $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. We define the **Paulion** $\sigma_{\vec{x}}$ by[1]

$$\sigma_{\vec{x}} = \sigma \cdot \vec{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \tag{11.9}$$

Claim 23

$$\sigma_{\vec{a}}\sigma_{\vec{b}} = \vec{a} \cdot \vec{b} + i\sigma_{\vec{a}\times\vec{b}} \tag{11.10}$$

As a consequence,

$$(\sigma_{\vec{a}})^2 = |\vec{a}|^2 \tag{11.11}$$

$$(\sigma_{\hat{a}})^2 = 1 \tag{11.12}$$

$$[\sigma_{\vec{a}}, \sigma_{\vec{b}}]_+ = 2(\vec{a} \cdot \vec{b}) \tag{11.13}$$

$$[\sigma_{\vec{a}}, \sigma_{\vec{b}}] = 2i\sigma_{\vec{a}\times\vec{b}} \tag{11.14}$$

proof: This follows directly from Eq.(11.3).
QED

Claim 24 *If* $\vec{a} \in \mathbb{R}^3$ *then*

$$e^{i\sigma_{\vec{a}}} = \cos|\vec{a}| + i\sigma_{\hat{a}}\sin|\vec{a}| \tag{11.15}$$

and

$$e^{i\beta\sigma_{\hat{a}}} = \cos(\beta) + i\sigma_{\hat{a}}\sin(\beta) \tag{11.16}$$

proof:

First show by Taylor expansion of the exponential that the following equation is true.

$$e^{i\beta\sigma_i} = \cos\beta + i\sigma_i\sin\beta \tag{11.17}$$

Then replace σ_i by $\sigma_{\hat{a}}$.
QED

---

[1]The term Paulion is my own. As far as I know, the construct $\sigma_{\vec{a}}$, although often used, doesn't have a common name.

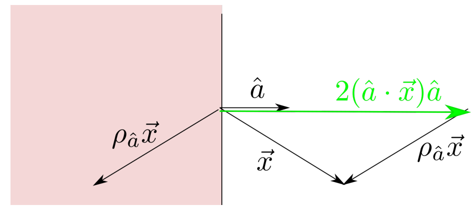


Figure 11.1: Reflection about plane with normal vector $\hat{a}$

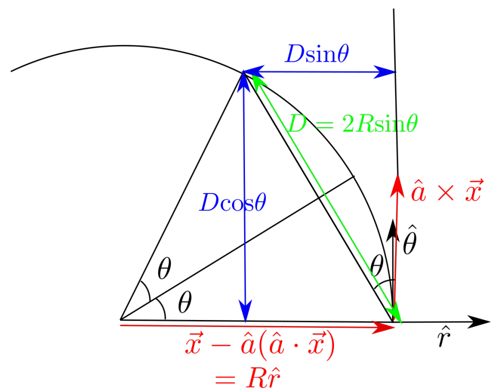


Figure 11.2: Geometry of $R_{\hat{a}}(\Theta)x$, where $\Theta = 2\theta$

Define

$$\left\langle \sigma_{\underline{\hat{a}}} \right\rangle (\vec{x}) = -\sigma_{\hat{a}}\sigma_{\vec{x}}\sigma_{\hat{a}} \tag{11.18}$$

Fig.11.1 illustrates reflection about plane with normal vector $\hat{a}$

Claim 25 *(Reflection)*

$$-\sigma_{\hat{a}}\sigma_{\vec{x}}\sigma_{\hat{a}} = \sigma_{\rho_{\hat{a}}\vec{x}} \tag{11.19}$$

where

$$\rho_{\hat{a}}\vec{x} = \vec{x} - 2(\hat{a}\cdot\vec{x})\hat{a} \tag{11.20}$$

proof:

$$\begin{aligned}
\sigma_{\hat{a}}\sigma_{\vec{x}}\sigma_{\hat{a}} &= [\hat{a}\cdot\vec{x} + i\sigma_{\hat{a}\times\vec{x}}]\,\sigma_{\hat{a}} && (11.21)\\
&= \hat{a}\cdot\vec{x}\sigma_{\hat{a}} + i\underbrace{(\hat{a}\times\vec{x})\cdot\hat{a}}_{=0} - \sigma\cdot\underbrace{(\hat{a}\times\vec{x})\times\hat{a}}_{\vec{x}-(\vec{x}\cdot\hat{a})\hat{a}} && (11.22)\\
&= -\sigma\cdot[\vec{x} - 2(\hat{a}\cdot\vec{x})\hat{a}] && (11.23)
\end{aligned}$$

QED

Claim 26 *(Rotation, Rodrigues formula)*
If

$$U = e^{-i\frac{\Theta}{2}\sigma_{\hat{a}}}, \quad \Theta = 2\theta \tag{11.24}$$

then

$$U\sigma_{\vec{x}}U^\dagger = \sigma_{R_{\hat{a}}(\Theta)\vec{x}} \tag{11.25}$$

where

$$\begin{aligned} R_{\hat{a}}(\Theta)\vec{x} &= \vec{x} + (2|\vec{x}|\sin\beta\sin\theta)(-\sin\theta\hat{r} + \cos\theta\hat{\theta}) & (11.26)\\ &= \vec{x} + (2\sin\theta)(-\sin\theta\vec{r} + \cos\theta\vec{\theta}) & (11.27)\end{aligned}$$

where (see Fig.11.2)

$$\beta = \angle(\vec{x},\vec{a}), \quad \hat{\theta} = \frac{1}{|\vec{x}|\sin\beta}\overbrace{(\hat{a}\times\vec{x})}^{\vec{\theta}}, \quad \hat{r} = \frac{1}{|\vec{x}|\sin\beta}\overbrace{[\vec{x} - \hat{a}(\hat{a}\cdot\vec{x})]}^{\vec{r}} \tag{11.28}$$

This immediately implies that $SU(2)_{\mathbb{R}}$ is a double cover of $SO(3)$.

$$SU(2)_{\mathbb{R}} \xrightarrow[\textit{2 to 1 map}]{} SO(3), \quad SU(2)_{\mathbb{R}}/\{1,-1\} \cong SO(3) \tag{11.29}$$

proof:
Let $C = \cos\theta$, $S = \sin\theta$. Then

$$\begin{aligned} U\sigma_{\vec{x}}U^\dagger &= (C - iS\sigma_{\hat{a}})\sigma_{\vec{x}}(C + iS\sigma_{\hat{a}}) & (11.30)\\ &= C^2\sigma_{\vec{x}} - iSC[\sigma_{\hat{a}},\sigma_{\vec{x}}] + S^2\sigma_{\hat{a}}\sigma_{\vec{x}}\sigma_{\hat{a}} & (11.31)\\ &= \sigma\cdot\left\{C^2\vec{x} + 2SC(\hat{a}\times\vec{x}) - S^2[\vec{x} - 2(\hat{a}\cdot\vec{x})\hat{a}]\right\} & (11.32)\\ &= \sigma\cdot\left\{\vec{x} + 2SC\vec{\theta} - 2S^2\vec{r}\right\} & (11.33)\end{aligned}$$

If U produces rotation R, then $-U$ gives the same rotation so 2 to 1 map:

$$(-U)\sigma_{\vec{x}}(-U)^\dagger = U\sigma_{\vec{x}}U^\dagger. \tag{11.34}$$

QED

11.2 Gammions

Suppose

$n_- = n$ for n even and $n_- = n - 1$ for n odd.
$int(x) =$ integer part of $x \in \mathbb{R}$. $int(n/2) = n_-/2$
$\mu = 1, 2, \ldots, n$
$a_\mu, b_\mu \in \mathbb{R}$ for each μ,
$\gamma_\mu \in \mathbb{C}^{d\times d}$ where $d = int(n/2)$.

$$a \cdot b = a_\mu b^\mu \tag{11.35}$$

We define the **Gammion** $\gamma_{\underline{a}}$ by[2]

$$\gamma \cdot a = \gamma_{\underline{a}} = \gamma_\mu a^\mu \tag{11.36}$$

We will use an underline in $\gamma_{\underline{a}}$ to distinguish a from an index.

The Clifford anticommutation relation is

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu} \tag{11.37}$$

where $\eta_{\mu\nu}$ is real valued[3] For $n = 4$, it equals the Euclidean metric $(1, 1, 1, 1)$ or the Lorentzian metric $(1, -1, -1, -1)$. The Clifford anticommutation relation can be expressed in terms of gammions thus:

$$\gamma_{\underline{a}}\gamma_{\underline{b}} = a \cdot b + \gamma_{\mu\nu} a^\mu b^\nu, \tag{11.38}$$

where bivector

$$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \tag{11.39}$$

is the generator of $\mathfrak{so}(n)_\mathbb{R}$ or $\mathfrak{so}(p, q)$.

Define

$$\left\langle \gamma_{\underline{\hat{a}}} \right\rangle (x) = -\gamma_{\underline{\hat{a}}}\gamma_{\underline{x}}\gamma_{\underline{\hat{a}}} \tag{11.40}$$

Claim 27 *(Reflection)*

$$-\gamma_{\underline{\hat{a}}}\gamma_{\underline{x}}\gamma_{\underline{\hat{a}}} = \gamma_{\underline{\rho_{\hat{a}} x}} \tag{11.41}$$

where

$$\rho_{\hat{a}} x = x - 2(\hat{a} \cdot x)\hat{a} \tag{11.42}$$

[2]The term Gammion, like the term Paulion, is my own. As far as I know, the construct $\gamma_{\underline{a}}$, although often used, doesn't have a common name.

[3]Since $\eta_{\mu\nu}$ is real, we consider later on in this chapter, the Clifford algebra $Cl(n)_\mathbb{R}$ instead of its complexification $Cl(n)_\mathbb{C}$

proof:
QED

Claim 28 *(Rotation)*
If

$$S(\omega) = e^{\,\omega^{\mu\nu}\gamma_{\mu\nu}} \tag{11.43}$$

for some $\omega_{\mu\nu} \in \mathbb{R}$*, then*

$$S(\omega)\gamma_{\underline{x}}S^{\dagger}(\omega) = \gamma_{Rx} \tag{11.44}$$

where $R \in SO(n)$ *(or* $SO(p,q)$*).*

proof:
QED

11.3 $Spin(n)_{\mathbb{R}}$

If $\gamma_\mu^\dagger = \gamma_\mu$ for $\mu = 1, 2, \ldots, n$ (this assumes the Euclidean metric which is $g_{\mu\nu} = diag(1,1,1,1)$ for $n = 4$)

$$Spin(n)_{\mathbb{R}} = \{e^{\omega^{\mu\nu}\gamma_{\mu\nu}} | \omega_{\mu\nu} = -\omega_{\nu\mu} \in \mathbb{R}\}\} \tag{11.45}$$

If $\gamma_0^\dagger = \gamma_0$, and $\gamma_i^\dagger = -\gamma_i$ (this assumes the mostly-minus metric which is $g_{\mu\nu} = diag(1,-1,-1,-1)$ for $n = 4$)

$$Spin(1,3)_{\mathbb{R}} = \{e^{-i\omega^{\mu\nu}\gamma_{\mu\nu}} | \omega_{\mu\nu} = -\omega_{\nu\mu}, \omega_{0,i} \in \mathbb{R}, \omega_{i,j} \in i\mathbb{R}\} \tag{11.46}$$

Spinors are the vectors in the vector space upon which the group $Spin(n)_{\mathbb{R}}$ acts.

Suppose $e_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots n$, and all components of e_i are zero except the ith one. Define

$$\gamma_i = \gamma_{\underline{e}_i} \tag{11.47}$$

Consider the Clifford anticommutator in the Euclidean metric:

$$[\gamma_i, \gamma_j]_+ = 2\delta_{i,j} \tag{11.48}$$

Define the Clifford algebra $Cl(n)_{\mathbb{R}}$ by

$$\Pi_0 = \{1\}, \quad |\Pi_0| = \binom{n}{0} = 1 \tag{11.49}$$

$$\Pi_1 = \{\gamma_i | i = 1, 2, \ldots, n\} \quad |\Pi_1| = \binom{n}{1} = n \tag{11.50}$$

$$\Pi_2 = \{\gamma_{i_1}\gamma_{i_2}|i_1 < i_2\}, \quad |\Pi_2| = \binom{n}{2} \tag{11.51}$$

$$\Pi_3 = \{\gamma_{i_1}\gamma_{i_2}\gamma_{i_3}|i_1 < i_2 < i_3\}, \quad |\Pi_3| = \binom{n}{3} \tag{11.52}$$

$$\Pi_n = \{\gamma_1\gamma_2 \ldots \gamma_n\}, \quad |\Pi_n| = \binom{n}{n} = 1 \tag{11.53}$$

$$\sum_{k=0}^{n} |\Pi_k| = \sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n \tag{11.54}$$

$$Cl(n)_{\mathbb{R}} = span_{\mathbb{R}} \left(\bigcup_{k=0,1,2,\ldots,n} \Pi_k \right) \tag{11.55}$$

$$Cl^0(n)_{\mathbb{R}} = span_{\mathbb{R}} \left(\bigcup_{k=0,2,4,\ldots,n_-} \Pi_k \right) \tag{11.56}$$

$$Cl^1(n)_{\mathbb{R}} = span_{\mathbb{R}} \left(\bigcup_{k=1,3,5,\ldots,n_-} \Pi_k \right) \tag{11.57}$$

$$Cl(n)_{\mathbb{R}} = Cl^0(n)_{\mathbb{R}} \oplus Cl^1(n)_{\mathbb{R}} \tag{11.58}$$

Since $e^{\omega^{\mu\nu}\gamma_{\mu\nu}}$ is the exponentiation of a bivector, its Taylor expansion only contains summands with an even number of gammas. Hence,

$$Spin(n)_{\mathbb{R}} \subset Cl^0(n)_{\mathbb{R}} \tag{11.59}$$

Define thee unit sphere in n dimensions by

$$S^n = \{\hat{a} \in \mathbb{R}^n | \hat{a}^2 = 1\} \tag{11.60}$$

Pin stands for "Product of involutions" An involution in this case is a unit vector (i.e., $\hat{a} \in \mathbb{R}^n$ such that $\hat{a} \cdot \hat{a} = \hat{a}^2 = 1$) The $Pin(n)_{\mathbb{R}}$ group is defined by

$$Pin(n)_{\mathbb{R}} \quad = \quad \cup_{k=1}^{\infty}\{\gamma_{\underline{\hat{a}}_1}\gamma_{\underline{\hat{a}}_1} \ldots \gamma_{\underline{\hat{a}}_k} | \hat{a}_i \in S^n \text{ for all } i\} \tag{11.61}$$

Note that

- One refection
$$det[\left\langle \gamma_{\underline{\hat{a}}} \right\rangle (x)] = -det(\gamma_{\underline{x}}\gamma_{\underline{\hat{a}}}^2) = -\det(\gamma_{\underline{x}}) \tag{11.62}$$
- Two reflections = a rotation
$$det[\left\langle \gamma_{\underline{\hat{a}}_1}\gamma_{\underline{\hat{a}}_2} \right\rangle (x)] = +\det(\gamma_{\underline{x}}) \tag{11.63}$$

Hence, $Pin(n)_{\mathbb{R}}$ is a double cover of $O(n)$ (because $det = \pm 1$)

$$Pin(n)_{\mathbb{R}} \underset{2 \text{ to } 1 \text{ map}}{\longrightarrow} O(n), \quad Pin(n)_{\mathbb{R}}/\{1,-1\} \cong O(n) \tag{11.64}$$

Previously, we defined $Spin(n)_{\mathbb{R}}$ by exponentiating its generators. $Spin(n)_{\mathbb{R}}$ can also be defined as follows, in terms of products of gammions instead of exponentiating generators:

$$Spin(n)_{\mathbb{R}} = \cup_{k=1}^{\infty}\{\gamma_{\underline{\hat{a}}_1}\gamma_{\underline{\hat{a}}_1}\ldots\gamma_{\underline{\hat{a}}_{2k}}|\hat{a}_i \in S^n \text{ for all } i\} \tag{11.65}$$

$Spin(n)_{\mathbb{R}}$ is a double cover of $SO(n)$ (because $\left\langle\gamma_{\underline{\hat{a}}}\right\rangle = \langle\gamma_{-\hat{a}}\rangle$)

$$Spin(n)_{\mathbb{R}} \underset{2 \text{ to } 1 \text{ map}}{\longrightarrow} SO(n), \quad Spin(n)_{\mathbb{R}}/\{1,-1\} \cong SO(n) \tag{11.66}$$

Define $Cl(n)_{\mathbb{F}}$ as a span over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $Spin(n)_{\mathbb{F}} \subset Cl^0(n)_{\mathbb{F}}$.
When you go from a vector space

$$V_{\mathbb{R}} = span_{\mathbb{R}}(e_1, e_2, \ldots, e_n) \tag{11.67}$$

to

$$V_{\mathbb{C}} = span_{\mathbb{C}}(e_1, e_2, \ldots, e_n), \tag{11.68}$$

this is called a **complexification** of the vector space. Lie algebras over $\mathbb{R}$ can be complexified. $Cl(n)_{\mathbb{C}}$ is the complexification of $Cl(n)_{\mathbb{R}}$.

$Spin(n)_{\mathbb{R}} \underset{2 \text{ to } 1 \text{ map}}{\longrightarrow} SO(n;\mathbb{R})$ and $Spin(n)_{\mathbb{C}} \underset{2 \text{ to } 1 \text{ map}}{\longrightarrow} SO(n;\mathbb{C})$. Rotations by a complex angle as in $SO(n)_{\mathbb{C}}$ are meaningless, even in Quantum Mechanics. Only group $Spin(n)_{\mathbb{R}}$ describes symmetries in physical systems. However, in Quantum Mechanics, we use complex representations of $Spin(n)_{\mathbb{R}}$ and a complex spinor space. The complex rep of $Spin(n)_{\mathbb{R}}$ and its complexification $Spin(n)_{\mathbb{C}}$ are isomorphic.

Careful. The following possibilities are not equivalent

1. representation of $Spin(n)_{\mathbb{F}}$ is real or complex
2. Spinor space on which $Spin(n)_{\mathbb{F}}$ acts is real or complex
3. γ matrices are real or complex

$n =$	0	1	2	3	4	5	6	7
$Cl(n)_{\mathbb{R}} \cong$	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$

Table 11.1: Sets isomorphic ($\cong$) to $Cl(n)_{\mathbb{R}}$. In this figure, $M_n(\mathbb{F}) = \mathbb{F}^{n\times n}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ where $\mathbb{H}$ are the quaternions.

$n=$	$2k$	$2k+1$
$Cl(n)_{\mathbb{C}} \cong$	$M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$	$M_{2^k}(\mathbb{C})$

Table 11.2: Sets isomorphic ($\cong$) to $Cl(n)_{\mathbb{C}}$. Table 11.1 collapses to this table when $Cl(n)_{\mathbb{R}}$ is complexified.

11.4 $Cl(n)_{\mathbb{R}}$ representations

Consider Table 11.1 which gives sets isomorphic to $Cl(n)_{\mathbb{R}}$ for all n.

Notes on Table 11.1

- Dimension check: the real dimension for the two isomorphic sets is equal to 2^n in every column. For example[4], $\dim_{\mathbb{R}} M_2(\mathbb{C}) = 8 = 2^3$.
- The pattern repeats with period 8 (**Bott Periodicity**).

$$Cl(n+8)_{\mathbb{R}} \cong Cl(n)_{\mathbb{R}} \otimes M_{16}(\mathbb{R}) \tag{11.69}$$

  Rather than depending on n, the pattern depends on[5] $n \mod 8$. For example

$$Cl(8)_{\mathbb{R}} \cong \underbrace{Cl(0)}_{\mathbb{R}} \otimes M_{16}(\mathbb{R}) \cong M_{16}(\mathbb{C}) \tag{11.70}$$

$$Cl(9)_{\mathbb{R}} \cong \underbrace{Cl(1)}_{\mathbb{R}\oplus\mathbb{R}} \otimes M_{16}(\mathbb{R}) \cong M_{16}(\mathbb{C}) \oplus M_{16}(\mathbb{C}) \tag{11.71}$$

  $M_{16}(\mathbb{C})$ are $2^{int(n/2)} \times 2^{int(n/2)}$ complex matrices where $n = 8$.

- The exact same table is valid for $Cl(p,q)_{\mathbb{R}}$, except with n replaced by $(p-q)\%8$. Note, however, that according to this, $Cl(1,3)_{\mathbb{R}}$ is congruent to column $(1-3)\%8 = 6$ of Table 11.1, whereas $Cl(3,1)_{\mathbb{R}}$ is congruent to column $(3-1)\%8 = 2$ and columns 2 and 6 are not isomorphic. Hence

$$Cl(p,q)_{\mathbb{R}} \ncong Cl(q,p)_{\mathbb{R}} \tag{11.72}$$

  However, one can show that

$$Cl(p,q)_{\mathbb{C}} \cong Cl(q,p)_{\mathbb{C}} \tag{11.73}$$

  and

[4] $dim_{\mathbb{R}}$= number of real degrees of freedom (dofs). $dim_{\mathbb{C}} = 2dim_{\mathbb{R}}$. For example, $dim_{\mathbb{R}}(\mathbb{H}) = 4$, where $\mathbb{H}$ are the quaternions.

[5] In Python $n \mod 8$ is denoted by $n\%8$, the remainder after dividing n by 8.

$$Spin(p,q)_{\mathbb{R}} \cong Spin(q,p)_{\mathbb{R}}, \quad \mathfrak{spin}(p,q)_{\mathbb{R}} \cong \mathfrak{spin}(q,p)_{\mathbb{R}} \tag{11.74}$$

$$SO(p,q)_{\mathbb{R}} \cong SO(q,p)_{\mathbb{R}}, \quad \mathfrak{so}(p,q)_{\mathbb{R}} \cong \mathfrak{so}(q,p)_{\mathbb{R}} \tag{11.75}$$

11.5 $Cl(n)_{\mathbb{C}}$ representations

Upon complexification, Table 11.1 collapses to Table 11.2. In fact, the representations of $Cl(n)_{\mathbb{C}}$ are all complex.

For $n = 2k$,

$$Cl(2k)_{\mathbb{C}} \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}) \supset Spin(2k)_{\mathbb{C}} \tag{11.76}$$

and for $n = 2k+1$,

$$Cl(2k+1)_{\mathbb{C}} \cong M_{2^k}(\mathbb{C}) \supset Spin(2k+1)_{\mathbb{C}} \tag{11.77}$$

Assuming complexification, if n is even, we can define a nontrivial chirality operator:

$$\Gamma_5 = i^{n/2}\gamma_1\gamma_2\cdots\gamma_n. \tag{11.78}$$

Γ_5 satisfies:

- anticommutes with all the γ_μ
- commutes with the elements of group $Spin(n)_{\mathbb{R}}$ (because this group consists of products of even number of gammions)
- $\Gamma_5^2 = 1$. Therefore, Γ_5 has eigenvalues $1, -1$.

Assuming complexification, if n is even, and $V = \mathbb{C}^{n/2}$, then

$$V = V_+ \oplus V_-, \tag{11.79}$$

where:

$$V_\pm = \{\psi \in V \mid \Gamma_5\psi = \pm\psi\} \tag{11.80}$$

Thus, there are the two Weyl (or chiral) spinors when n is even.

Assuming complexification, if n is odd, the expression $\gamma^1\cdots\gamma^n$ is proportional to the identity. Hence, there is no new operator that commutes with all the elements of $Spin(n)_{\mathbb{R}}$ and there is no chiral decomposition of vector space V. The space V is irreducible.

11.6 Examples

$Spin(2)_{\mathbb{R}}$

$$Cl(2)_{\mathbb{R}} = span_{\mathbb{R}}\{1, \sigma_1, \sigma_2, \underbrace{\sigma_1\sigma_2}_{i\sigma_3}\} \tag{11.81}$$

$$Cl^0(2)_{\mathbb{R}} = span_{\mathbb{R}}\{1, \underbrace{\sigma_1\sigma_2}_{i\sigma_3}\} = \{Ae^{i\theta\sigma_3} : \theta, A \in \mathbb{R}\} \cong \mathbb{C} \tag{11.82}$$

$$Cl^1(2)_{\mathbb{R}} = span_{\mathbb{R}}\{\sigma_1, \sigma_2\} \tag{11.83}$$

$$Cl(2)_{\mathbb{R}} = Cl^0(2)_{\mathbb{R}} \oplus Cl^1(2)_{\mathbb{R}} \tag{11.84}$$

$$Spin(2)_{\mathbb{R}} = \{e^{i\theta\sigma_3} : \theta \in \mathbb{R}\} \cong U(1) \tag{11.85}$$

Claim 29 *(Double Cover)*
$Spin(2)_{\mathbb{R}} \xrightarrow[\text{2 to 1 map}]{} SO(2)$.

proof:
If $U = e^{i\theta\sigma_3}$ and $\vec{x} = (x_1, x_2)$, then

$$U\sigma_{\vec{x}}U^\dagger = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & x_1 - ix_2 \\ x_1 + ix_2 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{11.86}$$

$$= \begin{pmatrix} 0 & z^* e^{i2\theta} \\ ze^{-i2\theta} & 0 \end{pmatrix} \quad (z = x_1 + ix_2) \tag{11.87}$$

$$= \begin{pmatrix} 0 & x_1' - ix_2' \\ x_1' + ix_2' & 0 \end{pmatrix} \tag{11.88}$$

$$x_1' = \mathrm{Re}(ze^{-i2\theta}) = x_1 \cos 2\theta + x_2 \sin\theta \tag{11.89}$$

$$x_2' = \mathrm{Im}(ze^{-i2\theta}) = -x_1 \sin 2\theta + x_2 \cos 2\theta \tag{11.90}$$

$$R_z(2\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \tag{11.91}$$

$$\vec{x} = (x_1, x_2)^T, \quad \vec{x'} = (x_1', x_2')^T \tag{11.92}$$

$$\vec{x'} = R_z(2\theta)\vec{x} \tag{11.93}$$

The 2 to 1 map:

$$e^{i\theta\sigma_3} \mapsto R_z(2\theta) \tag{11.94}$$

QED

$Spin(3)_{\mathbb{R}}$

$$Cl(3)_{\mathbb{R}} = span_{\mathbb{R}}\{1, \sigma_1, \sigma_2, \sigma_3, \underbrace{\sigma_1\sigma_2}_{i\sigma_3}, \underbrace{\sigma_2\sigma_3}_{i\sigma_1}, \underbrace{\sigma_3\sigma_1}_{i\sigma_2}, \underbrace{\sigma_1\sigma_2\sigma_3}_{i}\} \cong \mathbb{C}^{2\times 2} \tag{11.95}$$

$$Cl^0(3)_{\mathbb{R}} = span_{\mathbb{R}}\{1, i\sigma_1, i\sigma_2, i\sigma_3\} \cong \mathbb{H} \tag{11.96}$$

$$Cl^1(3)_{\mathbb{R}} = span_{\mathbb{R}}\{\sigma_1, \sigma_2, \sigma_3, i\} \cong -i\mathbb{H} \tag{11.97}$$

$$Cl(3)_{\mathbb{R}} = Cl^0(3)_{\mathbb{R}} \oplus Cl^1(3)_{\mathbb{R}} \cong \mathbb{H} \oplus -i\mathbb{H} \tag{11.98}$$

$$Spin(3)_{\mathbb{R}} = \{e^{i\sigma_{\vec{a}}} | \vec{a} \in \mathbb{R}^3\} \cong SU(2) \tag{11.99}$$

Claim 30

$$SL(2;\mathbb{C}) \supset SU(2) \cong Spin(3)_{\mathbb{R}} \underset{\textit{2 to 1 map}}{\longrightarrow} SO(3) \tag{11.100}$$

proof:

QED

Quaternions

The set of quaternions is called $\mathbb{H}$ in honor of Hamilton. ($\mathbb{Q}$ is used for the rationals). A **quaternion** q is an expression of the form

$$q = a + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{11.101}$$

where $a, x, y, z \in \mathbb{R}$. The multiplication table for quaternions is the same as that for Pauli matrices, if we map

$$i\sigma_1 \leftrightarrow \mathbf{i}, \qquad i\sigma_2 \leftrightarrow \mathbf{j}, \qquad i\sigma_3 \leftrightarrow \mathbf{k} \tag{11.102}$$

Quaternion multiplication rules:

$$(i\sigma_i)(i\sigma_j) = -(\delta_{ij} + \epsilon_{ijk} i\sigma_k) \tag{11.103}$$

Chapter 12

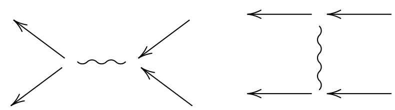
Quantum Shannon Information Theory: COMING SOON

Chapter 13

Recoupling Identities

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

In this chapter, we will refer to the following 2 birdtracks as s and t channels.[1]

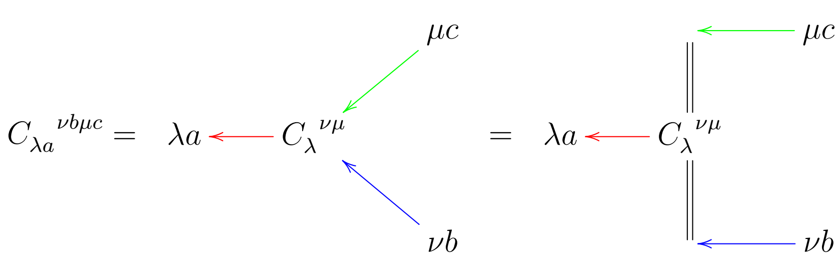


s-channel t-channel (13.1)

This terminology comes from High Energy Physics, where these birdtracks are used to define the so called Mandelstam variables. The Mandelstam variables measure the energy of particles in various birdtracks.

13.1 Parallel Channels to Sum of t-channels

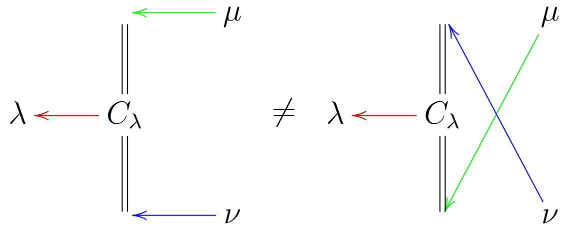
Clebsch-Gordan (CG) coefficients were introduced in Chapter 3. Define the CG coefficients node



(13.2)

Note that

[1] My mnemonic to remember which is which: **s-channel**: particle synergy, energy from particles coming together, **t-channel**: particle trade.



(13.3)

Note that we are defining the CG coefficient C_λ so that the λ rep particle is created in an s-channel by converging μ and ν rep particles. When we define the generators T^i_λ, the i (gluon, adj-rep particle) is in a t-channel emanating from incoming and outgoing def-rep particles. Another big difference between C_λ and T^i_λ is that T^i_λ is assumed to be Hermitian, whereas C_λ is not Hermitian in general. C_λ is not even a square matrix in general.

In this chapter, we won't use implicit summation over Greek indices.

In this section, sometimes instead of labelling arrows by a lower case Greek letter denoting its rep, we will disclose an arrow's rep by a color, according to the following rep-to-color code.

$$\lambda : red, \quad \mu : green, \quad \nu : blue \tag{13.4}$$

According to Chapter 3, the CG coefficient C_λ satisfies

$$C_\lambda C_\lambda^\dagger = P_\lambda \tag{13.5}$$

$$\mathrm{tr}(P_\lambda) = d_\lambda \tag{13.6}$$

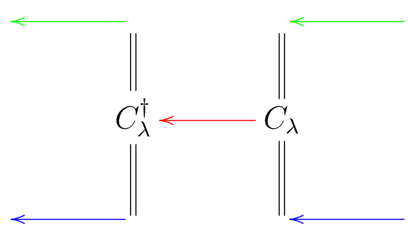
where P_λ is the projection operator onto the vector space of the rep λ and d_λ is the dimension of that vector space.

Note that if we divide C_λ by $\sqrt{d_\lambda}$, then

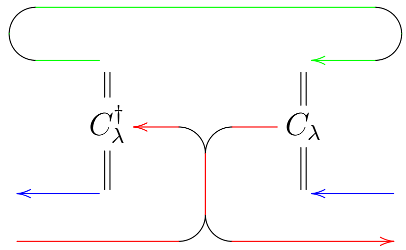
$$\mathrm{tr}\left(\frac{C_\lambda}{\sqrt{d_\lambda}}\frac{C_\lambda^\dagger}{\sqrt{d_\lambda}}\right) = 1 \tag{13.7}$$

Define

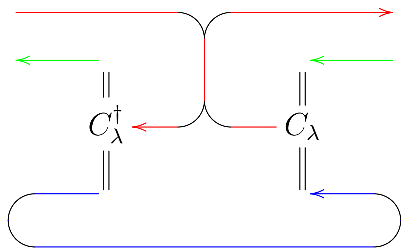
$P_\lambda =$



(13.8)

$$P_\mu = \frac{d_\mu}{d_\lambda}$$ 

(13.9)

$$P_\nu = \frac{d_\nu}{d_\lambda}$$ 

(13.10)

One can check that these operators are projection operators normalized to the dimennsion of their rep; i.e., for $\Omega \in \{\lambda, \mu, \nu\}$,

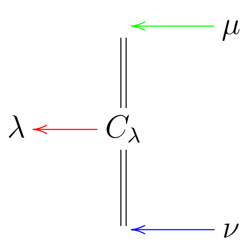
$$P_\Omega^2 = P_\Omega \tag{13.11}$$

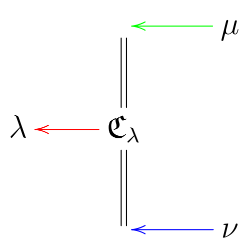
and

$$\mathrm{tr}(P_\Omega) = d_\Omega \tag{13.12}$$

The normalization of the projectors P_Ω can be remembered if one takes the denominator d_λ and splits it into two factors of $\sqrt{d_\lambda}$ and puts one $\sqrt{d_\lambda}$ under C_λ and the other under $C_\lambda^\dagger$. Then one "trades" $\frac{C_\lambda}{\sqrt{d_\lambda}}$ by $\frac{C_\mu}{\sqrt{d_\mu}}$ or $\frac{C_\nu}{\sqrt{d_\nu}}$.

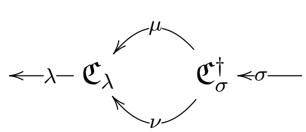
Next we define a scaled version of the CG coefficients C_λ as follows

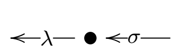


$$= \frac{1}{\sqrt{\kappa_\lambda{}^{\nu\mu}}}$$ 

(13.13)

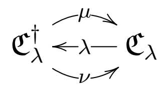
The **scaled CG coefficients** $\mathfrak{C}_\lambda$ satisfy



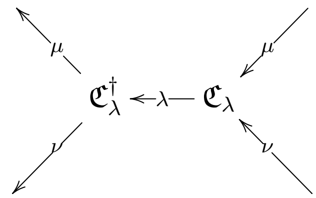
$$= \kappa_\lambda{}^{\nu\mu}\delta(\lambda,\sigma)$$ 

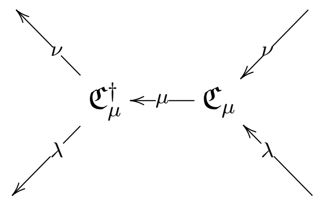
(13.14)

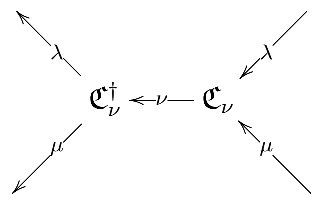
Therefore

 $= \kappa_\lambda{}^{\nu\mu} d_\lambda$ (13.15)

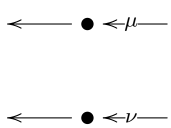
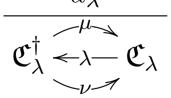
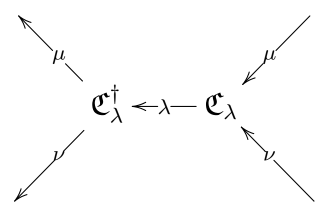
The projection operators P_Ω for $\Omega \in \{\lambda, \mu, \nu\}$ can be expressed in a more symmetrical form using nodes for the scaled CG coefficients as follows

$P_\lambda = \frac{1}{\kappa_\lambda{}^{\nu\mu}}$  (13.16)

$P_\mu = \frac{1}{\kappa_\mu{}^{\lambda\nu}}$  (13.17)

$P_\nu = \frac{1}{\kappa_\nu{}^{\mu\lambda}}$  (13.18)

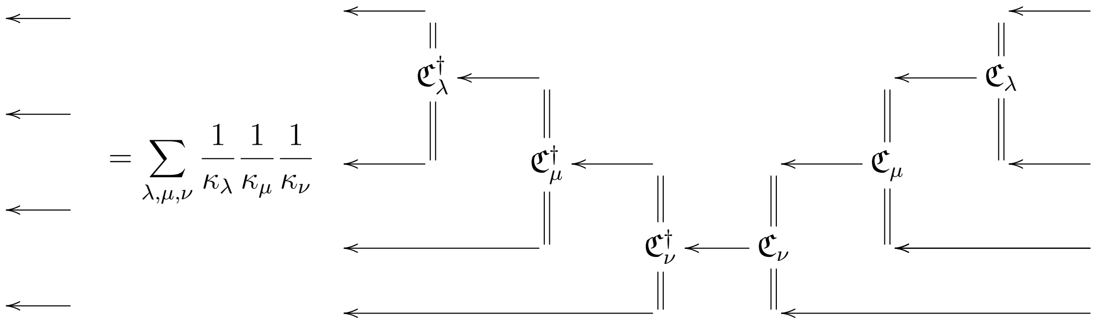
The CG series for $V_\mu \otimes V_\nu = \sum_\lambda V_\lambda$ can be expressed in terms of birdtracks as follows

 $= \sum_\lambda P_\lambda = \sum_\lambda \frac{d_\lambda}{}$   (13.19)

This CG series expresses two **parallel channels** as a sum of s-channels.

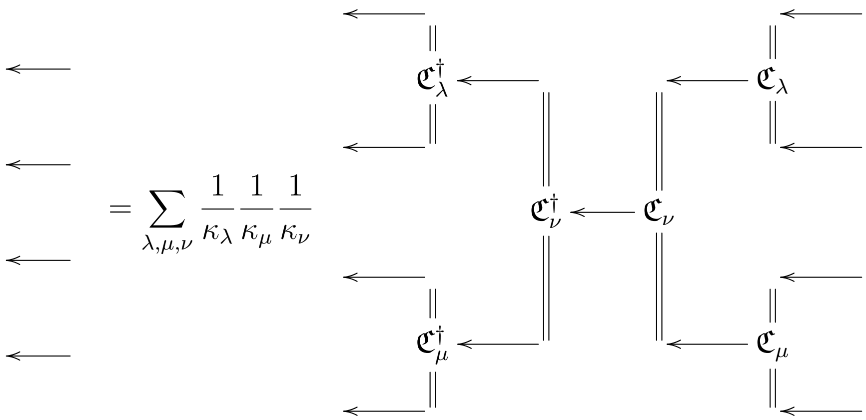
The CG series for $N > 2$ parallel channels $V_{\mu_1} \otimes V_{\mu_2} \otimes \ldots \otimes V_{\mu_N} = \sum_\lambda V_\lambda$ is obtained by combining pairs of vector spaces recursively. The series depends on what vector space pairs are chosen in what order. For example, we can use[2]

[2]For succinctness, we are dropping the rep labels μ, λ from $\kappa_\nu{}^{\mu\lambda}$, but the κ_ν still depends on them.



(13.20)

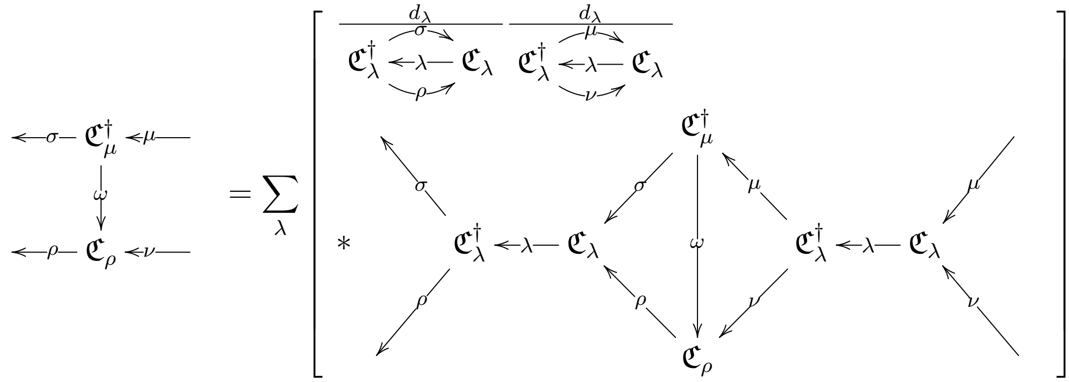
Another possibility is



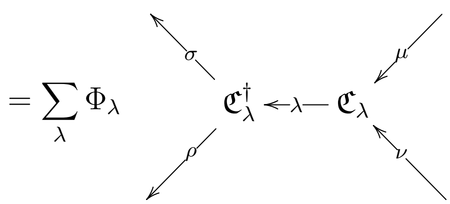
(13.21)

13.2 t-channel to Sum of s-channels

We can express a t-channel as a sum over s-channels as follows

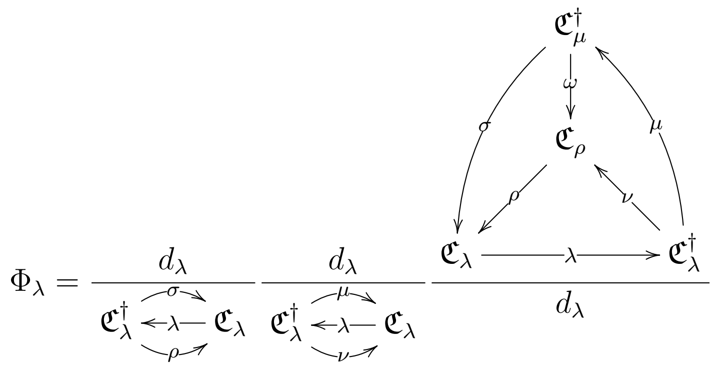


(13.22)

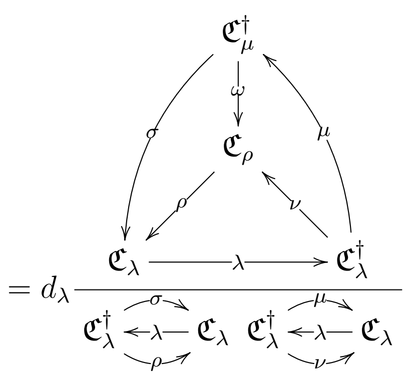


(13.23)

where



(13.24)

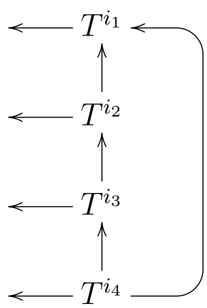


(13.25)

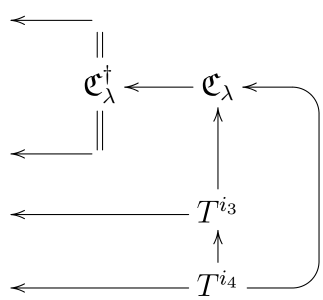
13.3 Wigner $3n - j$ Coefficients/DAGs

A DAG with no incoming or outgoing arrows is called an **isolated DAG**. Physicists sometimes call it a **vacuum bubble** also. On the right hand side of Eq.(13.25), the isolated DAG with two $\mathfrak{C}$ is called a $3j$ **coefficient/DAG**, and the one with 4 $\mathfrak{C}$ is called a $6j$ **coefficient/DAG**. So far we seen $3j$ and $6j$ coefficients/DAGs. Atomic physicists define **Wigner** $3n - j$ **coefficients/DAGs**, for $n = 1, 2, 3, \ldots$. They are called that because they describe how to "add" $3n$ angular momenta j. There is only one topological distinct $3j$ DAG but two $6j$ DAGs, five $9j$ DAGs, and so on.

In Chapter 1, we discussed Casimir suns. Next we show that they can always be expressed in terms of $3j$ and $6j$ coefficients and CG coefficients. We proceed as we did in Eq.(13.20) but here we use the most general t-channel to sum of s-channels conversion Eq.(13.23).

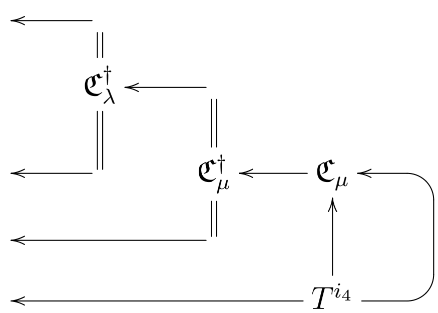


$= \sum_{\lambda,\mu} \Phi_\lambda \Phi_\mu$



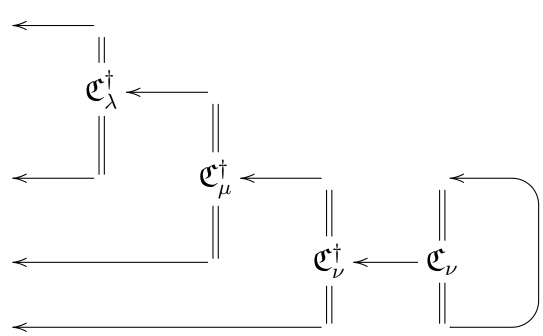
(13.26)

$= \sum_{\lambda,\mu} \Phi_\lambda \Phi_\mu$



(13.27)

$= \sum_{\lambda,\mu,\nu} \Phi_\lambda \Phi_\mu \Phi_\nu$



(13.28)

Chapter 14

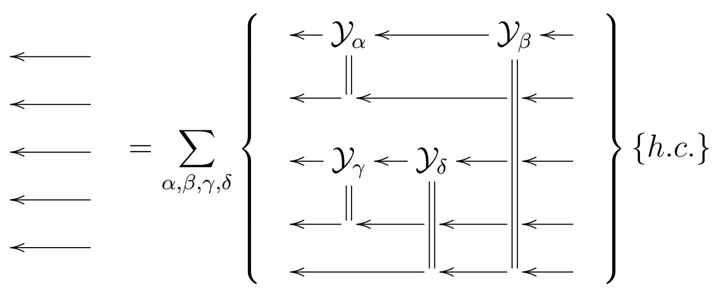
Recoupling Identities for $U(n)$

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

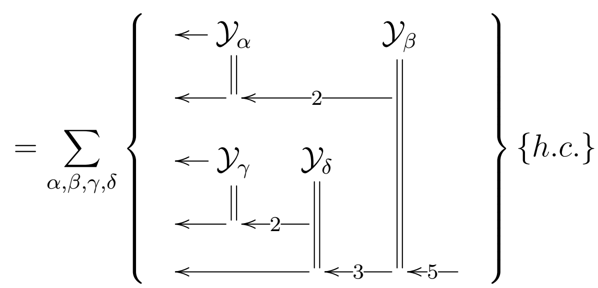
For $U(n)$ (as opposed to $SU(n)$), there are no antiparticles (i.e., one can use only lowered indices). A consequence of this is that for proper operators in $U(n)$, the total particle number is conserved.

Young Tableau are discussed in Chapter 23.

Clebsh-Gordan series for $U(n)$ can be written in terms of Standard Young Tableau (SYT). For example, the tensor decomposition of $V^{\otimes 5}$ is:[1]



(14.1)

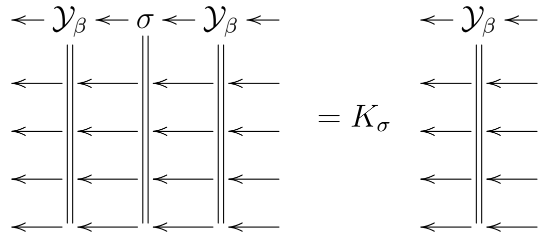


(14.2)

where $\xleftarrow{p}$ means p parallel arrows superimposed on each other.

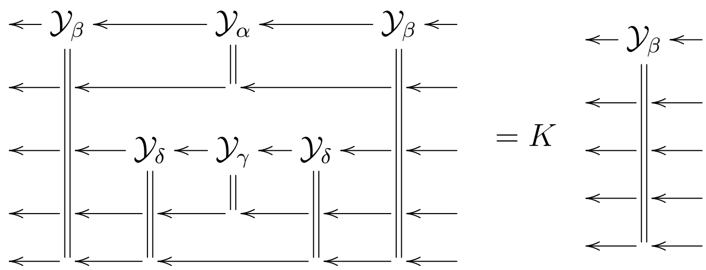
[1] $x(h.c.) = xx^{\dagger}$.

It's always true that



(14.3)

for some $K_\sigma \in \{-1, 0, 1\}$. More generally,



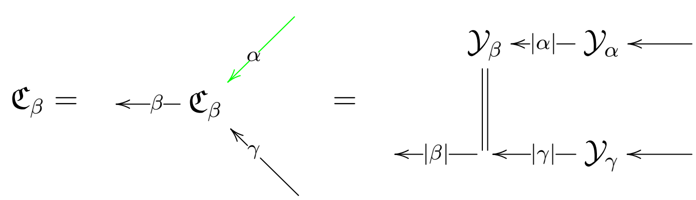
(14.4)

for some $K \in \mathbb{R}$ that is independent of n.

14.1 $3j$ Coefficients

Recall that $|\mathcal{Y}_\alpha|$ or $|\alpha|$ is the number of boxes (or number of outgoing legs in its birdtrack) in the YT $\mathcal{Y}_\alpha$.

Clebsch-Gordan (CG) coefficients are discussed in Chapter 3. One can define a CG coefficient $\mathfrak{C}_\beta$ in terms of Young Tableau as follows:

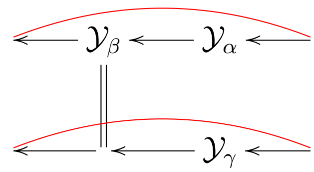


(14.5)

where $|\beta| = |\alpha| + |\gamma|$

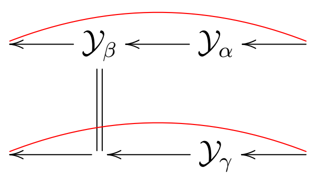
Claim 31 *(3j coefficient for $U(n)$ in terms of YT)*[2]

$$\mathrm{tr}(\mathfrak{C}^\dagger_\beta\mathfrak{C}_\beta) = \begin{tikzcd} \mathfrak{C}^\dagger_\beta \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\gamma"'] & \mathfrak{C}_\beta \arrow[l, "\beta"'] \end{tikzcd} = \tag{14.6}$$

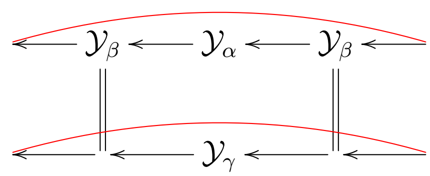


$$= dim(\mathcal{Y}_\beta) \tag{14.7}$$

proof:



$$= \tag{14.8}$$



$$= K dim(\mathcal{Y}_\beta) \tag{14.9}$$

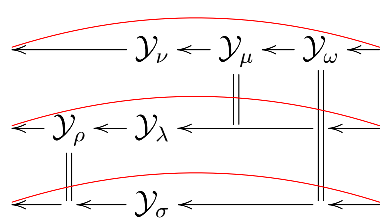
Ref. [1] shows that for this example, $K = 1$.

QED

14.2 $6j$ Coefficients

Claim 32 *(6j coefficient for $U(n)$ in terms of YT)*[3]

$$\begin{tikzcd} & \mathfrak{C}^\dagger_\rho \arrow[d, "\lambda"] \arrow[ddl, bend right, "\sigma"'] & \\ & \mathfrak{C}_\mu \arrow[dl, "\mu"'] & \\ \mathfrak{C}_\omega \arrow[rr, "\omega"] & & \mathfrak{C}^\dagger_\omega \arrow[uul, bend right, "\rho"'] \arrow[ul, "\nu"'] \end{tikzcd} = \tag{14.10}$$



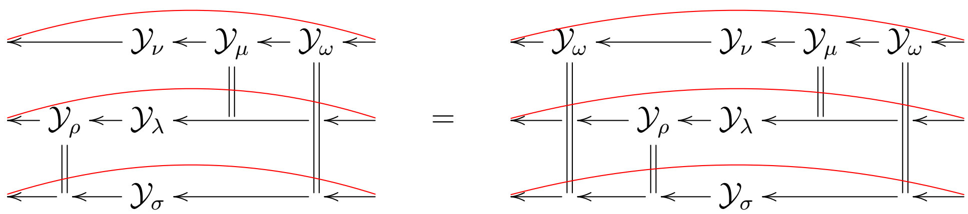
$$= K dim(\mathcal{Y}_\omega) \tag{14.11}$$

where K is independent of n

[2] Ref.[1], on which this claim is based, uses different labels for the Young projectors. Their labels and ours are related as follows:
$\mathcal{Y}_\alpha \to X$, $\mathcal{Y}_\beta \to Y$, $\mathcal{Y}_\gamma \to Z$.

[3] Ref.[1], on which this claim is based, uses different labels for the Young projectors. Their labels and ours are related as follows:
$\mathcal{Y}_\sigma \to X$, $\mathcal{Y}_\omega \to Y$, $\mathcal{Y}_\rho \to U$, $\mathcal{Y}_\lambda \to W$, $\mathcal{Y}_\nu \to V$, $\mathcal{Y}_\mu \to Z$.

proof: Replace each of the 6 Young projectors $\mathcal{Y}_\alpha$ of the right hand side (RHS) by its square. That gives 12 Young projectors on the RHS. Each of the 4 generators $\mathfrak{C}_\lambda$ on the left hand side (LHS) is composed of 3 Young projectors so there are 12 Young projectors on the LHS too.



(14.12)

$$= K dim(\mathcal{Y}_\omega) \tag{14.13}$$

QED

For example, Ref.[1] shows that if

$$\mathcal{Y}_\rho = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & \\ \cline{1-1}\end{array}, \qquad \mathcal{Y}_\nu = \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \qquad \mathcal{Y}_\lambda = \begin{array}{|c|}\hline 2 \\ \hline\end{array} \tag{14.14}$$

$$\mathcal{Y}_\sigma = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}, \qquad \mathcal{Y}_\omega = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} 4 & \\ \cline{1-1}\end{array}, \qquad \mathcal{Y}_\mu = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \tag{14.15}$$

then

$$K = \frac{1}{3}, \quad dim\mathcal{Y}_\omega = \frac{n(n^2-1)(n^2-2)}{8} \tag{14.16}$$

14.3 Sum Rules

Let

$SYT(n_b)$ = set of SYT with n_b boxes
$SYT = \bigcup_{n_b=1}^{\infty} SYT(n_b)$

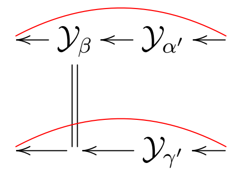
Claim 33

$$\sum_{\alpha',\gamma' \in SYT} \mathbb{1}(|\alpha'| + |\gamma'| = |\beta|)\, \mathfrak{C}_\beta^\dagger \overset{\alpha'}{\underset{\gamma'}{\leftarrow\beta-}} \mathfrak{C}_\beta = (|\beta|-1) dim(\mathcal{Y}_\beta) \tag{14.17}$$

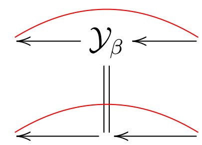
proof:

$$\sum_{\alpha' \in SYT(|\alpha|)} \mathcal{Y}_{\alpha'} = 1, \qquad \sum_{\gamma' \in SYT(|\gamma|)} \mathcal{Y}_{\gamma'} = 1 \tag{14.18}$$

$$\sum_{\alpha',\gamma'\in SYT} \mathbb{1}(|\alpha'|+|\gamma'|=|\beta|) \begin{tikzcd} \mathfrak{C}^\dagger_\beta \arrow[r, bend left, "\alpha'"] \arrow[r, bend right, "\gamma'"'] & \mathfrak{C}_\beta \arrow[l, "\beta"'] \end{tikzcd} = \sum_{|\alpha'|=1}^{|\beta|-1} \sum_{\substack{\alpha'\in SYT(|\alpha'|)\\ \gamma'\in SYT(|\beta|-|\alpha'|)}} \tag{14.19}$$



$$= \sum_{|\alpha'|=1}^{|\beta|-1} \tag{14.20}$$



$$= (|\beta|-1)dim(\mathcal{Y}_\beta) \tag{14.21}$$

QED

Let

$$A = \{\rho,\nu,\lambda,\sigma,\omega,\mu\}, \quad B = A - \{\omega\} \tag{14.22}$$

$$A' = \{\rho',\nu',\lambda',\sigma',\omega',\mu'\}, \quad B' = A' - \{\omega'\} \tag{14.23}$$

$$J(A) = \mathbb{1}\begin{pmatrix} |\sigma|+|\mu| = |\omega|, \\ |\nu|+|\rho| = |\omega|, \\ |\sigma|+|\lambda| = |\rho|, \\ |\lambda|+|\nu| = |\mu| \end{pmatrix} \tag{14.24}$$

Claim 34

$$\prod_{\alpha'\in B'}\left[\sum_{\alpha'\in SYT}\right] J(B',\omega) \begin{tikzcd} & \mathfrak{C}^\dagger_{\rho'} \arrow[d, "\lambda'"] \arrow[ddl, bend right, "\sigma'"'] & \\ & \mathfrak{C}_{\mu'} \arrow[dl, "\mu'"'] & \\ \mathfrak{C}_\omega \arrow[rr, "\omega"] & & \mathfrak{C}^\dagger_\omega \arrow[ul, "\nu'"'] \arrow[uul, bend right, "\rho'"'] \end{tikzcd} = \frac{1}{2}(|\omega|-1)(|\omega|-2)dim\mathcal{Y}_\omega \tag{14.25}$$

proof:

See Ref.[1] for proof.

QED

Chapter 15

Reducibility of Representations

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

15.1 Eigenvalue Projectors

Suppose $M \in \mathbb{C}^{d\times d}$ has eigenvalues λ_i with corresponding eigenvectors $|\lambda_i\rangle$

$$M|\lambda_i\rangle = \lambda_i|\lambda_i\rangle \tag{15.1}$$

for $i \in \mathbb{Z}_{[1,r]}$. The characteristic polynomial of M is defined as

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M-\lambda) = \prod_{i=1}^{r}(\lambda-\lambda_i)^{d_i} \tag{15.2}$$

It satisfies

$$cp(\lambda) = 0 \tag{15.3}$$

for $\lambda = \lambda_i$.

Note that if M is Hermitian ($M^\dagger = M$), then all its eigenvalues are real. (because $\lambda_i = \langle\lambda_i|M|\lambda_i\rangle \in \mathbb{R}$)

If M is Hermitian, then there exists a matrix C that is unitary ($CC^\dagger = C^\dagger C = 1$) and diagonalizes M

$$CMC^\dagger = \begin{bmatrix} \Lambda_{\lambda_1} & 0 & 0 & 0 \\ 0 & \Lambda_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Lambda_{\lambda_r} \end{bmatrix} \tag{15.4}$$

where

$$\Lambda_{\lambda_i} = \lambda_i \text{diag}\underbrace{(1,1,\ldots,1)}_{d_i \text{ times}} = \lambda_i I^{d_i\times d_i} \tag{15.5}$$

and

$$d = \sum_{i=1}^{r} d_i \tag{15.6}$$

As in Chapter 3, let us set

$$\pi_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_i \times d_i} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} \tag{15.7}$$

and

$$P_i = C^\dagger \pi_i C \tag{15.8}$$

For example, when $d = 2$,

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{15.9}$$

so

$$\pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2}, \quad P_1 = \frac{M - \lambda_2}{\lambda_1 - \lambda_2} \tag{15.10}$$

$$\pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1}, \quad P_2 = \frac{M - \lambda_1}{\lambda_2 - \lambda_1} \tag{15.11}$$

$\{\pi_1, \pi_2\}$ is a complete orthogonal set of projection operators, and so is $\{P_1, P_2\}$.

Similarly, for $d > 2$, we have

$$\pi_i = \prod_{j \neq i} \frac{CMC^\dagger - \lambda_j}{\lambda_i - \lambda_j}, \quad P_i = \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \tag{15.12}$$

$\{\pi_i\}_{i=1}^{r}$ is a complete set of orthogonal projection operators and $\{P_i\}_{i=1}^{r}$ is too.

Note that

$$d_i = \mathrm{tr}(\pi_i) = \mathrm{tr}(P_i) \tag{15.13}$$

15.2 $[P_i, M] = 0$ Consequences

From Eq.(15.12), it is clear that P_i and M commute

$$[P_i, M] = P_i M - M P_i = 0 \tag{15.14}$$

From the P_i's completeness and commutativity with M, we get

$$
\begin{aligned}
M &= \sum_{i=1}^{r}\sum_{j=1}^{r} P_i M P_j && (15.15)\\
&= \sum_{i=1}^{r} P_i M P_i && (15.16)
\end{aligned}
$$

Claim 35 *For all i,*

$$MP_i = \lambda_i P_i \textit{ (no } i \textit{ sum)} \tag{15.17}$$

proof:

$$
\begin{aligned}
MP_i &= [C^\dagger \Lambda C][C^\dagger \pi_i C] && (15.18)\\
&= \lambda_i [C^\dagger \pi_i C] && (15.19)\\
&= \lambda_i P_i && (15.20)
\end{aligned}
$$

QED

From the last claim, it immediately follows that if $f(x)$ can be expressed as a power series in x, then [1]

$$f(M)P_i = f(\lambda_i)P_i \text{ (no } i \text{ sum)} \tag{15.21}$$

15.3 Multiple Invariant Matrices

Suppose $M^{(1)}, M^{(2)} \in \mathbb{C}^{d\times d}$ are Hermitian matrices that commute

$$[M^{(1)}, M^{(2)}] = 0 \tag{15.22}$$

Use $M^{(1)}$ to decompose $V = \mathbb{C}^{d\times d}$ into a direct sum of vector spaces $\bigoplus_i V_i$. Then we can use $M^{(2)}$ to decompose V_i into $\bigoplus_j V_{i,j}$. If $M^{(1)}$ and $M^{(2)}$ don't commute, let $P_i^{(1)}$ be an eigenvalue projection operator of $M^{(1)}$. Then replace $M^{(2)}$ by $P_i^{(1)} M^{(2)} P_i^{(1)}$. Now

$$
\begin{aligned}
[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] &= \sum_j \lambda_j^{(1)} [P_j^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] && (15.23)\\
&= 0 && (15.24)
\end{aligned}
$$

[1] M must also satisfy some convergence conditions that we won't get into.

15.4 $[G, M] = 0$ Consequences

An invariant matrix (see Ch.7) commutes with all the elements G of a group $\mathcal{G}$

$$[G, M] = 0 \tag{15.25}$$

If P_i are the projection operators of M, then $P_i = f_i(M)$ so

$$[G, P_i] = 0 \tag{15.26}$$

for all $G \in \mathcal{G}$ and i. Hence,

$$G = 1G1 = \sum_i \sum_j P_i G P_j = \sum_j \underbrace{P_j G P_j}_{\stackrel{\text{def}}{=}\; G'_j} \tag{15.27}$$

Since $P_i = C^\dagger \pi_i C$,

$$[CGC^\dagger, \pi_i] = 0 \tag{15.28}$$

Hence

$$CGC^\dagger = 1G1 = \sum_i \sum_j \pi_i CGC^\dagger \pi_j = \sum_j \underbrace{\pi_j CGC^\dagger \pi_j}_{\stackrel{\text{def}}{=}\; G_j} = diag(G_1, G_2, \ldots, G_r) \tag{15.29}$$

Note that

$$C^\dagger G_j C = G'_j \tag{15.30}$$

A rep-matrix G'_i acts only on a d_i dimensional vector space $V^{d_i} = P_i V^d$. In this way, an invariant matrix $M \in \mathbb{C}^{d \times d}$ with r distinct eigenvalues, induces a decomposition of V^d into a direct sum of vector spaces

$$V^d \xrightarrow[M]{} V_1^{d_1} \oplus V_2^{d_2} \oplus \ldots \oplus V_r^{d_r} \tag{15.31}$$

If a rep-matrix G'_i cannot itself be reduced further, it is said to be an **irreducible representation (irrep)**.

Note that sometimes the term representation is used to refer to the vector space $V_i^{d_i}$ instead of the matrix G_i.

Chapter 16

Spinors

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

The reader should read Chapter 11 before reading this chapter. There he will learn about the group $Spin(n)_{\mathbb{R}}$.

Let

$$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \tag{16.1}$$

If $\gamma_\mu^\dagger = \gamma_\mu$ for $\mu = 1, 2, \ldots, n$ (this assumes the Euclidean metric which is $g_{\mu\nu} = diag(1, 1, 1, 1)$ for $n = 4$)

$$Spin(n)_{\mathbb{R}} = \{e^{\omega^{\mu\nu}\gamma_{\mu\nu}} | \omega_{\mu\nu} = -\omega_{\nu\mu} \in \mathbb{R}\}\} \tag{16.2}$$

If $\gamma_0^\dagger = \gamma_0$, and $\gamma_i^\dagger = -\gamma_i$ (this assumes the mostly-minus metric which is $g_{\mu\nu} = diag(1, -1, -1, -1)$ for $n = 4$)

$$Spin(1, 3)_{\mathbb{R}} = \{e^{-i\omega^{\mu\nu}\gamma_{\mu\nu}} | \omega_{\mu\nu} = -\omega_{\nu\mu}, \omega_{0,i} \in \mathbb{R}, \omega_{i,j} \in i\mathbb{R}\} \tag{16.3}$$

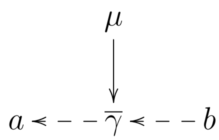
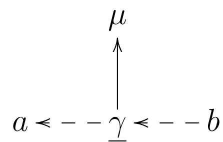
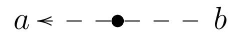
Spinors are the vectors in the vector space upon which the group $Spin(n)_{\mathbb{R}}$ acts.

n	1	2	3	4	5	6	7	8	9	10
$int(n/2)$	0	1	1	2	2	3	3	4	4	5
$d = 2^{int(n/2)}$	1	2	2	4	4	8	8	16	16	32

Table 16.1: $\gamma_\mu \in \mathbb{C}^{d\times d}$

Let $\mu \in \{1, 2, \ldots, n\}$
$int(x)$ = integer part of $x \in \mathbb{R}$
$a, b \in \{1, 2, \ldots, 2^{int(n/2)}\}$
$\gamma_\mu \in \mathbb{C}^{d\times d}$ where $d = 2^{int(n/2)}$ (See Table 16.1)

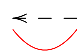
Define the following birdtracks

$\gamma^{\mu}_{ab} =$  , $(\gamma_\mu)_{ab} =$  , $\delta^b_a =$ 

(16.4)

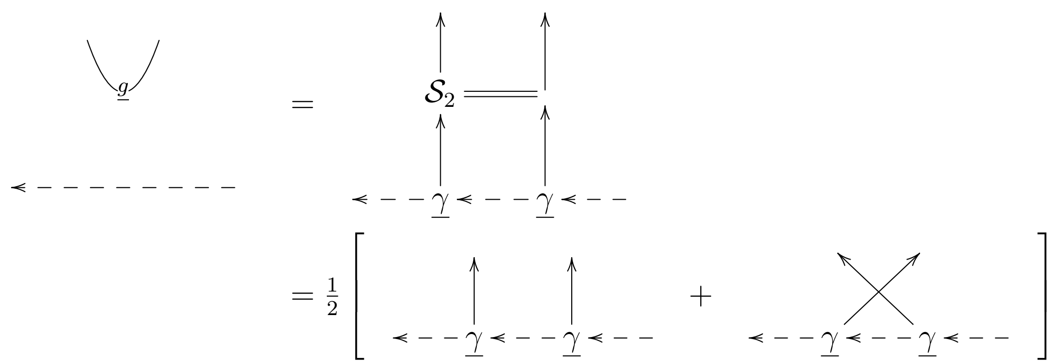
$$g_{\mu\nu} = \mu \longleftarrow \underline{g} \longrightarrow \nu\,, \quad g^{\mu\nu} = \mu \longrightarrow \overline{g} \longleftarrow \nu \tag{16.5}$$

Note that

$g_\mu{}^\mu =$  $= 2^{int(n/2)}$ (16.6)

Clifford Algebra anticommutator:

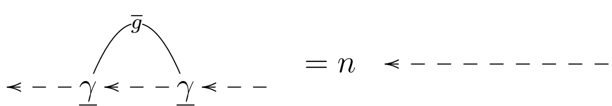
$$\boxed{g_{\mu,\nu}\mathbb{1} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]_+}$$



(16.7)

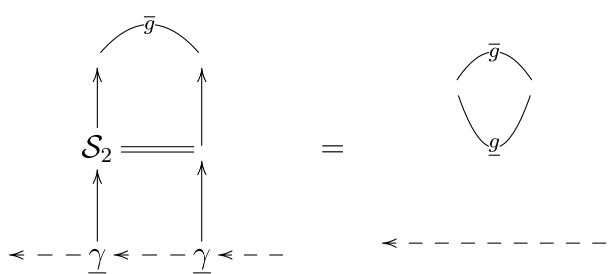
By virtue of Eq.(16.7), it is possible to replace all g's by swaps, and vice versa, in a linear combination of birdtracks.

Claim 36



(16.8)

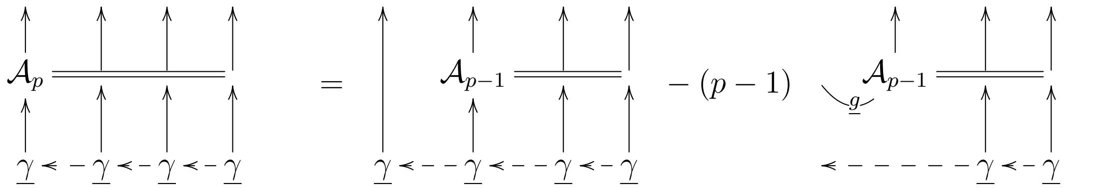
proof:



(16.9)

QED

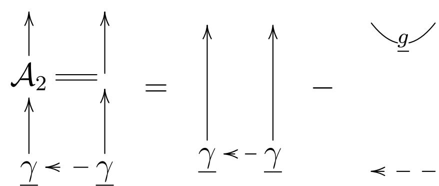
Claim 37



(16.10)

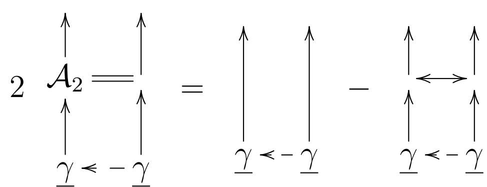
proof:

The claim for $p = 2$ is



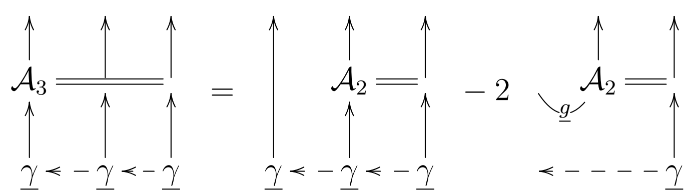
(16.11)

where we define

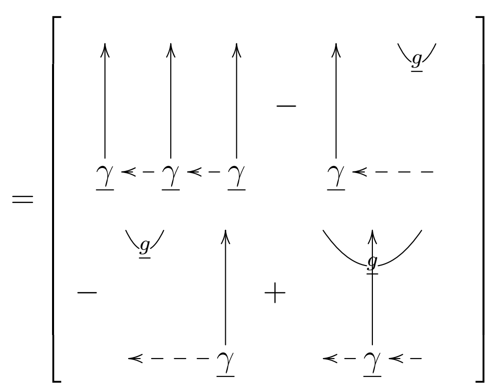


(16.12)

The claim for $p = 3$ is

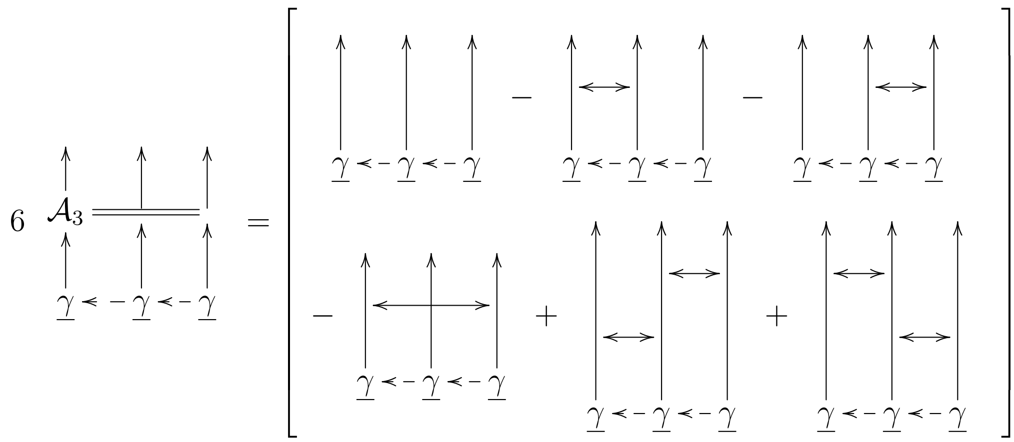


(16.13)



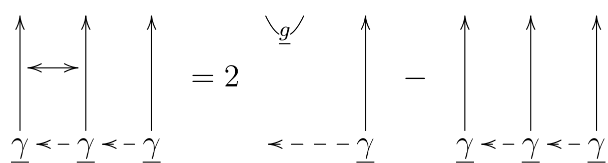
(16.14)

where we define

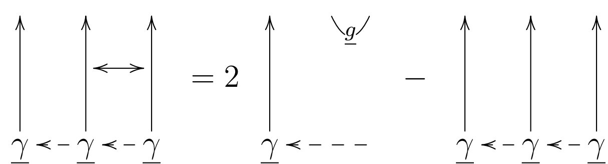


(16.15)

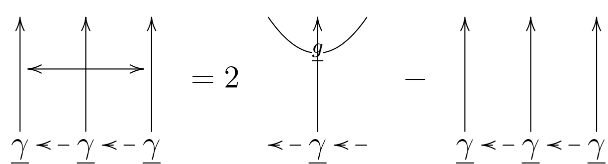
Now let us express each of the 6 birdtracks on the right hand side of Eq.(16.15) in terms of g's instead of swaps.



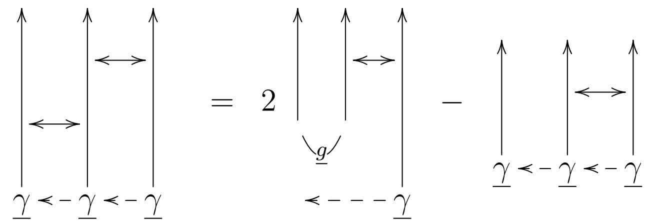
(16.16)



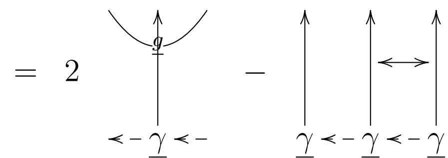
(16.17)



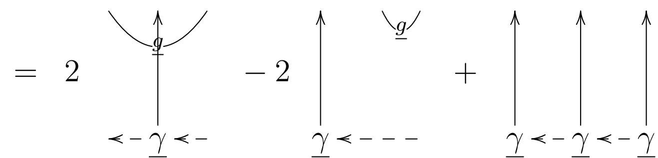
(16.18)



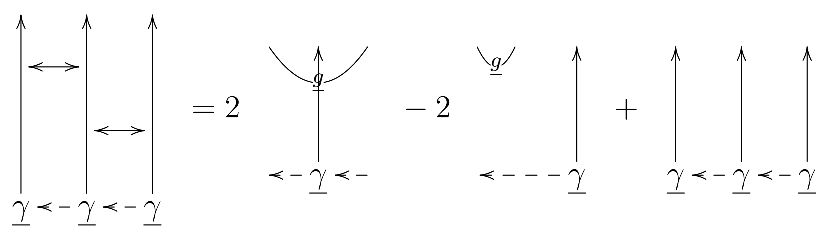
(16.19)



(16.20)



(16.21)



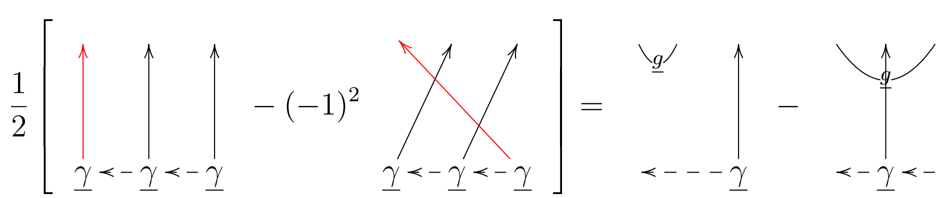
(16.22)

QED

Claim 38

$$\frac{1}{2}\left[\gamma_\nu \gamma_{\mu_1} \dots \gamma_{\mu_a} - (-1)^a \gamma_{\mu_1} \dots \gamma_{\mu_a} \gamma_\nu\right] = \sum_{k=1}^{a} (-1)^{k-1} g_{\nu\mu_k} \gamma_{\mu_1} \dots \widehat{\gamma_{\mu_k}} \dots \gamma_{\mu_a} \tag{16.23}$$

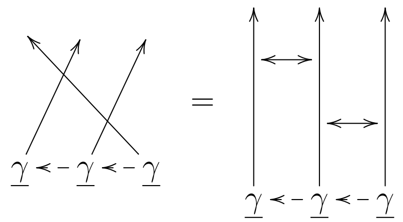
where a hat means omission. For $a = 2$,



(16.24)

proof:

Note that



(16.25)

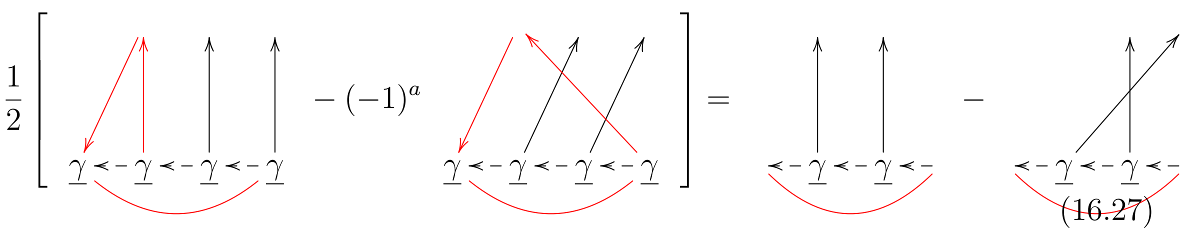
so this claim has been proven before for $a = 2$ in Eq.(16.22)
QED

Claim 39 *If $n \neq a$ and a is odd*

$$\mathrm{tr}(\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_a}) = 0 \tag{16.26}$$

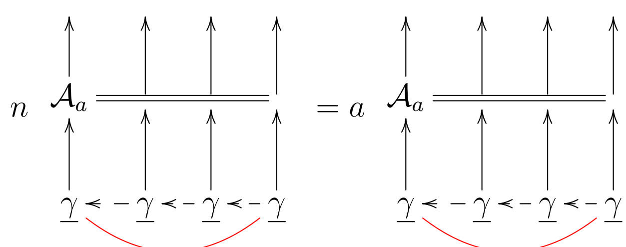
proof:

If we left-multiply both sides of Eq.(16.24) by γ_{μ_1} where μ_1 is the first index, and then we trace over the spinor indices, we get



(16.27)

Assuming a is odd, we get



(16.28)

Hence, if $n \neq a$ and a is odd, the trace of a gammas is zero.

The above proof is now complete and it works for odd or even n. However, let us mention that for even n, a chirality operator Γ_5, exists, and this claim can be proven using Γ_5. Indeed, for n even, let

$$\Gamma_5 = K\gamma_1\gamma_2\ldots\gamma_n \tag{16.29}$$

Γ_5 anti-commutes with all γ_μ:

$$\Gamma_5\gamma_\mu\Gamma_5^{-1} = -\gamma_\mu \tag{16.30}$$

Hence

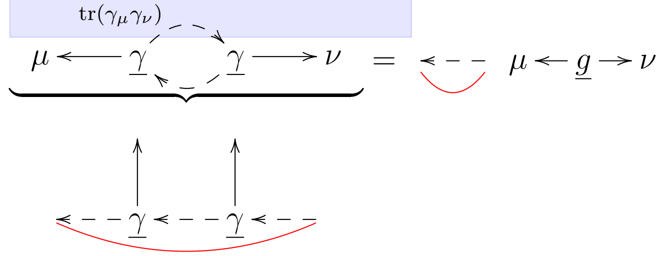
$$
\begin{aligned}
\mathrm{tr}(\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_a}) &= \mathrm{tr}(\Gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_a}\Gamma_5^{-1}) && (16.31)\\
&= (-1)^a\mathrm{tr}(\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_a}) && (16.32)
\end{aligned}
$$

So for odd a, $\mathrm{tr}(\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_a}) = 0$.

QED

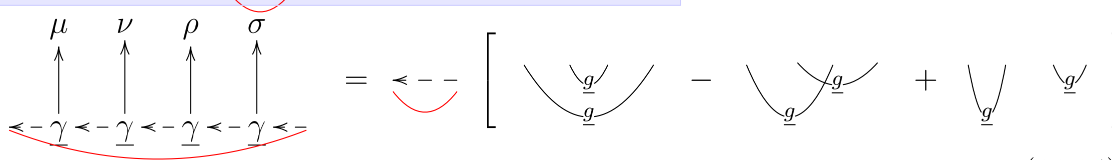
Claim 40

$$\boxed{\underbrace{\frac{1}{2}\mathrm{tr}([\gamma_\mu,\gamma_\nu]_+)}_{\mathrm{tr}(\gamma_\mu\gamma_\nu)} = \quad g_{\mu\nu}}$$



(16.33)

$$\boxed{\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = \quad (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\sigma\rho})}$$

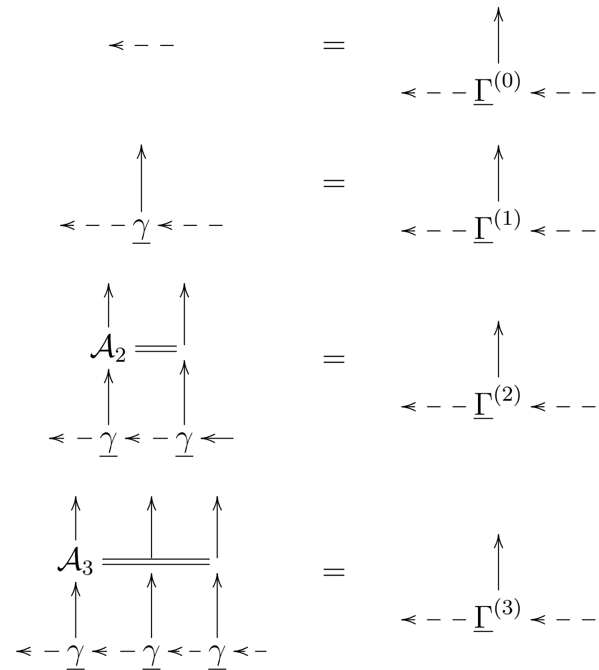


(16.34)

proof:
QED

Define the antisymmetrized gamma matrix products $\Gamma^{(k)}$ by:

$$\Gamma^{(0)} = 1, \qquad \Gamma^{(1)}_{\mu} = \gamma_\mu, \qquad \Gamma^{(2)}_{\mu\nu} = \tfrac{1}{2}[\gamma_\mu, \gamma_\nu], \qquad \Gamma^{(3)}_{\lambda\mu\nu} = \tfrac{1}{6}[\gamma_\lambda, \gamma_\mu, \gamma_\nu]$$

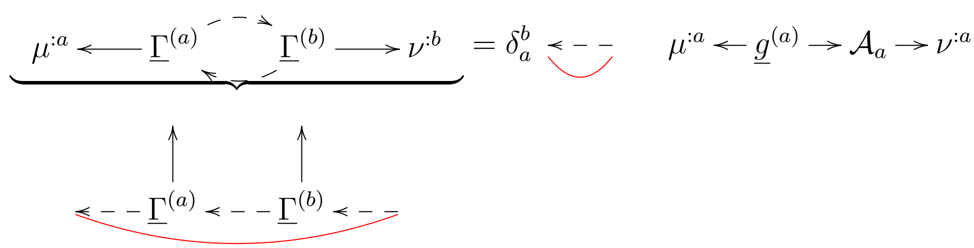


(16.35)

Henceforth, let

$$\underline{g}^{(t)} = \prod_{i=1}^{t} g_{\mu_i \nu_i} \tag{16.36}$$

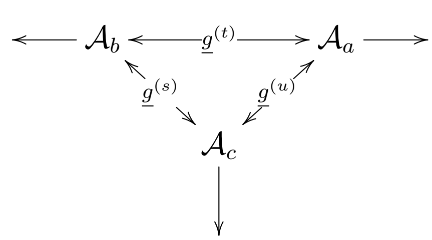
Claim 41



(16.37)

proof:
QED

Define χ_c by

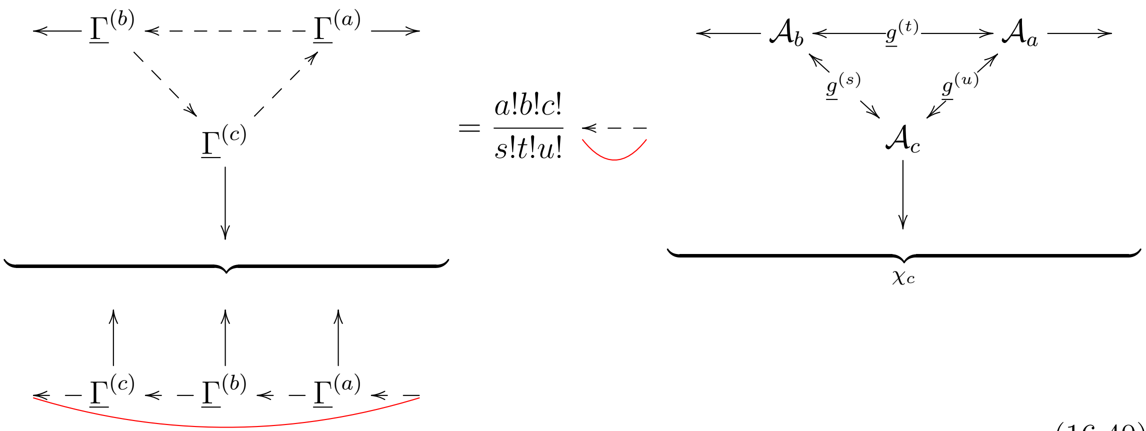
$\chi_c =$ 

(16.38)

where

$$
\begin{aligned}
s &= (b+c-a)/2 \\
t &= (b+a-c)/2 \\
u &= (a+c-b)/2
\end{aligned}
\tag{16.39}
$$

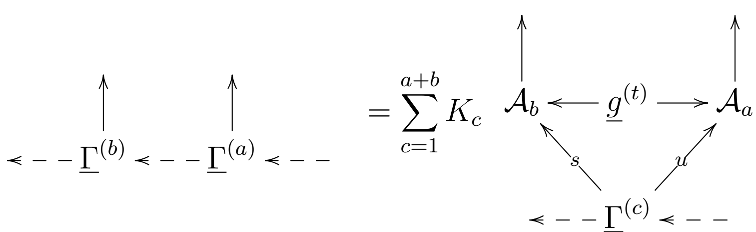
Claim 42



(16.40)

proof:
QED

Claim 43



(16.41)

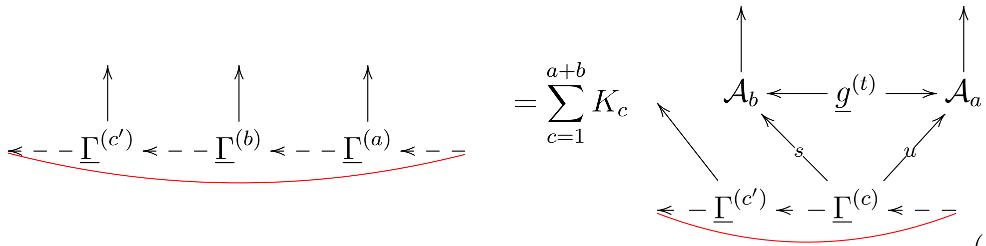
where $t = a - u = b - s$ *and*

$$
K_c = \frac{a!b!c!}{s!t!u!}
\tag{16.42}
$$

proof:

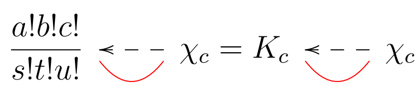
Eq.(16.41) follows because the $\Gamma^{(c)}$ span the vector space of $\Gamma^{(c)}$ products.

If we left-multiply both sides of Eq.(16.41) by $\Gamma^{(c')}$ and then we trace over the spinor indices, we get



(16.43)

so K_c satisfies



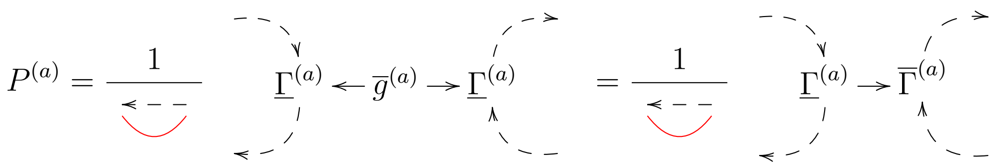
(16.44)

Hence,

$$K_c = \frac{a!b!c!}{s!t!u!} \tag{16.45}$$

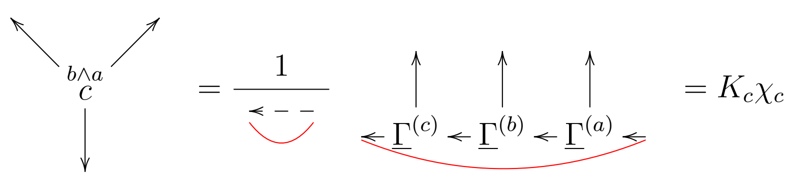
QED

Define the projector $P^{(a)}$ by



(16.46)

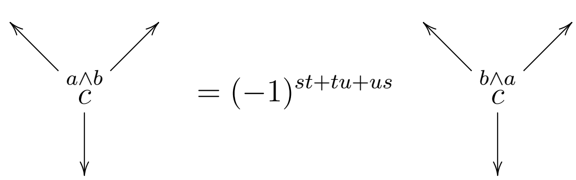
and the 3 leg vertex $\overset{b\wedge a}{c}$ by



(16.47)

Note that the vertex $\overset{b\wedge a}{c}$ is nonzero iff $a+b+c$ is even, and if a, b, and c satisfy the triangle inequalities $|a-b| \le c \le |a+b|$

Claim 44



(16.48)

proof:
QED

Note that

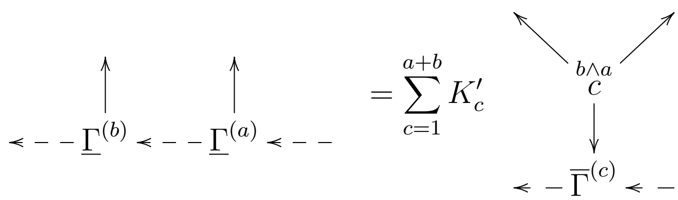
$$d^{(a)} = \mathrm{tr}(P^{(a)}) = \frac{1}{\leftarrow - -}\ \underline{\Gamma}^{(a)} \longrightarrow \overline{\Gamma}^{(a)} \tag{16.49}$$

$$= \longleftarrow \mathcal{A}_a \longleftarrow \tag{16.50}$$

$$= \begin{cases} \binom{n}{a} & \text{if } a \leq n \text{ (see Eq.(20.43))} \\ 0 & \text{otherwise} \end{cases} \tag{16.51}$$

$$\sum_{a=1}^{n} d^{(a)} = \sum_{a=1}^{n} \binom{n}{a} = (1+1)^2 = 2^a \tag{16.52}$$

Claim 45



(16.53)

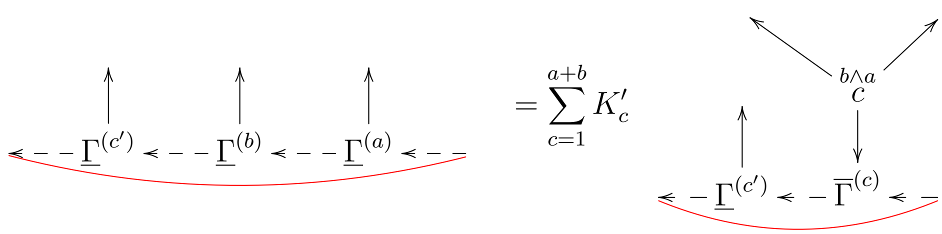
where

$$K'_c = \frac{a!b!c!}{s!t!u!} = K_c \tag{16.54}$$

proof:

Eq.(16.53) follows because the $\Gamma^{(c)}$ span the vector space of $\Gamma^{(c)}$ products.

If we left-multiply both sides of Eq.(16.53) by $\Gamma^{(c')}$ and then we trace over the spinor indices, we get



(16.55)

So K'_c satisfies

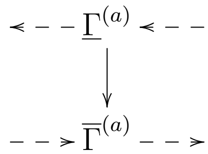
$$\frac{a!b!c!}{s!t!u!} \leftarrow - -\ \chi_c = K'_c \leftarrow - -\ \chi_c \tag{16.56}$$

Hence,

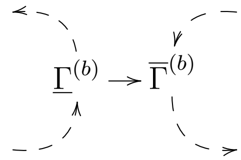
$$K'_c = \frac{a!b!c!}{s!t!u!} = K_c \tag{16.57}$$

QED

Claim 46 *(t-channel to sum of s-channels)*



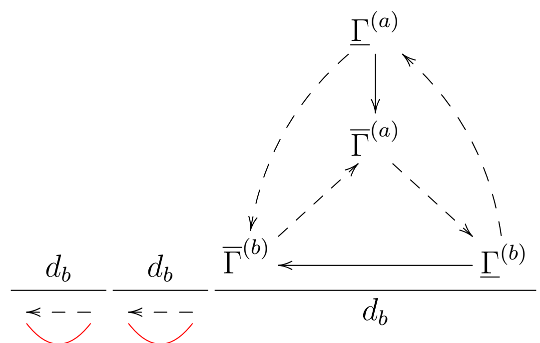
$$= \sum_b \Phi_b$$



(16.58)

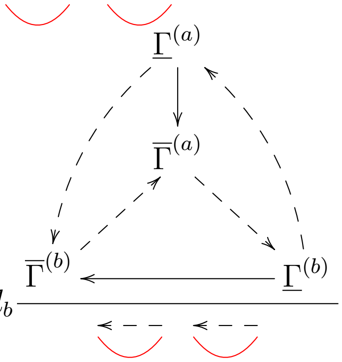
where

$$\Phi_b =$$



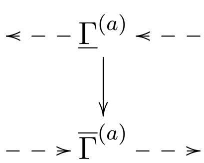
(16.59)

$$= d_b$$

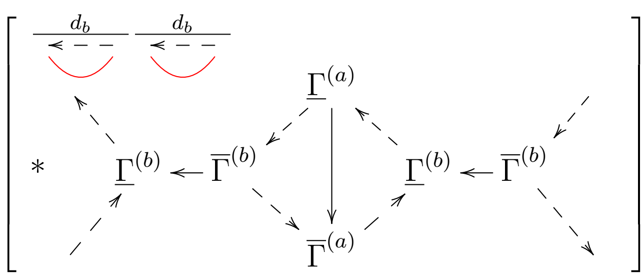


(16.60)

proof:

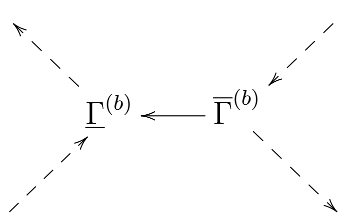


$$= \sum_b$$



(16.61)

$$= \sum_b \Phi_b$$



(16.62)

QED

Chapter 17

Spinors, Their Handedness

This chapter is based on an AI output and Ref.[1].

17.1 In 1+3 dim

Consider $SO(1,3)$ so $n = 1 + 3 = 4$

γ^{μ}_{ab}

$a, b \in \{1, 2, 3, 4\}$

$\mu = 1, 2, \ldots, n$

Mostly-plus metric (M+M) (a.k.a. East Coast metric) is $g_{\mu\nu} = diag(-1, 1, 1, 1)$

Mostly-minus metric (M-M) (a.k.a. West Coast metric) is $g_{\mu\nu} = diag(1, -1, -1, -1)$

$$[\gamma_\mu, \gamma_\nu]_+ = 2g^{\mu\nu} \quad \text{for } \mu, \nu = 0, 1, 2, 3 \tag{17.1}$$

$$\begin{array}{ll} \text{M–M}: & \gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i \quad \text{for } i = 1, 2, 3 \\ \text{M+M}: & \gamma_0^\dagger = -\gamma_0, \quad \gamma_i^\dagger = \gamma_i \quad \text{for } i = 1, 2, 3 \end{array} \tag{17.2}$$

The last equation is true iff

$$\begin{array}{ll} \text{M–M:} & \gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \\ \text{M+M:} & \gamma_\mu^\dagger = -\gamma_0 \gamma_\mu \gamma_0 \end{array} \tag{17.3}$$

In 1+3 dimensional relativistic quantum field theory, the **chirality operator** γ_5 is defined in terms of the four Dirac gamma matrices γ_μ, where $\mu = 0, 1, 2, 3$, as follows

$$\begin{array}{ll} \text{M–M:} & \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \\ \text{M+M:} & \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \end{array} \tag{17.4}$$

In the "Dirac representation",

$$\gamma_5 = \pm \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \tag{17.5}$$

The properties of γ_5 might change by a sign depending on which metric we are using. Henceforth, we will use the M–M

γ_5 properties:

- Anticommutes with γ_μ

$$\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5 \quad \text{for } \mu = 0, 1, 2, 3 \tag{17.6}$$

  This follows because $g_{\mu\nu}$ is diagonal so, by the Clifford algebra definition Eq.(17.1) (anti-cummutator of gammas), the γ_μ anticommutes with 3 of the 4 gamma matrices in γ_5

- Hermitian

$$\gamma_5^\dagger = \gamma_5 \tag{17.7}$$

  This follows because

$$\gamma_5^\dagger = -i(-\gamma_3)(-\gamma_2)(-\gamma_1)(+\gamma_0) = i\gamma_3\gamma_2\gamma_1\gamma_0 = (-1)^6 i(\gamma_0\gamma_1\gamma_2\gamma_3) = \gamma_5 \tag{17.8}$$

- Square is 1

$$\gamma_5^2 = I. \tag{17.9}$$

  Therefore its eigenvalues are $+1$ (positive chirality) and -1 (negative chirality). This follows because

$$(\gamma_5)^2 = -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 = -(-1)^6(\gamma_0)^2(\gamma_1)^2(\gamma_2)^2(\gamma_3)^2 = -g_{00}g_{11}g_{22}g_{33} = 1 \tag{17.10}$$

- Traceless

$$\mathrm{tr}(\gamma_5) = 0 \tag{17.11}$$

  This follows immediately from the expression for γ_5 in the Dirac representation. Alternatively, note that if spinor ψ satisfies

$$\gamma_5\psi = +\psi \tag{17.12}$$

  (has positive chirality), then spinor $\gamma_\mu\psi$ has negative chirality because

$$\gamma_5(\gamma_\mu\psi) = -\gamma_\mu\gamma_5\psi = -\gamma_\mu\psi \tag{17.13}$$

  This means that the eigenvalues of γ_5 come in ± 1 pairs. Hence the trace of γ_5 must be zero.

- $$\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}, \tag{17.14}$$

  with $\epsilon^{0123} = +1$

- If
  $$P_L = P_- = \frac{1}{2}(1-\gamma_5), \qquad P_R = P_+ = \frac{1}{2}(1+\gamma_5). \tag{17.15}$$
  where R= right handed, L = left handed then
  $$\psi = \underbrace{\psi_L}_{P_L\psi} + \underbrace{\psi_R}_{P_R\psi} \tag{17.16}$$

- If $\mathbb{P} = \gamma_0$ (Parity Operator) and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ (Lorentz generators)
  $$[\gamma_5, \sigma^{\mu\nu}] = 0, \quad \gamma_5\mathbb{P} = -\mathbb{P}\gamma_5 \tag{17.17}$$
  Thus γ_5 is **pseudoscalar** under Lorentz transformations (i.e., invariant under proper Lorentz transformations but flips sign under parity).

17.2 In $p+q$ dim

Consider $SO(p,q)$ so $n = p+q$

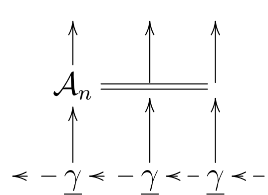
γ^μ_{ab}
$int(x)$ = integer part of $x \in \mathbb{R}$
$a, b \in \{1, 2, \dots, 2^{int(n/2)}\}$
$\mu = 1, 2, \dots, n$

n	1	2	3	4	5	6	7	8	9	10
$int(n/2)$	0	1	1	2	2	3	3	4	4	5
$d = 2^{int(n/2)}$	1	2	2	4	4	8	8	16	16	32

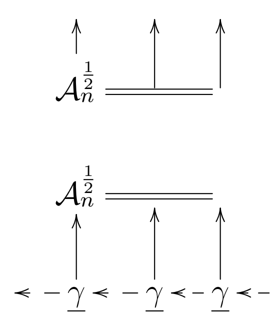
Table 17.1: $\gamma_\mu \in \mathbb{C}^{d\times d}$

$$[\gamma_\mu, \gamma_\nu]_+ = 2g^{\mu\nu} \quad \text{for } \mu, \nu = 0, 1, 2, \dots, n \tag{17.18}$$

$\Gamma^{(3)}_{\lambda\mu\nu} =$



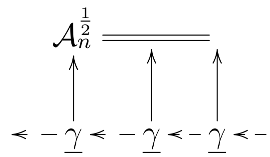
$=$



(17.19)

Generalize $\gamma_5 \to \Gamma_5$, from $SO(1,3)$ to $SO(p,q)$.

$$\Gamma_5 = \frac{1}{\sqrt{n!}} \tag{17.20}$$



$$= \frac{e^{i\phi}}{n!}\epsilon^{\mu_1\mu_2\ldots\mu_n}\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_n} = e^{i\phi}\gamma_1\gamma_2\ldots\gamma_n \tag{17.21}$$

Γ_5 properties

- For n odd, Γ_5 is a constant.

  The Lorentz group for odd n has only one irreducible spinor representation. For even n, it has two inequivalent spinor irreps (positive and negative chirality).

  Thus chirality does not exist in odd dimensions because the representation theory of the Lorentz group does not support it.

- Γ_5 anticommutes with γ_μ if n is even. Commutes with γ_μ if n is odd.

$$\begin{cases} \gamma_\mu\Gamma_5 = -\Gamma_5\gamma_\mu & (n \text{ even}) \\ \gamma_\mu\Gamma_5 = \Gamma_5\gamma_\mu & (n \text{ odd}) \end{cases} \tag{17.22}$$

  This follows because $g_{\mu\nu}$ is diagonal so, by the Clifford algebra definition Eq.(17.18) (anti-cummutator of gammas),

  - for n even, the γ_μ anticommutes with $n-1$ of the n gamma matrices in Γ_5
  - for n odd, Γ_5 is a constant

- Square is one

  If

$$e^{i\phi} = i^{\frac{n(n-1)}{2}}\sqrt{\prod_{\mu=1}^{n} g_{\mu\mu}} \tag{17.23}$$

  with $g_{\mu\mu} \in \{1, -1\}$, then

$$\Gamma_5^2 = 1 \tag{17.24}$$

  This follows because

$$\begin{aligned} \Gamma_5^2 &= e^{i2\phi}\gamma_1\gamma_2\cdots\gamma_n\gamma_1\gamma_2\cdots\gamma_n && (17.25)\\ &= e^{i2\phi}(-1)^{\frac{n(n-1)}{2}}(\gamma_1)^2(\gamma_2)^2\ldots(\gamma_n)^2 && (17.26)\\ &= e^{i2\phi}(-1)^{\frac{n(n-1)}{2}}\prod_{\mu=1}^{n} g_{\mu\mu} && (17.27) \end{aligned}$$

- If n is even and

$$P_{\pm} = \frac{1}{2}(1 \pm \Gamma_5), \tag{17.28}$$

then

$$P_+^2 = P_-^2 = 1, \quad P_+P_- = 0, \quad P_+ + P_- = 1 \tag{17.29}$$

$$\gamma_\mu P_+ = P_- \gamma_\mu \tag{17.30}$$

$$\gamma_\mu = P_+ \gamma_\mu P_- + P_- \gamma_\mu P_+ \tag{17.31}$$

17.3 Weyl and Majorana Spinors

Spinor Type	Condition	# of dofs in 4D	Exists in 4D?
Dirac	none	8	yes
Weyl	$\Gamma_5 \psi = \pm\psi$	4	yes
Majorana	$\psi^c = \psi$	4	yes
Majorana-Weyl	both	0	no

Table 17.2: Different types of spinors, their number of dofs (real degrees of freedom) in 4D and whether they exist in 4D.

- Weyl spinors

$$P_L = \frac{1}{2}(1 - \Gamma_5), \quad P_R = \frac{1}{2}(1 + \Gamma_5) \tag{17.32}$$

A **left-handed Weyl spinor** satisfies

$$\psi_L = P_L \psi, \quad \Gamma_5 \psi_L = -\psi_L \tag{17.33}$$

A **a right-handed Weyl spinor** satisfies

$$\psi_R = P_R \psi, \quad \Gamma_5 \psi_R = +\psi_R. \tag{17.34}$$

- Majorana Spinors

The **charge conjugation matrix** C is defined by

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T, \tag{17.35}$$

and exists in any dimension.

Given a Dirac spinor ψ, its **charge-conjugate** is

$$\psi^c = C\bar{\psi}^T \quad \text{where } \bar{\psi} = \psi^\dagger \gamma^0 \tag{17.36}$$

A **Majorana spinor** is one satisfying

$$\psi = \psi^c \tag{17.37}$$

Whether Majorana spinors exist depends on the spacetime dimension and signature. In 1+3 dimensions, Majorana spinors do exist because one can choose a representation where all gamma matrices are purely real or imaginary so that ψ can be chosen real.

A Dirac spinors has 4 complex components. A Majorana spinor has 4 real components.

The Weyl spinor and Majorana spinor constraints cannot both be imposed simultaneously in 1+3 dim. But in some dimensions, Majorana–Weyl spinors **do** exist. (see Table 17.3)

n	Dirac spinor	Weyl spinor	Majorana spinor	Majorana-Weyl spinor
1	Yes, $\dim_{\mathbb{R}} = 2$	No	No	No
2	Yes, $\dim_{\mathbb{R}} = 4$	Yes, $\dim_{\mathbb{R}} = 2$	Yes, $\dim_{\mathbb{R}} = 2$	Yes, $\dim_{\mathbb{R}} = 1$
3	Yes, $\dim_{\mathbb{R}} = 4$	No	Yes, $\dim_{\mathbb{R}} = 2$	No
4	Yes, $\dim_{\mathbb{R}} = 8$	Yes, $\dim_{\mathbb{R}} = 4$	Yes, $\dim_{\mathbb{R}} = 4$	No
5	Yes, $\dim_{\mathbb{R}} = 8$	No	No	No
6	Yes, $\dim_{\mathbb{R}} = 16$	Yes, $\dim_{\mathbb{R}} = 8$	No	No
7	Yes, $\dim_{\mathbb{R}} = 16$	No	No	No
8	Yes, $\dim_{\mathbb{R}} = 32$	Yes, $\dim_{\mathbb{R}} = 16$	No	No

Table 17.3: Spinor types in Lorentzian signature $(1, n-1)$ (mostly minus). Note that Majorana-Weyl spinors are possible for $n = 2, 10$ (10%8=2). This is why superstring theories are 10 dimensional.

Chapter 18

Squashed Entanglement: COMING SOON

Chapter 19

Symplectic Groups

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Throughout this chapter, we will assume that n is even. The **symplectic group** $Sp(n)$ is defined as

$$Sp(n) = \{G \in GL(n, \mathbb{C}) : G^\dagger G = 1, G^T f G = f\} \tag{19.1}$$

where f is the anti-symmetric matrix

$$f = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \tag{19.2}$$

Note that

$$f^T = -f, \quad f^T f = I_n, \quad f^2 = -I_n \tag{19.3}$$

f^2 is a primitive invariant matrix because f is, so f^2 must be proportional to the identity.

Claim 47 *If $G \in Sp(n)$, then $det(G) = 1$.*

proof:

Note that

$$\underbrace{det(G^T f G)}_{det^2(G) det(f)} = det(f) \tag{19.4}$$

$$det(f) = det(I_{n/2}) det(-I_{n/2}) = (-1)^{n/2} \tag{19.5}$$

Hence,

$$det^2(G) = 1 \implies det(G) = \pm 1 \tag{19.6}$$

$Sp(n)$ is connected, $I_n \in Sp(n)$ and $det(I_n) = 1$. Hence $det(G) = 1$.

QED

Define the **pseudo metric tensor** f_{ab} to be a antisymetric matrix that satisfies:

$$f_{ab} = -f_{ba} = [f]_{ab}, \quad f^{ab} = -f^{ba} = [f]_{ab}, \quad f_a{}^b = f_b{}^a = \delta_a^b \tag{19.7}$$

$$f_{ba}x^a = x_b, \quad (f^T)^{cb}x_b = x^c \quad (\text{so } f_{ba}(f^T)^{cb} = \delta_a^c) \tag{19.8}$$

where $a, b, c \in \{1, 2, \ldots, n\}$ and x_a is any tensor.

$Sp(n)$ leaves invariant the following skew symmetric quadratic form:

$$h(x) = f_{ab}x^a x^b \tag{19.9}$$

where $a, b \in \{1, \ldots, n\}$. Thus

$$h(Gx) = h(x) \tag{19.10}$$

$$f_{ab}G^a{}_{a'}G^b{}_{b'}x^{a'}x^{b'} = f_{a'b'}x^{a'}x^{b'} \implies f_{ab}G^a{}_{a'}G^b{}_{b'} = f_{a'b'} \implies G^T f G = f \tag{19.11}$$

In this chapter (and in this book), we will point the arrows in a birdtrack so that the birdtrack is a DAG. Cycles that make the birdtrack not acyclic will have a segment in red. Without that red segment, the birdtrack becomes acyclic. The reason we follow this arrow convention is that it promotes acyclic birdtracks which are more akin to bnets. We will eschew undirected birdtracks for the same reason: bnets are directed.

Let

$$\boxed{f_a{}^b = \delta_a^b}, \qquad \longleftarrow f \longleftarrow \;=\; \longleftarrow \tag{19.12}$$

$$\boxed{f^a{}_b = \delta_b^a}, \qquad \longrightarrow f \longrightarrow \;=\; \longrightarrow \tag{19.13}$$

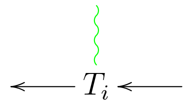
$$\boxed{f_{ac}(f^T)^{cb} = \delta_a^b} \qquad \longleftarrow \underline{f} \longrightarrow \overline{f}^T \longleftarrow \;=\; \longleftarrow \tag{19.14}$$

Note that we used

$$\underline{f} = [f_{ab}], \quad \overline{f} = [f^{ab}], \quad f^T = -f \tag{19.15}$$

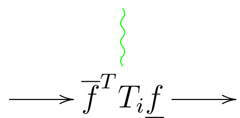
We could write Eq.(19.14) without the overline and underline on f. Those f-decorations are redundant as omitting them would not introduce any ambiguity. However, we will use them because they make spotting errors in the arrow directions easier.

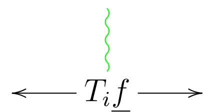
The generators of symplectic groups will be represented by:

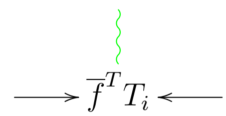
$$(T_i)_a{}^b =$$ 

(19.16)

We will also use

$(T_i)^a{}_b =$ 

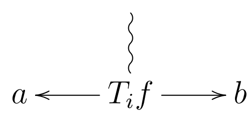
$(T_i)_{ab} =$ 

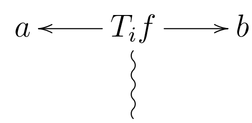
$(T_i)^{ab} =$ 

(19.17)

For $G \in Sp(n)$, $f^T G^T f G = 1$ with $G = e^{iT_i\epsilon_i}$ where $\epsilon_i \in \mathbb{R}$. Hence, the generators T_i must satisfy

$$\underbrace{f^T T_i^T f}_{(f^T T_i f)^T} = -T_i \implies T_i^T = f T_i f \implies (T_i f)^T = T_i f \tag{19.18}$$

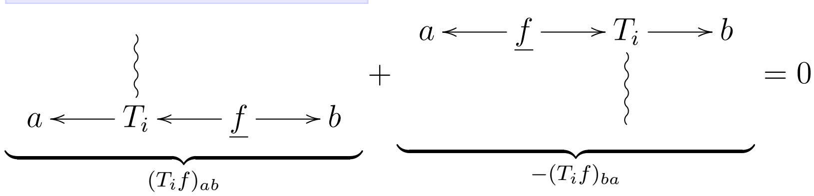
$\boxed{(T_i f)_{ab} = (T_i f)_{ba}}$ 

$=$ 

(19.19)

$f_a^b = \delta_a^b$ is obviously an invariant matrix. f_{ab} must be invariant too, so

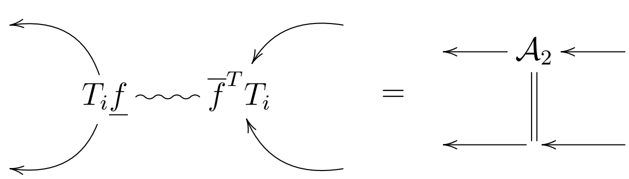
$$\boxed{\underbrace{(T_i)_a{}^c f_{cb} + (T_i)_b{}^c f_{ac} = 0}_{(T_i f)_{ab} = (T_i f)_{ba}}}$$



(19.20)

Hence, the invariance condition Eq.(19.20) reduces to to the statement that $(T_i f)_{ab}$ is symmetric.

The anti-symmetrizer $\mathcal{A}_2$ is an invariant tensor (see Section 20.3). Other projectors of the $V \otimes V$ are not invariant tensors. Therefore, we must have



(19.21)

For $SO(n)$ and $O(n)$, the dimension N of the adjoint rep (= number of generators) is

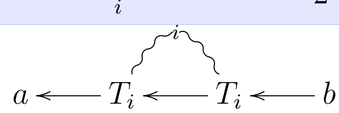
$$N = \frac{n(n-1)}{2} = \tag{19.22}$$

If you take an $n \times n$ matrix and remove its diagonal, this N is the number of entries in the upper (or lower) triangular sector of the matrix. Recall that for $U(n)$, $N = n^2$, and for $SU(n)$, $N = n^2 - 1$. So for $U(n)$ (or $SU(n)$), there is a generator for each entry (or each entry minus one) of an $n \times n$ matrix.

Claim 48

$$\boxed{\Gamma_{fun}\delta_a^b = \sum_i (T_i T_i)_a{}^b = \frac{n-1}{2}\delta_a^b} \tag{19.23}$$

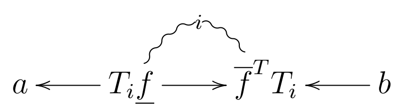
$\sum_i$



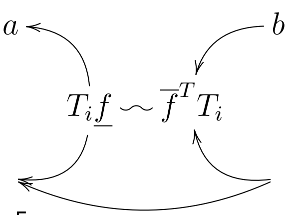
$$= \left(\tfrac{n-1}{2}\right) a \leftarrow\!\bullet\!\!-\!\!- b$$

proof:

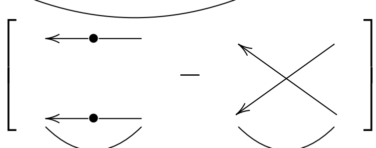
$$(T_i T_i)_a{}^b = \tag{19.24}$$



$$= \tag{19.25}$$



$$= \frac{1}{2}\Big[\quad - \quad \Big] \tag{19.26}$$



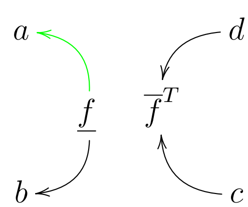
$$= \left(\frac{n-1}{2}\right) a \leftarrow\!\bullet\!\!-\!\!- b \tag{19.27}$$

QED

19.1 $V_{def} \otimes V_{def}$ Decomposition

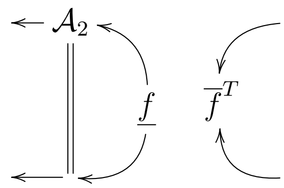
Define

$$M_{ab}{}^{cd} = \tag{19.28}$$

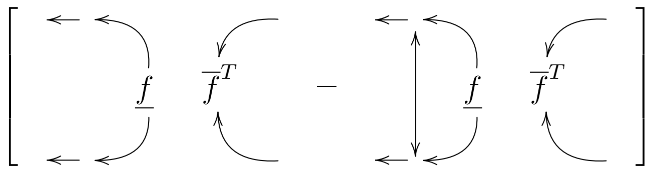


Note that M is antisymmetric:

$$\mathcal{A}_2 M \quad = \tag{19.29}$$



$$= \quad \frac{1}{2} \tag{19.30}$$



$$= \quad M \tag{19.31}$$

Since M is ant-symmetric, only the anti-symmetric space decomposes.

Note also that

$$M^2 = nM \tag{19.32}$$

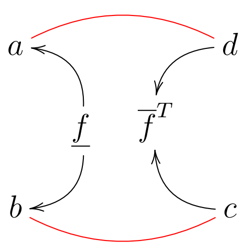
Hence, $(M - n)M = 0$ so M has two eigenvalues $\lambda = 0, n$.

Next we will use the following equation from Chapter 15 [1] to obtain a projection (PO) operator for each eigenvalue

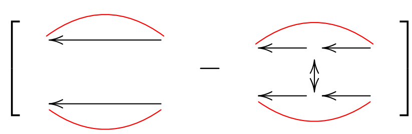
$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \tag{19.33}$$

Below, we use the following traces to evaluate the traces of our projection operators

$$\mathrm{tr}(M) = \quad = \mathrm{tr}(\underline{f}\,\overline{f}^T) = n \tag{19.34}$$

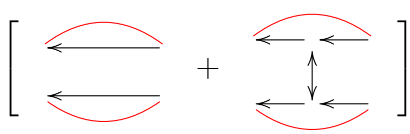


$$\mathrm{tr}(\mathcal{A}_2) \quad = \quad \frac{1}{2} \tag{19.35}$$



$$= \quad \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n-1) \tag{19.36}$$

$$\mathrm{tr}(\mathcal{S}_2) \quad = \quad \frac{1}{2} \tag{19.37}$$

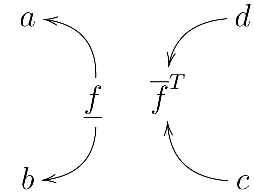


$$= \quad \frac{1}{2}(n^2 + n) = \frac{1}{2}n(n+1) \tag{19.38}$$

[1] Note that this equation projects to zero all eigenvalues except one.

1. Singlet (Anti-symmetric $\mathcal{A}_2 P_S = 0$) PO

$$P_S = \frac{1}{n}\underline{f}_{ab}(\overline{f}^T)^{cd} = \frac{1}{n} \tag{19.39}$$

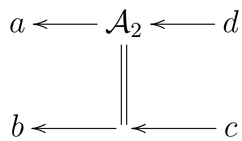


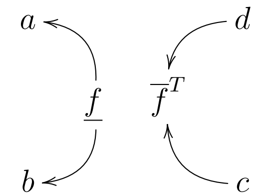
$$\mathrm{tr}(P_S) = \frac{1}{n}\mathrm{tr}(M) = 1 \tag{19.40}$$

2. Traceless Anti-symmetric PO[2]

$$P_{TA} = \frac{1}{2}(\delta_a{}^d\delta_b{}^c - \delta_b{}^d\delta_a{}^c) - \frac{1}{n}\underline{f}_{ab}(\overline{f}^T)^{cd} \tag{19.41}$$

$$= \qquad - \frac{1}{n} \tag{19.42}$$



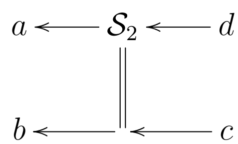


$$\mathrm{tr}(P_{TA}) = \frac{1}{2}(n^2 - n - 2) = \frac{1}{2}(n+1)(n-2) \tag{19.43}$$

3. Symmetric PO

$$P_{SYM} = \frac{1}{2}(\delta_a{}^d\delta_b{}^c + \delta_b{}^d\delta_a{}^c) \tag{19.44}$$

$$= \tag{19.45}$$



$$\mathrm{tr}(P_{SYM}) = \frac{1}{2}(n^2 + n) = \frac{1}{2}n(n+1) \tag{19.46}$$

Clearly,

$$P_{SYM}^2 = P_{SYM}, \quad P_S^2 = P_S \tag{19.47}$$

[2]Traceless here refers to $P_a{}^a{}_c{}^d V_d{}^c = (PV)_a{}^a = 0$ for any vector $V_d{}^c$. It does not refer to $P_a{}^b{}_b{}^a = 0$

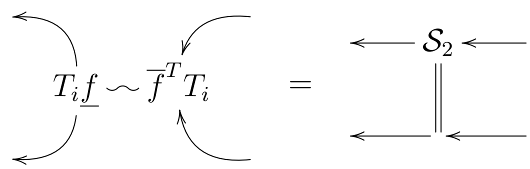
Note that

$$P_{TA}P_S = (\mathcal{A}_2 - P_S)P_S = P_S - P_S = 0 \tag{19.48}$$

Hence

$$P_{TA}^2 = (\mathcal{A}_2 - P_S)^2 = P_{TA} \tag{19.49}$$

P_{SYM} is the only of the 3 POs (P_{SYM}, P_S, P_{TA}) that is an invariant tensor so

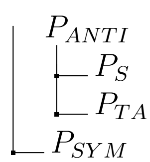


(19.50)

Claim 49 *The Clebsch-Gordan series for $V \otimes V$ (i.e., decomposition of $V \otimes V$) is*

$$\begin{array}{ccccccc} \overbrace{V \otimes V}^{\mathcal{V}} & = & P_S\mathcal{V} & \oplus & P_{SYM}\mathcal{V} & \oplus & P_{TA}\mathcal{V} \\ \square \otimes \square & = & \bullet & \oplus & \square\square & \oplus & \begin{array}{c}\square\\ \square\end{array} \\ n^2 & = & 1 & + & \frac{1}{2}n(n+1) & + & \frac{1}{2}(n+1)(n-2) \end{array} \tag{19.51}$$

The projection operator tree is



proof:
QED

Chapter 20

Symmetrization and Antisymmetrization

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

As preparation for this chapter, read Sec.A.9.

20.1 Symmetrizer

The set of permutations of 2 elements can be represented by the following $2! = 2$ birdtracks.

$$\mathbb{1}_{a_1,a_2}{}^{b_2,b_1} = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2} = \begin{array}{c} a_1 \leftarrow \bullet \leftarrow b_1 \\ \\ a_2 \leftarrow \bullet \leftarrow b_2 \end{array} \tag{20.1}$$

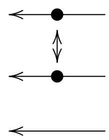
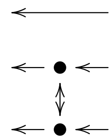
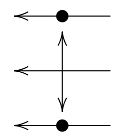
$$(\sigma_{(1,2)})_{a_1,a_2}{}^{b_2,b_1} = \delta^{b_2}_{a_1}\delta^{b_1}_{a_2} = \begin{array}{c} a_1 \leftarrow \bullet \leftarrow b_1 \\ \updownarrow \\ a_2 \leftarrow \bullet \leftarrow b_2 \end{array} \tag{20.2}$$

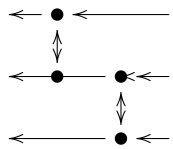
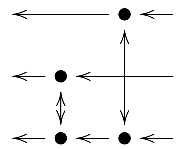
The vertical double headed arrow is called a **swap**. It moves an upstairs particle downstairs and a downstairs particle upstairs.

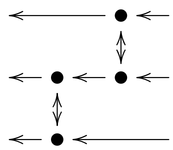
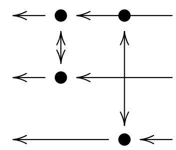
The set of values that a_i and b_i can assume can be anything, as long as, for some set V, $val(\underline{a}_i) = val(\underline{b}_i) = V$ for all i and $|V| = n$.

The set of permutations of 3 elements can be represented by the following $3! = 6$ birdtracks:

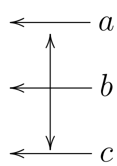
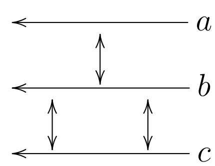
$$\mathbb{1} = \begin{array}{c} a_1 \longleftarrow \bullet \longleftarrow b_1 \\ \\ a_2 \longleftarrow \bullet \longleftarrow b_2 \\ \\ a_3 \longleftarrow \bullet \longleftarrow b_3 \end{array} \tag{20.3}$$

$\sigma_{(1,2)} =$  $\sigma_{(2,3)} =$  $\sigma_{(1,3)} =$  (20.4)

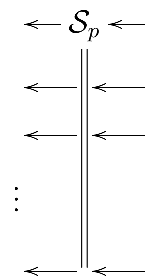
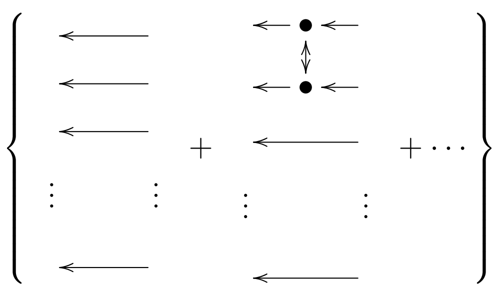
$\sigma_{(1,2,3)} =$  $=$  (20.5)

$\sigma_{(1,3,2)} =$  $=$  (20.6)

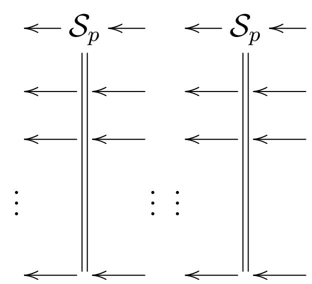
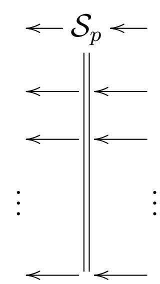
Note that

$\boxed{(c,a) = (b,c)(b,a)(b,c)}$  $=$  (20.7)

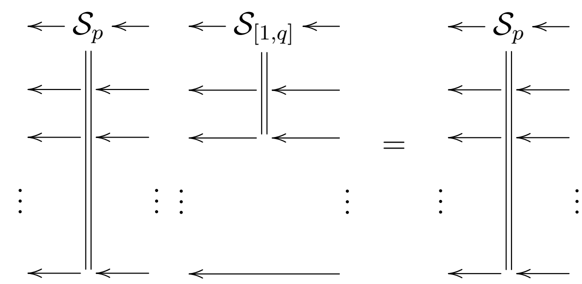
The p**-element symmetrizer** $\mathcal{S}_p$ is defined as the birdtrack

 $= \dfrac{1}{p!}$  (20.8)

Note that $\mathcal{S}_p$ satisfies the following identities:

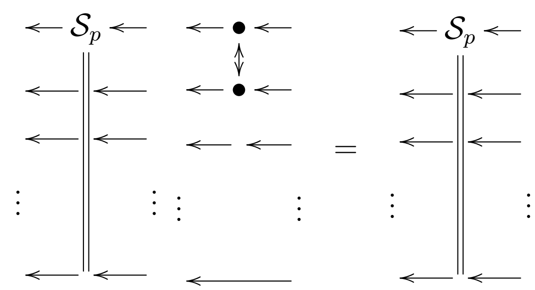
$\boxed{\mathcal{S}_p^2 = \mathcal{S}_p}$  $=$  (20.9)

$\boxed{\mathcal{S}_p \mathcal{S}_{[1,q]} = \mathcal{S}_p}$



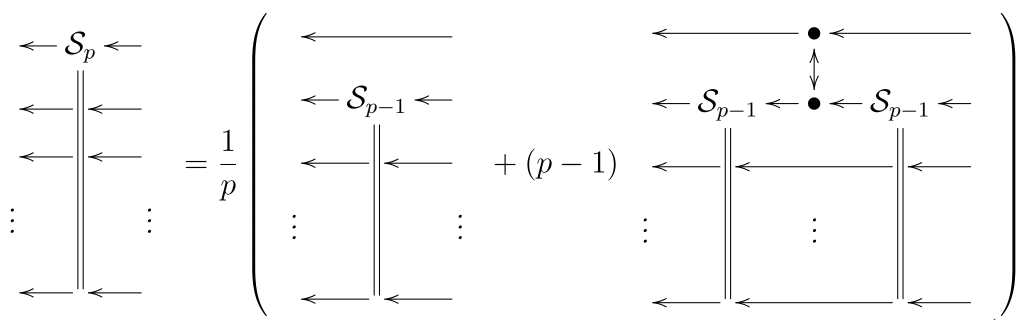
(20.10)

$\boxed{\mathcal{S}_p \sigma_{(1,2)} = \mathcal{S}_p}$



(20.11)

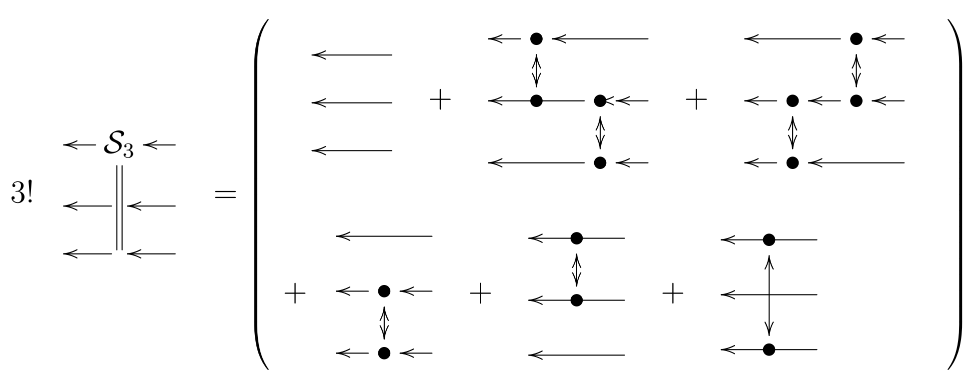
Claim 50



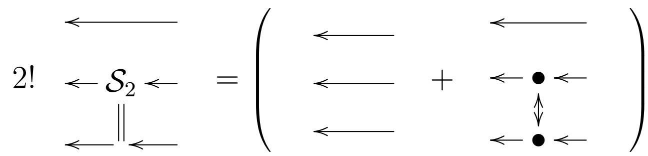
(20.12)

proof:

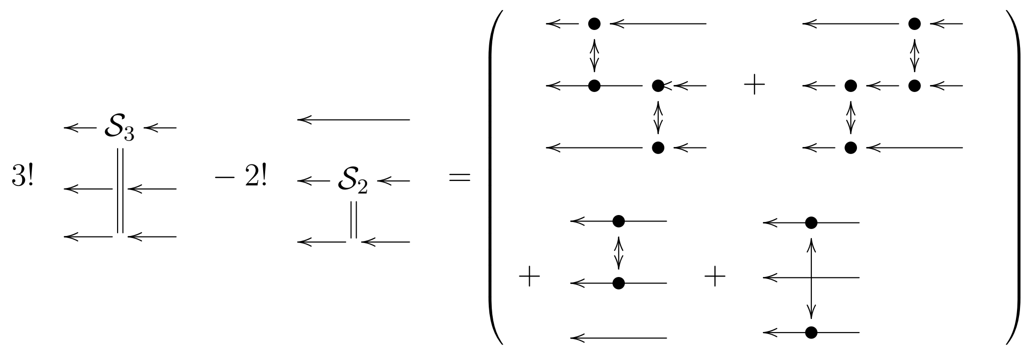
We only prove it for $p = 3$.



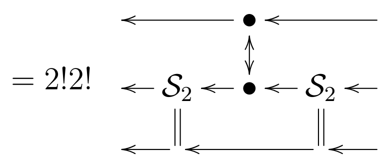
(20.13)



(20.14)



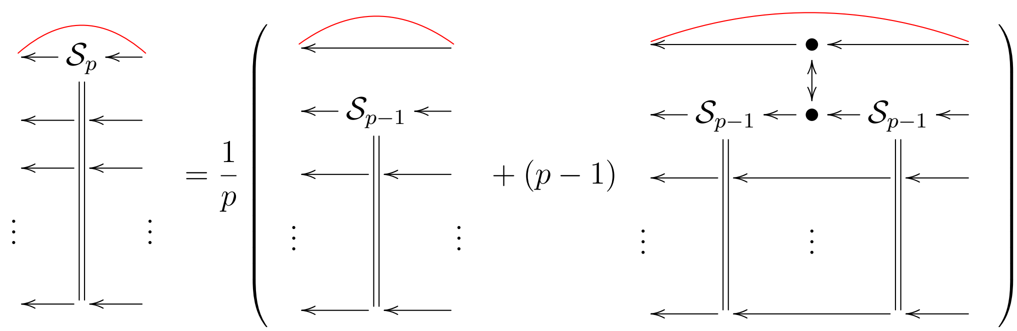
(20.15)



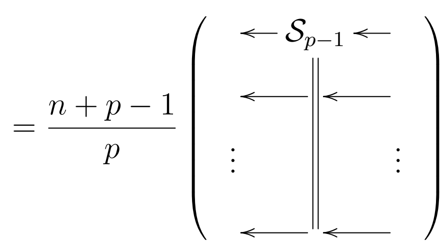
(20.16)

QED

Tracing over the identity of Claim 50, we get



(20.17)



(20.18)

Hence

$$\mathrm{tr}_{\underline{a}_1}\mathcal{S}_p = \frac{n+p-1}{p}\mathcal{S}_{p-1} \tag{20.19}$$

$$\mathrm{tr}_{\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_k}\mathcal{S}_p = \frac{(n+p-1)(n+p-2)\ldots(n+p-k)}{p(p-1)\ldots(p-k+1)}\mathcal{S}_{p-k} \tag{20.20}$$

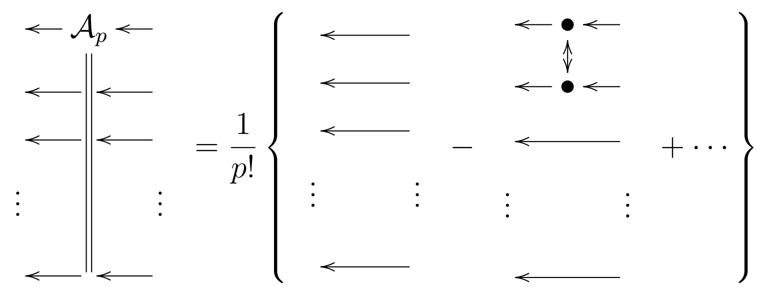
$$d_{\mathcal{S}_p} = \mathrm{tr}_{\underline{a}^p}\mathcal{S}_p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \tag{20.21}$$

For $p = 2$,

$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \tag{20.22}$$

20.2 Antisymmetrizer

The p-**element antisymmetrizer** $\mathcal{A}_p$ is defined as the birdtrack

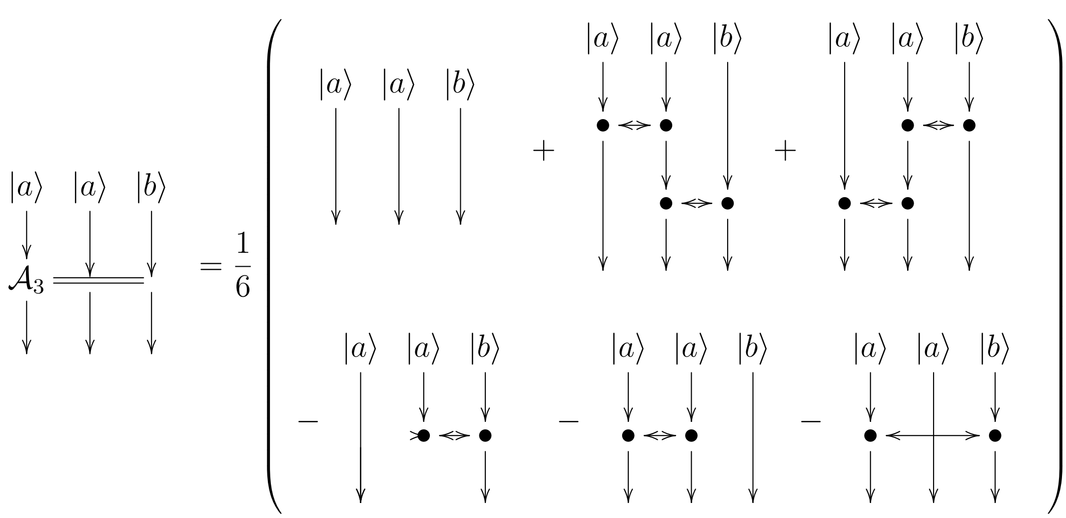


(20.23)

Note that

$$\mathcal{A}_p = 0 \text{ if } n < p \tag{20.24}$$

because when $n < p$, there must be two lines with the same value emerging from $\mathcal{A}_p x$, so $-\mathcal{A}_p x = \mathcal{A}_p x = 0$. For example, for $n = 2$ and $p = 3$



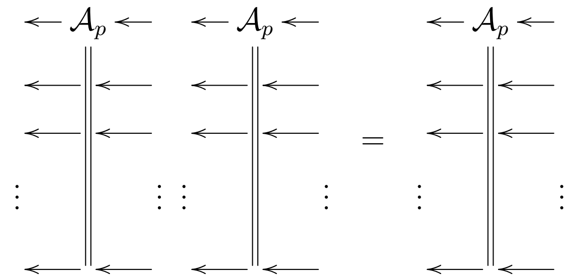
(20.25)

$$\mathcal{A}_3|a,a,b\rangle = \frac{1}{6}\left(\begin{array}{l} |a,a,b\rangle + |a,b,a\rangle + |b,a,a\rangle \\ -|a,b,a\rangle - |a,a,b\rangle - |b,a,a\rangle \end{array}\right) \tag{20.26}$$

$$= 0 \tag{20.27}$$

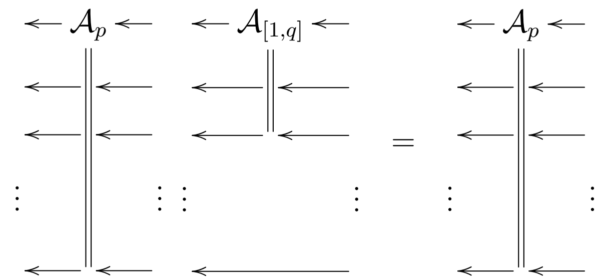
Note that $\mathcal{A}_p$ satisfies the following identities:

$$\boxed{\mathcal{A}_p^2 = \mathcal{A}_p}$$



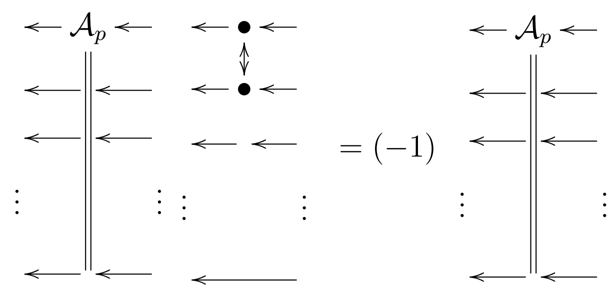
(20.28)

$$\boxed{\mathcal{A}_p\mathcal{A}_{[1,q]} = \mathcal{A}_p}$$



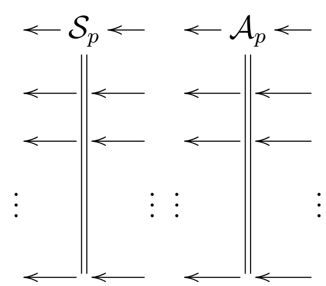
(20.29)

$$\boxed{\mathcal{A}_p\sigma_{(1,2)} = -\mathcal{A}_p}$$



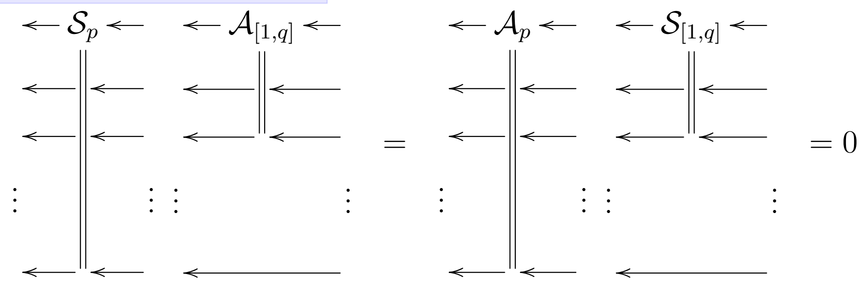
(20.30)

$$\boxed{\mathcal{S}_p\mathcal{A}_q = \mathcal{A}_p\mathcal{S}_q = 0}$$



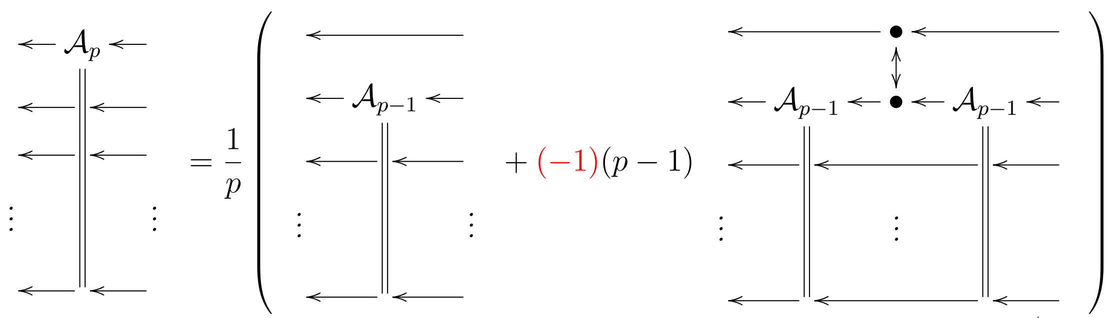
$= 0$ (20.31)

$\boxed{\mathcal{S}_p\mathcal{A}_{[1,q]} = \mathcal{A}_p\mathcal{S}_{[1,q]} = 0}$



(20.32)

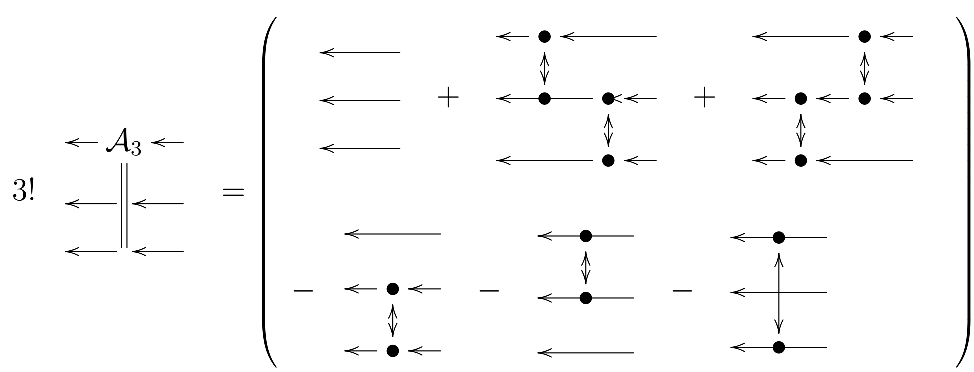
Claim 51



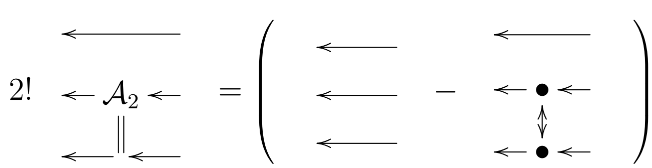
(20.33)

proof:

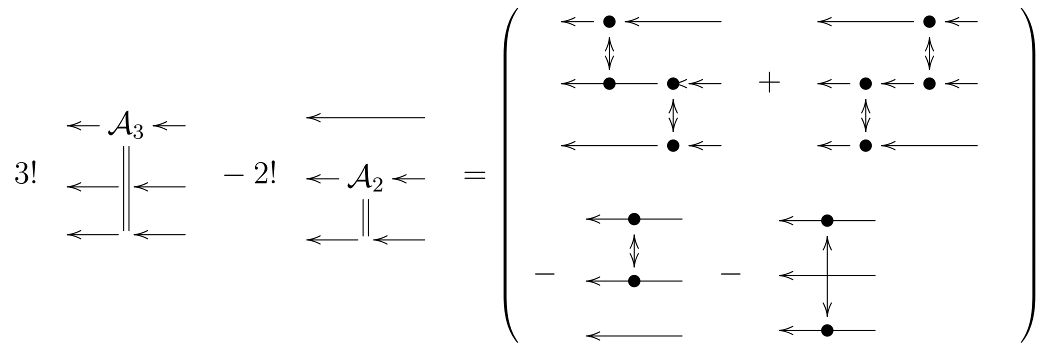
We only prove it for $p = 3$.



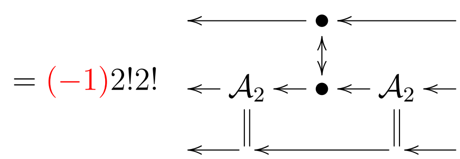
(20.34)



(20.35)



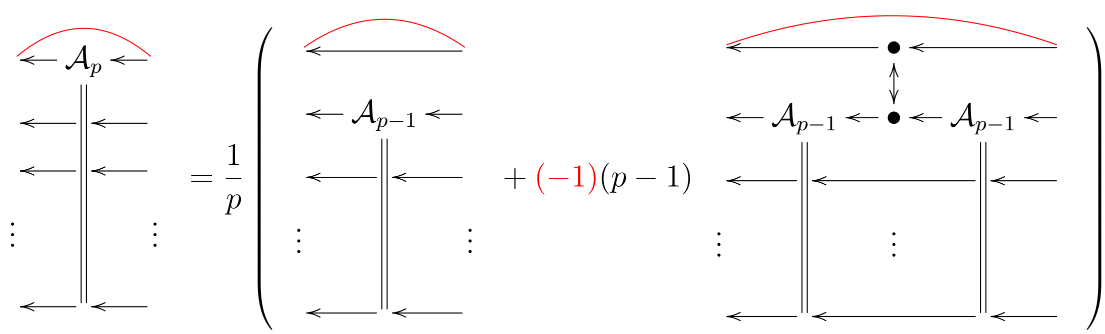
(20.36)



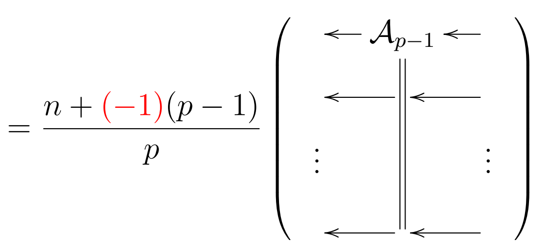
(20.37)

QED

Tracing over the identity of Claim 51, we get



(20.38)



(20.39)

Hence,

$$\mathrm{tr}_{\underline{a}_1}\mathcal{A}_p = \frac{n-p+1}{p}\mathcal{A}_{p-1} \tag{20.40}$$

$$\mathrm{tr}_{\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_k}\mathcal{A}_p = \frac{(n-p+1)(n-p+2)\ldots(n-p+k)}{p(p-1)\ldots(p-k+1)}\mathcal{A}_{p-k} \tag{20.41}$$

$$d_{\mathcal{A}_p} = \mathrm{tr}_{\underline{a}^p}\mathcal{A}_p = \frac{\prod_{i=n}^{n-p+1} i}{p!} \tag{20.42}$$

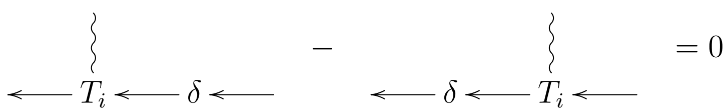
$$= \begin{cases} \frac{n!}{p!(n-p)!} = \binom{n}{p} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} \tag{20.43}$$

For $p = 2 \leq n$,

$$d_{\mathcal{A}_2} = \binom{n}{2} \tag{20.44}$$

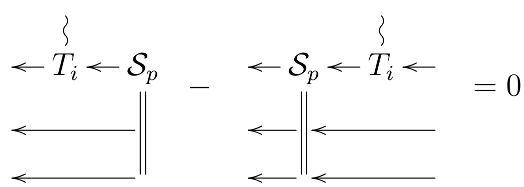
20.3 Invariance of $\mathcal{S}_p$ and $\mathcal{A}_p$

The Kronecker delta is obviously always an invariant matrix because

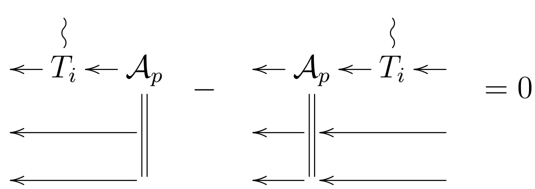


(20.45)

An immediate consequence of the the invariance of the Kronecker delta is that the symmetrizer $\mathcal{S}_p$ and anti-symmetrizer $\mathcal{A}_p$ are tensor invariants too. Indeed,



(20.46)



(20.47)

20.4 Levi-Civita Tensor

The **Levi-Civita tensor** $\epsilon_{a^{:p}}$ where $a_i \in \{1, 2, \ldots, p\}$ equals $+1$ (resp., -1) if $a^{:p}$ is an even (resp., odd) permutation of $(1, 2, \ldots, p)$. Thus

$$\epsilon_{123} = \epsilon^{123} = 1 \tag{20.48}$$

$$\epsilon_{213} = \epsilon^{213} = -1 \tag{20.49}$$

More generally,

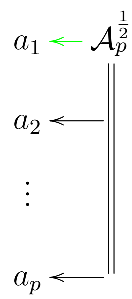
$$\epsilon^{123\ldots p} = \epsilon_{123\ldots p} = 1 \tag{20.50}$$

and

$$\epsilon_{rev(a^{:p})} = (-1)^{\binom{p}{2}} \epsilon_{a^{:p}} \tag{20.51}$$

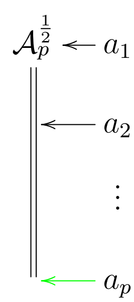
Define

$$(C_{\mathcal{A}_p})^1_{a^{:p}} = \frac{\epsilon_{a^{:p}}}{\sqrt{p!}} = \tag{20.52}$$



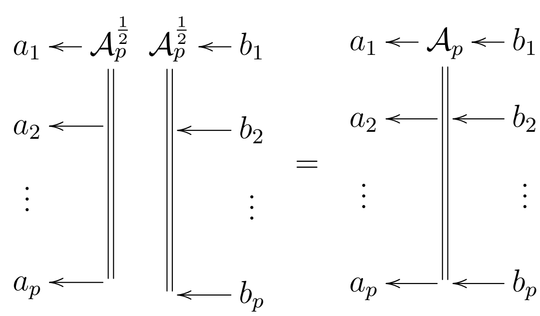
and

$$(C^\dagger_{\mathcal{A}_p})_1^{rev(a^{:p})} = \frac{\epsilon^{rev(a^{:p})}}{\sqrt{p!}} = \tag{20.53}$$



Then

$$\boxed{\frac{1}{p!}\epsilon_{a^{:p}}\epsilon^{rev(b^{:p})} = (\mathcal{A}_p)_{a^{:p}}{}^{rev(b^{:p})}}$$

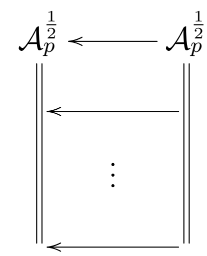


(20.54)

and

$$\boxed{e^{i2\phi}\frac{1}{p!}\epsilon^{rev(a^{:n})}\epsilon_{a^{:n}} = 1}$$

$e^{i2\phi}$



$= 1 \qquad (20.55)$

For the FL Convention, we will use $\phi = 0$.
For the CC Convention, we must choose

$$e^{i2\phi} = (-1)^{\binom{p}{2}} = e^{i\pi\frac{p(p-1)}{2}} \tag{20.56}$$

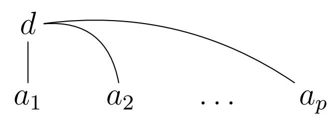
so

$$\phi = \frac{\pi}{4}p(p-1) \tag{20.57}$$

20.5 Fully-symmetric and Fully-antisymmetric Tensors

A **fully symmetric (FS) tensor** d

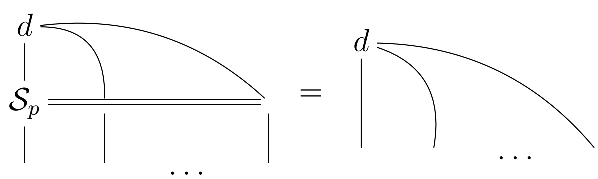
$d_{a_1 a_2 \ldots a_p} =$



(20.58)

is a tensor that satisfies

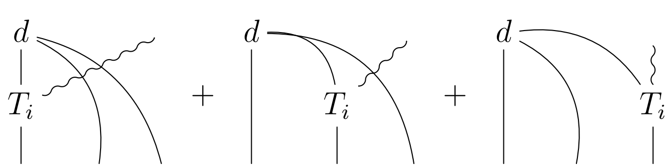
$$\boxed{\mathcal{S}_p d = d}$$



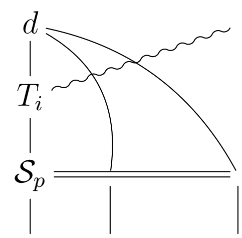
(20.59)

If d is a tensor invariant (see Chapter 7), it must satisfy

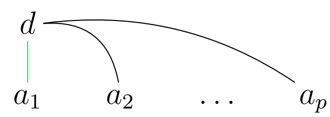
$0 =$



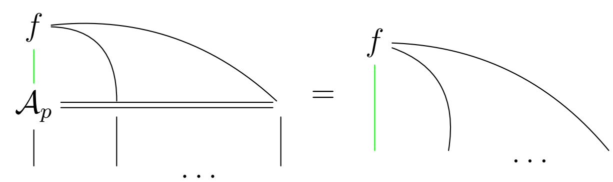
(20.60)

$$0 =$$  (20.61)

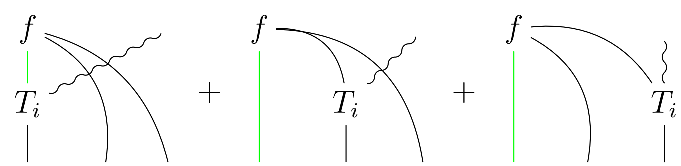
A **fully antisymmetric(FA) tensor** f

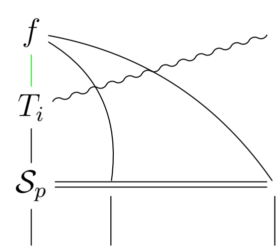
$$f_{a_1 a_2 \ldots a_p} =$$  (20.62)

is a tensor that satisfies

$$\boxed{\mathcal{A}_p f = f}$$  (20.63)

If f is a tensor invariant (see Chapter 7), it must satisfy

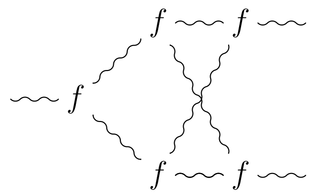
$$0 =$$  (20.64)

$$0 =$$  (20.65)

20.6 Identically Vanishing Birdtracks

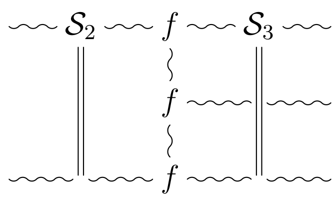
Identically vanishing (IV) birdtracks are birdtracks that vanish by virtue of their symmetrized or antisymmetrized components.

- Example of birdtrack that vanishes for any FA tensor f

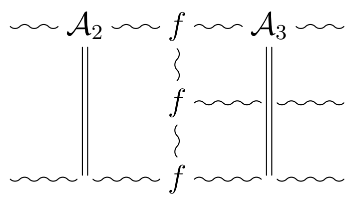


$= 0$ (Kuratowsky graph) (20.66)

- Example of birdtrack that vanishes for any f that is a structure constant of a Lie algebra



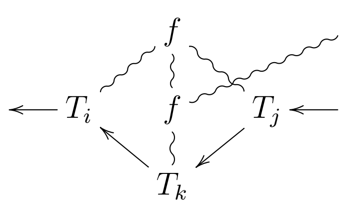
$= 0,$



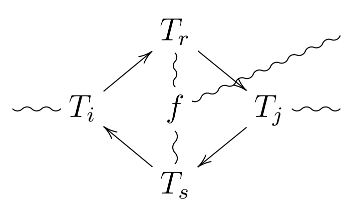
$= 0$

(20.67)

- Birdtrack that is zero for an irrep



$= 0,$



$= 0$

(20.68)

Chapter 21

Unitary Groups

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Let
n-dim General Linear group $GL(n;\mathbb{C}) = \{G \in \mathbb{C}^{n\times n} : det(G) \neq 0\}$
n-dim Special Linear group $SL(n;\mathbb{C}) = \{G \in GL(n;\mathbb{C}) : det(G) = 1\}$
n-dim Unitary group, $U(n) = \{G \in GL(n;\mathbb{C}) : GG^\dagger = G^\dagger G = 1\}$
n-dim Special Unitary group $SU(n) = \{G \in U(n) : det(G) = 1\}$
Chapter 23 on Young Tableau is closely connected to this chapter.

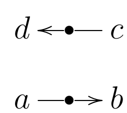
21.1 $SU(n)$

In $SU(n) \subset \mathbb{C}^{n\times n}$ in the defining rep, we have the quadratic form

$$m(p,q) = (p_b)^* \delta^a_b q_a \tag{21.1}$$

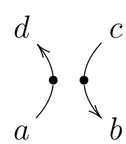
Let

$$\mathbb{1}_{d\ b}^{\ a\ c} = \delta^a_b \delta^c_d = \tag{21.2}$$



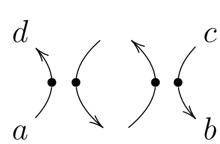
and

$$M^{d\ b}_{\ a\ c} = \delta^a_d \delta^c_b = \tag{21.3}$$

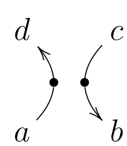


Note that

$$\boxed{M^2 = nM} \tag{21.4}$$



$= n$



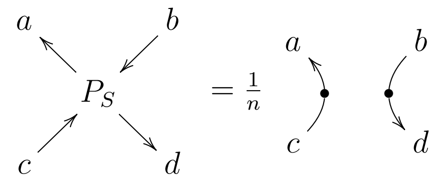
Hence, $(M-n)M = 0$ so M has two eigenvalues $\lambda = 0, n$.

Next we will use the following equation from Chapter 15 [1] to obtain a projection operator (PO) for each eigenvalue

$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \tag{21.5}$$

1. Singlet PO ($\lambda_S = 0$)

$$\boxed{P_S = \frac{M-0}{n-0} = \frac{1}{n}M}$$



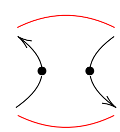
(21.6)

The singlet projection operator P_S projects the singlet part of a tensor x:

$$P_S x = \frac{1}{n} x^b{}_b \delta^c_a \tag{21.7}$$

P_S has dimension 1:

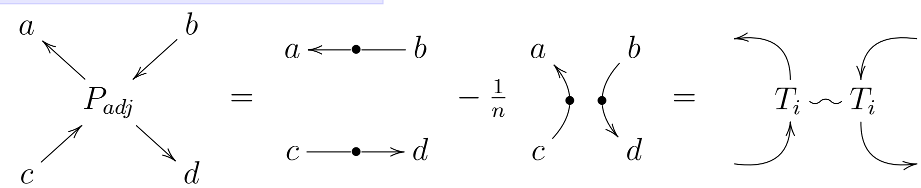
$$dim(P_S) = \mathrm{tr} P_S = \frac{1}{n} \tag{21.8}$$



$$= 1 \tag{21.9}$$

2. Adjoint PO

$$\boxed{P_{adj} = \frac{M-n}{0-n} = 1 - \frac{1}{n}M}$$



(21.10)

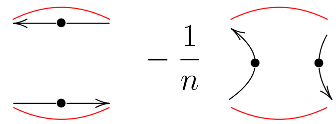
The adjoint projection operator P_{adj} projects the traceless part of a tensor x

$$P_{adj} x = x^a{}_c - \left(\frac{1}{n} x^b{}_b \delta^c_a\right) \tag{21.11}$$

[1]Note that this equation projects to zero all eigenvalues except one.

The P_{adj} has dimension $n^2 - 1$

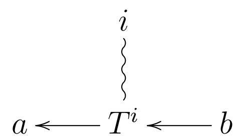
$$dim(P_{adj}) = \mathrm{tr} P_{adj} \quad = \qquad\qquad - \frac{1}{n} \qquad\qquad \tag{21.12}$$



$$= \quad n^2 - 1 \tag{21.13}$$

We will denote the generators T_i of $SU(n)$ by

$$(T_i)_a{}^b = \tag{21.14}$$

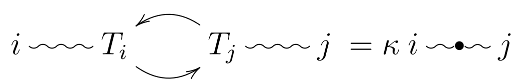


For $G \in U(n)$, $G^\dagger G = 1$ with $G = e^{iT_i \epsilon_i}$ where $\epsilon_i \in \mathbb{R}$. Hence, the generators T_i must be Hermitian

$$T_i^\dagger = T_i \tag{21.15}$$

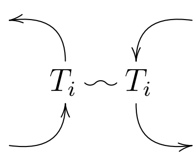
We will assume that they also satisfy

$$\boxed{\mathrm{tr}(T_i T_j) = \kappa \delta_i^j} \tag{21.16}$$

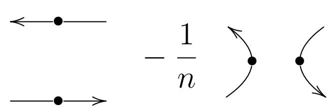


Usually, we set $\kappa = 1$ and, if necessary, restore the κ's at the end by dimensional analysis. (Replace each T_i in a κ-less equation by $T_i/\sqrt{\kappa}$.)

The adjoint projection operator for $SU(n)$ is

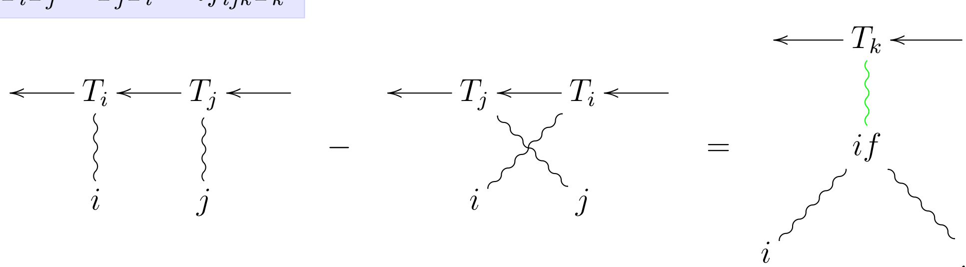


$$\stackrel{\text{def}}{=} \quad P_{adj} = \qquad\qquad - \frac{1}{n} \tag{21.17}$$



The Lie Algebra commutators for $SU(n)$ are

$$\boxed{T_i T_j - T_j T_i = i f_{ijk} T_k}$$

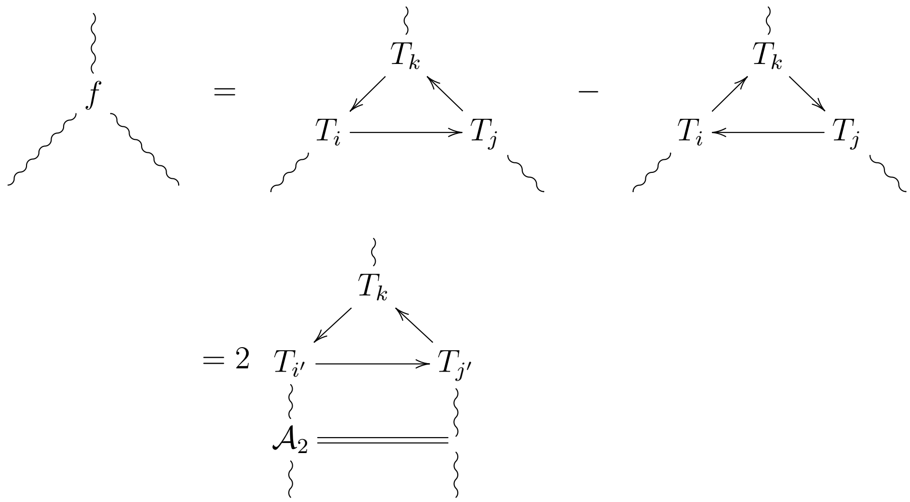


(21.18)

The structure constants f_{ijk} for $SU(n)$ is a totally antisymmetric tensor. In the CC convention, the first index of f_{ijk} corresponds to the green leg in the birdtracks.[2]

Multiplying Lie Algebra commutator by by T_k and taking the trace, we get

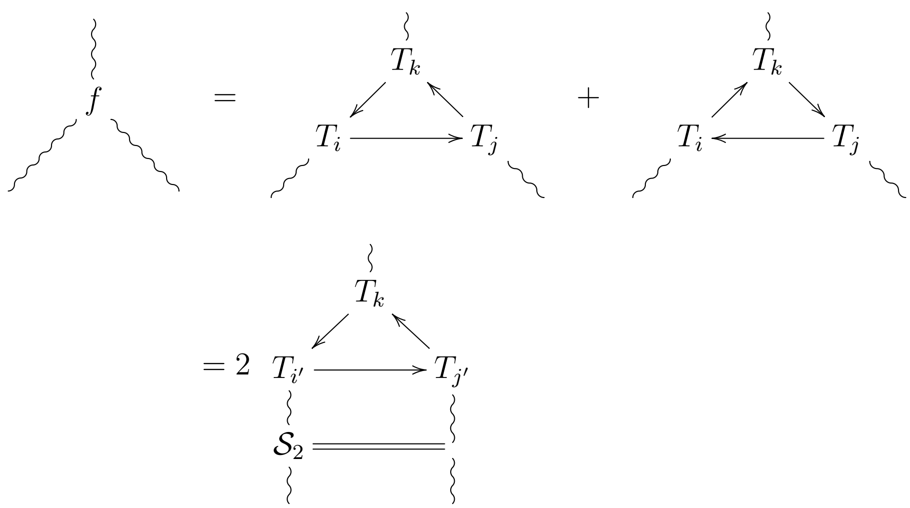
$$\boxed{if_{ijk} = \mathrm{tr}([T_i, T_j]T_k)}$$



(21.19)

One can define a totally symmetric tensor d_{ijk} analogously by

$$\boxed{d_{ijk} = \mathrm{tr}([T_i, T_j]_+ T_k)}$$



(21.20)

Claim 52 .

- $\mathrm{tr}([T_i, T_j]T_k)$ *is totally anti-symmetric*

[2]Actually, it doesn't matter which index is taken first. This is explained in Chapter B

- $\mathrm{tr}([T_i, T_j]_+ T_k)$ *is totally symmetric*

in the indices i, j, k

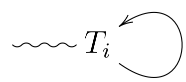
proof:

$$\mathrm{tr}([T_i, T_j]T_k) = -\mathrm{tr}([T_k, T_j]T_i) \tag{21.21}$$

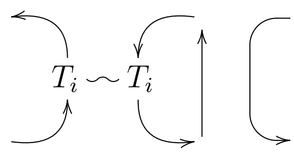
$$\mathrm{tr}([T_i, T_j]T_k) = +\mathrm{tr}([T_k, T_j]_+ T_i) \tag{21.22}$$

QED

Claim 53

$$\boxed{\mathrm{tr}(T_i) = 0}$$  $= 0$ (21.23)

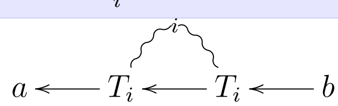
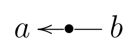
proof:

$0 = P_{adj}P_S =$  (21.24)

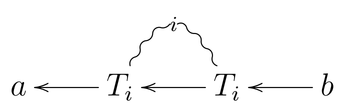
QED

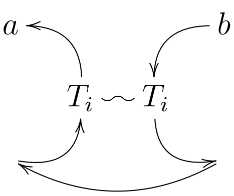
Claim 54

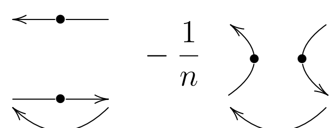
$$\boxed{\Gamma_{fun}\delta_a^b = \sum_i (T_i T_i)_a{}^b = \frac{n^2-1}{n}\delta_a^b}$$

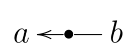
$\sum_i$  $= \left(\frac{n^2-1}{n}\right)$  (21.25)

proof:

$(T_i T_i)_a{}^b \quad =$  (21.26)

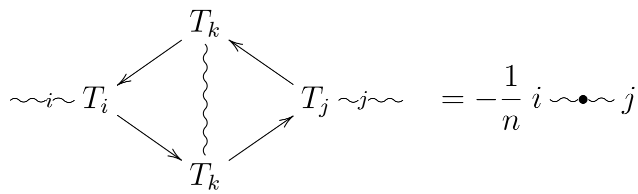
$=$  (21.27)

$=$  (21.28)

$= \left(n - \dfrac{1}{n}\right)$  (21.29)

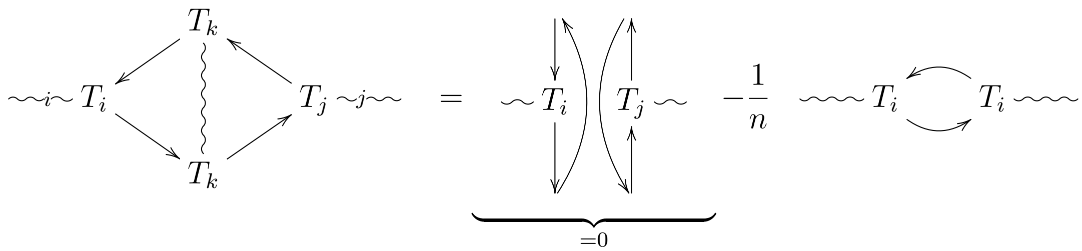
QED

Claim 55



(21.30)

proof:

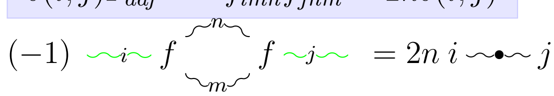


(21.31)

QED

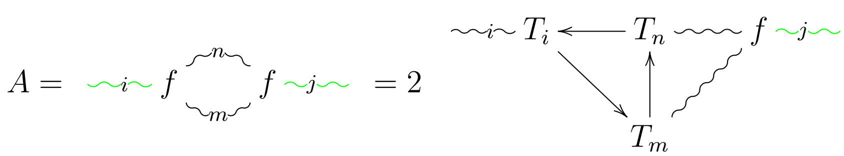
Claim 56

$$\boxed{\delta(i,j)\Gamma_{adj} = -f_{imn}f_{jnm} = 2n\delta(i,j)}$$

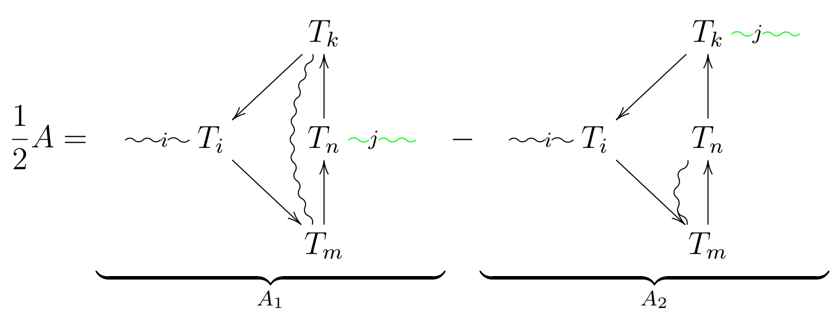


(21.32)

proof:



(21.33)



(21.34)

$$A_1 = \frac{n^2 - 1}{n}\delta(i, j) \tag{21.35}$$

$$A_2 = -\frac{1}{n}\delta(i, j) \tag{21.36}$$

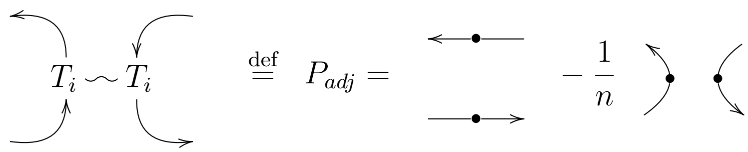
$$A = 2(A_1 - A_2) = 2n\delta(i, j) \tag{21.37}$$

QED

21.2 Differences Between $U(n)$ and $SU(n)$

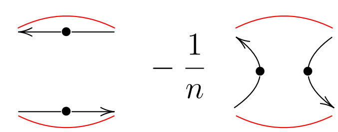
1. $SU(n)$

   primitive invariants: Kronecker delta, Levi-Civita tensor



(21.38)

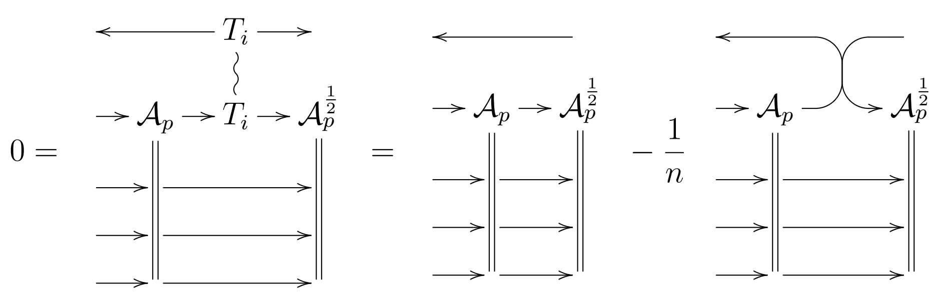
$dim(P_{adj}) = \mathrm{tr}P_{adj} \quad =$



(21.39)

$$= \quad n^2 - 1 \tag{21.40}$$

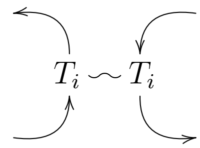
Since the Levi-Civita tensor is an invariant matrix for $SU(n)$, we must have



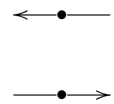
(21.41)

2. $U(n)$

   primitive invariants: Kronecker delta

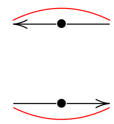


$$\overset{\text{def}}{=} \quad P_{adj} =$$



(21.42)

$$dim(P_{adj}) = \mathrm{tr}P_{adj} =$$



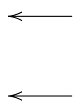
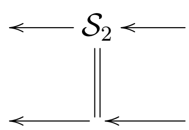
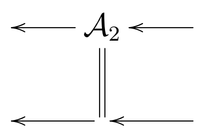
(21.43)

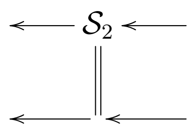
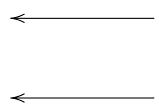
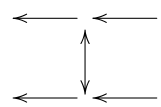
$$= n^2 \qquad (21.44)$$

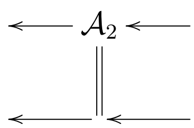
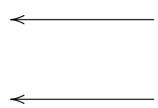
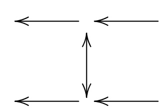
21.3 $V_{def} \otimes V_{def}$ Decomposition

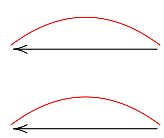
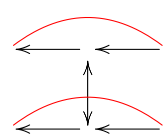
Let

$$V_{def} = V = \text{vector space in defining representation } \{|a\rangle\}_{a=1}^{n}.$$

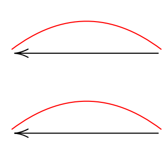
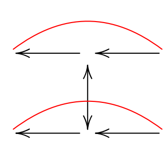
 $=$  $+$  (21.45)

 $= \frac{1}{2}\Big\{$  $+$  $\Big\}$ (21.46)

 $= \frac{1}{2}\Big\{$  $-$  $\Big\}$ (21.47)

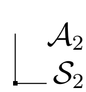
$dim(\mathcal{S}_2) = \frac{1}{2}\Big\{$  $+$  $\Big\}$ (21.48)

$$= \frac{n(n+1)}{2} \qquad (21.49)$$

$dim(\mathcal{A}_2) = \frac{1}{2}\Big\{$  $-$  $\Big\}$ (21.50)

$$= \frac{n(n-1)}{2} \qquad (21.51)$$

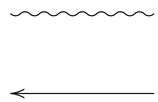
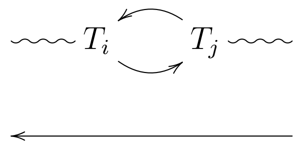
The projection operator tree is

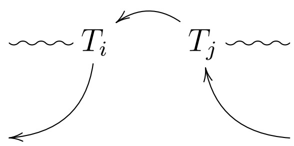
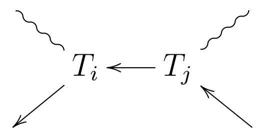


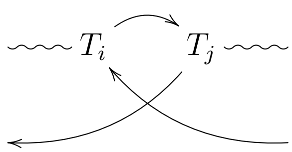
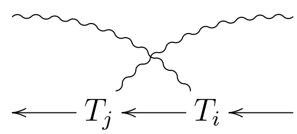
21.4 $V_{adj} \otimes V_{def}$ Decomposition

Let

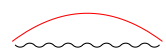
$V_{def} = V =$ vector space in defining representation $\{|a\rangle\}_{a=1}^{n}$.
$V_{adj} =$ vector space in adjoint representation $\{|i\rangle\}_{i=1}^{N}$.
$V_{adj} \otimes V \cong (V \otimes V^{\dagger}) \otimes V$

$e =$  $\cong$  (21.52)

$R =$  $=$  (21.53)

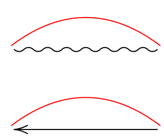
$Q =$  $=$  (21.54)

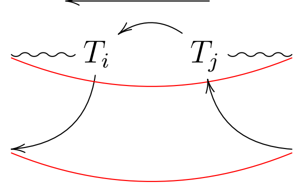
Recall that for $SU(n)$, the dimension N of the adjoint rep is

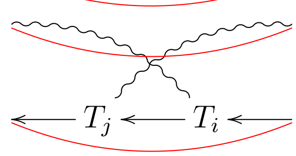
$N = n^2 - 1 =$  (21.55)

For example, for $SU(2)$, $N = 3$ and for $SU(3)$, $N = 8$.

Note that

$\mathrm{tr}(e) =$  $= Nn$ (21.56)

$\mathrm{tr}(R) =$  $= N$ (21.57)

$\mathrm{tr}(Q) =$  $= N$ (21.58)

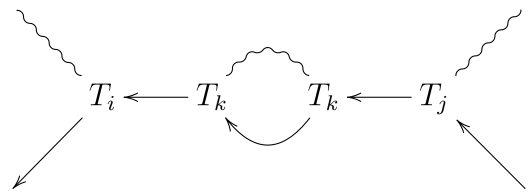
Claim 57

$$R^2 = \frac{n^2 - 1}{n} R \tag{21.59}$$

$$QR = RQ = -\frac{1}{n} R \tag{21.60}$$

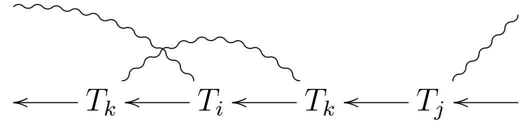
$$Q^2 - e = -\frac{1}{n}R \tag{21.61}$$

proof:

$R^2 \;=$ 

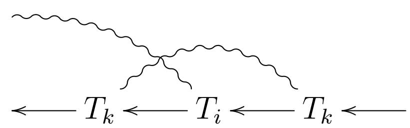
(21.62)

$$= \frac{n^2-1}{n}R \quad \text{(by Eq.(21.25))} \tag{21.63}$$

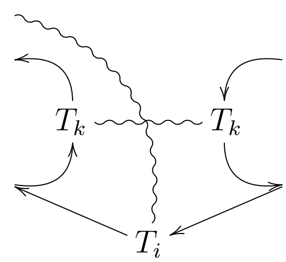
$QR \;=$ 

(21.64)

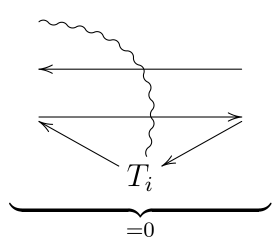
Define

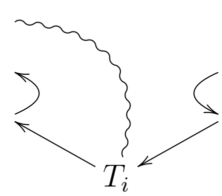
$X =$ 

(21.65)

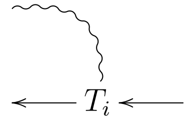
$X \;=$ 

(21.66)

$=$ 

$-\frac{1}{n}$ 

(21.67)

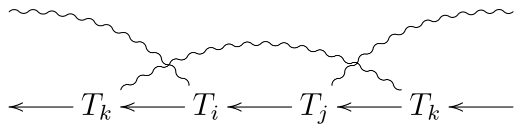
$= \;-\frac{1}{n}$ 

(21.68)

so

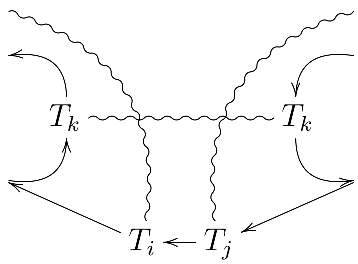
$$QR = RQ = -\frac{1}{n}R \tag{21.69}$$

$Q^2 =$



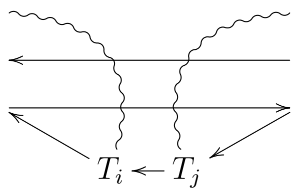
(21.70)

$=$

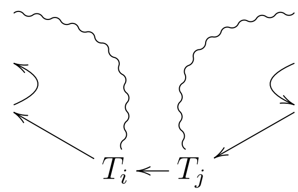


(21.71)

$=$

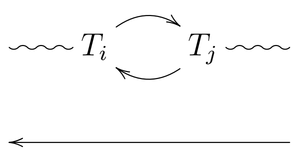


$-\frac{1}{n}$

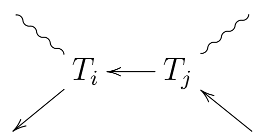


(21.72)

$=$



$-\frac{1}{n}$



(21.73)

$$= e - \frac{1}{n}R \tag{21.74}$$

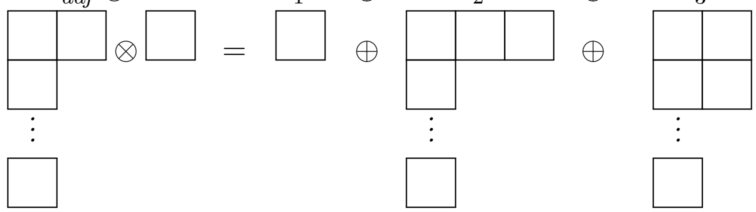
QED

Claim 58

$$\begin{aligned}
P_1 &= \frac{n}{n^2-1}R && (21.75)\\
P_2 &= \frac{1}{2}P_4(1+Q) = \frac{1}{2}\left[e + Q - \frac{1}{n+1}R\right] && (21.76)\\
P_3 &= \frac{1}{2}P_4(1-Q) = \frac{1}{2}\left[e - Q - \frac{1}{n-1}R\right] && (21.77)\\
P_4 &= 1 - P_1 && (21.78)
\end{aligned}$$

are projectors for $SU(n)$. The $V_{adj} \otimes V = \sum_\lambda V_\lambda$ Clebsch-Gordan series is given by

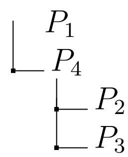
$$\overbrace{V_{adj} \otimes V}^{\mathcal{V}} = P_1\mathcal{V} \oplus P_2\mathcal{V} \oplus P_3\mathcal{V}$$



(21.79)

$$\begin{array}{ccccccc}
(n^2-1)n & = & n & + & \frac{n(n-1)(n+2)}{2} & + & \frac{n(n+1)(n-2)}{2} \\
SU(3): 8(3) & = & 3 & + & 15 & + & 6
\end{array}$$

The projection operator tree is



proof:

$$\mathrm{tr}(P_1) = \frac{n}{n^2-1}N = n \tag{21.80}$$

$$\begin{aligned}
\mathrm{tr}(P_2) &= \frac{N}{2}\left(n+1-\frac{1}{n+1}\right) &(21.81)\\
&= \frac{N}{2}\frac{n^2+2n}{n+1} &(21.82)\\
&= \frac{N}{2}\frac{n(n+2)}{n+1} &(21.83)\\
&= \frac{(n-1)n(n+2)}{2} &(21.84)
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(P_3) &= \frac{N}{2}\left(n-1-\frac{1}{n-1}\right) &(21.85)\\
&= \frac{N}{2}\frac{n^2-2n}{n-1} &(21.86)\\
&= \frac{N}{2}\frac{n(n-2)}{n-1} &(21.87)\\
&= \frac{(n+1)n(n-2)}{2} &(21.88)
\end{aligned}$$

From $R^2 = \frac{n^2-1}{n}R$,

$$P_1 = \frac{n}{n^2-1}R \tag{21.89}$$

Define

$$P_4 = e - P_1 \tag{21.90}$$

From $Q^2 - e = -\frac{1}{n}R$, we get

$$P_4(Q^2-1) = 0 \tag{21.91}$$

Let

$$P_2 = \frac{1}{2}P_4(1+Q), \quad P_3 = \frac{1}{2}P_4(1-Q) \tag{21.92}$$

and

$$a = \frac{n}{n^2-1} \tag{21.93}$$

Then

$$\begin{aligned} P_2 &= \frac{1}{2}P_4(1+Q) && (21.94)\\ &= \frac{1}{2}(e-aR)(1+Q) && (21.95)\\ &= \frac{1}{2}(e-aR+Q-aRQ) && (21.96)\\ &= \frac{1}{2}\left(e+\left(\frac{1}{n}-1\right)aR+Q\right) \quad (\text{use } QR = -\frac{1}{n}R) && (21.97)\end{aligned}$$

where

$$\begin{aligned} \left(\frac{1}{n}-1\right)a &= \frac{1-n}{n}\frac{n}{n^2-1} && (21.98)\\ &= -\frac{1}{n+1} && (21.99)\end{aligned}$$

Furthermore

$$\begin{aligned} P_3 &= \frac{1}{2}P_4(1-Q) && (21.100)\\ &= \frac{1}{2}(e-aR)(1-Q) && (21.101)\\ &= \frac{1}{2}(e-aR-Q+aRQ) && (21.102)\\ &= \frac{1}{2}\left(e-\left(\frac{1}{n}+1\right)aR-Q\right) \quad (\text{use } QR = -\frac{1}{n}R) && (21.103)\end{aligned}$$

where

$$\left(\frac{1}{n}+1\right)a=\frac{1}{n-1} \tag{21.104}$$

QED

Let $Q_1, Q_2, Q_3 = e, R, Q$

$$Q_\lambda|Q_j\rangle = |Q_\lambda Q_j\rangle = \sum_i A^\lambda_{ij}|Q_i\rangle \tag{21.105}$$

$$\langle Q_i|Q_\lambda|Q_j\rangle = A^\lambda_{ij} \tag{21.106}$$

If A^λ are diagonalized and divided by their eigenvalues, and they have a single non-zero eigenvalue, then they become a complete set of projectors with 1 or 0 along their diagonals.

Chapter 22

Wigner-Ekart Theorem

This chapter on the Wigner-Ekart (WE) Theorem is based on Cvitanovic's Birdtracks book Ref. [1].

22.1 WE in General

The birdtracks with no incomming or outgoing arrows are know as **reduced matrix elements, isolated DAGs and vacuum bubbles**

The following 3 claims are related. They reduce a tensor with 1, 2 and 3 indices

Claim 59 *(one index)*

If M is an invariant vector (i.e., $G_\lambda(g)M = M$ for all $g \in \mathcal{G}$), then

$$M_a = \sum_\lambda a \xleftarrow{\lambda} P_\lambda \xleftarrow{\lambda} M \tag{22.1}$$

$$= \sum_{\lambda \in SR} a \xleftarrow{\lambda} P_\lambda \xleftarrow{\lambda} M \tag{22.2}$$

where $P_\lambda = |\lambda\rangle\langle\lambda| = \mathfrak{C}_\lambda^\dagger \mathfrak{C}_\lambda$ and $SR =$ set of singlet representations.

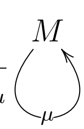
proof:
QED

Claim 60 *(Schur's Lemma) (2 indices)*

If μ and λ are irreps, and M is an invariant matrix, then

$$M_{\lambda a}{}^{\mu b} = a \xleftarrow{\lambda} M \xleftarrow{\mu} b \tag{22.3}$$

$$= \frac{1}{d_\mu}$$

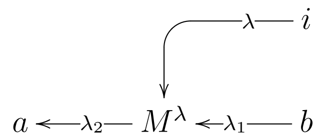


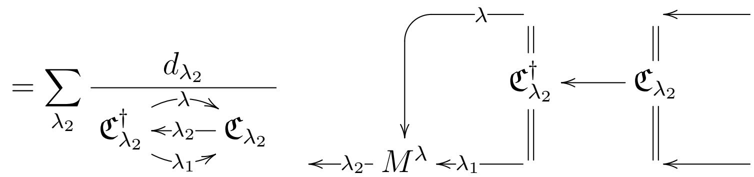
$$\delta(\mu, \lambda) \xleftarrow{\lambda} \tag{22.4}$$

proof:
QED

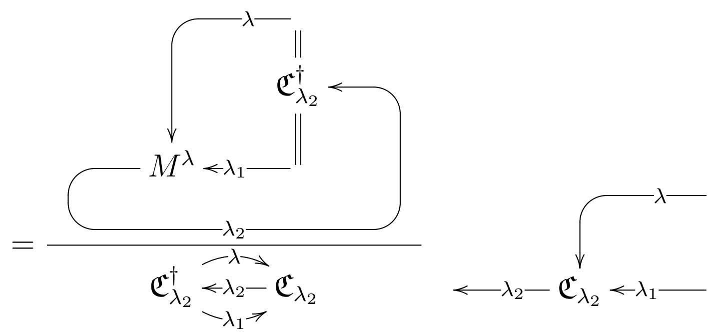
Claim 61 *(Wigner-Ekart (WE) Theorem) (3 indices)*
If M is an invariant 3 index tensor,

$$(M^{\lambda i})_{\lambda_2 a}{}^{\lambda_1 b} = \tag{22.5}$$





(22.6)

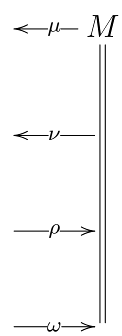


(22.7)

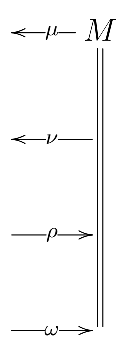
proof:
QED

What about 4 indices and beyond? Consider

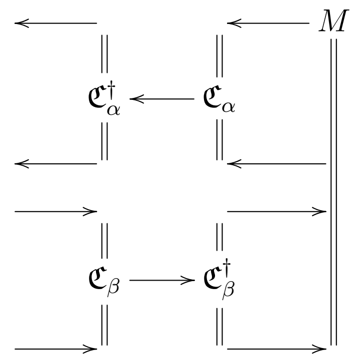
$$M = \tag{22.8}$$



Then

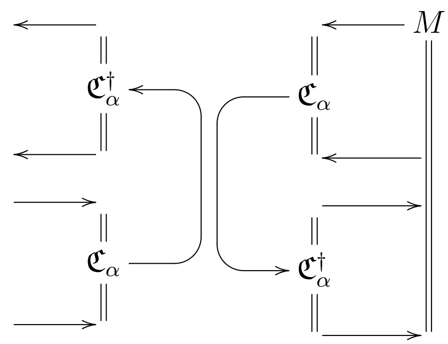


$$= \sum_{\alpha,\beta} \frac{1}{\kappa_\alpha \kappa_\beta}$$



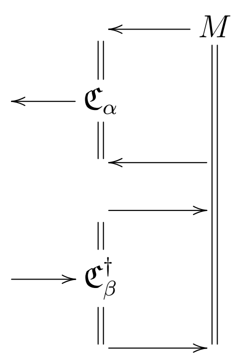
(22.9)

$$= \sum_{\alpha} \frac{1}{\kappa_\alpha^2 d_\alpha}$$

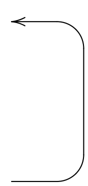


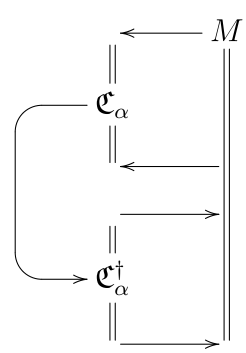
(22.10)

Above, we used



$$= \frac{\delta(\alpha, \beta)}{d_\alpha}$$





(22.11)

22.2 WE for Angular Momentum

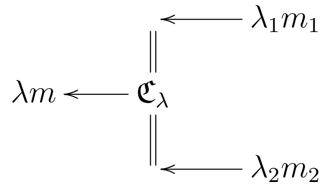
Let

$\lambda = J$, $\lambda_i = J_i$ for $i = 1, 2$. We will use Greek letters instead of J so as to keep convention of using Greek letters for rep labels.

$m, m' = -\lambda,\ -\lambda + 1, \ldots, \lambda - 1, \lambda$. Note that $d_\lambda = 2\lambda + 1$

for $i = 1, 2$, $m_i = -\lambda_i, -\lambda_i + 1, \ldots, \lambda_i - 1, \lambda_i$. Note that $d_{\lambda_i} = 2\lambda_i + 1$

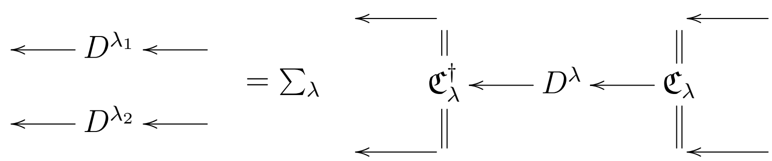
Define

$$\langle(\lambda_1\lambda_2)\lambda m|\lambda_1 m_1\lambda_2 m_2\rangle \quad = \quad$$  (22.12)

$$D^{\lambda}_{mm'}(g) \quad = \quad m \longleftarrow D^{\lambda} \longleftarrow m' \tag{22.13}$$

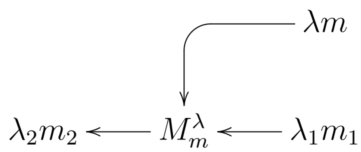
Then the Clebsch-Gordan decomposition of $D^{\lambda_1}\otimes D^{\lambda_2}$ is

$$\boxed{D^{\lambda_1}_{m_1m_1'}(g)D^{\lambda_2}_{m_2m_2'}(g) = \sum_{\lambda,m,m'} \langle\lambda_1 m_1\lambda_2 m_2|\lambda_1\lambda_2\lambda m\rangle\, D^{\lambda}_{mm'}(g)\, \langle\lambda_1\lambda_2\lambda m'|\lambda_1 m_1'\lambda_2 m_2'\rangle}$$



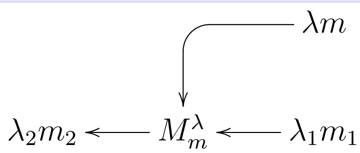
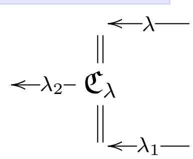
(22.14)

We will denote a **tensor operator** M^{λ}_{m} by the birdtrack

$$\left\langle\lambda_2 m_2|M^{\lambda}_{m}|\lambda_1 m_1\right\rangle =$$  (22.15)

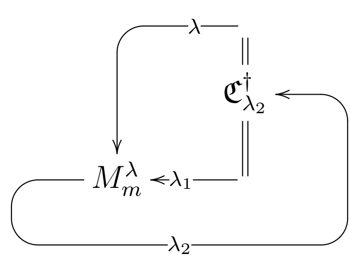
Claim 62 *(Wigner-Ekart for angular momentum)*

$$\boxed{\left\langle\lambda_2 m_2|M^{\lambda}_{m}|\lambda_1 m_1\right\rangle = \langle(\lambda\lambda_1)\lambda_2 m_2|\lambda m\lambda_1 m_1\rangle\, QR}$$

 $= QR$  (22.16)

where

$$Q(\lambda,\lambda_1,\lambda_2) \quad = \quad \frac{1}{d_{\lambda_2}}\sum_{m_1,m_2,m}\langle\lambda m\lambda_1 m_1|(\lambda\lambda_1)\lambda_2 m_2\rangle\left\langle\lambda_2 m_2|M^{\lambda}_{m}|\lambda_1 m_1\right\rangle \tag{22.17}$$

$$= \quad \frac{1}{d_{\lambda_2}}$$  (22.18)

and

$$R(\lambda, \lambda_1, \lambda_2) = \frac{d_{\lambda_2}}{\begin{tikzcd} \mathfrak{C}^{\dagger}_{\lambda_2} \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\lambda_1"'] & \mathfrak{C}_{\lambda_2} \arrow[l, "\lambda_2"] \end{tikzcd}} \tag{22.19}$$

proof:
QED

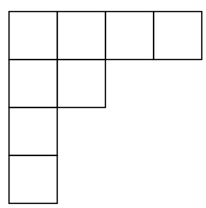
Chapter 23

Young Tableau

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

We recommend that the read Chapter 20 on symmetrizers and antisymetrizers before reading this one.

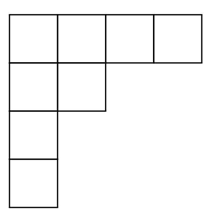
A **Young Diagram** (YD) $\mathcal{Y} = [\lambda_1, \lambda_2, \ldots, \lambda_D]$ consists of λ_1 left-aligned empty boxes (LAEB) over λ_2 LAEB, over λ_3 LAEB, up to λ_{NR} LAEB, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{NR} \geq 1$. NR = number of rows. For example,



$$= [4, 2, 1, 1] \tag{23.1}$$

We will call $[4, 2, 1, 1]$ the **row lengths** (RL) method of labeling YD.

A alternative method of labelling YD is called the **Dynkin (D) labels** or **row changes (RC)**. These labels list the change in number of columns as we go down the YD. For example,



$$= [2, 1, 0, 1, 0 \ldots]_D \tag{23.2}$$

A **Young Tableau** (YT) $\mathcal{Y}_\alpha$ is a YD in which integers from 1 to n where $n \leq n_b$ and n_b is the number boxes, are inserted according to some rules. The rules for insertion are that integers must increase when reading a row left to right and when reading a column from top to bottom. Obviously, for $n < n_b$, some integers are repeated.

A **Standard Young Tableau** (SYT) $\mathcal{Y}_\alpha$ is a YT such that $n = n_b$ and no integer is repeated. Fig.23.1 shows all SYT for $n_b = 1, 2, 3, 4$

- $n_b = 1$

$$\begin{ytableau} 1 \end{ytableau}$$

- $n_b = 2$

$$\begin{ytableau} 1 & 2 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \end{ytableau}$$

- $n_b = 3$

$$\begin{ytableau} 1 & 2 & 3 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$$

- $n_b = 4$

$$\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}$$

Figure 23.1: All SYT for $n_b = 1, 2, 3, 4$.

We will use $|\mathcal{Y}|$, or $|\mathcal{Y}_\alpha|$ or $|\alpha|$ to denote the number of boxes in a YD or YT.[1]

23.1 Symmetric Group S_{n_b}

Let

S_{n_b} = the symmetric group in n_b letters (or n_b boxes)
$irreps(S_{n_b})$ = the set of all irreps of S_{n_b}.
The **transpose of a YT** is defined as if it were a matrix. For example

$$transpose\left(\begin{ytableau} 1 & 3 & 5 \\ 2 & 6 \\ 4 \end{ytableau} \right) = \begin{ytableau} 1 & 2 & 4 \\ 3 & 6 \\ 5 \end{ytableau} \tag{23.3}$$

[1]For many authors and for us too, $|S|$ stands for the number of elements in a finite set S. This should not lead to confusion as a YD or YT are not sets.

n-dim General Linear group $GL(n;\mathbb{C}) = \{G \in \mathbb{C}^{n\times n} : det(G) \neq 0\}$
n-dim Special Linear group $SL(n;\mathbb{C}) = \{G \in GL(n;\mathbb{C}) : det(G) = 1\}$
n-dim Unitary group, $U(n) = \{G \in GL(n;\mathbb{C}) : GG^\dagger = G^\dagger G = 1\}$
n-dim Special Unitary group $SU(n) = \{G \in U(n) : det(G) = 1\}$
$YD(n_b)$ = set of YD with n_b boxes. $YD = \cup_{n_n=1}^{\infty} YD(n_b)$.
$SYD(n_b)$ = set of SYD with n_b boxes. $YT = \cup_{n_b=1}^{\infty} YT(n_b)$.
$SYT(n_b, NR)$ = set of STY with n_b boxes and NR rows.
$YT(\mathcal{Y})$ = set of YT with a YD $\mathcal{Y}$.
$SYT(\mathcal{Y})$ = set of SYT with a YD $\mathcal{Y}$.
$dim(\mathcal{Y}|S_{n_b})$ = dimension of irrep $\mathcal{Y}$ of S_{n_b}
$dim(\mathcal{Y}_\alpha|U(n))$ = dimension of irrep $\mathcal{Y}_\alpha$ of $U(n)$ or $SU(n)$.

Claim 63

1. *The YD with n_b boxes label all irreps of the symmetric group S_{n_b}.*

$$irreps(S_{n_b}) = YD(n_b) \tag{23.4}$$

2. *The SYT with n_b boxes and no more than n rows ($NR \leq n$), label the irreps of $GL(n)$ and of $U(n)$*

$$irreps(U(n)) = \bigcup_{n_b \leq n,\ NR \leq n} STY(n_b, NR) \tag{23.5}$$

3. *The SYT with n_b boxes and no more than $n-1$ rows ($NR \leq n-1$), label the irreps of $SL(n)$ and $SU(n)$.*

$$irreps(SU(n)) = \bigcup_{n_b \leq n,\ NR \leq n-1} STY(n_b, NR) \tag{23.6}$$

proof:
QED

23.1.1 $dim(\mathcal{Y}|S_{n_b})$

Claim 64

$$dim(\mathcal{Y}|S_{n_b}) = |SYT(\mathcal{Y})| \tag{23.7}$$

proof:
QED

For example, there are 3 irreps of S_4 associated withe the YD

$$\mathcal{Y} = \begin{ytableau} \; & \; \\ \; \\ \; \end{ytableau} \tag{23.8}$$

And each of those 3 irreps has dimension 3. Why? Because there are 3 possible SYT for this YD:

$$\begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}, \quad \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}, \quad \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} \implies dim(\mathcal{Y}|S_4) = 3 \tag{23.9}$$

Thus, we can denote the basis vectors of one of these 3 degenerate irreps by

$$|\begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}\rangle, \quad |\begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}\rangle, \quad |\begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau}\rangle \tag{23.10}$$

To compute $hook(\mathcal{Y})$:

1. Fill each box of the YD with the number of boxes below and to the right of the box, including the box itself.

2. Multiply the numbers in all the boxes.

For example,

$$\mathcal{Y} = \begin{ytableau} \; & \; & \; & \; \\ \; & \; & \; \\ \; & \; \end{ytableau} \implies hook(\mathcal{Y}) = \begin{ytableau} 6 & 5 & 3 & 1 \\ 4 & 3 & 1 \\ 2 & 1 \end{ytableau} = 6!3 \tag{23.11}$$

Claim 65 (**hook rule** *for computing* $dim(\mathcal{Y}|S_{n_b})$)

$$dim(\mathcal{Y}|S_{n_b}) = \frac{n_b!}{hook(\mathcal{Y})} \tag{23.12}$$

proof:
QED

For example

$$\mathcal{Y} = \begin{ytableau} \; & \; \\ \; \\ \; \end{ytableau} \implies hook(\mathcal{Y}) = \begin{ytableau} 4 & 1 \\ 2 \\ 1 \end{ytableau} = 8 \tag{23.13}$$

so

$$dim(\mathcal{Y}|S_4) = \frac{4!}{4(2)} = 3 \tag{23.14}$$

23.1.2 Regular Representation

The **regular representation** of the symmetric group S_{n_b} is defined as follows. For each permutation $\sigma \in S_{n_b}$, define an independent vector $|\sigma\rangle$ in a vector space $\mathcal{V} = \{|\sigma\rangle : \sigma \in S_{n_b}\} = \{|\sigma_i\rangle : i = 1, 2, \ldots, n_b!\}$. Let

$$|x\rangle = \sum_i x_i |\sigma_i\rangle \tag{23.15}$$

For any $\tau \in S_{n_b}$, suppose

$$\langle \sigma_j | \tau | \sigma_i \rangle = \langle \sigma_j \tau | \sigma_i \rangle \tag{23.16}$$

$$\langle \sigma_j | \tau | x \rangle = \langle \sigma_j \tau | x \rangle = \sum_i x_i \langle \sigma_j \tau | \sigma_i \rangle \tag{23.17}$$

Claim 66 *The regular rep is $n_b!$ dimensional and reducible. In the decomposition of the regular rep of S_{n_b}, each $\lambda \in irreps(S_{n_b})$ appears $dim(\lambda|S_{n_b})$ times.*

proof:
QED

From the last claim, it follows that

$$n_b! = |S_{n_b}| = \sum_{\lambda \in irreps(S_{n_b})} [dim(\lambda|S_{n_b})]^2 \tag{23.18}$$

$$= [n_b!]^2 \sum_{\mathcal{Y} \in YD(n_b)} \frac{1}{[hook(\mathcal{Y})]^2} \quad \text{(Because } |irreps(S_{n_b})| = |YD(n_b)|\text{)} \tag{23.19}$$

Hence,

$$1 = n_b! \sum_{\mathcal{Y} \in YD(n_b)} \frac{1}{[hook(\mathcal{Y})]^2} \tag{23.20}$$

The Clebsch-Gordan series for the regular rep of S_{n_b} is

$$1 = \sum_{\mathcal{Y} \in YD(n_b)} P_{\mathcal{Y}} \tag{23.21}$$

where each $P_{\mathcal{Y}}$ can be further decomposed into

$$P_{\mathcal{Y}} = \sum_{\mathcal{Y}_\alpha \in SYT(\mathcal{Y})} \underbrace{|\mathcal{Y}_\alpha\rangle\langle\mathcal{Y}_\alpha|}_{P_{\mathcal{Y}_\alpha}} \tag{23.22}$$

The projection operators

$$\{P_{\mathcal{Y}_\alpha} : \mathcal{Y}_\alpha \in STY(\mathcal{Y}), \mathcal{Y} \in YD(n_b)\} = \{P_{\mathcal{Y}_\alpha} : \mathcal{Y}_\alpha \in SYT(n_b)\} \tag{23.23}$$

are complete and orthogonal.

23.1.3 Tensor Product Decompositions

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \tag{23.24}$$

$$\begin{aligned} &\ydiagram{3} \otimes \ydiagram{2,1} = \\ &\ydiagram{5,1} \oplus \ydiagram{4,2} \oplus \ydiagram{4,1,1} \oplus \ydiagram{3,2,1} \end{aligned} \tag{23.25}$$

23.2 Unitary group $U(n)$

Let

$STY(n_b, NR < n')$ = set of STY with n_b boxes and number of rows $NR < n'$

Recall that[2]

$$irreps(U(n)) = \bigcup_{n_b \le n,\, NR \le n} STY(n_b, NR) = \bigcup_{n_b=1}^{n} STY(n_b, NR < n) \tag{23.26a}$$

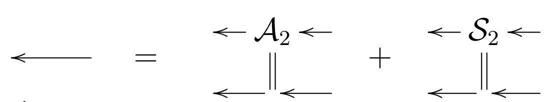
$$irreps(SU(n)) = \bigcup_{n_b \le n,\, NR \le n-1} STY(n_b, NR) = \bigcup_{n_b=1}^{n} STY(n_b, NR < n-1) \tag{23.26b}$$

A SYT with n_b boxes represents a tensor with n_b indices (n_b-particles state). Each index ranges from 1 to n.

$n_b = 1$: A 1-index, 1-box tensor is a 1-particle with n states. This corresponds to the fundamental representation.

$n_b = 2$: A 2-index, 2-box tensor is a 2-particle with n^2 states. These n^2 states break into two sets, symmetric and anti-symmetric.

$$\young(1) \otimes \young(1) = \young(1,2) \oplus \young(12) \tag{23.27}$$



(23.28)

[2]Note that $STY(n_b)$ only contains STY with $n_b \le n$ boxes, so the $n_b \le n$ constraint might seem redundant in Eqs.(23.26). It isn't redundant because by $\cup_{n_b \le n}$ we mean $\cup_{n_b=1}^{n}$.

The SYT of an irrep describes the symmetry of the indices of a tensor in that irrep. A single column SYT indicates a totally anti-symmetric tensor, a single row SYT indicates a totally symmetric tensor, and a SYT with more than one row or column indicates a mixed symmetry tensor. This is why we can't have more than n rows, because there are only n integers to fill all boxes so more than n rows would require a repetition of an integer in a column, and such a column, after antisymetrizing, would lead to zero.

23.2.1 Young Projection Operators

For each SYT $\mathcal{Y}_\alpha \in irreps(U(n))$, define the **Young projection operator**

$$P_{\mathcal{Y}_\alpha} = \mathcal{N}\left(\prod_i S_i\right)\left(\prod_j A_j\right) \tag{23.29}$$

for some normalization constant $\mathcal{N}$ yet to be determined. These projection operators are not unique.

Claim 67

$$\mathcal{N} = \frac{\left(\prod_i |S_i|!\right)\left(\prod_j |A_j|!\right)}{hook(\mathcal{Y})} \tag{23.30}$$

where $|S_i|$ and $|A_j|$ are the number of arrows entering the symmetrizer or anti-symmetrizer. Note that the normalization constant $\mathcal{N}$ depends only on the YD $\mathcal{Y}$. Furthermore, the operators $P_{\mathcal{Y}_\alpha}$ are idempotent (i.e., their square equals themselves) mutually orthogonal and complete:

$$P_{\mathcal{Y}_\alpha} P_{\mathcal{Y}_\beta} = P_{\mathcal{Y}_\alpha}\delta(\alpha,\beta) \tag{23.31}$$

$$1 = \sum_{\mathcal{Y}_\alpha \in SYT(n_b,\, NR<n')} P_{\mathcal{Y}_\alpha} \tag{23.32}$$

where

$$n' = \begin{cases} n & \text{for } U(n) \\ n-1 & \text{for } SU(n) \end{cases} \tag{23.33}$$

proof:

$$P_{\mathcal{Y}_\alpha} = \mathcal{N}\frac{1}{\prod_i |S_i|!\prod_j |A_j|!}\left(\underbrace{\begin{array}{c}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{array}}_{\mathbb{1}} + \cdots\right) \tag{23.34}$$

From Eq.(23.32)

$$
\begin{aligned}
\mathbb{1} &= \sum_{\mathcal{Y}_\alpha \in SYT(n_b, NR<n')} \mathcal{N} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} && (23.35)\\
&= \sum_{\mathcal{Y} \in YD(n_b)} \frac{n_b!}{hook(\mathcal{Y})} \mathcal{N} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} && (23.36)\\
&= \sum_{\mathcal{Y} \in YD(n_b)} \frac{n_b!}{[hook(\mathcal{Y})]^2} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} \quad \text{(if assume Eq.(23.30))} && (23.37)\\
&= \mathbb{1} \quad \text{(by Eq.(23.20))} && (23.38)
\end{aligned}
$$

QED

23.2.2 $dim(\mathcal{Y}_\alpha | U(n))$

Let $dim(\mathcal{Y}_\alpha|U(n))$ be the dimension of an irrep of $U(n)$ with STY given by $\mathcal{Y}_\alpha \in STY(n_b, NR < n)$.

Claim 68

$$dim(\mathcal{Y}_\alpha|U(n)) = |YT(\mathcal{Y})| \tag{23.39}$$

Note that the right hand side is independent of α, so this dimension is the same for all irreps α with the same YD $\mathcal{Y}$.

proof:
QED

Hence, $\{|\alpha\rangle : \alpha \in YT(\mathcal{Y})\}$ are a basis for the irrep $\mathcal{Y}_\alpha$ of $U(n)$. Note that the irreps of $U(n)$ are given by SYT $\mathcal{Y}_\alpha$, whereas the basis vectors of an irrep are given by YT (not just STY). For example, for

$$\mathcal{Y}_\alpha = \boxed{1 \mid 2} \tag{23.40}$$

the basis vectors are

$$|\boxed{1 \mid 1}\rangle, \quad |\boxed{1 \mid 2}\rangle, \quad |\boxed{2 \mid 2}\rangle \tag{23.41}$$

so

$$dim(\mathcal{Y}_\alpha|U(2)) = 3 \tag{23.42}$$

In Eq.(23.39) we gave a way of finding $dim(\mathcal{Y}_\alpha|U(n))$ A second way is by taking the trace of the corresponding projection operator

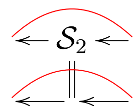
$$dim(\mathcal{Y}_\alpha|U(n)) = \mathrm{tr}(P_{\mathcal{Y}_\alpha}) \tag{23.43}$$

For example, if

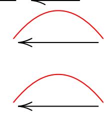
$$\mathcal{Y}_\alpha = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \tag{23.44}$$

then

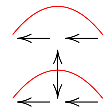
$$dim(\mathcal{Y}_\alpha|U(n)) \quad = \tag{23.45}$$



$$= \frac{1}{2}\Bigg(\tag{23.46}$$



$$+$$



$$\Bigg)$$

$$\begin{aligned} &= \frac{1}{2}(n^2+n) && (23.47) \\ &= 3 \text{ for } n = 2 && (23.48) \end{aligned}$$

A third way of computing $dim(\mathcal{Y}_\alpha|U(n))$ is by computing the hook and coat functions and using the formula

$$dim(\mathcal{Y}_\alpha|U(n)) = \frac{coat(\mathcal{Y})}{hook(\mathcal{Y})} \tag{23.49}$$

Note that right hand side is independent of α; it depends only on the YD. We've already discussed how to compute $hook(\mathcal{Y})$. $coat(\mathcal{Y})$ is calculated as follows.[3]

1. Fill $\mathcal{Y}$ with
   - n at the diagonal blocks
   - n increasing by 1 per block when reading from left to right
   - n decreasing by 1 per blockk when reading from top to bottom
2. multiply all the boxes

   Examples

$$dim(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, U(2)) = \frac{\begin{array}{|c|c|}\hline \text{n} & \text{n+1} \\ \hline \end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}} = \frac{n(n+1)}{2} \tag{23.50}$$

$$dim(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, U(2)) = \frac{\begin{array}{|c|}\hline \text{n} \\ \hline \text{n-1} \\ \hline \end{array}}{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}} = \frac{n(n-1)}{2} \tag{23.51}$$

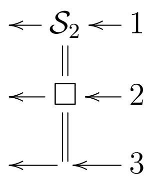
[3] I invented the name $coat(\mathcal{Y})$. I don't know if it has a name.

$$dim\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2} 7\\\cline{1-1}\end{array}, U(7)\right) = \frac{\begin{array}{|c|c|c|c|}\hline n & n+1 & n+2 & n+3\\\hline n-1 & n \\\cline{1-2} n-2\\\cline{1-1}\end{array}}{\begin{array}{|c|c|c|c|}\hline 6&4&2&1\\\hline 3&1\\\cline{1-2} 1\\\cline{1-1}\end{array}} = \frac{n^2(n^2-1)(n^2-4)(n+3)}{144} \qquad (23.52)$$

23.2.3 Young Projection Operators for $n_b = 1, 2, 3, 4$

Symmetrizers $\mathcal{S}_p$ and antisymmetriers $\mathcal{A}_p$ are discussed in Chapter 20.

In this section, we will use symmetrizers and antisymmetrizers with "holes" A hole, denoted by an empty square, will signify a particle that the symmetrizer or antisymmetrizer does not touch. For example

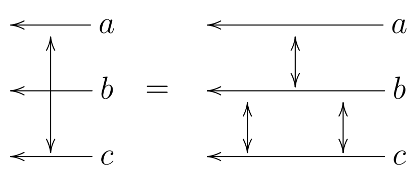


(23.53)

denotes a symmetrizer of the particles 1 and 3 but not 2.

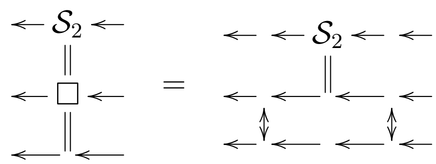
Note that

$(c, a) = (b, c)(b, a)(b, c)$



(23.54)

Similarly

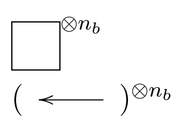


(23.55)

Hence, one can avoid using symmetrizers and antisymmetrizers with holes, if one is willing to use swaps instead of holes.

Below, we use holes, but keep in mind that those holes can we replaced by swaps.

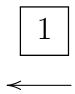
Below, we give the Clebsch-Gordan decomposition of



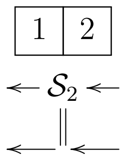
(23.56)

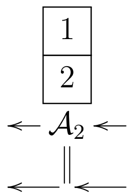
for $n_b = 1, 2, 3, 4$.

- $n_b = 1$

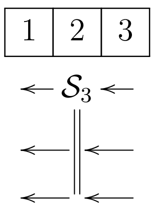


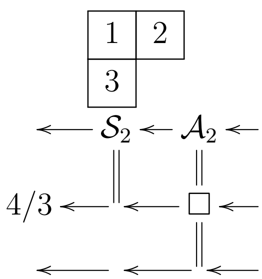
- $n_b = 2$

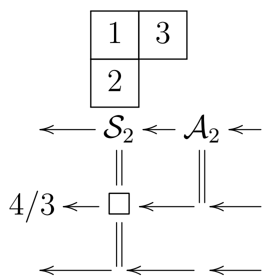


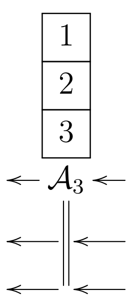


- $n_b = 3$

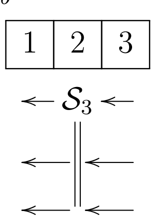


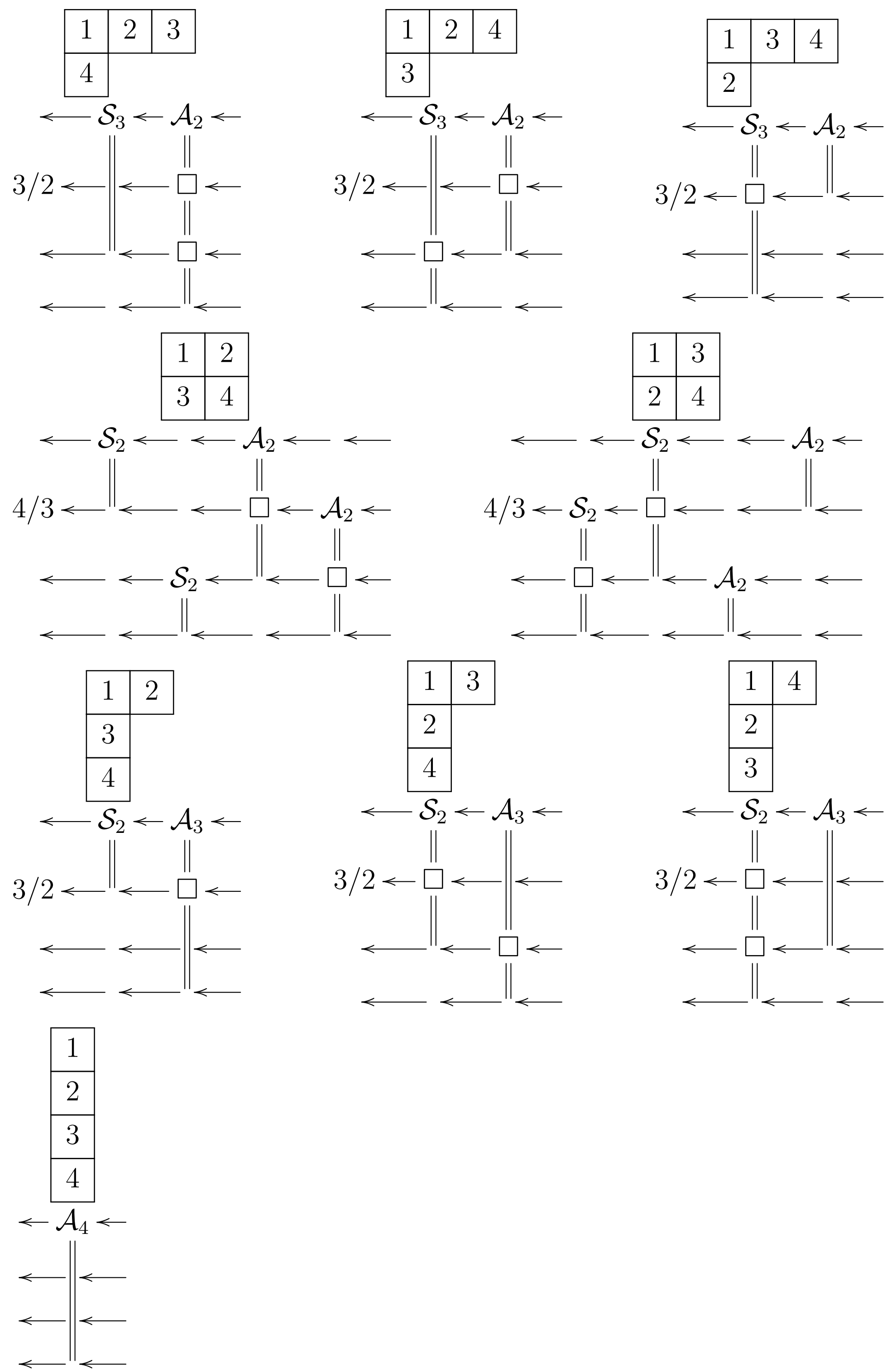






- $n_b = 4$




1 2 3 / 4
$\mathcal{S}_3$ $\mathcal{A}_2$
3/2
1 2 4 / 3
$\mathcal{S}_3$ $\mathcal{A}_2$
3/2
1 3 4 / 2
$\mathcal{S}_3$ $\mathcal{A}_2$
3/2
1 2 / 3 4
$\mathcal{S}_2$ $\mathcal{A}_2$
4/3
$\mathcal{A}_2$
$\mathcal{S}_2$
1 3 / 2 4
$\mathcal{S}_2$ $\mathcal{A}_2$
4/3 $\mathcal{S}_2$
$\mathcal{A}_2$
1 2 / 3 / 4
$\mathcal{S}_2$ $\mathcal{A}_3$
3/2
1 3 / 2 / 4
$\mathcal{S}_2$ $\mathcal{A}_3$
3/2
1 4 / 2 / 3
$\mathcal{S}_2$ $\mathcal{A}_3$
3/2
1 / 2 / 3 / 4
$\mathcal{A}_4$

23.2.4 Young Projection Operator with Swaps

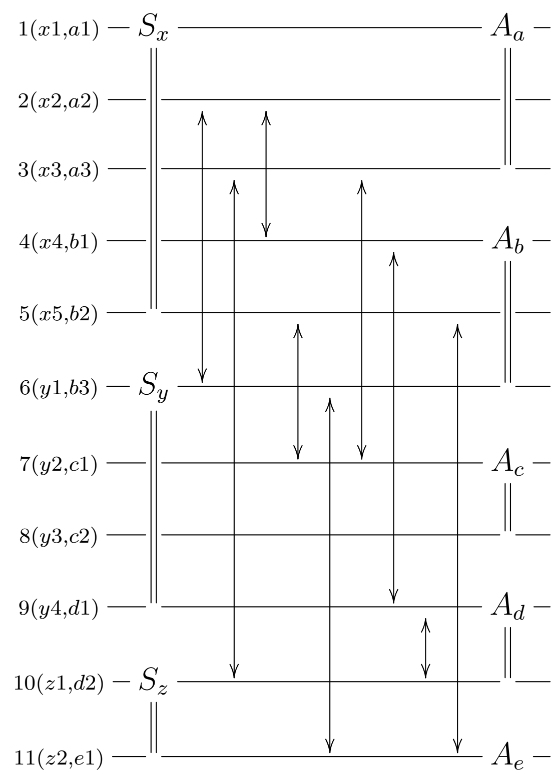
Eq.(23.57) gives a particular STY $\mathcal{Y}_\alpha$, and its projector $P_{\mathcal{Y}_\alpha}$. the projector is expressed using swaps instead of holes.

$\mathcal{Y}_\alpha =$

	A_a	A_b	A_c	A_d	A_e
S_x	1	2	3	4	5
S_y	6	7	8	9	
S_z	10	11			

$a1 \longleftrightarrow 1$	
$a2 \longleftrightarrow 6$	
$a3 \longleftrightarrow 10$	
$b1 \longleftrightarrow 2$	
$b2 \longleftrightarrow 7$	
$b3 \longleftrightarrow 11$	
$c1 \longleftrightarrow 3$	
$c2 \longleftrightarrow 8$	
$d1 \longleftrightarrow 4$	
$d2 \longleftrightarrow 9$	
$e1 \longleftrightarrow 5$	

$P_{\mathcal{Y}_\alpha} =$

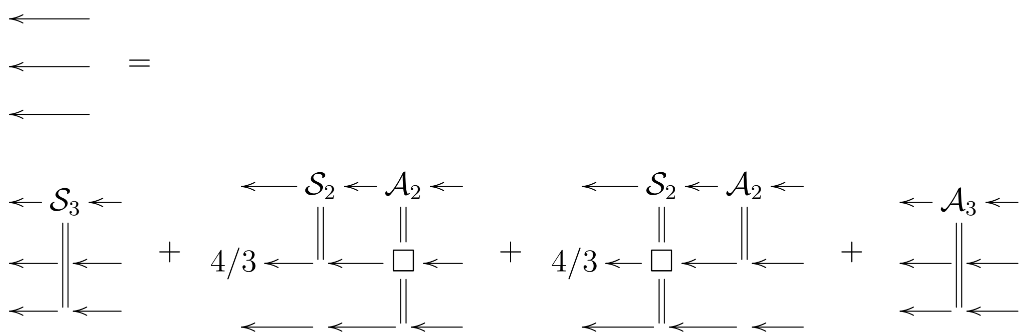


(23.57)

23.2.5 Tensor Product Decompositions

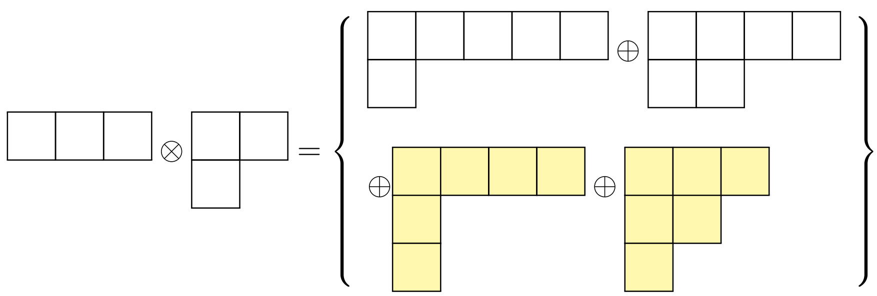
$$\boxed{1} \otimes \boxed{2} \otimes \boxed{3} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right) \otimes \boxed{3} \tag{23.58}$$

$$= \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \tag{23.59}$$



(23.60)

$$n^3 = \frac{n(n+1)(n+2)}{6} + \frac{n(n^2-1)}{3} + \frac{n(n^2-1)}{3} = \frac{(n-2)(n-1)n}{6} \tag{23.61}$$



(23.62)

For $SU(n)$, the yellow YDs are zero for $n = 2$, and non-zero for $n \geq 2$.

23.2.6 $SU(n)$

For $U(n)$ (as opposed to $SU(n)$), there are no antiparticles (i.e., one can use only lowered indices). A consequence of this is that for proper operators in $U(n)$, the total particle number is conserved.

The elements G of $SU(n)$ satisfy

$$\underbrace{\epsilon_{12\ldots n}}_{1} = \underbrace{G_1{}^{a'_1} G_2{}^{a'_2} \cdots G_n{}^{a'_n}\ \epsilon_{a'_1 a'_2 \ldots a'_n}}_{\det G} \tag{23.63}$$

$$\epsilon_{a_1 a_2 \ldots a_n} = G_{a_1}{}^{a'_1} G_{a_2}{}^{a'_2} \cdots G_{a_n}{}^{a'_n}\ \epsilon_{a'_1 a'_2 \ldots a'_n} \tag{23.64}$$

so the Levi-Civita tensor is a primitive invariant of $SU(n)$ (but not of $U(n)$)

This leads to 2 consequences.

1. YD for $SU(n)$ has a maximum of $n-1$ rows.

   For an example of this, see Fig.23.2. The yellow columns in that figure are singlets obtained by fully contracting Levi-Civita tensors. Hence, those yellow columns can removed.

2. Conjugate YD

   Given a YD $\mathcal{Y}$, its **conjugate YD** $conj(\mathcal{Y})$ is obtained as follows:

   - add yellow colored boxes to the original YD so that the resulting YD is rectangular and has n rows for each column.
   - keep only the yellow boxes, and rotate those clockwise by 180 degrees.

   See Fig.23.3 for an example of constructing a conjugate YD.

   This is possible because in the intermediate rectangular YD, the columns with n white and yellow boxes represent a fully contracted Levi-Civita tensor.

   Claim 69 *The reps corresponding to YDs $\mathcal{Y}$ and $conj(\mathcal{Y})$ have the same dimension.*

   proof:
   QED

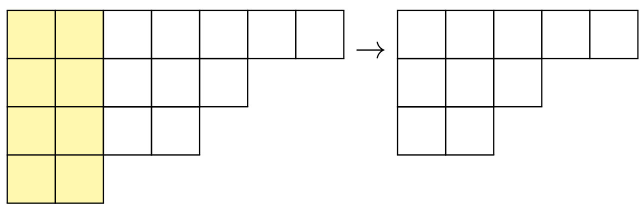


Figure 23.2: Iluustration of removal of columns 4 boxes long when dealing with $SU(4)$. In this case, the YD in Dynkin notation goes from $[2,1,2,2,0,\ldots]_D$ to $[2,1,2,0,\ldots]_D$

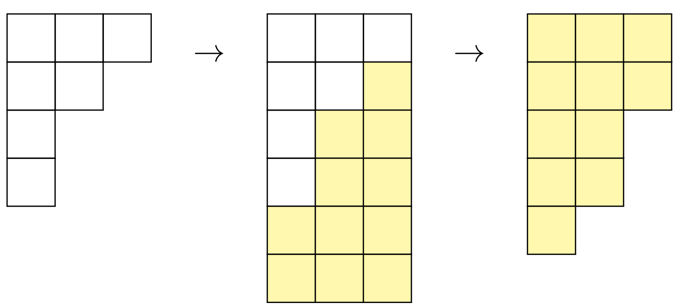


Figure 23.3: Construction of a conjugate YD for $SU(6)$

Besides the RL (row lengths) and RC/D (row change/Dynkin) methods discussed previously, a third method commonly used to label YDs for $SU(n)$ is as follows. Label them by their dimension, and then add a subscript or overline if there are more than one reps with a different YD but the same dimension. This method is used mostly by physicists for $SU(3)$ (The Eightfold Way). Note that all SYT with the same YD have the same dimension, so this really labels YD instead of YT. For example, for $SU(3)$ we have

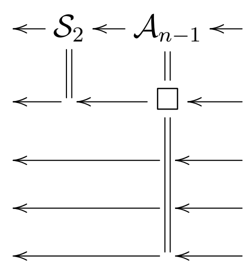
$$
\begin{aligned}
&\ydiagram{1} = [1,0]_D = 3 \qquad && \ydiagram{2} = [0,1]_D = \overline{3} \\
&\ydiagram{2} = [2,0]_D = 6 \qquad && \ydiagram{2,2} = [0,2]_D = \overline{6} \\
&\ydiagram{2,1} = [1,1]_D = 8 \qquad && \ydiagram{3,1} = [2,1]_D = 15
\end{aligned}
\tag{23.65}
$$

Using this notation, we have for $SU(n)$,

$$
\ydiagram{1} \otimes \left.\begin{array}{c}\ydiagram{1,1}\\ \vdots \\ \ydiagram{1}\end{array}\right\} n-1 \text{ rows} \quad = \quad 1 \oplus \left.\begin{array}{l}\ydiagram{2,1}\\ \vdots \\ \ydiagram{1}\end{array}\right\} n-1 \text{ rows} \tag{23.66}
$$

$$
\begin{aligned}
n \otimes \overline{n} &= 1 \oplus (n^2 - 1) && (23.67) \\
\text{fun rep} \otimes \text{conjugate rep} &= \text{singlet rep} \oplus \text{ adjoint rep} && (23.68)
\end{aligned}
$$

Adjoint representation

$$P_{adj} = \frac{2(n-1)}{n}$$ 

(23.69)

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[2] JP Elliott and PG Dawber. *Symmetry in Physics, vols. 1, 2*. Springer, 1979.

[3] Robert R. Tucci. Bayesuvius (free book). `https://github.com/rrtucci/Bayesuvius`.

[4] Robert R. Tucci. Quantum Bayesian nets. *International Journal of Modern Physics B*, 09(03):295–337, January 1995.