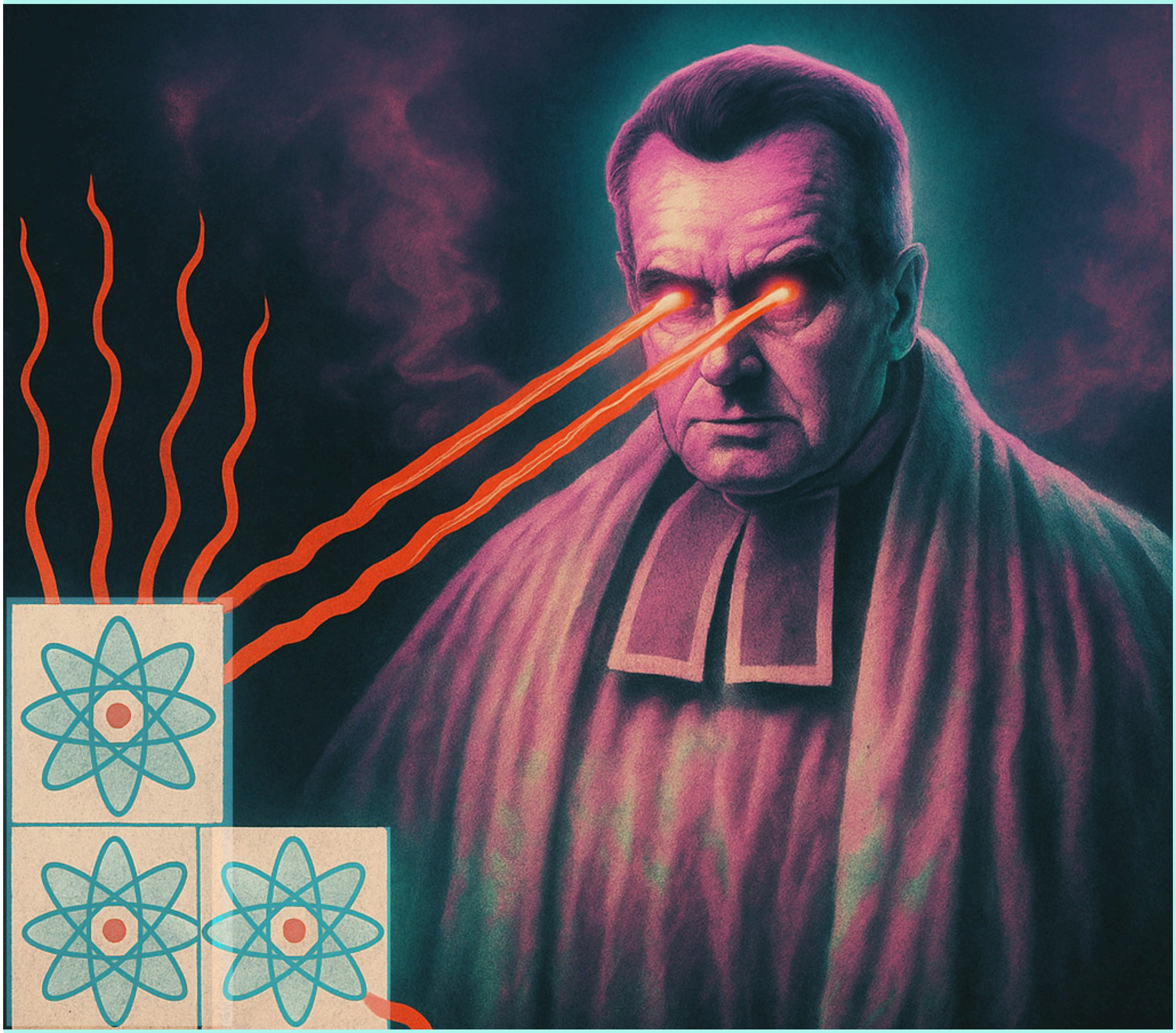
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



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Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayesuvius Quantico
by Robert R. Tucci

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Appendix A

Notational Conventions and Preliminaries

This book is a sequel to my book entitled "Bayesuvius" (see Ref.[3]). For consistency, I have tried to follow in this book the same notational conventions used in the prior book. If any notation is not defined in this book, check in the prior book. It might be defined there.

A.1 Set notation

The number of elements in any set S is denoted by $|S|$.

$\mathbb{Z}$ = integers
$\mathbb{Z}_{>0}$ = positive integers
$\mathbb{Z}_{[a,b]} = a, a+1, \ldots, b$ for some integers a, b such that $a \leq b$
$\mathbb{R}$ = reals
$\mathbb{C}$= complex numbers
$\mathbb{C}^{n\times m} = n \times m$ matrices of complex numbers

A.2 Group Theory References

Much of this book deals with Group Theory (GT).

GT is a vast subject. Who would have thought that the simple definition of a group would generate so many elegant and useful results.

GT books by mathematicians are very different from GT books by physicists, even though, of course, they agree on the definitions. Mathematicians are, as to be expected, more rigorous and abstract. But it goes much further than that. Physicists are much more interested in applications to physical systems, especially Quantum Mechanics (QM). Soon after QM was invented, it was realized that Linear Algebra (LA) and GT (especially Group Representation Theory, which combines GT and LA) are extremely relevant and useful in QM. Hermann Weyl, Eugene Wigner, Hans Bethe, Linus Pauling, etc. combined QM and GT to understand the spectra and chemistry of atoms and molecules, and later GT was heavily used in Quantum Field Theory and Particle Physics to devise the Standard Model. Condensed Matter physicists have also used it to understand crystalline solids and to predict quasi particles that can be detected in the lab.

My PhD is in physics so in this book I cover GT topics that are mainly of interests to physicists and engineers. Furthermore, I am nowhere as abstract and rigorous as mathematicians usually are.

My favorite books about GT for physicists are the Elliott & Dawber's (ED) 2 volume series Ref. [2] and Predrag Cvitanovic's Birdtracks book Ref.[1]. I highly recommend both of these references. I think both of them are excellent.

The Birdtracks book explains key concepts in GT representation theory using network diagrams (Cvitanovic calls such diagrams birdtracks) The ED books, on the other hand, do not use birdtracks. They use algebra instead. In fact, most GT books don't use birdtracks either. But since this is a book about visualization using network diagrams (quantum bnets), we use birdtracks. In fact, many of the chapters in this book were heavily influenced by Ref.[1] by Cvitanovic. I hope he doesn't mind. I really love his book.

A.3 Group

A **group** $\mathcal{G}$ is a set of elements with a multiplication map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ such that

1. the multiplication is **associative** ; i.e.,

$$(ab)c = a(bc) \tag{A.1}$$

   for $a, b, c \in \mathcal{G}$.

2. there exists an **identity element** $e \in \mathcal{G}$ such that

$$ea = ae = a \tag{A.2}$$

   for all $a \in \mathcal{G}$

3. for any $a \in \mathcal{G}$, there exists an **inverse** $a^{-1} \in \mathcal{G}$ such that

$$aa^{-1} = a^{-1}a = e \tag{A.3}$$

$|\mathcal{G}|$ (i.e., number of elements in $\mathcal{G}$) is called the **order** of the group.

If multiplication is **commutative** (i.e., $ab = ba$ for all $a, b \in \mathcal{G}$), the group is said to be **abelian**.

A **subgroup** $\mathcal{H}$ of $\mathcal{G}$ is a subset of $\mathcal{G}$ ($\mathcal{H} \subset \mathcal{G}$) which is also a group. It's easy to show that any $\mathcal{H} \subset \mathcal{G}$ is a group if it contains the identity and is **closed under multiplication** (i.e., $ab \in \mathcal{H}$ for all $a, b \in \mathcal{H}$)

A.4 Group Representation

A **group representation** (rep) of a group $\mathcal{G}$ is a map $\phi : \mathcal{G} \to \mathbb{C}^{n\times n}$[1] such that

$$\phi(a)\phi(b) = \phi(ab), \quad \phi(e) = I \tag{A.4}$$

where e is the identity of the group and I is the identity matrix. Such a map is called a **homomorphism** (because it preserves an operation). The map ϕ partitions $\mathcal{G}$ into disjoints subsets (equivalence classes), such that all elements of $\mathcal{G}$ in each disjoint subset are represented by the same matrix.

In this book, we will usually label reps by a Greek letter such as λ, and we will refer to $\phi(g) = G_\lambda(g) = G_\lambda$ as the **representation matrix** (rep-matrix) of $g \in \mathcal{G}$.

One way to specify a representation is to give the effect of each group element $a \in \mathcal{G}$ on a basis of vectors $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$.

$$\phi(a)|i\rangle = \sum_j M_{ij}|j\rangle \implies \langle i|\phi(a)|j\rangle = M_{ij} \tag{A.5}$$

If the map ϕ is 1-1, onto, we call it a **faithful representation**

The **trivial "representation"** represents all $g \in \mathcal{G}$ by $diag(1, 1, \ldots, 1)$. The dimension of this "rep" is $d_\lambda = 0$. This "rep" is often not considered a rep because it doesn't capture any of the group structure.

A **singlet representation** is any 1 dimensional rep. It represents each $g \in \mathcal{G}$ by $z(g)diag(1, 1, \ldots, 1)$ for some $z(g) \in \mathbb{C}$. $z(g)$ is a 1×1 matrix, a scalar, so the rep is one dimensional. For the singlet rep, $\langle i|\phi(a)|j\rangle = z(g)\delta(i, j)$. The dimension of this rep is $d_\lambda = 1$. The projection operator $\delta_a^b \delta_c^d$ when acting on ${G_c}^d$ gives a $z(G)diag(1, 1, \ldots, 1)$ where $z(G) = {G_c}^c = \mathrm{tr}(G)$, so it projects to a singlet rep.

When a group is defined using matrices, those matrices are called the **defining representation** (defrep). For example, the group of **General Linear Transformations** is defined by

$$GL(n, \mathbb{C}) = \{M \in \mathbb{C}^{n\times n} : \det M \neq 0\} \tag{A.6}$$

The **adjoint representation** (adjrep) is defined in terms of the structure constants of the Lie Algebra. If the Lie Algebra satisfies $[T^i, T^j] = if_{ijk}T^k$, then the adjrep is given by the matrices with i, j entries $M_{ij}^k = -i{f^k}_{ij}$. Let $|x\rangle = x_i|T^i\rangle$. Then

$$[|x\rangle, \cdot]|T^j\rangle = |[x, T^j]\rangle = ix_i f_{ijk}|T^k\rangle \implies \langle T^k|[|x\rangle, \cdot]|T^j\rangle = ix_i f_{ijk} \tag{A.7}$$

Irreducible representations (irreps) are defined in Ch. 13

The **fundamental representation** (funrep) is defined as the smallest irrep. The defrep equals the funrep for $SU(n), SO(n), SP(n)$, but not for E_8.

[1]More generally, the $\mathbb{C}^{n\times n}$ can be replaced by $\mathbb{R}^{n\times n}$ or by $\mathbb{F}^{n\times n}$ for any field $\mathbb{F}$

A.5 Vector Space and Algebra over a field $\mathbb{F}$

A **vector space (a.k.a. linear space) $\mathcal{V}$ over a field** $\mathbb{F}$ is defined as a set $\mathcal{V}$ endowed with two operations: vector addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and scalar multiplication $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$, such that

- $\mathcal{V}$ is an abelian group under $+$ with identity 0 and inverse of $x \in \mathcal{V}$ equal to $-x \in \mathcal{V}$
- For $\alpha, \beta \in \mathbb{F}$ and $x, y \in \mathcal{V}$

$$\alpha(x+y) = \alpha x + \alpha y \tag{A.8}$$

$$(\alpha+\beta)x = \alpha x + \beta x \tag{A.9}$$

$$\alpha(\beta x) = (\alpha\beta)x \tag{A.10}$$

$$1x = x \tag{A.11}$$

$$0x = 0 \tag{A.12}$$

In this book, we will always use either $\mathbb{C}$ or $\mathbb{R}$ for $\mathbb{F}$. Both of these fields are infinite but some fields are finite.

An **algebra** $\mathcal{A}$ is a vector space which, besides being endowed with vector addition and scalar multiplication as all vector spaces are, it has a **bilinear vector product**. A bilinear vector product is a product that is linear on both sides; i.e.,

$$(\alpha x + \beta y) \cdot z = \alpha x \cdot z + \beta y \cdot z \tag{A.13}$$

and

$$z \cdot (\alpha x + \beta y) = \alpha z \cdot x + \beta z \cdot y \tag{A.14}$$

for $x, y, z \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. The cross product (but not the dot product) for vectors in $\mathbb{R}^3$, the multiplication of 2 complex numbers, the matrix product or matrix commutator of 2 square matrices, are all good examples of bilinear vector products.

Let $B = \{\tau_i : i = 1, 2, \ldots, r\}$ be a basis for the vector space $\mathcal{A}$. Then note that $\mathcal{A}$ is closed under vector multiplication.

$$\tau_i \cdot \tau_j = \sum_k {c_{ij}}^k \tau_k \tag{A.15}$$

where ${c_{ij}}^k \in \mathbb{C}$. The ${c_{ij}}^k$ are called **structure constants** of $\mathcal{A}$. In Dirac notation

$$\tau_i|\tau_j\rangle = |\tau_i \cdot \tau_j\rangle = \sum_k {c_{ij}}^k |\tau_k\rangle \tag{A.16}$$

$$\langle\tau_k|\tau_i|\tau_j\rangle = {c_{ij}}^k \tag{A.17}$$

An **associative algebra** satisfies $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for $x, y, z \in \mathcal{A}$.

- Not associative: cross product for vectors in $\mathbb{R}^3$.
- Associative: the matrix product or matrix commutator of 2 square matrices and the product of complex numbers

A.6 Tensors

Let

$(x_a) = (x_1, x_2, \ldots, x_n) = x^{:n} \in V^n = \mathbb{C}^{n\times 1}$

Reverse of vector $rev(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_1)$

$x^b = \sum_a g^{ba} x_a$, g^{ab} is the **metric tensor**

$(y^b) = (y^1, y^2, \ldots, y^n) = y^{\dagger:n} \in V^{\dagger n} = \mathbb{C}^{n\times 1}$. V^n is the lower indices vector space and $V^{\dagger n}$ is its **dual vector space** (i.e., with upper indices).

${M_a}^b \in \mathbb{C}^{n\times n}$, $a, b \in \mathbb{Z}_{[1,n]}$

Implicit Summation Convention

$$ {M_a}^b x_b = \sum_{b=1}^{n} {M_a}^b x_b \tag{A.18} $$

The **Hermitian conjugate** $\dagger$ equals $*T$ where $*$ is complex conjugation and T is transpose. Hence

$$ {(M^T)_b}^a = {M_a}^b \tag{A.19} $$

$$ {(M^\dagger)_b}^a = ({M_a}^b)^* \tag{A.20} $$

To avoid confusion, follow the golden rule: <u>write $\dagger$ and T only before declaring the indices; and write the $*$ only after declaring the indices.</u> Note that $\dagger$ does 3 things:

1. reverse the horizontal order of the indices
2. reverse vertical positions of the indices; i.e., lower upper indices and raise lower indices.
3. replace the tensor components by their complex conjugates

Transposing only does items 1 and 2.

If M is a Hermitian matrix (i.e., $M^\dagger = M$), then

$$ {M_b}^a = ({M_a}^b)^* \tag{A.21} $$

Suppose $a_i, b_i, c_i, d_i \in \mathbb{Z}_{[1,n]}$. From Fig.A.1

$$ {y_{a^{:p}}}^{b^{:q}} = {{{M_{a^{:p}}}^{b^{:q}}}_{rev(c^{:q})}}^{rev(d^{:p})} {x_{d^{:p}}}^{c^{:q}} \tag{A.22} $$

If we define x_α and x^α by

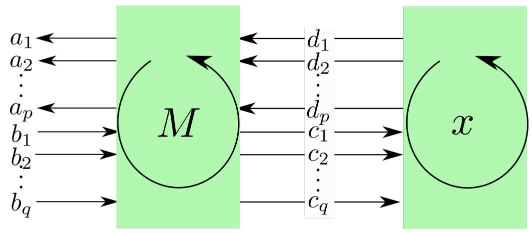

Figure A.1: Index labels for Mx where $M \in \mathbb{C}^{n^{p+q} \times n^{p+q}}$ and $x \in V^{n^p} \otimes V^{\dagger n^q}$. Note that we list indices in counterclockwise (CC) direction, starting at the top.

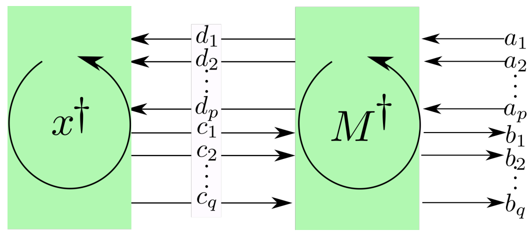

Figure A.2: Index labels for $x^\dagger M^\dagger$ corresponding to Fig.A.2. Note that we list indices in counterclockwise (CC) direction, starting at the top.

$$x_\alpha = x_{a^{:p}}{}^{b^{:q}}, \quad x^\alpha = x_{rev(b^{:q})}{}^{rev(a^{:p})} \tag{A.23}$$

then

$$x_\alpha = M_\alpha{}^\beta x_\beta \tag{A.24}$$

Hermitian conjugation (see Fig.A.2)

$$\begin{cases} (M^\dagger)_a{}^d = (M_d{}^a)^* \\ (M^\dagger)_\alpha{}^\delta = (M_{rev(\delta)}{}^{rev(\alpha)})^* \end{cases} \tag{A.25}$$

Note that $\dagger$ does 3 things to the birdtrack:

1. It flips the horizontal axis of the figure. (In the algebraic expression of the tensor, this corresponds to reversing the horizontal order of the indices.)

2. For each node, it changes incoming arrows to outgoing ones and vice versa. (In the algebraic expression of the tensor, this corresponds reversing the vertical positions of the indices; i.e., lowering upper indices and raising lower ones.)

3. It replaces the tensor component by its complex conjugate

Hermitian matrix

$$M^\dagger = M, \quad \begin{cases} (M_d{}^a)^* = M_a{}^d \\ (M_{rev(\delta)}{}^{rev(\alpha)})^* = M_\alpha{}^\delta \end{cases} \tag{A.26}$$

Unitary matrix

$$M^\dagger M = 1, \quad \begin{cases} (M_b{}^a)^* M_a{}^c = \delta_b^c \\ (M_{rev(\beta)}{}^{rev(\alpha)})^* M_\alpha{}^\gamma = \delta_{rev(\beta)}^\gamma \end{cases} \tag{A.27}$$

Note that for $x \in V^n$, $y \in V^{\dagger n}$, and $G \in \mathcal{G} \subset GL(n, \mathbb{C})$,

$$(x')_a (y')^b = G^b{}_c G_a{}^d x_d y^c \tag{A.28}$$

If $x \in V^{n^p} \otimes V^{\dagger n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a^{:p}}{}^{b^{:q}} = \mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}}, \quad (x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta) \tag{A.29}$$

where we define

$$\mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} \stackrel{\text{def}}{=} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \prod_{i=1}^{q} G^{\dagger b_i}{}_{c_i} \tag{A.30}$$

An issue that arises with tensors is this: When is it permissible to represent a tensor by M_{ab}^{cd}? If we define M_{ab}^{cd} by

$$M_{ab}^{cd} = M_{ab}{}^{cd} \tag{A.31}$$

then it's always permissible. Then one can define tensors like $M_a{}^{bcd}$ as

$$M_a{}^{bcd} = g^{bb'} M_{ab'}{}^{cd} = g^{bb'} M_{ab'}^{cd} \tag{A.32}$$

One drawback of using the notation M_{ab}^{cd} is that if one is interested in using several versions of M_{ab}^{cd} with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing $M_a{}^{bcd}$, you'll have to write $g^{bb'} M_{ab'}^{cd}$. This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too succinct.

A.7 Permutations

Some well known notation and results about permutations are these.

$(1, 2)$ stands for a **transposition**; i.e., a map that swaps 1 and 2:

$$\begin{pmatrix} 1 & 2 & 3 & \dots & p \\ \searrow\!\!\!\!\swarrow & & \downarrow & & \downarrow \\ 1 & 2 & 3 & \dots & p \end{pmatrix} \tag{A.33}$$

$(3, 2, 1)$ stands for a **permutation**; i.e., a map that maps $3 \to 2 \to 1 \to 3$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & p \\ \searrow & \swarrow & \swarrow & \downarrow & & \downarrow \\ 1 & 2 & 3 & 4 & \dots & p \end{pmatrix} \tag{A.34}$$

Any reordering of $(1, 2, 3, \ldots, p)$ is a permutation of p letters (or numbers or elements).

The set S_p of all permutation of p letters is called the **symmetric group in** p **letters**. It has $p!$ elements (i.e., $|S_p| = p!$) and is a group, where the group's product is map composition and the group's identity element is the identity map.

Any permutation can be expressed as a product of transpositions, For example, $(3, 2, 1) = (3, 2)(2, 1)$.

An **even permutation** such as $(3, 2, 1)$ can be expressed as a product of an even number of transpositions. An **odd permutation** can be expressed as a product of an odd number of transpositions.

Appendix B

Birdtracks

This chapter is based on Cvitanovic's Birdtracks book Ref. [1] and my paper Ref. [4]

The tensor notation discussed in Sec.A.6 is succinct and straightforward, but it's not visually illuminating. The birdtrack notation that we shall discuss in this chapter, is not as succinct as the tensor notation, and can lead to sign errors if you are careless, but it is very visually illuminating. Thus, the tensor and birdtrack notations complement each other well. We will often display results using both, side by side.

B.1 Classical Bayesian Networks and their Instantiations

Classical Bayesian Networks (bnets) are discussed exhaustively in the first book of this series, Ref.[3]. This is a brief section to remind the reader of how they are defined.

Let PD stand for probability distribution.

We call $P_{\underline{y}|\underline{x}} : val(\underline{y}) \times val(\underline{x}) \rightarrow [0,1]$ a **Transition Probability Matrix** (TPM)[1] if

$$\sum_{y \in val(\underline{y})} P_{\underline{y}|\underline{x}}(y|x) = 1 \tag{B.1}$$

In other words, a TPM is a conditional PD. A TPM of the form

$$P(y|x) = \delta(y, f(x)) \tag{B.2}$$

for some function $f : val(\underline{x}) \rightarrow val(\underline{y})$ is said to be **deterministic**.

A bnet is a **Directed Acyclic Graph** (DAG) with the nodes labelled by random variables[2]. Each bnet stands for a full PD of the node random variables expressed as a product of a TPM for each node. For example, the bnet

[1] A TPM is also known as a Conditional Probability Table (CPT).

[2] As in the first volume of this series, we indicate random variables by underlined letters

$$\mathcal{C} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ul] \arrow[ll] \end{tikzcd} \tag{B.3}$$

stands for the full PD

$$P(a,b,c) = P(c|b,a)P(b|a)P(a) \tag{B.4}$$

Bnets do not have free indices because their nodes are labelled by random variables. It is convenient to draw the DAG for a bnet but with the underlining removed from the random variables, and then to assign a numerical value to this new DAG. The resultant DAG now has free indices. We call it an **instantiation of the bnet**. For example, from the bnet $\mathcal{C}$ of Eq.(B.3), we get the instantiation[3]

$$P(a,b,c) = P(c|b,a)P(b|a)P(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ul] \arrow[ll] \end{tikzcd} \quad P(a) \tag{B.5}$$

Let $a^{:2} = (a_1, a_2)$. Based on the bnet $\mathcal{C}$ of Eq.(B.3), define a new bnet $\mathcal{C}'$ as follows

$$\mathcal{C}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^{:2} \arrow[ul, "\underline{a}_1"'] \arrow[ll, "\underline{a}_2"] \end{tikzcd} \tag{B.6}$$

$\mathcal{C}'$ represents the the full PD

$$P(a^{:2}, b, c) = P(c|b, a_2)P(a_2|a^{:2})P(b|a_1)P(a_1|a^{:2})P(a^{:2}) \tag{B.7}$$

The 2 new nodes $\underline{a}_1$ and $\underline{a}_2$ of bnet $\mathcal{C}'$ are called **marginalizer nodes**. We assign to them the following TPMs (printed in blue):

$$P[a'_i|\underline{a}^{:2} = (a_1, a_2)] = \delta(a'_i, a_i) \tag{B.8}$$

for $i = 1, 2$. We can also define an instantiation of $\mathcal{C}'$ as follows:

$$P'(a^{:2}, b, c) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^{:2} \arrow[ul, "a_1"'] \arrow[ll, "a_2"] \end{tikzcd} \quad P(a^{:2}) \tag{B.9}$$

[3]Note that we don't include the root node probabilities as part of the graph value. Thus, $P(a,b) = \underbrace{b \leftarrow a}_{P(b|a)} P(a)$

B.2 Quantum Bayesian Networks and their Instantiations

As far as I know, Quantum Bayesian Networks (qbnets) were invented by me in Ref.[4].

qbnets are closely analogous to classical bnets, but the TPM are replaced by **Transition Amplitude Matrices (TAM)**.

Let PA stand for probability amplitude.

We call $A_{\underline{y}|\underline{x}} : val(\underline{y}) \times val(\underline{x}) \rightarrow \mathbb{C}$ a TAM if

$$\sum_{y \in val(\underline{y})} |A(y|x)|^2 = 1 \tag{B.10}$$

Note that if A is the matrix with entries $\langle y|A|x\rangle = A(y|x)$, then

$$\left\langle x|A^\dagger A|x\right\rangle = \sum_{y \in val(\underline{y})} |A(y|x)|^2 = 1 \tag{B.11}$$

If A is a unitary matrix, then $A^\dagger A = AA^\dagger = 1$ so "half" ($A^\dagger A = 1$) of the definition of unitary matrix is satisfied by a TAM. If both halves were satisfied, A would have to be a square matrix.

A qbnet is a DAG with the nodes labelled by random variables. Each qbnet stands for a full PA of the node random variables expressed as a product of a TAM for each node. For example, the qbnet

$$\mathcal{Q} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ul] \arrow[ll] \end{tikzcd} \tag{B.12}$$

stands for the full PA

$$A(a,b,c) = A(c|b,a)A(b|a)A(a) \tag{B.13}$$

Qbnets do not have free indices because their nodes are labelled by random variables. It is convenient to draw the DAG for a qbnet but with the underlining removed from the random variables, and then to assign a numerical value to this new DAG. The resultant DAG now has free indices. We call it an **instantiation of the qbnet**. For example, from the bnet $\mathcal{Q}$ of Eq.(B.12), we get the instantiation

$$A(a,b,c) = A(c|b,a)A(b|a)A(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ul] \arrow[ll] \end{tikzcd} \quad A(a) \tag{B.14}$$

Let $a^{:2} = (a_1, a_2)$. Based on the qbnet $\mathcal{Q}$ of Eq.(B.12), define a new qbnet $\mathcal{Q}'$ as follows

$$\mathcal{Q}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^{:2} \arrow[ul, "\underline{a}_1"'] \arrow[ll, "\underline{a}_2"] \end{tikzcd} \tag{B.15}$$

$\mathcal{Q}'$ represents the the full PA

$$A(a^{:2}, b, c) = A(c|b, a_2)A(a_2|a^{:2})A(b|a_1)A(a_1|a^{:2})A(a^{:2}) \tag{B.16}$$

The 2 new nodes $\underline{a}_1$ and $\underline{a}_2$ of qbnet $\mathcal{Q}'$ are called **marginalizer nodes**. We assign to them the following TAMs (printed in blue):

$$A[a'_i|\underline{a}^{:2} = (a_1, a_2)] = \delta(a'_i, a_i) \tag{B.17}$$

for $i = 1, 2$. We can also define an instantiation of $\mathcal{Q}'$ as follows:

$$A(a^{:2}, b, c) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^{:2} \arrow[ul, "a_1"'] \arrow[ll, "a_2"] \end{tikzcd} \quad A(a^{:2}) \tag{B.18}$$

B.3 Birdtracks

Tensors written in **algebraic notation** such as $T_a{}^{bc}$ were already discussed in Section A.6

Birdtracks are a DAG used to represent algebraic tensor equations. The nodes of the DAG are labelled by tensors and the arrows are labeled by the indices of the tensors: upper indices of a tensor are pictured as incoming arrows of the node, and lower indices as outgoing arrows.

We've already discussed in Section A.6 what we will call the **Counter Clockwise (CC) convention** of drawing birdtrack nodes. Now that we have discussed classical and quantum bnets, we would like to introduce an equivalent, more bnet like, convention that we will call the **Fully Label (FL) convention**. Cvitanovic's birdtracks book Ref.[1] uses the CC convention. We will use both. No confusion will arise, as long as it is clear from context which convention is being used.

Next we review the CC convention and then describe the FL convention for the first time.

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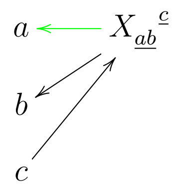
1. **CC convention**

   In the CC convention, we must specify for each the node, which arrow is first, and then the CC order in which the arrows enter or leave the node is drawn so that it reproduces the horizontal order of the indices in the algebraic notation for the tensor. We shall often indicate the first arrow by coloring it green.

   For example,

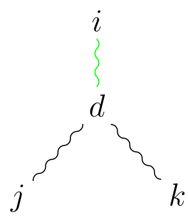
$$\delta(b, a) = \mathbb{1}(a = b) = \delta_a^b = \; a \leftarrow\!\bullet\!\!-\, b \tag{B.19}$$

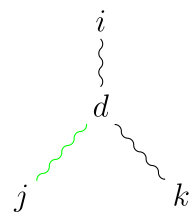
$$X_{ab}{}^{c} =$$ 

(B.20)

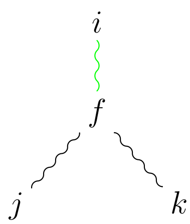
In this picture, the green arrow indicates which tensor index is first horizontally in the algebraic representation of the tensor.

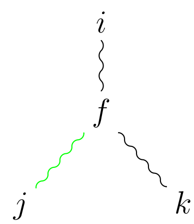
Sometimes there is no need to indicate which arrow is first by drawing it in green, because all choices give the same number. For example, in the birdtracks for δ_a^b, starting with the incoming arrow or the outgoing arrow leads to the same number. Likewise, with the totally symmetric tensor d_{ijk} (doesn't change sign under swap of any two indices) and the totally antisymmetric tensor f_{ijk} (changes sign under swap of any two indices), it doesn't matter if one starts at i, j or k. This is shown below.



$$= d_{ijk} = d_{jki} =$$


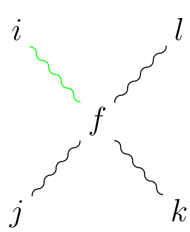
(B.21)

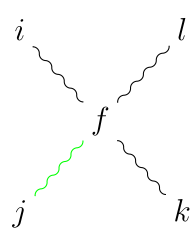


$$= f_{ijk} = f_{jki} =$$


(B.22)

Note that for a totally antisymmetic tensor with an even number of indices, the beginning arrow can change the sign. Indeed,



$$= f_{ijkl} = -f_{jkli} = (-1)$$


(B.23)

2. **FL convention**

In the FL convention, the arrows must be labelled by random (underlined) variables, and the names of the nodes must also indicate by underlined variables what is the the order of the indices

For example,

$$\delta(b,a) = \mathbb{1}(a=b) = \delta^b_a = \begin{tikzcd} a & \bullet \arrow[l] & b \arrow[l, dash] \end{tikzcd} \tag{B.24}$$

$$\langle a,b|X_{\underline{ab}}{}^{\underline{c}}|c\rangle = X_{ab}{}^{c} = \begin{tikzcd} \underline{a}=a & X_{\underline{ab}}{}^{\underline{c}} \arrow[l] \arrow[dl] \\ \underline{b}=b & \\ \underline{c}=c \arrow[uur] & \end{tikzcd} \tag{B.25}$$

Sometimes, we will denote this node simply by X. This is okay as long as we state that $X = X_{\underline{ab}}{}^{\underline{cd}}$, and we don't start using X to represent a different version of $X_{\underline{ab}}{}^{\underline{cd}}$ with some of the indices raised or lowered or their horizontal order changed.

Often, we will write simply a instead of $\underline{a}=a$. This is similar to the shorthand $P(\underline{a}=a) = P(a)$.

Note that, unlike in the CC convention, in the FL convention, the CC order in which the arrows enter or leave the node, is meaningless. All orders are equivalent. This is akin to the notation for bnets and qbnets.

If we don't follow either convention CC or FL, we won't be able to distinguish between the graphical representations of $T^{1,2,3}$ and $T^{2,1,3}$, for example.

Two other features of the CC and FL conventions that we would like to discuss before ending this section are how to indicate

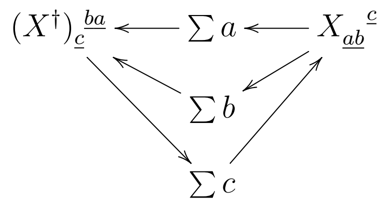
- **noncyclic index contractions**; i.e., index contractions (i.e., summations) that do not introduce cycles, and
- **traces**; i.e., index contractions that do introduce cycles.

Noncyclic index contractions will be indicated by an arrow connecting two nodes, with the symbol $\sum a$ midway in the arrow if the index a is being contracted. For simplicity, we often omit writing the $\sum a$ altogether.

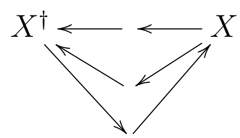
For example (in CC convention),

$$X_{ab}{}^{c} = \begin{tikzcd} a & X_{\underline{ab}}{}^{\underline{c}} \arrow[l] \arrow[dl] \\ b & \\ c \arrow[uur] & \end{tikzcd} \quad , \quad (X^\dagger)_c{}^{ba} = \begin{tikzcd} (X^\dagger)_{\underline{c}}{}^{\underline{ba}} & a \arrow[l] \\ & b \arrow[ul] \\ & c \arrow[uul] \end{tikzcd} \tag{B.26}$$

$$(X^\dagger)_c{}^{ba} X_{ab}{}^c \quad = \tag{B.27}$$



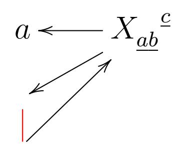
$$= \tag{B.28}$$



Birdtracks are DAGs until we are asked to take a trace of one of their indices. Tracing ruins their acyclicity. The acyclicity of DAGs is mandated by causality. The acyclicity of tracing hints to its acausal (or feedback) nature.

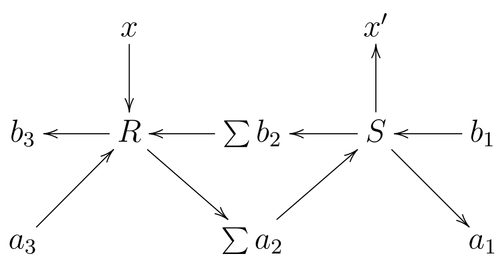
In this book, we will indicate tracing with a red undirected arrow. For example, in the CC convention,

$$\mathrm{tr}_{\underline{b}} X_{a\underline{b}}{}^{\underline{b}} = \sum_b X_{ab}{}^b = \tag{B.29}$$



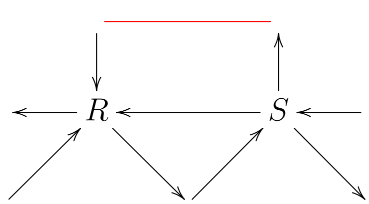
If

$$R^x{}_{b_3}{}^{a_3}{}_{a_2}{}^{b_2} S_{x'b_2}{}^{a_2}{}_{a_1}{}^{b_1} = \tag{B.30}$$



then

$$\mathrm{tr}_{\underline{x}} R^{\underline{x}}{}_{b_3}{}^{a_3}{}_{a_2}{}^{b_2} S_{\underline{x}b_2}{}^{a_2}{}_{a_1}{}^{b_1} = \tag{B.31}$$



When using the FL convention, it becomes clear that birdtracks can be understood as instantiations of qbnets, provided that we weaken slightly the definition of qbnets, by not requiring that the unitarity condition Eq.(B.10) be satisfied. Also, the outgoing arrows of the nodes of a birdtrack must be understood as the result of marginalizer nodes. For example, if the arrows leaving a node are labelled a_1 and a_2, then these two arrows must be understood as the result of marginalizing an arrow $a^{:2} = (a_1, a_2)$.

Chapter 1

Casimir Operators

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

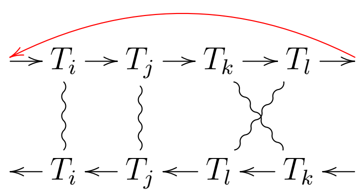
The term **Casimir operator** will be used to refer to 2 types of operators: a Casimir matrix or a Casimir sun

Casimir matrix

Examples:

$M_2 =$ 

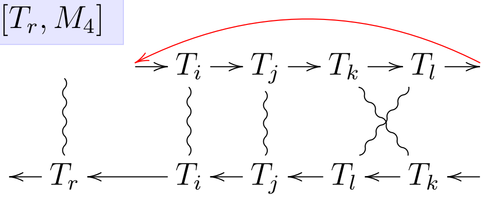
(1.1)

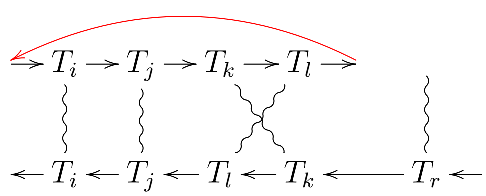
$M_4 =$ 

(1.2)

Casimir matrices are invariant matrices so they satisfy

$\boxed{0 = [T_r, M_4]}$

$0 =$ 

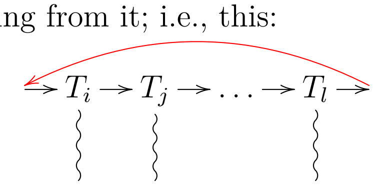
$-$ 

(1.3)

Because Casimir matrices are invariant matrices, they commute with each other. For example,

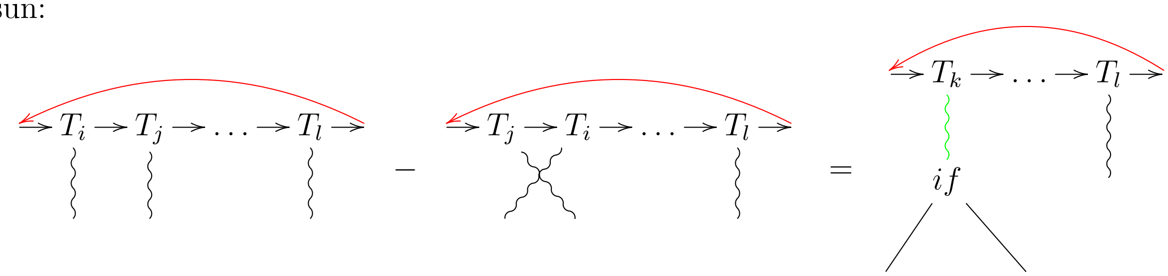
$$M_2 M_4 = M_4 M_2 \tag{1.4}$$

Casimir sun. By this we mean a tensor consisting of a loop of fundamental particles with gluons (rays) emanating from it; i.e., this:

$\mathrm{tr}(T_i T_j \dots T_l) =$ 

(1.5)

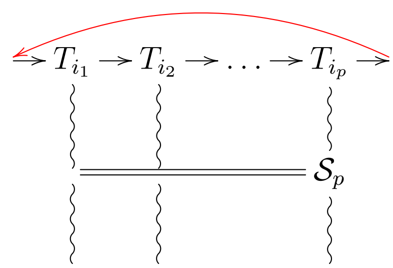
Note that the Lie Algebra commutation relations can be applies to a Casimir sun:



$$(1.6)$$

Note also that we can define a symmetrized version of a Casimir sun:

$$h_{i_1 i_2 \ldots i_p} = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} \mathrm{tr}(T_{\sigma(i_1)} T_{\sigma(i_2)} \ldots T_{\sigma(i_p)}) =$$

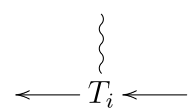


$$(1.7)$$

1.1 Independent Casimirs of Simple Lie Groups

So as not to have any gluon free indices, it is convenient to contract with a matrix M, all the outgoing gluons of a Casimir sun . Let

$$\boxed{M = \sum_i T_i x_i} \qquad \longleftarrow M \longleftarrow \; = \sum_i x_i$$



$$(1.8)$$

Then

$$\mathrm{tr}(M^k) \;=\; \leftarrow M \leftarrow M \quad \ldots \leftarrow M \leftarrow \tag{1.9}$$

$$= \sum_{i_1 i_2 \ldots i_k} \leftarrow T_{i_1} \leftarrow T_{i_2} \quad \ldots \leftarrow T_{i_k} \leftarrow \quad x_{i_1} x_{i_2} \ldots x_{i_k} \tag{1.10}$$

$$= \sum_{i_1 i_2 \ldots i_k} \underbrace{\begin{array}{c} \leftarrow T_{i'_1} \leftarrow T_{i'_2} \quad \ldots \leftarrow T_{i'_k} \leftarrow \\ \mathcal{S}_k \\ i_1 \quad i_2 \quad \quad i_k \end{array}}_{h_{i_1 i_2 \ldots i_k}} \quad x_{i_1} x_{i_2} \ldots x_{i_k} \tag{1.11}$$

Recall Eq.(2.22) for the general characteristic equation of a matrix M

$$0 \;=\; \sum_{k=0}^{n} (-1)^k \left(\mathrm{tr}_{1\ldots n-k} \mathcal{A}_{n-k} M^{\otimes n-k}\right) M^k \tag{1.12}$$

$$= \begin{cases} M^n \\ -M^{n-1}(\mathrm{tr}M) \\ +M^{n-2}(\mathrm{tr}_{1\ldots 2}\mathcal{A}_2 M^{\otimes 2}) \\ \ldots \\ (-1)^n det(M) \end{cases} \tag{1.13}$$

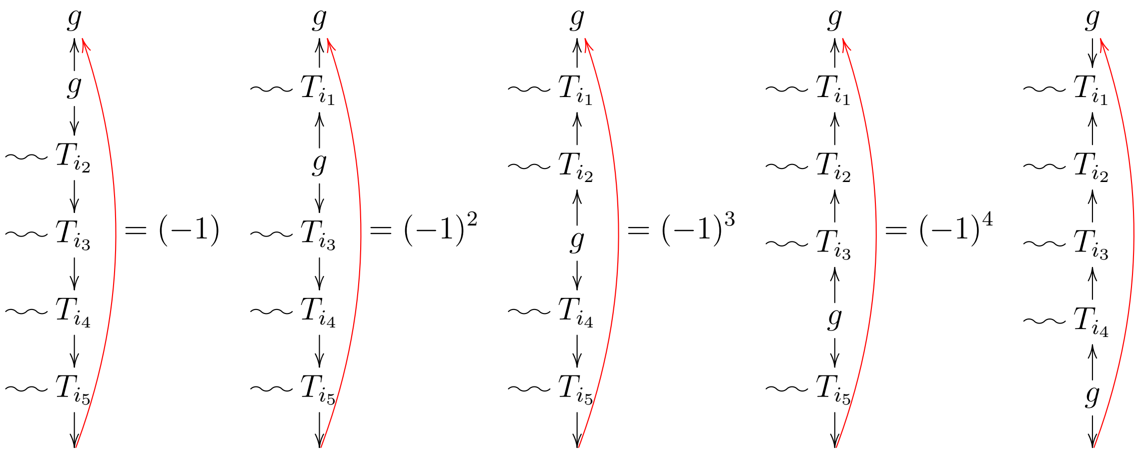
Note that that $\mathrm{tr}_{12}\mathcal{A}_2 M^{\otimes 2}$ can be expressed in terms of $\mathrm{tr}(M)$ and $\mathrm{tr}(M^2)$. Likewise, $\mathrm{tr}_{123}\mathcal{A}_3 M^{\otimes 3}$ can be expressed in terms of $\mathrm{tr}(M)$, $\mathrm{tr}(M^2)$ and $\mathrm{tr}(M^3)$. If we take the trace of the above equation, we get an equation constraining $\mathrm{tr}(M^k)$ for $k = 1, 2, \ldots n$.

The **Betti number** of the Casimir $\mathrm{tr}(M^k) \neq 0$ is the integer k. Table 1.1 gives all the Betti numbers for the simple Lie Algebras. Note that the Betti numbers in Table 1.1 are all even except for $SU(n)$.

For all simple Lie Groups except for $SU(n)$, there is a invertible symmetric or skew-symmetric bilinear invariant matrix g_{ab} satisfying $g_{ab}g^{bc} = \delta^c_a$. Hence

$A_r = \mathfrak{su}(r+1)$	$2, 3, \ldots, r+1$
$B_r = \mathfrak{so}(2r+1)$	$2, 4, 6, \ldots, 2r$
$C_r = \mathfrak{sp}(2r)$	$2, 4, 6, \ldots, 2r$
$D_r = \mathfrak{so}(2r)$	$2, 4, \ldots, 2r-2, 2r$
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$6, 8, 10, 12, 14, 18$
E_8	$8, 12, 14, 18, 20, 24, 30$

Table 1.1: Betti numbers for the simple Lie Algebras

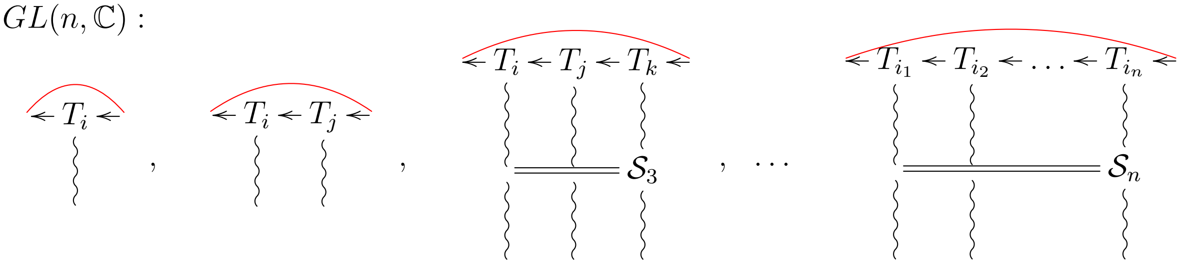


(1.14)

As illustrated in Eq.(1.14), if such a g^{ab} exists, a Casimir $\mathrm{tr}(M^k)$ equals itself times $(-1)^k$. Hence, only Casimirs with even k are non-zero.

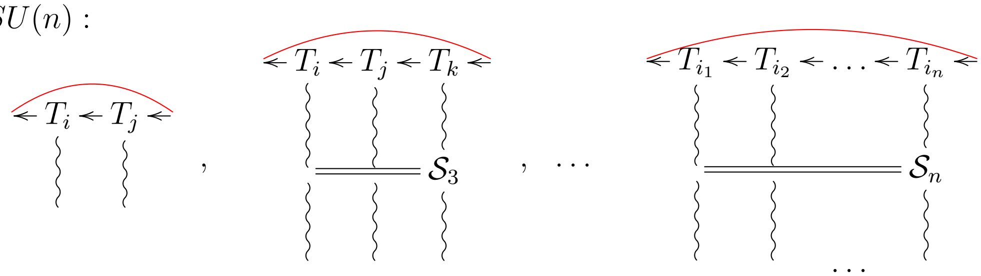
Claim 1 *The following are a complete set of Casimir operators for the given groups*

$GL(n, \mathbb{C})$:



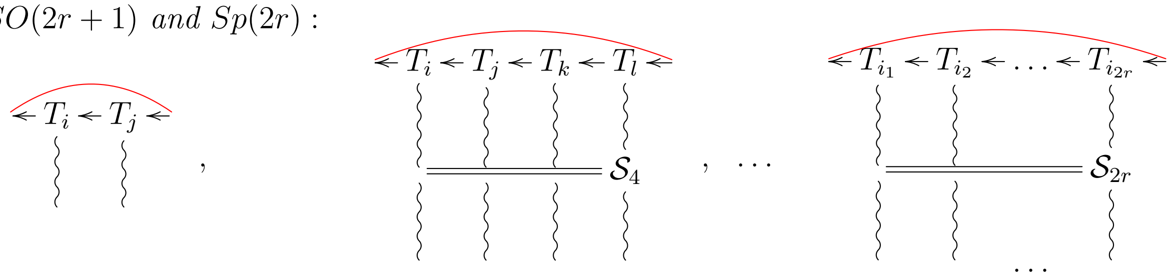
(1.15)

$SU(n):$



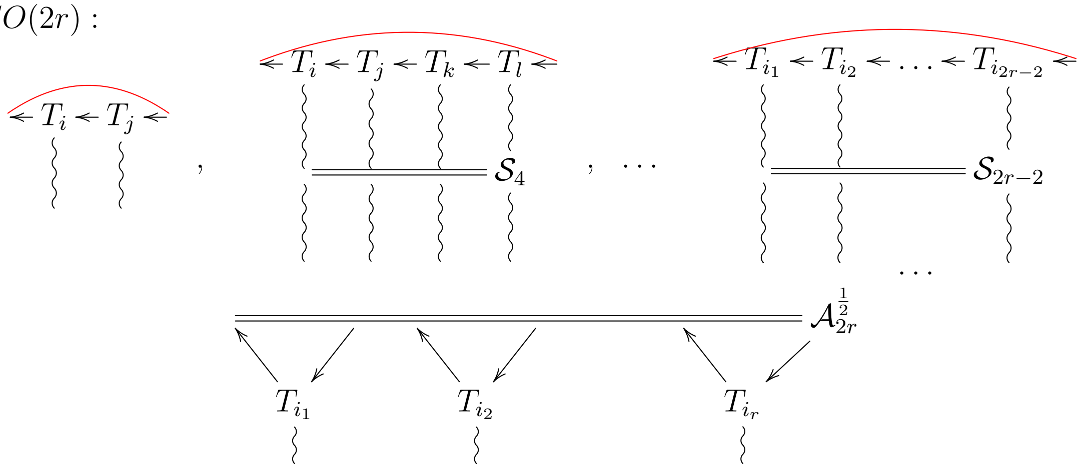
(1.16)

$SO(2r+1)$ *and* $Sp(2r):$



(1.17)

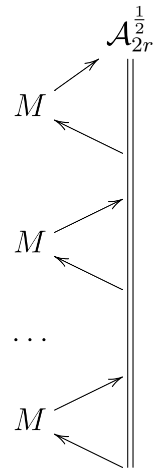
$SO(2r):$



(1.18)

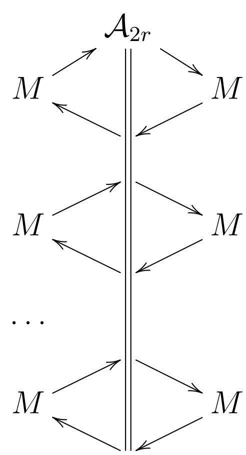
proof:

Define

$$I_r(x) = $$ 

(1.19)

where $\mathcal{A}^{\frac{1}{2}}$ is the Levi Civita tensor.(see Chapter 17). Then an expansion of $I_r^2(x)$ contains $\mathrm{tr}(M^{2r})$ among its summands.

$$I_r^2(x) = $$ 

$$= \mathrm{tr}(M^{2r}) + \dots \qquad (1.20)$$

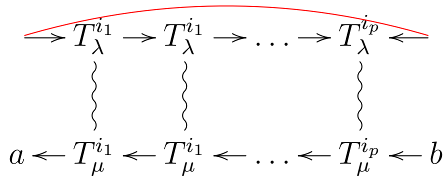
QED

1.2 Casimir matrix expressed in terms of $6j$ coefficients

Define the Casimir matrix I_p as

$$(I_p)_a{}^b = \mathrm{tr}(T_\lambda^{i_1} T_\lambda^{i_2} \dots T_\lambda^{i_p})(T_\mu^{i_1} T_\mu^{i_2} \dots T_\mu^{i_p})_a{}^b \tag{1.21}$$

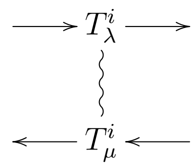
$$= \tag{1.22}$$



The goal of this section is to express I_p in terms of $6j$ coefficients.

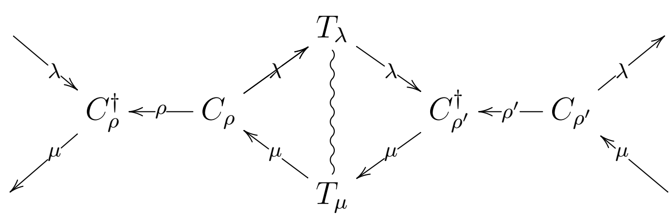
Let

$$M = \tag{1.23}$$

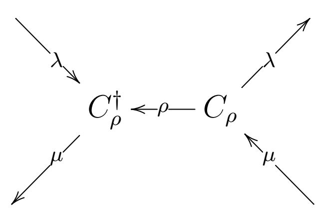


We will first decompose M in terms of $6j$ coefficients, and then use that result to decompose I_p for $p = 1, 2, 3, \dots$. Note that

$$M = \sum_{\rho,\rho'} \tag{1.24}$$

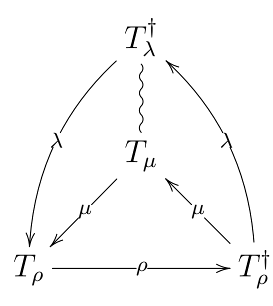


$$= \sum_{\rho} A(\lambda, \rho, \mu) \tag{1.25}$$



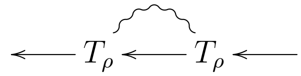
where

$$A(\lambda, \rho, \mu) = \frac{1}{d_\rho} \tag{1.26}$$



Claim 2 *If*

$$\Gamma_2(\rho) = \tag{1.27}$$

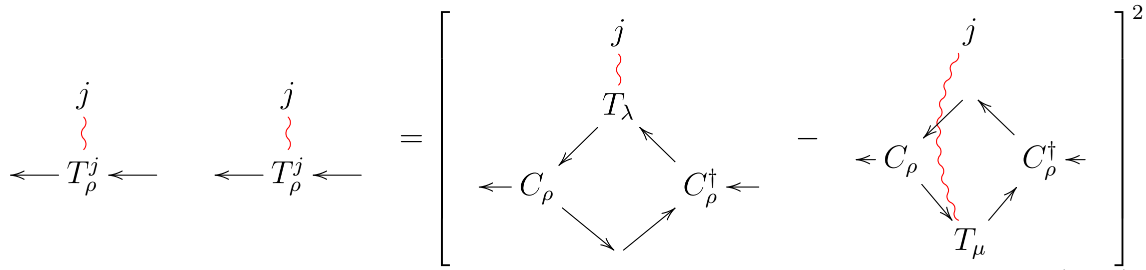


then

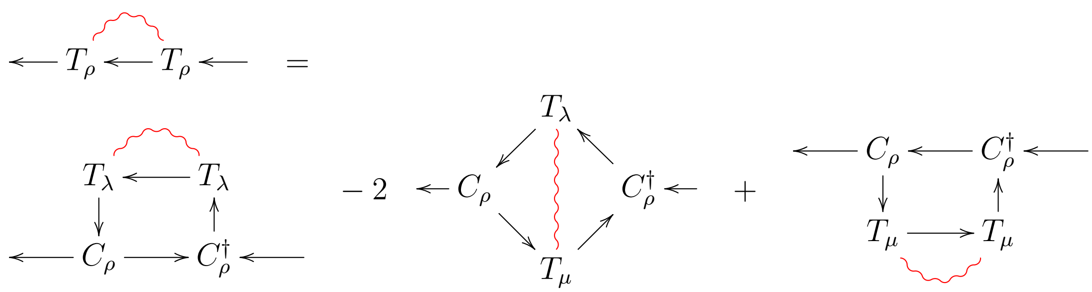
$$A(\lambda, \mu, \rho) = -\frac{1}{2}\left[\Gamma_2(\rho) - \Gamma_2(\lambda) - \Gamma_2(\mu)\right] \tag{1.28}$$

proof:

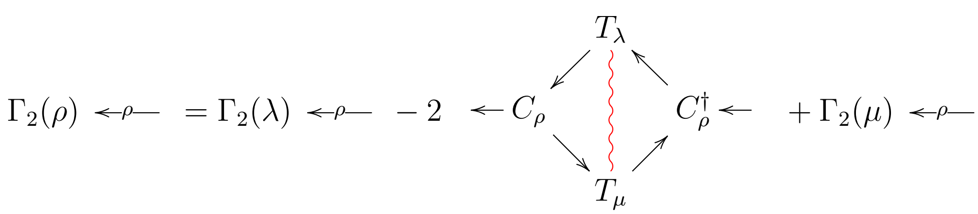
Recall Eq.(8.20). Square both sides of the equation.



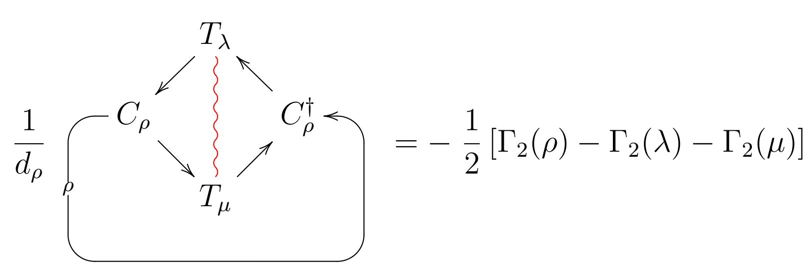
(1.29)



(1.30)



(1.31)



(1.32)

This is similar to assuming

$$\vec{J} = \vec{L} + \vec{S} \tag{1.33}$$

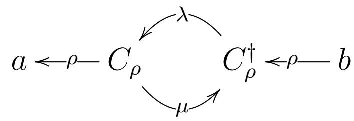
where $\vec{J}, \vec{L}, \vec{S}$ are the total, orbital and spin angular momentum. Then

$$\vec{L} \cdot \vec{S} = \frac{1}{2}\left[J^2 - L^2 - S^2\right] \tag{1.34}$$

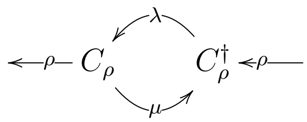
QED Next note that

$$\left(I_p\right)_a{}^b = \left(M^p\right)^c{}_a{}^b{}_c \tag{1.35}$$

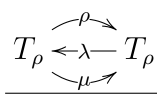
$$= \sum_{\rho \in irreps} [A(\lambda, \mu, \rho)]^p \tag{1.36}$$



If μ is an irrep,



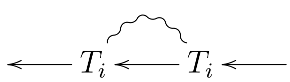
$$= \frac{\quad}{d_\mu} \xleftarrow{\mu} \tag{1.37}$$

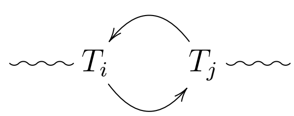


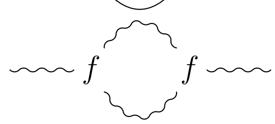
$$= \frac{d_\rho}{d_\lambda} \xleftarrow{\mu} \text{ (because } \rho \text{ is an irrep)} \tag{1.38}$$

1.3 $\mathrm{tr}(M^2)$ and $\mathrm{tr}(M^3)$

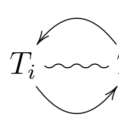
There are 3 quadratic Casimir ($\mathrm{tr}(M^2)$) matrices:

1. $\boxed{(T_i T_i)_a{}^b = \Gamma_{fun}\delta_a^b}$  $= \Gamma_{fun} \longleftarrow$ (1.39)

2. $\boxed{\mathrm{tr}(T_i T_j) = \kappa \delta_i^j}$  $= \kappa$ (1.40)

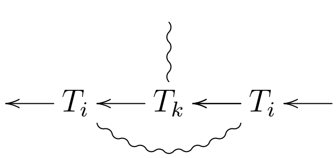
3. $\boxed{f_{ijk} f_{kji'} = \Gamma_{adj}\delta_i^{i'}}$  $= \Gamma_{adj}$ (1.41)

Note that

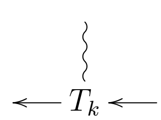


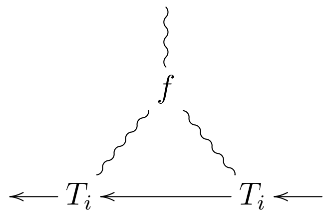
$$= n\Gamma_{fun} = N\kappa \tag{1.42}$$

Claim 3

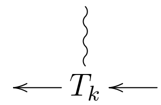


$$= \left(\frac{\kappa N}{n} - \frac{\Gamma_{adj}}{2}\right) \tag{1.43}$$

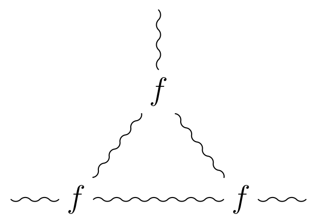




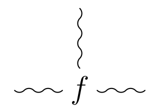
$$= \frac{\Gamma_{adj}}{2}$$



(1.44)



$$= \frac{\Gamma_{adj}}{2}$$

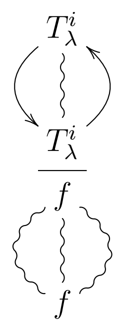


(1.45)

proof:
QED

1.4 Dynkin Index

$$DI_\lambda = \frac{\mathrm{tr}(T^i_\lambda T^i_\lambda)}{f^i_{jk} f^i_{kj}} =$$



(1.46)

Chapter 2

Characteristic Equations

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Let

$$M_a{}^b = \; a \leftarrow\!\!\!-\!\!- M \leftarrow\!\!\!-\!\!- b \tag{2.1}$$

for $a, b = 1, 2, \ldots, n$.

The goal of this chapter is to express the coefficients of the characteristic equation (i.e., $det(M - \lambda) = 0$) of M as traces.

For starters, note the difference between birdtracks for a matrix power and a tensor power of M.

$$M^2 = \;\longleftarrow M^2 \longleftarrow \; = \; \longleftarrow M \longleftarrow M \longleftarrow \tag{2.2}$$

$$M \otimes M = M^{\otimes 2} = \begin{array}{c} \longleftarrow M \longleftarrow \\ \\ \longleftarrow M \longleftarrow \end{array} \tag{2.3}$$

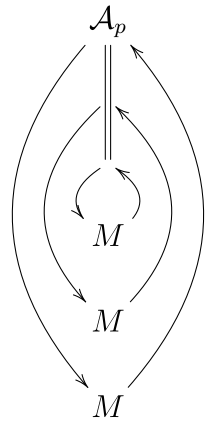
In general, $M^{\otimes p}$ is defined by

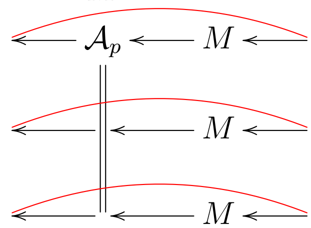
$$\boxed{(M^{\otimes p})_\alpha{}^\beta = (M^{\otimes p})_{a^{\cdot p}}{}^{rev(b^p)} = M_{a_1}{}^{b_1} M_{a_2}{}^{b_2} \ldots M_{a_p}{}^{b_p}}$$

$$\begin{array}{ccc} \longleftarrow & M^{\otimes p} & \longleftarrow \\ \longleftarrow & \Vert & \longleftarrow \\ \vdots & \Vert & \vdots \\ \longleftarrow & \Vert & \longleftarrow \end{array} = \begin{array}{c} \longleftarrow M \longleftarrow \\ \longleftarrow M \longleftarrow \\ \vdots \qquad \vdots \\ \longleftarrow M \longleftarrow \end{array} \tag{2.4}$$

where $a_i, b_i \in \mathbb{Z}_{[1,n]}$, and we define the anti-symmetrized trace of $M^{\otimes p}$ by

$$\mathrm{tr}_{1\dots p}\mathcal{A}[M^{\otimes p}] \;=\; \mathcal{A}_{a^{:p}}{}^{rev(b^{:p})}\prod_{i=1}^{p} M_{b_i}{}^{a_i} \tag{2.5}$$

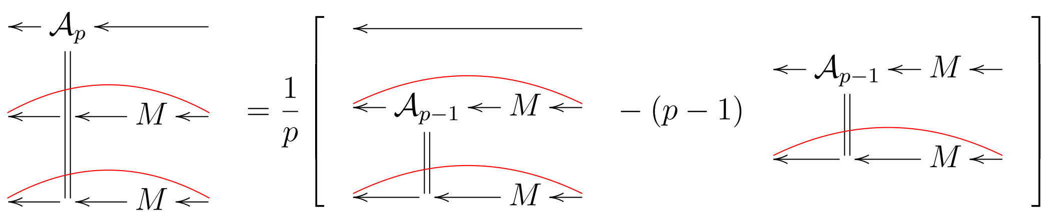
$=$  (Cvitanovic Drawing Style) (2.6)

$=$  (This book's drawing style) (2.7)

Note that the determinant of M is one of those traces

$$det M = \mathrm{tr}_{1\dots n}\mathcal{A}[M^{\otimes n}] \tag{2.8}$$

Claim 4



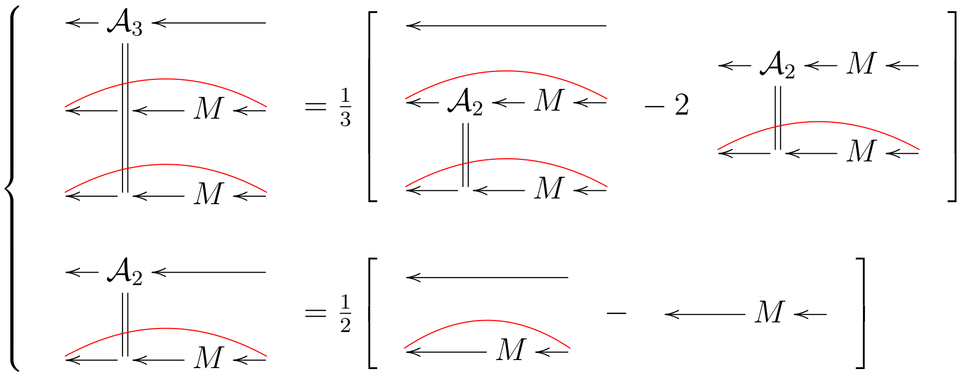
(2.9)

proof:

See Chapter 17.

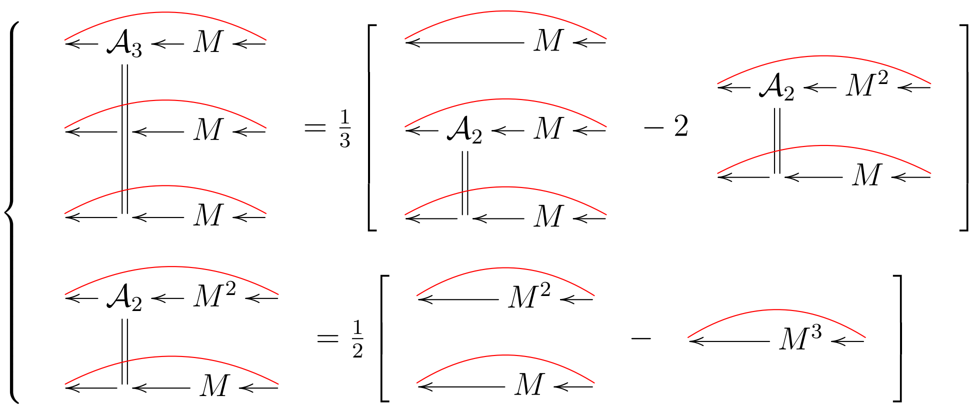
QED

Consider the above claim for $p = 2, 3$.



(2.10)

If we multiply from the right, by M^d for $d = 1, 2$, the first row of Eq.(2.10) and then take the trace of that row, we get



(2.11)

Let

$$\tau = \mathrm{tr}(M) \tag{2.12}$$

Then Eqs.(2.11) can be expressed algebraically by

$$\mathrm{tr}_{1,2,3}\mathcal{A}_3(M^{\otimes 3}) = \frac{1}{3}\left[\tau \mathrm{tr}_{1,2}\mathcal{A}_2(M^{\otimes 2}) - \mathrm{tr}(M^2)\tau + \mathrm{tr}M^3\right] \tag{2.13}$$

and

$$\mathrm{tr}_{1,2}\mathcal{A}_2 M^{\otimes 2} = \frac{1}{2}\left[\tau^2 - \mathrm{tr}(M^2)\right] \tag{2.14}$$

Therefore,

$$\begin{aligned} \mathrm{tr}_{1,2,3}\mathcal{A}_3(M^{\otimes 3}) &= \frac{1}{3}\left[\frac{1}{2}\tau^3 - \frac{3}{2}\mathrm{tr}(M^2)\tau + \mathrm{tr}M^3\right] && (2.15)\\ &= \frac{1}{3!}\left[\tau^3 - 3\mathrm{tr}(M^2)\tau + 2\mathrm{tr}M^3\right] && (2.16) \end{aligned}$$

In general,

$$\mathrm{tr}_{1\ldots p}\mathcal{A}_p M = \frac{1}{p}\sum_{k=1}^{p}(-1)^{k-1}\left(\mathrm{tr}_{1\ldots p-k}\mathcal{A}_{p-k}M^{\otimes p-k}\right)\mathrm{tr}(M^k) \tag{2.17}$$

Next note that

$$\mathcal{A}_p = 0 \quad \text{if } p > n \tag{2.18}$$

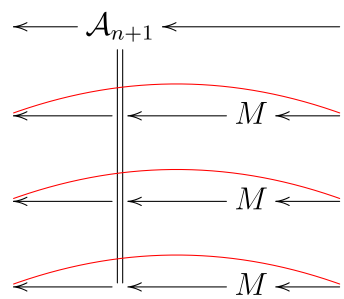
This follows because the Levi Civita tensor with more than n indices is zero.; i.e.,

$$\epsilon_{a_1,a_2,\ldots,a_{n+1}} = 0 \tag{2.19}$$

Indeed, two of the a_i must be equal, so that element of the ϵ tensor is zero

Let I be the $n\times n$ identity matrix. Then, since $\mathcal{A}_{n+1} = 0$, the following is true

$$\boxed{0 = \mathrm{tr}_{2\ldots n+1}\mathcal{A}_{n+1} I \otimes M^{\otimes n}} \qquad 0 = \tag{2.20}$$



We can now expand the right hand side of Eq.(2.20) using identity Eq.(2.17)

$$0 = \sum_{k=0}^{n}(-1)^k\left(\mathrm{tr}_{1\ldots n-k}\mathcal{A}_{n-k}M^{\otimes n-k}\right)M^k \tag{2.21}$$

$$= \left\{\begin{array}{l} M^n \\ -M^{n-1}(\mathrm{tr}M) \\ +M^{n-2}(\mathrm{tr}_{1\ldots 2}\mathcal{A}_2 M^{\otimes 2}) \\ \ldots \\ (-1)^n det(M) \end{array}\right. \tag{2.22}$$

Viola. The last equation is none other than the characteristic equation of M. As promised, the coefficients of this polynomial in M, are expressed as traces.

Chapter 3

Clebsch-Gordan Coefficients

This chapter is based on Cvitanovic's Birdtracks book Ref.[1].

Suppose that for some $M \in \mathbb{C}^{d\times d}$, we have

$$M = C^\dagger D C \tag{3.1}$$

where D is a diagonal matrix and $C \in \mathbb{C}^{d\times d}$ is unitary. Then one can partition C into rectangular submatrices C_λ that have d_λ rows with $d_\lambda < d$, such that we have one C_λ for each eigenvalue λ of C. Likewise, we can partition $C^\dagger$ into rectangular submatrices $C^\dagger_\lambda$ that have d_λ columns with $d_\lambda < d$, such that we have one $C^\dagger_\lambda$ for each eigenvalue λ of C. Thus, if $I^{d_\lambda\times d_\lambda}$ is the $d_\lambda \times d_\lambda$ identity matrix,

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda\times d} \\ 0 \end{bmatrix}^{d\times d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda\times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d\times d} C^{d\times d} \tag{3.2}$$

$$\begin{bmatrix} 0 & (C^\dagger)_\lambda^{d\times d_\lambda} & 0 \end{bmatrix}^{d\times d} = (C^\dagger)^{d\times d} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda\times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d\times d} \tag{3.3}$$

The matrices C_λ are called the **Clebsch-Gordan Coefficient** (CGC) matrices.

Let $b^{:nb} = (b_1, b_2, \dots, b_{nb})$ where $b_i \in Z_{[0,d_{\mu_i}]}$ and $a \in Z_{[1,d_\lambda]}$. Assume that

$$d_\lambda = \prod_{i=1}^{:nb} d_{\mu_i} \tag{3.4}$$

Now define the birdtracks

$$(C_\lambda)_a{}^{rev(b^{:nb})} = \begin{tikzcd} & & \mu_1 b_1 \arrow[dl] \\ \lambda a & C_\lambda \arrow[l] & \mu_2 b_2 \arrow[l] \\ & & \mu_{nb} b_{nb} \arrow[ul] \end{tikzcd} \tag{3.5}$$

and

$$
(C_\lambda^\dagger)^a{}_{b^{:nb}} = \quad
\begin{tikzcd}[row sep=small]
\mu_1 b_1 & \\
\mu_2 b_2 & (C_\lambda^\dagger) \arrow[l] \arrow[ul] \arrow[dl] & \lambda a \arrow[l] \\
\mu_{nb} b_{nb} &
\end{tikzcd}
\tag{3.6}
$$

We will assume there is no difference between when a Greek letter is lowered and when it is raised. Also, all summations over a Greek letter will be stated explicitly; i.e., no implicit summations over repeated Greek letters.

On the other hand, the Latin letter indices b_i, a of C_λ and $C_\lambda^\dagger$ may be lowered or raised and their arrows changed from outgoing to incoming or vice versa. Furthermore, we will use implicit summation over repeated Latin letters.

The Greek letters label representation of the group (not necessarily irreps). Each b_i labels a member of μ_i, and each a labels a member of λ.

Recall that if $|x\rangle$ for $x \in val(\underline{x})$ is a complete, orthonormal basis in Quantum Mechanics, then

$$
\langle x|y\rangle = \delta(x,y) \quad \text{(orthonormality)} \tag{3.7}
$$

and

$$
\sum_x |x\rangle\langle x| = 1 \quad \text{(completeness)} \tag{3.8}
$$

Furthermore, if we define

$$
\pi_x = |x\rangle\langle x| \tag{3.9}
$$

then π_x is a is a projection operator so

$$
\pi_x \pi_x = \pi_x \tag{3.10}
$$

and

$$
\pi_x|y\rangle = |y\rangle\delta(x,y), \quad \langle y|\pi_x = \langle y|\delta(x,y) \tag{3.11}
$$

If we identify C_λ with $\langle x|$, and $C_\lambda^\dagger$ with $|x\rangle$, then C_λ and $C_\lambda^\dagger$ satisfy analogous identities:

$$
\boxed{(C_\lambda)_a{}^{rev(b^{:nb})}(C_\mu^\dagger)^{a'}{}_{b^{:nb}} = \delta(\lambda,\mu)\delta_a^{a'}, \quad C_\lambda C_\mu^\dagger = \delta(\mu,\lambda)}
$$

$$
\begin{tikzcd}[row sep=small]
 & & \sum b_1 \arrow[dl] & & \\
a & C_\lambda \arrow[l] & \sum b_2 \arrow[l] & (C_\mu^\dagger) \arrow[ul] \arrow[l] \arrow[dl] & a' \arrow[l] \\
 & & \sum b_{nb} \arrow[ul] & &
\end{tikzcd}
= \delta(\mu,\lambda)\; a \longleftarrow\!\bullet\!\!-\!\!- a'
\tag{3.12}
$$

$$\boxed{\sum_\lambda (C_\lambda^\dagger)^a{}_{b^{:nb}} (C_\lambda)_a{}^{rev((b')^{:nb})} = \delta_{b^{:nb}}^{rev((b')^{:nb})}, \quad \sum_\lambda C_\lambda^\dagger C_\lambda = 1}$$

$$\begin{tikzcd}
 & b_1 & & & & b'_1 \arrow[dl] & & b_1 & \bullet \arrow[l] & b'_1 \arrow[l, dash] \\
\sum_\lambda & b_2 & (C_\lambda^\dagger) \arrow[l] \arrow[ul] \arrow[dl] & \sum a \arrow[l] & C_\lambda \arrow[l] & b'_2 \arrow[l] & = & b_2 & \bullet \arrow[l] & b'_2 \arrow[l, dash] \\
 & b_{nb} & & & & b'_{nb} \arrow[ul] & & b_{nb} & \bullet \arrow[l] & b'_{nb} \arrow[l, dash]
\end{tikzcd} \tag{3.13}$$

$$\boxed{(C_\lambda)_a{}^{rev((b')^{:nb})} (P_\mu)_{(b')^{:nb}}{}^{rev(b^{:nb})} = \delta(\mu,\lambda)(C_\mu)_a{}^{rev(b^{:nb})}, \quad C_\lambda P_\mu = \delta(\mu,\lambda) C_\mu}$$

$$\begin{tikzcd}
 & & \sum b'_1 \arrow[dl] & & & b_1 \arrow[dl] & & & & b_1 \arrow[dl] \\
a & C_\lambda \arrow[l] & \sum b'_2 \arrow[l] & & P_\mu \arrow[l] \arrow[ul] \arrow[dl] & b_2 \arrow[l] & = \delta(\mu,\lambda) & a & C_\lambda \arrow[l] & b_2 \arrow[l] \\
 & & \sum b'_{nb} \arrow[ul] & & & b_{nb} \arrow[ul] & & & & b_{nb} \arrow[ul]
\end{tikzcd} \tag{3.14}$$

$$\boxed{(P_\mu)_{b^{:nb}}{}^{rev((b')^{:nb})} (C_\lambda^\dagger)^a{}_{(b')^{:nb}} = \delta(\mu,\lambda)(C_\mu^\dagger)^a{}_{b^{:nb}}, \quad P_\mu C_\lambda^\dagger = \delta(\mu,\lambda) C_\mu^\dagger}$$

$$\begin{tikzcd}
b_1 & & \sum b'_1 \arrow[dl] & & & & & b_1 & & \\
b_2 & P_\mu \arrow[l] \arrow[ul] \arrow[dl] & \sum b'_2 \arrow[l] & & (C_\lambda^\dagger) \arrow[l] \arrow[ul] \arrow[dl] & a \arrow[l] & = \delta(\mu,\lambda) & b_2 & (C_\lambda^\dagger) \arrow[l] \arrow[ul] \arrow[dl] & a \arrow[l] \\
b_{nb} & & \sum b'_{nb} \arrow[ul] & & & & & b_{nb} & &
\end{tikzcd} \tag{3.15}$$

Chapter 4

Dynkin Diagrams: COMING SOON

Chapter 5

General Relativity Nets: COMING SOON

T.his chapter is based on Cvitanovic's Birdtracks book Ref. [1]

Chapter 6

Integrals over a Group

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

In this chapter, we will consider only non-trivial reps; i.e., reps with dimension $d_\lambda > 0$. Trivial "reps", i.e., those with $d_\lambda = 0$, will be excluded.

For a group $\mathcal{G}$, let SR = singlet reps (i.e., one dimensional reps= scalar reps), and SR^c = nonsinglet reps. Let g be an element of $\mathcal{G}$ with a rep-matrix G. The goal of this chapter is to show how to evaluate integrals over a group $\mathcal{G}$, of the form:

$$\int dg G_a{}^b G_c{}^d \dots (G^\dagger)_e{}^f (G^\dagger)_g{}^h \tag{6.1}$$

subject to the constraints that:

$$\int dg = 1 \tag{6.2}$$

and

$$\int dg\ G_\lambda = 0 \quad \text{if } \lambda \in SR^c \tag{6.3}$$

We will represent the rep-matrix G by

$$G_a{}^b = \ a \leftarrow G \leftarrow b\ , \quad (G^\dagger)_b{}^a = \ b \leftarrow G^\dagger \leftarrow a \tag{6.4}$$

Note that we will always take the out arrow as the first one (green).

We will assume that G is a unitary matrix

$$\boxed{G^\dagger G = G G^\dagger = 1} \qquad \leftarrow G^\dagger \leftarrow G \leftarrow \ = \ \leftarrow G \leftarrow G^\dagger \leftarrow \ = \ \leftarrow \bullet \leftarrow \tag{6.5}$$

Tensor products of G's will be represented thus

$$G \otimes G \otimes G^\dagger = \begin{array}{c} \leftarrow G \leftarrow \\ \leftarrow G \leftarrow \\ \leftarrow G^\dagger \leftarrow \end{array} \tag{6.6}$$

6.1 $\int dg\, G$

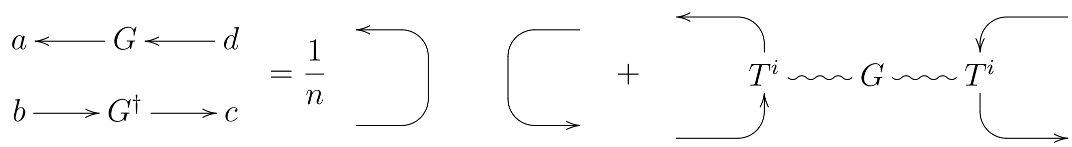
To evaluate $\int dg\, G$, we expand G in its Clebsch-Gordan series. Such series and the Clebsch-Gordan coefficients C_λ are discussed in Chapter 3.

$$\begin{aligned} \int dg\, G &= \sum_\lambda C_\lambda^\dagger \left[\int dg\, G_\lambda\right] C_\lambda && (6.7)\\ &= \sum_{\lambda\in SR} C_\lambda^\dagger C_\lambda && (6.8)\\ &= \sum_{\lambda\in SR} P_\lambda && (6.9)\end{aligned}$$

This result is valid for any group $\mathcal{G}$ and any rep-matrix G of that group.

6.2 $\int dg\, G\otimes G^\dagger$

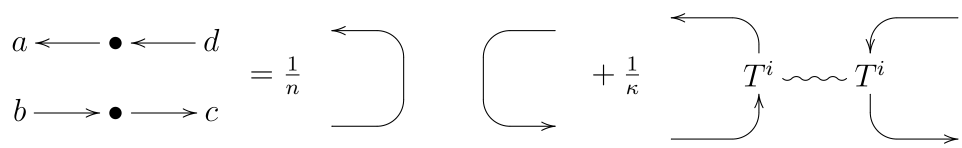
Claim 5 *For $G \in SU(n) \subset \mathbb{C}^{n\times n}$,*



(6.10)

proof:

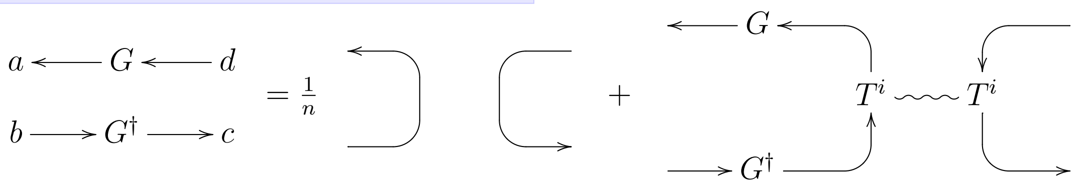
$$\boxed{\delta_a^d \delta_c^b = \frac{1}{n}\delta_a^b\delta_c^d + \frac{1}{\kappa}(T^i)_a{}^b (T^i)_c{}^d}$$



(6.11)

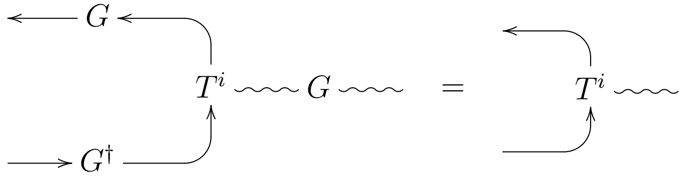
Will set $\kappa = 1$ from here on.

$$\boxed{G_a{}^d (G^\dagger)_c{}^b = \frac{1}{n}\delta_a^b\delta_c^d + (G^\dagger T^i G)_a{}^b (T^i)_c{}^d}$$



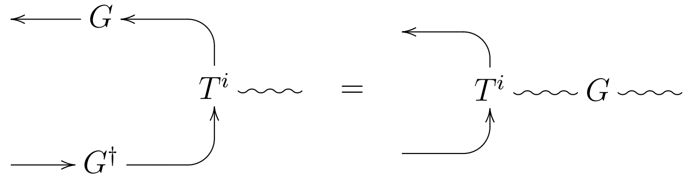
(6.12)

$$\boxed{G_a{}^{a'}(G^\dagger)_{b'}{}^b(T^{i'})_{a'}{}^{b'}G_{i'i} = (T^i)_a{}^b}$$



(6.13)

$$\boxed{G_a{}^{a'}(G^\dagger)_{b'}{}^b(T^i)_{a'}{}^{b'} = (T^i)_a{}^b G_{ii'}}$$

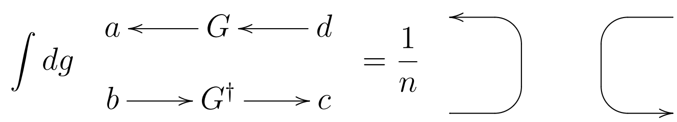


(6.14)

QED

Claim 6 *For* $G \in SU(n) \subset \mathbb{C}^{n\times n}$,

$$\boxed{\int dg\ G_a{}^d(G^\dagger)_b{}^c = \frac{1}{n}\delta_a^b\delta_c^d}$$



(6.15)

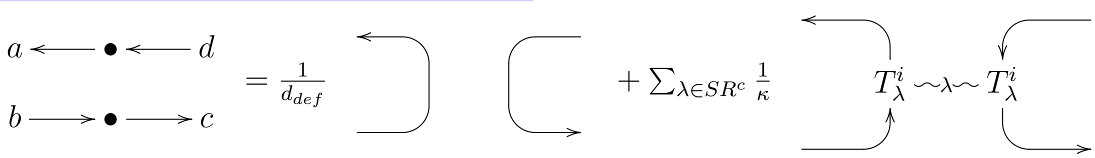
proof:

This claim follows immediately from the previous one.

QED

Claim 6 can be extended to any group $\mathcal{G}$ that has a single singlet rep. For such groups, we have, if $G \in \mathbb{C}^{d_{def}\times d_{def}}$ is the defining rep so that $a, b, c, , d \in \{1, 2, \ldots d_{def}\}$,

$$\boxed{\delta_a^d\delta_c^b = \frac{1}{d_{def}}\delta_a^b\delta_c^d + \sum_{\lambda\in SR^c}\frac{1}{\kappa}(T^i_\lambda)_a{}^b(T^i_\lambda)_c{}^d}$$



(6.16)

so Eq.(6.15) is valid with n replaced by d_{def}.

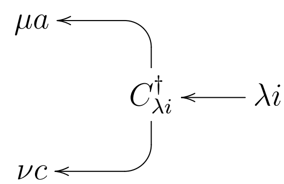
Claim 7 *For any group $\mathcal{G}$ with rep-matrices G_μ and G_ν (μ, ν are not necessarily irreps)*

$$\int dg\ (G_\mu)_{ab}(G_\nu)^{cd} = \sum_{\lambda \in SR} (P_\lambda)_{ab}{}^{cd} \tag{6.17}$$

proof:

Let

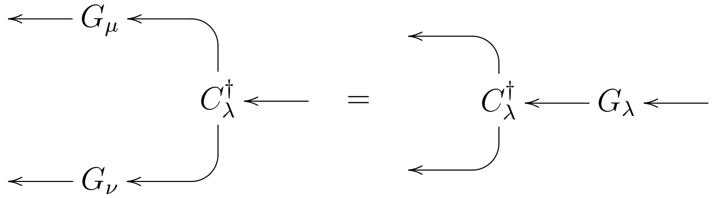
$$(C^\dagger_{\lambda i})_{ac} = \tag{6.18}$$



represent the Clebsch-Gordan coefficients for the Clebsch-Gordan series $\mu \otimes \nu = \sum \lambda$.

Since the C_λ are invariant tensors:

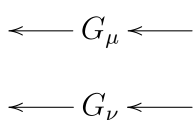
$$(G_\mu)_a{}^{a'}(G_\nu)_{b'}{}^{b}(C^\dagger_{\lambda i})_{a'}{}^{b'} = (C^\dagger_{\lambda i'})_a{}^{b}(G_\lambda)_{i'i}$$



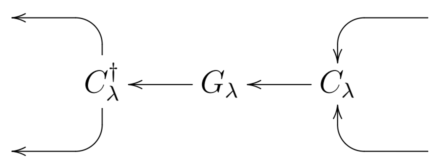
(6.19)

Therefore,

$$\int dg$$



$$= \int dg \sum_\lambda$$



(6.20)

$$= \sum_{ij}\sum_{\lambda}(C^\dagger_{\lambda i})_{ab}(C_{\lambda j})^{cd}\int dg (G_\lambda)_{ij} \tag{6.21}$$

$$= \sum_{i}\sum_{\lambda \in SR}(C^\dagger_{\lambda i})_{ab}(C_{\lambda i})^{cd} \tag{6.22}$$

$$= \sum_{\lambda \in SR}(P_\lambda)_{ab}{}^{cd} \tag{6.23}$$

QED

6.3 Character Orthonormality Relation

For any matrix rep-matrix G_λ in rep λ such that G_λ represents the group element g in the Group $\mathcal{G}$, define the **character of g in rep λ** by

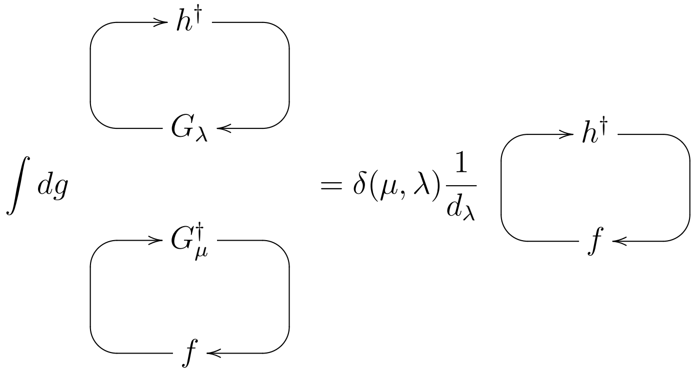
$$\chi_\lambda(g) = \chi_\lambda(G_\lambda) \stackrel{\text{def}}{=} \text{tr}G_\lambda = (G_\lambda)_a{}^a \tag{6.24}$$

Note that

$$\mathrm{tr}G^\dagger_\lambda = ({G_a}^a)^* = \chi_\lambda(g)^* \tag{6.25}$$

Claim 8 *Suppose G_λ and G_μ are rep-matrices in irreps λ and μ, respectively. Suppose $h, G_\lambda \in \mathbb{C}^{d_\lambda \times d_\lambda}$ and $f, G_\mu \in \mathbb{C}^{d_\mu \times d_\mu}$. Then*

$$\boxed{\int dg\ \chi_\lambda(h^\dagger G_\lambda)\chi^*_\mu(G^\dagger_\mu f) = \delta(\mu,\lambda)\frac{1}{d_\mu}\chi_\lambda(h^\dagger f)}$$



(6.26)

proof:

This claim follows from Eq.(6.16 once we prove that the left hand side of Eq.(6.26) is zero if $\lambda \neq \mu$. Because λ and μ are both irreps, there can be no matrix connecting G_μ and G_λ when $\lambda \neq \mu$, so the left hand side of Eq.(6.26) is indeed zero. Even when $\lambda = \mu$, there can only be one matrix, namely a Kronecker delta, connecting G_λ and G_μ, so the group must have only one singlet rep.

QED

Note that since the matrices $h, f \in \mathbb{C}^{d_\mu \times d_\mu}$ are arbitrary, differentiation can be used to retrieve G_μ from its character with various h:

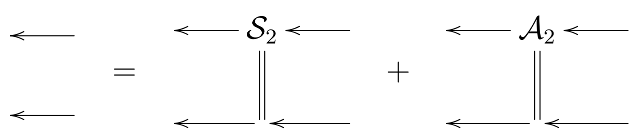
$${G_a}^b = \frac{d}{d{(h^\dagger)_b}^a}\underbrace{\chi_\mu(h^\dagger G)}_{{(h^\dagger)_b}^a{G_a}^b} \tag{6.27}$$

6.4 $SU(n)$ examples

In $SU(n)$, $n = d_{def}$, where d_{def} is the dimension of the defining rep. ($\mathcal{G} \subset \mathbb{C}^{n\times n}$). In this section, all matrices G are elements of $\mathbb{C}^{n\times n}$.

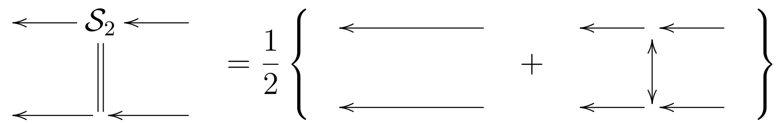
6.4.1 $\int dg\ G\otimes G$

Consider $V \otimes V$. We have



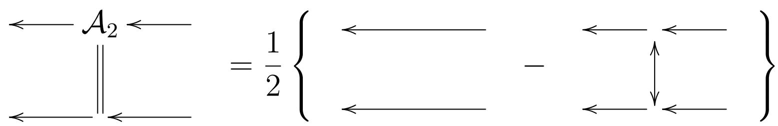
(6.28)

because



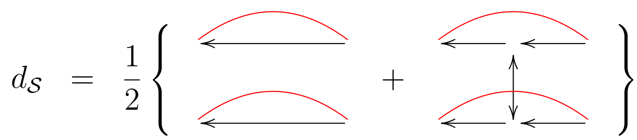
(6.29)

and



(6.30)

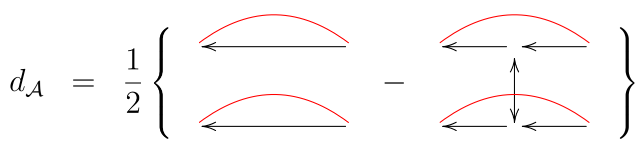
Thus



(6.31)

$$= \frac{n(n+1)}{2} \tag{6.32}$$

and



(6.33)

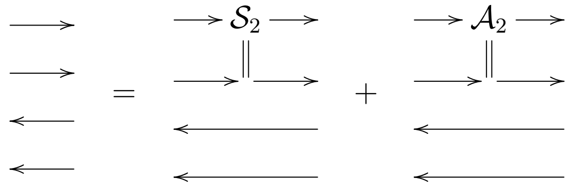
$$= \frac{n(n-1)}{2} \tag{6.34}$$

Note that $d_{\mathcal{S}} = 1$ iff $n = 1$, and $d_{\mathcal{A}} = 1$ iff $n = 2$. Therefore, for $SU(n)$

$$\int dg\; G \otimes G = 0 \quad \text{if } n > 2 \tag{6.35}$$

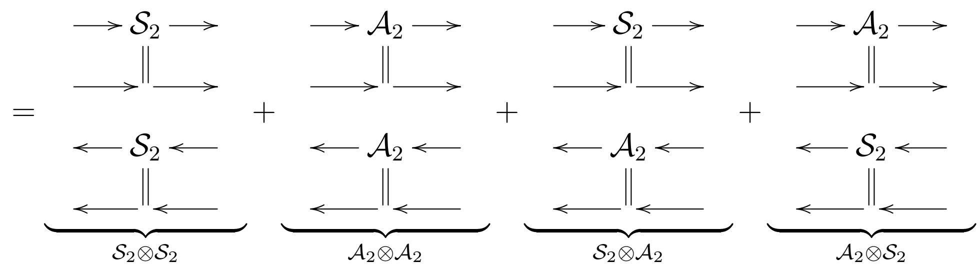
6.4.2 $\int dg\; G^{\dagger} \otimes G^{\dagger} \otimes G \otimes G$

Consider $V^{\dagger} \otimes V^{\dagger} \otimes V \otimes V$. We have



(6.36)

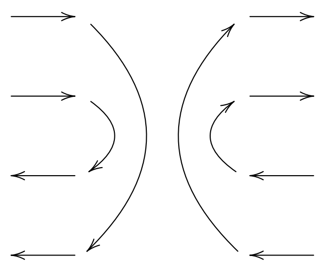
(6.37)



(6.38)

Let

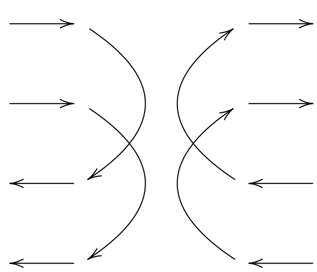
$$P_1 = \frac{1}{n^2}$$



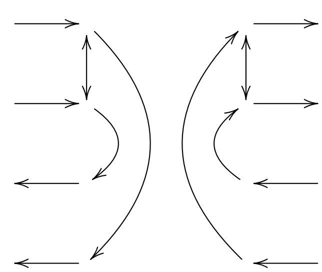
(6.39)

and

$$P_2 = \frac{1}{n^2}$$



$$= \frac{1}{n^2}$$



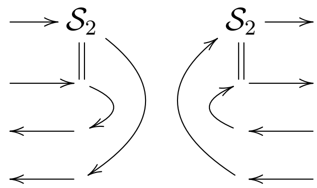
(6.40)

Then

$$P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1 P_2 \neq 0 \tag{6.41}$$

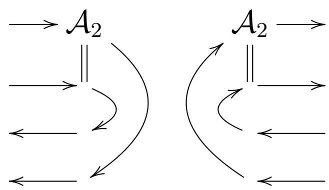
$$dim(P_1) = dim(P_2) = 1 \tag{6.42}$$

This hints to the possibility of two orthogonal projectors, if only we include terms where there is a single swap on either the right or the left side, but not on both sides as in Eq.(6.40). So define

$$\pi_{\mathcal{S}} = \frac{1}{d_{\mathcal{S}}}$$ 

$$\text{where } d_{\mathcal{S}} = \frac{n(n+1)}{2} \tag{6.43}$$

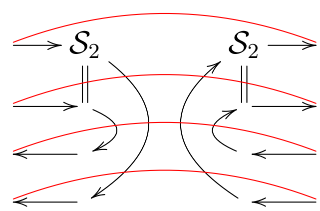
and

$$\pi_{\mathcal{A}} = \frac{1}{d_{\mathcal{A}}}$$ 

$$\text{where } d_{\mathcal{A}} = \frac{n(n-1)}{2} \tag{6.44}$$

Then

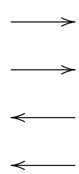
$$\pi_{\mathcal{A}}^2 = \pi_{\mathcal{A}}, \quad \pi_{\mathcal{S}}^2 = \pi_{\mathcal{S}}, \quad \pi_{\mathcal{A}}\pi_{\mathcal{S}} = 0 \tag{6.45}$$

$$dim(\pi_{\mathcal{S}}) = \mathrm{tr}\pi_{\mathcal{S}} = \frac{1}{d_{\mathcal{S}}}$$ 

$$= 1 \tag{6.46}$$

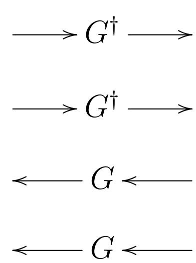
$$dim(\pi_{\mathcal{A}}) = 1 \tag{6.47}$$

Thus



$$= \pi_{\mathcal{S}} + \pi_{\mathcal{A}} + \text{ non-singlet projectors} \tag{6.48}$$

Hence

$$\int dg$$ 

$$= \pi_{\mathcal{S}} + \pi_{\mathcal{A}} \tag{6.49}$$

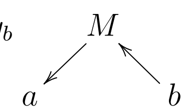
Chapter 7

Invariant Tensors

This chapter is based on Cvitanovic's Birdtracks book Ref.[1].

A **bilinear form** is a linear function $m : V^{\dagger n} \times V^n \to \mathbb{C}$ with $V^{\dagger n}, V^n = \mathbb{C}^n$. For example,

$$m(x^{\dagger :n}, y^{:n}) = x^{\dagger a} M_a{}^b y_b \tag{7.1}$$

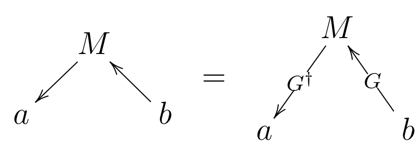


$m()$ is said to be invariant if

$$m(x^{\dagger :n}, y^{:n}) = m(x^{\dagger :n} G^\dagger, G y^{:n}) \tag{7.2}$$

$m()$ is invariant iff matrix M is an **invariant matrix**; i.e., iff

$$\boxed{M_a{}^b = (G^\dagger)_a{}^{a'} G_{b'}{}^b M_{a'}{}^{b'}} \tag{7.3}$$



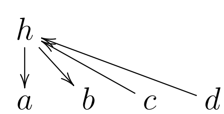
$$M = G^\dagger M G \tag{7.4}$$

If G is unitary,

$$GM = MG, \quad [G, M] = 0 \tag{7.5}$$

A **multilinear form** is a linear function $h : V^{\dagger n^p} \times V^{n^q} \to \mathbb{C}$ with $V^{\dagger d}, V^d = \mathbb{C}^d$. For example,

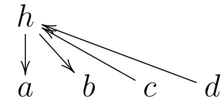
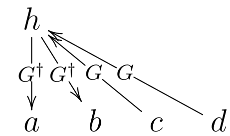
$$h(w^\dagger, x^\dagger, y, z) = h_{ab}{}^{cd} w^{\dagger a} x^{\dagger b} y_c z_d \tag{7.6}$$



$h()$ is said to be invariant if

$$h(w^\dagger, x^\dagger, y, z) = h(w^\dagger G^\dagger, x^\dagger G^\dagger, G y, G z) \tag{7.7}$$

$h()$ is invariant iff tensor $h_{ab}{}^{cd}$ is a **invariant tensor** (IT); i.e., iff

$$\boxed{h_{ab}{}^{cd} = (G^\dagger)_a{}^{a'}(G^\dagger)_b{}^{b'} h_{a'b'}{}^{c'd'} G_{c'}{}^{c} G_{d'}{}^{d}}$$  $=$  (7.8)

A **composed IT** is a IT that can be written as a product or contraction of ITs.

A **tree IT** is a composed ITs without any loops.

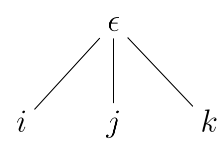
A **primitive IT** is a IT that can be expressed as a linear combination of a finite number of tree ITs.

The **primitiveness assumption**: All IT are primitive.

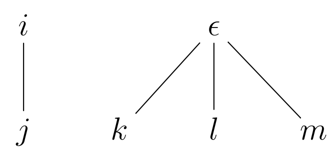
Examples. Suppose $x, y, z \in \mathbb{R}^3$ and $i, j, k \in \{1, 2, 3\}$.

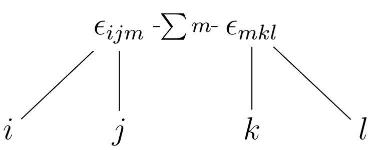
- Primitive ITs

$$length(x) = \delta_{ij} x_i x_i \quad volume(x, y, z) = \epsilon_{ijk} x_i y_j z_k \tag{7.9}$$

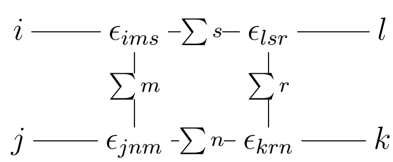
$$\delta_{ij} = \; i \text{ —— } j \;, \quad \epsilon_{ijk} = $$  (7.10)

- Tree ITs

$$\delta_{ij}\epsilon_{klm} =$$  (7.11)

$$\epsilon_{ijm}\delta_{mn}\epsilon_{nkl} =$$  (7.12)

- Non-tree IT

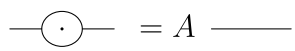
$$\epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{lsr} =$$  (7.13)

$$\epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{lsr} = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}$$

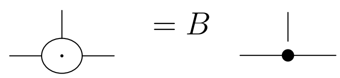
$$\begin{array}{ccccccc} i \text{ ——— } \epsilon_{ims} \text{ -}\sum{}^{s}\text{– } \epsilon_{lsr} \text{ ——— } l & & i & & l & & i \text{ ——— } l \\ \quad\; \sum{}^{m} \qquad\quad \sum{}^{r} & = & | & & | & + & \\ j \text{ ——— } \epsilon_{jnm} \text{ -}\sum{}^{n}\text{– } \epsilon_{krn} \text{ ——— } k & & j & & k & & j \text{ ——— } k \end{array} \tag{7.14}$$

- Primitiveness Assumption

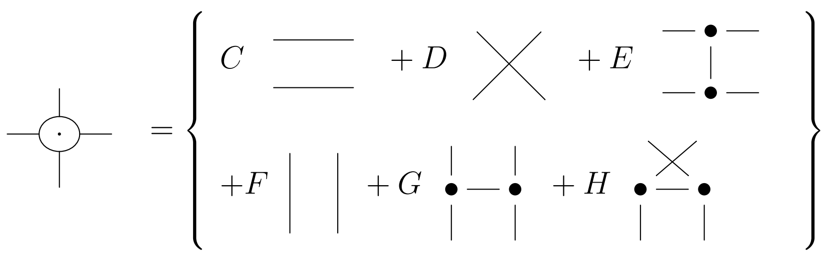
  Suppose $\mathcal{P} = \{\delta_{ij}, f_{ijk}\}$ where f_{ijk} is not ϵ_{ijk}. For some $A, B, C, \ldots H \in \mathbb{C}$, one has



(7.15)



(7.16)



(7.17)

Let $\mathcal{P} = (p_1, p_2, \ldots, p_k)$ be a **full set of primitive ITs**. By "full", we mean no others exist. $\mathcal{P}$ is a basis for an **algebra of invariants**.[1]

An **invariance group** $\mathcal{G}$ is the set of all linear transformation $G \in \mathcal{G}$ such that

$$\begin{aligned} p_1(x^\dagger, y) &= p_1(x^\dagger G^\dagger, Gy) && (7.18) \\ p_2(w^\dagger, x^\dagger, y, z) &= p_2(w^\dagger G^\dagger, x^\dagger G^\dagger, Gy, Gz) && (7.19) \\ &\quad \text{etc.} && (7.20) \end{aligned}$$

Example. Consider an invariance group with a single primitive IT $p()$ defined by

$$p(x^\dagger, y) = \delta_a^b {x^\dagger}^a y_b = {x^\dagger}^b y_b \tag{7.21}$$

Then

$$(x')^{\dagger a}(y')_a = {x^\dagger}^b (G^\dagger G)_b{}^c y_c = {x^\dagger}^b y_b \tag{7.22}$$

[1] An algebra over a field is defined in Sec.A.5

so G must be unitary

$$G^\dagger G = 1 \tag{7.23}$$

The group of n dimensional unitary matrices is called $U(n)$

Chapter 8

Lie Algebras

This chapter is based on Ref.[1].

8.1 Generators (infinitesimal transformations)

For some group $\mathcal{G}$, assume that any group element $G \in \mathcal{G}$ that is infinitesimal close to the identity 1 can be parametrized by

$$G = 1 + i\sum_i \epsilon_i T^i \tag{8.1}$$

where $T^i \in \mathbb{C}^{n\times n}$ for $i = 1, 2, \ldots, N$, $\epsilon_i \in \mathbb{R}$ and $|\epsilon_i| << 1$.

The T^i matrices are called the **generators** of infinitesimal transformations for group $\mathcal{G}$. The generators of a group $\mathcal{G}$ span a vector space called a Lie algebra $\mathfrak{g}$.[1] For example, the generators of the group SU(2) span the **Lie algebra** $\mathfrak{su}(2)$.

Assume that the T^i matrices are Hermitian and that they satisfy

$$\mathrm{tr}(T^iT^j) = \kappa\delta(i,j) \tag{8.2}$$

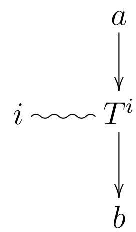
A Lie algebra that satisfies Eq.(8.2) is called a **simple Lie algebra**. $g_{ij} = \mathrm{tr}(T_i^\dagger T_j)$ is called the **Cartan-Killing form**. A **semi-simple Lie algebra** is a direct sum of simple Lie algebras.

It's customary to choose generators so that $\kappa = \frac{1}{2}$.[2] However, we will often set $\kappa = 1$ for intermediate calculations and restore $\kappa \neq 1$ at the end by dimensional analysis. Just remember that each T^j scales as $\sqrt{\kappa}$. For example, given the equation $\mathrm{tr}(T^iT^j) = \delta(i,j)$, we know that when $\kappa \neq 1$, $\mathrm{tr}(T^iT^j) = \kappa\delta(i,j)$ so both sides of the equation scale as κ.

We will use the following scaled version of T^j as a birdtrack. Define

[1] See Sec.A.5 for the definition of an algebra over a field.

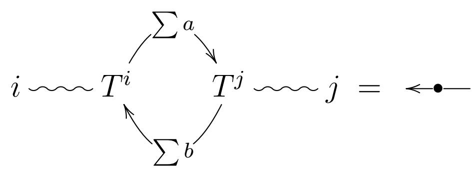
[2] For $SU(2)$, it is customary to choose $T^i = \frac{1}{2}\sigma_i$, where σ_i for $i = 1, 2, 3$ are the Pauli matrices. For $SU(3)$, it is customary to choose $T^i = \frac{1}{2}\lambda_i$ where λ_i for $i = 1, 2, \ldots, 8$ are the Gell-Mann matrices. For both of these choices, $\kappa = \frac{1}{2}$.

$$(C^i_{Adj})_b{}^a = \frac{1}{\sqrt{\kappa}}(T^i)_b{}^a = \frac{1}{\sqrt{\kappa}}$$  (8.3)

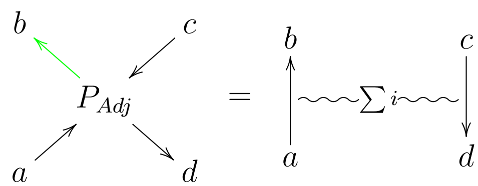
In the CC convention, we will always start reading the indices of this node at the wavy undirected leg.

Adj stands the Adjoint. In this node (vertex), an adjoint representation (adjrep) particle (wavy line, gluon) is generated (released) by a defining representation (defrep) particle (straight solid line, arrow).

In terms of birdtracks, Eq.(8.2) becomes

$$\boxed{(T^i)^b{}_a(T^j)^a{}_b = \mathrm{tr}(T^iT^j) = \delta(i,j)}$$  (8.4)

We can now define the projection operator for the adrep (gluon exchange between 2 defrep particles)

$$\boxed{(P_{Adj})_b{}^a{}_d{}^c = \sum_i (T^i)_b{}^a(T^i)_d{}^c}$$  (8.5)

The green arrow is the first index in the CC convention.

Note that if $x \in V^n \otimes V^{\dagger n}$, then

$$(P_{Adj})_b{}^a{}_d{}^c x_c{}^d = \sum_i (T^i)_b{}^a \underbrace{\left[(T^i)_d{}^c x_c{}^d\right]}_{\epsilon_i \in \mathbb{R}} \tag{8.6}$$

Recall Eq.(A.29). If $x \in V^{n^p} \otimes V^{\dagger n^q}$, and $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a^{:p}}{}^{b^{:q}} = \mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}}, \quad x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta \tag{8.7}$$

where we define

$$\mathbb{G}_\alpha{}^\beta \stackrel{\text{def}}{=} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \prod_{i=1}^{q} G^{\dagger b_i}{}_{c_i} \tag{8.8}$$

If $\mathbb{G}$ is infinitesimally close to the identity, then we can parametrize it as

$$\begin{aligned}
\mathbb{G}_\alpha{}^\beta &= 1 + i\sum_j \epsilon_j (M^j)_\alpha{}^\beta && (8.9)\\
G_{a_i}{}^{d_i} &= 1 + i\sum_j \epsilon_j (T^j)_{a_i}{}^{d_i} && (8.10)\\
G^{\dagger b_i}{}_{c_i} &= 1 - i\sum_j \epsilon_j (T^j)^{b_i}{}_{c_i} && (8.11)
\end{aligned}$$

Define

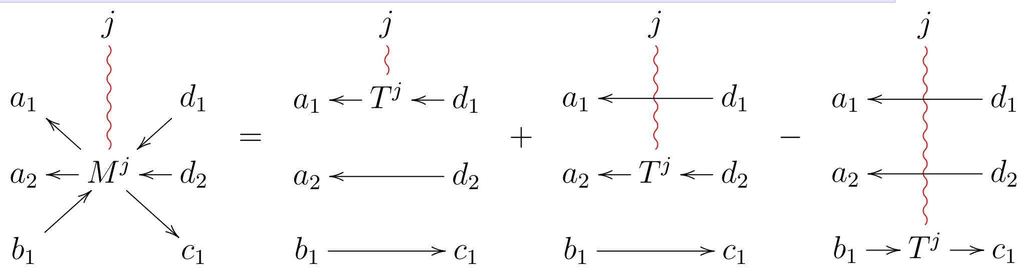
$$(M^j)_\alpha{}^\beta = \left[(T^j)_{a_i}{}^{d_i}\frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}{}_{c_i}\frac{1}{\delta_{c_i}^{b_i}}\right]\delta_{a^{:p}}^{d^{:p}}\delta_{c^{:q}}^{b^{:q}} \tag{8.12}$$

When $x'_\alpha = x_\alpha$, to first order in ϵ_i,

$$0 = (M^j)_\alpha{}^\beta x_\beta = \left[(T^j)_{a_i}{}^{d_i}\frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}{}_{c_i}\frac{1}{\delta_{c_i}^{b_i}}\right]\delta_{a^{:p}}^{d^{:p}}\delta_{c^{:q}}^{b^{:q}} x_{d^{:p}}{}^{c^{:q}} \tag{8.13}$$

For example, if we define

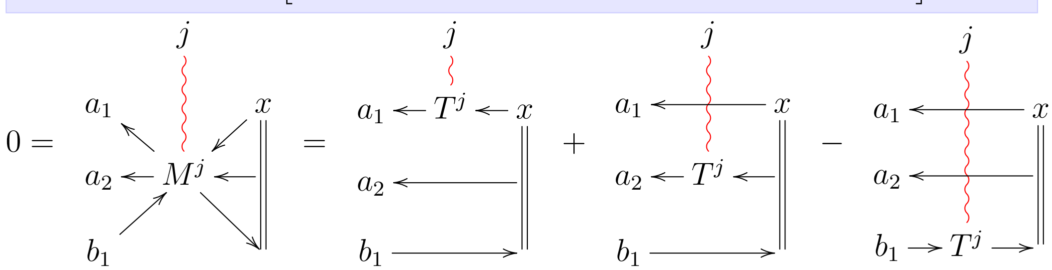
$$\boxed{(M^j)_{a_1a_2}{}^{b_1}{}_{c_1}{}^{d_2d_1} = (T^j)_{a_1}{}^{d_1}\delta_{a_2}^{d_2}\delta_{c_1}^{b_1} + \delta_{a_1}^{d_1}(T^j)_{a_2}{}^{d_2}\delta_{c_1}^{b_1} - \delta_{a_1}^{d_1}\delta_{a_2}^{d_2}(T^j)^{b_1}{}_{c_1}}$$



(8.14)

then

$$\boxed{0 = (M^jx)_{a_1a_2}{}^{b_1} = \left[(T^j)_{a_1}{}^{d_1}\delta_{a_2}^{d_2}\delta_{c_1}^{b_1} + \delta_{a_1}^{d_1}(T^j)_{a_2}{}^{d_2}\delta_{c_1}^{b_1} - \delta_{a_1}^{d_1}\delta_{a_2}^{d_2}(T^j)^{b_1}{}_{c_1}\right] x_{d_1d_2}{}^{c_1}}$$



(8.15)

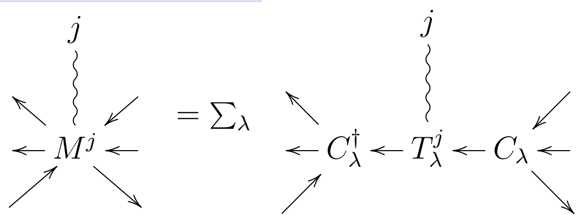
8.2 Clebsch-Gordan Coefficients

The Clebsch Gordan coefficients (CBC) are introduced in Ch.3. Note that the generators ${(T^i)_a}^b$ are a simple kind of CGC matrix, one with

- a gluon (adjrep) particle instead of a general λ rep particle emanating from the i index,
- a particle of the defrep entering and another leaving the node, instead of any number of defrep particles entering and leaving.

Since $\mathbb{G} = 1 + i\sum_j \epsilon_j M^j$, generators decompose in the same way as the group elements

$$\boxed{M^j = \sum_\lambda C_\lambda^\dagger T_\lambda^j C_\lambda}$$



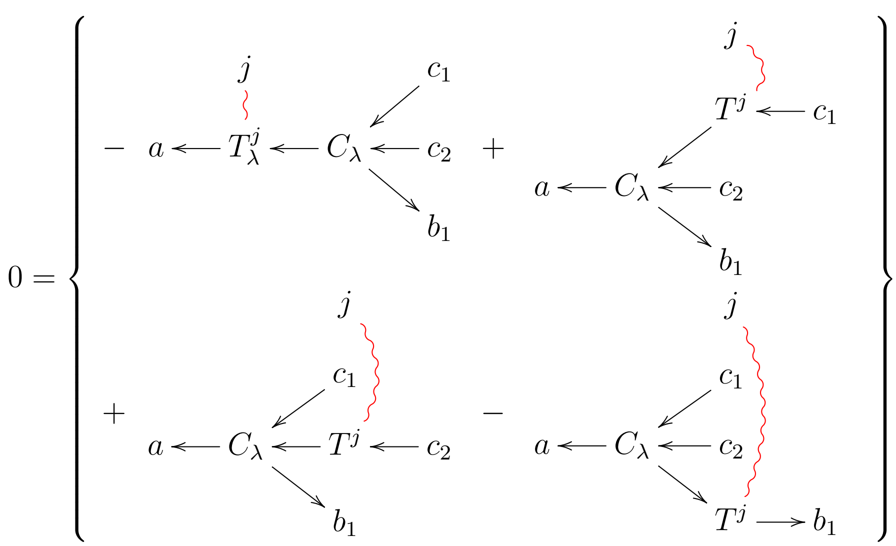
(8.16)

The CGC matrices are invariant matrices.

$$C_\lambda = G_\lambda^\dagger C_\lambda G \tag{8.17}$$

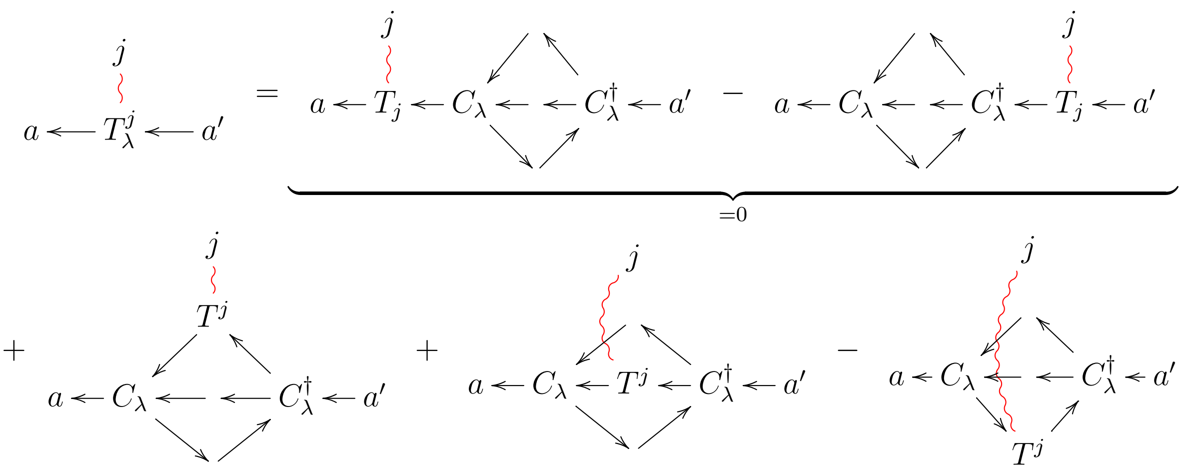
Hence,

$$0 = -T_\lambda^j C_\lambda + C_\lambda T^j \tag{8.18}$$



(8.19)

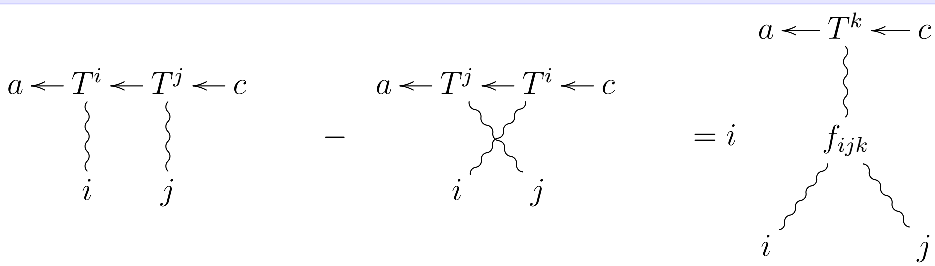
Multiplying on the left by $C_\lambda^\dagger$, we obtain an expression for the generator T_λ^i in term the generators T^j (and C_λ CGC matrices).



(8.20)

8.3 Structure Constants (3 gluon vertex)

$$\underbrace{T^iT^j - T^jT^i}_{[T^i,T^j]} = i f_{ijk} T^k \quad \textbf{(Lie Algebra commutation relations)}$$



(8.21)

The f_{ijk} tensors are called the **structure constants** of the Lie Algebra. They define a 3 gluon vertex in term of the generators T^i.[3]

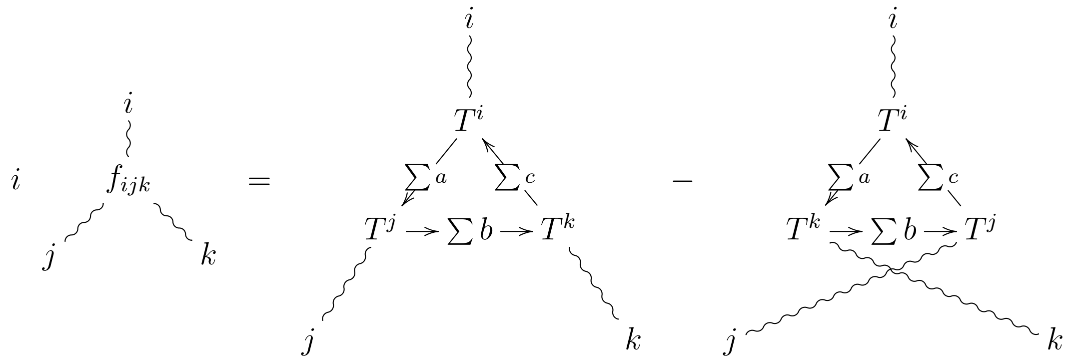
If $(T^j)_a{}^b$ are the rep-matrices (in the defrep) of the generators of a group $\mathcal{G}$, then Eq.(8.21) shows that the matrices $(M^k)_{ij} = -iC_{ijk}$ are also a rep-matrix (in the adrep) of the generators of $\mathcal{G}$.

Since $\mathrm{tr}(T^kT^{k'}) = \delta(k,k')$, Eq.(8.21) implies

$$\mathrm{tr}([T^i,T^j]T^k) = i f_{ijk} \tag{8.22}$$

[3]It's possible to distinguish between upper and lower gluon indices (i.e., to give the gluon arrows a direction. In that case, the Lie Algebra commutation relations would be $[T^i,T^j] = f^{ij}{}_k T^k$ and the gluon indices could be lowered and raised using the metric (called the **Cartan-Killing form**) $g_{ij} = \mathrm{tr}((T^i)^\dagger T^j)$. But since we are assuming $g_{ij} = \kappa\delta_i^j$, there is no need to do this.

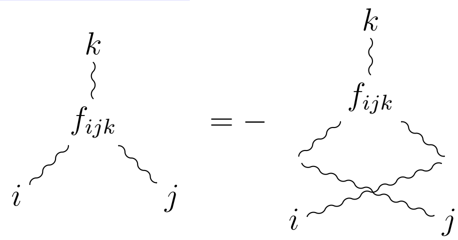
$$\boxed{if_{ijk} = \mathrm{tr}([T^i, T^j]T^k) = (T^i)_a{}^c (T^j)_b{}^a (T^k)_c{}^b - (T^j)_c{}^b (T^i)_a{}^c (T^k)_b{}^a}$$



(8.23)

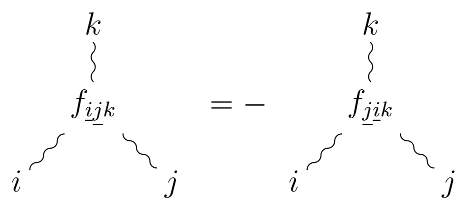
Note that

$$\boxed{f_{ijk} = -f_{jik}}$$



(Convention CC)

(8.24)



(Convention L)

In fact, the tensor f_{ijk} is **totally antisymmetric** (i.e., it changes sign under a transposition of any two indices).

Claim 9 *f_{ijk} is a real number.*

proof:

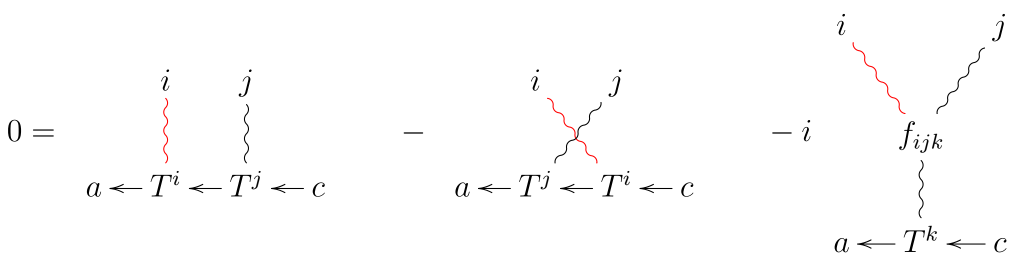
$$\left[i\mathrm{tr}([T^i, T^j]T^k)\right]^\dagger = (-i)\mathrm{tr}(T^k[T^j, T^i]) \tag{8.25}$$

$$= (-i)\mathrm{tr}([T^j, T^i]T^k) \tag{8.26}$$

$$= i\mathrm{tr}([T^j, T^k]T^k) \tag{8.27}$$

QED

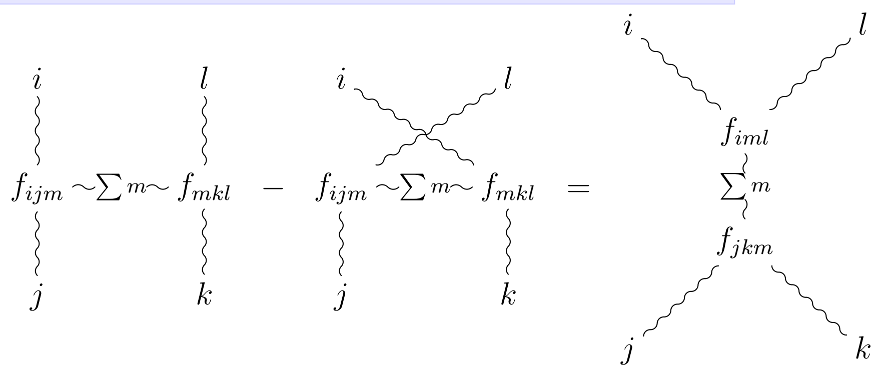
Note that the birdtrack for the Lie Algebra commutation relations Eq.(8.21) can be understood as the statement that the generators T^j are invariant matrices. Below we restate Eq.(8.21) to make that obvious



(8.28)

Claim 10

$$\boxed{f_{ijm}f_{mkl} - f_{ljm}f_{mki} = f_{iml}f_{jkm} \quad \textbf{(Jacobi identity)}}$$



(8.29)

proof:

Note that

$$\begin{aligned} \mathrm{tr}\left([[T^i,T^j],T^k]T^l\right) &= \mathrm{tr}\left(f_{ijm}[T^m,T^k]\right) && (8.30)\\ &= \mathrm{tr}\left(f_{ijm}f_{mkl'}T^{l'}T^l\right) && (8.31)\\ &= f_{ijm}f_{mkl} && (8.32) \end{aligned}$$

so the Jacobi identity can be restated as

$$\mathrm{tr}\left(\left\{[[T^i,T^j],T^k] + [[T^j,T^k],T^i] + [[T^k,T^i],T^j]\right\}T^l\right) = 0 \tag{8.33}$$

Hence, the claim follows if we can prove that

$$\underbrace{[[T^i,T^j],T^k] + [[T^j,T^k],T^i] + [[T^k,T^i],T^j]}_{\text{cyclic permutations of } ijk} = 0 \tag{8.34}$$

If we expand the left hand side on Eq.(8.34), we find 6 terms that cancel in pairs.
QED

Note Claim 10 can be undertood as the Lie Algebra commutation relations Eq.(8.21), but stated in the adrep instead of the defrep. Indeed, if

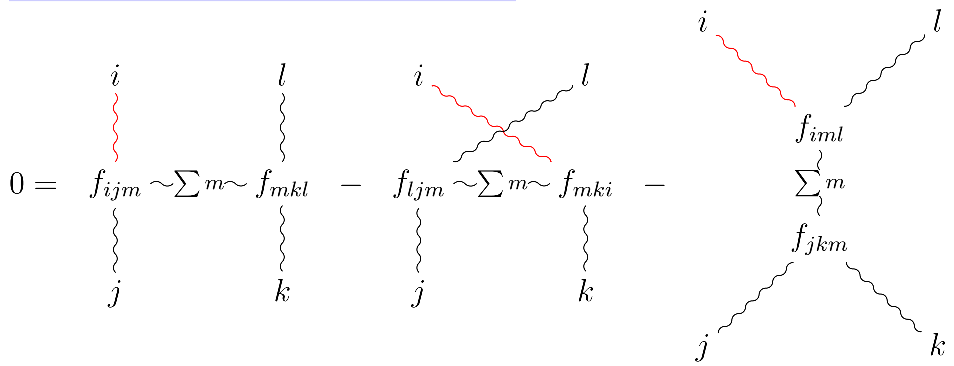
$$M^i_{jk} = -i f_{ijk} \tag{8.35}$$

then Claim 10 becomes

$$(M^i M^l - M^l M^i)_{jk} = i C_{ilm} (M^m)_{jk} \tag{8.36}$$

Note that Claim 10 can be understood as a statement of the fact that f_{ijk} is an invariant tensor.

$$\boxed{0 = f_{ijm} f_{mkl} - f_{ljm} f_{mki} - f_{iml} f_{jkm}}$$

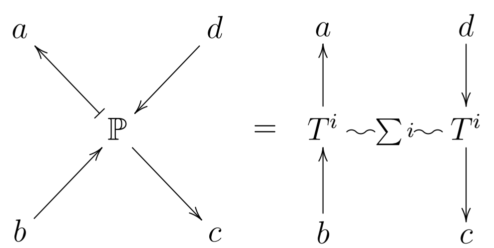


(8.37)

8.4 Two types of gluon exchanges

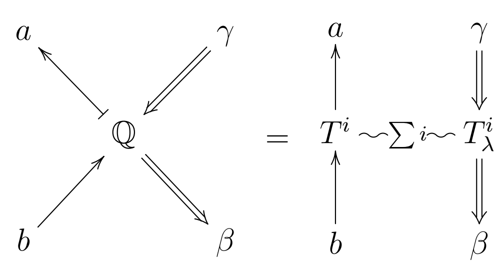
Consider the following two gluon exchange operators. Note that $\mathbb{P}^2 = \mathbb{P}$, but $\mathbb{Q}^2 \neq \mathbb{Q}$, so $\mathbb{P}$ is a bonafide projection operator but not $\mathbb{Q}$. $\mathbb{Q}\mathbb{Q}^\dagger = \mathbb{P}$ so $\mathbb{Q}$ is like half of a projection operator.

$$\boxed{\mathbb{P}_{a\ c}^{\ b\ d} = \sum_i (T^i)_a^{\ b} (T^i)_c^{\ d}}$$



(8.38)

$$\boxed{\mathbb{Q}_{a\ \beta}^{\ b\ \gamma} = \sum_i (T^i)_a^{\ b} (T^i_\lambda)_\beta^{\ \gamma}}$$



(8.39)

Claim 11 *If $\mathbb{Q}_b{}^a$ is the matrix with (ν, γ) entries $\mathbb{Q}_b{}^a{}_\nu{}^\gamma$, then*

$$[\mathbb{Q}_b{}^a, \mathbb{Q}_d{}^c] = \mathbb{P}_{b'}{}^a{}_d{}^c \mathbb{Q}_b{}^{b'} - \mathbb{Q}_{a'}{}^a \mathbb{P}_b{}^{a'}{}_c{}^d \tag{8.40}$$

proof:

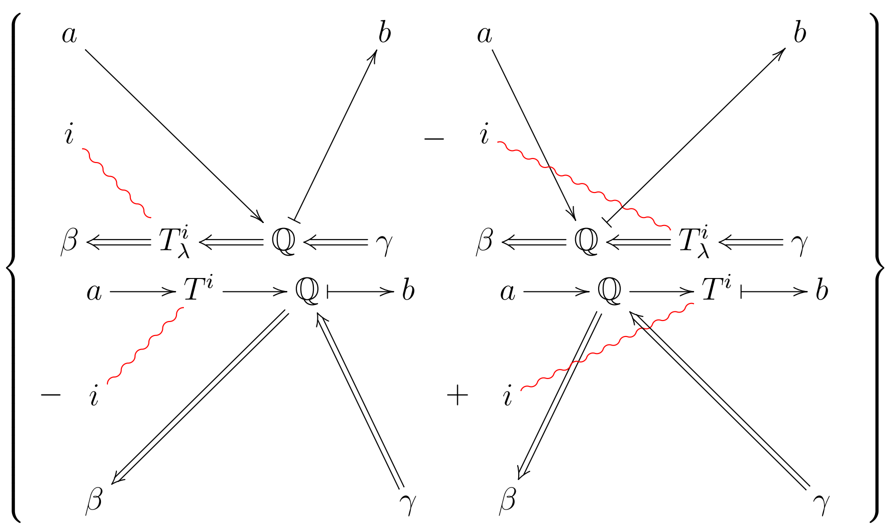
$$(T^j)_b{}^a (T^i)_d{}^c [T^j_\lambda, T^i_\lambda] = \left[(T^i)_{b'}{}^a (T^k)_b{}^{b'} - (T^k)_{a'}{}^a (T^i)_b{}^{a'} \right] (T^i)_d{}^c T^k_\lambda \tag{8.41}$$

$$(T^j)_b{}^a (T^i)_d{}^c i f_{jik} T^k_\lambda = i f_{ikj} (T^j)_b{}^a (T^i)_d{}^c T^k_\lambda \tag{8.42}$$

QED

This claim can be visualized as follows. $\mathbb{Q}$ is an invariant tensor so

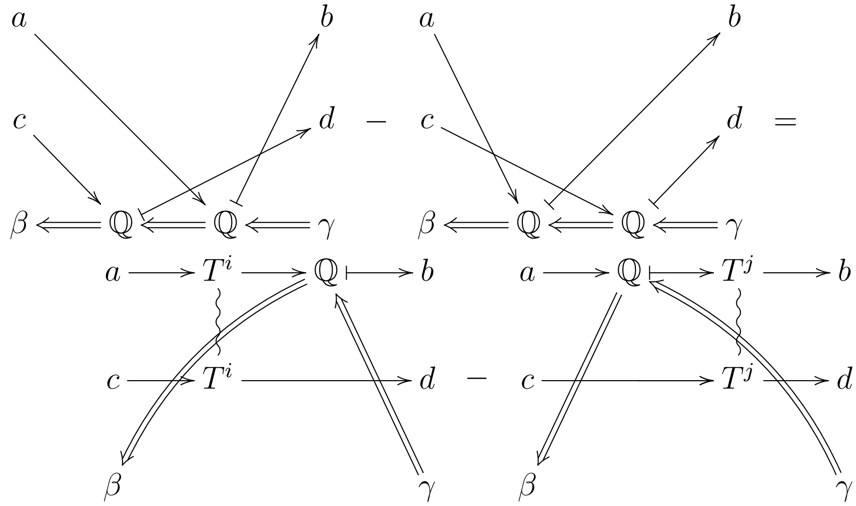
$0 =$



(8.43)

Now multiplying by $(T^i)_c{}^d$, we get

$$\boxed{\mathbb{Q}_d{}^c{}_\beta{}^\nu \mathbb{Q}_b{}^a{}_\nu{}^\gamma - \mathbb{Q}_b{}^a{}_\beta{}^\nu \mathbb{Q}_d{}^c{}_\nu{}^\gamma = \mathbb{P}_{a'}{}^a{}_d{}^c \mathbb{Q}_b{}^{a'}{}_\beta{}^\gamma - \mathbb{Q}_{b'}{}^a{}_\beta{}^\gamma \mathbb{P}_b{}^{b'}{}_d{}^c}$$



(8.44)

Chapter 9

Orthogonal Groups: COMING SOON

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Chapter 10

Quantum Shannon Information Theory: COMING SOON

Chapter 11

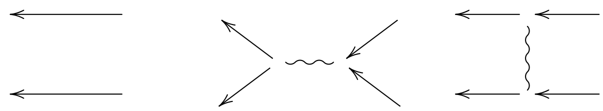
Recoupling Identities

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Mandelstam variables

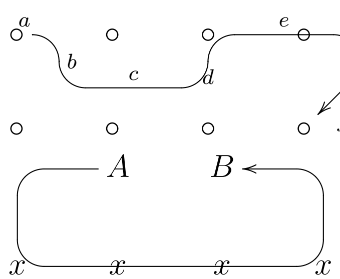
s-channel, particles shag (have sex),

t-channel, particles have tea



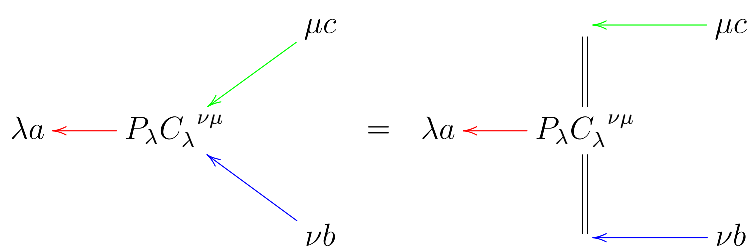
parallel channels s-channel t-channel (11.1)

11.1 Parallel channels to sum of t-channels



$$\lambda = red, \quad \mu = green, \quad \nu = blue \tag{11.2}$$

no implicit sum over Greek indices

$P_\lambda C_{\lambda a}{}^{\nu b \mu c} =$ 

(11.3)

$$\mathcal{C}_\lambda{}^{\nu\mu} = P_\lambda C_\lambda{}^{\nu\mu} \tag{11.4}$$

$$\mathcal{C}_\lambda \mathcal{C}_\lambda^\dagger = P_\lambda^2 = P_\lambda \tag{11.5}$$

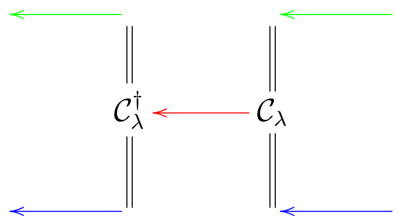
$$\mathrm{tr}(P_\lambda) = d_\lambda \tag{11.6}$$

where d_λ is the dimension of rep λ. Actually, $\mathcal{C}_\lambda = P_\lambda C_\lambda = C_\lambda$, but we make the P_λ explicit for pedagogical purposes.

Note that if we divide $\mathcal{C}_\lambda$ by $\sqrt{d_\lambda}$, then

$$\mathrm{tr}\left(\frac{\mathcal{C}_\lambda}{\sqrt{d_\lambda}}\frac{\mathcal{C}_\lambda^\dagger}{\sqrt{d_\lambda}}\right) = 1 \tag{11.7}$$

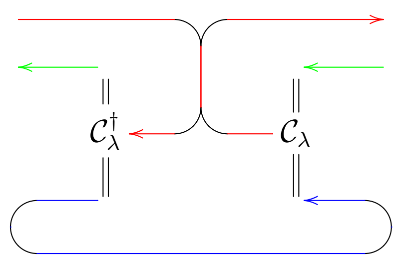
$\mathcal{P}_\lambda =$



(11.8)

$$\mathcal{P}_\lambda^2 = \mathcal{P}_\lambda \tag{11.9}$$

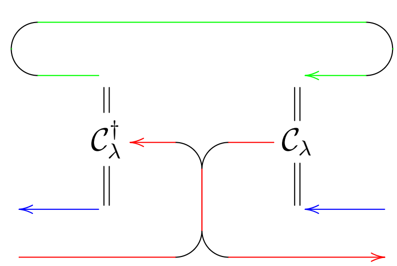
$\mathcal{P}_\nu = \frac{d_\nu}{d_\lambda}$



(11.10)

$$\mathcal{P}_\nu^2 = \mathcal{P}_\nu \tag{11.11}$$

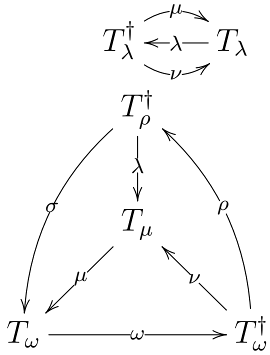
$\mathcal{P}_\mu = \frac{d_\mu}{d_\lambda}$



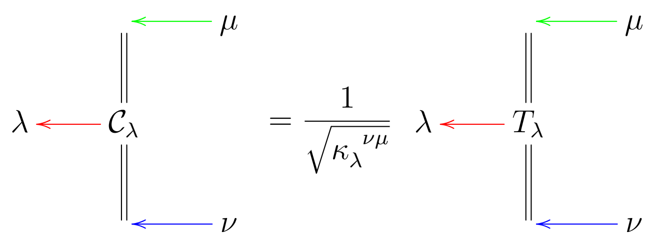
(11.12)

$$\mathcal{P}_\mu^2 = \mathcal{P}_\mu \tag{11.13}$$

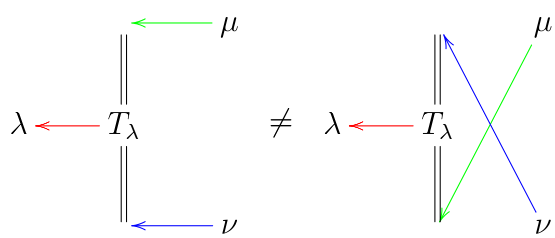
The normalization of the projectors $\mathcal{P}_\lambda, \mathcal{P}_\nu, \mathcal{P}_\mu$ can be remembered if one takes the denominator d_λ and splits it into two factors of $\sqrt{d_\lambda}$ and puts one $\sqrt{d_\lambda}$ under $\mathcal{C}_\lambda$ and the other under $\mathcal{C}^\dagger_\lambda$. Then one "trades" $\frac{\mathcal{C}_\lambda}{\sqrt{d_\lambda}}$ by $\frac{\mathcal{C}_\nu}{\sqrt{d_\nu}}$ or $\frac{\mathcal{C}_\mu}{\sqrt{d_\mu}}$.



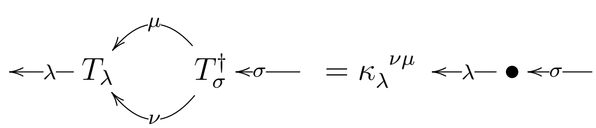
arrow directions for specific case being considered. they can be changed



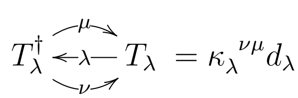
(11.14)



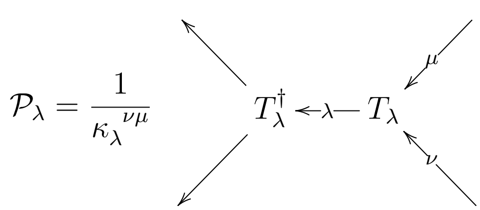
(11.15)



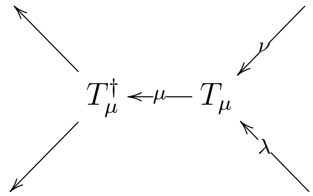
(11.16)

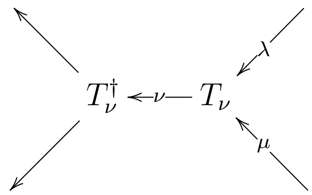


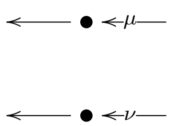
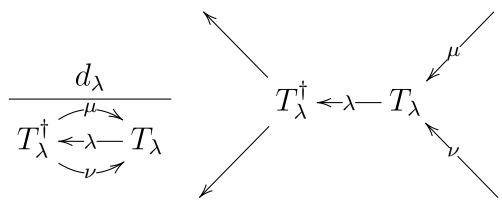
(11.17)

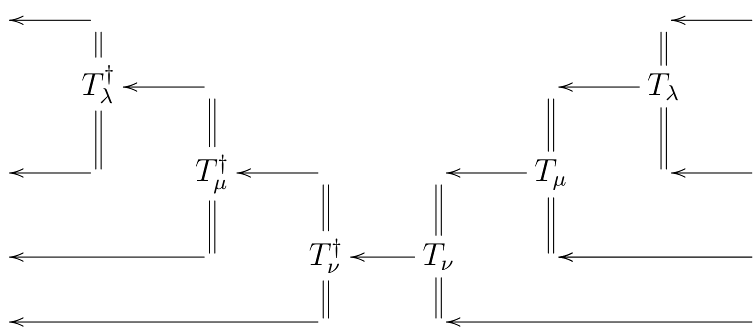


(11.18)

$$\mathcal{P}_\mu = \frac{1}{{\kappa_\mu}^{\lambda\nu}}$$  (11.19)

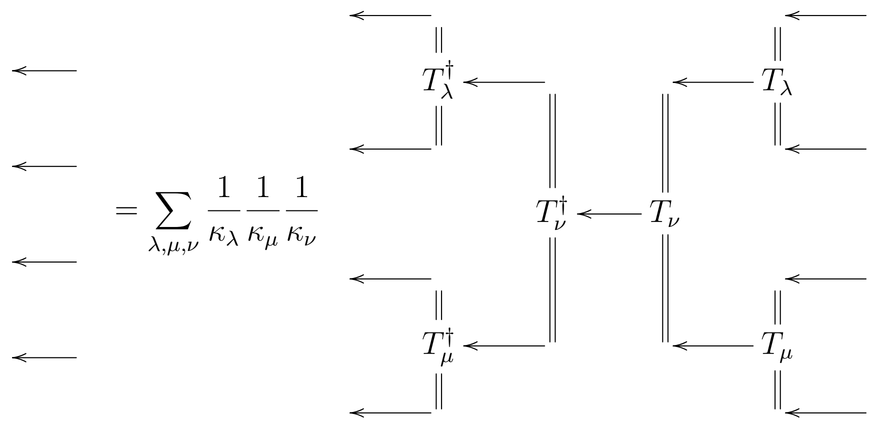
$$\mathcal{P}_\nu = \frac{1}{{\kappa_\nu}^{\mu\lambda}}$$  (11.20)

 $$= \sum_\lambda \mathcal{P}_\lambda = \sum_\lambda$$  (11.21)

 $$= \sum_{\lambda,\mu,\nu} \frac{1}{\kappa_\lambda} \frac{1}{\kappa_\mu} \frac{1}{\kappa_\nu}$$ 

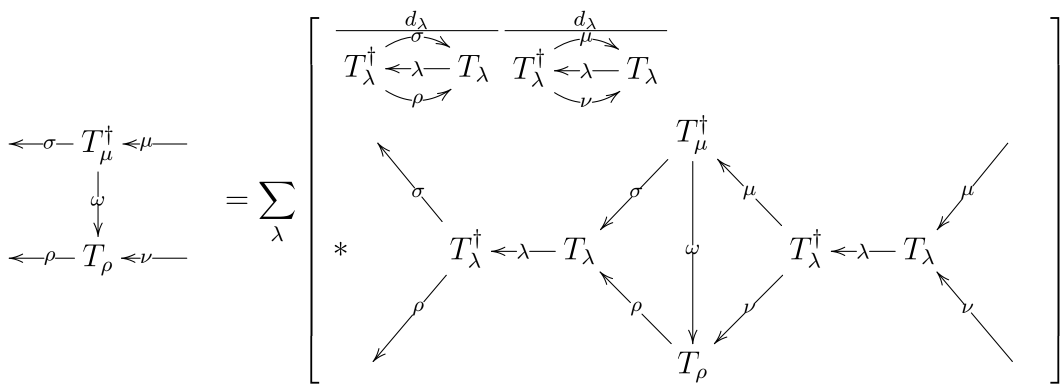
(11.22)

Dropping two upper representation labels from ${\kappa_\nu}^{\mu\lambda}$ for succinctness, but their dependance still there.

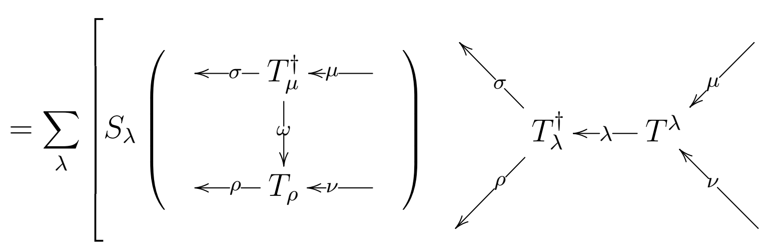


(11.23)

11.2 t-channel to sum of s-channels



(11.24)



(11.25)

$$
S_\lambda \left(\begin{tikzcd} {} & T^\dagger_\mu \arrow[l, "\sigma"'] \arrow[d, "\omega"] & {} \arrow[l, "\mu"'] \\ {} & T_\rho \arrow[l, "\rho"'] & {} \arrow[l, "\nu"'] \end{tikzcd} \right) = \frac{d_\lambda}{\begin{tikzcd} T^\dagger_\lambda \arrow[r, bend left, "\sigma"] \arrow[r, bend right, "\rho"'] & T_\lambda \arrow[l, "\lambda"'] \end{tikzcd}} \frac{d_\lambda}{\begin{tikzcd} T^\dagger_\lambda \arrow[r, bend left, "\mu"] \arrow[r, bend right, "\nu"'] & T_\lambda \arrow[l, "\lambda"'] \end{tikzcd}} \frac{\begin{tikzcd} & T^\dagger_\mu \arrow[d, "\omega"] \arrow[ddl, bend right, "\sigma"'] & \\ & T_\rho \arrow[dl, "\rho"'] & \\ T_\lambda \arrow[rr, "\lambda"] & & T^\dagger_\lambda \arrow[ul, "\nu"'] \arrow[uul, bend right, "\mu"'] \end{tikzcd}}{d_\lambda} \tag{11.26}
$$

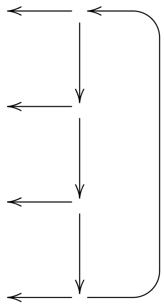
$$
= d_\lambda \frac{\begin{tikzcd} & T^\dagger_\mu \arrow[d, "\omega"] \arrow[ddl, bend right, "\sigma"'] & \\ & T_\rho \arrow[dl, "\rho"'] & \\ T_\lambda \arrow[rr, "\lambda"] & & T^\dagger_\lambda \arrow[ul, "\nu"'] \arrow[uul, bend right, "\mu"'] \end{tikzcd}}{\begin{tikzcd} T^\dagger_\lambda \arrow[r, bend left, "\sigma"] \arrow[r, bend right, "\rho"'] & T_\lambda \arrow[l, "\lambda"'] & T^\dagger_\lambda \arrow[r, bend left, "\mu"] \arrow[r, bend right, "\nu"'] & T_\lambda \arrow[l, "\lambda"'] \end{tikzcd}} \tag{11.27}
$$

11.3 Wigner $3n - j$ coefficients

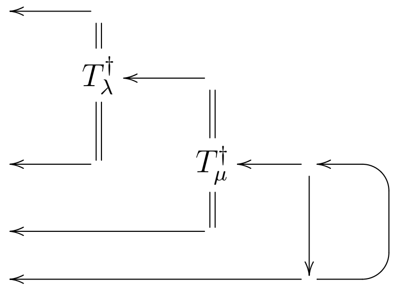
we will refer to x as a $3j$ symbol, and to x as a $6j$ symbol. Atomic physicsists also define $3n - j$ symbols, for $n = 1, 2, 3, \ldots$. They are called that because they describe how to "add" $3n$ angular momenta j. There is only one $3j$ but two $6j$'s . five $9 - j$s, and so on. We only show one $3j$ and one $6j$.

Similar to Eq.(11.22) but here we use the most general t-channel to sum of s-channels conversion Eq.(11.25)

$$
\begin{tikzcd} {} & {} \arrow[l] \arrow[d] & {} \arrow[l] \\ {} & {} \arrow[l] \arrow[d] & \\ {} & {} \arrow[l] \arrow[d] & \\ {} & {} \arrow[l] \arrow[r, dash] & {} \arrow[uuu, dash] \end{tikzcd}
= \sum_{\lambda,\mu,\nu} S_\lambda S_\mu S_\nu
\begin{tikzcd} {} & {} \arrow[l] \arrow[d, equal] & & & & \\ & T^\dagger_\lambda \arrow[d, equal] & {} \arrow[l] \arrow[d, equal] & & & \\ {} & {} \arrow[l] & T^\dagger_\mu \arrow[d, equal] & {} \arrow[l] \arrow[d, equal] & {} \arrow[d, equal] & {} \arrow[l] \arrow[dd, dash] \\ {} & & {} \arrow[ll] & T^\dagger_\nu \arrow[d, equal] & T_\nu \arrow[l] \arrow[d, equal] & \\ {} & & & {} \arrow[lll] & {} \arrow[r, dash] & {} \end{tikzcd} \tag{11.28}
$$



$$= \sum_{\lambda,\mu} S_\lambda S_\mu$$



(11.29)

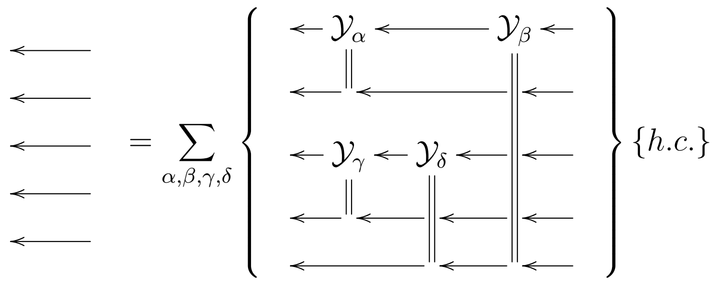
$$3n - j \text{ coefficient} = \sum (\text{product of } S_\lambda\text{'s})(\text{tree graph with one-point loop}) \quad (11.30)$$

4-point loop $\rightarrow$ one-point loop
one-point loop , self energy loop?
zero point loop = vacuum bubble

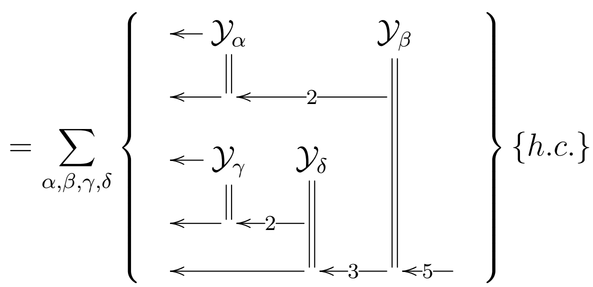
Chapter 12

Recoupling Identities for $U(n)$

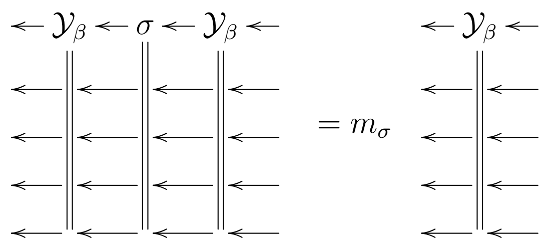
This chapter is based on Cvitanovic's Birdtracks book Ref. [1].



(12.1)

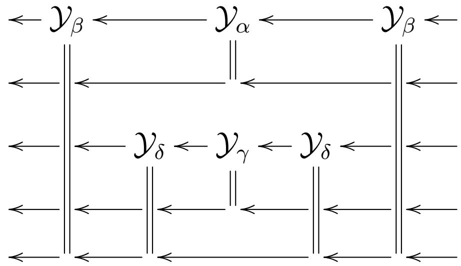


(12.2)

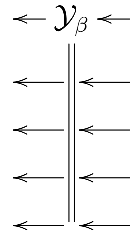


(12.3)

where $m_\sigma \in \{-1, 0, 1\}$.



$= M$



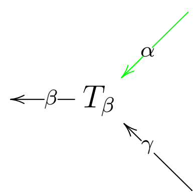
(12.4)

M is independent on n for $U(n)$.

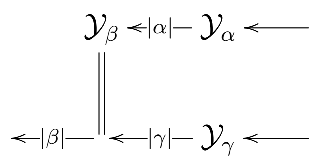
12.1 $3j$ coefficients

Recall that $|\mathcal{Y}_\alpha|$ or $|\alpha|$ is the number of boxes (or number of outgoing legs in its birdtrack) in the YT $\mathcal{Y}_\alpha$.

$T_\beta =$



$=$

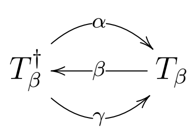


(12.5)

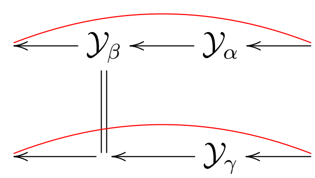
where $|\beta| = |\alpha| + |\gamma|$

Claim 12 *For $U(n)$, we have*[1]

$\mathrm{tr}(T_\beta^\dagger T_\beta) =$



$=$

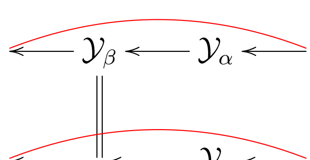


(12.6)

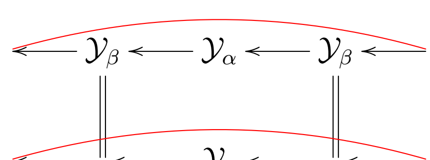
$$= M dim(\mathcal{Y}_\beta) \tag{12.7}$$

where M is independent of n.

proof:



$=$



(12.8)

$$= M dim(\mathcal{Y}_\beta) \tag{12.9}$$

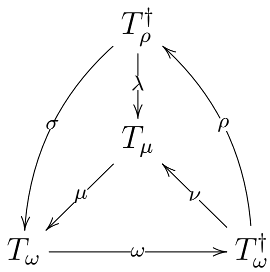
[1]Ref.[1], on which this claim is based, uses different labels for the Young projectors. Their labels and ours are related as follows:
$\mathcal{Y}_\alpha \to X$, $\mathcal{Y}_\beta \to Y$, $\mathcal{Y}_\gamma \to Z$.

QED

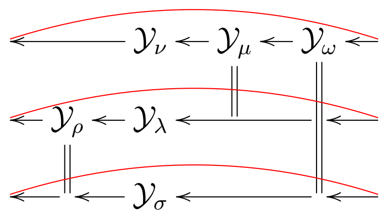
Ref. [1] shows that for this example, $M = 1$.

12.2 $6j$ coefficients

Claim 13 *For $U(n)$, we have*[2]



$=$

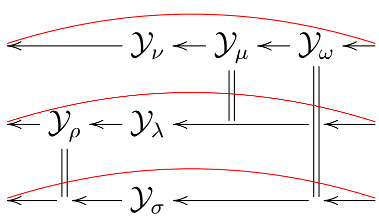


(12.10)

$$= M dim(\mathcal{Y}_\omega) \tag{12.11}$$

where M is independent of n

proof: Replace each of the 6 Young projectors $\mathcal{Y}_\alpha$ of the right hand side (RHS) by its square. That gives 12 Young projectors on the RHS. Each of the 4 generators T_λ on the left hand side (LHS) is composed of 3 Young projectors so there are 12 Young projectors on the LHS too.



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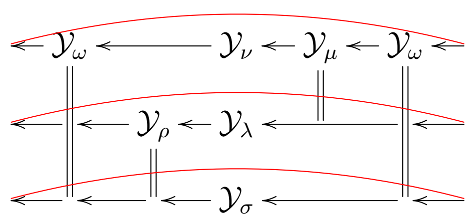


(12.12)

$$= M dim(\mathcal{Y}_\omega) \tag{12.13}$$

QED

For example, Ref.[1] shows that for

$$\mathcal{Y}_\rho = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \qquad \mathcal{Y}_\nu = \boxed{1}, \qquad \mathcal{Y}_\lambda = \boxed{2} \tag{12.14}$$

[2]Ref.[1], on which this claim is based, uses different labels for the Young projectors. Their labels and ours are related as follows:
$\mathcal{Y}_\sigma \to X$, $\mathcal{Y}_\omega \to Y$, $\mathcal{Y}_\rho \to U$, $\mathcal{Y}_\lambda \to W$, $\mathcal{Y}_\nu \to V$, $\mathcal{Y}_\mu \to Z$.

$$\mathcal{Y}_\sigma = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}, \qquad \mathcal{Y}_\omega = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} 4 & \\ \cline{1-1}\end{array}, \qquad \mathcal{Y}_\mu = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \tag{12.15}$$

we have

$$M = \frac{1}{3}, \quad dim\mathcal{Y}_\omega = \frac{n(n^2-1)(n^2-2)}{8} \tag{12.16}$$

12.3 Sum rules

Let

$STY(n_b)$ = set of STY with n_b boxes

STY_+ = set of STY with one or more boxes. Hence, SYT $\mathcal{Y}_\alpha$ with $|\alpha| = 0$ is excluded.

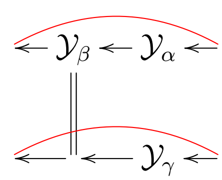
Claim 14

$$\sum_{\alpha,\gamma \in SYT_+} \mathbb{1}(|\alpha| + |\gamma| = |\beta|)\, T_\beta^\dagger \begin{smallmatrix}\xrightarrow{\alpha} \\ \xleftarrow{\beta} \\ \xrightarrow{\gamma}\end{smallmatrix} T_\beta \; = (|\beta| - 1)dim(\mathcal{Y}_\beta) \tag{12.17}$$

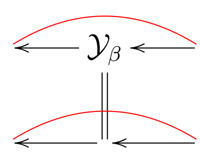
proof:

$$\sum_{\alpha \in SYT(|\alpha_0|)} \mathcal{Y}_\alpha = 1, \qquad \sum_{\gamma \in SYT(|\gamma_0|)} \mathcal{Y}_\gamma = 1 \tag{12.18}$$

$$\sum_{\alpha,\gamma \in SYT_+} \mathbb{1}(|\alpha| + |\gamma| = |\beta|)\, T_\beta^\dagger \begin{smallmatrix}\xrightarrow{\alpha} \\ \xleftarrow{\beta} \\ \xrightarrow{\gamma}\end{smallmatrix} T_\beta \; = \sum_{|\alpha|=1}^{|\beta|-1} \sum_{\substack{\alpha \in SYT(|\alpha|) \\ \gamma \in SYT(|\beta|-|\alpha|)}} \tag{12.19}$$



$$= \sum_{|\alpha|=1}^{|\beta|-1} \tag{12.20}$$



$$= (|\beta| - 1)dim(\mathcal{Y}_\beta) \tag{12.21}$$

QED

Let

$$A = \{\rho, \nu, \lambda, \sigma, \omega, \mu\}, \quad A(\omega) = A - \{\omega\} \tag{12.22}$$

$$J(A) = \mathbb{1}\left(\begin{array}{l} |\sigma| + |\mu| = |\omega|, \\ |\nu| + |\rho| = |\omega|, \\ |\sigma| + |\lambda| = |\rho|, \\ |\lambda| + |\nu| = |\mu| \end{array}\right) \tag{12.23}$$

Claim 15

$$\prod_{\alpha \in A(\omega)} \left[\sum_{\alpha \in SYT(|\alpha|)}\right] J(A) \quad \begin{tikzcd} & T_\rho^\dagger \arrow[d, "\lambda"] \arrow[ddl, bend right, "\sigma"'] & \\ & T_\mu \arrow[dl, "\mu"'] & \\ T_\omega \arrow[rr, "\omega"] & & T_\omega^\dagger \arrow[ul, "\nu"'] \arrow[uul, bend right, "\rho"'] \end{tikzcd} \quad = \frac{1}{2}(|\omega| - 1)(|\omega| - 2) dim \mathcal{Y}_\omega \tag{12.24}$$

proof:

See Ref.[1] for proof.

QED

Chapter 13

Reducibility of Representations

This chapter is based on Ref.[1].

13.1 Eigenvalue Projectors

Suppose $M \in \mathbb{C}^{d\times d}$ has eigenvalues λ_i with corresponding eigenvectors $|\lambda_i\rangle$

$$M|\lambda_i\rangle = \lambda_i|\lambda_i\rangle \tag{13.1}$$

for $i \in \mathbb{Z}_{[1,r]}$. The characteristic polynomial of M is defined as

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M-\lambda) = \prod_{i=1}^{r}(\lambda-\lambda_i)^{d_i} \tag{13.2}$$

It must satisfy

$$cp(\lambda) = 0 \tag{13.3}$$

Note that if M is Hermitian ($M^\dagger = M$), then all its eigenvalues are real. (because $\lambda_i = \langle\lambda_i|M|\lambda_i\rangle \in \mathbb{R}$)

If M is a Hermitian, then there exists a matrix C that is a unitary ($CC^\dagger = C^\dagger C = 1$) and diagonalizes M

$$CMC^\dagger = \begin{bmatrix} D_{\lambda_1} & 0 & 0 & 0 \\ 0 & D_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{\lambda_r} \end{bmatrix} \tag{13.4}$$

where

$$D_{\lambda_i} = \mathrm{diag}\underbrace{(\lambda_i, \lambda_i, \ldots, \lambda_i)}_{d_i \text{ times}} \tag{13.5}$$

$$d = \sum_{i=1}^{r} d_i \tag{13.6}$$

For example, when $d = 2$,

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{13.7}$$

Note that for $d = 2$,

$$CP_1C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2} \tag{13.8}$$

$$CP_2C^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1} \tag{13.9}$$

P_1 and P_2 are a set of complete orthogonal projection operators

$$P_1 + P_2 = 1 \tag{13.10}$$

$$P_1^2 = P_1,\ P_2^2 = P_2,\ P_1P_2 = P_2P_1 = 0 \tag{13.11}$$

Similarly, for $d > 2$, we can define one projection operator P_i for each eigenvalue λ_i. If $I^{d_i \times d_i}$ is the d_i dimensional unit matrix, then

$$\begin{aligned} P_i &= C^\dagger diag(0, \ldots, 0, I^{d_i \times d_i}, 0, \ldots, 0)C && (13.12) \\ &= \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} && (13.13) \end{aligned}$$

As for $d = 2$, the P_i just defined are a complete set of orthogonal projection operators:

$$\sum_{i=1}^{r} P_i = 1 \quad \text{(completeness)} \tag{13.14}$$

$$P_iP_j = P_i\delta(i,j) \quad \text{(orthonormality)} \tag{13.15}$$

for all $i, j \in \mathbb{Z}_{[1,r]}$

Note that

$$\begin{aligned} d_i &= \text{tr}[C^\dagger P_i C] && (13.16) \\ &= \text{tr}P_i && (13.17) \end{aligned}$$

Note that the P_i's are Hermitian ($P_i^\dagger = P_i$) because M is Hermitian and its eigenvalues are real.

13.2 $[P_i, M] = 0$ consequences

Note that for any i, P_i and M commute

$$[P_i, M] = P_i M - M P_i = 0 \tag{13.18}$$

From the P_i's completeness and commutativity with M, we get

$$\begin{aligned} M &= \sum_{i=1}^{r} \sum_{j=1}^{r} P_i M P_j && (13.19) \\ &= \sum_{i=1}^{r} P_i M P_i && (13.20) \end{aligned}$$

Claim 16 *For all i,*

$$M P_i = \lambda_i P_i \text{ (no } i \text{ sum)} \tag{13.21}$$

proof: We only show it for $d = 2$

$$\begin{aligned} C M P_1 C^\dagger &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} && (13.22) \\ &= \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} && (13.23) \\ &= \lambda_i C P_i C^\dagger && (13.24) \end{aligned}$$

QED

From the last claim, it immediately follows that if $f(x)$ can be expressed as a power series in x, then [1]

$$f(M)P_i = f(\lambda_i)P_i \text{ (no } i \text{ sum)} \tag{13.25}$$

Suppose $M^{(1)}, M^{(2)} \in \mathbb{C}^{d\times d}$ are Hermitian matrices that commute

$$[M^{(1)}, M^{(2)}] = 0 \tag{13.26}$$

Use $M^{(1)}$ to decompose $V = \mathbb{C}^{d\times d}$ into a direct sum of vector spaces $\bigoplus_i V_i$. Then we can use $M^{(2)}$ to decompose V_i into $\bigoplus_j V_{i,j}$. If $M^{(1)}$ and $M^{(2)}$ don't commute, let $P_i^{(1)}$ be an eigenvalue projection operator of $M^{(1)}$. Then replace $M^{(2)}$ by $P_i^{(1)} M^{(2)} P_i^{(1)}$. Now

$$[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] = 0 \tag{13.27}$$

[1] M must also satisfy some convergence conditions that we won't get into.

13.3 $[G, M] = 0$ consequences

An invariant matrix (see Ch.7) commutes with all the elements G of a group $\mathcal{G}$

$$[G, M] = 0 \tag{13.28}$$

If P_i are the projection operators of M, then $P_i = f_i(M)$ so

$$[G, P_i] = 0 \tag{13.29}$$

for all $G \in \mathcal{G}$ and i.

$$G = 1G1 = \sum_i \sum_j P_i G P_j = \sum_j \underbrace{P_j G P_j}_{\stackrel{\text{def}}{=} G_j} \tag{13.30}$$

Claim 17

$$G = C^\dagger diag(G_1, G_2, \ldots)C \tag{13.31}$$

$$G = \sum_i C_i^\dagger G_i C_i \tag{13.32}$$

where the matrices C_i are the Clebsch Gordan matrices of M (see Ch. 3)

proof:

$$C_i G C_i^\dagger = \sum_j C_i P_j G P_j C_i^\dagger = C_i G_i C_i^\dagger = G_i \tag{13.33}$$

QED

A rep-matrix G_i acts only on a d_i dimensional vector space $V^{d_i} = P_i V^d$. In this way, an invariant matrix $M \in \mathbb{C}^{d \times d}$ with r distinct eigenvalues, induces a decomposition of V^d into a direct sum of vector spaces

$$V^d \xrightarrow[M]{} V_1^{d_1} \oplus V_2^{d_2} \oplus \ldots \oplus V_r^{d_r} \tag{13.34}$$

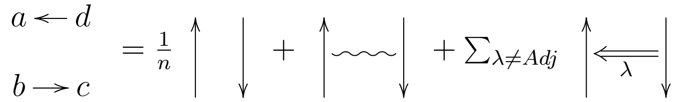
If a rep-matrix G_i cannot itself be reduced further, it is said to be an **irreducible representation (irrep)**.

Note that sometimes the term representation is used to refer to the vector space $V_i^{d_i}$ instead of the matrix G_i.

We've considered the decomposition of V^d into irreps. An example of such a decomposition is the decomposition of $V^n \otimes V^{\dagger n}$

$$\boxed{1 = \frac{1}{n} \uparrow\downarrow + P_{Adj} + \sum_{\lambda \neq Adj} P_{\lambda},}$$

$$\boxed{\delta^a_d \delta^c_d = \frac{1}{n} \delta^a_b \delta^c_d + (P_{Adj})_a{}^b{}_c{}^d + \sum_{\lambda \neq Adj} (P_\lambda)_a{}^b{}_c{}^d} \tag{13.35}$$



Chapter 14

Spinors: COMING SOON

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Chapter 15

Squashed Entanglement: COMING SOON

Chapter 16

Symplectic Groups: COMING SOON

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Chapter 17

Symmetrization and Antisymmetrization

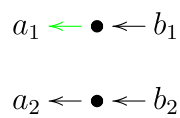
This chapter is based on Ref.[1].

As preparation for this chapter, read Sec.A.7.

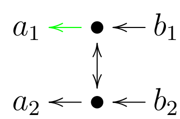
17.1 Symmetrizer

The set of permutations of 2 elements can be represented by the following $2! = 2$ birdtracks[1]

$$\mathbb{1}_{a_1,a_2}{}^{b_2,b_1} = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2} = \tag{17.1}$$

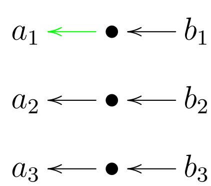


$$(\sigma_{(1,2)})_{a_1,a_2}{}^{b_2,b_1} = \delta^{b_2}_{a_1}\delta^{b_1}_{a_2} = \tag{17.2}$$

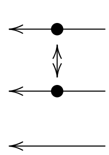


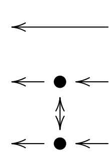
The set of permutations of 3 elements can be represented by the following $3! = 6$ birdtracks:

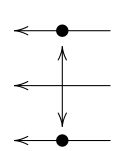
$$\mathbb{1} = \tag{17.3}$$



$$\sigma_{(1,2)} = \quad \sigma_{(2,3)} = \quad \sigma_{(1,3)} = \tag{17.4}$$

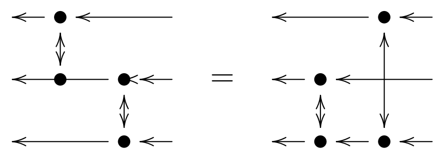






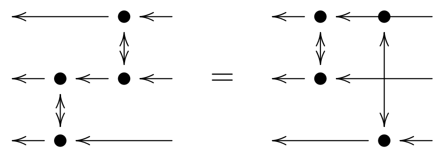
[1]Note that the set of values that a_i and b_i can assume can be anything, as long as, for some set V, $val(\underline{a}_i) = val(\underline{b}_i) = V$ for all i.

$$\sigma_{(1,2,3)} =$$



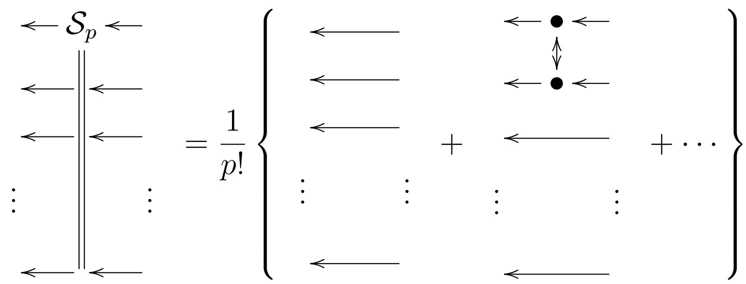
(17.5)

$$\sigma_{(1,3,2)} =$$



(17.6)

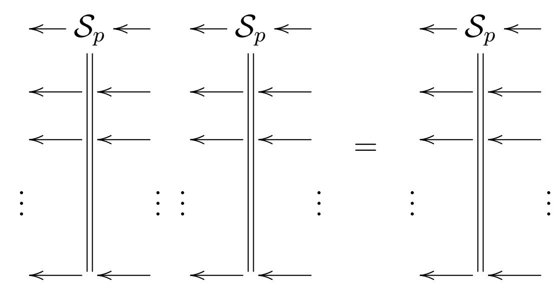
The p**-element symmetrizer** $\mathcal{S}_p$ is defined as the birdtrack



(17.7)

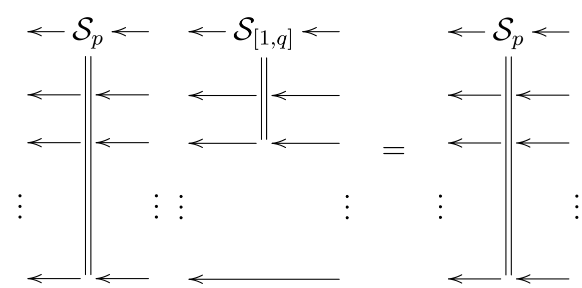
Note that $\mathcal{S}_p$ satisfies the following identities

$$\boxed{\mathcal{S}_p^2 = \mathcal{S}_p}$$

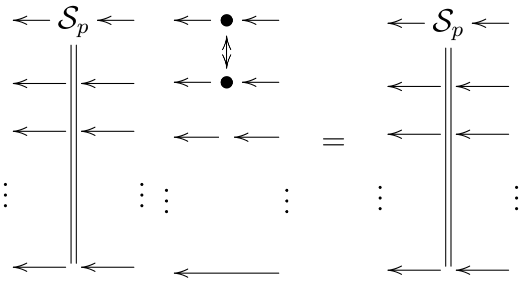


(17.8)

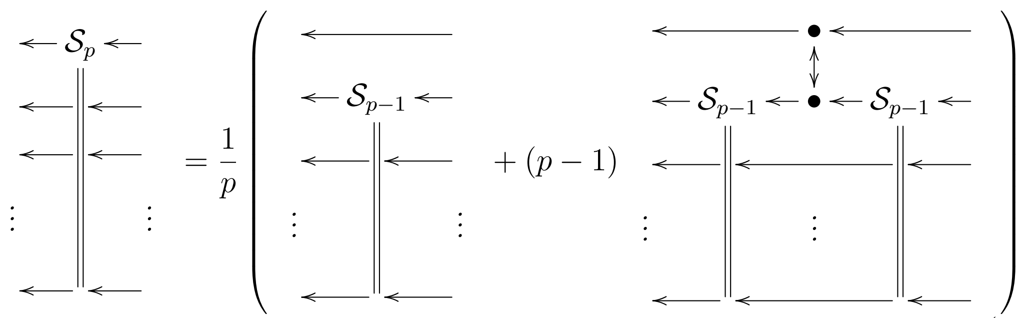
$$\boxed{\mathcal{S}_p\mathcal{S}_{[1,q]} = \mathcal{S}_p}$$



(17.9)

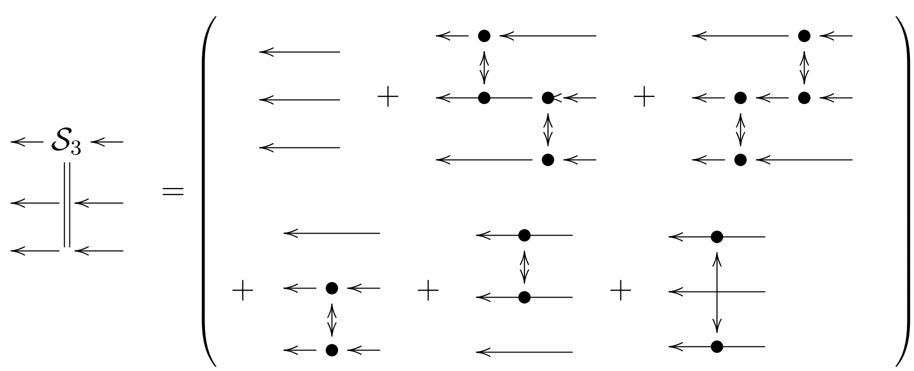
$\boxed{\mathcal{S}_p\sigma_{(1,2)} = \mathcal{S}_p}$  (17.10)

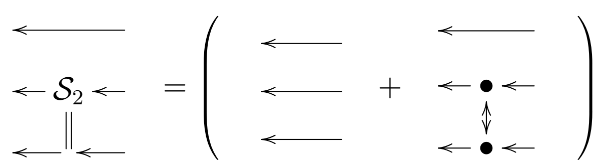
Claim 18

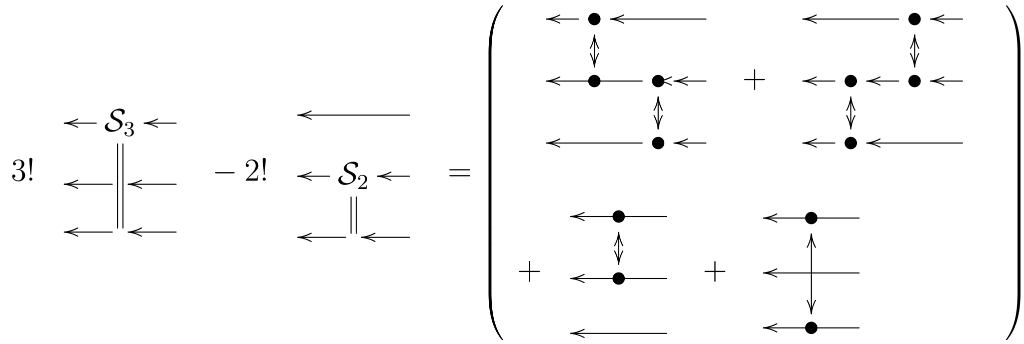


(17.11)

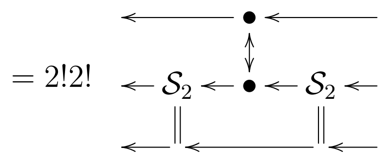
proof: We only prove it for $p = 3$.

3!  (17.12)

2!  (17.13)



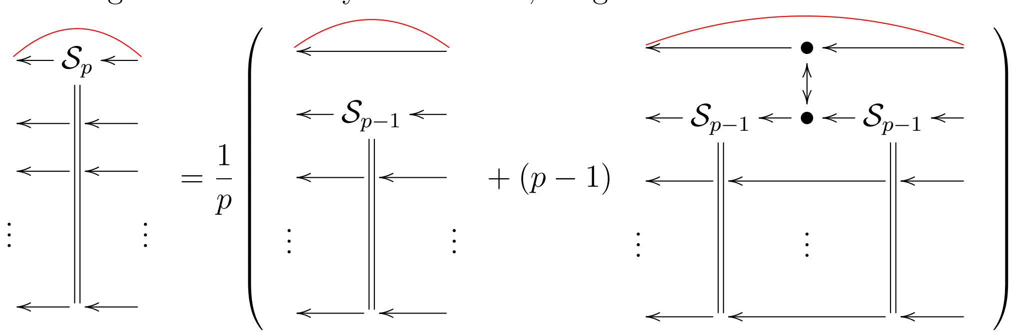
(17.14)



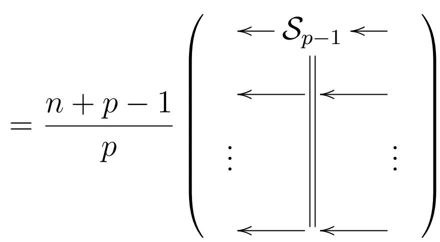
(17.15)

QED

Tracing over the identity of Claim 18, we get



(17.16)



(17.17)

Hence

$$\mathrm{tr}_{\underline{a}_1}\mathcal{S}_p = \frac{n+p-1}{p}\mathcal{S}_{p-1} \tag{17.18}$$

$$\mathrm{tr}_{\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_k}\mathcal{S}_p = \frac{(n+p-1)(n+p-2)\ldots(n=p-k)}{p(p-1)\ldots(p-k+1)}\mathcal{S}_{p-k} \tag{17.19}$$

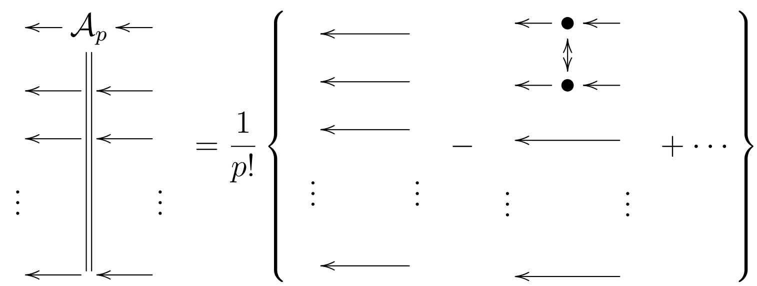
$$d_{\mathcal{S}_p} = \mathrm{tr}_{\underline{a}^p}\mathcal{S}_p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \tag{17.20}$$

For $p = 2$,

$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \tag{17.21}$$

17.2 Antisymmetrizer

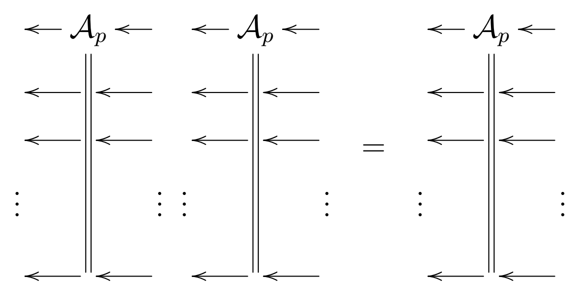
The p-**element antisymmetrizer** $\mathcal{A}_p$ is defined as the birdtrack



(17.22)

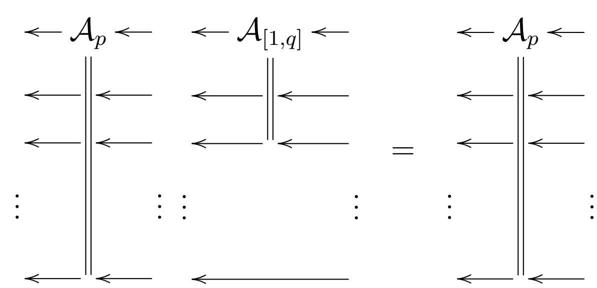
Note that $\mathcal{A}_p$ satisfies the following identities

$\boxed{\mathcal{A}_p^2 = \mathcal{A}_p}$



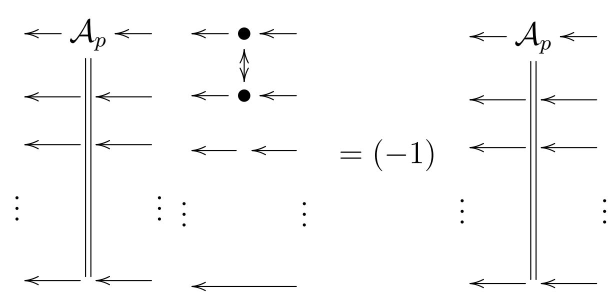
(17.23)

$\boxed{\mathcal{A}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p}$



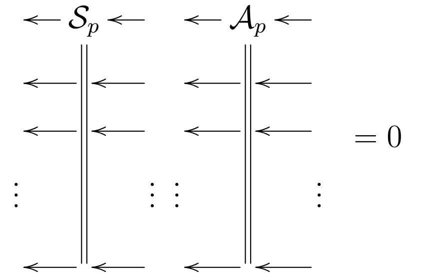
(17.24)

$\boxed{\mathcal{A}_p \sigma_{(1,2)} = -\mathcal{A}_p}$



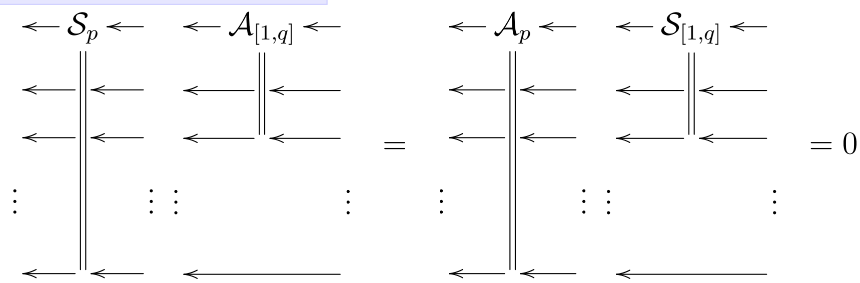
(17.25)

$\boxed{\mathcal{S}_p\mathcal{A}_q = \mathcal{A}_p\mathcal{S}_q = 0}$



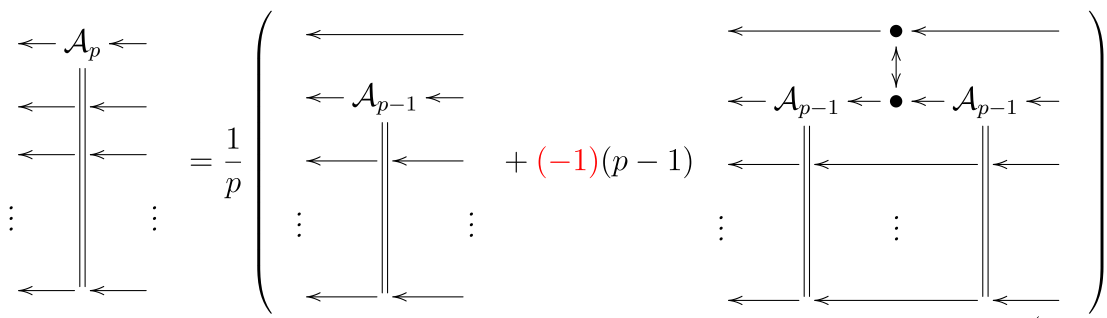
(17.26)

$\boxed{\mathcal{S}_p\mathcal{A}_{[1,q]} = \mathcal{A}_p\mathcal{S}_{[1,q]} = 0}$



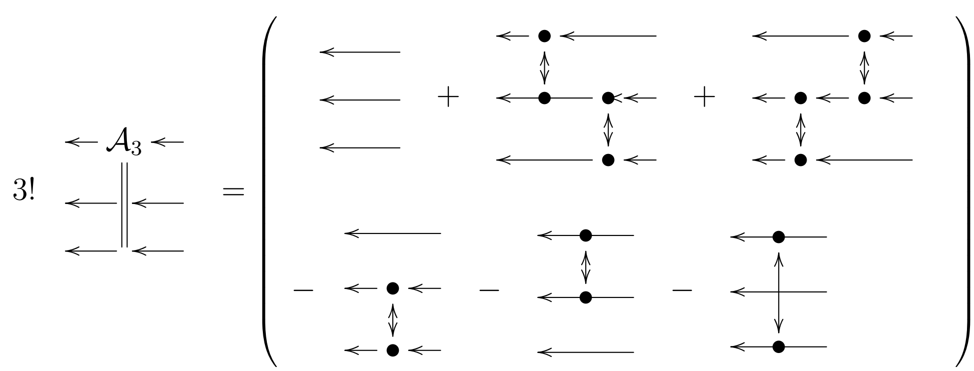
(17.27)

Claim 19

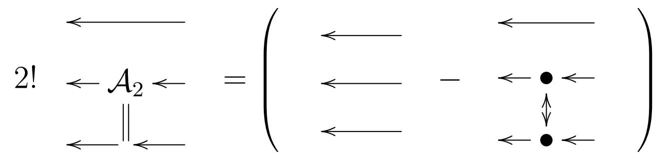


(17.28)

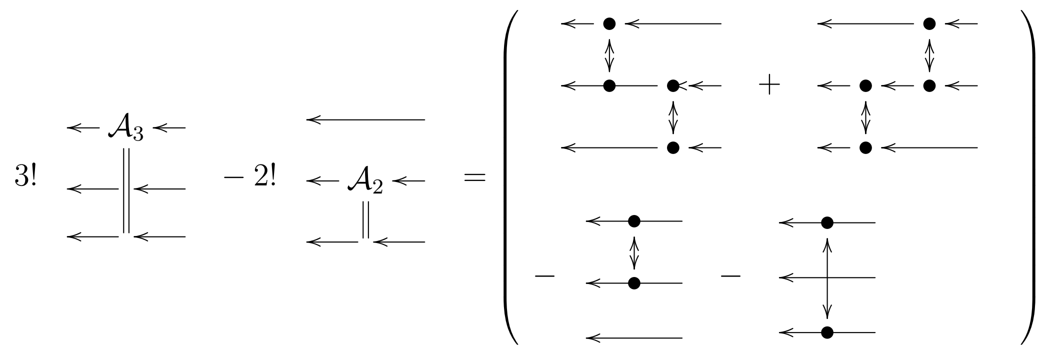
proof: We only prove it for $p = 3$.



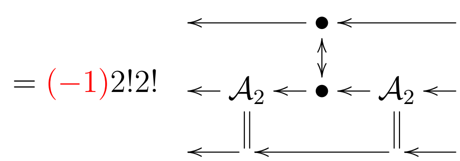
(17.29)



(17.30)



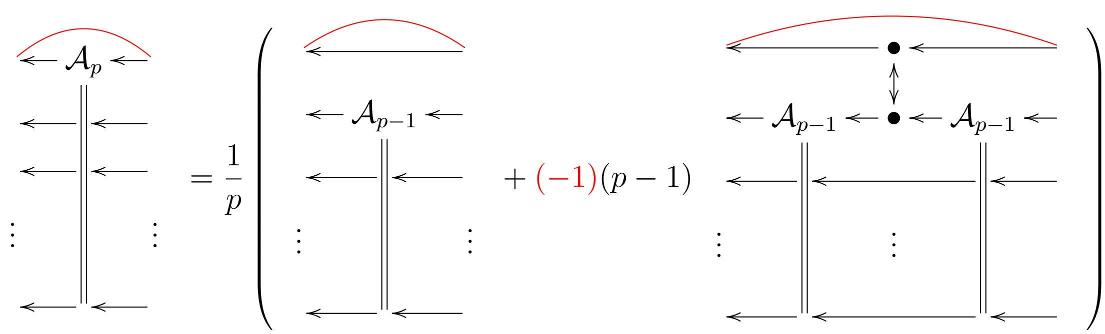
(17.31)



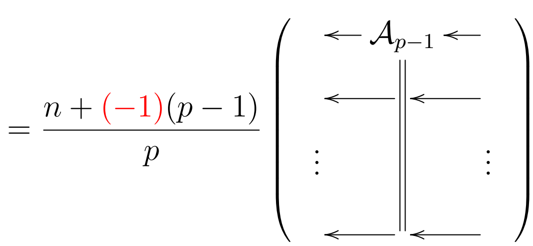
(17.32)

QED

Tracing over the identity of Claim 19, we get



(17.33)



(17.34)

Hence,

$$\mathrm{tr}_{\underline{a}_1}\mathcal{A}_p = \frac{n-p+1}{p}\mathcal{A}_{p-1} \tag{17.35}$$

$$\mathrm{tr}_{\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_k}\mathcal{A}_p = \frac{(n-p+1)(n-p+2)\ldots(n-p+k)}{p(p-1)\ldots(p-k+1)}\mathcal{A}_{p-k} \tag{17.36}$$

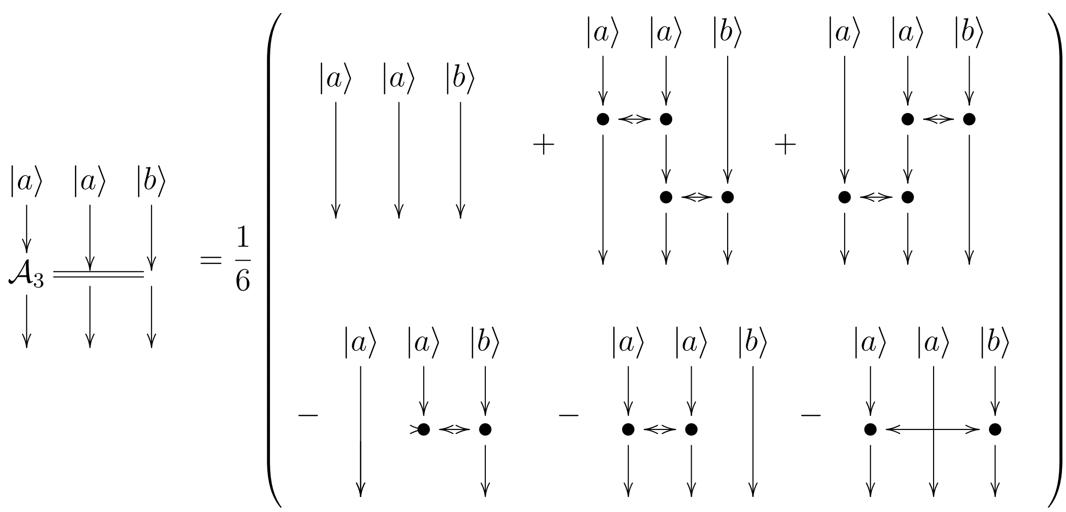
$$\begin{aligned} d_{\mathcal{A}_p} = \mathrm{tr}_{\underline{a}^p}\mathcal{A}_p &= \frac{\prod_{i=n-p+1}^{n} i}{p!} && (17.37)\\ &= \frac{\prod_{i=n}^{n-p+1} i}{p!} && (17.38)\\ &= \begin{cases} \frac{n!}{p!(n-p)!} = \binom{n}{p} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} && (17.39) \end{aligned}$$

For $p = 2 \leq n$,

$$d_{\mathcal{A}_2} = \binom{n}{2} \tag{17.40}$$

$$\mathcal{A}_p = 0 \text{ if } n < p \tag{17.41}$$

For example, for $n = 2$ and $p = 3$



(17.42)

$$\begin{aligned} \mathcal{A}_3|a,a,b\rangle &= \frac{1}{6}\begin{pmatrix} |a,a,b\rangle + |a,b,a\rangle + |b,a,a\rangle \\ -|a,b,a\rangle - |a,a,b\rangle - |b,a,a\rangle \end{pmatrix} && (17.43)\\ &= 0 && (17.44) \end{aligned}$$

17.3 Levi-Civita Tensor

The **Levi-Civita tensor** $\epsilon_{a^{:p}}$ where $a_i \in \{1, 2, \ldots, p\}$ equals $+1$ (resp., -1) if $a^{:p}$ is an even (resp., odd) permutation of $(1, 2, \ldots, p)$. Thus

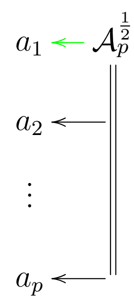
$$\epsilon^{123\ldots p} = \epsilon_{123\ldots p} = 1 \tag{17.45}$$

and

$$\epsilon_{rev(a^{:p})} = (-1)^{\binom{p}{2}} \epsilon_{a^{:p}} \tag{17.46}$$

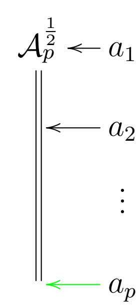
Define

$$(C_{\mathcal{A}_p})^1_{a^{:p}} = e^{i\phi} \frac{\epsilon_{a^{:p}}}{\sqrt{p!}} = \tag{17.47}$$



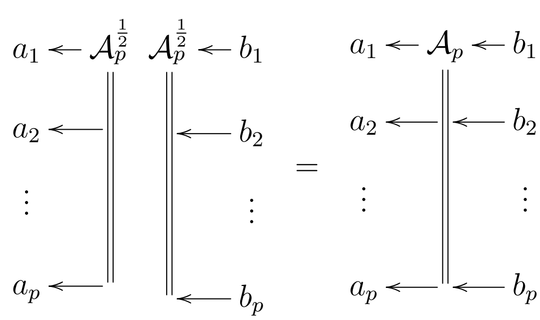
and

$$(C^\dagger_{\mathcal{A}_p})_1^{rev(a^{:p})} = e^{-i\phi} \frac{\epsilon^{rev(a^{:p})}}{\sqrt{p!}} = \tag{17.48}$$



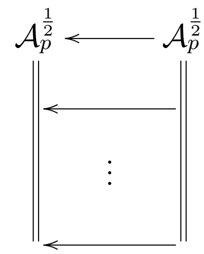
Then

$$\boxed{\frac{1}{p!}\epsilon_{a^{:p}}\epsilon^{rev(b^{:p})} = (\mathcal{A}_p^{\frac{1}{2}})_{a^{:p}}{}^{rev(b^{:p})}} \tag{17.49}$$



and

$$\boxed{e^{i2\phi}\frac{1}{p!}\epsilon^{rev(a^{:n})}\epsilon_{a^{:n}} = \delta^1_1 = 1}$$



$$= 1 \tag{17.50}$$

For the L Convention, we will use $\phi = 0$.
For the CC Convention, we must choose

$$e^{i2\phi} = (-1)^{\binom{p}{2}} = e^{i\pi\frac{p(p-1)}{2}} \tag{17.51}$$

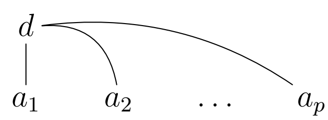
so

$$\phi = \frac{\pi}{4}p(p-1) \tag{17.52}$$

17.4 Fully-symmetric and Fully-antisymmetric tensors

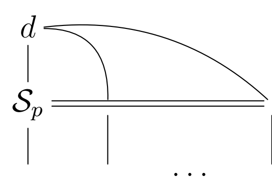
fully symmetric (FS) tensor d

$$d_{a_1 a_2 \ldots a_p} =$$

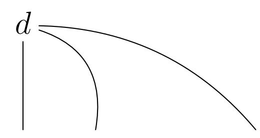


(17.53)

$$\boxed{\mathcal{S}_p d = d}$$

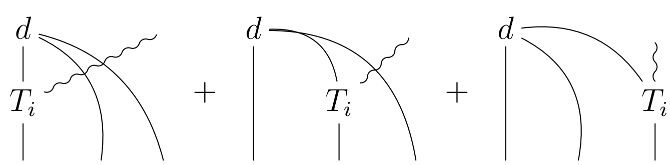


$=$



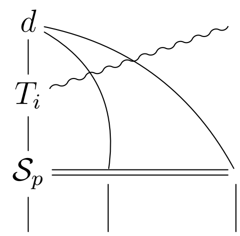
(17.54)

$0 =$



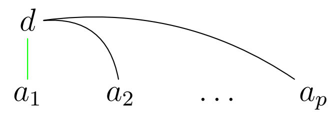
(17.55)

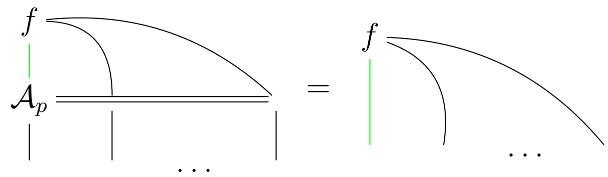
$0 =$

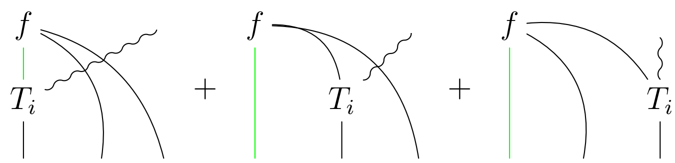


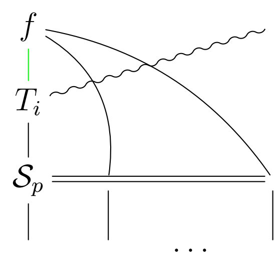
(17.56)

Fully antisymmetric(FA) tensor (FA) f

$f_{a_1a_2\ldots a_p} =$  (17.57)

$\boxed{\mathcal{A}_p f = f}$  (17.58)

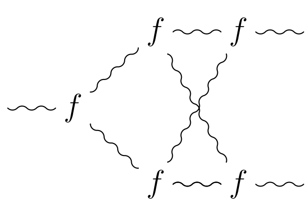
$0 =$  (17.59)

$0 =$  (17.60)

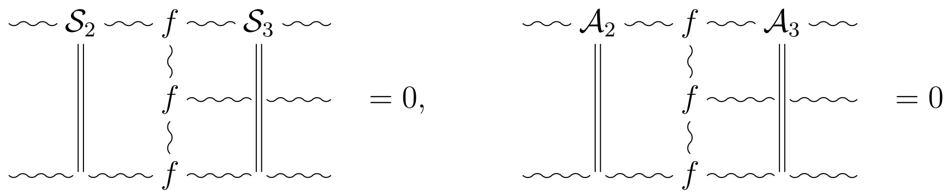
17.5 Identically Vanishing Tensors

Identically vanishing (IV) tensors

- Example of birdtrack that vanishes for any FA tensor f

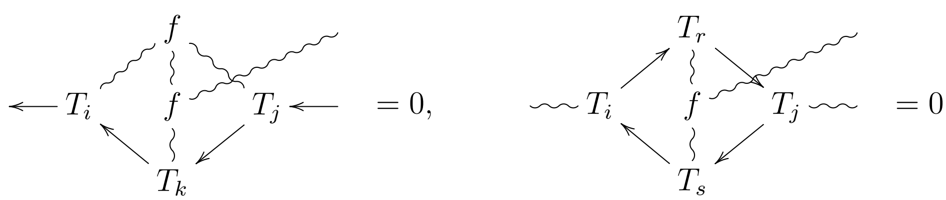
 $= 0$ (Kuratowsky graph) (17.61)

- Example of birdtrack that vanishes for any f that is a structure constant of a Lie algebra



(17.62)

- Birdtrack that is zero for an irrep



(17.63)

Chapter 18

Unitary Groups

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

For $SU(n)$ and $U(n)$

$n = d_{def}$ = dimension of defining rep

$N = d_{ad}$ = dimension of adjoint representation

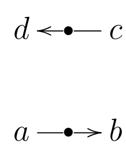
d_λ = dimension of rep-matrix G in rep λ

18.1 $SU(n)$

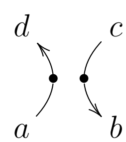
In $SU(n) \subset \mathbb{C}^{n\times n})$ in defining rep

$$m(p,q) = \delta^a_b \sum_{a=1}^{n} (p_a)^* q_a \tag{18.1}$$

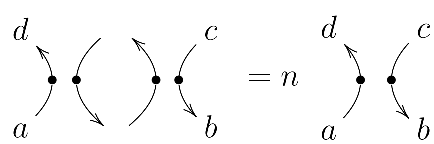
$$\mathbb{1}^{a,c}_{d,b} = \delta^a_b \delta^c_d = \tag{18.2}$$



$$M^{d\ \ b}_{\ a\ \ c} = \delta^a_d \delta^c_b = \tag{18.3}$$



$$M^2 = nM \tag{18.4}$$

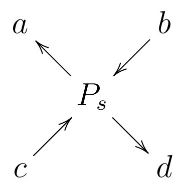


$$P_i = \sum_{j\neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \tag{18.5}$$

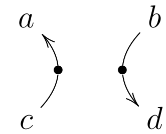
$\lambda = 0, n$

$\lambda_s = 0$

$$\boxed{P_s = \frac{M-0}{n-0} = \frac{1}{n}M}$$



$= \frac{1}{n}$

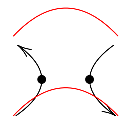


(18.6)

singlet rep part

$$P_s x = \frac{1}{n} x^b{}_b \delta^c_a \tag{18.7}$$

$$dim(P_s) = \mathrm{tr} P_s \quad = \quad \frac{1}{n}$$

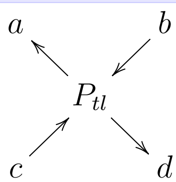


(18.8)

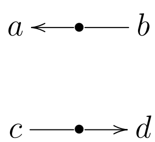
$$= \quad 1 \tag{18.9}$$

$\lambda_{tl} = n$

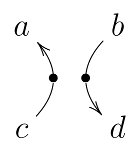
$$\boxed{P_{tl} = \frac{M-n}{0-n} = 1 - \frac{1}{n}M}$$



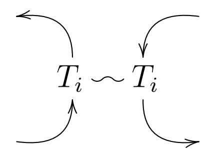
$=$



$- \frac{1}{n}$



$=$

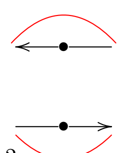


(18.10)

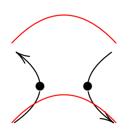
traceless part of x

$$P_{tl} x = x^a{}_c - \left(\frac{1}{n} x^b{}_b \delta^c_a \right) \tag{18.11}$$

$$dim(P_{tl}) = \mathrm{tr} P_{tl} \quad =$$



$- \frac{1}{n}$

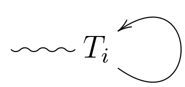


(18.12)

$$= \quad n^2 - 1 \tag{18.13}$$

Claim 20

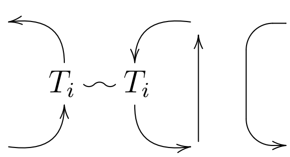
$$\boxed{\mathrm{tr}(T_i) = 0}$$



$= 0$ (18.14)

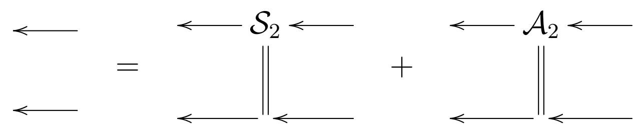
proof:

$$0 = P_{tl} P_s =$$

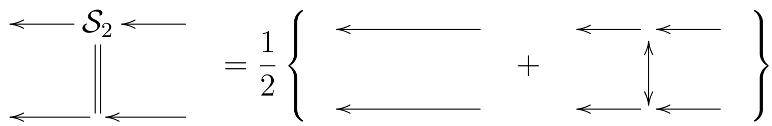


(18.15)

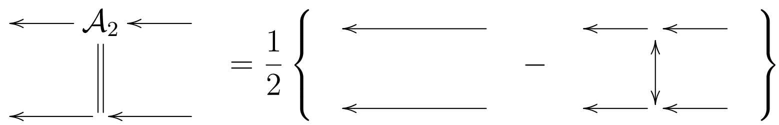
QED



(18.16)

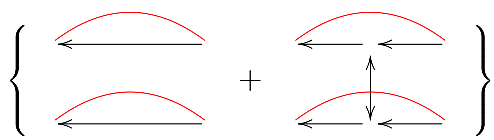


(18.17)



(18.18)

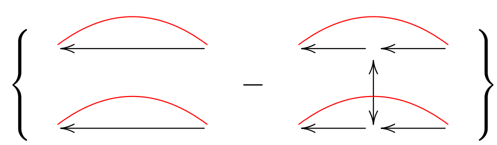
$$dim(\mathcal{S}_2) = \frac{1}{2}$$



(18.19)

$$= \frac{n(n+1)}{2} \tag{18.20}$$

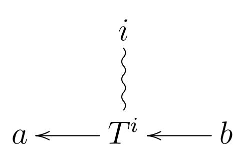
$$dim(\mathcal{A}_2) = \frac{1}{2}$$



(18.21)

$$= \frac{n(n-1)}{2} \tag{18.22}$$

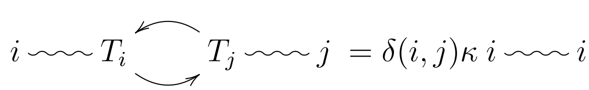
$$(T_i)^b{}_a =$$



(18.23)

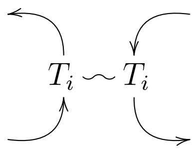
$$T_i^\dagger = T_i \tag{18.24}$$

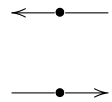
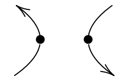
$$\boxed{\mathrm{tr}(T_i T_j) = \kappa\delta(i,j)}$$



(18.25)

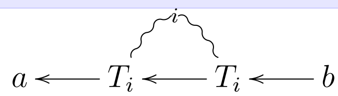
Usually set $\kappa = 1$



$$\overset{\text{def}}{=} \; P_{tl} =$$  $$-\frac{1}{n}$$  (18.26)

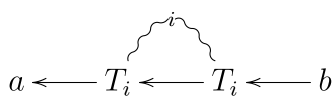
Claim 21

$$\boxed{\Gamma_{fun}\delta_a^b = \sum_i (T_iT_i)_a{}^b = \frac{n^2-1}{n}\delta_a^b}$$

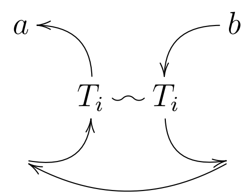
$\sum_i$ 

$= \left(\frac{n^2-1}{n}\right) \; a \leftarrow\!\bullet\!- b$ (18.27)

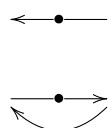
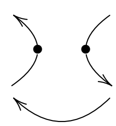
proof:

$$(T_iT_i)_a{}^b = \sum_i$$ 

(18.28)

$=$ 

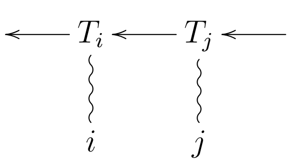
(18.29)

$=$  $-\frac{1}{n}$  (18.30)

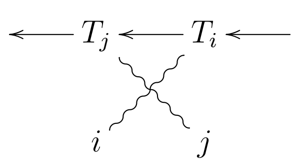
$$= \left(n - \frac{1}{n}\right) \; a \leftarrow\!\bullet\!- b \tag{18.31}$$

QED

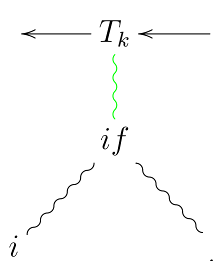
$$\boxed{T_iT_j - T_jT_i = if_{ijk}T_k}$$



$-$



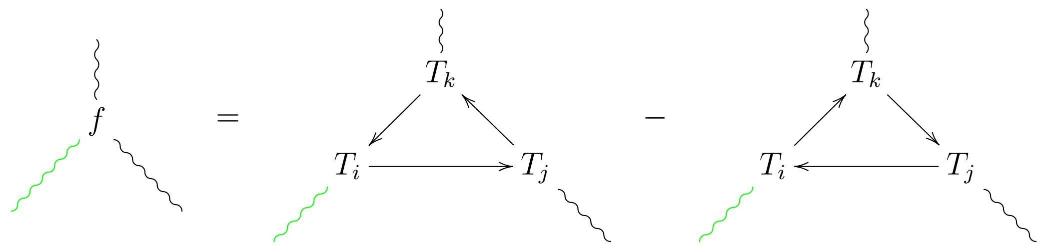
$=$



(18.32)

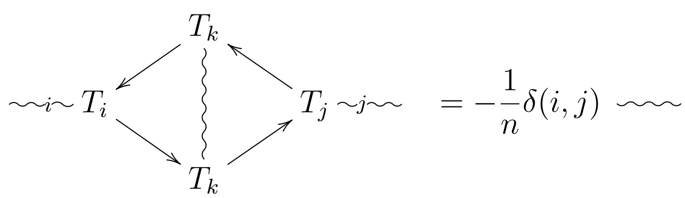
f_{ijk} is totally antisymmetric. First index in green

$$\boxed{if_{ijk} = \mathrm{tr}(T_i T_j T_k) - \mathrm{tr}(T_j T_i T_k)}$$



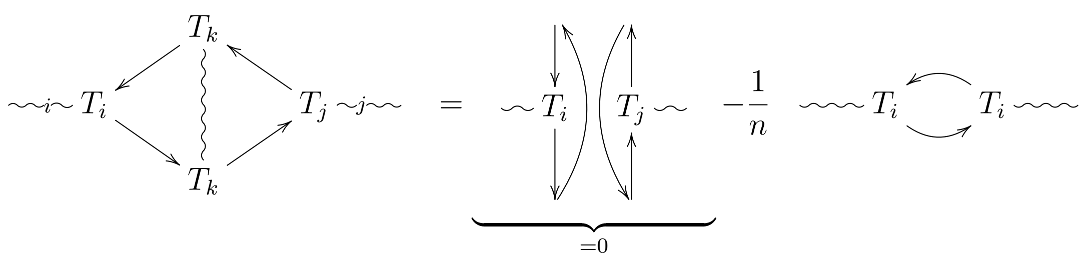
(18.33)

Claim 22



(18.34)

proof:

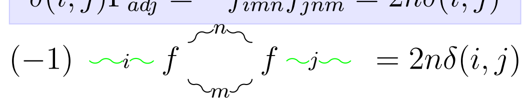


(18.35)

QED

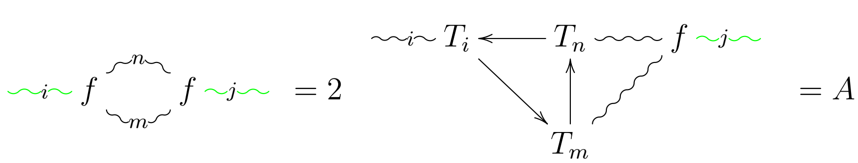
Claim 23

$$\boxed{\delta(i,j)\Gamma_{adj} = -f_{imn} f_{jnm} = 2n\delta(i,j)}$$

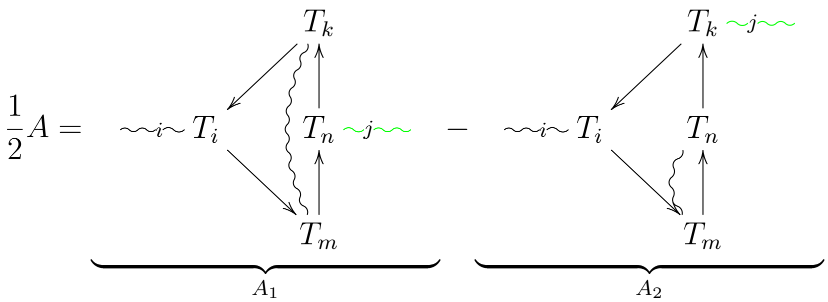


(18.36)

proof:



(18.37)



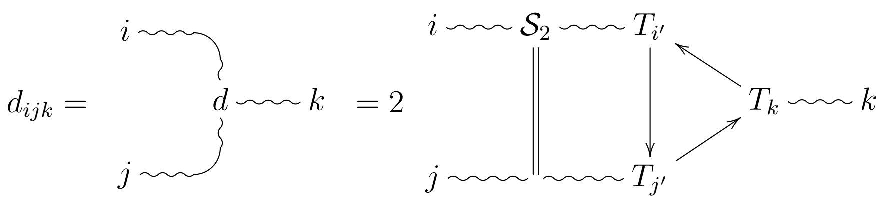
(18.38)

$$A_1 = \frac{n^2 - 1}{n}\delta(i, j) \tag{18.39}$$

$$A_2 = -\frac{1}{n}\delta(i, j) \tag{18.40}$$

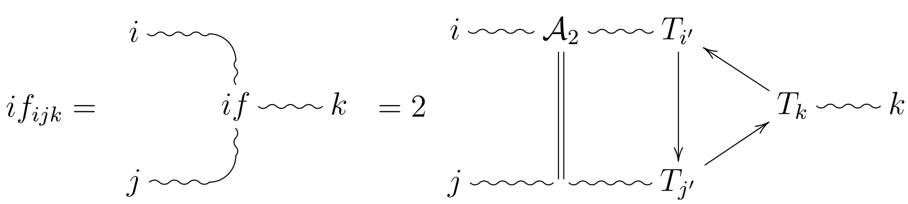
$$A = 2(A_1 - A_2) = 2n\delta(i, j) \tag{18.41}$$

QED



(18.42)

Multiplying Jacobi identity by T_k and taking the trace, we get

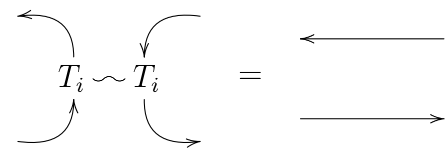


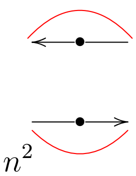
(18.43)

18.2 $U(n)$

- $U(n)$ primitive invariants: Kronecker delta
- $SU(n)$ primitive invariants: Kronecker delta, Levi-Civita tensor

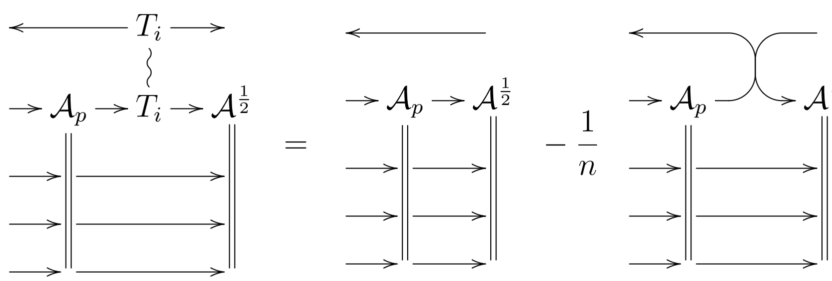
$P_{tl} \to P_{adj}$

$P_{adj} =$  (18.44)

$dim(P_{adj}) = \mathrm{tr}P_{adj} \;=$  (18.45)

$= \; n^2$ (18.46)

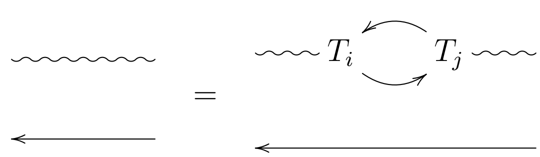
Claim 24 *The Levi-Civita tensor is an invariant matrix for $SU(n)$.*

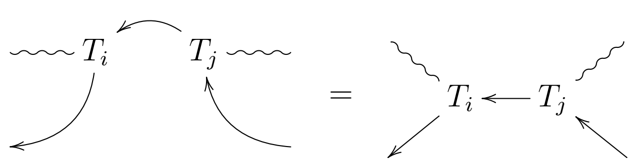
 $= 0$ (18.47)

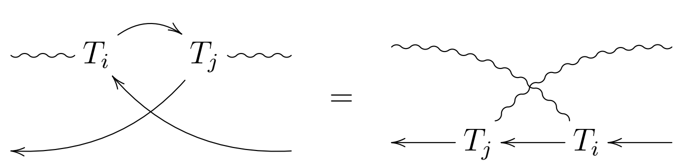
proof:
QED

18.3 $V \otimes V_{adj}$ decomposition

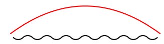
$V \otimes V_{adj} \sim V \otimes V \otimes V^\dagger$

$e =$  (18.48)

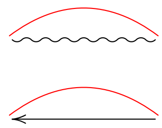
$R =$  (18.49)

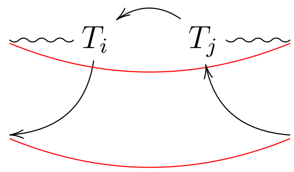
$Q =$  (18.50)

Recall that for $SU(n)$, the dimension of the adjoint space N is

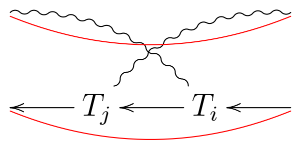
$N = n^2 - 1 =$  (18.51)

For example, for $SU(2)$, $N = 3$ and for $SU(3)$, $N = 8$.

$\mathrm{tr}(e) =$  $= Nn$ (18.52)

$\mathrm{tr}(R) =$ 

$= N$ (18.53)

$\mathrm{tr}(Q) =$ 

$= N$ (18.54)

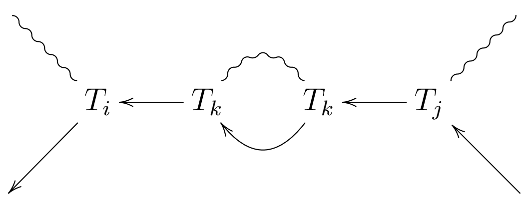
Claim 25

$$R^2 = \frac{n^2 - 1}{n} R \tag{18.55}$$

$$QR = RQ = -\frac{1}{n} R \tag{18.56}$$

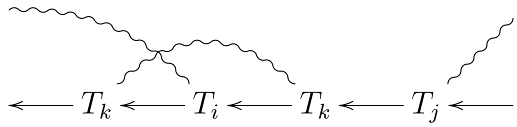
$$Q^2 - e = -\frac{1}{n} R \tag{18.57}$$

proof:

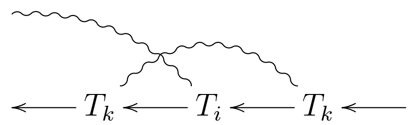
$R^2 \quad =$ 

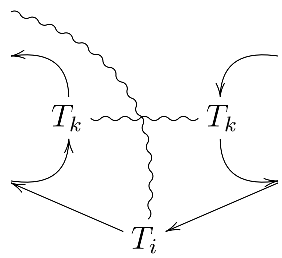
(18.58)

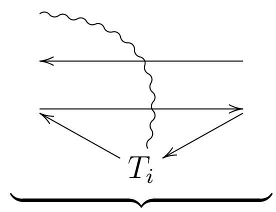
$$= \quad \frac{n^2 - 1}{n} R \quad \text{(by Eq.(18.27))} \tag{18.59}$$

$QR \quad =$ 

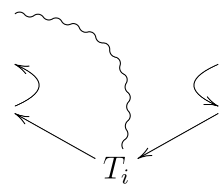
(18.60)

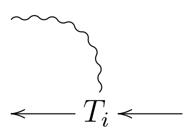
$X =$  (18.61)

$X \quad =$  (18.62)

$=$ 

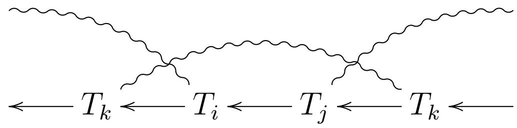
=0

$-\frac{1}{n}$  (18.63)

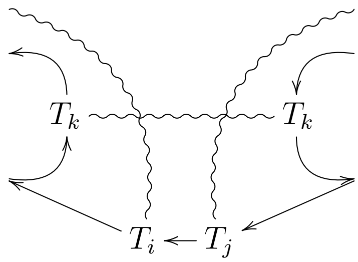
$= \quad -\frac{1}{n}$  (18.64)

so

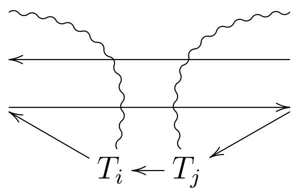
$$QR = RQ = -\frac{1}{n}R \tag{18.65}$$

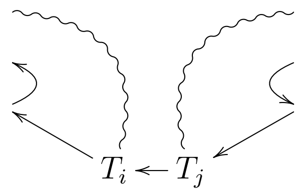
$Q^2 =$ 

(18.66)

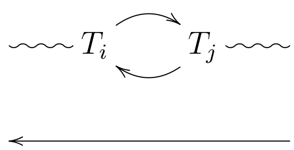
$=$ 

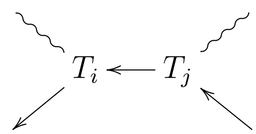
(18.67)

$=$ 

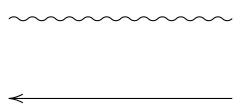
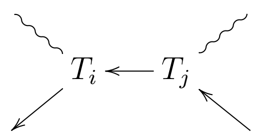
$-\frac{1}{n}$ 

(18.68)

$=$ 

$-\frac{1}{n}$ 

(18.69)

$=$ 
$-\frac{1}{n}$ 

(18.70)

$$= e - \frac{1}{n}R \tag{18.71}$$

QED

Claim 26

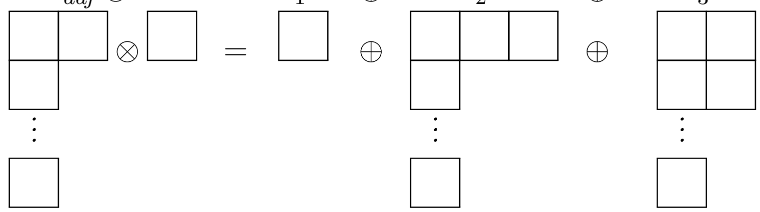
$$P_1 = \frac{n}{n^2-1}R \tag{18.72}$$

$$P_2 = \frac{1}{2}\left[e + Q - \frac{1}{n+1}R\right] \tag{18.73}$$

$$P_3 = \frac{1}{2}\left[e - Q - \frac{1}{n-1}R\right] \tag{18.74}$$

ae projectors such that for $SU(n)$, the $V \otimes V_{adj}$ Clebsch-Gordan series is given by

$$
\begin{array}{ccccccc}
V_{adj} \otimes V & = & P_1 V & \oplus & P_2 V & \oplus & P_3 V
\end{array}
$$



$$
\begin{array}{ccccccc}
(n^2-1)n & = & n & + & \frac{n(n-1)(n+2)}{2} & + & \frac{n(n+1)(n-2)}{2} \\
SU(3) : 8(3) & = & 3 & + & 15 & + & 6
\end{array}
\qquad (18.75)
$$

proof:

$$
\mathrm{tr}(P_1) = \frac{n}{n^2-1} nN = \frac{n^2(n-1)}{n^2-1} = \frac{n^2}{n+1} \qquad (18.76)
$$

$$
\begin{aligned}
\mathrm{tr}(P_2) &= \frac{N}{2}\left(n+1-\frac{1}{n+1}\right) && (18.77)\\
&= \frac{N}{2}\frac{n^2+2n}{n+1} && (18.78)\\
&= \frac{N}{2}\frac{n(n+2)}{n+1} && (18.79)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}(P_3) &= \frac{N}{2}\left(n-1-\frac{1}{n-1}\right) && (18.80)\\
&= \frac{N}{2}\frac{n^2-2n}{n-1} && (18.81)\\
&= \frac{N}{2}\frac{n(n-2)}{n-1} && (18.82)
\end{aligned}
$$

From $R^2 = \frac{n^2-1}{n} R$,

$$
P_1 = \frac{n}{n^2-1} R \qquad (18.83)
$$

$$
P_4 = e - P_1 \qquad (18.84)
$$

From $Q^2 - e = -\frac{1}{n} R$ get

$$
P_4(Q^2-1) = 0 \qquad (18.85)
$$

$$
P_2 = \frac{1}{2} P_4 (1+Q), \quad P_3 = \frac{1}{2} P_4 (1+Q) \qquad (18.86)
$$

$$\begin{aligned}
P_2 &= \frac{1}{2}P_4(1+Q) && (18.87)\\
&= \frac{1}{2}(e-aR)(1+Q) \quad (\text{ where } a = \frac{n}{n^2-1}) && (18.88)\\
&= \frac{1}{2}(e-aR+Q-aRQ) && (18.89)\\
&= \frac{1}{2}\left(e+\left(\frac{1}{n}-1\right)aR+Q\right) && (18.90)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{n}-1\right)a &= \frac{1-n}{n}\frac{n}{n^2-1} && (18.91)\\
&= -\frac{1}{n+1} && (18.92)
\end{aligned}$$

$$\begin{aligned}
P_3 &= \frac{1}{2}P_4(1-Q) && (18.93)\\
&= \frac{1}{2}(e-aR)(1-Q) && (18.94)\\
&= \frac{1}{2}(e-aR-Q+aRQ) && (18.95)\\
&= \frac{1}{2}\left(e-\left(\frac{1}{n}+1\right)aR-Q\right) && (18.96)
\end{aligned}$$

$$\left(\frac{1}{n}+1\right)a = \frac{1}{n-1} \tag{18.97}$$

QED

Claim 27

proof:

QED

Let $Q_1, Q_2, Q_3 = e, R, Q$

$$Q_\lambda|Q_j\rangle = |Q_\lambda Q_j\rangle = \sum_i A^\lambda_{ij}|Q_i\rangle \tag{18.98}$$

$$\langle Q_i|Q_\lambda|Q_j\rangle = A^\lambda_{ij} \tag{18.99}$$

If A^λ are diagonalized and divided by their eigenvalues, and they have a single non-zero eigenvalue, then they become a complete set of projectors with 1 or 0 along their diagonals.

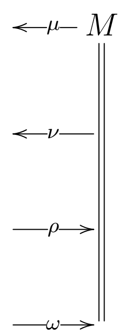
Chapter 19

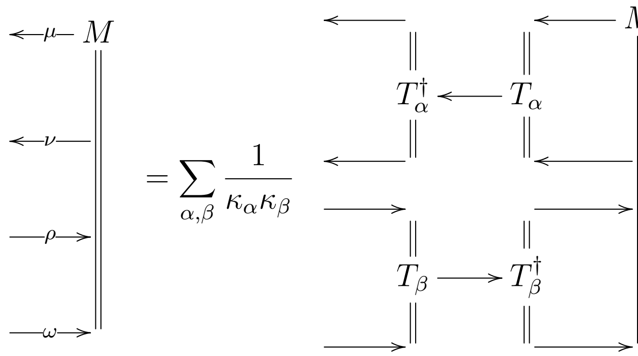
Wigner-Ekart Theorem

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

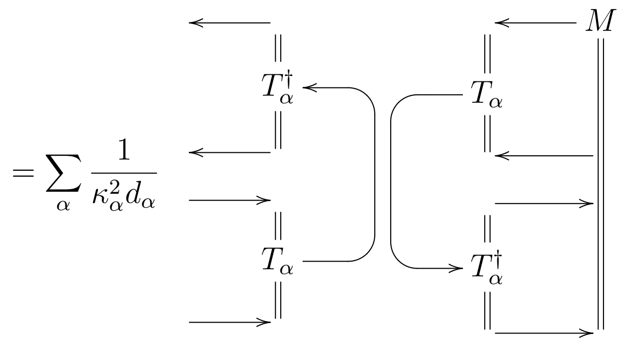
19.1 WE in general

$$M = \tag{19.1}$$

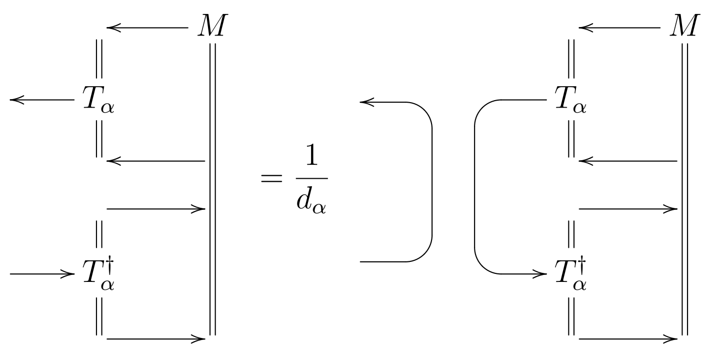




(19.2)



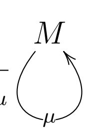
(19.3)

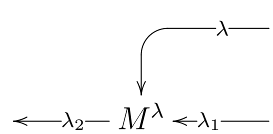


(19.4)

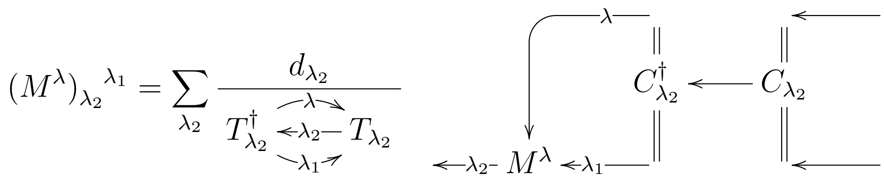
$$M_a = \sum_{\lambda} \lambda a \longleftarrow T_{\lambda} \tag{19.5}$$

$$M_{\lambda a}{}^{\mu b} = \xleftarrow{\;\lambda\;} M \xleftarrow{\;\mu\;} \tag{19.6}$$

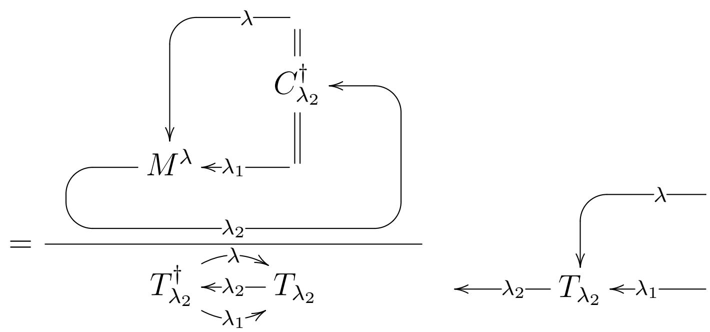
$$(M^{\lambda i})_{\lambda_2 a}{}^{\lambda_1 b} = \frac{1}{d_\mu}$$  $$\delta(\mu, \lambda) \tag{19.7}$$

$$(M^{\lambda i})_{\lambda_2 a}{}^{\lambda_1 b} =$$ 

(19.8)



$$(19.9)$$



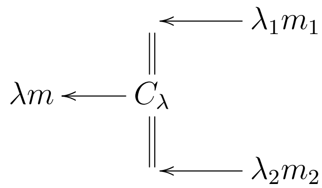
$$(19.10)$$

19.2 WE for angular momentum

$\lambda = J$, $\lambda_i = J_i$ for $i = 1, 2$. We will use Greek letters instead of J so as to keep convention of using Greek letters for rep labels.

$m, m' = -\lambda,\ -\lambda+1, \ldots, \lambda-1, \lambda$. Note $d_\lambda = 2\lambda+1$

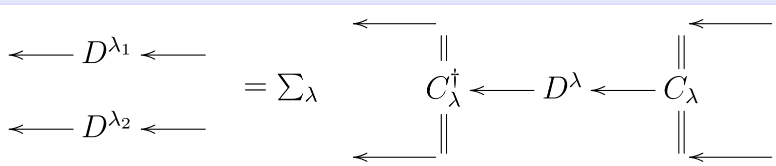
for $i = 1, 2$, $m_i = -\lambda_i, -\lambda_i+1, \ldots, \lambda_i - 1, \lambda_i$. Note $d_{\lambda_i} = 2\lambda_i + 1$



$$= \langle(\lambda_1\lambda_2)\lambda m|\lambda_1 m_1 \lambda_2 m_2\rangle \qquad (19.11)$$

$$m \longleftarrow D^\lambda \longleftarrow m' \quad = \quad D^\lambda_{mm'}(g) \qquad (19.12)$$

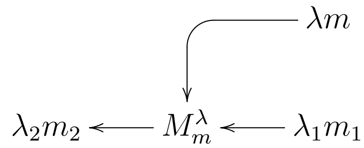
$$\boxed{D^{\lambda_1}_{m_1 m_1'}(g) D^{\lambda_2}_{m_2 m_2'}(g) = \sum_{\lambda, m, m'} \langle\lambda_1 m_1 \lambda_2 m_2|\lambda_1\lambda_2\lambda m\rangle\, D^\lambda_{mm'}(g)\, \langle\lambda_1\lambda_2\lambda m'|\lambda_1 m_1' \lambda_2 m_2'\rangle}$$



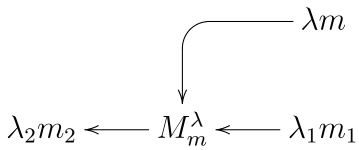
$$(19.13)$$

tensor operator M^λ_m

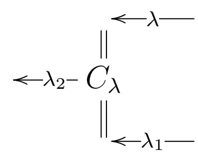
$$\left\langle \lambda_2 m_2 | M_m^\lambda | \lambda_1 m_1 \right\rangle = \tag{19.14}$$



$$\boxed{\left\langle \lambda_2 m_2 | M_m^\lambda | \lambda_1 m_1 \right\rangle = \langle (\lambda\lambda_1)\lambda_2 m_2 | \lambda m \lambda_1 m_1 \rangle \, q(\lambda, \lambda_1, \lambda_2)}$$

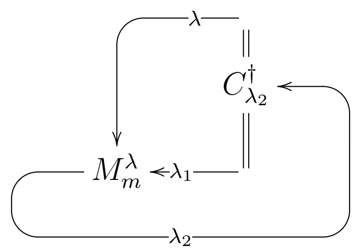


$$= q(\lambda, \lambda_1, \lambda_2) \tag{19.15}$$



$$q(\lambda, \lambda_1, \lambda_2) \quad = \quad \frac{1}{d_{\lambda_2}} \sum_{m_1, m_2, m} \langle \lambda m \lambda_1 m_1 | (\lambda\lambda_1)\lambda_2 m_2 \rangle \left\langle \lambda_2 m_2 | M_m^\lambda | \lambda_1 m_1 \right\rangle \tag{19.16}$$

$$= \quad \frac{1}{d_{\lambda_2}} \tag{19.17}$$



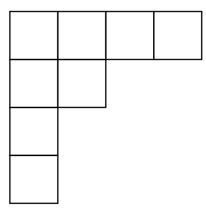
Chapter 20

Young Tableau

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

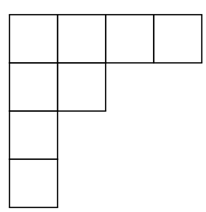
A **Young Diagram** (YD) $\mathcal{Y} = [\lambda_1, \lambda_2, \ldots, \lambda_D]$ consists of λ_1 left-aligned empty boxes over λ_2, over λ_3 left-aligned empty boxes, up to λ_D boxes, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_D \geq 1$. For example,

$$[4, 2, 1, 1] = \tag{20.1}$$



We will call $[4, 2, 1, 1]$ the **row lengths** (RL) method of labeling YD.

A alternative method of labelling YD is called the **Dynkin (D) labels** or **row changes (RC)**. These labels list the change in number of columns as we go down the YD. For example,



$$= [2, 1, 0, 1, 0 \ldots]_D \tag{20.2}$$

A **Young Tableau** (YT) $\mathcal{Y}_\alpha$ is a YD in which integers from 1 to n where $n \leq n_b$ and n_b is the number boxes, are inserted according to some rules. The rules for insertion are that integers must increase when reading a row left to right and when reading a column from top to bottom. Obviously, for $n < n_b$, some integers are repeated.

A **Standard Young Tableau** (SYT) $\mathcal{Y}_\alpha$ is a YT such that $n = n_b$ and no integer is repeated.

We will use $|\mathcal{Y}|$, or $|\mathcal{Y}_\alpha|$ or $|\alpha|$ to denote the number of boxes in a YD or YT. Sometimes $|S|$ means the number of elements in a finite set S, but that should not lead to confusion.

- $n_b = 1$

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

- $n_b = 2$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

- $n_b = 3$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

- $n_b = 4$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

Figure 20.1: SYT for $n_b = 1, 2, 3, 4$.

20.1 Symmetric group S_{n_b}

Let

S_{n_b} = the symmetric group in n_b letters (or n_b boxes)
$irreps(S_{n_b})$ = the set of all irreps of S_{n_b}.
The **transpose of a YT** is defined as if it were a matrix. For example

$$transpose\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \tag{20.3}$$

n-dim General Linear group $GL(n) = \{M \in \mathbb{C}^{n\times n} : det(M) \neq 0\}$
n-dim Special Linear group $SL(n) = \{M \in GL(n) : det(M) = 1\}$
n-dim Unitary group, $U(n) = \{M \in GL(n) : MM^\dagger = M^\dagger M = 1\}$
n-dim Special Unitary group $SU(n) = \{M \in U(n) : det(M) = 1\}$

Claim 28

1. *The YD with n_b boxes label all irreps of the symmetric group S_{n_b}.*
2. *The SYT with n_b boxes and no more than n_b rows, label the irreps of $GL(n_b)$ and of $U(n_b)$*
3. *The SYT with n_b boxes and no more than $n_b - 1$ rows, label the irreps of $SL(n_b)$ and $SU(n_b)$.*

proof:
QED

$YD(n_b)$ = set of YD with n_b boxes
$SYD(n_b)$ = set of SYD with n_b boxes
The irreps of S_{n_b} are labelled by the $\mathcal{Y} \in YD(n_b)$. Hence, there is a 1-1 onto map between $irreps(S_{n_b})$ and $YD(n_b)$

$$irreps(S_{n_b}) = YD(n_b) \tag{20.4}$$

$YT(\mathcal{Y})$ = set of YT for a $\mathcal{Y} \in YD(n_b)$.
$SYT(\mathcal{Y})$ = set of SYT for a $\mathcal{Y} \in YD(n_b)$.
$dim(\mathcal{Y})$ = dimension of irrep $\mathcal{Y}$.

$$dim(\mathcal{Y}) = |SYT(\mathcal{Y})| \tag{20.5}$$

For example, the irrep

$$\mathcal{Y} = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{20.6}$$

has dimension 3 because there are 3 possible SYT for this YD:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \implies dim(\mathcal{Y}) = 3 \tag{20.7}$$

Thus, we can denote the basis vectors of this irrep by

$$|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\rangle, \quad |\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\rangle, \quad |\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\rangle \tag{20.8}$$

To compute $hook(\mathcal{Y})$:

1. Fill each box of the YD with the number of boxes below and to the right of the box, including the box itself.
2. Multiply the numbers in all the boxes.

For example,

$$\mathcal{Y} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \multicolumn{1}{c}{} \\ \cline{1-3} & & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \implies hook(\mathcal{Y}) = \begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} = 6!3 \tag{20.9}$$

Claim 29 *(***hook rule** *for computing dim($\mathcal{Y}$))*

$$dim(\mathcal{Y}) = \frac{n_b!}{hook(\mathcal{Y})} \tag{20.10}$$

proof:
QED

For example

$$\mathcal{Y} = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \implies hook(\mathcal{Y}) = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 8 \tag{20.11}$$

so

$$dim(\mathcal{Y}) = \frac{4!}{4(2)} = 3 \tag{20.12}$$

The **regular representation** of the symmetric group S_{n_b} is defined as follows. For each permutation $\sigma \in S_{n_b}$, define an independent vector $|\sigma\rangle$ in a vector space $\mathcal{V} = \{|\sigma\rangle : \sigma \in S_{n_b}\} = \{|\sigma_i\rangle : i = 1, 2, \ldots, n_b!\}$. Let

$$|x\rangle = \sum_i x_i |\sigma_i\rangle \tag{20.13}$$

For any $\tau \in S_{n_b}$, suppose

$$\langle \sigma_j | \tau | \sigma_i \rangle = \left\langle \tau^{-1} \sigma_j | \sigma_i \right\rangle \tag{20.14}$$

$$\langle \sigma_j | \tau | x \rangle = \left\langle \tau^{-1} \sigma_j | x \right\rangle \tag{20.15}$$

The regular rep is $n_b!$ dimensional and reducible.

Claim 30 *The regular rep of S_{n_b} decomposes into each $\lambda \in rreps(S_{n_b})$, appearing $dim(\lambda)$ times. Thus*

$$n_b! = |S_{n_b}| = \sum_{\lambda \in irreps(S_{n_b})} [dim(\lambda)]^2 \tag{20.16}$$

$$= [n_b!]^2 \sum_{\mathcal{Y} \in YD(n_b)} \frac{1}{[hook(\mathcal{Y})]^2} \quad \textit{(Because } irreps(S_{n_b}) = YD(n_b)\textit{)} \tag{20.17}$$

Hence,

$$1 = n_b! \sum_{\mathcal{Y} \in YD(n_b)} \frac{1}{[hook(\mathcal{Y})]^2} \tag{20.18}$$

proof:
QED

$$1 = \sum_{\mathcal{Y} \in YD(n_b)} P_{\mathcal{Y}} \tag{20.19}$$

$$P_{\mathcal{Y}} = \sum_{\mathcal{Y}_\alpha \in SYT(\mathcal{Y})} |\mathcal{Y}_\alpha\rangle \langle \mathcal{Y}_\alpha| \tag{20.20}$$

The projection operators are complete and orthogonal.

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \tag{20.21}$$

$$\ydiagram{3} \otimes \ydiagram{2,1} =$$

$$\ydiagram{5,1} \oplus \ydiagram{4,2} \oplus \ydiagram{4,1,1} \oplus \ydiagram{3,2,1} \tag{20.22}$$

20.2 Unitary group $U(n)$

$SYT(n_b, R \le n)$ = SYTs with n_b boxes and less then n rows

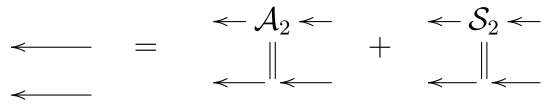
$$irreps(U(n)) = \cup_{n_b \le n} SYT(n_b, R \le n) \tag{20.23}$$

A SYT with n_b boxes represents a tensor with n_b indices (n_b-particles state). Each index ranges from 1 to n.

$n_b = 1$: A 1-index, 1-box tensor is a 1-particle with n states. This corresponds to the fundamental representation.

$n_b = 2$: A 2-index, 2-box tensor is a 2-particle with n^2 states. These n^2 states break into two sets, symmetric and anti-symmetric.

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array} \tag{20.24}$$



(20.25)

The SYT of an irrep describes the symmetry of the indices of a tensor in that irrep. A single column SYT indicates a totally anti-symmetric tensor, a single row SYT indicates a totally symmetric tensor, and a neither single column nor single row SYT indicates mixed symmetry. This is why we can't have more than n rows, because there are only n_b integers to fill all boxes so more than n rows would require a repetition in of an integer in a column, and such a column, after antisymetrizing, equals zero.

$YD(n_b)$ = set of YD with n_b boxes

$YT(\mathcal{Y})$ = set of YT for a $\mathcal{Y} \in YD(n_b)$.

If $\mathcal{Y}_\alpha$ is a SYT in $irreps(U(n))$ and the YD of $\mathcal{Y}_\alpha$ is $\mathcal{Y}$, then

$$dim(\mathcal{Y}_\alpha) = |YT(\mathcal{Y})| \tag{20.26}$$

Hence, $\{|\alpha\rangle : \alpha \in YT(\mathcal{Y})\}$ are a basis for the irrep $\mathcal{Y}_\alpha$. Note that the irreps of $U(n)$ are given by SYT $\mathcal{Y}_\alpha$, whereas the basis vectors of an irrep are given by YT (not just STY). For example, for

$$\mathcal{Y}_\alpha = \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array} \tag{20.27}$$

the basis vectors are

$$\left|\begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}\right\rangle, \quad \left|\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right\rangle, \quad \left|\begin{array}{|c|c|}\hline 2 & 2\\\hline\end{array}\right\rangle \tag{20.28}$$

so

$$dim(\mathcal{Y}_\alpha) = 3 \tag{20.29}$$

20.3 Young Projection operators

For each in the SYT $\mathcal{Y}_\alpha \in irreps(U(n))$, we have

$$P_{\mathcal{Y}_\alpha} = \mathcal{N}_{\mathcal{Y}} \left(\prod_i S_i\right) \left(\prod_j A_j\right) \tag{20.30}$$

Note that the normalization constant $\mathcal{N}_{\mathcal{Y}}$ depends only on the YD $\mathcal{Y}$.

These projection operators are not unique.

Claim 31 *Let*

$$\mathcal{N}_{\mathcal{Y}} = \frac{\left(\prod_i |S_i|!\right)\left(\prod_j |A_j|!\right)}{hook(\mathcal{Y})} \tag{20.31}$$

where $|S_i|$ and $|A_j|$ are the number of arrows entering the symmetrizer or anti-symmetrizer. Then the operators $P_{\mathcal{Y}_\alpha}$ are idempotent (i.e., their square equals themselves) mutually orthogonal and complete:

$$P_{\mathcal{Y}_\alpha} P_{\mathcal{Y}_\beta} = P_{\mathcal{Y}_\alpha} \delta(\alpha, \beta) \tag{20.32}$$

$$1 = \sum_{\mathcal{Y}_\alpha \in SYT(n_b, R<n)} P_{\mathcal{Y}_\alpha} \tag{20.33}$$

proof:

$$P_{\mathcal{Y}_\alpha} = \mathcal{N}_{\mathcal{Y}} \frac{1}{\prod_i |S_i|! \prod_j j|A_j|!} \left(\underbrace{\begin{matrix} \longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow \end{matrix}}_{\mathbb{1}} + \cdots \right) \tag{20.34}$$

From Eq.(20.33)

$$\begin{aligned}
\mathbb{1} &= \sum_{\mathcal{Y}_\alpha \in SYT(n_b, R<n)} \mathcal{N}_{\mathcal{Y}} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} && (20.35) \\
&= \sum_{\mathcal{Y}} \frac{n_b!}{hook(\mathcal{Y})} \mathcal{N}_{\mathcal{Y}} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} && (20.36) \\
&= \sum_{\mathcal{Y}} \frac{n_b!}{[hook(\mathcal{Y})]^2} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} \quad \text{(if assume Eq.(20.31))} && (20.37) \\
&= \mathbb{1} \quad \text{(by Eq.(20.18))} && (20.38)
\end{aligned}$$

QED

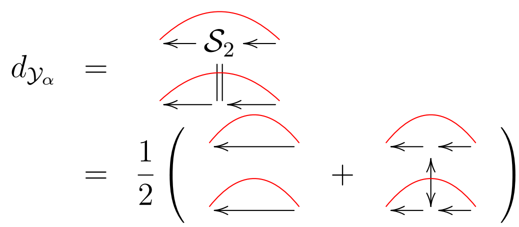
Let $dim(\mathcal{Y}_\alpha)$ be the dimension of an irrep of $U(n)$ with STY given by $\mathcal{Y}_\alpha \in STY(n_{\underline{b}}, R < n)$. In Eq.(20.26) we gave a way of finding $dim(\mathcal{Y}_\alpha)$ A second way is by taking the trace of the corresponding projection operator

$$dim(\mathcal{Y}_\alpha) = \mathrm{tr}(P_{\mathcal{Y}_\alpha}) \tag{20.39}$$

For example, if

$$\mathcal{Y}_\alpha = \boxed{1 \mid 2} \tag{20.40}$$

then



(20.41)

(20.42)

$$= \frac{1}{2}(n^2 + n) \tag{20.43}$$

$$= 3 \text{ for } U(n = 2) \tag{20.44}$$

A third way of computing $dim(\mathcal{Y}_\alpha)$ is by computing the hook and coat functions and using the formula

$$dim(\mathcal{Y}_\alpha) = \frac{coat(\mathcal{Y})}{hook(\mathcal{Y})} \tag{20.45}$$

Note that right hand side is independent of α; it depends only on the YD. We've already discussed how to compute $hook(\mathcal{Y})$. $coat(\mathcal{Y})$ is calculated as follows.[1]

1. Fill $\mathcal{Y}$ with
   - n at the diagonal blocks
   - n increments increasing by 1 when reading from left to right
   - n increments decreasing by 1 when reading from top to bottom
2. multiply all the boxes

   Examples

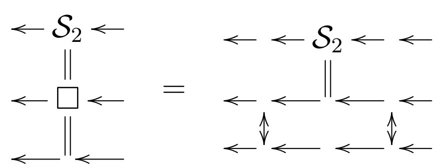
$$dim(\boxed{1 \mid 2}) = \frac{\boxed{\mathrm{n} \mid \mathrm{n+1}}}{\boxed{2 \mid 1}} = \frac{n(n+1)}{2} \tag{20.46}$$

[1] I invented that name. I don't know if it has a name.

$$dim\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\right) = \frac{\begin{array}{|c|}\hline n\\ \hline n\text{-}1\\ \hline\end{array}}{\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}} = \frac{n(n-1)}{2} \tag{20.47}$$

$$dim\left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}\right) = \frac{\begin{array}{|c|c|c|c|}\hline n & n+1 & n+2 & n+3\\ \hline n\text{-}1 & n \\ \cline{1-2} n\text{-}2 \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1\\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}} = \frac{n^2(n^2-1)(n^2-4)(n+3)}{144} \tag{20.48}$$

20.4 Young Projection operators for $n_b = 1, 2, 3, 4$



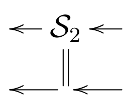
(20.49)

- $n_b = 1$

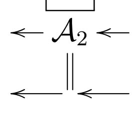
$\begin{array}{|c|}\hline 1\\ \hline\end{array}$

- $n_b = 2$

$\begin{array}{|c|c|}\hline 1 & 2\\ \hline\end{array}$

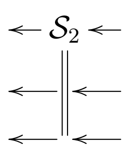


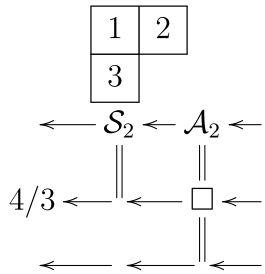
$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$

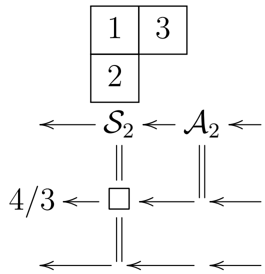


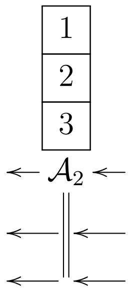
- $n_b = 3$

$\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\ \hline\end{array}$

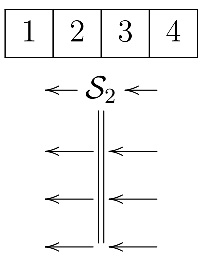


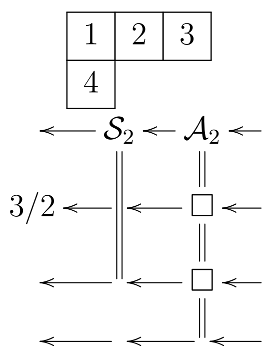

1 2
3
$\mathcal{S}_2$ $\mathcal{A}_2$
4/3

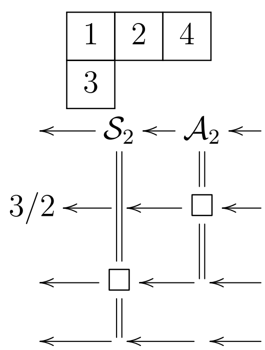

1 3
2
$\mathcal{S}_2$ $\mathcal{A}_2$
4/3

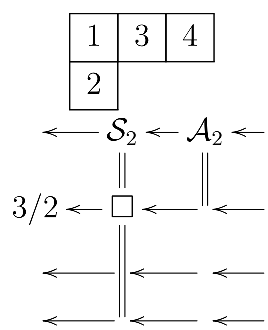

1
2
3
$\mathcal{A}_2$

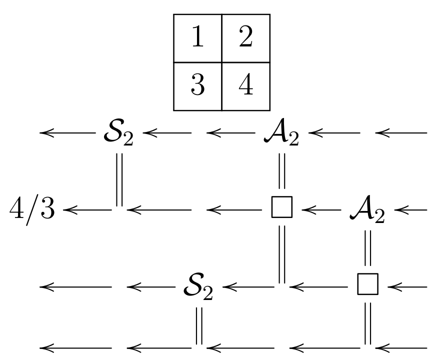
- $n_b = 4$

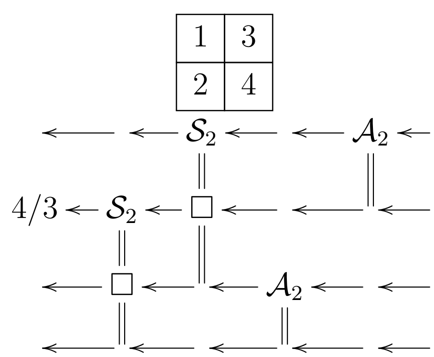

1 2 3 4
$\mathcal{S}_2$

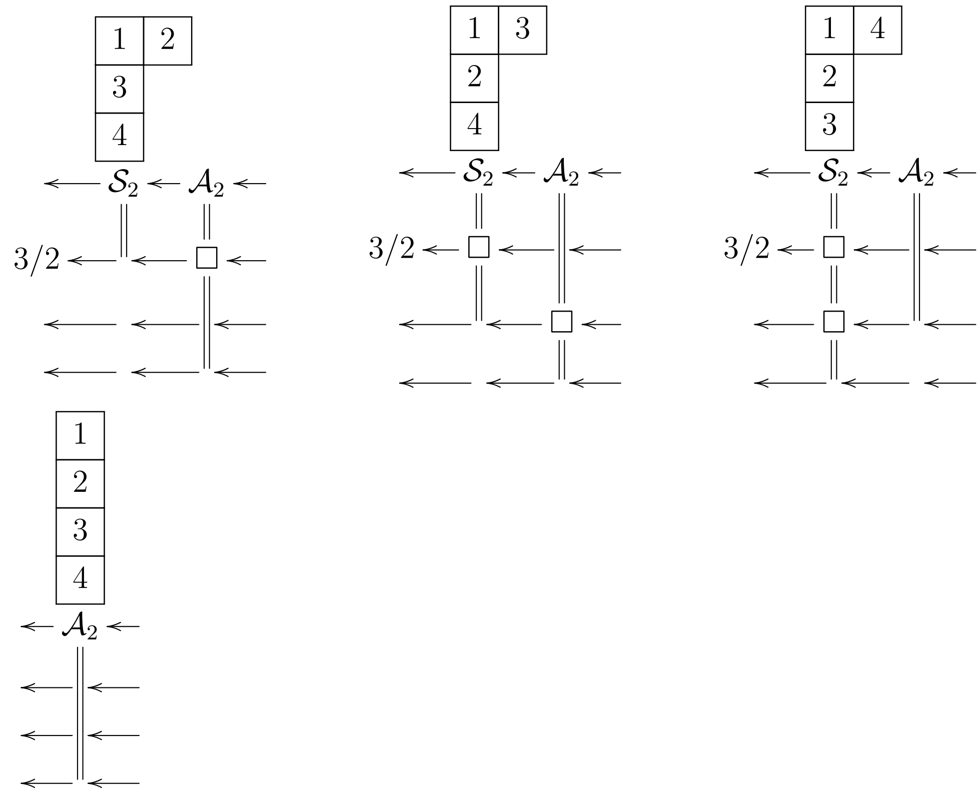

1 2 3
4
$\mathcal{S}_2$ $\mathcal{A}_2$
3/2


1 2 4
3
$\mathcal{S}_2$ $\mathcal{A}_2$
3/2


1 3 4
2
$\mathcal{S}_2$ $\mathcal{A}_2$
3/2


1 2
3 4
$\mathcal{S}_2$ $\mathcal{A}_2$
4/3
$\mathcal{A}_2$
$\mathcal{S}_2$


1 3
2 4
$\mathcal{S}_2$ $\mathcal{A}_2$
4/3
$\mathcal{S}_2$
$\mathcal{A}_2$



SYT and corresponding unnormalized projection operators for $n_b = 1, 2, 3, 4$

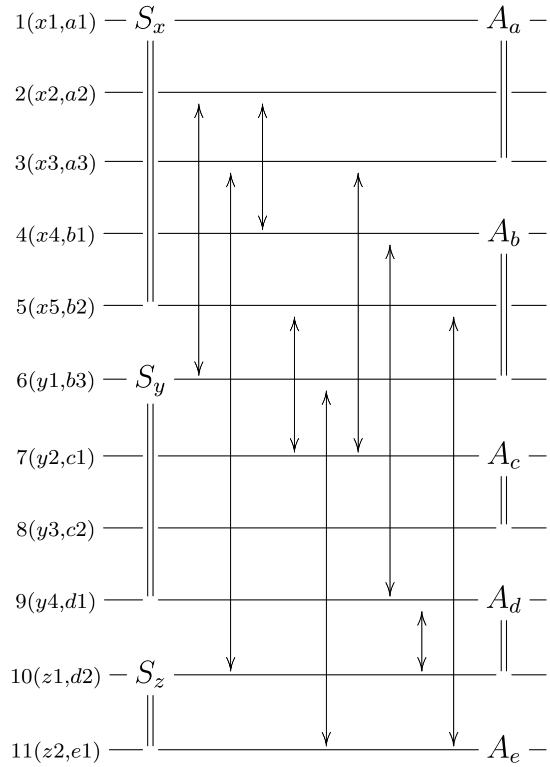
20.5 Young Projection Operator with swaps

$\mathcal{Y}_\alpha =$

	A_a	A_b	A_c	A_d	A_e
S_x	1	2	3	4	5
S_y	6	7	8	9	
S_z	10	11			

$a1 \longleftrightarrow 1$	
$a2 \longleftrightarrow 6$	
$a3 \longleftrightarrow 10$	
$b1 \longleftrightarrow 2$	
$b2 \longleftrightarrow 7$	
$b3 \longleftrightarrow 11$	
$c1 \longleftrightarrow 3$	
$c2 \longleftrightarrow 8$	
$d1 \longleftrightarrow 4$	
$d2 \longleftrightarrow 9$	
$e1 \longleftrightarrow 5$	

$P_{\mathcal{Y}_\alpha} =$

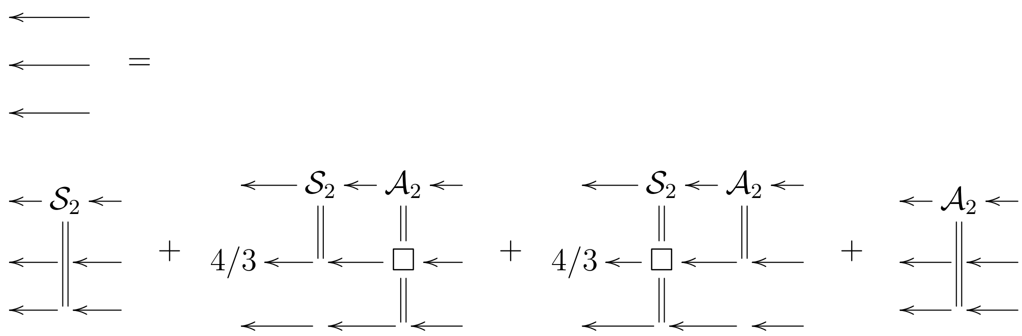


(20.50)

20.6 Tensor product decompositions

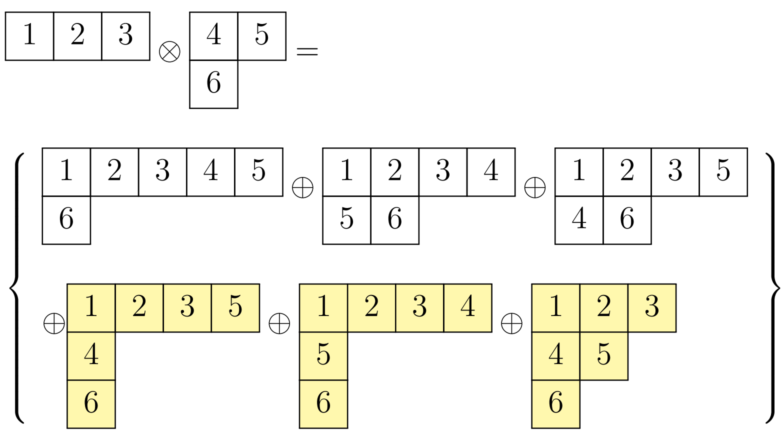
$$\boxed{1} \otimes \boxed{2} \otimes \boxed{3} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right) \otimes \boxed{3} \tag{20.51}$$

$$= \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \tag{20.52}$$



(20.53)

$$n^3 = \frac{n(n+1)(n+2)}{6} + \frac{n(n^2-1)}{3} + \frac{n(n^2-1)}{3} = \frac{(n-2)(n-1)n}{6} \tag{20.54}$$



(20.55)

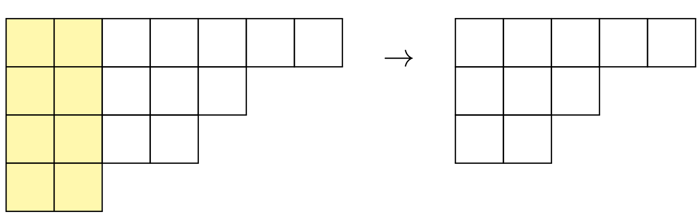
For $U(n)$, the blue YT are zero for $n_b = 2$, and non-zero otherwise.

20.7 $SU(n)$

$\mathfrak{s}u(n)$ A_n

$$\epsilon_{a_1 a_2 \dots a_n} = G_{a_1}{}^{a'_1} G_{a_2}{}^{a'_2} \cdots G_{a_n}{}^{a'_n}\, \epsilon_{a'_1 a'_2 \dots a'_n} \tag{20.56}$$

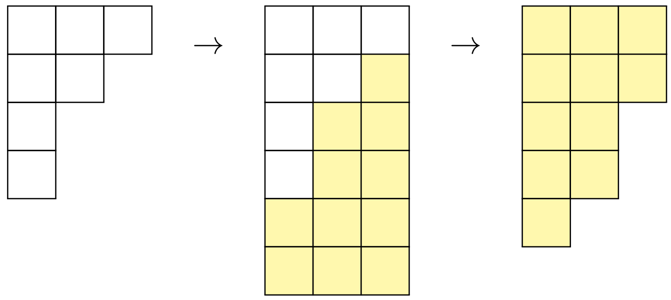
YD for $SU(n)$ has a maximum of $n-1$ rows. For $SU(4)$



(20.57)

$[5,3,2]_{RL}$ row lengths, $[2,1,2,0,\dots]_D$ row changes (Dynkin labels).

$[b_1, b_2, \ldots, b_{n-1}]$
For $SU(6)$,



(20.58)

Besides RL and RC, a third way of labelling YD (reps of $SU(n)$) is by their dimension, and then adding a subscript or overline if there are more than one reps with a different YD but the same dimension. This method is used mostly by physicists for $SU(3)$ (The Eightfold Way). Note that all YT with the same YD have the same dimension, so this really labels YD instead of YT. For example, for $SU(3)$ we have

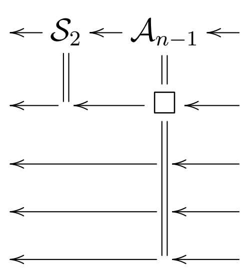
$$\begin{array}{ll} \ydiagram{1} = [1,0]_D = 3 & \ydiagram{1,1} = [0,1]_D = \overline{3} \\ \ydiagram{2} = [2,0]_D = 6 & \ydiagram{2,2} = [0,2]_D = \overline{6} \\ \ydiagram{2,1} = [1,1]_D = 8 & \ydiagram{3,1} = [2,1]_D = 15 \end{array} \tag{20.59}$$

$$\ydiagram{1} \otimes \left.\begin{array}{c}\ydiagram{1}\\ \ydiagram{1}\\ \vdots\\ \ydiagram{1}\end{array}\right\} n-1 \quad = \quad 1 \oplus \left.\begin{array}{l}\ydiagram{2}\\ \ydiagram{1}\\ \vdots\\ \ydiagram{1}\end{array}\right\} n-1 \tag{20.60}$$

$$n \otimes \overline{n} \quad = \quad 1 \oplus (n^2 - 1) \tag{20.61}$$

$$\text{fun rep} \otimes \text{conjugate rep} \quad = \quad \text{singlet rep} \oplus \text{ adjoint rep} \tag{20.62}$$

Adjoint representation

$$P_{adj} = \frac{2(n-1)}{n}$$ 

(20.63)

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[2] JP Elliott and PG Dawber. *Symmetry in Physics, vols. 1, 2.* Springer, 1979.

[3] Robert R. Tucci. Bayesuvius (free book). `https://github.com/rrtucci/Bayesuvius`.

[4] Robert R. Tucci. Quantum Bayesian nets. *International Journal of Modern Physics B*, 09(03):295–337, January 1995.