

# BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF  
QUANTUM BAYESIAN NETWORKS



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# **Bayesuvious Quantico,** a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download  
the latest version, go to

<https://github.com/rrtucci/bayes-quantico>

## **Bayes Quantico**

by Robert R. Tucci

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# Appendices

# Chapter 18

## Symmetrization and Antisymmetrization: COMING SOON

$(1, 2)$  transposition, swaps 1 and 2,  $1 \rightarrow 2 \rightarrow 1$ .  $(3, 2, 1)$  means  $3 \rightarrow 2 \rightarrow 1 \rightarrow 3$ . A reordering of  $(1, 2, 3, \dots, p)$  is a permutation on  $p$  letters. A permutation can be expressed as a product of transpositions  $(3, 2, 1) = (3, 2)(2, 1)$  is an even permutation because it can be expressed as a product of an even number of transpositions. An odd permutation can be expressed as a product of an odd number of permutations.

### 18.1 Symmetrization

$$\mathbb{I}_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} = \begin{array}{l} a_1 \leftarrow b_1 \\ a_2 \leftarrow b_2 \end{array} \quad (18.1)$$

$$(\sigma_{(1,2)})_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} = \begin{array}{c} a_1 \leftarrow \bullet \leftarrow b_1 \\ \updownarrow \\ a_2 \leftarrow \bullet \leftarrow b_2 \end{array} \quad (18.2)$$

$$\mathbb{I} = \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \quad (18.3)$$

$$\sigma_{(1,2)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \end{array} \quad \sigma_{(2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad \sigma_{(1,3)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \end{array} \quad (18.4)$$

$$\sigma_{(1,2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \end{array} \quad (18.5)$$

$$\sigma_{(1,3,2)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad (18.6)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p!} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} + \dots \right\} \quad (18.7)$$

$$\mathcal{S}_p^2 = \mathcal{S}_p \quad (18.8)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (18.9)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \begin{array}{c} \leftarrow \mathcal{S}_{[1,q]} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (18.10)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (18.11)$$



### Claim 1

$$= \frac{1}{p} \left( \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \right. + (p-1) \left( \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \end{array} \right) \quad (18.12)$$

**proof:** We only prove it for  $p = 3$ .

$$3! \left( \begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \\ + \\ \text{Diagram 3} \\ + \\ \text{Diagram 4} \\ + \\ \text{Diagram 5} \\ + \\ \text{Diagram 6} \\ + \\ \text{Diagram 7} \\ + \\ \text{Diagram 8} \end{array} \right) = \dots \quad (18.13)$$

$$2! \begin{array}{c} \leftarrow \\ \leftarrow \mathcal{S}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left( \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} + \begin{array}{c} \leftarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \updownarrow \bullet \leftarrow \end{array} \right) \quad (18.14)$$

$$3! \begin{array}{c} \leftarrow \mathcal{S}_3 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} - 2! \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} = \left( \begin{array}{cc} \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} & + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \\ + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \end{array} & + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \end{array} \end{array} \right) \quad (18.15)$$

$$= 2!2! \quad (18.16)$$

QED

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p} \left( \begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} + (p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \quad \parallel \\ \leftarrow \leftarrow \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \leftarrow \leftarrow \end{array} \right) \quad (18.17)$$

$$= \frac{n+p-1}{p} \left( \begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} \right) \quad (18.18)$$

$$\text{tr}_{\underline{a}_1} \mathcal{S}_p = \frac{n+p-1}{p} \mathcal{S}_{p-1} \quad (18.19)$$

$$\text{tr}_{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k} \mathcal{S}_p = \frac{(n+p-1)(n+p-2) \dots (n=p-k)}{p(p-1) \dots (p-k+1)} \mathcal{S}_{p-k} \quad (18.20)$$

$$d_{\mathcal{S}_p} = \text{tr}_{\underline{a}^p} \mathcal{S}_p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \quad (18.21)$$

For  $p = 2$ ,

$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \quad (18.22)$$

## 18.2 Antisymmetrization

$$\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p!} \left\{ \begin{array}{c} \leftarrow \leftarrow \quad \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \quad \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \quad \leftarrow \leftarrow \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \leftarrow \leftarrow \quad \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} + \dots \right\} \quad (18.23)$$

$$\mathcal{A}_p^2 = \mathcal{A}_p \quad (18.24)$$

$$\begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = \quad (18.25)$$

$$\begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_{[1,q]} \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = \quad (18.26)$$

$$\begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \begin{array}{c} \bullet \\ \updownarrow \\ \bullet \end{array} \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = (-1) \quad (18.27)$$

$$\mathcal{S}_p \mathcal{A}_q = \mathcal{A}_p \mathcal{S}_q = 0 \quad (18.28)$$

$$\begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow & \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \\ \vdots & \vdots \vdots & \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \end{array} = 0 \quad (18.29)$$

$$\begin{array}{ccc}
\leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{A}_{[1,q]} \leftarrow & \\
\leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \\
\leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \\
\vdots & \vdots \vdots & \vdots \\
\leftarrow \parallel \leftarrow & \leftarrow & 
\end{array} = 
\begin{array}{ccc}
\leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{S}_{[1,q]} \leftarrow & \\
\leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \\
\leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \\
\vdots & \vdots \vdots & \vdots \\
\leftarrow \parallel \leftarrow & \leftarrow & 
\end{array} = 0 \quad (18.30)$$

**Claim 2**

$$\begin{array}{ccc}
\leftarrow \mathcal{A}_p \leftarrow & & \\
\leftarrow \parallel \leftarrow & & \\
\leftarrow \parallel \leftarrow & & \\
\vdots & \vdots & \\
\leftarrow \parallel \leftarrow & & 
\end{array} = \frac{1}{p} \left( \begin{array}{ccc} \leftarrow & & \leftarrow \bullet \leftarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow & & \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow & & \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \\ \vdots & & \vdots \\ \leftarrow \parallel \leftarrow & & \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \end{array} + (-1)(p-1) \begin{array}{ccc} \leftarrow & & \leftarrow \bullet \leftarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow & & \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow & & \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \\ \vdots & & \vdots \\ \leftarrow \parallel \leftarrow & & \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \end{array} \right) \quad (18.31)$$

**proof:** We only prove it for  $p = 3$ .

$$3! \begin{array}{ccc} \leftarrow \mathcal{A}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left( \begin{array}{ccc} \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \bullet \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \\ + & \leftarrow \bullet \leftarrow \bullet \leftarrow & + \leftarrow \bullet \leftarrow \bullet \leftarrow \\ - & \leftarrow \bullet \leftarrow & - \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \end{array} \right) \quad (18.32)$$

$$2! \begin{array}{ccc} \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left( \begin{array}{ccc} \leftarrow & & \leftarrow \\ \leftarrow & & \leftarrow \bullet \leftarrow \\ \leftarrow & & \leftarrow \bullet \leftarrow \end{array} \right) \quad (18.33)$$

$$\begin{aligned}
3! \begin{array}{c} \leftarrow \mathcal{A}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} - 2! \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} &= \left( \begin{array}{cc} \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} & + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \\ - \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} & - \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \end{array} \right) \quad (18.34)
\end{aligned}$$

$$= (-1)2!2! \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{A}_2 \leftarrow \bullet \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} \quad (18.35)$$

QED

$$\begin{aligned}
\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} &= \frac{1}{p} \left( \begin{array}{c} \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} + (-1)(p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} \right) \quad (18.36)
\end{aligned}$$

$$= \frac{n + (-1)(p-1)}{p} \left( \begin{array}{c} \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} \right) \quad (18.37)$$

$$\text{tr}_{\underline{a}_1} \mathcal{A}_p = \frac{n-p+1}{p} \mathcal{A}_{p-1} \quad (18.38)$$

$$\text{tr}_{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k} \mathcal{A}_p = \frac{(n-p+1)(n-p+2) \dots (n-p+k)}{p(p-1) \dots (p-k+1)} \mathcal{A}_{p-k} \quad (18.39)$$

$$d_{\mathcal{A}_p} = \text{tr}_{\underline{a}^p} \mathcal{A}_p = \frac{\prod_{i=n-p+1}^n i}{p!} \quad (18.40)$$

$$= \frac{\prod_{i=n}^{n-p+1} i}{p!} \quad (18.41)$$

$$= \begin{cases} \frac{n!}{p!(n-p)!} = \binom{n}{p} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} \quad (18.42)$$

For  $p = 2 \leq n$ ,

$$d_{\mathcal{A}_2} = \binom{n}{2} \quad (18.43)$$

$$\mathcal{A}_p = 0 \text{ if } n < p \quad (18.44)$$

For example, for  $n = 2$  and  $p = 3$

$$\begin{array}{c} |a\rangle \\ \downarrow \\ |a\rangle \\ \downarrow \\ |a\rangle \\ \downarrow \\ |b\rangle \\ \downarrow \\ |b\rangle \\ \downarrow \\ |b\rangle \end{array} \mathcal{A}_3 = \frac{1}{6} \left( \begin{array}{c} \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \right) \\ - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \leftarrow \bullet \rightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \end{array} \quad (18.45)$$

$$\mathcal{A}_3|a, a, b\rangle = \frac{1}{6} \left( \begin{array}{c} |a, a, b\rangle + |a, b, a\rangle + |b, a, a\rangle \\ -|a, b, a\rangle - |a, a, b\rangle - |b, a, a\rangle \end{array} \right) \quad (18.46)$$

$$= 0 \quad (18.47)$$

# Bibliography