

BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF
QUANTUM BAYESIAN NETWORKS



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Bayesuvious Quantico, a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download
the latest version, go to

<https://github.com/rrtucci/bayes-quantico>

Bayes Quantico

by Robert R. Tucci

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Appendices

Appendix A

Spectral Decomposition and Eigenvalue Projection Operators: COMING SOON

$$M \in \mathbb{C}^{d \times d}$$

$$M|v\rangle = \lambda|v\rangle \quad (\text{A.1})$$

If M is Hermitian ($H^\dagger = H$), its eigenvalues are real. ($\lambda = \langle \lambda | M | \lambda \rangle \in \mathbb{R}$)

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M - \lambda) = 0 \quad (\text{A.2})$$

If M is a Hermitain matrix, then there exists a unitary matrix ($CC^\dagger = C^\dagger C = 1$) such that

$$CMC^\dagger = \begin{bmatrix} D_{\lambda_1} & 0 & 0 & 0 \\ 0 & D_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{\lambda_r} \end{bmatrix} \quad (\text{A.3})$$

where

$$D_{\lambda_i} = \text{diag}(\underbrace{\lambda_i, \lambda_i, \dots, \lambda_i}_{d_i \text{ times}}) \quad (\text{A.4})$$

$$d = \sum_{i=1}^r d_i \quad (\text{A.5})$$

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (\text{A.6})$$

$$CP_1C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2} \quad (\text{A.7})$$

$$CP_2C^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1} \quad (\text{A.8})$$

If $I^{d_i \times d_i}$ is the d_i dimensional unit matrix,

$$P_i = C^\dagger \text{diag}(0, \dots, 0, I^{d_i \times d_i}, 0, \dots, 0)C \quad (\text{A.9})$$

$$= \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (\text{A.10})$$

Note that P_i are Hermitian ($P_i^\dagger = P_i$) because M is Hermitian and its eigenvalues are real.)

Note that P_i and M commute

$$[P_i, M] = P_i M - M P_i = 0 \quad (\text{A.11})$$

orthogonal

$$P_i P_j = \delta(i, j) P_j \quad (\text{A.12})$$

complete

$$\sum_i P_i = 1 \quad (\text{A.13})$$

$$M = \sum_{i=1}^r P_i M P_i \quad (\text{A.14})$$

$$d_i = \text{tr} P_i \quad (\text{A.15})$$

$$C M P_1 C^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.16})$$

$$= \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.17})$$

$$M P_i = \lambda_i P_i \text{ (no } i \text{ sum)} \quad (\text{A.18})$$

$$f(M) P_i = f(\lambda_i) P_i \text{ (no } i \text{ sum)} \quad (\text{A.19})$$

$M^{(1)}, M^{(2)}$

$$[M^{(1)}, M^{(2)}] = 0 \quad (\text{A.20})$$

Use $M^{(1)}$ to decompose V into $\bigoplus_i V_i$. Use $M^{(2)}$ to decompose V_i into $\bigoplus_j V_{i,j}$. If $M^{(1)}$ and $M^{(2)}$ don't commute, let $P_i^{(1)}$ be the eigenvalue projection operators of $M^{(1)}$. The replace $M^{(2)}$ by $P_i^{(1)} M^{(2)} P_i^{(1)}$

$$[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] = 0 \quad (\text{A.21})$$

Appendix B

Birdtracks: COMING SOON

Cvitanovic Birdtracks book [1]

Elliott-Dawber book [2]

My paper “Quantum Bayesian Nets” [3]

B.1 Classical Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $P(y|x) \in [0, 1]$ where $x \in \text{val}(\underline{x})$ and $y \in \text{val}(\underline{y})$

$$\sum_{y \in \text{val}(\underline{y})} P(y|x) = 1 \quad (\text{B.1})$$

$$\mathcal{C} = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow & \underline{a} \end{array} \quad (\text{B.2})$$

$$\mathcal{C}(a, b, c) = P(c|b, a)P(b|a)P(a) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow & a \end{array} P(a) \quad (\text{B.3})$$

$$a^2 = (a_1, a_2)$$

$$\mathcal{C}' = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow & \underline{a}^2 \end{array} \quad (\text{B.4})$$

$$\mathcal{C}'(a^2, b, c) = P(c|b, a_2)P(a_2|a^2)P(b|a_1)P(a_1|a^2)P(a^2) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow & a^2 \end{array} P(a^2) \quad (\text{B.5})$$

Marginalizer nodes \underline{a}_1 and \underline{a}_2 have the TPMs

$$P(a'_i | \underline{a}^2 = (a_1, a_2)) = \delta(a'_i, a_i) \quad (\text{B.6})$$

for $i = 1, 2$

B.2 Quantum Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $A(y|x) \in \mathbb{C}$ where $x \in \text{val}(\underline{x})$ and $y \in \text{val}(\underline{y})$

$$\sum_{y \in \text{val}(\underline{y})} |A(y|x)|^2 = 1 \quad (\text{B.7})$$

$$\mathcal{Q} = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow & \underline{a} \end{array} \quad (\text{B.8})$$

$$\mathcal{Q}(a, b, c) = A(c|b, a)A(b|a)A(a) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow & a \end{array} A(a) \quad (\text{B.9})$$

$$a^2 = (a_1, a_2)$$

$$\mathcal{Q}' = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow \underline{a}_2 & \underline{a}^2 \end{array} \quad (\text{B.10})$$

$$\mathcal{Q}'(a^2, b, c) = A(c|b, a_2)A(a_2|a^2)A(b|a_1)A(a_1|a^2)A(a^2) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow a_2 & a^2 \end{array} A(a^2) \quad (\text{B.11})$$

Marginalizer nodes \underline{a}_1 and \underline{a}_2 have the TAMs

$$A(a'_i | \underline{a}^2 = (a_1, a_2)) = \delta(a'_i, a_i) \quad (\text{B.12})$$

for $i = 1, 2$

B.3 Birdtracks

$$\delta(b, a) = \mathbb{1}(a = b) = \delta_a^b = a \leftarrow \bullet \rightarrow b \quad (\text{B.13})$$

$$\langle a, b | X_{\underline{ab}}^{\underline{cd}} | c, d \rangle = X_{ab}^{cd} = \begin{array}{c} \underline{a} = a \leftarrow X_{\underline{ab}}^{\underline{cd}} \\ \swarrow \quad \nearrow \\ \underline{b} = b \\ \swarrow \quad \nearrow \\ \underline{c} = c \\ \swarrow \quad \nearrow \\ \underline{d} = d \end{array} \quad (\text{B.14})$$

$$\begin{array}{c} a \leftarrow X_{\underline{ab}}^{\underline{cd}} \\ \swarrow \quad \nearrow \\ b \\ \swarrow \quad \nearrow \\ c \\ \swarrow \quad \nearrow \\ d \end{array} \rightarrow \begin{array}{c} a, b \leftarrow X_{\underline{ab}}^{\underline{cd}} \\ \swarrow \quad \nearrow \\ a, b \\ \swarrow \quad \nearrow \\ c \\ \swarrow \quad \nearrow \\ d \end{array} \quad (\text{B.15})$$

$X_{\underline{ab}}^{\underline{cd}} \in V^2 \otimes V_2$. Sometimes, we will omit denote this node simply by X . This is okay as long as we are not using, X to also denote a different version of $X_{\underline{ab}}^{\underline{cd}}$ with some of the indices raised or lowered or their order has been changed. ¹

$$(X^\dagger)_{dc}^{ba} = \begin{array}{c} (X^\dagger)_{dc}^{ba} \leftarrow \underline{a} = a \\ \swarrow \quad \nearrow \\ \underline{b} = b \\ \swarrow \quad \nearrow \\ \underline{c} = c \\ \swarrow \quad \nearrow \\ \underline{d} = d \end{array} \quad (\text{B.16})$$

¹For matrices, $(A^\dagger)_{i,j} = (A_{j,i})^*$ so taking a Hermitian conjugate involves both taking the complex conjugate of the matrix element and reversing the left-to-right (L2R) order of its indices. This generalizes to $(X^\dagger)_{dc}^{ba} = (X_{ab}^{cd})^*$. Besides raising and lowering indices, we reverse their L2R order.

$$\begin{array}{c}
(X^\dagger)_{dc}^{ba} \longleftarrow \sum a \longleftarrow X_{ab}^{cd} \\
\swarrow \quad \searrow \quad \swarrow \quad \searrow \\
\sum b \quad \sum c \\
\searrow \quad \swarrow \\
\sum d
\end{array}
\quad (B.17)$$

$$\begin{array}{c}
X^\dagger \longleftarrow X \\
\swarrow \quad \searrow \quad \swarrow \quad \searrow \\
\sum b \quad \sum c \\
\searrow \quad \swarrow \\
\sum d
\end{array}
\quad (B.18)$$

Birdtracks originated as a graphical way to represent the tensors in General Relativity (Gravitation). In General Relativity, one deals with tensors such as $T_a^b{}_c$ which have some indices raised and some lowered. One can use the metric $g^{a,b}$ to raise all the lowered indices to get T^{abc} . If we represent this graphically as a node with incoming arrows a, b, c , we need to follow one of the following 2 conventions: either

1. label the arrows as $\underline{a}, \underline{b}, \underline{c}$, and define the node as $T^{\underline{abc}}$, or
2. instead of labelling the arrows explicitly $\underline{a}, \underline{b}, \underline{c}$, indicate in the node where is the first arrow \underline{a} , and draw the arrows $\underline{a}, \underline{b}, \underline{c}$ so that they enter the node in **counterclockwise** (CC) order. The **left-to-right** (L2R) order of the indices on T corresponds the CC order of the arrows.

If we don't do either 1 or 2, we won't be able to distinguish between the graphical representations of $T^{1,2,3}$ and $T^{2,1,3}$, for example. Cvitanovic's Birdtracks book Ref.[1] follows Convention 2, but most of the time, in this book, we will follow Convention 1² The reason I chose to do so is for the sake of consistency: Convention 2 is closer to the quantum bnet conventions.

Another issue that arises in using birdtracks is this. When is it permissible to represent a tensor by T_{ab}^{cd} ? If we define T_{ab}^{cd} by

$$T_{ab}^{cd} = T_{ab}{}^{cd} \quad (B.19)$$

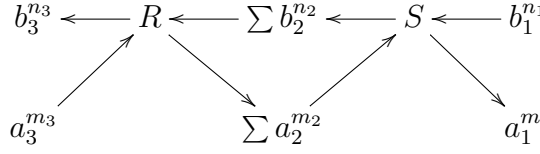
then it's always permissible. Then one can define tensors like $T_a{}^{bcd}$ as

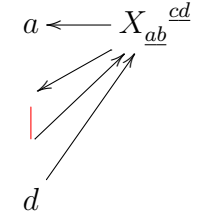
$$T_a{}^{bcd} = g^{bb'} T_{ab'}{}^{cd} = g^{bb'} T_{ab'}^{cd} \quad (B.20)$$

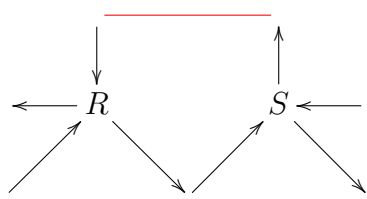
²If we follow Convention 1, we don't need to reverse the L2R order of the indices when taking a Hermitian conjugate. Thus, $(X^\dagger)^{\underline{ab}}_{\underline{cd}} = X_{\underline{ab}}^{\underline{cd}} = X^{\underline{dc}}_{\underline{ba}}$. As long as $\underline{a}, \underline{b}$ are lower indices and $\underline{c}, \underline{d}$ are upper indices of X , any L2R order of $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ is equivalent under Convention 1.

Hence, one drawback of using the notation T_{ab}^{cd} is that if one is interested in using versions of T_{ab}^{cd} with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing T_a^{bcd} , you'll have to write $g^{bb'}T_{ab'}^{cd}$. This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too elegantly concise.

$$a^m \in \mathbb{Z}_+^m$$

$$R_{b_3^{n_3}, a_2^{m_2}}^{a_3^{m_3}, b_2^{n_2}} S_{b_2^{n_2}, a_1^{m_1}}^{a_2^{m_2}, b_1^{n_1}} =$$

(B.21)

$$\text{tr}_{\underline{b}} X_{\underline{a}\underline{b}}^{\underline{b}\underline{d}} = \sum_b X_{ab}^{bd} =$$

(B.22)


(B.23)

Appendix C

Clebsch-Gordan Coefficients: COMING SOON

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d \times d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} C^{d \times d} \quad (\text{C.1})$$

Let $b^{nb} = (b_1, b_2, \dots, b_{nb})$ where $b_i \in Z_{[0, db_i]}$ and $a \in Z_{[1, d_\lambda]}$. Hence,

$$d_\lambda = \prod_{i=1}^{nb} db_i \quad (\text{C.2})$$

$$(C_\lambda)_{a^{b^{nb}}} = a \longleftarrow C_\lambda \begin{matrix} \swarrow b_1 \\ \longleftarrow b_2 \\ \searrow b_{nb} \end{matrix} \quad (\text{C.3})$$

$$\begin{bmatrix} 0 & (C^\dagger)_\lambda^{d \times d_\lambda} & 0 \end{bmatrix}^{d \times d} = (C^\dagger)^{d \times d} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} \quad (\text{C.4})$$

$$(C_\lambda^\dagger)_{b^{nb}}^a = \begin{matrix} b_1 \swarrow \\ b_2 \longleftarrow \\ b_{nb} \searrow \end{matrix} (C_\lambda^\dagger) \longleftarrow a \quad (\text{C.5})$$

More generally, some of the b_i indices may be lowered and their arrows changed to outgoing instead of ingoing. Each b_i represents a different rep (or irrep)

$(C_\lambda^\dagger)_a^{b^{nb}} (C_\lambda)_a^{(b')^{nb}} = (P_\lambda)_{(b')^{nb}}^{b^{nb}}$

$$b_1 \leftarrow b'_1$$

$$b_2 \leftarrow (C_\lambda^\dagger) \leftarrow \sum a \leftarrow C_\lambda \leftarrow b'_2 = b^{nb} \leftarrow P_\lambda \leftarrow (b')^{nb}$$

$$b_{nb} \leftarrow b'_{nb}$$

(C.6)

$(C_\lambda)_{b^{nb}}^{(a')^{na}} (C_\mu^\dagger)_a^{b^{nb}} = \delta(\lambda, \mu) \delta_a^{(a')^{na}}$

$$a \leftarrow C_\lambda \leftarrow \sum b_2 \leftarrow (C_\mu^\dagger) \leftarrow a' = \delta(\mu, \lambda) a \leftarrow \bullet \rightarrow a'$$

$$\sum b_1 \leftarrow (C_\mu^\dagger)$$

$$\sum b_{nb} \leftarrow (C_\mu^\dagger)$$

(C.7)

Chapter 1

**Casimir Operators: COMING
SOON**

Chapter 2

Determinants: COMING SOON

Chapter 3

**General Relativity Nets: COMING
SOON**

Chapter 4

Group Integrals: COMING SOON

Chapter 5

Invariants: COMING SOON

Chapter 6

Levi-Civita Tensor

$$\epsilon^{123\dots p} = \epsilon_{123\dots p} = 1 \quad (6.1)$$

$$\epsilon_{rev(a^p)} = (-1)^{\binom{p}{2}} \epsilon_{a^p} \quad (6.2)$$

where $rev(a^p)$ is the reverse of a^p . $rev(a_1, a_2, \dots, a_p) = (a_p, a_{p-1}, \dots, a_1)$

$$(C_{\mathcal{A}_p})_1^{a^p} = e^{i\phi} \frac{\epsilon^{a^p}}{\sqrt{p!}} = \mathcal{A}_p \leftarrow \begin{array}{c} a_1 \\ \leftarrow a_2 \\ \vdots \\ \leftarrow a_p \end{array} \quad (6.3)$$

$$(C_{\mathcal{A}_p}^\dagger)_{a^p}^1 = e^{-i\phi} \frac{\epsilon_{a^p}}{\sqrt{p!}} = \begin{array}{c} a_1 \leftarrow \mathcal{A}_p \\ a_2 \leftarrow \\ \vdots \\ a_p \leftarrow \end{array} \quad (6.4)$$

$$\boxed{\frac{1}{p!} \epsilon_{a^p} \epsilon^{b^p} = \mathcal{A}_{p a^p}^{b^p}} \quad \begin{array}{c} a_1 \leftarrow \mathcal{A}_p \\ a_2 \leftarrow \\ \vdots \\ a_p \leftarrow \end{array} \begin{array}{c} \mathcal{A}_p \leftarrow b_1 \\ \leftarrow b_2 \\ \vdots \\ \leftarrow b_p \end{array} = \begin{array}{c} a_1 \leftarrow \mathcal{A}_p \leftarrow b_1 \\ a_2 \leftarrow \leftarrow b_2 \\ \vdots \\ a_p \leftarrow \leftarrow b_p \end{array} \quad (6.5)$$

$$\boxed{e^{i2\phi} \frac{1}{p!} \epsilon^{a^n} \epsilon_{a^n} = \delta_1^1 = 1} \quad \left(\begin{array}{c} \mathcal{A}_p \longleftarrow \mathcal{A}_p \\ \hline \longleftarrow \\ \hline \vdots \\ \hline \longleftarrow \\ \hline \end{array} \right) = 1 \quad (6.6)$$

For Convention 1, we will use $\phi = 0$.

For Convention 2, we must choose

$$e^{i2\phi} = (-1)^{\binom{p}{2}} = e^{i\pi \frac{p(p-1)}{2}} \quad (6.7)$$

so

$$\phi = \frac{\pi}{4} p(p-1) \quad (6.8)$$

Chapter 7

**Lie Algebra Definition: COMING
SOON**

Chapter 8

**Lie Algebra Classification, Dynkin
Diagrams: COMING SOON**

Chapter 9

**Orthogonal Groups: COMING
SOON**

Chapter 10

Quantum Shannon Information Theory: COMING SOON

Chapter 11

**Recoupling Equations: COMING
SOON**

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Reducibility: COMING SOON

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Spinors: COMING SOON

Chapter 14

Squashed Entanglement: COMING SOON

Chapter 15

**Symplectic Groups: COMING
SOON**

Chapter 16

Symmetrization and Antisymmetrization: COMING SOON

$(1, 2)$ transposition, swaps 1 and 2, $1 \rightarrow 2 \rightarrow 1$. $(3, 2, 1)$ means $3 \rightarrow 2 \rightarrow 1 \rightarrow 3$. A reordering of $(1, 2, 3, \dots, p)$ is a permutation on p letters. A permutation can be expressed as a product of transpositions $(3, 2, 1) = (3, 2)(2, 1)$ is an even permutation because it can be expressed as a product of an even number of transpositions. An odd permutation can be expressed as a product of an odd number of permutations.

16.1 Symmetrization

$$\mathbb{I}_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} = \begin{array}{c} a_1 \leftarrow b_1 \\ a_2 \leftarrow b_2 \end{array} \quad (16.1)$$

$$(\sigma_{(1,2)})_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} = \begin{array}{c} a_1 \leftarrow \bullet \leftarrow b_1 \\ \updownarrow \\ a_2 \leftarrow \bullet \leftarrow b_2 \end{array} \quad (16.2)$$

$$\mathbb{I} = \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \quad (16.3)$$

$$\sigma_{(1,2)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \end{array} \quad \sigma_{(2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad \sigma_{(1,3)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \end{array} \quad (16.4)$$

$$\sigma_{(1,2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \end{array} \quad (16.5)$$

$$\sigma_{(1,3,2)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad (16.6)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p!} \left\{ \begin{array}{cc} \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \updownarrow \\ \leftarrow & \leftarrow \bullet \leftarrow \\ \vdots & \vdots \\ \leftarrow & \leftarrow \end{array} + \begin{array}{cc} \leftarrow \bullet \leftarrow & \\ \updownarrow & \\ \leftarrow \bullet \leftarrow & \\ \vdots & \vdots \\ \leftarrow & \end{array} + \dots \right\} \quad (16.7)$$

$$\boxed{\mathcal{S}_p^2 = \mathcal{S}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{S}_p \leftarrow \\ \parallel & \parallel & \parallel \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (16.8)$$

$$\boxed{\mathcal{S}_p \mathcal{S}_{[1,q]} = \mathcal{S}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{S}_{[1,q]} \leftarrow & \leftarrow \mathcal{S}_p \leftarrow \\ \parallel & \parallel & \parallel \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (16.9)$$

$$\boxed{\mathcal{S}_p \sigma_{(1,2)} = \mathcal{S}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \bullet \leftarrow \\ \parallel & \updownarrow \\ \leftarrow \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow \\ \vdots & \vdots \\ \leftarrow \leftarrow & \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (16.10)$$

Claim 1

$$\begin{aligned}
& \left(\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} \right) = \frac{1}{p} \left(\begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} \right) + (p-1) \left(\begin{array}{c} \leftarrow \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \leftarrow \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \leftarrow \leftarrow \end{array} \right) \quad (16.11)
\end{aligned}$$

proof: We only prove it for $p = 3$.

$$3! \mathcal{S}_3 = \left(\begin{array}{c} \begin{array}{c} \leftarrow \mathcal{S}_3 \rightarrow \\ \leftarrow \parallel \rightarrow \\ \leftarrow \parallel \rightarrow \end{array} \\ + \begin{array}{c} \begin{array}{ccc} \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \end{array} & \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \end{array} & \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \end{array} \\ \downarrow \quad \downarrow \quad \downarrow \\ \begin{array}{ccc} \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \end{array} \end{array} \end{array} \right) \quad (16.12)$$

$$2! \begin{array}{c} \leftarrow \\ \leftarrow \mathcal{S}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} + \begin{array}{c} \leftarrow \\ \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \right) \quad (16.13)$$

$$3! \begin{array}{c} \leftarrow \mathcal{S}_3 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} - 2! \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \leftarrow \end{array} = \left(\begin{array}{cc} \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} & \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \\ + & + \\ \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} & \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \end{array} \right) \quad (16.14)$$

$$= 2!2! \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{S}_2 \leftarrow \bullet \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \qquad \parallel \\ \leftarrow \qquad \leftarrow \end{array} \quad (16.15)$$

QED

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p} \left(\begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} + (p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \quad \parallel \\ \leftarrow \leftarrow \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \leftarrow \leftarrow \end{array} \right) \quad (16.16)$$

$$= \frac{n+p-1}{p} \left(\begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} \right) \quad (16.17)$$

$$\text{tr}_{\underline{a}_1} \mathcal{S}_p = \frac{n+p-1}{p} \mathcal{S}_{p-1} \quad (16.18)$$

$$\text{tr}_{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k} \mathcal{S}_p = \frac{(n+p-1)(n+p-2) \dots (n=p-k)}{p(p-1) \dots (p-k+1)} \mathcal{S}_{p-k} \quad (16.19)$$

$$d_{\mathcal{S}_p} = \text{tr}_{\underline{a}^p} \mathcal{S}_p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \quad (16.20)$$

For $p = 2$,

$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \quad (16.21)$$

16.2 Antisymmetrization

$$\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p!} \left\{ \begin{array}{c} \leftarrow \leftarrow \quad \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \quad \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \quad \leftarrow \leftarrow \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \leftarrow \leftarrow \quad \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \\ \vdots \quad \vdots \\ \leftarrow \leftarrow \end{array} + \dots \right\} \quad (16.22)$$

$$\boxed{\mathcal{A}_p^2 = \mathcal{A}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \end{array} = \quad (16.23)$$

$$\boxed{\mathcal{A}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{A}_{[1,q]} \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow & \leftarrow \parallel \leftarrow \end{array} = \quad (16.24)$$

$$\boxed{\mathcal{A}_p \sigma_{(1,2)} = -\mathcal{A}_p} \quad \begin{array}{ccc} \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \updownarrow \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \leftarrow & \leftarrow \parallel \leftarrow \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow & \leftarrow \parallel \leftarrow \end{array} = (-1) \quad (16.25)$$

$$\boxed{\mathcal{S}_p \mathcal{A}_q = \mathcal{A}_p \mathcal{S}_q = 0} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{A}_p \leftarrow & \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & = 0 \\ \vdots & \vdots \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \end{array} \quad (16.26)$$

$$\boxed{\mathcal{S}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p \mathcal{S}_{[1,q]} = 0} \quad \begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{A}_{[1,q]} \leftarrow & \leftarrow \mathcal{A}_p \leftarrow & \leftarrow \mathcal{S}_{[1,q]} \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \\ \vdots & \vdots \vdots & \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \end{array} = \quad = 0 \quad (16.27)$$

Claim 2

$$\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} = \frac{1}{p} \left(\begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} + (-1)(p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \end{array} \right) \quad (16.28)$$

proof: We only prove it for $p = 3$.

$$3! \begin{array}{c} \leftarrow \mathcal{A}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \bullet \leftarrow + \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \leftarrow \\ - \leftarrow \bullet \leftarrow \leftarrow - \leftarrow \bullet \leftarrow \leftarrow - \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (16.29)$$

$$2! \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \leftarrow \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow - \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (16.30)$$

$$3! \begin{array}{c} \leftarrow \mathcal{A}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \end{array} - 2! \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} = \left(\begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow + \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow + \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \\ - \leftarrow \bullet \leftarrow \leftarrow - \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \bullet \leftarrow \leftarrow \leftarrow \bullet \leftarrow \leftarrow \end{array} \right) \quad (16.31)$$

$$= (-1)2!2! \begin{array}{c} \leftarrow \bullet \leftarrow \leftarrow \\ \leftarrow \mathcal{A}_2 \leftarrow \bullet \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \leftarrow \parallel \leftarrow \end{array} \quad (16.32)$$

QED

$$\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} = \frac{1}{p} \left(\begin{array}{c} \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} + (-1)(p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} \right) \quad (16.33)$$

$$= \frac{n + (-1)(p-1)}{p} \left(\begin{array}{c} \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} \right) \quad (16.34)$$

$$\text{tr}_{\underline{a}_1} \mathcal{A}_p = \frac{n-p+1}{p} \mathcal{A}_{p-1} \quad (16.35)$$

$$\text{tr}_{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k} \mathcal{A}_p = \frac{(n-p+1)(n-p+2) \dots (n-p+k)}{p(p-1) \dots (p-k+1)} \mathcal{A}_{p-k} \quad (16.36)$$

$$d_{\mathcal{A}_p} = \text{tr}_{\underline{a}^p} \mathcal{A}_p = \frac{\prod_{i=n-p+1}^n i}{p!} \quad (16.37)$$

$$= \frac{\prod_{i=n}^{n-p+1} i}{p!} \quad (16.38)$$

$$= \begin{cases} \frac{n!}{p!(n-p)!} = \binom{n}{p} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} \quad (16.39)$$

For $p = 2 \leq n$,

$$d_{\mathcal{A}_2} = \binom{n}{2} \quad (16.40)$$

$$\mathcal{A}_p = 0 \text{ if } n < p \quad (16.41)$$

For example, for $n = 2$ and $p = 3$

$$\begin{array}{c} |a\rangle \\ \downarrow \\ \mathcal{A}_3 \\ \downarrow \end{array} \begin{array}{c} |a\rangle \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} |b\rangle \\ \downarrow \\ \downarrow \end{array} = \frac{1}{6} \left(\begin{array}{c} \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \right. \\ \left. - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \Leftrightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} - \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \leftarrow \rightarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \right) \quad (16.42)
\end{array}$$

$$\mathcal{A}_3|a, a, b\rangle = \frac{1}{6} \left(\begin{array}{c} |a, a, b\rangle + |a, b, a\rangle + |b, a, a\rangle \\ -|a, b, a\rangle - |a, a, b\rangle - |b, a, a\rangle \end{array} \right) \quad (16.43)$$

$$= 0 \quad (16.44)$$

Chapter 17

Unitary Groups: COMING SOON

17.1 SU(n)

$$m(p, q) = \delta_b^a \sum_{a=1}^n (p_a)^* q_a \quad (17.1)$$

$$\mathbb{1}_{d,b}^{a,c} = \delta_b^a \delta_d^c = \begin{array}{c} d \leftarrow \bullet \longrightarrow c \\ a \longrightarrow \bullet \longrightarrow b \end{array} \quad (17.2)$$

$$\uparrow\downarrow_{d,b}^{a,c} = \delta_d^a \delta_b^c = \begin{array}{cc} d & c \\ \uparrow & \downarrow \\ \bullet & \bullet \\ \downarrow & \uparrow \\ a & b \end{array} \quad (17.3)$$

$$\boxed{\uparrow\downarrow^2 = n \uparrow\downarrow} \quad \begin{array}{c} d \\ \uparrow \\ \bullet \\ \downarrow \\ a \end{array} \quad \begin{array}{c} \curvearrowright \\ \bullet \\ \curvearrowleft \end{array} \quad \begin{array}{c} c \\ \downarrow \\ \bullet \\ \uparrow \\ b \end{array} = n \quad \begin{array}{c} d \\ \uparrow \\ \bullet \\ \downarrow \\ a \end{array} \quad \begin{array}{c} c \\ \downarrow \\ \bullet \\ \uparrow \\ b \end{array} \quad (17.4)$$

$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (17.5)$$

$$\lambda_1 = n$$

$$\boxed{P_1 = \frac{\uparrow\downarrow - n}{0 - n} = 1 - \frac{1}{n} \uparrow\downarrow} \quad \begin{array}{ccc} a & & b \\ & \searrow \quad \swarrow & \\ & P_1 & \\ & \swarrow \quad \searrow & \\ c & & d \end{array} = \begin{array}{ccc} a & \leftarrow \bullet \longrightarrow & b \\ & & \\ c & \leftarrow \bullet \longrightarrow & d \end{array} - \frac{1}{n} \quad \begin{array}{c} a \\ \uparrow \\ \bullet \\ \downarrow \\ c \end{array} \quad \begin{array}{c} b \\ \downarrow \\ \bullet \\ \uparrow \\ d \end{array} \quad (17.6)$$

$$\lambda_2 = 0$$

$$\boxed{P_2 = \frac{\uparrow\downarrow - 0}{n - 0} = \frac{1}{n} \uparrow\downarrow} \quad
\begin{array}{c} a \\ \searrow \\ P_2 \\ \swarrow \\ c \end{array}
\begin{array}{c} b \\ \swarrow \\ P_2 \\ \searrow \\ d \end{array}
= \frac{1}{n}
\begin{array}{c} a \\ \uparrow \\ \bullet \\ \downarrow \\ c \end{array}
\begin{array}{c} b \\ \downarrow \\ \bullet \\ \uparrow \\ d \end{array}
\quad (17.7)$$

$$\text{tr} P_1 = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} - \frac{1}{n} \begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \bullet \downarrow \end{array} \quad (17.8)$$

$$= n^2 - 1 \quad (17.9)$$

$$\text{tr} P_2 = \frac{1}{n} \begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \bullet \downarrow \end{array} \quad (17.10)$$

$$= 1 \quad (17.11)$$

$$\begin{array}{c} b \\ \downarrow \\ (T_i)_a^b = i \text{---} T_i \\ \downarrow \\ a \end{array} \quad (17.12)$$

$$T_i^\dagger = T_i \quad (17.13)$$

Claim 3

$$C_F \delta_a^b = (T_i T_i)_a^b = \frac{n^2 - 1}{n} \delta_a^b \quad (17.14)$$

proof:

$$(T_i T_i)_a^b = \sum_i \begin{array}{c} \text{---} \bullet \text{---} \\ \downarrow \\ a \end{array} \begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \\ b \end{array} \quad (17.15)$$

$$= \sum_i \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \quad (17.16)$$

QED

Chapter 18

**Wigner Coefficients: COMING
SOON**

Chapter 19

**Wigner-Ekart Theorem: COMING
SOON**

Chapter 20

Young Tableau: COMING SOON

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