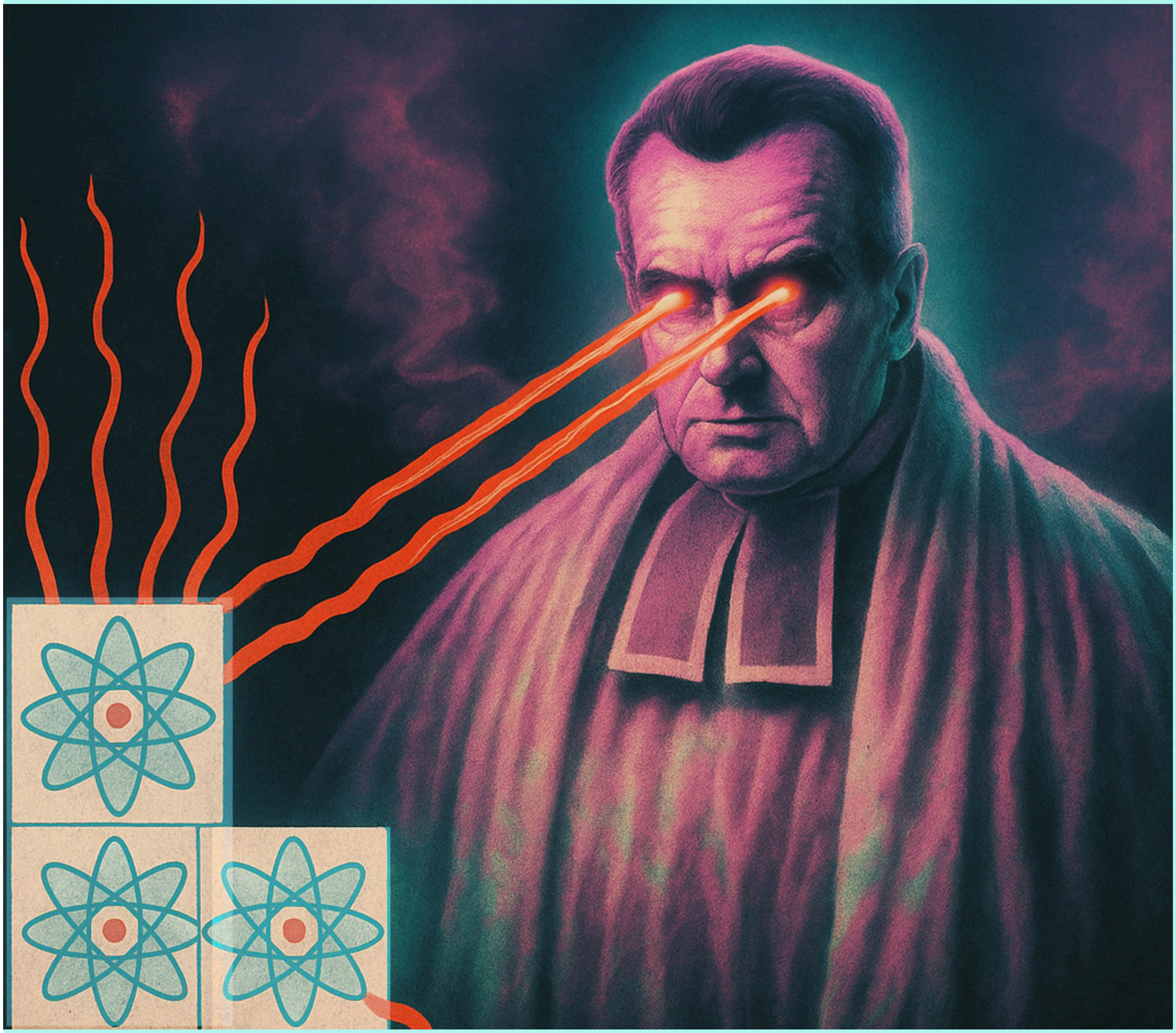
# BAYESUVIUS QUANTICO

## A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

**Bayesuvius Quantico**,
a visual dictionary of Quantum Bayesian Networks

Robert R. Tucci
www.ar-tiste.xyz

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

**Bayes Quantico**
by Robert R. Tucci

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# Notational Conventions and Preliminaries

## A.1 Group

A **group** $\mathcal{G}$ is a set of elements with a multiplication map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ such that

1. the multiplication is **associative** ; i.e.,

$$(ab)c = a(bc) \tag{A.1}$$

   for $a, b, c \in \mathcal{G}$.

2. there exists an **identity element** $e \in \mathcal{G}$ such that

$$ea = ae = a \tag{A.2}$$

   for all $a \in \mathcal{G}$

3. for any $g \in \mathcal{G}$, there exists an **inverse** $a^{-1} \in \mathcal{G}$ such that

$$aa^{-1} = a^{-1}a = e \tag{A.3}$$

The number of elements in any set $S$ is denoted by $|S|$. $|\mathcal{G}|$ is called the **order** of the group.

If multiplication is **commutative** (i.e., $ab = ba$ for all $a, b \in \mathcal{G}$, the group is said to be **abelian**.

A **subgroup** $\mathcal{H}$ of $\mathcal{G}$ is a subset of $\mathcal{G}$ ($\mathcal{H} \subset \mathcal{G}$) which is also a group. It's easy to show that any $\mathcal{H} \subset \mathcal{G}$ is a group if it contains the identity and is **closed under multiplication** (i.e., $ab \in \mathcal{H}$ for all $a, b \in \mathcal{H}$)

## A.2 Group Representation

A **group representation** of a group $\mathcal{G}$ is a map $\phi : \mathcal{G} \to \mathbb{C}^{n\times n}$[1] such that

$$\phi(a)\phi(b) = \phi(ab) \tag{A.4}$$

Such a map is called a **homomorphism**. When a group is defined using matrices, those matrices are called the **defining representation**. The map $\phi$ partitions $\mathcal{G}$ into disjoints subsets (equivalence classes), such that all elements of $\mathcal{G}$ in a disjoint set are represented by the same matrix.

For example, the group of **General Linear Transformations** is defined by

$$GL(n, \mathbb{C}) = \{M \in \mathbb{C}^{n\times n} : \det M \neq 0\} \tag{A.5}$$

## A.3 Vector Space and Algebra over a field $\mathbb{F}$

A vector (or linear) space $\mathcal{V}$ is defined as a set endowed with two operations: vector addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and scalar multiplication $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$, such that

- $\mathcal{V}$ is an abelian group under $+$ with identity $0$ and inverse of $x \in \mathcal{V}$ equal to $-x \in \mathcal{V}$
- For $\alpha, \beta \in \mathbb{F}$ and $x, y \in \mathcal{V}$

$$\begin{aligned}
\alpha(x+y) &= \alpha x + \alpha y && \text{(A.6)}\\
(\alpha+\beta)x &= \alpha x + \beta y && \text{(A.7)}\\
\alpha(\beta x) &= (\alpha\beta)x && \text{(A.8)}\\
1x &= x && \text{(A.9)}\\
0x &= 0 && \text{(A.10)}
\end{aligned}$$

In this book, we will always use either $\mathbb{C}$ or $\mathbb{R}$ for $\mathbb{F}$. Both of these fields are infinite but some fields are finite.

An **algebra** $\mathcal{A}$ is a vector space which, besides being endowed with vector addition and scalar multiplication with which all vector spaces are, it has a **bilinear vector product**. A bilinear vector product is a product that is linear on both sides; i.e.,

$$(\alpha x + \beta y) \cdot z = \alpha x \cdot z + \beta y \cdot z \tag{A.11}$$

and

$$z \cdot (\alpha x + \beta y) = \alpha z \cdot x + \beta z \cdot y \tag{A.12}$$

[1] More generally, the $\mathbb{C}^{n\times n}$ can be replaced by $\mathbb{R}^{n\times n}$ or by $\mathbb{F}^{n\times n}$ for any field $\mathbb{F}$

for $x, y, z \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. The cross product (but not the dot product) for vectors in $\mathbb{R}^3$, the multiplication of 2 complex numbers, and the commutator for square matrices, are all good examples of bilinear vector products.

Let $B = \{\tau_i : i = 1, 2, \ldots, r\}$ be a basis for the vector space $\mathcal{A}$. Then note that $B$ is closed under vector multiplication.

$$\tau_i \cdot \tau_j = \sum_k c_{ij}{}^k \tau_k \tag{A.13}$$

where $c_{ij}{}^k \in \mathbb{C}$. The $c_{ij}{}^k$ are called **structure constants** of $B$.

An **associative algebra** satisfies $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for $x, y, z \in \mathcal{A}$.

- Not associative: cross product for vectors in $\mathbb{R}^3$.
- Associative: the commutator for square matrices and product of complex numbers

## A.4 Tensors

$(x_1, x_2, \ldots, x_n) = x^{:n} \in V^n = \mathbb{C}^{n\times 1}$

Reverse of vector $rev(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_1)$

$y^b = \sum_b g^{ba} x^{:n}$

$(y^1, y^2, \ldots, y^n) = \bar{y}^{:n} \in \bar{V}^n = \mathbb{C}^{n\times 1}$. $V^n$ and $\bar{V}^n$ are **dual vector spaces**.

$M_a{}^b \in \mathbb{C}^{n\times n}$, $a, b \in \mathbb{Z}_{[1,n]}$

Implicit Summation Convention

$$M_a{}^b x_b = \sum_{b=1}^{n} M_a{}^b x_b \tag{A.14}$$

$$\begin{aligned} (M^\dagger)_b{}^a &= (M^*)_a{}^b && \text{(A.15)} \\ &= M_b{}^a \quad \text{(only if } M \text{ is a unitary matrix)} && \text{(A.16)} \end{aligned}$$

For $x_a \in V^n$,

$$(x')_a = M_a{}^b x_b \tag{A.17}$$

For $x^a \in \bar{V}^n$,

$$\begin{aligned} (x'^*)^a &= x^{*b} (M^*)^a{}_b && \text{(A.18)} \\ &= x^{*b} (M^\dagger)_b{}^a && \text{(A.19)} \end{aligned}$$

so

$$(M^{\dagger})_b{}^a = (M^*)^a{}_b \tag{A.20}$$

If the Hermitian conjugate $\dagger$ equals $*T$ where $*$ is complex conjugation and $T$ is transpose,

$$(M^T)_b{}^a = M^a{}_b \tag{A.21}$$

This corresponds to flipping $M$ along its horizontal.

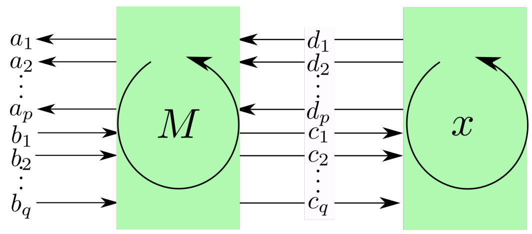


Figure A.1: Index labels for $Mx$ where $M \in \mathbb{C}^{n^{p+q} \times n^{p+q}}$ and $x \in V^{n^p} \otimes \bar{V}^{n^q}$. Note that we list indices in counterclockwise (CC) direction, starting at the top.

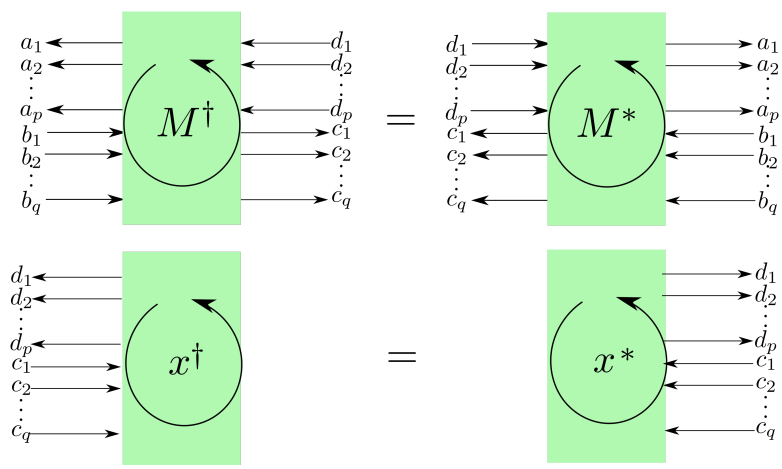


Figure A.2: Index labels for $M^\dagger$ where $M \in \mathbb{C}^{n^{p+q} \times n^{p+q}}$. Note that we list indices in counterclockwise (CC) direction, starting at the top.

Suppose $a_i, b_i, c_i, d_i \in \mathbb{Z}_{[1,n]}$. From Fig.A.1

$$y_{a^{:p}}{}^{b^{:q}} = M_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}} \tag{A.22}$$

$$X_\alpha = X_{a^{:p}}{}^{b^{:q}}, \quad X^\alpha = X_{rev(b^{:q})}{}^{rev(a^{:p})} \tag{A.23}$$

$$x_\alpha = M_\alpha{}^\beta x_\beta \tag{A.24}$$

Hermitian conjugation (see Fig.A.2)

$$\begin{cases} (M^\dagger)_a{}^d = (M^d{}_a)^* \\ (M^\dagger)_\alpha{}^\delta = (M^{rev(\delta)}{}_{rev(\alpha)})^* \end{cases} \tag{A.25}$$

Hermitian matrix

$$M^\dagger = M, \quad \begin{cases} (M^d{}_a)^* = M_a{}^d \\ (M^{rev(\delta)}{}_{rev(\alpha)})^* = M_\alpha{}^\delta \end{cases} \tag{A.26}$$

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Note that for $x \in V^n$, $y \in \bar{V}^n$, and $G \in \mathcal{G} \subset GL(n, \mathbb{C})$,

$$(x')_a (y')^b = G^b{}_c G_a{}^d x_d y^c \tag{A.27}$$

If $x \in V^{n^p} \otimes \bar{V}^{n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a^{\cdot p}}{}^{b^{\cdot q}} = \mathbb{G}_{a^{\cdot p}}{}^{b^{\cdot q}}{}_{rev(c^{\cdot q})}{}^{rev(d^{\cdot p})} x_{d^{\cdot p}}{}^{c^{\cdot q}}, \quad (x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta) \tag{A.28}$$

where we define

$$\mathbb{G}_{a^{\cdot p}}{}^{b^{\cdot q}}{}_{rev(c^{\cdot q})}{}^{rev(d^{\cdot p})} \stackrel{\text{def}}{=} \prod_{i=1}^{q} G^{b_i}{}_{c_i} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \tag{A.29}$$

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An issue that arises with tensors is this: When is it permissible to represent a tensor by $T^{cd}_{ab}$? If we define $T^{cd}_{ab}$ by

$$T^{cd}_{ab} = T_{ab}{}^{cd} \tag{A.30}$$

then it's always permissible. Then one can define tensors like $T_a{}^{bcd}$ as

$$T_a{}^{bcd} = g^{bb'} T_{ab'}{}^{cd} = g^{bb'} T^{cd}_{ab'} \tag{A.31}$$

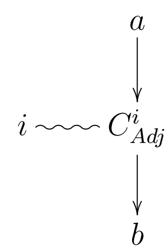
Hence, one drawback of using the notation $T^{cd}_{ab}$ is that if one is interested in using versions of $T^{cd}_{ab}$ with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing $T_a{}^{bcd}$, you'll have to write $g^{bb'} T^{cd}_{ab'}$. This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too elegantly concise.

# Chapter 9

# Lie Algebras

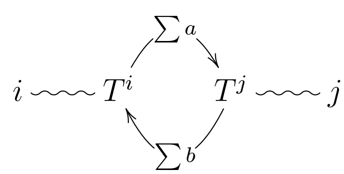
$i \in \mathbb{Z}_{[1,N]}$, $a, b \in \mathbb{Z}_{[1,n]}$

$$(C^i_{Adj})_b{}^a = \frac{1}{\sqrt{K}} (T^i)_b{}^a = \tag{9.1}$$



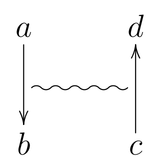
Note that we list the indices of $T^i$ in the counter-clockwise (CC) direction, starting at the $i$ leg. The matrices $T^i$ are called the generators. It's customary to choose them so that they are Hermitian and $K = \frac{1}{2}$. [1]

$$\boxed{(T^i)_b{}^a (T^j)_a{}^b = \mathrm{tr}(T^i T^j) = K\delta(i,j)} \tag{9.2}$$



$= K$ 

$$(P_{Adj})^{a,c}_{b,d} = \sum_i \frac{1}{K} (T^i)_b{}^a (T^i)_d{}^c = \frac{1}{K} \tag{9.3}$$



$H \in V^n \otimes \bar{V}^n$

$$(P_{Adj})^{ac}_{bd} H_c{}^d = \sum_i (T^i)_b{}^a \underbrace{\left[ \frac{1}{K} (T^i)_d{}^c H_c{}^d \right]}_{h_i \in \mathbb{R}} \tag{9.4}$$

[1] For $SU(2)$, it is customary to use $T^i = \frac{1}{2}\sigma_i$, where $\sigma_i$ for $i = 1, 2, 3$ are the Pauli matrices. For $SU(3)$, it is customary to choose $T^i = \frac{1}{2}\lambda_i$ where $\lambda_i$ for $i = 1, 2, \ldots, 8$ are the Gelll-Mann matrices.

$G = 1 + iD \in \mathcal{G}$
$\epsilon_i \in \mathbb{R}$, $|\epsilon_i| << 1$
$D = \sum_i \epsilon_i T^i = V^n \otimes \bar{V}^n$
Recall Eq.(A.28). If $x \in V^{n^p} \otimes \bar{V}^{n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a^{:p}}{}^{b^{:q}} = \mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}}, \quad (x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta) \tag{9.5}$$

where we define

$$\mathbb{G}_\alpha{}^\beta \stackrel{\text{def}}{=} \prod_{i=1}^{q} G^{b_i}{}_{c_i} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \tag{9.6}$$

$$\begin{aligned} \mathbb{G}_\alpha{}^\beta &= 1 + i \sum_j \epsilon_j (\mathbb{T}^j)_\alpha{}^\beta & (9.7) \\ G^{b_i}{}_{c_i} &= 1 + i \sum_j \epsilon_j (T^j)^{b_i}{}_{c_i} & (9.8) \\ G_{a_i}{}^{d_i} &= (G^*)^{d_i}{}_{a_i} = 1 - i \sum_j \epsilon_j (T^{j*})^{d_i}{}_{a_i} = 1 - i \sum_j \epsilon_j (T^j)_{a_i}{}^{d_i} & (9.9) \end{aligned}$$

When $x'_\alpha = x_\alpha$, to first order in $\epsilon_i$,

$$0 = (\mathbb{T}^j)_\alpha{}^\beta x_\beta = \left[ (T^j)^{b_i}{}_{c_i} \frac{1}{\delta^{b_i}_{c_i}} - (T^j)_{a_i}{}^{d_i} \frac{1}{\delta^{d_i}_{a_i}} \right] \delta^{b^{:q}}_{c^{:q}} \delta^{d^{:p}}_{a^{:p}} x_{d^{:p}}{}^{c^{:q}} \tag{9.10}$$

$$\boxed{(\mathbb{T}^j)_\alpha{}^\beta = \left[ (T^j)^{b_i}{}_{c_i} \frac{1}{\delta^{b_i}_{c_i}} - (T^j)_{a_i}{}^{d_i} \frac{1}{\delta^{d_i}_{a_i}} \right] \delta^{b^{:q}}_{c^{:q}} \delta^{d^{:p}}_{a^{:p}}} \tag{9.11}$$

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