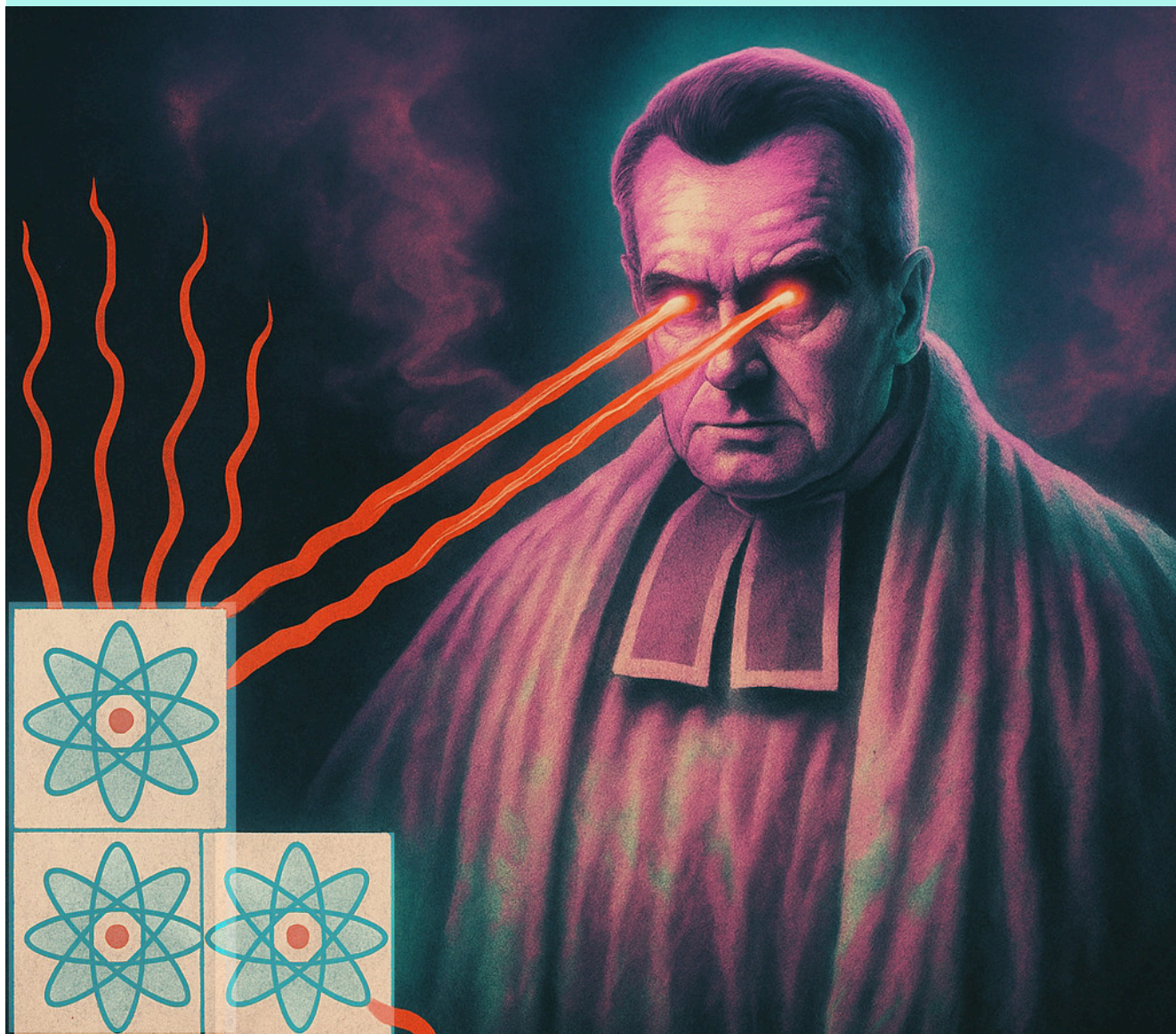


# BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF  
QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

# **Bayesuvious Quantico,** a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download  
the latest version, go to

<https://github.com/rrtucci/bayes-quantico>

## **Bayesuvius Quantico**

by Robert R. Tucci

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# Appendices



# Appendix A

## Notational Conventions and Preliminaries

This book is a sequel to my book entitled “Bayesuvius” (see Ref.[4]). For consistency, I have tried to follow in this book the same notational conventions used in the prior book. If any notation is not defined in this book, check in the prior book. It might be defined there.

### A.1 Set notation

Definitions

$|S|$  = the number of elements in a set  $S$ . (known as its **order, size, length, cardinality**)

$\mathbb{Z}$  = integers

$\mathbb{Z}_{>0}$  = positive integers

$\mathbb{Z}_{[a,b]}$  =  $a, a+1, \dots, b$  for some integers  $a, b$  such that  $a \leq b$

$\mathbb{R}$  = reals

$\mathbb{C}$  = complex numbers

$\mathbb{C}^{n \times m}$  =  $n \times m$  matrices of complex numbers

### A.2 Commutator and Anti-commutator

Let

**commutator of  $A$  and  $B$**

$$[A, B] = AB - BA \tag{A.1}$$

**Anti-commutator of  $A$  and  $B$**

$$[A, B]_+ = AB + BA \tag{A.2}$$

## A.3 Group Theory References

Much of this book deals with Group Theory (GT).

GT is a vast subject. Who would have thought that the simple definition of a group would generate so many elegant and useful results.

GT books by mathematicians are very different from GT books by physicists, even though, of course, they agree on the definitions. Mathematicians are, as to be expected, more rigorous and abstract. But it goes much further than that. Physicists are much more interested in applications to physical systems, especially Quantum Mechanics (QM). Soon after QM was invented, it was realized that Linear Algebra (LA) and GT (especially Group Representation Theory, which combines GT and LA) are extremely relevant and useful in QM. Hermann Weyl, Eugene Wigner, Hans Bethe, Linus Pauling, etc. combined QM and GT to understand the spectra and chemistry of atoms and molecules, and later GT was heavily used in Quantum Field Theory and Particle Physics to devise the Standard Model. Condensed Matter physicists have also used it to understand crystalline solids and to predict quasi particles that can be detected in the lab.

My PhD is in physics so in this book I cover GT topics that are mainly of interests to physicists and engineers. Furthermore, I am nowhere as abstract and rigorous as mathematicians usually are.

My favorite books about GT for physicists are the Elliott & Dawber's (ED) 2 volume series Ref. [2] and Predrag Cvitanovic's Birdtracks book Ref.[1]. I highly recommend both of these references. I think both of them are excellent.

The Birdtracks book explains key concepts in GT representation theory using network diagrams (Cvitanovic calls such diagrams birdtracks) The ED books, on the other hand, do not use birdtracks. They use algebra instead. In fact, most GT books don't use birdtracks either. But since this is a book about visualization using network diagrams (quantum bnets), we use birdtracks. In fact, many of the chapters in this book were heavily influenced by Ref.[1] by Cvitanovic. I hope he doesn't mind. I really love his book.

## A.4 Group

A **group**  $\mathcal{G}$  is a set of elements with a multiplication map  $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$  such that

1. the multiplication is **associative** ; i.e.,

$$(ab)c = a(bc) \tag{A.3}$$

for  $a, b, c \in \mathcal{G}$ .

2. there exists an **identity element**  $e \in \mathcal{G}$  such that

$$ea = ae = a \quad (\text{A.4})$$

for all  $a \in \mathcal{G}$

3. for any  $a \in \mathcal{G}$ , there exists an **inverse**  $a^{-1} \in \mathcal{G}$  such that

$$aa^{-1} = a^{-1}a = e \quad (\text{A.5})$$

$|\mathcal{G}|$  (i.e., number of elements in  $\mathcal{G}$ ) is called the **order** of the group.

If multiplication is **commutative** (i.e.,  $ab = ba$  for all  $a, b \in \mathcal{G}$ ), the group is said to be **abelian**.

A **subgroup**  $\mathcal{H}$  of  $\mathcal{G}$  is a subset of  $\mathcal{G}$  ( $\mathcal{H} \subset \mathcal{G}$ ) which is also a group. It's easy to show that any  $\mathcal{H} \subset \mathcal{G}$  is a group if it contains the identity and is **closed under multiplication** (i.e.,  $ab \in \mathcal{H}$  for all  $a, b \in \mathcal{H}$ )

## A.5 Group Representation

A **group representation** (rep) of a group  $\mathcal{G}$  is a map  $\phi : \mathcal{G} \rightarrow \mathbb{C}^{n \times n1}$  such that

$$\phi(a)\phi(b) = \phi(ab), \quad \phi(e) = I \quad (\text{A.6})$$

where  $e$  is the identity of the group and  $I$  is the identity matrix. Such a map is called a **homomorphism** (because it preserves an operation). The map  $\phi$  partitions  $\mathcal{G}$  into disjoint subsets (equivalence classes), such that all elements of  $\mathcal{G}$  in each disjoint subset are represented by the same matrix.

In this book, we will usually label reps by a Greek letter such as  $\lambda$ , and we will refer to  $\phi(g) = G_\lambda(g) = G_\lambda$  as the **representation matrix** (rep-matrix) of  $g \in \mathcal{G}$ .

One way to specify a representation is to give the effect of each group element  $a \in \mathcal{G}$  on a basis of vectors  $\{|1\rangle, |2\rangle, \dots, |n\rangle\}$ .

$$\phi(a)|i\rangle = \sum_j M_{ij}|j\rangle \implies \langle i|\phi(a)|j\rangle = M_{ij} \quad (\text{A.7})$$

If the map  $\phi$  is 1-1, onto, we call it a **faithful representation**

A **singlet (or invariant or conserved) quantity of group  $\mathcal{G}$**  is a quantity that is invariant under the group transformations  $g \in \mathcal{G}$ . The **trivial or singlet representation** is the rep with  $\phi(g) = 1 = [1]^{1 \times 1}$  for all  $g \in \mathcal{G}$ . The dimension of this rep is  $d_\lambda = 1$ . If  $\phi(g) = \text{diag}(1, 1)$ , this is referred to as two identical copies of a singlet rep. The **singlet projection operator**  $\delta_a^b \delta_c^d$  when acting on  $z_c^d$  gives a  $\text{tr}(z) \text{diag}(1, 1, \dots, 1)$  where  $\text{tr}(z) = z_c^c$ , so it projects to out a singlet quantity. A singlet projection operator  $P_\lambda$  is associated with a singlet rep  $\lambda$  with rep-matrices

---

<sup>1</sup>More generally, the  $\mathbb{C}^{n \times n}$  can be replaced by  $\mathbb{R}^{n \times n}$  or by  $\mathbb{F}^{n \times n}$  for any field  $\mathbb{F}$

$G_\lambda(g) = 1$ . For example,  $P_\lambda = \delta_a^b \delta_c^d$  is associated with a rep  $\lambda$  with rep-matrices  $G_\lambda \otimes G_\lambda^\dagger = 1$

A **1-dimensional (1-D or 1dim) representation** assigns a complex number to each  $g \in \mathcal{G}$ .<sup>2</sup> For example, the rep with  $\phi(g) = e^{i\beta(g)} = [e^{i\beta(g)}]^{1 \times 1}$  for all  $g \in \mathcal{G}$ , where  $\beta(g) \in \mathbb{R}$ . The trivial/singlet rep is a special 1-dim rep. The dimension of this rep is  $d_\lambda = 1$ .

When a group is defined using matrices, those matrices are called the **defining representation** (def-rep). For example, the group of **General Linear Transformations** is defined by

$$GL(n; \mathbb{C}) = \{M \in \mathbb{C}^{n \times n} : \det M \neq 0\} \quad (\text{A.8})$$

The **adjoint representation** (adj-rep) is defined in terms of the structure constants of the Lie Algebra. If the Lie Algebra satisfies  $[T^i, T^j] = if_{ijk}T^k$ , then the adj-rep is given by the matrices with  $i, j$  entries  $M_{ij}^k = -if_{ij}^k$ . Let  $|x\rangle = x_i|T^i\rangle$ . Then

$$[|x\rangle, \cdot]|T^j\rangle = |[x, T^j]\rangle = ix_i f_{ijk}|T^k\rangle \implies \langle T^k|[|x\rangle, \cdot]|T^j\rangle = ix_i f_{ijk} \quad (\text{A.9})$$

**Irreducible representations** (irreps) are defined in Ch. 16

The **fundamental representation** (fun-rep) is defined as the smallest irrep. The def-rep equals the fun-rep for  $SU(n), SO(n), SP(n)$ , but not for  $E_8$ .

The **regular representation** is defined in Chapter 26 for the symmetric group on  $n_b$  letters (or  $n_b$  boxes)  $S_{n_b}$ .

## A.6 Dimensions

In Physics/Math, the term “dimension” can mean various things. For example, it might mean

1. (vector space dimension) the number of vectors in a basis of a vector space
2. (matrix row or column dimensions) the number of rows or columns in a matrix  $M_{a,b}$ .
3. (vector dimension) the number of components of a vector  $x_a$

These 3 uses of the term “dimension” are all closely related but not the same. Sometimes, there are several dimensions at play in the same conversation.

Let MD stand for matrix dimension. A rep  $\lambda$  with rep-matrices  $G_\lambda$  has 2 MDs associated with it that we must distinguish:

---

<sup>2</sup>Note that a 1-dim rep and a tensor with one index  $x_a$ , where  $a = 1, 2, \dots, n$  are not the same thing.  $x_a$  is not even a rep.  $x_a$  is often referred to as an  $n$ -dim vector.  $x_a$  might transform as the  $n$ -dim rep with rep-matrices  $G_b^a$  where  $b = 1, 2, \dots, n$ . Always associate the dim of a rep with the matrix dimension of a square matrix.

1. (adjoint rep MD) the number  $N$  of generators  $T_i$ , where  $i = 1, 2, \dots, N$ .
2. (def rep MD) the number of rows and columns of the square rep-matrix  $G_\lambda$

For example, the Pauli matrices are 3  $2 \times 2$  matrices.

For  $SU(n)$  and  $U(n)$

$n = d_{def}$  = MD of rep-matrices  $G$  in defining rep of  $U(n)$  or  $SU(n)$ . This MD equals  $n$  for both  $U(n)$  and  $SU(n)$ .

$N = d_{adj}$  = MD of rep-matrices  $G$  in adjoint rep of  $U(n)$  or  $SU(n)$ . As we shall prove in Chapter 23,  $N = n^2$  for  $U(n)$  but  $N = n^2 - 1$  for  $SU(n)$ .

## A.7 Vector Space and Algebra Over a Field $\mathbb{F}$

A **vector space (a.k.a. linear space)**  $\mathcal{V}$  over a field  $\mathbb{F}$  is defined as a set  $\mathcal{V}$  endowed with two operations: vector addition  $+: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ , and scalar multiplication  $\mathbb{F} \times \mathcal{V} \rightarrow \mathcal{V}$ , such that

- $\mathcal{V}$  is an abelian group under  $+$  with identity  $0$  and inverse of  $x \in \mathcal{V}$  equal to  $-x \in \mathcal{V}$
- For  $\alpha, \beta \in \mathbb{F}$  and  $x, y \in \mathcal{V}$

$$\alpha(x + y) = \alpha x + \alpha y \quad (\text{A.10})$$

$$(\alpha + \beta)x = \alpha x + \beta x \quad (\text{A.11})$$

$$\alpha(\beta x) = (\alpha\beta)x \quad (\text{A.12})$$

$$1x = x \quad (\text{A.13})$$

$$0x = 0 \quad (\text{A.14})$$

In this book, we will always use either  $\mathbb{C}$  or  $\mathbb{R}$  for  $\mathbb{F}$ . Both of these fields are infinite but some fields are finite.

An **algebra**  $\mathcal{A}$  is a vector space which, besides being endowed with vector addition and scalar multiplication as all vector spaces are, it has a **bilinear vector product**. A bilinear vector product is a product that is linear on both sides; i.e.,

$$(\alpha x + \beta y) \cdot z = \alpha x \cdot z + \beta y \cdot z \quad (\text{A.15})$$

and

$$z \cdot (\alpha x + \beta y) = \alpha z \cdot x + \beta z \cdot y \quad (\text{A.16})$$

for  $x, y, z \in \mathcal{A}$  and  $\alpha, \beta \in \mathbb{C}$ . The cross product (but not the dot product) for vectors in  $\mathbb{R}^3$ , the multiplication of 2 complex numbers, the matrix product or matrix commutator of 2 square matrices, are all good examples of bilinear vector products.

Let  $B = \{\tau_i : i = 1, 2, \dots, r\}$  be a basis for the vector space  $\mathcal{A}$ . Then note that  $\mathcal{A}$  is closed under vector multiplication.

$$\tau_i \cdot \tau_j = \sum_k c_{ij}^k \tau_k \quad (\text{A.17})$$

where  $c_{ij}^k \in \mathbb{C}$ . The  $c_{ij}^k$  are called **structure constants** of  $\mathcal{A}$ . In Dirac notation

$$\tau_i |\tau_j\rangle = |\tau_i \cdot \tau_j\rangle = \sum_k c_{ij}^k |\tau_k\rangle \quad (\text{A.18})$$

$$\langle \tau_k | \tau_i | \tau_j \rangle = c_{ij}^k \quad (\text{A.19})$$

An **associative algebra** satisfies  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$  for  $x, y, z \in \mathcal{A}$ .

- Not associative: cross product for vectors in  $\mathbb{R}^3$ .
- Associative: the matrix product or matrix commutator of 2 square matrices and the product of complex numbers

## A.8 Tensors

Let

$$(x_a) = (x_1, x_2, \dots, x_n) = x^{\cdot n} \in V^n = \mathbb{C}^{n \times 1}$$

$$\textbf{Reverse of vector } rev(x_1, x_2, \dots, x_n) = (x_n, x_{n-1}, \dots, x_1)$$

$$x^b = \sum_a g^{ba} x_a, g^{ab} \text{ is the } \textbf{metric tensor}$$

$(y^b) = (y^1, y^2, \dots, y^n) = y^{\dagger \cdot n} \in V^{\dagger n} = \mathbb{C}^{n \times 1}$ .  $V^n$  is the lower indices vector space and  $V^{\dagger n}$  is its **dual vector space** (i.e., with upper indices).

$$M_a^b \in \mathbb{C}^{n \times n}, a, b \in \mathbb{Z}_{[1, n]}$$

**Implicit Summation Convention**

$$M_a^b x_b = \sum_{b=1}^n M_a^b x_b \quad (\text{A.20})$$

The **Hermitian conjugate**  $\dagger$  equals  $*T$  where  $*$  is complex conjugation and  $T$  is transpose. Hence

$$(M^T)_a^b = M_b^a \quad a \longleftarrow M^T \longleftarrow b = b \longleftarrow M \longleftarrow a \quad (\text{A.21})$$

$$(M^\dagger)_a^b = (M_b^a)^* \quad a \longleftarrow M^\dagger \longleftarrow b = b \longleftarrow M^* \longleftarrow a \quad (\text{A.22})$$

To avoid confusion, follow the golden rule: write  $\dagger$  and  $T$  only before declaring the indices; and write the  $*$  only after declaring the indices. Note that  $\dagger$  does 3 things:

1. reverse the horizontal order of the indices

2. reverse vertical positions of the indices; i.e., lower upper indices and raise lower indices.
3. replace the tensor components by their complex conjugates

Transposing only does items 1 and 2.

If  $M$  is a Hermitian matrix (i.e.,  $M^\dagger = M$ ), then

$$M_a^b = (M_b^a)^* \quad a \longleftarrow M \longleftarrow b = b \longleftarrow M^* \longleftarrow a \quad (\text{A.23})$$

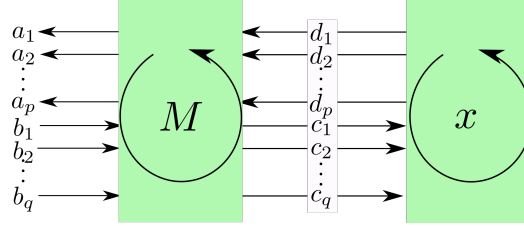


Figure A.1: Index labels for  $Mx$  where  $M \in \mathbb{C}^{n^{p+q} \times n^{p+q}}$  and  $x \in V^{n^p} \otimes V^{\dagger n^q}$ . Note that we list indices in counterclockwise (CC) direction, starting at the top.

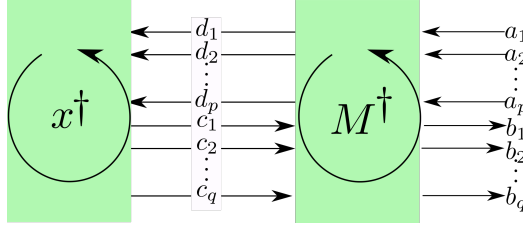


Figure A.2: Index labels for  $x^\dagger M^\dagger$  corresponding to Fig.A.2. Note that we list indices in counterclockwise (CC) direction, starting at the top.

Suppose  $a_i, b_i, c_i, d_i \in \mathbb{Z}_{[1,n]}$ . From Fig.A.1

$$y_{a:p}^{b:q} = M_{a:p}^{b:q}{}_{rev(c:q)}^{rev(d:p)} x_{d:p}^{c:q} \quad (\text{A.24})$$

If we define  $x_\alpha$  and  $x^\alpha$  by

$$x_\alpha = x_{a:p}^{b:q}, \quad x^\alpha = x_{rev(b:q)}^{rev(a:p)} \quad (\text{A.25})$$

then

$$x_\alpha = M_\alpha^\beta x_\beta \quad (\text{A.26})$$

---

Hermitian conjugation (see Fig.A.2)

$$\begin{cases} (M^\dagger)_a^d = (M_d^a)^* \\ (M^\dagger)_\alpha^\delta = (M_{rev(\delta)}^{rev(\alpha)})^* \end{cases} \quad \alpha \longleftarrow M^\dagger \longleftarrow \delta = rev(\delta) \longleftarrow M^* \longleftarrow rev(\alpha) \quad (\text{A.27})$$

Note that  $\dagger$  does 3 things to the birdtrack:

1. It flips the horizontal axis of the figure. (In the algebraic expression of the tensor, this corresponds to reversing the horizontal order of the indices.)
2. For each node, it changes incoming arrows to outgoing ones and vice versa. (In the algebraic expression of the tensor, this corresponds reversing the vertical positions of the indices; i.e., lowering upper indices and raising lower ones.)
3. It replaces the tensor component by its complex conjugate

Hermitian matrix

$$M^\dagger = M, \quad \begin{cases} M_a^d = (M_d^a)^* \\ M_\alpha^\delta = (M_{rev(\delta)}^{rev(\alpha)})^* \end{cases} \quad (\text{A.28})$$

Unitary matrix

$$M^\dagger M = 1, \quad \begin{cases} (M_b^a)^* M_a^c = \delta_b^c \\ (M_{rev(\beta)}^{rev(\alpha)})^* M_\alpha^\gamma = \delta_{rev(\beta)}^\gamma \end{cases} \quad (\text{A.29})$$

---

Note that for  $x \in V^n$ ,  $y \in V^{\dagger n}$ , and  $G \in \mathcal{G} \subset GL(n; \mathbb{C})$ ,

$$(x')_a (y')^b = G^b_c G_a^d x_d y^c \quad (\text{A.30})$$

If  $x \in V^{n^p} \otimes V^{\dagger n^q}$ ,  $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}; \mathbb{C})$ ,

$$(x')_{a:p}^{b:q} = \mathbb{G}_{a:p}^{b:q}{}_{rev(c:q)}^{rev(d:p)} x_{d:p}^{c:q}, \quad (x'_\alpha = \mathbb{G}_\alpha^\beta x_\beta) \quad (\text{A.31})$$

where we define

$$\mathbb{G}_{a:p}^{b:q}{}_{rev(c:q)}^{rev(d:p)} \stackrel{\text{def}}{=} \prod_{i=1}^p G_{a_i}^{d_i} \prod_{i=1}^q G^{\dagger b_i}_{c_i} \quad (\text{A.32})$$

---

An issue that arises with tensors is this: When is it permissible to represent a tensor by  $M_{ab}^{cd}$ ? If we define  $M_{ab}^{cd}$  by

$$M_{ab}^{cd} = M_{ab}{}^{cd} \quad (\text{A.33})$$

then it's always permissible. Then one can define tensors like  $M_a{}^{bcd}$  as

$$M_a{}^{bcd} = g^{bb'} M_{ab'}{}^{cd} = g^{bb'} M_{ab'}^{cd} \quad (\text{A.34})$$



One drawback of using the notation  $M_{ab}^{cd}$  is that if one is interested in using several versions of  $M_{ab}^{cd}$  with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing  $M_a^{bcd}$ , you'll have to write  $g^{bb'} M_{ab'}^{cd}$ . This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too succinct.

## A.9 Permutations

Some well known notation and results about permutations are these.

$(1, 2)$  stands for a **transposition**; i.e., a map that swaps 1 and 2:

$$\left( \begin{array}{cccccc} 1 & 2 & 3 & \dots & p \\ & \searrow & \downarrow & & \downarrow \\ & 1 & 2 & \dots & p \end{array} \right) \quad (\text{A.35})$$

$(3, 2, 1)$  stands for a **permutation**; i.e., a map that maps  $3 \rightarrow 2 \rightarrow 1 \rightarrow 3$ .

$$\left( \begin{array}{cccccc} 1 & 2 & 3 & 4 & \dots & p \\ & \searrow & \nearrow & \downarrow & & \downarrow \\ & 1 & 2 & 3 & \dots & p \end{array} \right) \quad (\text{A.36})$$

Any reordering of  $(1, 2, 3, \dots, p)$  is a permutation of  $p$  letters (or numbers or elements).

The set  $S_p$  of all permutation of  $p$  letters is called the **symmetric group in  $p$  letters**. It has  $p!$  elements (i.e.,  $|S_p| = p!$ ) and is a group, where the group's product is map composition and the group's identity element is the identity map.

Any permutation can be expressed as a product of transpositions, For example,  $(3, 2, 1) = (3, 2)(2, 1)$ .

An **even permutation** such as  $(3, 2, 1)$  can be expressed as a product of an even number of transpositions. An **odd permutation** can be expressed as a product of an odd number of transpositions.

# Appendix B

## Birdtracks

This chapter is based on Cvitanovic's Birdtracks book Ref. [1] and my paper Ref. [5]

The tensor notation discussed in Sec.A.8 is succinct and straightforward, but it's not visually illuminating. The birdtrack notation that we shall discuss in this chapter, is not as succinct as the tensor notation, and can lead to sign errors if you are careless, but it is very visually illuminating. Thus, the tensor and birdtrack notations complement each other well. We will often display results using both, side by side.

### B.1 Classical Bayesian Networks and their Instantiations

Classical Bayesian Networks (bnets) are discussed exhaustively in the first book of this series, Ref.[4]. This is a brief section to remind the reader of how they are defined.

Let PD stand for probability distribution.

We call  $P_{\underline{y}|\underline{x}} : val(\underline{y}) \times val(\underline{x}) \rightarrow [0, 1]$  a **Transition Probability Matrix** (TPM)<sup>1</sup> if

$$\sum_{y \in val(\underline{y})} P_{\underline{y}|\underline{x}}(y|x) = 1 \quad (\text{B.1})$$

In other words, a TPM is a conditional PD. A TPM of the form

$$P(y|x) = \delta(y, f(x)) \quad (\text{B.2})$$

for some function  $f : val(\underline{x}) \rightarrow val(\underline{y})$  is said to be **deterministic**.

A bnet is a **Directed Acyclic Graph** (DAG) with the nodes labelled by random variables<sup>2</sup>. Each bnet stands for a full PD of the node random variables expressed as a product of a TPM for each node. For example, the bnet

---

<sup>1</sup>A TPM is also known as a Conditional Probability Table (CPT).

<sup>2</sup>As in the first volume of this series, we indicate random variables by underlined letters

$$\mathcal{C} = \begin{array}{ccc} & \underline{b} & \\ \swarrow & & \searrow \\ \underline{c} & \longleftarrow & \underline{a} \end{array} \quad (\text{B.3})$$

stands for the full PD

$$P(a, b, c) = P(c|b, a)P(b|a)P(a) \quad (\text{B.4})$$

Bnets do not have free indices because their nodes are labelled by random variables. It is convenient to draw the DAG for a bnet but with the underlining removed from the random variables, and then to assign a numerical value to this new DAG. The resultant DAG now has free indices. We call it an **instantiation of the bnet**. For example, from the bnet  $\mathcal{C}$  of Eq.(B.3), we get the instantiation<sup>3</sup>

$$P(a, b, c) = P(c|b, a)P(b|a)P(a) = \begin{array}{ccc} & b & \\ \swarrow & & \searrow \\ c & \longleftarrow & a \end{array} P(a) \quad (\text{B.5})$$

Let  $a^{:2} = (a_1, a_2)$ . Based on the bnet  $\mathcal{C}$  of Eq.(B.3), define a new bnet  $\mathcal{C}'$  as follows

$$\mathcal{C}' = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow \\ \underline{c} & \longleftarrow \underline{a_2} & \underline{a^{:2}} \end{array} \quad (\text{B.6})$$

$\mathcal{C}'$  represents the the full PD

$$P(a^{:2}, b, c) = P(c|b, a_2)P(a_2|a^{:2})P(b|a_1)P(a_1|a^{:2})P(a^{:2}) \quad (\text{B.7})$$

The 2 new nodes  $\underline{a_1}$  and  $\underline{a_2}$  of bnet  $\mathcal{C}'$  are called **marginalizer nodes**. We assign to them the following TPMs (printed in blue):

$$P[a'_i|\underline{a^{:2}} = (a_1, a_2)] = \delta(a'_i, a_i) \quad (\text{B.8})$$

for  $i = 1, 2$ . We can also define an instantiation of  $\mathcal{C}'$  as follows:

$$P'(a^{:2}, b, c) = \begin{array}{ccc} & b & \\ \swarrow & & \nwarrow \\ c & \longleftarrow \underline{a_2} & \underline{a^{:2}} \end{array} P(a^{:2}) \quad (\text{B.9})$$

---

<sup>3</sup>Note that we don't include the root node probabilities as part of the graph value. Thus,  

$$P(a, b) = \underbrace{b \longleftarrow a}_{P(b|a)} P(a)$$

## B.2 Quantum Bayesian Networks and their Instantiations

As far as I know, Quantum Bayesian Networks (qbnets) were invented by me in Ref.[5].

qbnets are closely analogous to classical bnets, but the TPM are replaced by **Transition Amplitude Matrices (TAM)**.

Let PA stand for probability amplitude.

We call  $A_{y|x} : \text{val}(\underline{y}) \times \text{val}(\underline{x}) \rightarrow \mathbb{C}$  a TAM if

$$\sum_{y \in \text{val}(\underline{y})} |A(y|x)|^2 = 1 \quad (\text{B.10})$$

Note that if  $A$  is the matrix with entries  $\langle y|A|x \rangle = A(y|x)$ , then

$$\langle x|A^\dagger A|x \rangle = \sum_{y \in \text{val}(\underline{y})} |A(y|x)|^2 = 1 \quad (\text{B.11})$$

If  $A$  is a unitary matrix, then  $A^\dagger A = AA^\dagger = 1$  so “half” ( $A^\dagger A = 1$ ) of the definition of unitary matrix is satisfied by a TAM. If both halves were satisfied,  $A$  would have to be a square matrix.

A qbnet is a DAG with the nodes labelled by random variables. Each qbnet stands for a full PA of the node random variables expressed as a product of a TAM for each node. For example, the qbnet

$$\mathcal{Q} = \begin{array}{c} b \\ \swarrow \quad \searrow \\ \underline{c} \longleftarrow \underline{a} \end{array} \quad (\text{B.12})$$

stands for the full PA

$$A(a, b, c) = A(c|b, a)A(b|a)A(a) \quad (\text{B.13})$$

Qbnets do not have free indices because their nodes are labelled by random variables. It is convenient to draw the DAG for a qbnet but with the underlining removed from the random variables, and then to assign a numerical value to this new DAG. The resultant DAG now has free indices. We call it an **instantiation of the qbnet**. For example, from the bnet  $\mathcal{Q}$  of Eq.(B.12), we get the instantiation

$$A(a, b, c) = A(c|b, a)A(b|a)A(a) = \begin{array}{c} b \\ \swarrow \quad \searrow \\ c \longleftarrow a \end{array} A(a) \quad (\text{B.14})$$

Let  $a^{:2} = (a_1, a_2)$ . Based on the qbnet  $\mathcal{Q}$  of Eq.(B.12), define a new qbnet  $\mathcal{Q}'$  as follows

$$\mathcal{Q}' = \begin{array}{c} b \\ \swarrow \quad \nwarrow \\ \underline{c} \xleftarrow{a_2} \underline{a}^{i2} \end{array} \quad (\text{B.15})$$

$\mathcal{Q}'$  represents the the full PA

$$A(a^{i2}, b, c) = A(c|b, a_2)A(a_2|a^{i2})A(b|a_1)A(a_1|a^{i2})A(a^{i2}) \quad (\text{B.16})$$

The 2 new nodes  $\underline{a}_1$  and  $\underline{a}_2$  of qbnet  $\mathcal{Q}'$  are called **marginalizer nodes**. We assign to them the following TAMs (printed in blue):

$$A[a'_i|\underline{a}^{i2} = (a_1, a_2)] = \delta(a'_i, a_i) \quad (\text{B.17})$$

for  $i = 1, 2$ . We can also define an instantiation of  $\mathcal{Q}'$  as follows:

$$A(a^{i2}, b, c) = \begin{array}{c} b \\ \swarrow \quad \nwarrow \\ \underline{c} \xleftarrow{a_2} \underline{a}^{i2} \end{array} A(a^{i2}) \quad (\text{B.18})$$

### B.3 Birdtracks

Tensors written in **algebraic notation** such as  $T_a^{bc}$  were already discussed in Section A.8

Birdtracks are a DAG used to represent algebraic tensor equations. The nodes of the DAG are labelled by tensors and the arrows are labeled by the indices of the tensors: upper indices of a tensor are pictured as incoming arrows of the node, and lower indices as outgoing arrows.

We've already discussed in Section A.8 what we will call the **Counter Clock-wise (CC) convention** of drawing birdtrack nodes. Now that we have discussed classical and quantum bnets, we would like to introduce an equivalent, more bnet like, convention that we will call the **Fully Label (FL) convention**. Cvitanovic's birdtracks book Ref.[1] uses the CC convention. We will use both. No confusion will arise, as long as it is clear from context which convention is being used.

Next we review the CC convention and then describe the FL convention for the first time.

#### 1. CC convention

In the CC convention, we must specify for each the node, which arrow is first, and then the CC order in which the arrows enter or leave the node is drawn so that it reproduces the horizontal order of the indices in the algebraic notation for the tensor. We shall often indicate the first arrow by coloring it green.

For example,

$$\delta(b, a) = \mathbb{1}(a = b) = \delta_a^b = a \leftarrow \bullet \rightarrow b \quad (\text{B.19})$$

$$X_{ab}{}^c = \begin{array}{c} a \xleftarrow{\text{green}} X_{\underline{ab}}{}^{\underline{c}} \\ \swarrow \nearrow \\ b \quad c \end{array} \quad (\text{B.20})$$

In this picture, the green arrow indicates which tensor index is first horizontally in the algebraic representation of the tensor.

Sometimes there is no need to indicate which arrow is first by drawing it in green, because all choices give the same number. For example, in the birdtracks for  $\delta_a^b$ , starting with the incoming arrow or the outgoing arrow leads to the same number. Likewise, with the totally symmetric tensor  $d_{ijk}$  (doesn't change sign under swap of any two indices) and the totally antisymmetric tensor  $f_{ijk}$  (changes sign under swap of any two indices), it doesn't matter if one starts at  $i$ ,  $j$  or  $k$ . This is shown below.

$$\begin{array}{c} i \\ \text{green wavy} \\ d \\ \swarrow \searrow \\ j \quad k \end{array} = d_{ijk} = d_{jki} = \begin{array}{c} i \\ \text{wavy} \\ d \\ \swarrow \searrow \\ j \quad k \end{array} \quad (\text{B.21})$$

$$\begin{array}{c} i \\ \text{green wavy} \\ f \\ \swarrow \searrow \\ j \quad k \end{array} = f_{ijk} = f_{jki} = \begin{array}{c} i \\ \text{wavy} \\ f \\ \swarrow \searrow \\ j \quad k \end{array} \quad (\text{B.22})$$

Note that for a totally antisymmetric tensor with an even number of indices, the beginning arrow can change the sign. Indeed,

$$\begin{array}{c} i \quad l \\ \text{green wavy} \quad \text{wavy} \\ f \\ \swarrow \searrow \\ j \quad k \end{array} = f_{ijkl} = -f_{jkli} = (-1) \begin{array}{c} i \quad l \\ \text{wavy} \quad \text{wavy} \\ f \\ \swarrow \searrow \\ j \quad k \end{array} \quad (\text{B.23})$$

## 2. FL convention

In the FL convention, the arrows must be labelled by random (underlined) variables, and the names of the nodes must also indicate by underlined variables what is the the order of the indices

For example,

$$\delta(b, a) = \mathbb{1}(a = b) = \delta_a^b = a \leftarrow \bullet b \quad (\text{B.24})$$

$$\langle a, b | X_{ab}^c | c \rangle = X_{ab}^c = \begin{array}{c} \underline{a} = a \leftarrow X_{ab}^c \\ \swarrow \quad \nearrow \\ \underline{b} = b \\ \nwarrow \quad \nearrow \\ \underline{c} = c \end{array} \quad (\text{B.25})$$

Sometimes, we will denote this node simply by  $X$ . This is okay as long as we state that  $X = X_{ab}^{cd}$ , and we don't start using  $X$  to represent a different version of  $X_{ab}^{cd}$  with some of the indices raised or lowered or their horizontal order changed.

Often, we will write simply  $a$  instead of  $\underline{a} = a$ . This is similar to the shorthand  $P(\underline{a} = a) = P(a)$ .

Note that, unlike in the CC convention, in the FL convention, the CC order in which the arrows enter or leave the node, is meaningless. All orders are equivalent. This is akin to the notation for bnets and qbnets.

---

If we don't follow either convention CC or FL, we won't be able to distinguish between the graphical representations of  $T^{1,2,3}$  and  $T^{2,1,3}$ , for example.

Two other features of the CC and FL conventions that we would like to discuss before ending this section are how to indicate

- **noncyclic index contractions**; i.e., index contractions (i.e., summations) that do not introduce cycles, and
- **traces**; i.e., index contractions that do introduce cycles.

Noncyclic index contractions will be indicated by an arrow connecting two nodes, with the symbol  $\sum a$  midway in the arrow if the index  $a$  is being contracted. For simplicity, we often omit writing the  $\sum a$  altogether.

For example (in CC convention),

$$X_{ab}^c = \begin{array}{c} a \leftarrow X_{ab}^c \\ \swarrow \quad \nearrow \\ b \\ \nwarrow \quad \nearrow \\ c \end{array}, \quad (X^\dagger)_c^{ba} = \begin{array}{c} (X^\dagger)_c^{ba} \leftarrow a \\ \nwarrow \quad \nearrow \\ b \\ \swarrow \quad \nwarrow \\ c \end{array} \quad (\text{B.26})$$

$$(X^\dagger)_c^{ba} X_{ab}^c = \begin{array}{c} (X^\dagger)_c^{ba} \xleftarrow{\quad} \sum a \xleftarrow{\quad} X_{ab}^c \\ \quad \swarrow \quad \searrow \quad \nearrow \\ \quad \quad \sum b \quad \quad \\ \quad \searrow \quad \nearrow \\ \quad \quad \sum c \end{array} \quad (\text{B.27})$$

$$= \begin{array}{c} X^\dagger \xleftarrow{\quad} X \\ \quad \swarrow \quad \searrow \quad \nearrow \\ \quad \quad \quad \end{array} \quad (\text{B.28})$$

Birdtracks are DAGs until we are asked to take a trace of one of their indices. Tracing ruins their acyclicity. The acyclicity of DAGs is mandated by causality. The acyclicity of tracing hints to its acausal (or feedback) nature.

In this book, we will indicate tracing with a red undirected arrow. For example, in the CC convention,

$$\text{tr}_{\underline{b}} X_{ab}^{\underline{b}} = \sum_b X_{ab}^b = \begin{array}{c} a \xleftarrow{\quad} X_{ab}^c \\ \quad \swarrow \quad \searrow \\ \quad \quad \quad \end{array} \quad (\text{B.29})$$

If

$$R_{b_3 a_2}^{x a_3 b_2} S_{x' b_2 a_1}^{a_2 b_1} = \begin{array}{c} x \downarrow \quad \quad \quad x' \uparrow \\ b_3 \leftarrow R \xleftarrow{\quad} \sum b_2 \xleftarrow{\quad} S \leftarrow b_1 \\ \quad \nearrow \quad \searrow \quad \nearrow \quad \searrow \\ a_3 \quad \quad \quad \sum a_2 \quad \quad \quad a_1 \end{array} \quad (\text{B.30})$$

then

$$\text{tr}_{\underline{x}} R_{b_3 a_2}^{\underline{x} a_3 b_2} S_{\underline{x} b_2 a_1}^{a_2 b_1} = \begin{array}{c} \quad \quad \quad \text{---} \quad \quad \quad \\ \downarrow \quad \quad \quad \uparrow \\ \leftarrow R \xleftarrow{\quad} S \leftarrow \\ \quad \nearrow \quad \searrow \quad \nearrow \quad \searrow \end{array} \quad (\text{B.31})$$

When using the FL convention, it becomes clear that birdtracks can be understood as instantiations of qbnets, provided that we weaken slightly the definition



of qbnets, by not requiring that the unitarity condition Eq.(B.10) be satisfied. Also, the outgoing arrows of the nodes of a birdtrack must be understood as the result of marginalizer nodes. For example, if the arrows leaving a node are labelled  $a_1$  and  $a_2$ , then these two arrows must be understood as the result of marginalizing an arrow  $a{:}^2 = (a_1, a_2)$ .

# Appendix C

## Clebsch-Gordan Series Tables

In this Appendix, we present jpgs of some interesting tables from the Birdtracks book Ref.[1] by Cvitanovic. The tables give Clebsch-Gordan series (tensor product decompositions) of  $SU(n)$  and  $SO(n)$

A black (white) filled rectangle with  $p$  incoming and  $p$  outgoing legs represents the anti-symmetrizer  $\mathcal{A}_p$  (symmetrizer  $\mathcal{S}_p$ ).

Thick lines represent particles in the defining representation. These lines range over  $1, 2, \dots, n$ .




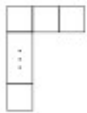

Thin lines represent particles in the adjoint rep. (In this book we represent such particles by a wavy line  $\sim\sim\sim$ .)

$Y_a$	$P_{Y_a}$	$d_{Y_a}$
$\begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline \end{array}$		$\frac{n(n+1)(n+2)}{6}$
$\begin{array}{ c c } \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}$		$\frac{n(n^2-1)}{3}$
$\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}$		
$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$		$\frac{(n-2)(n-1)n}{6}$
$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$		$n^3$

Table 9.1 Reduction of 3-index tensor. The last row shows the direct sum of the Young tableaux, the sum of the dimensions of the irreps adding up to  $n^3$ , and the sum of the projection operators adding up to the identity as verification of completeness (3.51).

Young tableaux	$\square \times \square = \bullet + \begin{array}{ c } \hline \square \\ \hline \end{array} + \begin{array}{ c c } \hline \square & \square \\ \hline \end{array}$
Dynkin labels	$(10 \dots) \times (10 \dots) = (00 \dots) + (010 \dots) + (20 \dots)$
Dimensions	$n^2 = 1 + \frac{n(n-1)}{2} + \frac{(n+2)(n-1)}{2}$
Dynkin indices	$2n \frac{1}{n-2} = 0 + 1 + \frac{n+2}{n-2}$
Projectors	$\begin{array}{ c } \hline \text{---} \\ \hline \end{array} = \frac{1}{n} \begin{array}{ c } \hline \text{---} \\ \hline \end{array} + \begin{array}{ c } \hline \text{---} \\ \hline \end{array} + \left\{ \begin{array}{ c } \hline \text{---} \\ \hline \end{array} - \frac{1}{n} \begin{array}{ c } \hline \text{---} \\ \hline \end{array} \right\}$

Table 10.1  $SO(n)$  Clebsch-Gordan series for  $V \otimes V$ .

	$\underline{A} \otimes \underline{q} =$	$\underline{V}_1$	$\oplus$	$\underline{V}_2$	$\oplus$	$\underline{V}_3$			
Dynkin labels	$(10 \dots 1) \otimes (10 \dots) =$	$(10 \dots)$	$\oplus$	$(200 \dots 01)$	$\oplus$	$(010 \dots 01)$			
		$\otimes$		$=$		$+$		$+$	
Dimensions:	$(n^2 - 1)n =$	$n$	$+$	$\frac{n(n-1)(n+2)}{2}$	$+$	$\frac{n(n+1)(n-2)}{2}$			
Indices:	$n + \frac{n^2-1}{2n} =$	$\frac{1}{2n}$	$+$	$\frac{(n+2)(3n-1)}{4n}$	$+$	$\frac{(n-2)(3n+1)}{4n}$			

---

SU(3) example:

Dimensions:	$8 \cdot 3 =$	$3$	$+$	$15$	$+$	$6$
Indices:	$13/3 =$	$1/6$	$+$	$10/3$	$+$	$5/6$

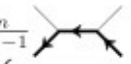
---

SU(4) example:

Dimensions:	$15 \cdot 4 =$	$4$	$+$	$36$	$+$	$20$
Indices:	$47/8 =$	$1/8$	$+$	$33/8$	$+$	$13/8$

---

Projection operators:

$P_1 = \frac{n}{n^2-1}$ 


$P_2 = \frac{1}{2} \left\{ \begin{array}{l} \text{Diagram of two parallel lines} \\ \text{Diagram of two lines crossing} \end{array} + \frac{1}{n+1} \begin{array}{l} \text{Diagram of two lines with a loop} \end{array} \right\}$

$P_2 = \frac{1}{2} \left\{ \begin{array}{l} \text{Diagram of two parallel lines} \\ \text{Diagram of two lines crossing} \end{array} - \frac{1}{n-1} \begin{array}{l} \text{Diagram of two lines with a loop} \end{array} \right\}$

Table 9.3  $SU(n)$   $V \otimes A$  Clebsch-Gordan series.













Young tableaux		$\times$		$=$		$+$		$+$								
Dynkin labels	$(010 \dots) \times (100 \dots)$	$=$	$(100 \dots)$	$+$	$(0010 \dots)$	$+$	$(110 \dots)$									
Dimensions	$\frac{n^2(n-1)}{2}$	$=$	$n$	$+$	$\frac{n(n-1)(n-2)}{6}$	$+$	$\frac{n(n^2-4)}{3}$									
$SO(3)$	<u>9</u>	$=$	<u>3</u>	$+$	<u>1</u>	$+$	<u>5</u>									
$SO(4)$	<u>24</u>	$=$	<u>4</u>	$+$	<u>4</u>	$+$	<u>16</u>									
Projectors		$=$	$\frac{2}{n-1}$		$+$	$\frac{1}{3!} \left\{$		$- 2$		$\left. \right\} + \frac{2}{3!} \left\{$		$+$		$- \frac{3}{n-1}$		$\left. \right\}$

Table 10.2  $SO(n)$   $A \otimes V$  Clebsch-Gordan series.

# Chapter 1

## Casimir Operators

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

The term **Casimir operator** will be used to refer to 2 types of operators: a Casimir matrix or a Casimir sun

**Casimir matrix**

Examples:

$$M_2 = \begin{array}{c} \text{~~~~~} \\ \leftarrow T_i \text{~~~~~} T_i \leftarrow \end{array} \quad (1.1)$$

$$M_4 = \begin{array}{c} \begin{array}{ccccccc} & \text{~~~~~} & & & & & \\ \leftarrow & T_i & \rightarrow & T_j & \rightarrow & T_k & \rightarrow & T_l & \rightarrow \\ & \text{~~~~~} & & & & & \\ \leftarrow & T_i & \leftarrow & T_j & \leftarrow & T_l & \leftarrow & T_k & \leftarrow \end{array} \end{array} \quad (1.2)$$

Casimir matrices are invariant matrices so they satisfy

$$0 = [T_r, M_4] \quad \begin{array}{c} \text{~~~~~} \\ \leftarrow T_r \text{~~~~~} T_i \text{~~~~~} T_j \text{~~~~~} T_l \text{~~~~~} T_k \leftarrow \end{array} - \begin{array}{c} \text{~~~~~} \\ \leftarrow T_i \text{~~~~~} T_j \text{~~~~~} T_l \text{~~~~~} T_k \text{~~~~~} T_r \leftarrow \end{array} \quad (1.3)$$

Because Casimir matrices are invariant matrices, they commute with each other. For example,

$$M_2 M_4 = M_4 M_2 \quad (1.4)$$

**Casimir sun.** By this we mean a tensor consisting of a loop of fundamental particles with gluons (rays) emanating from it; i.e., this:

$$\text{tr}(T_i T_j \dots T_l) = \begin{array}{c} \begin{array}{ccccccc} & \text{~~~~~} & & & & & \\ \leftarrow & T_i & \rightarrow & T_j & \rightarrow & \dots & \rightarrow & T_l & \rightarrow \\ & \text{~~~~~} & & & & & \\ & \text{~~~~~} & & & & & \end{array} \end{array} \quad (1.5)$$

Note that the Lie Algebra commutation relations can be applied to a Casimir sun:

$$\begin{aligned} \left( \begin{array}{c} \xrightarrow{\quad} T_i \rightarrow T_j \rightarrow \dots \rightarrow T_l \rightarrow \\ \text{wavy lines} \end{array} \right) - \left( \begin{array}{c} \xrightarrow{\quad} T_j \rightarrow T_i \rightarrow \dots \rightarrow T_l \rightarrow \\ \text{crossed wavy lines} \end{array} \right) = \left( \begin{array}{c} \xrightarrow{\quad} T_k \rightarrow \dots \rightarrow T_l \rightarrow \\ \text{crossed wavy lines with 'if' label} \end{array} \right) \end{aligned} \quad (1.6)$$

Note also that we can define a symmetrized version of a Casimir sun:

$$h_{i_1 i_2 \dots i_p} = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} \text{tr}(T_{\sigma(i_1)} T_{\sigma(i_2)} \dots T_{\sigma(i_p)}) = \begin{array}{c} \xrightarrow{\quad} T_{i_1} \rightarrow T_{i_2} \rightarrow \dots \rightarrow T_{i_p} \rightarrow \\ \text{wavy lines} \\ \text{double line } \mathcal{S}_p \end{array} \quad (1.7)$$

## 1.1 Independent Casimirs of Simple Lie Groups

So as not to have any gluon free indices, it is convenient to contract with a matrix  $M$ , all the outgoing gluons of a Casimir sun. Let

$$M = \sum_i T_i x_i \quad \leftarrow M \leftarrow = \sum_i x_i \quad \leftarrow \begin{array}{c} \text{wavy line} \\ T_i \end{array} \quad (1.8)$$

Then

$$\text{tr}(M^k) = \overleftarrow{M} \leftarrow M \quad \dots \leftarrow M \leftarrow \quad (1.9)$$

$$= \sum_{i_1 i_2 \dots i_k} \overleftarrow{T_{i_1} \leftarrow T_{i_2} \quad \dots \leftarrow T_{i_k} \leftarrow} \quad x_{i_1} x_{i_2} \dots x_{i_k} \quad (1.10)$$

$$= \sum_{i_1 i_2 \dots i_k} \overleftarrow{T_{i'_1} \leftarrow T_{i'_2} \quad \dots \leftarrow T_{i'_k} \leftarrow} \quad \begin{array}{c} \text{=} \\ \text{=} \\ \text{=} \end{array} \mathcal{S}_k \quad x_{i_1} x_{i_2} \dots x_{i_k} \quad (1.11)$$

$\underbrace{i_1 \quad i_2 \quad \dots \quad i_k}_{h_{i_1 i_2 \dots i_k}}$

Recall Eq.(2.22) for the general characteristic equation of a matrix  $M$

$$0 = \sum_{k=0}^n (-1)^k \left( \text{tr}_{1\dots n-k} \mathcal{A}_{n-k} M^{\otimes n-k} \right) M^k \quad (1.12)$$

$$= \begin{cases} M^n \\ -M^{n-1}(\text{tr} M) \\ +M^{n-2}(\text{tr}_{1\dots 2} \mathcal{A}_2 M^{\otimes 2}) \\ \dots \\ (-1)^n \det(M) \end{cases} \quad (1.13)$$

Note that that  $\text{tr}_{12} \mathcal{A}_2 M^{\otimes 2}$  can be expressed in terms of  $\text{tr}(M)$  and  $\text{tr}(M^2)$ . Likewise,  $\text{tr}_{123} \mathcal{A}_3 M^{\otimes 3}$  can be expressed in terms of  $\text{tr}(M)$ ,  $\text{tr}(M^2)$  and  $\text{tr}(M^3)$ . If we take the trace of the above equation, we get an equation constraining  $\text{tr}(M^k)$  for  $k = 1, 2, \dots n$ .

The **Betti number** of the Casimir  $\text{tr}(M^k) \neq 0$  is the integer  $k$ . Table 1.1 gives all the Betti numbers for the simple Lie Algebras. Note that the Betti numbers in Table 1.1 are all even except for  $SU(n)$ .

For all simple Lie Groups except for  $SU(n)$ , there is a invertible symmetric or skew-symmetric bilinear invariant matrix  $g_{ab}$  satisfying  $g_{ab} g^{bc} = \delta_a^c$ . Hence

$A_r = \mathfrak{su}(r+1)$	$2, 3, \dots, r+1$
$B_r = \mathfrak{so}(2r+1)$	$2, 4, 6, \dots, 2r$
$C_r = \mathfrak{sp}(2r)$	$2, 4, 6, \dots, 2r$
$D_r = \mathfrak{so}(2r)$	$2, 4, \dots, 2r-2, 2r$
$G_2$	$2, 6$
$F_4$	$2, 6, 8, 12$
$E_6$	$2, 5, 6, 8, 9, 12$
$E_7$	$6, 8, 10, 12, 14, 18$
$E_8$	$8, 12, 14, 18, 20, 24, 30$

Table 1.1: Betti numbers for the simple Lie Algebras

$$\begin{array}{c}
g \\
\uparrow \\
g \\
\downarrow \\
\sim T_{i_2} \\
\downarrow \\
\sim T_{i_3} \\
\downarrow \\
\sim T_{i_4} \\
\downarrow \\
\sim T_{i_5} \\
\downarrow
\end{array}
= (-1)
\begin{array}{c}
g \\
\uparrow \\
\sim T_{i_1} \\
\uparrow \\
g \\
\downarrow \\
\sim T_{i_3} \\
\downarrow \\
\sim T_{i_4} \\
\downarrow \\
\sim T_{i_5} \\
\downarrow
\end{array}
= (-1)^2
\begin{array}{c}
g \\
\uparrow \\
\sim T_{i_1} \\
\uparrow \\
\sim T_{i_2} \\
\uparrow \\
g \\
\downarrow \\
\sim T_{i_4} \\
\downarrow \\
\sim T_{i_5} \\
\downarrow
\end{array}
= (-1)^3
\begin{array}{c}
g \\
\uparrow \\
\sim T_{i_1} \\
\uparrow \\
\sim T_{i_2} \\
\uparrow \\
\sim T_{i_3} \\
\uparrow \\
g \\
\downarrow \\
\sim T_{i_5} \\
\downarrow
\end{array}
= (-1)^4
\begin{array}{c}
g \\
\downarrow \\
\sim T_{i_1} \\
\uparrow \\
\sim T_{i_2} \\
\uparrow \\
\sim T_{i_3} \\
\uparrow \\
\sim T_{i_4} \\
\uparrow \\
g \\
\downarrow
\end{array}
\quad (1.14)$$

As illustrated in Eq.(1.14), if such a  $g^{ab}$  exists, a Casimir  $\text{tr}(M^k)$  equals itself times  $(-1)^k$ . Hence, only Casimirs with even  $k$  are non-zero.

**Claim 1** *The following are a complete set of Casimir operators for the given groups*

$GL(n; \mathbb{C})$  :

$$\begin{array}{c}
\leftarrow T_i \leftarrow \\
\vdots
\end{array},
\quad
\begin{array}{c}
\leftarrow T_i \leftarrow T_j \leftarrow \\
\vdots \quad \vdots
\end{array},
\quad
\begin{array}{c}
\leftarrow T_i \leftarrow T_j \leftarrow T_k \leftarrow \\
\vdots \quad \vdots \quad \vdots \\
\hline
\mathcal{S}_3 \\
\vdots \quad \vdots \quad \vdots
\end{array},
\quad \dots \quad
\begin{array}{c}
\leftarrow T_{i_1} \leftarrow T_{i_2} \leftarrow \dots \leftarrow T_{i_n} \leftarrow \\
\vdots \quad \vdots \quad \vdots \\
\hline
\mathcal{S}_n \\
\vdots \quad \vdots \quad \vdots
\end{array}
\quad \dots \quad (1.15)$$



$SU(n) :$

$$\begin{array}{c}
 \leftarrow T_i \leftarrow T_j \leftarrow \\
 \vdots \quad \vdots \\
 \leftarrow T_i \leftarrow T_j \leftarrow T_k \leftarrow \\
 \vdots \quad \vdots \quad \vdots \\
 \text{---} S_3 \\
 \vdots \quad \vdots \quad \vdots \\
 \leftarrow T_{i_1} \leftarrow T_{i_2} \leftarrow \dots \leftarrow T_{i_n} \leftarrow \\
 \vdots \quad \vdots \quad \vdots \\
 \text{---} S_n \\
 \vdots \quad \vdots \quad \vdots
 \end{array}
 , \quad \dots$$

(1.16)

$SO(2r+1)$  and  $Sp(2r) :$

$$\begin{array}{c}
 \leftarrow T_i \leftarrow T_j \leftarrow \\
 \vdots \quad \vdots \\
 \leftarrow T_i \leftarrow T_j \leftarrow T_k \leftarrow T_l \leftarrow \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 \text{---} S_4 \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 \leftarrow T_{i_1} \leftarrow T_{i_2} \leftarrow \dots \leftarrow T_{i_{2r}} \leftarrow \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 \text{---} S_{2r} \\
 \vdots \quad \vdots \quad \vdots \quad \vdots
 \end{array}
 , \quad \dots$$

(1.17)

$SO(2r) :$

$$\begin{array}{c}
 \leftarrow T_i \leftarrow T_j \leftarrow \\
 \vdots \quad \vdots \\
 \leftarrow T_i \leftarrow T_j \leftarrow T_k \leftarrow T_l \leftarrow \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 \text{---} S_4 \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 \leftarrow T_{i_1} \leftarrow T_{i_2} \leftarrow \dots \leftarrow T_{i_{2r-2}} \leftarrow \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 \text{---} S_{2r-2} \\
 \vdots \quad \vdots \quad \vdots \quad \vdots \\
 \text{---} A_{2r}^{\frac{1}{2}} \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 T_{i_1} \quad T_{i_2} \quad \dots \quad T_{i_r} \\
 \vdots \quad \vdots \quad \vdots
 \end{array}$$

(1.18)

**proof:**

Define

$$I_r(x) = \begin{array}{c} \mathcal{A}_{2r}^{\frac{1}{2}} \\ \nearrow M \\ \nwarrow \\ \nearrow M \\ \nwarrow \\ \dots \\ \nearrow M \\ \nwarrow \end{array} \quad (1.19)$$

where  $\mathcal{A}^{\frac{1}{2}}$  is the Levi Civita tensor.(see Chapter 22). Then an expansion of  $I_r^2(x)$  contains  $\text{tr}(M^{2r})$  among its summands.

$$I_r^2(x) = \begin{array}{c} \mathcal{A}_{2r} \\ \nearrow M \quad \nwarrow M \\ \nwarrow \quad \nearrow \\ \nearrow M \quad \nwarrow M \\ \nwarrow \quad \nearrow \\ \dots \\ \nearrow M \quad \nwarrow M \\ \nwarrow \quad \nearrow \end{array} = \text{tr}(M^{2r}) + \dots \quad (1.20)$$

**QED**

## 1.2 Casimir Matrix Expressed in Terms of $6j$ Coefficients

Define the Casimir matrix  $I_p$  as

$$(I_p)_a^b = \text{tr}(T_\lambda^{i_1} T_\lambda^{i_2} \dots T_\lambda^{i_p})(T_\mu^{i_1} T_\mu^{i_2} \dots T_\mu^{i_p})_a^b \quad (1.21)$$

$$= \begin{array}{c} \xrightarrow{\quad} T_\lambda^{i_1} \xrightarrow{\quad} T_\lambda^{i_2} \xrightarrow{\quad} \dots \xrightarrow{\quad} T_\lambda^{i_p} \xleftarrow{\quad} \\ \text{~~~~~} \text{~~~~~} \text{~~~~~} \text{~~~~~} \text{~~~~~} \\ \text{~~~~~} \text{~~~~~} \text{~~~~~} \text{~~~~~} \text{~~~~~} \\ a \xleftarrow{\quad} T_\mu^{i_1} \xleftarrow{\quad} T_\mu^{i_2} \xleftarrow{\quad} \dots \xleftarrow{\quad} T_\mu^{i_p} \xleftarrow{\quad} b \end{array} \quad (1.22)$$

The goal of this section is to express  $I_p$  in terms of  $6j$  coefficients.

Let

$$M = \begin{array}{c} \longrightarrow T_\lambda^i \longrightarrow \\ \text{~~~~~} \text{~~~~~} \\ \longleftarrow T_\mu^i \longleftarrow \end{array} \quad (1.23)$$

We will first decompose  $M$  in terms of  $6j$  coefficients, and then use that result to decompose  $I_p$  for  $p = 1, 2, 3, \dots$ . Note that

$$M = \sum_{\rho, \rho'} \begin{array}{c} \swarrow \lambda \searrow \mu \\ C_\rho^\dagger \xleftarrow{\rho} C_\rho \xrightarrow{\lambda} T_\lambda \xrightarrow{\mu} T_\mu \xleftarrow{\mu} C_\mu^\dagger \xleftarrow{\rho'} C_{\rho'} \xrightarrow{\lambda} \swarrow \lambda \searrow \mu \end{array} \quad (1.24)$$

$$= \sum_{\rho} A(\lambda, \rho, \mu) \begin{array}{c} \swarrow \lambda \searrow \mu \\ C_\rho^\dagger \xleftarrow{\rho} C_\rho \xrightarrow{\lambda} \swarrow \lambda \searrow \mu \end{array} \quad (1.25)$$

where

$$A(\lambda, \rho, \mu) = \frac{1}{d_\rho} \begin{array}{c} T_\lambda^\dagger \\ \text{~~~~~} \text{~~~~~} \\ \text{~~~~~} \text{~~~~~} \\ T_\rho \xrightarrow{\rho} T_\rho^\dagger \end{array} \quad (1.26)$$

**Claim 2** *If*

$$\Gamma_2(\rho) = \longleftarrow T_\rho \text{~~~~~} T_\rho \longrightarrow \quad (1.27)$$

then

$$A(\lambda, \mu, \rho) = -\frac{1}{2} [\Gamma_2(\rho) - \Gamma_2(\lambda) - \Gamma_2(\mu)] \quad (1.28)$$

**proof:**

Recall Eq.(9.22). Square both sides of the equation.

$$\begin{array}{c} j \\ \text{red wavy line} \\ \leftarrow T_\rho^j \leftarrow \end{array} \quad \begin{array}{c} j \\ \text{red wavy line} \\ \leftarrow T_\rho^j \leftarrow \end{array} = \left[ \begin{array}{c} j \\ \text{red wavy line} \\ T_\lambda \\ \swarrow \quad \nwarrow \\ \leftarrow C_\rho \quad C_\rho^\dagger \leftarrow \\ \searrow \quad \swarrow \\ T_\mu \end{array} - \begin{array}{c} j \\ \text{red wavy line} \\ \leftarrow C_\rho \quad C_\rho^\dagger \leftarrow \\ \swarrow \quad \searrow \\ T_\mu \end{array} \right]^2 \quad (1.29)$$

$$\begin{array}{c} \text{red wavy line} \\ \leftarrow T_\rho \leftarrow T_\rho \leftarrow \end{array} = \begin{array}{c} T_\lambda \\ \swarrow \quad \nwarrow \\ \leftarrow C_\rho \quad C_\rho^\dagger \leftarrow \\ \searrow \quad \swarrow \\ T_\mu \end{array} - 2 \begin{array}{c} T_\lambda \\ \swarrow \quad \nwarrow \\ \leftarrow C_\rho \quad C_\rho^\dagger \leftarrow \\ \searrow \quad \swarrow \\ T_\mu \end{array} + \begin{array}{c} \leftarrow C_\rho \leftarrow C_\rho^\dagger \leftarrow \\ \downarrow \quad \uparrow \\ T_\mu \text{red wavy line} T_\mu \end{array} \quad (1.30)$$

$$\Gamma_2(\rho) \leftarrow \rho = \Gamma_2(\lambda) \leftarrow \rho - 2 \begin{array}{c} T_\lambda \\ \swarrow \quad \nwarrow \\ \leftarrow C_\rho \quad C_\rho^\dagger \leftarrow \\ \searrow \quad \swarrow \\ T_\mu \end{array} + \Gamma_2(\mu) \leftarrow \rho \quad (1.31)$$

$$\frac{1}{d_\rho} \begin{array}{c} T_\lambda \\ \swarrow \quad \nwarrow \\ C_\rho \quad C_\rho^\dagger \\ \searrow \quad \swarrow \\ T_\mu \end{array} = -\frac{1}{2} [\Gamma_2(\rho) - \Gamma_2(\lambda) - \Gamma_2(\mu)] \quad (1.32)$$

This is similar to assuming

$$\vec{J} = \vec{L} + \vec{S} \quad (1.33)$$

where  $\vec{J}, \vec{L}, \vec{S}$  are the total, orbital and spin angular momentum. Then

$$\vec{L} \cdot \vec{S} = \frac{1}{2} [J^2 - L^2 - S^2] \quad (1.34)$$

**QED** Next note that

$$(I_p)_a^b = (M^p)_a^c {}^b_c \quad (1.35)$$

$$= \sum_{\rho \in \text{irreps}} [A(\lambda, \mu, \rho)]^p \quad a \xleftarrow{\rho} C_\rho \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\mu} \end{array} C_\rho^\dagger \xleftarrow{\rho} b \quad (1.36)$$

If  $\mu$  is an irrep,

$$\xleftarrow{\rho} C_\rho \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\mu} \end{array} C_\rho^\dagger \xleftarrow{\rho} = \frac{T_\rho \begin{array}{c} \xleftarrow{\rho} \\ \xrightarrow{\mu} \end{array} T_\rho}{d_\mu} \xleftarrow{\mu} \quad (1.37)$$

$$= \frac{d_\rho}{d_\lambda} \xleftarrow{\mu} \quad (\text{because } \rho \text{ is an irrep}) \quad (1.38)$$

### 1.3 $\text{tr}(M^2)$ and $\text{tr}(M^3)$

There are 3 quadratic Casimir ( $\text{tr}(M^2)$ ) matrices:

$$1. \quad (T_i T_i)_a^b = \Gamma_{fun} \delta_a^b \quad \xleftarrow{\quad} T_i \begin{array}{c} \text{wavy} \\ \xleftarrow{\quad} \end{array} T_i \xleftarrow{\quad} = \Gamma_{fun} \xleftarrow{\quad} \quad (1.39)$$

$$2. \quad \text{tr}(T_i T_j) = \kappa \delta_i^j \quad \text{wavy} T_i \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} T_j \text{wavy} = \kappa \text{wavy} \quad (1.40)$$

$$3. \quad f_{ijk} f_{kji'} = \Gamma_{adj} \delta_i^{i'} \quad \text{wavy} f \begin{array}{c} \text{wavy} \\ \text{wavy} \end{array} f \text{wavy} = \Gamma_{adj} \text{wavy} \quad (1.41)$$

Note that

$$T_i \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} T_i = n \Gamma_{fun} = N \kappa \quad (1.42)$$

**Claim 3**

$$\xleftarrow{\quad} T_i \begin{array}{c} \text{wavy} \\ \text{wavy} \end{array} T_k \xleftarrow{\quad} T_i \xleftarrow{\quad} = \left( \frac{\kappa N}{n} - \frac{\Gamma_{adj}}{2} \right) \xleftarrow{\quad} T_k \xleftarrow{\quad} \quad (1.43)$$

$$\begin{array}{c} \text{wavy line} \\ | \\ \text{triangle with wavy lines} \\ | \\ \text{wavy line} \end{array} \quad = \quad \frac{\Gamma_{adj}}{2} \quad \begin{array}{c} \text{wavy line} \\ | \\ \leftarrow T_k \leftarrow \end{array} \quad (1.44)$$

$$\begin{array}{c} \text{wavy line} \\ | \\ \text{triangle with wavy lines} \\ | \\ \text{wavy line} \end{array} \quad = \quad \frac{\Gamma_{adj}}{2} \quad \begin{array}{c} \text{wavy line} \\ | \\ \text{wavy line} \end{array} \quad (1.45)$$

proof:  
QED

## 1.4 Dynkin Index

$$DI_\lambda = \frac{\text{tr}(T_\lambda^i T_\lambda^i)}{f_{jk}^i f_{kj}^i} = \frac{\begin{array}{c} T_\lambda^i \\ \text{wavy line} \\ T_\lambda^i \end{array}}{\begin{array}{c} f \\ \text{wavy line} \\ f \end{array}} \quad (1.46)$$

# Chapter 2

## Characteristic Equations

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Let

$$M_a^b = a \longleftarrow M \longleftarrow b \quad (2.1)$$

for  $a, b = 1, 2, \dots, n$ .

The goal of this chapter is to express the coefficients of the characteristic equation (i.e.,  $\det(M - \lambda) = 0$ ) of  $M$  as traces.

For starters, note the difference between birdtracks for a matrix power and a tensor power of  $M$ .

$$M^2 = \longleftarrow M^2 \longleftarrow = \longleftarrow M \longleftarrow M \longleftarrow \quad (2.2)$$

$$M \otimes M = M^{\otimes 2} = \begin{array}{c} \longleftarrow M \longleftarrow \\ \longleftarrow M \longleftarrow \end{array} \quad (2.3)$$

In general,  $M^{\otimes p}$  is defined by

$$\begin{array}{c} (M^{\otimes p})_{\alpha}^{\beta} = (M^{\otimes p})_{a:p}^{rev(b^p)} = M_{a_1}^{b_1} M_{a_2}^{b_2} \dots M_{a_p}^{b_p} \\ \begin{array}{ccc} \longleftarrow M^{\otimes p} \longleftarrow & & \longleftarrow M \longleftarrow \\ \longleftarrow \parallel \longleftarrow & & \longleftarrow M \longleftarrow \\ \vdots & = & \vdots \\ \longleftarrow \parallel \longleftarrow & & \longleftarrow M \longleftarrow \end{array} \end{array} \quad (2.4)$$

where  $a_i, b_i \in \mathbb{Z}_{[1,n]}$ , and we define the anti-symmetrized trace of  $M^{\otimes p}$  by

$$\text{tr}_{1\dots p} \mathcal{A}[M^{\otimes p}] = \mathcal{A}_{a:p}^{\text{rev}(b:p)} \prod_{i=1}^p M_{b_i}^{a_i} \quad (2.5)$$

$$= \begin{array}{c} \mathcal{A}_p \\ \downarrow \\ \text{---} M \text{---} \\ \uparrow \\ M \\ \downarrow \\ M \end{array} \quad (\text{Cvitanovic Drawing Style}) \quad (2.6)$$

$$= \begin{array}{c} \text{---} \mathcal{A}_p \text{---} M \text{---} \\ \downarrow \\ \text{---} M \text{---} \\ \downarrow \\ \text{---} M \text{---} \end{array} \quad (\text{This book's drawing style}) \quad (2.7)$$

Note that the determinant of  $M$  is one of those traces

$$\det M = \text{tr}_{1\dots n} \mathcal{A}[M^{\otimes n}] \quad (2.8)$$

**Claim 4**

$$\begin{array}{c} \text{---} \mathcal{A}_p \text{---} \\ \downarrow \\ \text{---} M \text{---} \\ \downarrow \\ \text{---} M \text{---} \end{array} = \frac{1}{p} \left[ \begin{array}{c} \text{---} \\ \downarrow \\ \text{---} \mathcal{A}_{p-1} \text{---} M \text{---} \\ \downarrow \\ \text{---} M \text{---} \end{array} - (p-1) \begin{array}{c} \text{---} \mathcal{A}_{p-1} \text{---} M \text{---} \\ \downarrow \\ \text{---} M \text{---} \end{array} \right] \quad (2.9)$$

**proof:**

See Chapter 22.

**QED**



Consider the above claim for  $p = 2, 3$ .

$$\left\{ \begin{array}{l} \begin{array}{c} \leftarrow \mathcal{A}_3 \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \end{array} = \frac{1}{3} \left[ \begin{array}{c} \leftarrow \leftarrow \\ \parallel \\ \leftarrow \mathcal{A}_2 \leftarrow M \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \end{array} - 2 \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow M \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \end{array} \right] \\ \\ \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \end{array} = \frac{1}{2} \left[ \begin{array}{c} \leftarrow \leftarrow \\ \parallel \\ \leftarrow M \leftarrow \end{array} - \leftarrow M \leftarrow \right] \end{array} \right. \quad (2.10)$$

If we multiply from the right, by  $M^d$  for  $d = 1, 2$ , the first row of Eq.(2.10) and then take the trace of that row, we get

$$\left\{ \begin{array}{l} \begin{array}{c} \leftarrow \mathcal{A}_3 \leftarrow M \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \end{array} = \frac{1}{3} \left[ \begin{array}{c} \leftarrow M \leftarrow \\ \parallel \\ \leftarrow \mathcal{A}_2 \leftarrow M \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \end{array} - 2 \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow M^2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \end{array} \right] \\ \\ \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow M^2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow M \leftarrow \end{array} = \frac{1}{2} \left[ \begin{array}{c} \leftarrow M^2 \leftarrow \\ \parallel \\ \leftarrow M \leftarrow \end{array} - \leftarrow M^3 \leftarrow \right] \end{array} \right. \quad (2.11)$$

Let

$$\tau = \text{tr}(M) \quad (2.12)$$

Then Eqs.(2.11) can be expressed algebraically by

$$\text{tr}_{1,2,3} \mathcal{A}_3(M^{\otimes 3}) = \frac{1}{3} [\tau \text{tr}_{1,2} \mathcal{A}_2(M^{\otimes 2}) - \text{tr}(M^2)\tau + \text{tr} M^3] \quad (2.13)$$

and

$$\text{tr}_{1,2} \mathcal{A}_2 M^{\otimes 2} = \frac{1}{2} [\tau^2 - \text{tr}(M^2)] \quad (2.14)$$

Therefore,

$$\text{tr}_{1,2,3} \mathcal{A}_3(M^{\otimes 3}) = \frac{1}{3} \left[ \frac{1}{2} \tau^3 - \frac{3}{2} \text{tr}(M^2)\tau + \text{tr} M^3 \right] \quad (2.15)$$

$$= \frac{1}{3!} [\tau^3 - 3 \text{tr}(M^2)\tau + 2 \text{tr} M^3] \quad (2.16)$$

In general,

$$\text{tr}_{1\dots p} \mathcal{A}_p M = \frac{1}{p} \sum_{k=1}^p (-1)^{k-1} \left( \text{tr}_{1\dots p-k} \mathcal{A}_{p-k} M^{\otimes p-k} \right) \text{tr}(M^k) \quad (2.17)$$

Next note that

$$\mathcal{A}_p = 0 \quad \text{if } p > n \quad (2.18)$$

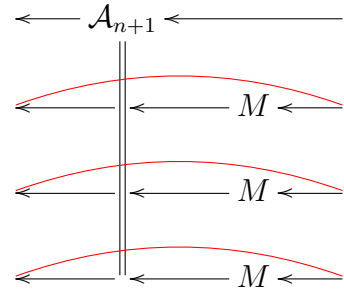
This follows because the Levi Civita tensor with more than  $n$  indices is zero.; i.e.,

$$\epsilon_{a_1, a_2, \dots, a_{n+1}} = 0 \quad (2.19)$$

Indeed, two of the  $a_i$  must be equal, so that element of the  $\epsilon$  tensor is zero

Let  $I$  be the  $n \times n$  identity matrix. Then, since  $\mathcal{A}_{n+1} = 0$ , the following is true

$0 = \text{tr}_{2\dots n+1} \mathcal{A}_{n+1} I \otimes M^{\otimes n}$

$0 =$ 


$(2.20)$

We can now expand the right hand side of Eq.(2.20) using identity Eq.(2.17)

$$0 = \sum_{k=0}^n (-1)^k \left( \text{tr}_{1\dots n-k} \mathcal{A}_{n-k} M^{\otimes n-k} \right) M^k \quad (2.21)$$

$$= \begin{cases} M^n \\ -M^{n-1}(\text{tr} M) \\ +M^{n-2}(\text{tr}_{1\dots 2} \mathcal{A}_2 M^{\otimes 2}) \\ \dots \\ (-1)^n \det(M) \end{cases} \quad (2.22)$$

Viola. The last equation is none other than the characteristic equation of  $M$ . As promised, the coefficients of this polynomial in  $M$ , are expressed as traces.

# Chapter 3

## Clebsch-Gordan Coefficients

This chapter is based on Cvitanovic's Birdtracks book Ref.[1].

Recall that if  $|x\rangle$  for  $x \in \text{val}(\underline{x})$  is a complete, orthonormal basis in Quantum Mechanics, then

$$\langle x|y\rangle = \delta(x, y) \quad (\text{orthonormality}) \quad (3.1)$$

and

$$\sum_x |x\rangle\langle x| = 1 \quad (\text{completeness}) \quad (3.2)$$

Furthermore, if we define

$$\pi_x = |x\rangle\langle x| \quad (3.3)$$

then  $\pi_x$  is a projection operator so

$$\pi_x \pi_x = \pi_x \quad (3.4)$$

and

$$\pi_x |y\rangle = |y\rangle \delta(x, y), \quad \langle y| \pi_x = \langle y| \delta(x, y) \quad (3.5)$$

Below, we will define matrices  $C_\lambda = \langle \lambda|$  and  $C_\lambda^\dagger = |\lambda\rangle$ . If we identify  $\langle \lambda|$  with  $\langle x|$ , and  $|\lambda\rangle$  with  $|x\rangle$ , then  $\langle \lambda|$  and  $|\lambda\rangle$  satisfy identities similar to those satisfied by  $\langle x|$  and  $|x\rangle$ . We will show this in this chapter.

### 3.1 CB Coefficients as Matrices

Suppose that  $M \in \mathbb{C}^{d \times d}$  is a Hermitian matrix. Then we have

$$M = C^\dagger \Lambda C \quad (3.6)$$

where  $C \in \mathbb{C}^{d \times d}$  is a unitary matrix, and  $\Lambda$  is a diagonal matrix.

One can partition  $C$  into rectangular submatrices  $\langle\lambda|$  that have  $d_\lambda$  rows with  $d_\lambda < d$ , such that we have one  $\langle\lambda|$  for each eigenvalue  $\lambda$  of  $C$ . Likewise, we can partition  $C^\dagger$  into rectangular submatrices  $C_\lambda^\dagger$  that have  $d_\lambda$  columns with  $d_\lambda < d$ , such that we have one  $|\lambda\rangle$  for each eigenvalue  $\lambda$  of  $C$ . Thus, if  $I^{d_\lambda \times d_\lambda}$  is the  $d_\lambda \times d_\lambda$  identity matrix,

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d \times d} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\pi_\lambda}^{d \times d} C^{d \times d} \quad (3.7)$$

$$\begin{bmatrix} 0 & (C_\lambda^\dagger)^{d \times d_\lambda} & 0 \end{bmatrix}^{d \times d} = (C^\dagger)^{d \times d} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\pi_\lambda}^{d \times d} \quad (3.8)$$

Henceforth in this chapter, we will use  $C_\lambda$  and  $\langle\lambda|$  interchangeably. Likewise, we will use  $C_\lambda^\dagger$  and  $|\lambda\rangle$  interchangeably. The matrices  $C_\lambda = \langle\lambda|$  are called the **Clebsch-Gordan (CG) coefficients** for  $M$ .

The matrices  $\pi_\lambda$  obviously form a complete orthogonal set of projection operators:

$$\sum_\lambda \pi_\lambda = 1, \quad \pi_\lambda \pi_\mu = \pi_\lambda \delta(\lambda, \mu) \quad (3.9)$$

We now have

$$\pi_\lambda C = \langle\lambda|, \quad C^\dagger \pi_\lambda = |\lambda\rangle \quad (3.10)$$

$$\langle\lambda||\lambda\rangle = \pi_\lambda C C^\dagger \pi_\lambda \quad (3.11)$$

$$= \pi_\lambda \quad (3.12)$$

$$M = C^\dagger \Lambda C \quad (3.13)$$

$$= C^\dagger \left( \sum_\lambda \lambda \pi_\lambda \right) C \quad (3.14)$$

$$= \sum_\lambda \lambda |\lambda\rangle \langle\lambda| \quad (3.15)$$

$$I^{d \times d} = C^\dagger C \quad (3.16)$$

$$= \sum_\lambda C^\dagger \pi_\lambda C \quad (3.17)$$

$$= \sum_\lambda \underbrace{|\lambda\rangle \langle\lambda|}_{P_\lambda} \quad (3.18)$$

We will call Eq.(3.18) a **Clebsch-Gordan (CG) series or decomposition** either in that form, or after multiplying by a vector space  $V$ , as in

$$V = \sum_{\lambda} P_{\lambda} V \quad (3.19)$$

A simple example of a CG series is

$$\vec{r} = \hat{x} \oplus \hat{y} \oplus \hat{z} \quad (3.20)$$

for a vector  $\vec{r} \in \mathbb{R}^3$ . In this expression, the vectors  $\hat{x}, \hat{y}, \hat{z}$  constitute a complete (i.e., basis) orthonormal set for the vectors acted upon by  $SO(3)$ . Any generic vector  $\vec{r} \in \mathbb{R}^3$  can be expressed as<sup>1</sup>

$$\vec{r} = a\hat{x} + b\hat{y} + c\hat{z} \quad (3.21)$$

for some  $a, b, c \in \mathbb{R}$ .

So far, we have established that

$$P_{\lambda} = |\lambda\rangle\langle\lambda| = C^{\dagger}\pi_{\lambda}C, \quad (3.22)$$

$$\pi_{\lambda} = \langle\lambda||\lambda\rangle \quad (3.23)$$

$$1 = \sum_{\lambda} \underbrace{|\lambda\rangle\langle\lambda|}_{P_{\lambda}} = \sum_{\lambda} \underbrace{\langle\lambda||\lambda\rangle}_{\pi_{\lambda}} \quad (3.24)$$

In fact, the  $P_{\lambda}$  form a complete orthogonal set of projection operators, just like the  $\pi_{\lambda}$ .

$$\sum_{\lambda} P_{\lambda} = 1, \quad P_{\lambda}P_{\mu} = P_{\lambda}\delta(\mu, \nu) \quad (3.25)$$

Whereas the  $\pi_{\lambda}$  satisfy

$$\pi_{\lambda}C = \langle\lambda|, \quad C^{\dagger}\pi_{\lambda} = |\lambda\rangle \quad (3.26)$$

the  $P_{\lambda}$  satisfy

$$CP_{\lambda} = \langle\lambda|, \quad P_{\lambda}C^{\dagger} = |\lambda\rangle \quad (3.27)$$

Since we are assuming  $M$  is Hermitian, its eigenvalues are real. Thus, we can absorb the eigenvalue  $\lambda$  into the CG coefficients by defining

$$\mathcal{C}_{\lambda} = \sqrt{\lambda}\langle\lambda| \quad (3.28)$$

and writing

$$M = \sum_{\lambda} \mathcal{C}_{\lambda}^{\dagger} \mathcal{C}_{\lambda} \quad (3.29)$$

---

<sup>1</sup>Recall that a direct sum of two vector spaces  $V = V_1 \oplus V_2$  means  $V_1 \cap V_2 = \{0\}$

Here is an example of a CG series. One can decompose  $V^n \otimes V^{\dagger n} = \sum_{\lambda} V_{\lambda}$  as follows

$$\begin{aligned}
1 &= \frac{1}{n} P_S + P_{adj} + \sum_{\lambda \neq Adj} P_{\lambda}, \\
\delta_d^a \delta_d^c &= \frac{1}{n} \delta_b^a \delta_d^c + (P_{adj})_{a \ c}^{b \ d} + \sum_{\lambda \neq Adj} (P_{\lambda})_{a \ c}^{b \ d}
\end{aligned} \tag{3.30}$$

$$\begin{array}{c}
a \leftarrow d \\
b \rightarrow c
\end{array}
= \frac{1}{n} \begin{array}{c} \uparrow \\ \downarrow \end{array} + \begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} + \sum_{\lambda \neq Adj} \begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array}$$

## 3.2 Generalization From Matrices to Tensors

Let  $b^{:nb} = (b_1, b_2, \dots, b_{nb})$  where  $b_i \in Z_{[0, d_{\mu_i}]}$  and  $a \in Z_{[1, d_{\lambda}]}$ . Assume that

$$d_{\lambda} = \prod_{i=1}^{:nb} d_{\mu_i} \tag{3.31}$$

Now define the birdtracks

$$(\langle \lambda |)_a^{rev(b^{:nb})} = \lambda a \leftarrow \langle \lambda | \begin{array}{l} \nwarrow \mu_1 b_1 \\ \leftarrow \mu_2 b_2 \\ \swarrow \mu_{nb} b_{nb} \end{array} \tag{3.32}$$

and

$$(|\lambda \rangle)_a^{b^{:nb}} = \begin{array}{l} \mu_1 b_1 \\ \nwarrow \mu_2 b_2 \\ \swarrow \mu_{nb} b_{nb} \end{array} \leftarrow |\lambda \rangle \leftarrow \lambda a \tag{3.33}$$

We will assume there is no difference between when a Greek letter is lowered and when it is raised. Also, all summations over a Greek letter will be stated explicitly; i.e., no implicit summations over repeated Greek letters.

On the other hand, the Latin letter indices  $b_i, a$  of  $\langle \lambda |$  and  $|\lambda \rangle$  may be lowered or raised and their arrows changed from outgoing to incoming or vice versa. Furthermore, we will use implicit summation over repeated Latin letters.

The Greek letters label representation of the group (not necessarily irreps). Each  $b_i$  labels a member of  $\mu_i$ , and each  $a$  labels a member of  $\lambda$ .

$$(\langle \lambda |)_a^{rev((b')^{:nb})} (P_\mu)_{(b')^{:nb}}^{rev(b^{:nb})} = \delta(\mu, \lambda) (\langle \mu |)_a^{rev(b^{:nb})}, \quad \langle \lambda | P_\mu = \delta(\mu, \lambda) \langle \mu |$$

$$\begin{array}{c}
 \begin{array}{ccc}
 & \sum b'_1 & b_1 \\
 & \swarrow \quad \searrow & \\
 a \leftarrow \langle \lambda | & \leftarrow \sum b'_2 & \leftarrow P_\mu \leftarrow b_2 \\
 & \swarrow \quad \searrow & \\
 & \sum b'_{nb} & b_{nb}
 \end{array}
 = \delta(\mu, \lambda)
 \begin{array}{ccc}
 & b_1 & \\
 & \swarrow \quad \searrow & \\
 a \leftarrow \langle \lambda | & \leftarrow b_2 & \\
 & \swarrow \quad \searrow & \\
 & b_{nb} &
 \end{array}
 \end{array}
 \quad (3.34)$$

$$(P_\mu)_{b^{:nb}}^{rev((b')^{:nb})} (|\lambda\rangle)^a_{(b')^{:nb}} = \delta(\mu, \lambda) (|\mu\rangle)^a_{b^{:nb}}, \quad P_\mu |\lambda\rangle = \delta(\mu, \lambda) |\mu\rangle$$

$$\begin{array}{c}
 \begin{array}{ccc}
 b_1 & \leftarrow \sum b'_1 & \\
 & \swarrow \quad \searrow & \\
 b_2 \leftarrow P_\mu & \leftarrow \sum b'_2 & \\
 & \swarrow \quad \searrow & \\
 b_{nb} & \leftarrow \sum b'_{nb} &
 \end{array}
 \leftarrow |\lambda\rangle \leftarrow a
 = \delta(\mu, \lambda)
 \begin{array}{ccc}
 b_1 & \leftarrow & \\
 & \swarrow \quad \searrow & \\
 b_2 \leftarrow |\lambda\rangle & \leftarrow a & \\
 & \swarrow \quad \searrow & \\
 b_{nb} & \leftarrow &
 \end{array}
 \end{array}
 \quad (3.35)$$

$$(\langle \lambda |)_a^{rev(b^{:nb})} (|\mu\rangle)^{a'}_{b^{:nb}} = \delta(\lambda, \mu) \delta_a^{a'}, \quad \langle \lambda | |\mu\rangle = \delta(\mu, \lambda)$$

$$\begin{array}{c}
 \begin{array}{ccc}
 & \sum b_1 & \\
 & \swarrow \quad \searrow & \\
 a \leftarrow \langle \lambda | & \leftarrow \sum b_2 & \\
 & \swarrow \quad \searrow & \\
 & \sum b_{nb} &
 \end{array}
 \leftarrow |\mu\rangle \leftarrow a'
 = \delta(\mu, \lambda)
 a \leftarrow \bullet \leftarrow a'
 \end{array}
 \quad (3.36)$$

$$\sum_\lambda (|\lambda\rangle)^a_{b^{:nb}} (\langle \lambda |)_a^{rev((b')^{:nb})} = \delta_{b^{:nb}}^{rev((b')^{:nb})}, \quad \sum_\lambda |\lambda\rangle \langle \lambda| = 1$$

$$\begin{array}{c}
 \begin{array}{ccc}
 b_1 & \leftarrow \sum a & \leftarrow \langle \lambda | \leftarrow b'_1 \\
 & \swarrow \quad \searrow & \\
 b_2 \leftarrow |\lambda\rangle & \leftarrow \sum a & \leftarrow \langle \lambda | \leftarrow b'_2 \\
 & \swarrow \quad \searrow & \\
 b_{nb} & \leftarrow \sum a & \leftarrow \langle \lambda | \leftarrow b'_{nb}
 \end{array}
 =
 \begin{array}{ccc}
 b_1 \leftarrow \bullet \leftarrow b'_1 & & \\
 b_2 \leftarrow \bullet \leftarrow b'_2 & & \\
 b_{nb} \leftarrow \bullet \leftarrow b'_{nb} & &
 \end{array}
 \end{array}
 \quad (3.37)$$

# Chapter 4

## Dynkin Diagrams

This chapter is based on Ref.[2], section 20.4.

This chapter is an overview of the classification of simple Lie algebras, a classification that was invented mainly by Killing, Cartan and Dynkin, in that historical order. The classification is valid for Lie algebras over  $\mathbb{C}$ <sup>1</sup> This caveat is important because there are more simple Lie algebras over  $\mathbb{R}$  than over  $\mathbb{C}$ . When defining the generators of Lie algebras in other chapters, we defined a real Lie algebra  $\mathfrak{g}_{\mathbb{R}}$  (a vector space over  $\mathbb{R}$ ) which exponentiates to a group  $\mathcal{G}_{\mathbb{R}} = \exp(\mathfrak{g}_{\mathbb{R}})$ . This chapter refers to the complexification  $\mathfrak{g}_{\mathbb{C}}$  (a vector space over  $\mathbb{C}$ ) of  $\mathfrak{g}_{\mathbb{R}}$  and to the group  $\mathcal{G}_{\mathbb{C}} = \exp(\mathfrak{g}_{\mathbb{C}})$

Suppose  $\mathfrak{g}_{\mathbb{C}}$  has generators  $X_s$  for  $s \in S = \{1, 2, \dots, \mathcal{D}\}$  where

$\mathcal{D} = \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{C}} = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}$  = half the number of real degrees of freedom of the Lie algebra.

By definition, the generators  $X_s$  are closed under commutation so

$$[X_q, X_p] = \sum_t f_{qp}^t X_t \quad (4.1)$$

for some  $f_{qp}^t \in \mathbb{C}$ .

Define

$$g_{qs} = \sum_{p,t} f_{qp}^t f_{st}^p = \quad q \text{ --- } f \text{ --- } s \quad (4.2)$$

If  $\det g = 0$ , we can find disjoint sets  $S_1, S_2$  so that  $S = S_1 \cup S_2$  and

$$[X_a, X_b] = 0 \quad \text{for } a, b \in S_1 \quad (4.3)$$

and

$$[X_q, X_p] = \sum_t f_{qp}^t X_t \quad \text{for } p, q, t \in S_2 \quad (4.4)$$

**Cartan Criterion:**  $\det g \neq 0$

---

<sup>1</sup>More generally, Cartan's classification is valid for Lie algebras over  $\mathbb{F}$ , where  $\mathbb{F}$  is an algebraically closed field of characteristic zero.  $\mathbb{C}$  satisfies both of these constraints.  $\mathbb{R}$  has characteristic zero but is not algebraically closed.



We will assume that the CC is satisfied. This implies that the Lie algebra is semi-simple and that we can and will assume that  $g_{st}$  is diagonal.

$$g_{st} = \delta(s, t) = \text{~~~~~} \quad (4.5)$$

$$f_{qp}{}^t = f_{qpt} \quad (4.6)$$

Will not, however, assume that  $f_{qpt}$  is totally antisymmetric, as is often assumed.

---

Cartan-Weyl commutators for  $\mathfrak{su}(2)$

$$E_- = \frac{1}{\sqrt{2}}|2\rangle\langle 1|, \quad E_+ = \frac{1}{\sqrt{2}}|1\rangle\langle 2|, \quad H = \frac{1}{2}[|1\rangle\langle 1| - |2\rangle\langle 2|] \quad (4.7)$$

$$\begin{array}{ccc} & H & \\ -E_- \swarrow & & \searrow E_+ \\ E_- & \xleftarrow{H} & E_+ \end{array} \quad \left\{ \begin{array}{l} [E_+, E_-] = H \\ [H, E_+] = E_+ \\ [H, E_-] = -E_- \end{array} \right. \quad (4.8)$$

---

Cartan-Weyl commutators for any semi-simple Lie algebra

For any lower case latin letter  $q$  and Greek letter  $\alpha$ , let

$q_- = 1, 2, \dots, \mathcal{R}$

$\vec{\alpha} = \mathcal{R} + 1, \mathcal{R} + 2, \dots, \mathcal{D}$

$q$  = either  $q_-$  or  $\vec{\alpha}$  but not both.

Let  $\{H_{i_-}\}_{i_-=1}^{\mathcal{R}}$  be the largest possible set of mutually commuting  $X_p$ .  $\mathcal{R}$  is called the **rank** of the group.

$$\boxed{[H_{i_-}, H_{j_-}] = 0} \quad (4.9)$$

Let  $E_{\vec{\alpha}}$  be eigenvectors of  $H_{i_-}$  for a commutator “product” instead of a matrix multiplication product

$$\boxed{[H_{i_-}, E_{\vec{\alpha}}] = \alpha_{i_-} E_{\vec{\alpha}}} \quad (4.10)$$

The vectors  $\vec{\alpha} \in \mathbb{R}^{\mathcal{R}}$  of eigenvalues are called **root vectors**.

From the properties of a commutator bracket<sup>2</sup>

$$[H_{i_-}, [E_{\vec{\alpha}}, E_{\vec{\beta}}]] = [[H_{i_-}, E_{\vec{\alpha}}], E_{\vec{\beta}}] + [E_{\vec{\alpha}}, [H_{i_-}, E_{\vec{\beta}}]] \quad (4.11)$$

$$= (\alpha_{i_-} + \beta_{i_-})[E_{\vec{\alpha}}, E_{\vec{\beta}}] \quad (4.12)$$

If  $\vec{\alpha} + \vec{\beta} = 0$ ,  $[H_{i_-}, [E_{\vec{\alpha}}, E_{\vec{\beta}}]] = 0$  so

$$\boxed{[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \sum_{i_-} \xi_{i_-} H_{i_-}} \quad (4.13)$$

---

<sup>2</sup>The commutator  $[x, y] = xy - yx$  acts like a derivative operator:  $[x[a, b]] = [[x, a], b] + [a, [x, b]]$

**Claim 5** *If we assume the normalization of  $\mathfrak{su}(2)$  operators used in Eq.(4.7), then*

$$\xi_{i_-} = \frac{\alpha_{i_-}}{\vec{\alpha} \cdot \vec{\alpha}} \quad (4.14)$$

**proof:**

Applying  $[E_{\vec{\alpha}}, \cdot]$  to both sides of Eq.(4.13)

$$[E_{\vec{\alpha}}, [E_{\vec{\alpha}}, E_{-\vec{\alpha}}]] = -\vec{\xi} \cdot \vec{\alpha} E_{\vec{\alpha}} \quad (4.15)$$

But for  $\mathfrak{su}(2)$

$$[E_+, [E_+, E_-]] = [E_+, H] = -E_+ \quad (4.16)$$

If set  $E_{\vec{\alpha}} = E_+$ ,  $E_{-\vec{\alpha}} = E_-$ , then

$$\vec{\xi} \cdot \vec{\alpha} = 1 \quad (4.17)$$

**QED**

If  $\vec{\alpha} + \vec{\beta} \neq 0$ ,

$$\boxed{[E_{\vec{\alpha}}, E_{\vec{\beta}}] = N_{\vec{\alpha}, \vec{\beta}} E_{\vec{\alpha} + \vec{\beta}} \quad \text{if } \vec{\alpha} + \vec{\beta} \neq 0} \quad (4.18)$$

for some  $N_{\vec{\alpha}, \vec{\beta}} \in \mathbb{C}$ .

The roots (**root system**) of a semisimple Lie algebra form a real  $\mathcal{R}$  dimensional vector space.

A **positive/negative (P/N) root**  $\vec{\alpha}$  is a root for which the first component  $\alpha_1$  is positive/negative. If the first component of  $\vec{\alpha}$  is zero, then decide the root's sign from its second component  $\alpha_2$ . And so on. Note that the definition of P/N root is basis dependent.

A P root  $\vec{\alpha}$  is a **simple positive (SP) root** if there are no P roots  $\vec{\rho}$  and  $\vec{\sigma}$  such that  $\vec{\alpha} = \vec{\rho} + \vec{\sigma}$ . Hence, SP roots are like the atoms or indivisible constituents of the root system.

Properties of root vectors  $\vec{\alpha}, \vec{\beta}, \dots \in \mathbb{R}^{\mathcal{R}}$

1. If  $\vec{\alpha}$  is a root, then  $-\vec{\alpha}$  is too.
2. We can find a basis of SP roots for the root space.
3. **Claim 6** *If  $\vec{\alpha}$  and  $\vec{\rho}$  are SP roots, then  $\vec{\alpha} - \vec{\rho}$  is not a root of any kind.*

**proof:**

Assume  $\vec{\alpha} - \vec{\rho}$  is either a P root or an N root.

If  $\vec{\alpha} - \vec{\rho}$  is a P root  $\vec{\sigma}$ , then  $\vec{\alpha} = \vec{\rho} + \vec{\sigma}$  so  $\vec{\alpha}$  is not a SP root.

Likewise, if  $\vec{\alpha} - \vec{\rho}$  is an N root  $\vec{\sigma}$ , then  $-\vec{\sigma}$  is a P root and  $\vec{\alpha} = \vec{\rho} + (-\vec{\sigma})$ .

**QED**

4. If  $\vec{\alpha}$  and  $\vec{\beta}$  are SP, then there can be roots

$$\{\vec{\beta} + n\vec{\alpha} | i = 0, 1, 2, \dots, n\} \quad (4.19)$$

for some terminal integer  $n \geq 0$  defined by (see Fig.4.1)

$$n = \frac{-2\vec{\alpha} \cdot \vec{\beta}}{\vec{\alpha} \cdot \vec{\alpha}} \quad (4.20)$$

The following roots are also possible

$$\{\vec{\alpha} + p\vec{\beta} | i = 0, 1, 2, \dots, p\} \quad (4.21)$$

for some terminal integer  $p \geq 0$  defined by (see Fig.4.1)

$$p = \frac{-2\vec{\alpha} \cdot \vec{\beta}}{\vec{\beta} \cdot \vec{\beta}} \quad (4.22)$$

5. angle constraint

Multiplying Eqs.(4.20) and (4.22), we get

$$-\sqrt{\frac{np}{4}} = \hat{\alpha} \cdot \hat{\beta} \in [-1, 0] \quad (4.23)$$

This angle constraint implies that the angle between the two SP roots  $\vec{\alpha}$  and  $\vec{\beta}$  can only have one of the 4 possible values listed in Table 4.1.

6. length ratio constraint

Dividing Eqs.(4.20) and (4.22), we get

$$\sqrt{\frac{n}{p}} = \frac{|\vec{\beta}|}{|\vec{\alpha}|} \quad (4.24)$$

Table 4.2 gives the possible length ratios implied by Eq.(4.24).

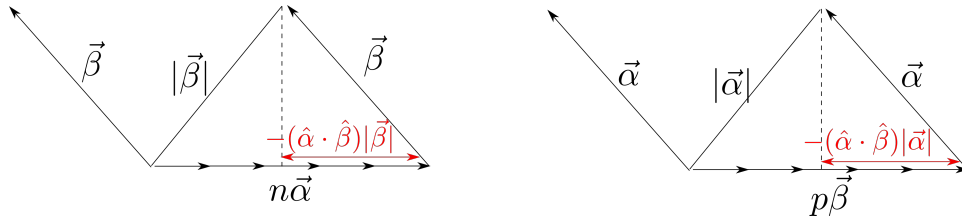


Figure 4.1: Pictorial representation of Eqs.(4.20) and (4.22).

$np$	$\sqrt{np/4}$	$\angle(\vec{\alpha}, \vec{\beta}) = \arccos(-\sqrt{np/4})$
0	0	$\frac{\pi}{2} = 90^\circ$
1	$\frac{1}{2}$	$\frac{2\pi}{3} = 120^\circ$
2	$\frac{1}{\sqrt{2}}$	$\frac{3\pi}{4} = 135^\circ$
3	$\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6} = 150^\circ$

Table 4.1: The 4 possible angles between two PS roots, as dictated by the angle constraint Eq.(4.23).

$np$	$(n, p)$	$ \vec{\beta} / \vec{\alpha} $
0	(0, 1) or (1,0)	0 or $\infty$
1	(1,1)	1
2	(1,2) or (2,1)	$1/\sqrt{2}$ or $\sqrt{2}$
3	(1,3) or (3,1)	$1/\sqrt{3}$ or $\sqrt{3}$

Table 4.2: Possible length ratios between two PS roots, as dictated by the length ratio constraint Eq.(4.24).

Using the definition of SP roots, and the angle and length constraints on SP roots, one can show that all possible simple Lie algebras have one of the root systems given by Fig.4.2. In that figure, the parameters of a root system are specified via Dynkin diagrams.

#### Rules for drawing **Dynkin Diagrams (DD)**

1. One dot for each SP root.  $k$  subscript in  $A_k, B_k, C_k, D_k$  is number of dots
2. Dots connected by  $np$  number of lines
3. If  $np > 1$ , draw arrowhead (i.e., greater-than sign  $>$ ) pointing from bigger to smaller root.
4. Draw one connected diagram (CD) for a simple Lie algebra. Draw multiple disconnected CDs for a semisimple Lie algebra.

This follows because the Lie algebra of a semisimple Lie algebra  $\mathfrak{g}$  is a direct sum  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \mathfrak{g}_t$  of simple Lie algebras  $\mathfrak{g}_i$  and the root vectors of any two of those Lie algebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are orthogonal so  $np = 0$  and there is no line connecting the roots of  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$ .

## 4.1 Examples

- DD for  $SO(3)$  and its double cover  $SU(2)$  is a single dot

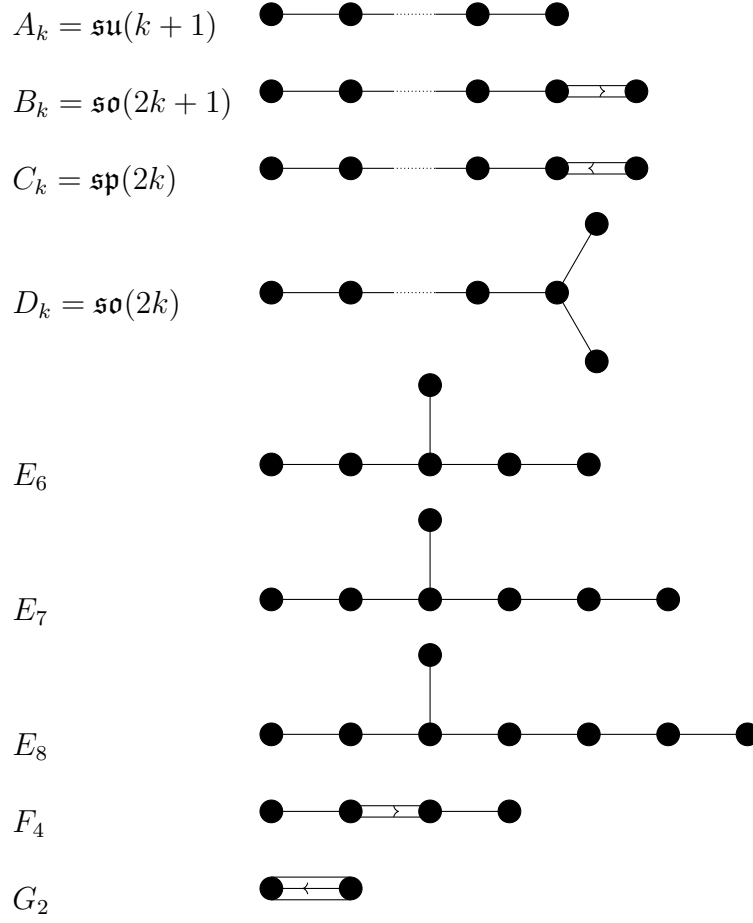


Figure 4.2: Dynkin diagrams for the simple Lie groups.  $n$  subscript in  $A_k, B_k, C_k, D_k$  is the number of dots, which equals the number of SP roots.

- $SO(4) \cong SO(3) \times SO(3)$  is not a simple Lie algebra. Its DD is two disconnected dots
- For  $SU(3)$ , the DD is  $\bullet \text{ --- } \bullet$

$$H_1 = T_z, \quad H_2 = \frac{\sqrt{3}}{2}Y \quad (4.25)$$

$$E_{\vec{\alpha}} = \frac{1}{\sqrt{2}}T_+, \quad E_{\vec{\beta}} = \frac{1}{\sqrt{2}}U_+, \quad E_{\vec{\alpha}+\vec{\beta}} = \frac{1}{\sqrt{2}}V_- \quad (4.26)$$

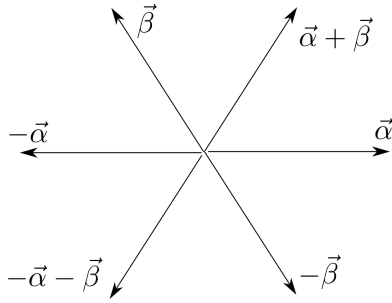


Figure 4.3: Root system for  $SU(3)$

## Chapter 5

# General Relativity Nets: COMING SOON

This chapter is based on Cvitanovic's Birdtracks book Ref. [1]

# Chapter 6

## Integrals over a Group

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

For a group  $\mathcal{G}$ , let

$SR$  = set of singlet reps, and  $SR^c$  = set of nonsinglet reps,

Let  $g$  be an element of  $\mathcal{G}$  with a rep-matrix  $G$ . The goal of this chapter is to show how to evaluate integrals over a group  $\mathcal{G}$ , of the form:

$$\int dg G_a^b G_c^d \dots (G^\dagger)_e^f (G^\dagger)_g^h \quad (6.1)$$

subject to the constraints that:

$$\int dg = 1 \quad (6.2)$$

and

$$\int dg G_\lambda = 0 \quad \text{if } \lambda \in SR^c \quad (6.3)$$

We will represent the rep-matrix  $G$  by

$$G_a^b = a \xleftarrow{\text{green}} G \longleftarrow b, \quad (G^\dagger)_b^a = b \xleftarrow{\text{green}} G^\dagger \longleftarrow a \quad (6.4)$$

Note that we will always take the out arrow (green) as the first one.

We will assume that  $G$  is a unitary matrix

$$G^\dagger G = GG^\dagger = 1 \quad \longleftarrow G^\dagger \longleftarrow G \longleftarrow = \longleftarrow G \longleftarrow G^\dagger \longleftarrow = \longleftarrow \bullet \longleftarrow \quad (6.5)$$

Tensor products of  $G$ 's will be represented thus

$$G \otimes G \otimes G^\dagger = \begin{array}{c} \longleftarrow G \longleftarrow \\ G \otimes G \otimes G^\dagger = \longleftarrow G \longleftarrow \\ \longleftarrow G^\dagger \longleftarrow \end{array} \quad (6.6)$$



## 6.1 $\int dg G$

To evaluate  $\int dg G$ , we expand  $G$  in its Clebsch-Gordan series. Such series and the Clebsch-Gordan coefficients  $C_\lambda$  are discussed in Chapter 3.

$$\int dg G = \sum_{\lambda} C_{\lambda}^{\dagger} \left[ \int dg G_{\lambda} \right] C_{\lambda} \quad (6.7)$$

$$= \sum_{\lambda \in SR} C_{\lambda}^{\dagger} C_{\lambda} \quad (6.8)$$

$$= \sum_{\lambda \in SR} P_{\lambda} \quad (6.9)$$

This result is valid for any group  $\mathcal{G}$  and any rep-matrix  $G$  of that group.

## 6.2 $\int dg G \otimes G^{\dagger}$

**Claim 7** For  $G \in SU(n) \subset \mathbb{C}^{n \times n}$ ,

$$\begin{array}{c} a \longleftarrow G \longleftarrow d \\ b \longrightarrow G^{\dagger} \longrightarrow c \end{array} = \frac{1}{n} \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \quad \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} + \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} T^i \text{---} G \text{---} T^i \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \quad (6.10)$$

**proof:** Recall that

$$\delta_a^d \delta_c^b = \frac{1}{n} \delta_a^b \delta_c^d + \frac{1}{\kappa} (T^i)_a^b (T^i)_c^d$$

$$\begin{array}{c} a \longleftarrow \bullet \longleftarrow d \\ b \longrightarrow \bullet \longrightarrow c \end{array} = \frac{1}{n} \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \quad \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} + \frac{1}{\kappa} \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} T^i \text{---} T^i \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \quad (6.11)$$

Will set  $\kappa = 1$  from here on. Multiplying both sides from the left by  $G \otimes G^{\dagger}$ , we get

$$G_a^d (G^{\dagger})_c^b = \frac{1}{n} \delta_a^b \delta_c^d + (G^{\dagger} T^i G)_a^b (T^i)_c^d$$

$$\begin{array}{c} a \longleftarrow G \longleftarrow d \\ b \longrightarrow G^{\dagger} \longrightarrow c \end{array} = \frac{1}{n} \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \quad \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} + \begin{array}{c} \longleftarrow G \longleftarrow \\ \longrightarrow G^{\dagger} \longrightarrow \end{array} T^i \text{---} T^i \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} \quad (6.12)$$

Since the generators  $T^i$  are invariant tensors,

$$\begin{array}{c}
 \boxed{G_a^{a'} (G^\dagger)_{b'}^b (T^i)_{a'}^{b'} G_{i'i} = (T^i)_a^b} \\
 \begin{array}{ccc}
 \leftarrow G & \leftarrow & \\
 & \uparrow & \\
 & T^i & \\
 & \downarrow & \\
 \rightarrow G^\dagger & \rightarrow &
 \end{array}
 \end{array}
 \begin{array}{c}
 \text{~~~~~} G \text{~~~~~} \\
 = \\
 \begin{array}{c}
 \leftarrow \\
 \uparrow \\
 T^i \\
 \downarrow \\
 \rightarrow
 \end{array}
 \end{array}
 \quad (6.13)$$

Hence,

$$\begin{array}{c}
 \boxed{G_a^{a'} (G^\dagger)_{b'}^b (T^i)_{a'}^{b'} = (T^i)_a^b G_{ii'}} \\
 \begin{array}{ccc}
 \leftarrow G & \leftarrow & \\
 & \uparrow & \\
 & T^i & \\
 & \downarrow & \\
 \rightarrow G^\dagger & \rightarrow &
 \end{array}
 \end{array}
 \begin{array}{c}
 \text{~~~~~} \\
 = \\
 \begin{array}{c}
 \leftarrow \\
 \uparrow \\
 T^i \\
 \downarrow \\
 \rightarrow
 \end{array}
 \text{~~~~~} G \text{~~~~~}
 \end{array}
 \quad (6.14)$$

**QED**

**Claim 8** For  $G \in SU(n) \subset \mathbb{C}^{n \times n}$ ,

$$\begin{array}{c}
 \boxed{\int dg G_a^d (G^\dagger)_b^c = \frac{1}{n} \delta_a^b \delta_c^d} \\
 \int dg \begin{array}{ccc} a \leftarrow G \leftarrow d \\ b \rightarrow G^\dagger \rightarrow c \end{array} = \frac{1}{n} \begin{array}{c} \leftarrow \\ \uparrow \\ \downarrow \\ \rightarrow \end{array} \begin{array}{c} \leftarrow \\ \uparrow \\ \downarrow \\ \rightarrow \end{array}
 \end{array}
 \quad (6.15)$$

**proof:**

This claim follows immediately from the previous one.

**QED**

Claim 8 can be extended to any group  $\mathcal{G}$  that has a single singlet rep. For such groups, we have, if  $G \in \mathbb{C}^{d_{def} \times d_{def}}$  is the defining rep so that  $a, b, c, d \in \{1, 2, \dots, d_{def}\}$ ,

$$\begin{array}{c}
 \boxed{\delta_a^d \delta_c^b = \frac{1}{d_{def}} \delta_a^b \delta_c^d + \sum_{\lambda \in SR^c} \frac{1}{\kappa} (T_\lambda^i)_a^b (T_\lambda^i)_c^d} \\
 \begin{array}{ccc}
 a \leftarrow \bullet \leftarrow d \\
 b \rightarrow \bullet \rightarrow c
 \end{array}
 = \frac{1}{d_{def}} \begin{array}{c} \leftarrow \\ \uparrow \\ \downarrow \\ \rightarrow \end{array} \begin{array}{c} \leftarrow \\ \uparrow \\ \downarrow \\ \rightarrow \end{array}
 + \sum_{\lambda \in SR^c} \frac{1}{\kappa} \begin{array}{c} \leftarrow \\ \uparrow \\ T_\lambda^i \\ \downarrow \\ \rightarrow \end{array} \text{~~~~~} \begin{array}{c} \leftarrow \\ \uparrow \\ T_\lambda^i \\ \downarrow \\ \rightarrow \end{array}
 \end{array}
 \quad (6.16)$$

so Eq.(6.15) is valid with  $n$  replaced by  $d_{def}$ .

**Claim 9** For any group  $\mathcal{G}$  with rep-matrices  $G_\mu$  and  $G_\nu$  ( $\mu, \nu$  are not necessarily irreps)

$$\int dg (G_\mu)_{ab} (G_\nu)^{cd} = \sum_{\lambda \in SR} (P_\lambda)_{ab}^{cd} \quad (6.17)$$

**proof:**

Let

$$(C_{\lambda i}^\dagger)_{ac} = \begin{array}{c} \mu a \leftarrow \\ \leftarrow C_{\lambda i}^\dagger \\ \nu c \leftarrow \end{array} \leftarrow \lambda i \quad (6.18)$$

represent the Clebsch-Gordan coefficients for the Clebsch-Gordan series  $V_\mu \otimes V_\nu = \sum_\lambda V_\lambda$ .

Since the  $C_\lambda$  are invariant tensors:

$$\begin{array}{c} (G_\mu)_a^{a'} (G_\nu)_{b'}^b (C_{\lambda i}^\dagger)_{a'}^{b'} = (C_{\lambda i'}^\dagger)_a^b (G_\lambda)_{i' i} \end{array}$$

$$\begin{array}{c} \leftarrow G_\mu \leftarrow \\ \leftarrow C_\lambda^\dagger \leftarrow \\ \leftarrow G_\nu \leftarrow \end{array} = \begin{array}{c} \leftarrow C_\lambda^\dagger \leftarrow G_\lambda \leftarrow \\ \leftarrow C_\lambda^\dagger \leftarrow \end{array} \quad (6.19)$$

Therefore,

$$\int dg \begin{array}{c} \leftarrow G_\mu \leftarrow \\ \leftarrow G_\nu \leftarrow \end{array} = \int dg \sum_\lambda \begin{array}{c} \leftarrow C_\lambda^\dagger \leftarrow G_\lambda \leftarrow C_\lambda \\ \leftarrow C_\lambda^\dagger \leftarrow \end{array} \quad (6.20)$$

$$= \sum_{ij} \sum_\lambda (C_{\lambda i}^\dagger)_{ab} (C_{\lambda j})^{cd} \int dg (G_\lambda)_{ij} \quad (6.21)$$

$$= \sum_i \sum_{\lambda \in SR} (C_{\lambda i}^\dagger)_{ab} (C_{\lambda i})^{cd} \quad (6.22)$$

$$= \sum_{\lambda \in SR} (P_\lambda)_{ab}^{cd} \quad (6.23)$$

**QED**

### 6.3 Character Orthonormality Relation

For any rep-matrix  $G_\lambda$  in rep  $\lambda$  such that  $G_\lambda$  represents the group element  $g$  in the Group  $\mathcal{G}$ , define the **character of  $g$  in rep  $\lambda$**  by

$$\chi_\lambda(g) = \chi_\lambda(G_\lambda) \stackrel{\text{def}}{=} \text{tr} G_\lambda = (G_\lambda)_a^a \quad (6.24)$$

Note that

$$\text{tr} G_\lambda^\dagger = (G_\lambda^a)^* = \chi_\lambda(g)^* \quad (6.25)$$

**Claim 10** Suppose  $G_\lambda$  and  $G_\mu$  are rep-matrices in irreps  $\lambda$  and  $\mu$ , respectively. Suppose  $h, G_\lambda \in \mathbb{C}^{d_\lambda \times d_\lambda}$  and  $f, G_\mu \in \mathbb{C}^{d_\mu \times d_\mu}$ . Then

$$\int dg \chi_\lambda(h^\dagger G_\lambda) \chi_\mu^*(G_\mu^\dagger f) = \delta(\mu, \lambda) \frac{1}{d_\mu} \chi_\lambda(h^\dagger f) \quad (6.26)$$

**proof:**

This claim follows from Eq.(6.16) once we prove that the left hand side of Eq.(6.26) is zero if  $\lambda \neq \mu$ . Because  $\lambda$  and  $\mu$  are both irreps, there can be no matrix connecting  $G_\mu$  and  $G_\lambda$  when  $\lambda \neq \mu$ , so the left hand side of Eq.(6.26) is indeed zero. Even when  $\lambda = \mu$ , there can only be one matrix, namely a Kronecker delta, connecting  $G_\lambda$  and  $G_\mu$ , so the group must have only one singlet rep.

**QED**

Note that since the matrices  $h, f \in \mathbb{C}^{d_\mu \times d_\mu}$  are arbitrary, differentiation can be used to retrieve  $G_\mu$  from its character with various  $h$ :

$$G_a^b = \frac{d}{d(h^\dagger)_b^a} \underbrace{\chi_\mu(h^\dagger G)}_{(h^\dagger)_b^a G_a^b} \quad (6.27)$$

### 6.4 $SU(n)$ Examples

In  $SU(n)$ ,  $n = d_{def}$ , where  $d_{def}$  is the dimension of the defining rep. ( $\mathcal{G} \subset \mathbb{C}^{n \times n}$ ). In this section, all matrices  $G$  are elements of  $\mathbb{C}^{n \times n}$ .

### 6.4.1 $\int dg G \otimes G$

Consider  $V \otimes V$ . We have

$$\begin{array}{c} \leftarrow \\ \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} \quad (6.28)$$

because

$$\begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} = \frac{1}{2} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \end{array} \right\} \quad (6.29)$$

and

$$\begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} = \frac{1}{2} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \end{array} \right\} \quad (6.30)$$

Thus

$$d_{\mathcal{S}} = \frac{1}{2} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \end{array} \right\} \quad (6.31)$$

$$= \frac{n(n+1)}{2} \quad (6.32)$$

and

$$d_{\mathcal{A}} = \frac{1}{2} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \end{array} \right\} \quad (6.33)$$

$$= \frac{n(n-1)}{2} \quad (6.34)$$

Note that  $d_{\mathcal{S}} = 1$  iff  $n = 1$ , and  $d_{\mathcal{A}} = 1$  iff  $n = 2$ . Therefore, for  $SU(n)$

$$\int dg G \otimes G = 0 \quad \text{if } n > 2 \quad (6.35)$$

### 6.4.2 $\int dg G^\dagger \otimes G^\dagger \otimes G \otimes G$

Consider  $V^\dagger \otimes V^\dagger \otimes V \otimes V$ . We have

$$\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longleftarrow \\
\longleftarrow
\end{array}
=
\begin{array}{c}
\longrightarrow \mathcal{S}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \longleftarrow \\
\longleftarrow \longleftarrow
\end{array}
+
\begin{array}{c}
\longrightarrow \mathcal{A}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \longleftarrow \\
\longleftarrow \longleftarrow
\end{array}
\quad (6.36)$$

$$\begin{array}{c}
\longrightarrow \mathcal{S}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \mathcal{S}_2 \longleftarrow \\
\parallel \\
\longleftarrow \longleftarrow \\
\hline
\mathcal{S}_2 \otimes \mathcal{S}_2
\end{array}
+
\begin{array}{c}
\longrightarrow \mathcal{A}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \mathcal{A}_2 \longleftarrow \\
\parallel \\
\longleftarrow \longleftarrow \\
\hline
\mathcal{A}_2 \otimes \mathcal{A}_2
\end{array}
+
\begin{array}{c}
\longrightarrow \mathcal{S}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \mathcal{A}_2 \longleftarrow \\
\parallel \\
\longleftarrow \longleftarrow \\
\hline
\mathcal{S}_2 \otimes \mathcal{A}_2
\end{array}
+
\begin{array}{c}
\longrightarrow \mathcal{A}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \mathcal{S}_2 \longleftarrow \\
\parallel \\
\longleftarrow \longleftarrow \\
\hline
\mathcal{A}_2 \otimes \mathcal{S}_2
\end{array}
\quad (6.37)$$

$$\begin{array}{c}
\longrightarrow \mathcal{S}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \mathcal{S}_2 \longleftarrow \\
\parallel \\
\longleftarrow \longleftarrow \\
\hline
\mathcal{S}_2 \otimes \mathcal{S}_2
\end{array}
+
\begin{array}{c}
\longrightarrow \mathcal{A}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \mathcal{A}_2 \longleftarrow \\
\parallel \\
\longleftarrow \longleftarrow \\
\hline
\mathcal{A}_2 \otimes \mathcal{A}_2
\end{array}
+
\begin{array}{c}
\longrightarrow \mathcal{S}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \mathcal{A}_2 \longleftarrow \\
\parallel \\
\longleftarrow \longleftarrow \\
\hline
\mathcal{S}_2 \otimes \mathcal{A}_2
\end{array}
+
\begin{array}{c}
\longrightarrow \mathcal{A}_2 \longrightarrow \\
\parallel \\
\longrightarrow \longrightarrow \\
\longleftarrow \mathcal{S}_2 \longleftarrow \\
\parallel \\
\longleftarrow \longleftarrow \\
\hline
\mathcal{A}_2 \otimes \mathcal{S}_2
\end{array}
\quad (6.38)$$

Let

$$P_1 = \frac{1}{n^2}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longleftarrow \\
\longleftarrow
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longleftarrow \\
\longleftarrow
\end{array}
\quad (6.39)$$

and

$$P_2 = \frac{1}{n^2}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longleftarrow \\
\longleftarrow
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longleftarrow \\
\longleftarrow
\end{array}
= \frac{1}{n^2}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longleftarrow \\
\longleftarrow
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longleftarrow \\
\longleftarrow
\end{array}
\quad (6.40)$$

Then

$$P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1 P_2 \neq 0 \quad (6.41)$$

$$\dim(P_1) = \dim(P_2) = 1 \quad (6.42)$$

This hints to the possibility of two orthogonal projectors, if only we include terms where there is a single swap on either the right or the left side, but not on both sides as in Eq.(6.40). So define

$$\pi_{\mathcal{S}} = \frac{1}{d_{\mathcal{S}}} \begin{array}{c} \longrightarrow \mathcal{S}_2 \\ \longrightarrow \parallel \\ \longleftarrow \\ \longleftarrow \end{array} \begin{array}{c} \longrightarrow \mathcal{S}_2 \\ \longrightarrow \parallel \\ \longleftarrow \\ \longleftarrow \end{array} \quad \text{where } d_{\mathcal{S}} = \frac{n(n+1)}{2} \quad (6.43)$$

and

$$\pi_{\mathcal{A}} = \frac{1}{d_{\mathcal{A}}} \begin{array}{c} \longrightarrow \mathcal{A}_2 \\ \longrightarrow \parallel \\ \longleftarrow \\ \longleftarrow \end{array} \begin{array}{c} \longrightarrow \mathcal{A}_2 \\ \longrightarrow \parallel \\ \longleftarrow \\ \longleftarrow \end{array} \quad \text{where } d_{\mathcal{A}} = \frac{n(n-1)}{2} \quad (6.44)$$

Then

$$\pi_{\mathcal{A}}^2 = \pi_{\mathcal{A}}, \quad \pi_{\mathcal{S}}^2 = \pi_{\mathcal{S}}, \quad \pi_{\mathcal{A}}\pi_{\mathcal{S}} = 0 \quad (6.45)$$

$$\dim(\pi_{\mathcal{S}}) = \text{tr}\pi_{\mathcal{S}} = \frac{1}{d_{\mathcal{S}}} \begin{array}{c} \longrightarrow \mathcal{S}_2 \\ \longrightarrow \parallel \\ \longleftarrow \\ \longleftarrow \end{array} \begin{array}{c} \longrightarrow \mathcal{S}_2 \\ \longrightarrow \parallel \\ \longleftarrow \\ \longleftarrow \end{array} = 1 \quad (6.46)$$

$$\dim(\pi_{\mathcal{A}}) = 1 \quad (6.47)$$

Thus

$$\begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow \end{array} = \pi_{\mathcal{S}} + \pi_{\mathcal{A}} + \text{non-singlet projectors} \quad (6.48)$$

Hence

$$\int dg \begin{array}{c} \longrightarrow G^\dagger \longrightarrow \\ \longrightarrow G^\dagger \longrightarrow \\ \longleftarrow G \longleftarrow \\ \longleftarrow G \longleftarrow \end{array} = \pi_{\mathcal{S}} + \pi_{\mathcal{A}} \quad (6.49)$$

# Chapter 7

## Invariant Tensors

This chapter is based on Cvitanovic's Birdtracks book Ref.[1].

A **bilinear form** is a linear function  $m : V^{\dagger n} \times V^n \rightarrow \mathbb{C}$  usually with  $V^{\dagger n}, V^n = \mathbb{C}^n$ . For example,

$$m(x^{\dagger n}, y^n) = x^{\dagger a} M_a^b y_b \quad \begin{array}{c} M \\ \swarrow \quad \searrow \\ a \quad \quad b \end{array} \quad (7.1)$$

$m()$  is said to be invariant if

$$m(x^{\dagger n}, y^n) = m(x^{\dagger n} G^{\dagger}, G y^n) \quad (7.2)$$

$m()$  is invariant iff matrix  $M$  is an **invariant matrix**; i.e., iff

$$M_a^b = (G^{\dagger})_a^{a'} G_{b'}^b M_{a'}^{b'} \quad \begin{array}{c} M \\ \swarrow \quad \searrow \\ a \quad \quad b \end{array} = \begin{array}{c} M \\ \swarrow \quad \searrow \\ a \xrightarrow{G^{\dagger}} \quad \xleftarrow{G} b \end{array} \quad (7.3)$$

$$M = G^{\dagger} M G \quad (7.4)$$

If  $G$  is unitary,

$$GM = MG, \quad [G, M] = 0 \quad (7.5)$$

A **multilinear form** is a linear function  $h : V^{\dagger n^p} \times V^{n^q} \rightarrow \mathbb{C}$ , usually with  $V^{\dagger}, V = \mathbb{C}$ . For example,

$$h(w^{\dagger}, x^{\dagger}, y, z) = h_{ab}^{cd} w^{\dagger a} x^{\dagger b} y_c z_d \quad \begin{array}{c} h \\ \downarrow \quad \swarrow \quad \searrow \quad \searrow \\ a \quad \quad b \quad \quad c \quad \quad d \end{array} \quad (7.6)$$

$h()$  is said to be invariant if

$$h(w^{\dagger}, x^{\dagger}, y, z) = h(w^{\dagger} G^{\dagger}, x^{\dagger} G^{\dagger}, G y, G z) \quad (7.7)$$



$h()$  is invariant iff tensor  $h_{ab}{}^{cd}$  is a **invariant tensor** (IT); i.e., iff

$$h_{ab}{}^{cd} = (G^\dagger)_a{}^{a'} (G^\dagger)_b{}^{b'} h_{a'b'}{}^{c'd'} G_{c'}{}^c G_{d'}{}^d \quad \begin{array}{c} h \\ \swarrow \downarrow \searrow \\ a \quad b \quad c \quad d \end{array} = \begin{array}{c} h \\ \swarrow \downarrow \searrow \\ G^\dagger \quad G^\dagger \quad G \quad G \\ \downarrow \downarrow \downarrow \downarrow \\ a \quad b \quad c \quad d \end{array} \quad (7.8)$$

A **composed IT** is an IT that can be written as a product or contraction of ITs.

A **tree IT** is a composed IT without any loops.

A **primitive IT** is an IT that can be expressed as a linear combination of a finite number of tree ITs.

The **primitiveness assumption**: All IT are primitive.

Examples. Suppose  $x, y, z \in \mathbb{R}^3$  and  $i, j, k \in \{1, 2, 3\}$ .

- Primitive ITs

$$length(x) = \delta_{ij} x_i x_j \quad volume(x, y, z) = \epsilon_{ijk} x_i y_j z_k \quad (7.9)$$

$$\delta_{ij} = i \text{ --- } j, \quad \epsilon_{ijk} = \begin{array}{c} \epsilon \\ \swarrow \downarrow \searrow \\ i \quad j \quad k \end{array} \quad (7.10)$$

- Tree ITs

$$\delta_{ij} \epsilon_{klm} = \begin{array}{c} i \\ | \\ j \end{array} \quad \begin{array}{c} \epsilon \\ \swarrow \downarrow \searrow \\ k \quad l \quad m \end{array} \quad (7.11)$$

$$\epsilon_{ijm} \delta_{mn} \epsilon_{nkl} = \begin{array}{c} \epsilon_{ijm} \text{---} \sum^m \text{---} \epsilon_{mkl} \\ \swarrow \downarrow \quad \downarrow \quad \swarrow \searrow \\ i \quad j \quad k \quad l \end{array} \quad (7.12)$$

- Non-tree IT

$$\epsilon_{ims} \epsilon_{jnm} \epsilon_{krn} \epsilon_{lsr} = \begin{array}{c} i \text{ --- } \epsilon_{ims} \text{---} \sum^s \text{---} \epsilon_{lsr} \text{ --- } l \\ \quad \downarrow \quad \quad \downarrow \\ \quad \sum^m \quad \quad \sum^r \\ j \text{ --- } \epsilon_{jnm} \text{---} \sum^n \text{---} \epsilon_{krn} \text{ --- } k \end{array} \quad (7.13)$$

$$\epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{lsr} = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}$$

$$\begin{array}{c} i \text{ --- } \epsilon_{ims} \text{ --- } \sum_m^s \text{ --- } \epsilon_{lsr} \text{ --- } l \\ j \text{ --- } \epsilon_{jnm} \text{ --- } \sum_n^r \text{ --- } \epsilon_{krn} \text{ --- } k \end{array} = \begin{array}{c} i \\ | \\ j \end{array} \begin{array}{c} l \\ | \\ k \end{array} + \begin{array}{c} i \text{ --- } l \\ j \text{ --- } k \end{array} \quad (7.14)$$

- Primitiveness Assumption

Suppose  $\mathcal{P} = \{\delta_{ij}, f_{ijk}\}$  where  $f_{ijk}$  is not  $\epsilon_{ijk}$ . For some  $A, B, C, \dots H \in \mathbb{C}$ , one has

$$\text{---} \bigcirc \text{---} = A \text{ ---} \quad (7.15)$$

$$\text{---} \bigcirc \text{---} = B \text{ ---} \bullet \text{---} \quad (7.16)$$

$$\text{---} \bigcirc \text{---} = \left\{ \begin{array}{l} C \text{ ---} \text{ ---} + D \text{ ---} \times \text{ ---} + E \text{ ---} \bullet \text{ ---} \\ + F \text{ ---} \text{ ---} + G \text{ ---} \bullet \text{ ---} \bullet \text{ ---} + H \text{ ---} \bullet \text{ ---} \times \text{ ---} \bullet \end{array} \right\} \quad (7.17)$$

---

Let  $\mathcal{P} = (p_1, p_2, \dots, p_k)$  be a **full set of primitive ITs**. By “full”, we mean no others exist.  $\mathcal{P}$  is a basis for an **algebra of invariants**.<sup>1</sup>

An **invariance group**  $\mathcal{G}$  with a full set of primitive ITs  $\mathcal{P} = \{p_1, p_2, \dots, p_k\}$  is the set of all linear transformation  $G \in \mathcal{G}$  such that

$$p_1(x^\dagger, y) = p_1(x^\dagger G^\dagger, Gy) \quad (7.18)$$

$$p_2(w^\dagger, x^\dagger, y, z) = p_2(w^\dagger G^\dagger, x^\dagger G^\dagger, Gy, Gz) \quad (7.19)$$

$$\text{etc.} \quad (7.20)$$

Example. Consider an invariance group with a single primitive IT  $p()$  defined by

$$p(x^\dagger, y) = \delta_a^b x^{\dagger a} y_b = x^{\dagger b} y_b \quad (7.21)$$

Then

$$(x')^{\dagger a} (y')_a = x^{\dagger b} (G^\dagger G)_b^c y_c = x^{\dagger b} y_b \quad (7.22)$$

---

<sup>1</sup>An algebra over a field is defined in Sec.A.7

so  $G$  must be unitary

$$G^\dagger G = 1 \tag{7.23}$$

The group of  $n$  dimensional unitary matrices is called  $U(n)$

# Chapter 8

## Lagrange's Theorem and Orbit-Stabilizer Theorem

### 8.1 Lagrange's Theorem

Let  $H$  be a subgroup of a finite group  $G$  (i.e.,  $H \leq G$ )

$gH = \{gh|h \in H\} = \text{left coset}$  with coset representative  $g$

$G/H = \{gH|g \in G\} = \{gH|g \in G : H\}$

$G : H = \text{set of coset representatives}$

The **Lagrange table** is a table with rows labelled by  $g \in G : H$ , columns labelled by  $h \in H$ , and table entries equal to  $gh$ .

**Claim 11** *Every element of  $G$  appears exactly once in the Lagrange Table. Therefore<sup>1</sup>*

$$|G| = |G : H| |H| \quad (8.1)$$

**proof:**

If  $x \in G$ , then  $x \in xH = gH$  for some  $g \in G : H$ . So every  $x \in G$  appears at least in one coset.

Suppose  $x$  appears in 2 cosets  $g_1H$  and  $g_2H$ .

$$g_1h_1 = g_2h_2 \quad (8.2)$$

so

$$g_2^{-1}g_1 = h_2h_1^{-1} \in H \quad (8.3)$$

Hence,

$$g_2^{-1}g_1H = H \implies g_2H = g_1H \quad (8.4)$$

**QED**

---

<sup>1</sup>Eq.(8.1) is for finite groups. For infinite Lie groups, the order of each group is replaced by the dimension of the Lie algebra vector space:  $\dim(\mathfrak{g}) = \dim(\mathfrak{a}) + \dim(\mathfrak{h})$ , where  $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$

A group  $G$  is **abelian** if  $g_1g_2 = g_2g_1$  for all  $g_1, g_2 \in G$ .

$H$  is a **normal** subgroup of  $G$  (i.e.,  $H \trianglelefteq G$ ) iff  $gH = Hg$  (i.e.,  $gHg^{-1} = H$ ) for all  $g \in G$ . All subgroups  $H$  of an abelian group  $G$  are normal subgroups.

$G$  = all symmetries.

$H$  = symmetries for a given reference state

$G : H$  = a set of reference states that yield all distinct left cosets, often called the **index set**

$G/H$  = set of left cosets, often called a **quotient space or coset space**. Also called **quotient group or coset group** if  $H$  is a normal subgroup

Examples:

---

1. ( $H$  not normal subgroup)

$G = S_3 = \{e, (12), (13), (23), (123), (132)\}$  The symmetric group on 3 letters,

$H = \{e, (12)\}$

Cosets

$$\begin{cases} eH &= \{e, (12)\} \\ (13)H &= \{(13), (123)\} \\ (23)H &= \{(23), (132)\} \end{cases} \quad (8.5)$$

$G : H = \{e, (13), (23)\}$  coset representatives.

$$|G| = 6, \quad |G : H| = 3, \quad |H| = 2 \quad (8.6)$$

In this example,  $H$  is not a normal subgroup of  $G$  because

$$(13)H = \{(13), \underbrace{(13)(12)}_{(123)}\} \quad (8.7)$$

and

$$H(13) = \{(13), \underbrace{(12)(13)}_{(213)}\} \quad (8.8)$$

are not equal.

---

2. ( $H$  normal because  $G$  is abelian)

$G = Z_6 = \{0, 1, 2, 3, 4, 5\}$  under addition mod 6.

$H = \{0, 3\}$

$G$  is abelian so every subgroup of  $G$  is normal

$G : H = \{0, 1, 2\}$ , coset representatives

$$\begin{cases} 0 + H = \{0, 3\} \\ 1 + H = \{1, 4\} \\ 2 + H = \{2, 5\} \end{cases} \quad (8.9)$$

3. ( $H$  is normal subgroup even though  $G$  is not abelian)

$S_3^e = \{e, (123), (132)\}$ , even number of cycles

$S_3^o = \{(12), (13), (23)\}$ , odd number of cycles

$G = S_3 = S_3^o \cup S_3^e$

$H = S_3^e$ .

$S_3^e \trianglelefteq S_3$  because  $\sigma S_3^e = S_3^e \sigma = S_3^o$  for all  $\sigma \in S_3$

$G : H = \{e, (12)\}$  coset representatives

Cosets

$$\begin{cases} eH &= S_3^e \\ (12)H &= S_3^o \end{cases} \quad (8.10)$$

$$S_3/S_3^e \cong Z_2 \quad (8.11)$$

## 8.2 Orbit-Stabilizer Theorem

Let a group  $G$  act on a set  $X$  by an action  $\odot$ . This means that  $\odot : G \times X \rightarrow X$  and  $(g, x) \mapsto g \odot x$ . Usually the **action**  $\odot$  is **group conjugation**.  $g \odot x = gxg^{-1}$ . In the special case of Lagrange's theorem,  $X = G$  and  $g \odot x = gx$ .

$Stab(x) = \{g \in G | g \odot x = x\}$  = **Stabilizer of  $x$** , the symmetries of  $x$ , group operations that keep  $x$  fixed or "stabilized".

$O(x) = \{g \odot x | g \in G\}$  = **Orbit of  $x$** , all possible projections of  $x$  via  $g \in G$ .

Fig.8.1 represents  $O(x)$  and  $Stab(x)$  pictorially.

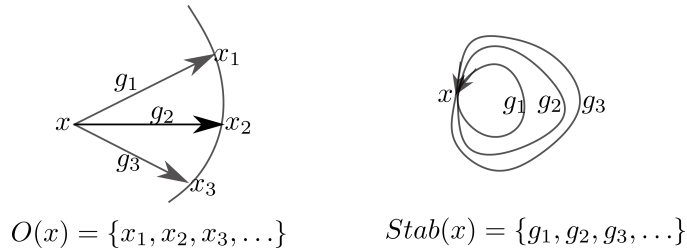


Figure 8.1: Pictorial representation of  $O(x)$  and  $Stab(x)$

**Claim 12** (*Orbit-Stabilizer theorem for single point  $x \in X$* )

$$|G| = |O(x)| |Stab(x)| \quad (8.12)$$

**proof:**

**QED**

$$Z(G) = \{z \in G | gzg^{-1} = z \text{ for all } g \in G\} = \textbf{center of } G$$

Suppose  $H$  is a subgroup of  $G$  (i.e.,  $H \leq G$ ) and the action is group conjugation. There are two ways of generalizing  $Stab(x)$ , to define it for a subgroup  $H$ , instead of a single point  $x$

$$\begin{cases} \cup_{h \in H} Stab(h) = \cup_{h \in H} \{g \in G | ghg^{-1} = h\} = C_G(H) & \textbf{Centralizer} \\ Stab(H) = \{g \in G | gHg^{-1} = H\} = N_G(H) & \textbf{Normalizer} \end{cases} \quad (8.13)$$

$N_G(H)$  is the biggest subset of  $G$  that leaves the subgroup  $H$  fixed under conjugation.  $N_G(H)$  is always a subgroup of  $G$ . If  $H$  is a normal subgroup of  $G$ , then  $N_G(H) = G$ .

$C_G(H)$  is a subgroup of  $N_G(H)$

$$\begin{pmatrix} Z(G) \\ H \end{pmatrix} \leq C_G(H) \leq N_G(H) \leq G \quad (8.14)$$

$O(H) = \cup_{h \in H} O(h)$  = projections (orbits) of structure(subgroup)  $H$  instead of single point  $H$

**Claim 13** (*Orbit-Stabilizer theorem for subgroup  $H \leq G$* )

$$|G| = |O(H)| \underbrace{|Stab(H)|}_{N_G(H)} \quad (8.15)$$

**proof:**

**QED**

Examples:

---

1.  $G = S_3$

$$x = (12)$$

$$Stab(x) = \{e, (12)\}$$

$$O(x) = \{(12), (13)(12)(13), (23)(12)(23)\} = \{(12), (23), (13)\}$$

$$|G| = 6, |O(x)| = 3, |Stab(x)| = 2 \implies |G| = |O(x)| |Stab(x)| \quad (8.16)$$


---

- error correction intuition

Pauli group  $= \{1, \sigma_x, \sigma_y, \sigma_z\} \times \{\pm 1, \pm i\}$ . Has 16 elements.

Clifford group = group generated by  $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  and  $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ . Has 24 elements

$G$  = Pauli or Clifford group

$x$  = code state

$O(x)$  = orbit (i.e., projections) of code state  $x$

$Stab(x)$  = subgroup of  $G$  that preserves (doesn't change, stabilizes, fixes) code state  $x$

$H$  = code space, subgroup of  $G$

$Stab(H)$  = subgroup of  $G$  that fixes code space  $H$

- thermodynamic intuition

$G$  = group

$x$  = single microstate

$O(x)$  = set of equivalent microstates = single macrostate

$\ln |O(x)|$  = entropy

$Stab(x)$  = subgroup of  $G$  that preserves (doesn't change, stabilizes, fixes) microstate  $x$ , residual symmetry group

- Lie groups intuition

$G$  = Lie group

$H$  = stabilizer, principal fibre

$G : H$  = base space

$\pi(g) = gH$  = projection, fibre

$\pi : G \rightarrow G/H = \{gH : g \in G\}$  fibre bundle

Example

$G = SU(2)$

$H = U(1)$  fibre

$G : H = S^2$  = two sphere, base space



# Chapter 9

## Lie Algebras

This chapter is based on Ref.[1].

### 9.1 Generators of Infinitesimal Transformations

For some group  $\mathcal{G}$ , assume that any group element  $G \in \mathcal{G}$  that is infinitesimal close to the identity 1 can be parametrized by

$$G = 1 + i \sum_i \epsilon_i T_i \quad (9.1)$$

where  $T_i \in \mathbb{C}^{n \times n}$  for  $i = 1, 2, \dots, N$ ,  $\epsilon_i \in \mathbb{R}$  and  $|\epsilon_i| \ll 1$ <sup>1</sup>.

The  $T_i$  matrices are called the **generators of infinitesimal transformations** for group  $\mathcal{G}$ . The generators of a group  $\mathcal{G}$  span a vector space called a Lie algebra  $\mathfrak{g}$ .<sup>2</sup> For example, the generators of the group  $SU(2)$  span the **Lie algebra**  $\mathfrak{su}(2)$ .

The tensor

$$g_{ij} = \text{tr}(T_i^\dagger T_j) \quad (9.2)$$

is called the **Cartan-Killing form**. This tensor can be used to raise and lower the the adjoint rep indices  $i, j, k$  in a tensor such as  $M_{ijk}$ :

$$M_{jk}^i = g^{ii'} M_{i'jk} \quad (9.3)$$

Assume that the  $T_i$  matrices are Hermitian and that they satisfy

$$g_{ij} = \text{tr}(T_i T_j) = \kappa \delta(i, j) \quad (9.4)$$

A Lie algebra that satisfies Eq.(9.4) is called a **simple Lie algebra**.

A **semi-simple Lie algebra** is a direct sum of simple Lie algebras.

---

<sup>1</sup>Note that the  $\epsilon_i$  are real, not complex.

<sup>2</sup>See Sec.A.7 for the definition of an algebra over a field.

It's customary to choose generators so that  $\kappa = \frac{1}{2}$ .<sup>3</sup> However, we will often set  $\kappa = 1$  for intermediate calculations and restore  $\kappa \neq 1$  at the end by dimensional analysis. Just remember that each  $T^j$  scales as  $\sqrt{\kappa}$ . For example, given the equation  $\text{tr}(T^i T^j) = \delta(i, j)$ , we know that when  $\kappa \neq 1$ ,  $\text{tr}(T^i T^j) = \kappa \delta(i, j)$  so both sides of the equation scale as  $\kappa$ .

We will use the following scaled version of  $T^j$  as a birdtrack. Define

$$(C_{adj}^i)_b^a = \frac{1}{\sqrt{\kappa}} (T^i)_b^a = \frac{1}{\sqrt{\kappa}} \begin{array}{c} a \\ \downarrow \\ i \text{ --- } T^i \\ \downarrow \\ b \end{array} \quad (9.5)$$

In the CC convention, we will always start reading the indices of this node at the wavy undirected green leg. *adj* stands the adjoint. In this node (vertex), an adj-rep particle (wavy line, gluon) is generated (released) by a def-rep particle (straight solid line, arrow).

In terms of birdtracks, Eq.(9.4) becomes

$$(T^i)_a^b (T^j)_b^a = \text{tr}(T^i T^j) = \delta(i, j) \quad i \text{ --- } T^i \begin{array}{c} \xrightarrow{\Sigma a} \\ \xleftarrow{\Sigma b} \end{array} T^j \text{ --- } j = \leftarrow \bullet \text{ ---} \quad (9.6)$$

We can now define the projection operator for the adj-rep. This projection operator represent a gluon exchange between 2 def-rep particles.

$$(P_{adj})_b^a{}_d^c = \sum_i (T^i)_b^a (T^i)_d^c \quad \begin{array}{c} b \text{ (green arrow)} \\ \swarrow \\ P_{adj} \\ \nwarrow \\ c \end{array} \quad \begin{array}{c} a \nearrow \\ \searrow \\ d \end{array} = \begin{array}{c} b \\ \uparrow \\ T^i \end{array} \text{ --- } \sum_i \begin{array}{c} \text{--- } T^i \\ \text{--- } \end{array} \begin{array}{c} c \\ \downarrow \\ T^i \\ \uparrow \\ d \end{array} \quad (9.7)$$

The green arrow is the first index in the CC convention.

Note that if  $x \in V^n \otimes V^{\dagger n}$ , then

$$(P_{adj})_b^a{}_d^c x_c^d = \sum_i (T^i)_b^a \underbrace{\left[ (T^i)_d^c x_c^d \right]}_{\epsilon_i \in \mathbb{R}} \quad (9.8)$$

---

<sup>3</sup>For  $SU(2)$ , it is customary to choose  $T^i = \frac{1}{2}\sigma_i$ , where  $\sigma_i$  for  $i = 1, 2, 3$  are the Pauli matrices. For  $SU(3)$ , it is customary to choose  $T^i = \frac{1}{2}\lambda_i$  where  $\lambda_i$  for  $i = 1, 2, \dots, 8$  are the Gell-Mann matrices. For both of these choices,  $\kappa = \frac{1}{2}$ .

## 9.2 Tensor Invariance Conditions

Recall Eq.(A.31). If  $x \in V^{n^p} \otimes V^{\dagger n^q}$ , and  $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}; \mathbb{C})$ ,

$$(x')_{a:p}^{b:q} = \mathbb{G}_{a:p}^{b:q}{}_{rev(c:q)}^{rev(d:p)} x_{d:p}^{c:q}, \quad x'_\alpha = \mathbb{G}_\alpha^\beta x_\beta \quad (9.9)$$

where we define

$$\mathbb{G}_\alpha^\beta \stackrel{\text{def}}{=} \prod_{i=1}^p G_{a_i}^{d_i} \prod_{i=1}^q G^{\dagger b_i}_{c_i} \quad (9.10)$$

If  $\mathbb{G}$  is infinitesimally close to the identity, then we can parametrize it as

$$\mathbb{G}_\alpha^\beta = 1 + i \sum_j \epsilon_j (M^j)_\alpha^\beta \quad (9.11)$$

$$G_{a_i}^{d_i} = 1 + i \sum_j \epsilon_j (T^j)_{a_i}^{d_i} \quad (9.12)$$

$$G^{\dagger b_i}_{c_i} = 1 - i \sum_j \epsilon_j (T^j)^{b_i}_{c_i} \quad (9.13)$$

Define

$$(M^j)_\alpha^\beta = \left[ (T^j)_{a_i}^{d_i} \frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}_{c_i} \frac{1}{\delta_{c_i}^{b_i}} \right] \delta_{a:p}^{d:p} \delta_{c:q}^{b:q} \quad (9.14)$$

When  $x'_\alpha = x_\alpha$ , to first order in  $\epsilon_i$ ,

$$0 = (M^j)_\alpha^\beta x_\beta = \left[ (T^j)_{a_i}^{d_i} \frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}_{c_i} \frac{1}{\delta_{c_i}^{b_i}} \right] \delta_{a:p}^{d:p} \delta_{c:q}^{b:q} x_{d:p}^{c:q} \quad (9.15)$$

For example, if we define

$$(M^j)_{a_1 a_2}^{b_1 c_1}{}_{d_2 d_1} = (T^j)_{a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{c_1}^{b_1} + \delta_{a_1}^{d_1} (T^j)_{a_2}^{d_2} \delta_{c_1}^{b_1} - \delta_{a_1}^{d_1} \delta_{a_2}^{d_2} (T^j)^{b_1}_{c_1}$$

$$= \begin{array}{c} \begin{array}{ccc} a_1 & \leftarrow & d_1 \\ a_2 & \leftarrow & d_2 \\ b_1 & \leftarrow & c_1 \end{array} \\ \begin{array}{ccc} a_1 & \leftarrow & T^j & \leftarrow & d_1 \\ a_2 & \leftarrow & d_2 \\ b_1 & \longrightarrow & c_1 \end{array} \end{array} + \begin{array}{c} \begin{array}{ccc} a_1 & \leftarrow & d_1 \\ a_2 & \leftarrow & T^j & \leftarrow & d_2 \\ b_1 & \longrightarrow & c_1 \end{array} \\ \begin{array}{ccc} a_1 & \leftarrow & T^j & \leftarrow & d_1 \\ a_2 & \leftarrow & d_2 \\ b_1 & \longrightarrow & c_1 \end{array} \end{array} - \begin{array}{c} \begin{array}{ccc} a_1 & \leftarrow & d_1 \\ a_2 & \leftarrow & d_2 \\ b_1 & \longrightarrow & T^j & \longrightarrow & c_1 \end{array} \end{array} \quad (9.16)$$

then

$$0 = (M^j x)_{a_1 a_2}^{b_1} = \left[ (T^j)_{a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{c_1}^{b_1} + \delta_{a_1}^{d_1} (T^j)_{a_2}^{d_2} \delta_{c_1}^{b_1} - \delta_{a_1}^{d_1} \delta_{a_2}^{d_2} (T^j)^{b_1}_{c_1} \right] x_{d_1 d_2}^{c_1}$$
(9.17)

We will refer to identities such as Eq.(9.16) and (9.17) as **tensor invariance conditions**.

### 9.3 Clebsch-Gordan Coefficients

The Clebsch Gordan (CG) coefficients are introduced in Ch.3. Note that the generators  $(T^i)_a^b$  are a simple kind of CG coefficient, one with

- a gluon (adj-rep) particle instead of a general  $\lambda$  rep particle emanating from the  $i$  index,
- a particle of the def-rep entering and another leaving the node, instead of any number of def-rep particles entering and leaving.

Since  $\mathbb{G} = 1 + i \sum_j \epsilon_j M^j$ , generators decompose in the same way as the group elements

$$M^j = \sum_{\lambda} C_{\lambda}^{\dagger} T_{\lambda}^j C_{\lambda}$$
(9.18)

The CG coefficients are invariant tensors.

$$C_{\lambda} = G_{\lambda}^{\dagger} C_{\lambda} G$$
(9.19)

Hence,

$$0 = -T_{\lambda}^j C_{\lambda} + C_{\lambda} T^j$$
(9.20)

Note that in the last equation,  $T_\lambda^j$  and  $T^j$  are different. In terms of birdtracks, we might have, for example,

$$0 = \left\{ \begin{array}{l} - \begin{array}{c} j \\ \textcolor{red}{\vdots} \\ a \leftarrow \cdots T_\lambda^j \leftarrow \cdots C_\lambda \begin{array}{l} \swarrow c_1 \\ \searrow c_2 \\ \searrow b_1 \end{array} \end{array} + \begin{array}{c} j \\ \textcolor{red}{\vdots} \\ T^j \leftarrow c_1 \\ \swarrow C_\lambda \begin{array}{l} \leftarrow c_2 \\ \searrow b_1 \end{array} \end{array} \\ + \begin{array}{c} j \\ \textcolor{red}{\vdots} \\ a \leftarrow \cdots C_\lambda \leftarrow T^j \leftarrow c_2 \\ \swarrow c_1 \\ \searrow b_1 \end{array} - \begin{array}{c} j \\ \textcolor{red}{\vdots} \\ a \leftarrow \cdots C_\lambda \begin{array}{l} \swarrow c_1 \\ \searrow c_2 \\ \searrow T^j \rightarrow b_1 \end{array} \end{array} \end{array} \right\} \quad (9.21)$$

Multiplying Eq.(9.21) on the left by  $C_\lambda^\dagger$ , and moving the first tem to the right side, we obtain an expression for the generator  $T_\lambda^i$  in term the generators  $T^j$  (and  $C_\lambda$  CG coefficients).

$$\begin{aligned} a \leftarrow \cdots T_\lambda^j \leftarrow \cdots a' &= \underbrace{a \leftarrow \cdots T_j \leftarrow C_\lambda \begin{array}{c} \swarrow \nwarrow \\ \searrow \swarrow \end{array} C_\lambda^\dagger \leftarrow a' - a \leftarrow C_\lambda \begin{array}{c} \swarrow \nwarrow \\ \searrow \swarrow \end{array} C_\lambda^\dagger \leftarrow T_j \leftarrow a'}_{=0} \\ + \begin{array}{c} j \\ \textcolor{red}{\vdots} \\ T^j \\ \swarrow \nwarrow \\ \searrow \swarrow \end{array} a \leftarrow C_\lambda \begin{array}{c} \swarrow \nwarrow \\ \searrow \swarrow \end{array} C_\lambda^\dagger \leftarrow a' &+ \begin{array}{c} j \\ \textcolor{red}{\vdots} \\ a \leftarrow C_\lambda \leftarrow T^j \leftarrow C_\lambda^\dagger \leftarrow a' - a \leftarrow C_\lambda \begin{array}{c} \swarrow \nwarrow \\ \searrow \swarrow \end{array} C_\lambda^\dagger \leftarrow T^j \end{array} \end{aligned} \quad (9.22)$$

The term with the underbrace in Eq.(9.22) does not come from Eq.(9.21). I included it to demonstrate to the reader that Eq.(9.22) is just another tensor invariance condition that touches all the incoming and outgoing arrows.

## 9.4 Structure Constants (3 gluon vertex)

A **Lie Algebra** is an algebra over the field  $\mathbb{C}$  such that its vector product is the matrix commutator (see Section A.7). Simply put, a Lie Algebra is a set of square Hermitian matrices  $\{T^i\}_{i=1}^N$  that satisfy

$$\underbrace{T^i T^j - T^j T^i}_{[T^i, T^j]} = i f_{ijk} T^k \quad (\text{Lie Algebra commutation relations})$$

$$\begin{array}{c} a \leftarrow T^i \leftarrow T^j \leftarrow c \\ \text{\scriptsize $\downarrow$} \qquad \text{\scriptsize $\downarrow$} \\ i \qquad j \end{array} \quad - \quad \begin{array}{c} a \leftarrow T^j \leftarrow T^i \leftarrow c \\ \text{\scriptsize $\swarrow$} \quad \text{\scriptsize $\searrow$} \\ i \qquad j \end{array} = i \quad \begin{array}{c} a \leftarrow T^k \leftarrow c \\ \text{\scriptsize $\downarrow$} \\ f_{ijk} \\ \text{\scriptsize $\swarrow$} \quad \text{\scriptsize $\searrow$} \\ i \qquad j \end{array} \quad (9.23)$$

The  $f_{ijk}$  tensors are called the **structure constants** of the Lie Algebra. They define a 3 gluon vertex in term of the generators  $T^i$ .<sup>4</sup>

If  $(T^j)_a{}^b$  are the rep-matrices (in the def-rep) of the generators of a group  $\mathcal{G}$ , then Eq.(9.23 ) shows that the matrices  $(M^k)_{ij} = if_{ijk}$  are also a rep-matrix (in the adj-rep) of the generators of  $\mathcal{G}$ .

Since  $\text{tr}(T^k T^{k'}) = \delta(k, k')$ , Eq.(9.23) implies

$$if_{ijk} = \text{tr}([T^i, T^j]T^k) = (T^i)_a{}^c (T^k)_c{}^b (T^j)_b{}^a - (T^i)_a{}^c (T^j)_c{}^b (T^k)_b{}^a$$

$$\begin{array}{c} i \\ \vdots \\ f_{ijk} \\ \swarrow \quad \searrow \\ j \quad k \end{array} = \begin{array}{c} i \\ \vdots \\ T^i \\ \swarrow \quad \searrow \\ \sum^a \quad \sum^c \\ \swarrow \quad \searrow \\ T^j \quad T^k \\ \swarrow \quad \searrow \\ j \quad k \end{array} - \begin{array}{c} i \\ \vdots \\ T^i \\ \swarrow \quad \searrow \\ \sum^a \quad \sum^c \\ \swarrow \quad \searrow \\ T^k \quad T^j \\ \swarrow \quad \searrow \\ j \quad k \end{array} \quad (9.24)$$

Note that

<sup>4</sup>It's possible to distinguish between upper and lower gluon indices (i.e., to give the gluon arrows a direction. In that case, the Lie Algebra commutation relations would be  $[T^i, T^j] = f^{ij}_k T^k$  and the gluon indices could be lowered and raised using the metric (called the **Cartan-Killing form**)  $g_{ij} = \text{tr}((T^i)^\dagger T^j)$ . But since we are assuming  $g_{ij} = \kappa \delta^j_i$ , there is no need to do this.

$f_{ijk} = -f_{jik}$

$$\begin{array}{ccc}
\begin{array}{c} k \\ \text{---} \\ f_{ijk} \\ \text{---} \\ i \quad j \end{array} & = - & \begin{array}{c} k \\ \text{---} \\ f_{ijk} \\ \text{---} \\ i \quad j \end{array} \quad (\text{Convention CC}) \\
\begin{array}{c} k \\ \text{---} \\ f_{ijk} \\ \text{---} \\ i \quad j \end{array} & = - & \begin{array}{c} k \\ \text{---} \\ f_{jik} \\ \text{---} \\ i \quad j \end{array} \quad (\text{Convention FL})
\end{array} \tag{9.25}$$

In fact, the tensor  $f_{ijk}$  is **totally antisymmetric** (i.e., it changes sign under a transposition of any two indices).

**Claim 14**  $f_{ijk}$  is a real number.

**proof:**

$$[i\text{tr}([T^i, T^j]T^k)]^\dagger = (-i)\text{tr}(T^k[T^j, T^i]) \tag{9.26}$$

$$= (-i)\text{tr}([T^j, T^i]T^k) \tag{9.27}$$

$$= i\text{tr}([T^i, T^j]T^k) \tag{9.28}$$

**QED**

Note that the birdtrack for the Lie Algebra commutation relations Eq.(9.23) can be understood as the statement that the generators  $T^j$  are invariant matrices. Below we restate Eq.(9.23) to make that obvious

$$0 = \begin{array}{c} a \leftarrow T^i \leftarrow T^j \leftarrow c \\ \text{---} \quad \text{---} \\ i \quad j \end{array} - \begin{array}{c} a \leftarrow T^j \leftarrow T^i \leftarrow c \\ \text{---} \quad \text{---} \\ i \quad j \end{array} - i \begin{array}{c} a \leftarrow T^k \leftarrow c \\ \text{---} \\ f_{ijk} \\ \text{---} \\ i \quad j \end{array} \tag{9.29}$$

**Claim 15**

$$f_{ijm}f_{mkl} - f_{ljm}f_{mki} = f_{iml}f_{jkm} \quad (\text{Jacobi identity})$$

$$\begin{array}{ccc}
 \begin{array}{c} i \\ \vdots \\ f_{ljm} \sim \sum_m \sim f_{mki} \\ \vdots \\ j \end{array} & - & \begin{array}{c} i \quad l \\ \diagdown \quad \diagup \\ \quad \quad \quad \\ \vdots \quad \quad \quad \vdots \\ j \quad \quad \quad k \end{array} \\
 & & = \begin{array}{c} i \quad l \\ \diagdown \quad \diagup \\ \quad \quad \quad \\ f_{iml} \\ \sum_m \\ f_{jkm} \\ \diagup \quad \diagdown \\ j \quad \quad \quad k \end{array}
 \end{array} \quad (9.30)$$

**proof:**

Note that

$$\text{tr} \left( [[T^i, T^j], T^k] T^l \right) = \text{tr} \left( f_{ijm} [T^m, T^k] T^l \right) \quad (9.31)$$

$$= \text{tr} \left( f_{ijm} f_{mkl} T^l T^l \right) \quad (9.32)$$

$$= f_{ijm} f_{mkl} \quad (9.33)$$

so the Jacobi identity can be restated as

$$\text{tr} \left( \left\{ [[T^i, T^j], T^k] + [[T^j, T^k], T^i] + [[T^k, T^i], T^j] \right\} T^l \right) = 0 \quad (9.34)$$

Hence, the claim follows if we can prove that

$$\underbrace{[[T^i, T^j], T^k] + [[T^j, T^k], T^i] + [[T^k, T^i], T^j]}_{\text{cyclic permutations of } ijk} = 0 \quad (9.35)$$

If we expand the left hand side on Eq.(9.35), we find 6 terms that cancel in pairs.

**QED**

Note Claim 15 can be understood as the Lie Algebra commutation relations Eq.(9.23), but stated in the adj-rep instead of the def-rep. Indeed, if

$$M_{jk}^i = i f_{ijk} \quad (9.36)$$

then Claim 15 becomes

$$(M^i M^l - M^l M^i)_{jk} = i f_{ilm} (M^m)_{jk} \quad (9.37)$$

Note that Claim 15 can be understood as a statement of the fact that  $f_{ijk}$  is an invariant tensor.



$$0 = f_{ijm}f_{mkl} - f_{ljm}f_{mki} - f_{iml}f_{jkm}$$

$$0 = \begin{array}{c} i \\ \text{wavy} \\ f_{ijm} \sim \sum^m \sim f_{mkl} \\ \text{wavy} \\ j \end{array} \begin{array}{c} l \\ \text{wavy} \\ f_{mkl} \\ \text{wavy} \\ k \end{array} - \begin{array}{c} i \\ \text{wavy} \\ f_{ljm} \sim \sum^m \sim f_{mki} \\ \text{wavy} \\ j \end{array} \begin{array}{c} l \\ \text{wavy} \\ f_{mki} \\ \text{wavy} \\ k \end{array} - \begin{array}{c} i \\ \text{wavy} \\ f_{iml} \\ \text{wavy} \\ \sum^m \\ f_{jkm} \\ \text{wavy} \\ j \end{array} \begin{array}{c} l \\ \text{wavy} \\ f_{jkm} \\ \text{wavy} \\ k \end{array} \quad (9.38)$$

## 9.5 Other Forms of Lie Algebra Commutators

Consider the following two gluon exchange operators. Note that  $\mathbb{P}^2 = \mathbb{P}$ , but  $\mathbb{Q}^2 \neq \mathbb{Q}$ , so  $\mathbb{P}$  is a bonafide projection operator but  $\mathbb{Q}$  isn't.  $\mathbb{Q}\mathbb{Q}^\dagger = \mathbb{P}$  so  $\mathbb{Q}$  behaves like half of a projection operator.

$$\mathbb{P}_a^b{}_c^d = \sum_i (T^i)_a^b (T^i)_c^d \quad \begin{array}{c} a \quad d \\ \swarrow \quad \searrow \\ \mathbb{P} \\ \swarrow \quad \searrow \\ b \quad c \end{array} = \begin{array}{c} a \quad d \\ \uparrow \quad \downarrow \\ T^i \sim \sum^i \sim T^i \\ \uparrow \quad \downarrow \\ b \quad c \end{array} \quad (9.39)$$

$$\mathbb{Q}_a^b{}_Y^X = \sum_i (T^i)_a^b (T_\lambda^i)_Y^X \quad \begin{array}{c} a \quad X \\ \swarrow \quad \searrow \\ \mathbb{Q} \\ \swarrow \quad \searrow \\ b \quad Y \end{array} = \begin{array}{c} a \quad X \\ \uparrow \quad \Downarrow \\ T^i \sim \sum^i \sim T_\lambda^i \\ \uparrow \quad \Downarrow \\ b \quad Y \end{array} \quad (9.40)$$

**Claim 16** If  $\mathbb{Q}_b^a$  is the matrix with  $(Z, X)$  entries  $\mathbb{Q}_b^a{}_Z^X$ , then

$$[\mathbb{Q}_b^a, \mathbb{Q}_d^c] = \mathbb{P}_{c'}^{ca} \mathbb{Q}_d^{c'} - \mathbb{Q}_{d'}^c \mathbb{P}_d^{d'a}{}_b \quad (9.41)$$

**proof:**

This claim can be visualized as follows.  $\mathbb{Q}$  is an invariant tensor so

$$0 = \left\{ \begin{array}{c} \begin{array}{ccc} c & & d \\ & \searrow & \nearrow \\ & i & \\ Y \leftarrow T_\lambda^i \leftarrow Q \leftarrow X & & \\ c \rightarrow T^i \rightarrow Q \rightarrow d & & \\ & \nearrow & \nwarrow \\ & i & \\ Y & & X \end{array} & - & \begin{array}{ccc} c & & d \\ & \searrow & \nearrow \\ & i & \\ Y \leftarrow Q \leftarrow T_\lambda^i \leftarrow X & & \\ c \rightarrow Q \rightarrow T^i \rightarrow d & & \\ & \nearrow & \nwarrow \\ & i & \\ Y & & X \end{array} \end{array} \right\} \quad (9.42)$$

Now multiplying by  $(T^i)_a^b$ , we get

$$\begin{array}{c} \mathbb{Q}_b^a \mathbb{Q}_d^c \mathbb{Q}_Y^Z \mathbb{Q}_d^c \mathbb{Q}_b^a - \mathbb{Q}_d^c \mathbb{Q}_Y^Z \mathbb{Q}_b^a \mathbb{Q}_d^c = \mathbb{P}_{c'}^{ca} \mathbb{Q}_d^{c'} \mathbb{Q}_Y^X - \mathbb{Q}_{d'}^c \mathbb{Q}_Y^X \mathbb{P}_d^{d'a} \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} c & & d \\ & \searrow & \nearrow \\ & a & \\ Y \leftarrow Q \leftarrow Q \leftarrow X & & \\ c \rightarrow T^i \rightarrow Q \rightarrow d & & \\ & \nearrow & \nwarrow \\ & a & \\ Y & & X \end{array} & - & \begin{array}{ccc} c & & d \\ & \searrow & \nearrow \\ & a & \\ Y \leftarrow Q \leftarrow Q \leftarrow X & & \\ c \rightarrow Q \rightarrow T^j \rightarrow d & & \\ & \nearrow & \nwarrow \\ & a & \\ Y & & X \end{array} \end{array} = \quad (9.43)$$

Finally, if we hide the capital letter indices to obtain a statement about matrices with capital letter indices, we get

$$\mathbb{Q}_b^a \mathbb{Q}_d^c - \mathbb{Q}_d^c \mathbb{Q}_b^a = \mathbb{P}_{c'}^{ca} \mathbb{Q}_d^{c'} - \mathbb{Q}_{d'}^c \mathbb{P}_d^{d'a} \quad (9.44)$$

**QED**

# Chapter 10

## Lie Algebras of Classical Groups

In this chapter, we present an overview of the Lie algebras for the classical simple Lie groups.

Recall from Chapter 4 that the Lie algebras of the simple Lie groups can be divided into two classes: the Classical and the Exceptional. The Lie algebras of the classical simple Lie groups are

- $A_k = \mathfrak{su}(k+1)$
- $B_k = \mathfrak{so}(2k+1)$
- $C_k = \mathfrak{sp}(2k)$
- $D_k = \mathfrak{so}(2k)$

and the Lie algebras of Exceptional simple Lie groups are:

- $E_6, E_7, E_8$
- $F_4$
- $G_2$

### 10.1 $SU(n)$

$$SU(n) = \{U \in \mathbb{C}^{n \times n} : U^\dagger U = I, \det U = 1\} \quad (10.1)$$

Note that

$$1 = U^\dagger U = e^{X^\dagger} e^X \implies X^\dagger = -X \quad (10.2)$$

$$1 = \det U = \det e^X = e^{\text{tr} X} \implies \text{tr} X = 0 \quad (10.3)$$

Thus:

$$\mathfrak{su}(n) = \{X \in \mathbb{C}^{n \times n} : X^\dagger = -X, \text{tr} X = 0\} \quad (10.4)$$

**Claim 17** (*Real dimension*)

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1 \quad (10.5)$$

**proof:**

Real parameter count:

- +1 for each diagonal entry
- +1 for the real part of each entry above the diagonal
- +1 for the imaginary part of each entry below the diagonal
- -1 for the zero trace constraint.

This adds up to  $n^2 - 1$ .

**QED**

Let  $X_j$  be the generators of the Lie algebra, and set

$$X_j = -iT_j. \quad (10.6)$$

By definition, the generators  $X_j$  are closed under commutation

$$[X_i, X_j] = f_{ijk}X_k \implies [T_i, T_j] = if_{ijk}T_k \quad (10.7)$$

where  $f_{ijk}$  are the real structure constants of  $\mathfrak{su}(n)$

$$X^\dagger = -X, \quad \text{tr} X = 0 \implies (T_i)^\dagger = T_i, \quad \text{tr}(T_i) = 0 \quad (10.8)$$

It's also possible to assume

$$\text{tr}(T_i T_j) = \frac{1}{2} \delta_{ij} \quad (10.9)$$

For  $SU(2)$ ,  $T^i = \frac{1}{2} \sigma_i$  (Pauli matrices)

For  $SU(3)$ ,  $T^i = \frac{1}{2} \lambda_i$  (Gel-Mann matrices)

## 10.2 $SO(n)$

$$SO(n) = \{G \in \mathbb{R}^{n \times n} : G^T G = 1, \quad \det G = 1\} \quad (10.10)$$

$$\mathfrak{so}(n) = \{X \in \mathbb{R}^{n \times n} | X^T = -X\} \quad (10.11)$$

Note that  $X^T = -X$  implies that the diagonal of  $X$  is zero, so  $\text{tr} X = 0$  automatically.

**Claim 18**

$$\dim_{\mathbb{R}} \mathfrak{so}(n) = \frac{n(n-1)}{2} \quad (10.12)$$

**proof:**

Real parameter count

- +1 for each entry above the diagonal

This adds up to  $\frac{n^2-n}{2} = \frac{n(n-1)}{2}$ .

**QED**

Define the generator basis  $\{L_{ij}|i < j\}$ , where

$$(L_{ij})_{ab} = \delta_{ia}\delta_{jb} - \delta_{ja}\delta_{ib} = \begin{array}{c} i \quad j \\ \searrow \quad \swarrow \\ a \quad b \end{array} - \begin{array}{c} i \quad j \\ \swarrow \quad \searrow \\ a \quad b \end{array} = \begin{array}{c} i \quad j \\ \searrow \quad \swarrow \\ a \leftarrow L \leftarrow b \end{array} \quad (10.13)$$

Note that  $L_{ij}^T = L_{ji} = -L_{ij}$  and  $L_{ii} = 0$ . Hence these generators satisfy  $X^T = -X$ . Physicists often define generators  $T_{ij} = iL_{ij}$  that are Hermitian.

**Claim 19**

$$[L_{ij}, L_{kl}] = \delta_{jk}L_{il} - \delta_{ik}L_{jl} - \delta_{jl}L_{ik} + \delta_{il}L_{jk}$$

$$= \left\{ \begin{array}{c} \begin{array}{c} i \quad j \quad k \quad l \\ \searrow \quad \swarrow \quad \searrow \quad \swarrow \\ \leftarrow L \leftarrow L \leftarrow \end{array} - \begin{array}{c} k \quad l \quad i \quad j \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \leftarrow L \leftarrow L \leftarrow \end{array} \\ + \begin{array}{c} i \quad j \quad k \quad l \\ \searrow \quad \swarrow \quad \searrow \quad \swarrow \\ \leftarrow L \leftarrow \end{array} - \begin{array}{c} i \quad j \quad k \quad l \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \leftarrow L \leftarrow \end{array} \\ - \begin{array}{c} i \quad j \quad k \quad l \\ \searrow \quad \swarrow \quad \searrow \quad \swarrow \\ \leftarrow L \leftarrow \end{array} - \begin{array}{c} i \quad j \quad k \quad l \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \leftarrow L \leftarrow \end{array} \end{array} \right\} \quad (10.14)$$

**proof:**

Just calculate the commutator  $[L_{ij}, L_{kl}]$  using the definition Eq.(10.13) of  $L_{ij}$

$$L_{ij}L_{kl} = \left[ \begin{array}{c} i \quad j \\ \searrow \quad \swarrow \\ \end{array} - \begin{array}{c} i \quad j \\ \swarrow \quad \searrow \\ \end{array} \right] \left[ \begin{array}{c} k \quad l \\ \searrow \quad \swarrow \\ \end{array} - \begin{array}{c} k \quad l \\ \swarrow \quad \searrow \\ \end{array} \right] \quad (10.15)$$

$$= (i \quad j \quad k \quad l) + (i \quad j \quad k \quad l) - (i \quad j \quad k \quad l) - (i \quad j \quad k \quad l) \quad (10.16)$$

$$L_{kl}L_{ij} = \left[ \begin{array}{c} k \quad l \\ \searrow \quad \swarrow \\ \end{array} - \begin{array}{c} k \quad l \\ \swarrow \quad \searrow \\ \end{array} \right] \left[ \begin{array}{c} i \quad j \\ \searrow \quad \swarrow \\ \end{array} - \begin{array}{c} i \quad j \\ \swarrow \quad \searrow \\ \end{array} \right] \quad (10.17)$$

$$= (k \quad l \quad i \quad j) + (k \quad l \quad i \quad j) - (k \quad l \quad i \quad j) - (k \quad l \quad i \quad j) \quad (10.18)$$

$$(i \frown j \frown k \frown l) - (k \frown l \frown i \frown j) = \{i \frown j \frown k \frown l\} = \delta_{jk} L_{il} \quad (10.19)$$

$$[L_{ij}, L_{kl}] = \{i \frown j \frown k \frown l\} + \{i \frown j \frown k \frown l\} - \{i \frown j \frown k \frown l\} - \{i \frown j \frown k \frown l\} \quad (10.20)$$

**QED**

### 10.3 $Sp(n)$

Assume  $n$  is even. Then

$$Sp(n) = \{U \in \mathbb{C}^{n \times n} : U^\dagger U = I, \quad U^T J U = J\} \quad (10.21)$$

where

$$J = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \quad (10.22)$$

Note that  $J$  satisfies

$$J^T = -J, \quad J^T J = 1 \quad (10.23)$$

Note that

$$1 = U^\dagger U = e^{X^\dagger} e^X \implies X^\dagger = -X. \quad (10.24)$$

$$U^T J U = J \implies 1 = J^T U^T J U = e^{J^T X^T J} e^X \implies J^T X^T J = -X \quad (10.25)$$

Hence

$$\mathfrak{sp}(n) = \{X \in \mathbb{C}^{n \times n} : X^\dagger = -X, \quad J^T X^T J = -X\} \quad (10.26)$$

**Claim 20**  $X^\dagger = -X$  and  $J^T X^T J = -X$  iff  $X$  has the **A-B form**

$$X = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \quad (10.27)$$

where

$$A^\dagger = -A, \quad B^T = B \quad (10.28)$$

**proof:**

Let

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (10.29)$$

Then

$$J^T X^T J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (10.30)$$

$$= \begin{pmatrix} -B^T & -D^T \\ A^T & C^T \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (10.31)$$

$$= \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \quad (10.32)$$

$$J^T X^T J + X = \begin{pmatrix} D^T + A & -B^T + B \\ -C^T + C & A^T + D \end{pmatrix} = 0 \quad (10.33)$$

Hence

$$\boxed{B^T = B, \quad C^T = C, \quad A^T = -D} \quad (10.34)$$

Also

$$X^\dagger + X = \begin{pmatrix} A^\dagger + A & C^\dagger + B \\ B^\dagger + C & D^\dagger + D \end{pmatrix} = 0 \quad (10.35)$$

Hence

$$\boxed{A^\dagger = -A, \quad D^\dagger = -D, \quad C^\dagger = -B} \quad (10.36)$$

Combining Eqs.(10.34) and (10.36), we get

$$C = -B^*, \quad D = A^* \quad (10.37)$$

**QED**

Let

$$X = X_A + \epsilon_B \quad (10.38)$$

where

$$X_A = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}, \quad X_A^\dagger = -X_A \quad (10.39)$$

and

$$\epsilon_B = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \quad \epsilon_B^\dagger = -\epsilon_B \quad (10.40)$$

Note that

$$J^T \epsilon_B^\dagger J = -\epsilon_B, \quad J^T X_A^\dagger J = -X_A \quad (10.41)$$

**Claim 21** (*Lie Algebra is closed*)

$$[X_1, X_2] \in \mathfrak{sp}(n), \quad (10.42)$$

**proof:**

If  $X, Y \in \mathfrak{sp}(n)$ ,

$$[X, Y]^\dagger = [Y^\dagger, X^\dagger] = [-Y, -X] = -[X, Y] \quad (10.43)$$

$$X_A \epsilon_B = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \quad (10.44)$$

$$= \begin{pmatrix} 0 & AB \\ -(AB)^* & 0 \end{pmatrix} \quad (10.45)$$

$$= \epsilon_{AB} \quad (10.46)$$

$$\epsilon_B X_A = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \quad (10.47)$$

$$= \begin{pmatrix} 0 & BA^* \\ -B^* A & 0 \end{pmatrix} \quad (10.48)$$

$$= \epsilon_{BA^*} \quad (10.49)$$

$$X_{A_1} X_{A_2} = X_{A_1 A_2} \quad (10.50)$$

$$\epsilon_{B_1} \epsilon_{B_2} = \begin{pmatrix} 0 & B_1 \\ -B_1^* & 0 \end{pmatrix} \begin{pmatrix} 0 & B_2 \\ -B_2^* & 0 \end{pmatrix} \quad (10.51)$$

$$= \begin{pmatrix} -B_1 B_2^* & 0 \\ 0 & -B_1^* B_2 \end{pmatrix} \quad (10.52)$$

$$= X_{-B_1 B_2^*} \quad (10.53)$$

$$[X_A, \epsilon_B] = \epsilon_{AB - BA^*} \quad (10.54)$$

$$[X_{A_1}, X_{A_2}] = X_{[A_1, A_2]} \quad (10.55)$$

$$[\epsilon_{B_1}, \epsilon_{B_2}] = X_{-B_1 B_2^* - B_2 B_1^*} \quad (10.56)$$

**QED**



**Claim 22** (*Real dimension*)

$$\dim_{\mathbb{R}} \mathfrak{sp}(n) = n(2n + 1) \quad (10.57)$$

**proof:**

$A^\dagger = -A$  so  $A$  contributes  $n^2$  real parameters.

$B^T = B$  so  $B$  contributes

$$\frac{(n^2 - n)}{2} + n = \frac{(n^2 + n)}{2} = \frac{n(n + 1)}{2} \quad (10.58)$$

complex parameters.

This adds up to

$$n^2 + n(n + 1) = n(2n + 1) \quad (10.59)$$

real parameters.

**QED**

# Chapter 11

## Orthogonal Groups

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

$O(n)$  and  $SO(n)$  are defined as the following groups of real matrices:

$$O(n) = \{G \in \mathbb{R}^{n \times n} : G^T G = 1\} \quad (11.1)$$

$$SO(n) = \{G \in O(n) : \det G = 1\} \quad (11.2)$$

$O(n)$  contains orthogonal matrices  $G$  ( $G^T G = G G^T = 1$ ) with  $\det G \in Z_2 = \{-1, 1\}$ .  $SO(n)$  only contains those with  $\det G = 1$ .  $O(n) \cong Z_2 \times SO(n)$  where  $Z_2 = \{-1, 1\}$  corresponds to the sign of  $\det G$ . Hence,  $O(n)$  is a **double cover** of  $SO(n)$ .  $O(n)$  consists of 2 **connected components** (CC), whereas  $SO(n)$  has only one CC.

An example of a  $G \in O(n)$  that is not in  $SO(n)$  is a reflection  $Gx = -x$  for odd  $n$ . Another example for  $O(2)$  is

$$\text{rotation: } G = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \det G = 1 \quad (11.3)$$

$$\text{improper rotation: } G = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \quad \det G = -1 \quad (11.4)$$

The irreps of  $SO(n)$  depend on whether  $n$  is even or odd, and whether the rep is spinor or non-spinor. This chapter deals with **non-spinor reps** (with either even or odd  $n$ ). Chapter 17 deals with the **spinor reps**.

We assume that the **metric tensor**  $g_{\mu\nu}$  is a primitive invariant that satisfies:

$$g_{\mu\nu} = g_{\nu\mu} = [g]_{\mu\nu}, \quad g^{\mu\nu} = g^{\nu\mu} = [g]_{\mu\nu}, \quad g_\mu{}^\nu = g_\nu{}^\mu = \delta_\mu^\nu \quad (11.5)$$

$$g_{\nu\mu} x^\mu = x_\nu, \quad g^{\rho\nu} x_\nu = x^\rho \quad (\text{so } g_{\nu\mu} g^{\rho\nu} = \delta_\mu^\rho) \quad (11.6)$$

where  $\mu, \nu, \rho \in \{1, 2, \dots, n\}$  and  $x_\mu$  is any tensor.

In this section, we will call **orthogonal groups** the group of matrices under which the following symmetric quadratic form is invariant

$$h(x) = g_{\mu\nu} x^\mu x^\nu \quad (11.7)$$

where  $\mu, \nu \in \{1, \dots, n\}$ . Thus

$$h(Gx) = h(x) \quad (11.8)$$

$$g_{\mu\nu} G^\mu{}_\alpha G^\nu{}_\beta x^\alpha x^\beta = g_{\alpha\beta} x^\alpha x^\beta \implies g_{\mu\nu} G^\mu{}_\alpha G^\nu{}_\beta = g_{\alpha\beta} \implies G^T g G = g \quad (11.9)$$

This condition guarantees that  $G \in O(n)$  is orthogonal for  $g_{\mu\nu} = \delta_\mu^\nu$  but not that  $\det G = 1$ . When  $g_{\mu\nu}$  is not the Kronecker delta function, we get a different group. For example, if  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  and  $\det G = 1$ , we get the **Lorentz group**  $SO(1, 3)$  used in Special Relativity.

In this chapter (and in this book), we will point the arrows in a birdtrack so that the birdtrack is a DAG. Cycles that make the birdtrack not acyclic will have a segment in red. Without that red segment, the birdtrack becomes acyclic. The reason we follow this arrow convention is that it promotes acyclic birdtracks which are more akin to bnets. We will eschew undirected birdtracks for the same reason: bnets are directed.

Let

$$g_\mu{}^\nu = \delta_\mu^\nu, \quad \longleftarrow g \longleftarrow = \longleftarrow \quad (11.10)$$

$$g^\mu{}_\nu = \delta_\nu^\mu, \quad \longrightarrow g \longrightarrow = \longrightarrow \quad (11.11)$$

$$g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu^\nu, \quad \longleftarrow \underline{g} \longrightarrow \bar{g} \longleftarrow = \longleftarrow \quad (11.12)$$

Note that we used

$$\underline{g} = [g_{\mu\nu}], \quad \bar{g} = [g^{\mu\nu}] \quad (11.13)$$

We could write Eq.(11.12) without the overline and underline on  $g$ . Those  $g$ -decorations are redundant as omitting them would not introduce any ambiguity. However, we will use them because they make spotting errors in the arrow directions easier.

The generators of orthogonal groups will be represented by:

$$(T_i)_\mu{}^\nu = \begin{array}{c} \textcolor{violet}{\curvearrowright} \\ \longleftarrow T_i \longleftarrow \end{array} \quad (11.14)$$

We will also use

$$(T_i)^\mu{}_\nu = \begin{array}{c} \text{green wavy line} \\ \longrightarrow \bar{g}T_i \underline{g} \longrightarrow \end{array} \quad (T_i)_{\mu\nu} = \begin{array}{c} \text{green wavy line} \\ \longleftarrow T_i \underline{g} \longrightarrow \end{array} \quad (T_i)^{\mu\nu} = \begin{array}{c} \text{green wavy line} \\ \longrightarrow \bar{g}T_i \longleftarrow \end{array} \quad (11.15)$$

For  $G \in O(n)$ ,  $G^T G = 1$  with  $G = e^{iT_i \epsilon_i}$  where  $\epsilon_i \in \mathbb{R}$ . Hence, the generators  $T_i$  must be anti-symmetric ( $T_i^T = -T_i$ ).

$$\boxed{(T_i)_{\mu\nu} = -(T_i)_{\nu\mu}} \quad \begin{array}{c} \text{wavy line} \\ \mu \longleftarrow T_i \longrightarrow \nu \end{array} = - \begin{array}{c} \mu \longleftarrow T_i \longrightarrow \nu \\ \text{wavy line} \end{array} \quad (11.16)$$

$g_\mu^\nu = \delta_\mu^\nu$  is obviously an invariant matrix.  $g_{\mu\nu}$  must be invariant too, so

$$\boxed{\underbrace{(T_i)_\mu{}^\sigma g_{\sigma\nu} + (T_i)_\nu{}^\sigma g_{\mu\sigma}}_{(T_i)_{\mu\nu} = -(T_i)_{\nu\mu}} = 0} \quad \begin{array}{c} \text{wavy line} \\ \mu \longleftarrow T_i \longleftarrow \underline{g} \longrightarrow \nu \end{array} + \begin{array}{c} \mu \longleftarrow \underline{g} \longrightarrow T_i \longrightarrow \nu \\ \text{wavy line} \end{array} = 0 \quad (11.17)$$

$\underbrace{\hspace{10em}}_{(T_i)_{\mu\nu}} \quad \underbrace{\hspace{10em}}_{(T_i)_{\nu\mu}}$

Hence, the invariance condition Eq.(11.17) reduces to to the statement that  $(T_i)_{\mu\nu}$  is antisymmetric.

The anti-symmetrizer  $\mathcal{A}_2$  is an invariant tensor (see Section 22.3). Other projectors of the  $V \otimes V$  are not invariant tensors. Therefore, we must have

$$\begin{array}{c} \text{curved arrows} \\ T_i \underline{g} \text{---} \text{wavy line} \text{---} \bar{g} T_i \end{array} = \begin{array}{c} \longleftarrow \mathcal{A}_2 \longleftarrow \\ \parallel \\ \longleftarrow \end{array} \quad (11.18)$$

For  $SO(n)$  and  $O(n)$ , the dimension  $N$  of the adjoint rep (= number of generators) is

$$N = \frac{n(n-1)}{2} = \begin{array}{c} \text{red curved arrow} \\ \text{wavy line} \end{array} \quad (11.19)$$

If you take an  $n \times n$  matrix and remove its diagonal, this  $N$  is the number of entries in the upper (or lower) triangular sector of the matrix. Recall that for  $U(n)$ ,  $N = n^2$ , and for  $SU(n)$ ,  $N = n^2 - 1$ . So for  $U(n)$  (or  $SU(n)$ ), there is a generator for each entry (or each entry minus one) of an  $n \times n$  matrix.

### Claim 23

$$\Gamma_{fun} \delta_\mu^\nu = \sum_i (T_i T_i)_\mu{}^\nu = \frac{n-1}{2} \delta_\mu^\nu$$

$$\sum_i \mu \longleftarrow T_i \overset{i}{\longleftrightarrow} T_i \longleftarrow \nu = \left(\frac{n-1}{2}\right) \mu \longleftarrow \bullet \longleftarrow \nu$$

**proof:**

$$(T_i T_i)_\mu{}^\nu = \mu \longleftarrow T_i \overbrace{g} \longrightarrow \bar{g} T_i \longleftarrow \nu \quad (11.21)$$

$$= \begin{array}{c} \mu \leftarrow \quad \quad \quad \nu \\ \quad \quad \quad \downarrow \quad \quad \downarrow \\ T_i \underline{g} \rightsquigarrow \bar{g} T_i \\ \quad \quad \quad \uparrow \quad \quad \uparrow \\ \quad \quad \quad \leftarrow \quad \quad \rightarrow \end{array} \quad (11.22)$$

$$= \frac{1}{2} \left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right] - \left[ \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \right] \quad (11.23)$$

$$= \binom{n-1}{2} \mu \leftarrow \bullet \rightarrow \nu \quad (11.24)$$

QED

### 11.1 $V_{def} \otimes V_{def}$ Decomposition

Let

$$V_{def} = V = \text{vector space in defining representation } \{|\mu\rangle\}_{\mu=1}^n.$$

Note that the symmetrizer  $\mathcal{S}_2$  originally has two upper and two lower indices.

Its two upper indices can be lowered using the metric tensor:

$$(\mathcal{S}_2)_{\mu\nu,\rho\sigma} = g_{\rho\rho'} g_{\sigma\sigma'} (\mathcal{S}_2)_{\mu\nu}{}^{\rho'\sigma'} \quad (11.25)$$

$$= \frac{1}{2} (g_{\mu\nu} g_{\nu\rho} + g_{\mu,\rho} g_{\mu\sigma}) \quad (11.26)$$

$$= \frac{\leftarrow \mathcal{S}_2 \leftarrow \underline{g} \rightarrow}{\leftarrow \parallel \leftarrow g \rightarrow} \quad (11.27)$$

$$= \frac{\leftarrow \mathcal{S}_2 \leftarrow \underline{g} \rightarrow}{\leftarrow \parallel \leftarrow g \rightarrow} \quad (11.28)$$

Likewise

$$\begin{array}{c} \leftarrow \underline{g} \leftarrow \\ \leftarrow \underline{g} \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \underline{g} \rightarrow \\ \parallel \\ \leftarrow \underline{g} \rightarrow \end{array} + \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \underline{g} \rightarrow \\ \parallel \\ \leftarrow \underline{g} \rightarrow \end{array} \quad (11.29)$$

Define tensor  $M$  by

$$M_{\mu\nu}{}^{\rho\sigma} = g_{\mu\nu}g^{\rho\sigma} = \begin{array}{c} \mu \leftarrow \\ \leftarrow \nu \end{array} \underline{g} \begin{array}{c} \rho \\ \downarrow \\ \uparrow \\ \sigma \end{array} \quad (11.30)$$

Note that

$$M^2 = \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \underline{g} \begin{array}{c} \downarrow \\ \uparrow \end{array} \underline{\bar{g}} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \underline{g} \begin{array}{c} \downarrow \\ \uparrow \end{array} \underline{\bar{g}} = nM \quad (11.31)$$

Hence,  $(M - n)M = 0$  so  $M$  has two eigenvalues  $\lambda = 0, n$ .

Next we will use the following equation from Chapter 16<sup>1</sup> to obtain a projection (PO) operator for each eigenvalue

$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (11.32)$$

1. Singlet PO

$$(P_S)_{\mu\nu}{}^{\rho\sigma} = \frac{1}{n} \begin{array}{c} \mu \leftarrow \\ \leftarrow \nu \end{array} \underline{g} \begin{array}{c} \rho \\ \downarrow \\ \uparrow \\ \sigma \end{array} \quad (11.33)$$

$$\dim(P_S) = \frac{1}{n} \begin{array}{c} \leftarrow \quad \leftarrow \\ \underline{g} \quad \underline{\bar{g}} \\ \rightarrow \quad \rightarrow \end{array} \quad (11.34)$$

$$= 1 \quad (11.35)$$

2. Traceless Symmetric PO<sup>2</sup>

---

<sup>1</sup>Note that this equation projects to zero all eigenvalues except one.

<sup>2</sup>Traceless here refers to  $P_a{}^a{}_c{}^c V_d{}^d = (PV)_a{}^a = 0$  for any vector  $V_d{}^d$ . It does not refer to  $P_a{}^b{}_b{}^a = 0$

$$(P_{TS})_{\mu\nu}{}^{\rho\sigma} = \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} - \frac{1}{n} \begin{array}{c} \leftarrow \quad \quad \leftarrow \\ \quad \quad \parallel \\ \leftarrow \quad \quad \leftarrow \end{array} \begin{array}{c} \leftarrow \\ \quad \quad \parallel \\ \leftarrow \end{array} \begin{array}{c} \leftarrow \\ \quad \quad \parallel \\ \leftarrow \end{array} \quad (11.36)$$

$$\dim(P_{TS}) = \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} - \frac{1}{n} \begin{array}{c} \leftarrow \quad \quad \leftarrow \\ \quad \quad \parallel \\ \leftarrow \quad \quad \leftarrow \end{array} \begin{array}{c} \leftarrow \\ \quad \quad \parallel \\ \leftarrow \end{array} \begin{array}{c} \leftarrow \\ \quad \quad \parallel \\ \leftarrow \end{array} \quad (11.37)$$

$$= \frac{1}{2}(n^2 + n) - 1 \quad (11.38)$$

$$= \frac{1}{2}(n+2)(n-1) \quad (11.39)$$

### 3. Anti-symmetric PO

$$(P_A)_{\mu\nu}{}^{\rho\sigma} = \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} \quad (11.40)$$

$$\dim(P_A) = \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} \quad (11.41)$$

$$= \frac{1}{2}(n^2 - n) \quad (11.42)$$

$$= \frac{1}{2}n(n-1) \quad (11.43)$$

**Claim 24** *The Clebsch-Gordan series for  $V \otimes V$  (i.e., decomposition of  $V \otimes V$ ) is*

$$\begin{array}{lcl} \overbrace{V \otimes V}^{\mathcal{V}} & = & P_S \mathcal{V} \oplus P_{TS} \mathcal{V} \oplus P_A \mathcal{V} \\ \begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} & = & \bullet \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \\ n^2 & = & 1 + \frac{1}{2}(n+2)(n-1) + \frac{1}{2}n(n-1) \end{array} \quad (11.44)$$

The projection operator tree is

$$\begin{array}{c} P_A \\ \downarrow \\ P_{SYM} \\ \downarrow \\ P_S \\ \downarrow \\ P_{TS} \end{array}$$

where  $P_{SYM} = \mathcal{S}_2$ .

**proof:**

**QED**

## 11.2 $V_{adj} \otimes V_{def}$ Decomposition

Let

$V_{def} = V$  = vector space in defining representation  $\{|\mu\rangle\}_{\mu=1}^n$ .

$V_{adj}$  = vector space in adjoint representation  $\{|i\rangle\}_{i=1}^N$ .

$V_{adj} \otimes V \cong (V \otimes V^\dagger) \otimes V$

$$e = \begin{array}{c} \text{~~~~~} \\ \longleftarrow \end{array} \cong \begin{array}{c} \text{~~~~~} T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} T_j \text{~~~~~} \\ \longleftarrow \end{array} \quad (11.45)$$

$$R = \begin{array}{c} \text{~~~~~} T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} T_j \text{~~~~~} \\ \curvearrowleft \quad \curvearrowright \end{array} = \begin{array}{c} \text{~~~~~} \\ \swarrow \quad \searrow \\ T_i \longleftarrow T_j \\ \swarrow \quad \searrow \end{array} \quad (11.46)$$

$$Q = \begin{array}{c} \text{~~~~~} T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} T_j \text{~~~~~} \\ \curvearrowleft \quad \curvearrowright \end{array} = \begin{array}{c} \text{~~~~~} \\ \swarrow \quad \searrow \\ \text{~~~~~} \\ \swarrow \quad \searrow \\ \longleftarrow T_j \longleftarrow T_i \longleftarrow \end{array} \quad (11.47)$$

Recall that for  $SO(n)$  and  $O(n)$ , the dimension  $N$  of the adjoint rep is

$$N = \frac{n(n-1)}{2} = \begin{array}{c} \text{~~~~~} \\ \curvearrowright \end{array} \quad (11.48)$$

For example, for  $SO(3)$ ,  $N = 3$ .

Note that

$$\text{tr}(e) = \begin{array}{c} \text{~~~~~} \\ \curvearrowright \end{array} = Nn \quad (11.49)$$

$$\text{tr}(R) = \begin{array}{c} \text{~~~~~} T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} T_j \text{~~~~~} \\ \curvearrowleft \quad \curvearrowright \end{array} = N \quad (11.50)$$

$$\text{tr}(Q) = \begin{array}{c} \text{~~~~~} \\ \swarrow \quad \searrow \\ \text{~~~~~} \\ \swarrow \quad \searrow \\ \longleftarrow T_j \longleftarrow T_i \longleftarrow \end{array} = N \quad (11.51)$$

**Claim 25**

$$R^2 = \frac{n-1}{2} R \quad (11.52)$$



$$QR = RQ = \frac{1}{2}R \quad (11.53)$$

$$Q^2 = \frac{e - Q}{2} \quad (11.54)$$

**proof:**

$$R^2 = \begin{array}{c} \text{Diagram: } T_i \leftarrow T_k \text{ (loop)} T_k \leftarrow T_j \\ \text{with external wavy lines and arrows} \end{array} \quad (11.55)$$

$$= \frac{n-1}{2}R \quad (\text{by Eq.(11.20)}) \quad (11.56)$$

$$QR = \begin{array}{c} \text{Diagram: } T_k \leftarrow T_i \leftarrow T_k \leftarrow T_j \\ \text{with external wavy lines and arrows} \end{array} \quad (11.57)$$

Define

$$X = \begin{array}{c} \text{Diagram: } T_k \leftarrow T_i \leftarrow T_k \leftarrow \\ \text{with external wavy lines and arrows} \end{array} \quad (11.58)$$

$$X = \begin{array}{c} \text{Diagram: A wavy line enters from the top left, splits into two wavy lines. The left wavy line has a curved arrow labeled } T_k \underline{g} \text{ pointing left. The right wavy line has a curved arrow labeled } \bar{g} T_k \text{ pointing right. Both wavy lines meet a central vertical wavy line. From the bottom of this central line, two straight lines diverge downwards and outwards, labeled } \bar{g} T_i \underline{g} \text{ at their ends.} \end{array} \quad (11.59)$$

$$= \frac{1}{2} \left[ \underbrace{\begin{array}{c} \text{Diagram: Similar to (11.59), but with two horizontal straight lines crossing the central vertical wavy line. The left horizontal line has an arrow pointing left, and the right horizontal line has an arrow pointing right. The bottom diverging lines are labeled } \bar{g} T_i \underline{g} \text{ and are underlined with a brace below labeled } =0. \end{array}} - \begin{array}{c} \text{Diagram: Similar to (11.59), but with two horizontal straight lines crossing the central vertical wavy line. The left horizontal line has an arrow pointing left, and the right horizontal line has an arrow pointing right. The bottom diverging lines are labeled } \bar{g} T_i \underline{g} \text{ and are underlined with a brace below labeled } =0. \end{array} \right] \quad (11.60)$$

$$= \frac{1}{2} \begin{array}{c} \text{Diagram: A wavy line enters from the top left, splits into two wavy lines. The left wavy line has a curved arrow labeled } T_i \text{ pointing left. The right wavy line has a curved arrow labeled } \bar{g} T_i \underline{g} \text{ pointing right. Both wavy lines meet a central vertical wavy line. From the bottom of this central line, two straight lines diverge downwards and outwards, labeled } \bar{g} T_i \underline{g} \text{ at their ends.} \end{array} \quad (11.61)$$

so

$$QR = RQ = \frac{1}{2}R \quad (11.62)$$

$$Q^2 = \begin{array}{c} \text{Diagram: A wavy line at the top with two vertices. From the left vertex, a line goes left to $T_k$. From the right vertex, a line goes right to $T_k$. Between the vertices, a line goes left to $T_i$ and then right to $T_j$.} \\ \leftarrow T_k \leftarrow T_i \leftarrow T_j \leftarrow T_k \leftarrow \end{array} \quad (11.63)$$

$$= \begin{array}{c} \text{Diagram: Similar to (11.63) but with labels $T_k \underline{g}$ and $\bar{g} T_k$ on the outer lines, and $\bar{g} T_i \underline{g} \rightarrow \bar{g} T_j \underline{g}$ on the bottom line.} \\ T_k \underline{g} \quad \bar{g} T_k \\ \bar{g} T_i \underline{g} \rightarrow \bar{g} T_j \underline{g} \end{array} \quad (11.64)$$

$$= \frac{1}{2} \left[ \begin{array}{c} \text{Diagram 1: Similar to (11.64) but with horizontal lines connecting the vertices.} \\ \bar{g} T_i \underline{g} \rightarrow \bar{g} T_j \underline{g} \end{array} - \begin{array}{c} \text{Diagram 2: Similar to (11.64) but with crossed lines connecting the vertices.} \\ \bar{g} T_i \underline{g} \rightarrow \bar{g} T_j \underline{g} \end{array} \right] \quad (11.65)$$

$$= \frac{1}{2} \left[ \begin{array}{c} \text{Diagram 3: Similar to (11.64) but with curved lines connecting the vertices.} \\ \bar{g} T_i \underline{g} \rightarrow \bar{g} T_j \underline{g} \end{array} - \begin{array}{c} \text{Diagram 4: Similar to (11.64) but with a different connection.} \\ \bar{g} T_i \underline{g} \leftarrow \bar{g} T_j \underline{g} \end{array} \right] \quad (11.66)$$

$$= \frac{e - Q}{2} \quad (11.67)$$

**QED**

**Claim 26**

$$P_1 = \frac{2}{n-1} R \quad (11.68)$$

$$P_2 = \frac{1}{3} P_4 (1 - 2Q) = \frac{1}{3} [e - 2Q] \quad (11.69)$$

$$P_3 = \frac{2}{3} P_4 (1 + Q) = \frac{2}{3} \left[ e + Q - \frac{3}{n-1} R \right] \quad (11.70)$$

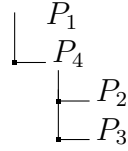
$$P_4 = e - P_1 \quad (11.71)$$

are projectors for  $O(n)$  and  $SO(n)$ . The  $V_{adj} \otimes V = \sum_{\lambda} V_{\lambda}$  Clebsch-Gordan series is given by

$$\begin{array}{rcl}
\overbrace{V_{adj} \otimes V}^{\mathcal{V}} & = & P_1 \mathcal{V} \oplus \\
\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} & = & \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \oplus \\
& & \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \oplus \\
& & \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}
\end{array} \quad (11.72)$$

$$\begin{array}{rclclcl}
\frac{1}{2}n^2(n-1) & = & n & + & \frac{1}{6}n(n-1)(n-2) & + & \frac{1}{3}n(n+2)(n-2) \\
SO(3) : 9 & = & 3 & + & 1 & + & 5 \\
SO(4) : 24 & = & 4 & + & 4 & + & 16
\end{array}$$

The projection operator tree is



**proof:**

$$\text{tr}(P_1) = \frac{2}{n-1}N \quad (11.73)$$

$$= \frac{1}{n-1}n(n-1) \quad (11.74)$$

$$= n \quad (11.75)$$

$$\text{tr}(P_2) = \frac{N}{3}(n-2) \quad (11.76)$$

$$= \frac{n(n-1)}{6}(n-2) \quad (11.77)$$

$$\text{tr}(P_3) = \frac{2N}{3} \left( n+1 - \frac{3}{n-1} \right) \quad (11.78)$$

$$= \frac{n}{3}(n^2-4) \quad (11.79)$$

From  $R^2 = \frac{2}{n-1}R$ ,

$$P_1 = \frac{2}{n-1}R \quad (11.80)$$

Define

$$P_4 = e - P_1 \quad (11.81)$$

From  $Q^2 = \frac{1}{2}(1 - Q)$ , we get

$$2Q^2 + Q - 1 = (2Q - 1)(Q + 1) = 0 \quad (11.82)$$

Let

$$P_2 = \frac{1}{3}P_4(1 - 2Q) \quad P_3 = \frac{2}{3}P_4(1 + Q) \quad (11.83)$$

and

$$a = \frac{2}{n - 1} \quad (11.84)$$

Then

$$P_3 = \frac{2}{3}P_4(1 + Q) \quad (11.85)$$

$$= \frac{2}{3}(e - aR)(1 + Q) \quad (11.86)$$

$$= \frac{2}{3}(e - aR + Q - aRQ) \quad (11.87)$$

$$= \frac{2}{3}\left(e - \frac{3}{2}aR + Q\right) \quad (\text{use } QR = \frac{1}{2}R) \quad (11.88)$$

$$= \frac{2}{3}\left(e - \frac{3}{n - 1}R + Q\right) \quad (11.89)$$

Furthermore

$$P_2 = \frac{1}{3}P_4(1 - 2Q) \quad (11.90)$$

$$= \frac{1}{3}(e - aR)(1 - 2Q) \quad (11.91)$$

$$= \frac{1}{3}(e - aR - 2Q + 2aRQ) \quad (11.92)$$

$$= \frac{1}{3}(e - 2Q) \quad (\text{use } QR = \frac{1}{2}R) \quad (11.93)$$

**QED**

# Chapter 12

## Paulions and Gammions

### 12.1 Paulions

Let

$$\vec{a} \in \mathbb{R}^3, \hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$i = 1, 2, 3, \mu = 0, 1, 2, 3$$

The Pauli matrices are defined by

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (12.1)$$

Note that they are Hermitian

$$\sigma_i^\dagger = \sigma_i \quad (12.2)$$

and their multiplication table is given by

$$\sigma_i \sigma_j = \delta_{ij} I_2 + i \epsilon_{ijk} \sigma_k \quad (12.3)$$

This multiplication table implies

- Square is 1 (unitary too because Hermitian)

$$\sigma_i^2 = 1 \quad (12.4)$$

- Different ones anticommute and proportional to third

$$\sigma_x \sigma_y = -\sigma_y \sigma_x \quad (12.5)$$

$$\sigma_x \sigma_y = i \sigma_z \quad (12.6)$$

It's common to write

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \quad (12.7)$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_\mu = (\sigma_0, \vec{\sigma}) \quad (12.8)$$

Suppose  $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ . We define the **Paulion**  $\sigma_{\vec{x}}$  by<sup>1</sup>

$$\sigma_{\vec{x}} = \sigma \cdot \vec{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \quad (12.9)$$

**Claim 27**

$$\sigma_{\vec{a}}\sigma_{\vec{b}} = \vec{a} \cdot \vec{b} + i\sigma_{\vec{a} \times \vec{b}} \quad (12.10)$$

As a consequence,

$$(\sigma_{\vec{a}})^2 = |\vec{a}|^2 \quad (12.11)$$

$$(\sigma_{\hat{a}})^2 = 1 \quad (12.12)$$

$$[\sigma_{\vec{a}}, \sigma_{\vec{b}}]_+ = 2(\vec{a} \cdot \vec{b}) \quad (12.13)$$

$$[\sigma_{\vec{a}}, \sigma_{\vec{b}}] = 2i\sigma_{\vec{a} \times \vec{b}} \quad (12.14)$$

**proof:** This follows directly from Eq.(12.3).

**QED**

**Claim 28** If  $\vec{a} \in \mathbb{R}^3$  then

$$e^{i\sigma_{\vec{a}}} = \cos |\vec{a}| + i\sigma_{\hat{a}} \sin |\vec{a}| \quad (12.15)$$

and

$$e^{i\beta\sigma_{\hat{a}}} = \cos(\beta) + i\sigma_{\hat{a}} \sin(\beta) \quad (12.16)$$

**proof:**

First show by Taylor expansion of the exponential that the following equation is true.

$$e^{i\beta\sigma_i} = \cos \beta + i\sigma_i \sin \beta \quad (12.17)$$

Then replace  $\sigma_i$  by  $\sigma_{\hat{a}}$ .

**QED**

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<sup>1</sup>The term Paulion is my own. As far as I know, the construct  $\sigma_{\vec{a}}$ , although often used, doesn't have a common name.

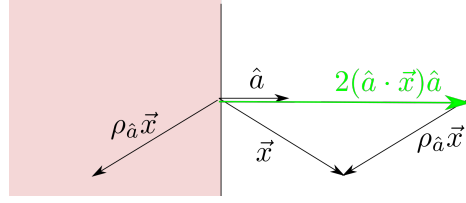


Figure 12.1: Reflection about plane with normal vector  $\hat{a}$

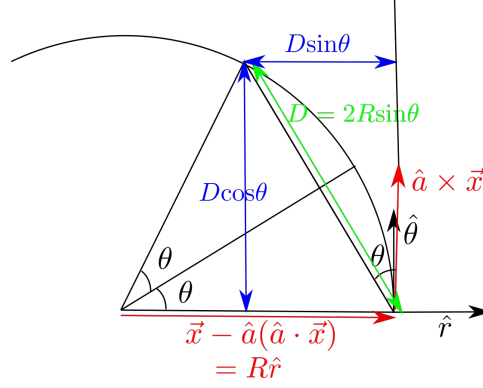


Figure 12.2: Geometry of  $R_{\hat{a}}(\Theta)x$ , where  $\Theta = 2\theta$

Define

$$\langle \sigma_{\hat{a}} \rangle (\vec{x}) = -\sigma_{\hat{a}} \sigma_{\vec{x}} \sigma_{\hat{a}} \quad (12.18)$$

Fig.12.1 illustrates reflection about plane with normal vector  $\hat{a}$

**Claim 29** (*Reflection*)

$$-\sigma_{\hat{a}} \sigma_{\vec{x}} \sigma_{\hat{a}} = \sigma_{\rho_{\hat{a}}\vec{x}} \quad (12.19)$$

where

$$\rho_{\hat{a}}\vec{x} = \vec{x} - 2(\hat{a} \cdot \vec{x})\hat{a} \quad (12.20)$$

**proof:**

$$\sigma_{\hat{a}} \sigma_{\vec{x}} \sigma_{\hat{a}} = [\hat{a} \cdot \vec{x} + i\sigma_{\hat{a} \times \vec{x}}] \sigma_{\hat{a}} \quad (12.21)$$

$$= \hat{a} \cdot \vec{x} \sigma_{\hat{a}} + i \underbrace{(\hat{a} \times \vec{x}) \cdot \hat{a}}_{=0} - \sigma \cdot \underbrace{(\hat{a} \times \vec{x}) \times \hat{a}}_{\vec{x} - (\vec{x} \cdot \hat{a})\hat{a}} \quad (12.22)$$

$$= -\sigma \cdot [\vec{x} - 2(\hat{a} \cdot \vec{x})\hat{a}] \quad (12.23)$$

**QED**



**Claim 30** (*Rotation, Rodrigues formula*)

If

$$U = e^{-i\frac{\Theta}{2}\sigma_{\hat{a}}}, \quad \Theta = 2\theta \quad (12.24)$$

then

$$U\sigma_{\vec{x}}U^\dagger = \sigma_{R_{\hat{a}}(\Theta)\vec{x}} \quad (12.25)$$

where

$$R_{\hat{a}}(\Theta)\vec{x} = \vec{x} + (2|\vec{x}|\sin\beta\sin\theta)(-\sin\theta\hat{r} + \cos\theta\hat{\theta}) \quad (12.26)$$

$$= \vec{x} + (2\sin\theta)(-\sin\theta\vec{r} + \cos\theta\vec{\theta}) \quad (12.27)$$

where (see Fig.12.2)

$$\beta = \angle(\vec{x}, \vec{a}), \quad \hat{\theta} = \frac{1}{|\vec{x}|\sin\beta} \overbrace{(\hat{a} \times \vec{x})}^{\vec{\theta}}, \quad \hat{r} = \frac{1}{|\vec{x}|\sin\beta} \overbrace{[\vec{x} - \hat{a}(\hat{a} \cdot \vec{x})]}^{\vec{r}} \quad (12.28)$$

This immediately implies that  $SU(2)_{\mathbb{R}}$  is a double cover of  $SO(3)$ .

$$SU(2)_{\mathbb{R}} \xrightarrow[2 \text{ to } 1 \text{ map}]{} SO(3), \quad SU(2)_{\mathbb{R}}/\{1, -1\} \cong SO(3) \quad (12.29)$$

**proof:**

Let  $C = \cos\theta$ ,  $S = \sin\theta$ . Then

$$U\sigma_{\vec{x}}U^\dagger = (C - iS\sigma_{\hat{a}})\sigma_{\vec{x}}(C + iS\sigma_{\hat{a}}) \quad (12.30)$$

$$= C^2\sigma_{\vec{x}} - iSC[\sigma_{\hat{a}}, \sigma_{\vec{x}}] + S^2\sigma_{\hat{a}}\sigma_{\vec{x}}\sigma_{\hat{a}} \quad (12.31)$$

$$= \sigma \cdot \{C^2\vec{x} + 2SC(\hat{a} \times \vec{x}) - S^2[\vec{x} - 2(\hat{a} \cdot \vec{x})\hat{a}]\} \quad (12.32)$$

$$= \sigma \cdot \{\vec{x} + 2SC\vec{\theta} - 2S^2\vec{r}\} \quad (12.33)$$

If  $U$  produces rotation  $R$ , then  $-U$  gives the same rotation so 2 to 1 map:

$$(-U)\sigma_{\vec{x}}(-U)^\dagger = U\sigma_{\vec{x}}U^\dagger. \quad (12.34)$$

**QED**

## 12.2 Gammions

Suppose

$$\begin{aligned} n_- &= n \text{ for } n \text{ even and } n_- = n - 1 \text{ for } n \text{ odd.} \\ \text{int}(x) &= \text{integer part of } x \in \mathbb{R}. \text{ int}(n/2) = n_-/2 \\ \mu &= 1, 2, \dots, n \\ a_\mu, b_\mu &\in \mathbb{R} \text{ for each } \mu, \\ \gamma_\mu &\in \mathbb{C}^{d \times d} \text{ where } d = \text{int}(n/2). \end{aligned}$$

$$a \cdot b = a_\mu b^\mu \quad (12.35)$$

We define the **Gammion**  $\gamma_{\underline{a}}$  by<sup>2</sup>

$$\gamma \cdot a = \gamma_{\underline{a}} = \gamma_\mu a^\mu \quad (12.36)$$

We will use an underline in  $\gamma_{\underline{a}}$  to distinguish  $a$  from an index.

The Clifford anticommutation relation is

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu} \quad (12.37)$$

where  $\eta_{\mu\nu}$  is real valued<sup>3</sup> For  $n = 4$ , it equals the Euclidean metric  $(1, 1, 1, 1)$  or the Lorentzian metric  $(1, -1, -1, -1)$ . The Clifford anticommutation relation can be expressed in terms of gammions thus:

$$\gamma_{\underline{a}}\gamma_{\underline{b}} = a \cdot b + \gamma_{\mu\nu} a^\mu b^\nu, \quad (12.38)$$

where bivector

$$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \quad (12.39)$$

is the generator of  $\mathfrak{so}(n)_\mathbb{R}$  or  $\mathfrak{so}(p, q)$ .

Define

$$\langle \gamma_{\hat{a}} \rangle(x) = -\gamma_{\hat{a}}\gamma_{\underline{x}}\gamma_{\hat{a}} \quad (12.40)$$

**Claim 31** (*Reflection*)

$$-\gamma_{\hat{a}}\gamma_{\underline{x}}\gamma_{\hat{a}} = \gamma_{\underline{\rho_{\hat{a}}x}} \quad (12.41)$$

where

$$\rho_{\hat{a}}x = x - 2(\hat{a} \cdot x)\hat{a} \quad (12.42)$$

---

<sup>2</sup>The term Gammion, like the term Paulion, is my own. As far as I know, the construct  $\gamma_{\underline{a}}$ , although often used, doesn't have a common name.

<sup>3</sup>Since  $\eta_{\mu\nu}$  is real, we consider later on in this chapter, the Clifford algebra  $Cl(n)_\mathbb{R}$  instead of its complexification  $Cl(n)_\mathbb{C}$

**proof:**  
**QED**

**Claim 32** (*Rotation*)

*If*

$$S(\omega) = e^{\omega^{\mu\nu}\gamma_{\mu\nu}} \quad (12.43)$$

*for some  $\omega_{\mu\nu} \in \mathbb{R}$ , then*

$$S(\omega)\gamma_{\underline{x}}S^\dagger(\omega) = \gamma_{Rx} \quad (12.44)$$

*where  $R \in SO(n)$  (or  $SO(p, q)$ ).*

**proof:**  
**QED**

### 12.3 $Spin(n)_\mathbb{R}$

If  $\gamma_\mu^\dagger = \gamma_\mu$  for  $\mu = 1, 2, \dots, n$  (this assumes the Euclidean metric which is  $g_{\mu\nu} = \text{diag}(1, 1, 1, 1)$  for  $n = 4$ )

$$Spin(n)_\mathbb{R} = \{e^{\omega^{\mu\nu}\gamma_{\mu\nu}} | \omega_{\mu\nu} = -\omega_{\nu\mu} \in \mathbb{R}\} \quad (12.45)$$

If  $\gamma_0^\dagger = \gamma_0$ , and  $\gamma_i^\dagger = -\gamma_i$  (this assumes the mostly-minus metric which is  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  for  $n = 4$ )

$$Spin(1, 3)_\mathbb{R} = \{e^{-i\omega^{\mu\nu}\gamma_{\mu\nu}} | \omega_{\mu\nu} = -\omega_{\nu\mu}, \omega_{0,i} \in \mathbb{R}, \omega_{i,j} \in i\mathbb{R}\} \quad (12.46)$$

**Spinors** are the vectors in the vector space upon which the group  $Spin(n)_\mathbb{R}$  acts.

Suppose  $e_i \in \mathbb{R}^n$  for  $i = 1, 2, \dots, n$ , and all components of  $e_i$  are zero except the  $i$ th one. Define

$$\gamma_i = \gamma_{e_i} \quad (12.47)$$

Consider the Clifford anticommutator in the Euclidean metric:

$$[\gamma_i, \gamma_j]_+ = 2\delta_{i,j} \quad (12.48)$$

Define the Clifford algebra  $Cl(n)_\mathbb{R}$  by

$$\Pi_0 = \{1\}, \quad |\Pi_0| = \binom{n}{0} = 1 \quad (12.49)$$

$$\Pi_1 = \{\gamma_i | i = 1, 2, \dots, n\} \quad |\Pi_1| = \binom{n}{1} = n \quad (12.50)$$

$$\Pi_2 = \{\gamma_{i_1} \gamma_{i_2} | i_1 < i_2\}, \quad |\Pi_2| = \binom{n}{2} \quad (12.51)$$

$$\Pi_3 = \{\gamma_{i_1} \gamma_{i_2} \gamma_{i_3} | i_1 < i_2 < i_3\}, \quad |\Pi_3| = \binom{n}{3} \quad (12.52)$$

$$\Pi_n = \{\gamma_1 \gamma_2 \dots \gamma_n\}, \quad |\Pi_n| = \binom{n}{n} = 1 \quad (12.53)$$

$$\sum_{k=0}^n |\Pi_k| = \sum_{k=0}^n \binom{n}{k} = (1+1)^n = 2^n \quad (12.54)$$

$$Cl(n)_{\mathbb{R}} = span_{\mathbb{R}} \left( \bigcup_{k=0,1,2,\dots,n} \Pi_k \right) \quad (12.55)$$

$$Cl^0(n)_{\mathbb{R}} = span_{\mathbb{R}} \left( \bigcup_{k=0,2,4,\dots,n-} \Pi_k \right) \quad (12.56)$$

$$Cl^1(n)_{\mathbb{R}} = span_{\mathbb{R}} \left( \bigcup_{k=1,3,5,\dots,n-} \Pi_k \right) \quad (12.57)$$

$$Cl(n)_{\mathbb{R}} = Cl^0(n)_{\mathbb{R}} \oplus Cl^1(n)_{\mathbb{R}} \quad (12.58)$$

Since  $e^{\omega^{\mu\nu} \gamma_{\mu\nu}}$  is the exponentiation of a bivector, its Taylor expansion only contains summands with an even number of gammas. Hence,

$$Spin(n)_{\mathbb{R}} \subset Cl^0(n)_{\mathbb{R}} \quad (12.59)$$

Define the unit sphere in  $n$  dimensions by

$$S^n = \{\hat{a} \in \mathbb{R}^n | \hat{a}^2 = 1\} \quad (12.60)$$

$Pin$  stands for “Product of involutions” An involution in this case is a unit vector (i.e.,  $\hat{a} \in \mathbb{R}^n$  such that  $\hat{a} \cdot \hat{a} = \hat{a}^2 = 1$ ) The  $Pin(n)_{\mathbb{R}}$  group is defined by

$$Pin(n)_{\mathbb{R}} = \bigcup_{k=1}^{\infty} \{\gamma_{\hat{a}_1} \gamma_{\hat{a}_2} \dots \gamma_{\hat{a}_k} | \hat{a}_i \in S^n \text{ for all } i\} \quad (12.61)$$

Note that

- One reflection

$$det[\langle \gamma_{\hat{a}} \rangle (x)] = -det(\gamma_{\underline{x}} \gamma_{\hat{a}}^2) = -det(\gamma_{\underline{x}}) \quad (12.62)$$

- Two reflections = a rotation

$$det[\langle \gamma_{\hat{a}_1} \gamma_{\hat{a}_2} \rangle (x)] = +det(\gamma_{\underline{x}}) \quad (12.63)$$

Hence,  $Pin(n)_{\mathbb{R}}$  is a double cover of  $O(n)$  (because  $\det = \pm 1$ )

$$Pin(n)_{\mathbb{R}} \xrightarrow{2 \text{ to } 1 \text{ map}} O(n), \quad Pin(n)_{\mathbb{R}}/\{1, -1\} \cong O(n) \quad (12.64)$$

Previously, we defined  $Spin(n)_{\mathbb{R}}$  by exponentiating its generators.  $Spin(n)_{\mathbb{R}}$  can also be defined as follows, in terms of products of gammions instead of exponentiating generators:

$$Spin(n)_{\mathbb{R}} = \cup_{k=1}^{\infty} \{\gamma_{\hat{a}_1} \gamma_{\hat{a}_2} \dots \gamma_{\hat{a}_k} | \hat{a}_i \in S^n \text{ for all } i\} \quad (12.65)$$

$Spin(n)_{\mathbb{R}}$  is a double cover of  $SO(n)$  (because  $\langle \gamma_{\hat{a}} \rangle = \langle \gamma_{-\hat{a}} \rangle$ )

$$Spin(n)_{\mathbb{R}} \xrightarrow{2 \text{ to } 1 \text{ map}} SO(n), \quad Spin(n)_{\mathbb{R}}/\{1, -1\} \cong SO(n) \quad (12.66)$$

Define  $Cl(n)_{\mathbb{F}}$  as a span over  $\mathbb{F} = \mathbb{R}, \mathbb{C}$  and  $Spin(n)_{\mathbb{F}} \subset Cl^0(n)_{\mathbb{F}}$ .  
When you go from a vector space

$$V_{\mathbb{R}} = span_{\mathbb{R}}(e_1, e_2, \dots, e_n) \quad (12.67)$$

to

$$V_{\mathbb{C}} = span_{\mathbb{C}}(e_1, e_2, \dots, e_n), \quad (12.68)$$

this is called a **complexification** of the vector space. Lie algebras over  $\mathbb{R}$  can be complexified.  $Cl(n)_{\mathbb{C}}$  is the complexification of  $Cl(n)_{\mathbb{R}}$ .

$Spin(n)_{\mathbb{R}} \xrightarrow{2 \text{ to } 1 \text{ map}} SO(n; \mathbb{R})$  and  $Spin(n)_{\mathbb{C}} \xrightarrow{2 \text{ to } 1 \text{ map}} SO(n; \mathbb{C})$ . Rotations by a complex angle as in  $SO(n)_{\mathbb{C}}$  are meaningless, even in Quantum Mechanics. Only group  $Spin(n)_{\mathbb{R}}$  describes symmetries in physical systems. However, in Quantum Mechanics, we use complex representations of  $Spin(n)_{\mathbb{R}}$  and a complex spinor space. The complex rep of  $Spin(n)_{\mathbb{R}}$  and its complexification  $Spin(n)_{\mathbb{C}}$  are isomorphic.

Careful. The following possibilities are not equivalent

1. representation of  $Spin(n)_{\mathbb{F}}$  is real or complex
2. Spinor space on which  $Spin(n)_{\mathbb{F}}$  acts is real or complex
3.  $\gamma$  matrices are real or complex

$n =$	0	1	2	3	4	5	6	7
$Cl(n)_{\mathbb{R}} \cong$	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$

Table 12.1: Sets isomorphic ( $\cong$ ) to  $Cl(n)_{\mathbb{R}}$ . In this figure,  $M_n(\mathbb{F}) = \mathbb{F}^{n \times n}$  for  $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  where  $\mathbb{H}$  are the quaternions.

$n =$	$2k$	$2k + 1$
$Cl(n)_{\mathbb{C}} \cong$	$M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$	$M_{2^k}(\mathbb{C})$

Table 12.2: Sets isomorphic ( $\cong$ ) to  $Cl(n)_{\mathbb{C}}$ . Table 12.1 collapses to this table when  $Cl(n)_{\mathbb{R}}$  is complexified.

## 12.4 $Cl(n)_{\mathbb{R}}$ representations

Consider Table 12.1 which gives sets isomorphic to  $Cl(n)_{\mathbb{R}}$  for all  $n$ .

Notes on Table 12.1

- Dimension check: the real dimension for the two isomorphic sets is equal to  $2^n$  in every column. For example<sup>4</sup>,  $\dim_{\mathbb{R}} M_2(\mathbb{C}) = 8 = 2^3$ .
- The pattern repeats with period 8 (**Bott Periodicity**).

$$Cl(n+8)_{\mathbb{R}} \cong Cl(n)_{\mathbb{R}} \otimes M_{16}(\mathbb{R}) \quad (12.69)$$

Rather than depending on  $n$ , the pattern depends on<sup>5</sup>  $n \bmod 8$ . For example

$$Cl(8)_{\mathbb{R}} \cong \underbrace{Cl(0)}_{\mathbb{R}} \otimes M_{16}(\mathbb{R}) \cong M_{16}(\mathbb{C}) \quad (12.70)$$

$$Cl(9)_{\mathbb{R}} \cong \underbrace{Cl(1)}_{\mathbb{R} \oplus \mathbb{R}} \otimes M_{16}(\mathbb{R}) \cong M_{16}(\mathbb{C}) \oplus M_{16}(\mathbb{C}) \quad (12.71)$$

$M_{16}(\mathbb{C})$  are  $2^{int(n/2)} \times 2^{int(n/2)}$  complex matrices where  $n = 8$ .

- The exact same table is valid for  $Cl(p, q)_{\mathbb{R}}$ , except with  $n$  replaced by  $(p - q) \% 8$ . Note, however, that according to this,  $Cl(1, 3)_{\mathbb{R}}$  is congruent to column  $(1 - 3) \% 8 = 6$  of Table 12.1, whereas  $Cl(3, 1)_{\mathbb{R}}$  is congruent to column  $(3 - 1) \% 8 = 2$  and columns 2 and 6 are not isomorphic. Hence

$$Cl(p, q)_{\mathbb{R}} \not\cong Cl(q, p)_{\mathbb{R}} \quad (12.72)$$

However, one can show that

$$Cl(p, q)_{\mathbb{C}} \cong Cl(q, p)_{\mathbb{C}} \quad (12.73)$$

and

---

<sup>4</sup> $\dim_{\mathbb{R}}$  = number of real degrees of freedom (dofs).  $\dim_{\mathbb{C}} = 2\dim_{\mathbb{R}}$ . For example,  $\dim_{\mathbb{R}}(\mathbb{H}) = 4$ , where  $\mathbb{H}$  are the quaternions.

<sup>5</sup>In Python  $n \bmod 8$  is denoted by  $n \% 8$ , the remainder after dividing  $n$  by 8.

$$Spin(p, q)_{\mathbb{R}} \cong Spin(q, p)_{\mathbb{R}}, \quad \mathfrak{spin}(p, q)_{\mathbb{R}} \cong \mathfrak{spin}(q, p)_{\mathbb{R}} \quad (12.74)$$

$$SO(p, q)_{\mathbb{R}} \cong SO(q, p)_{\mathbb{R}}, \quad \mathfrak{so}(p, q)_{\mathbb{R}} \cong \mathfrak{so}(q, p)_{\mathbb{R}} \quad (12.75)$$

## 12.5 $Cl(n)_{\mathbb{C}}$ representations

Upon complexification, Table 12.1 collapses to Table 12.2. In fact, the representations of  $Cl(n)_{\mathbb{C}}$  are all complex.

For  $n = 2k$ ,

$$Cl(2k)_{\mathbb{C}} \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}) \supset Spin(2k)_{\mathbb{C}} \quad (12.76)$$

and for  $n = 2k + 1$ ,

$$Cl(2k + 1)_{\mathbb{C}} \cong M_{2^k}(\mathbb{C}) \supset Spin(2k + 1)_{\mathbb{C}} \quad (12.77)$$

Assuming complexification, if  $n$  is even, we can define a nontrivial chirality operator:

$$\Gamma_5 = i^{n/2} \gamma_1 \gamma_2 \cdots \gamma_n. \quad (12.78)$$

$\Gamma_5$  satisfies:

- anticommutes with all the  $\gamma_\mu$
- commutes with the elements of group  $Spin(n)_{\mathbb{R}}$  (because this group consists of products of even number of gammons)
- $\Gamma_5^2 = 1$ . Therefore,  $\Gamma_5$  has eigenvalues  $1, -1$ .

Assuming complexification, if  $n$  is even, and  $V = \mathbb{C}^{n/2}$ , then

$$V = V_+ \oplus V_-, \quad (12.79)$$

where:

$$V_{\pm} = \{\psi \in V \mid \Gamma_5 \psi = \pm \psi\} \quad (12.80)$$

Thus, there are the two Weyl (or chiral) spinors when  $n$  is even.

Assuming complexification, if  $n$  is odd, the expression  $\gamma^1 \cdots \gamma^n$  is proportional to the identity. Hence, there is no new operator that commutes with all the elements of  $Spin(n)_{\mathbb{R}}$  and there is no chiral decomposition of vector space  $V$ . The space  $V$  is irreducible.

## 12.6 Examples

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$Spin(2)_{\mathbb{R}}$

$$Cl(2)_{\mathbb{R}} = span_{\mathbb{R}}\{1, \sigma_1, \sigma_2, \underbrace{\sigma_1\sigma_2}_{i\sigma_3}\} \quad (12.81)$$

$$Cl^0(2)_{\mathbb{R}} = span_{\mathbb{R}}\{1, \underbrace{\sigma_1\sigma_2}_{i\sigma_3}\} = \{Ae^{i\theta\sigma_3} : \theta, A \in \mathbb{R}\} \cong \mathbb{C} \quad (12.82)$$

$$Cl^1(2)_{\mathbb{R}} = span_{\mathbb{R}}\{\sigma_1, \sigma_2\} \quad (12.83)$$

$$Cl(2)_{\mathbb{R}} = Cl^0(2)_{\mathbb{R}} \oplus Cl^1(2)_{\mathbb{R}} \quad (12.84)$$

$$Spin(2)_{\mathbb{R}} = \{e^{i\theta\sigma_3} : \theta \in \mathbb{R}\} \cong U(1) \quad (12.85)$$

**Claim 33** (*Double Cover*)

$$Spin(2)_{\mathbb{R}} \xrightarrow[2 \text{ to } 1 \text{ map}]{\quad} SO(2).$$

**proof:**

If  $U = e^{i\theta\sigma_3}$  and  $\vec{x} = (x_1, x_2)$ , then

$$U\sigma_{\vec{x}}U^{\dagger} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & x_1 - ix_2 \\ x_1 + ix_2 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \quad (12.86)$$

$$= \begin{pmatrix} 0 & z^*e^{i2\theta} \\ ze^{-i2\theta} & 0 \end{pmatrix} \quad (z = x_1 + ix_2) \quad (12.87)$$

$$= \begin{pmatrix} 0 & x'_1 - ix'_2 \\ x'_1 + ix'_2 & 0 \end{pmatrix} \quad (12.88)$$

$$x'_1 = \operatorname{Re}(ze^{-i2\theta}) = x_1 \cos 2\theta + x_2 \sin 2\theta \quad (12.89)$$

$$x'_2 = \operatorname{Im}(ze^{-i2\theta}) = -x_1 \sin 2\theta + x_2 \cos 2\theta \quad (12.90)$$

$$R_z(2\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \quad (12.91)$$

$$\vec{x} = (x_1, x_2)^T, \quad \vec{x}' = (x'_1, x'_2)^T \quad (12.92)$$

$$\vec{x}' = R_z(2\theta)\vec{x} \quad (12.93)$$



The 2 to 1 map:

$$e^{i\theta\sigma_3} \mapsto R_z(2\theta) \quad (12.94)$$

**QED**

---

$Spin(3)_{\mathbb{R}}$

$$Cl(3)_{\mathbb{R}} = span_{\mathbb{R}}\{1, \sigma_1, \sigma_2, \sigma_3, \underbrace{\sigma_1\sigma_2}_{i\sigma_3}, \underbrace{\sigma_2\sigma_3}_{i\sigma_1}, \underbrace{\sigma_3\sigma_1}_{i\sigma_2}, \underbrace{\sigma_1\sigma_2\sigma_3}_i\} \cong \mathbb{C}^{2 \times 2} \quad (12.95)$$

$$Cl^0(3)_{\mathbb{R}} = span_{\mathbb{R}}\{1, i\sigma_1, i\sigma_2, i\sigma_3\} \cong \mathbb{H} \quad (12.96)$$

$$Cl^1(3)_{\mathbb{R}} = span_{\mathbb{R}}\{\sigma_1, \sigma_2, \sigma_3, i\} \cong -i\mathbb{H} \quad (12.97)$$

$$Cl(3)_{\mathbb{R}} = Cl^0(3)_{\mathbb{R}} \oplus Cl^1(3)_{\mathbb{R}} \cong \mathbb{H} \oplus -i\mathbb{H} \quad (12.98)$$

$$Spin(3)_{\mathbb{R}} = \{e^{i\sigma_{\vec{a}}} | \vec{a} \in \mathbb{R}^3\} \cong SU(2) \quad (12.99)$$

**Claim 34**

$$SL(2; \mathbb{C}) \supset SU(2) \cong Spin(3)_{\mathbb{R}} \xrightarrow[2 \text{ to } 1 \text{ map}]{\quad} SO(3) \quad (12.100)$$

**proof:**

**QED**

---

Quaternions

The set of quaternions is called  $\mathbb{H}$  in honor of Hamilton. ( $\mathbb{Q}$  is used for the rationals). A **quaternion**  $q$  is an expression of the form

$$q = a + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (12.101)$$

where  $a, x, y, z \in \mathbb{R}$ . The multiplication table for quaternions is the same as that for Pauli matrices, if we map

$$i\sigma_1 \leftrightarrow \mathbf{i}, \quad i\sigma_2 \leftrightarrow \mathbf{j}, \quad i\sigma_3 \leftrightarrow \mathbf{k} \quad (12.102)$$

Quaternion multiplication rules:

$$(i\sigma_i)(i\sigma_j) = -(\delta_{ij} + \epsilon_{ijk}i\sigma_k) \quad (12.103)$$

## Chapter 13

# Quantum Shannon Information Theory: COMING SOON

# Chapter 14

## Recoupling Identities

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

In this chapter, we will refer to the following 2 birdtracks as s and t channels.<sup>1</sup>

$$\begin{array}{cc}
 \begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \swarrow \end{array} & \begin{array}{c} \leftarrow \quad \leftarrow \\ \text{---} \\ \leftarrow \quad \leftarrow \end{array} \\
 \text{s-channel} & \text{t-channel}
 \end{array} \tag{14.1}$$

This terminology comes from High Energy Physics, where these birdtracks are used to define the so called Mandelstam variables. The Mandelstam variables measure the energy of particles in various birdtracks.

### 14.1 Parallel Channels to Sum of t-channels

Clebsch-Gordan (CG) coefficients were introduced in Chapter 3. Define the CG coefficients node

$$\begin{array}{c} \mu c \\ \nearrow \\ C_{\lambda a}^{\nu b \mu c} = \lambda a \leftarrow C_{\lambda}^{\nu \mu} \\ \nwarrow \\ \nu b \end{array} = \begin{array}{c} \mu c \leftarrow \\ \parallel \\ C_{\lambda}^{\nu \mu} \\ \parallel \\ \nu b \leftarrow \end{array} \tag{14.2}$$

Note that

---

<sup>1</sup>My mnemonic to remember which is which: **s-channel**: particle synergy, energy from particles coming together, **t-channel**: particle trade.

$$\lambda \leftarrow C_\lambda \begin{array}{c} \leftarrow \mu \\ \parallel \\ \leftarrow \nu \end{array} \neq \lambda \leftarrow C_\lambda \begin{array}{c} \nearrow \mu \\ \parallel \\ \searrow \nu \end{array} \quad (14.3)$$

Note that we are defining the CG coefficient  $C_\lambda$  so that the  $\lambda$  rep particle is created in an s-channel by converging  $\mu$  and  $\nu$  rep particles. When we define the generators  $T_\lambda^i$ , the  $i$  (gluon, adj-rep particle) is in a t-channel emanating from incoming and outgoing def-rep particles. Another big difference between  $C_\lambda$  and  $T_\lambda^i$  is that  $T_\lambda^i$  is assumed to be Hermitian, whereas  $C_\lambda$  is not Hermitian in general.  $C_\lambda$  is not even a square matrix in general.

In this chapter, we won't use implicit summation over Greek indices.

In this section, sometimes instead of labelling arrows by a lower case Greek letter denoting its rep, we will disclose an arrow's rep by a color, according to the following rep-to-color code.

$$\lambda : red, \quad \mu : green, \quad \nu : blue \quad (14.4)$$

According to Chapter 3, the CG coefficient  $C_\lambda$  satisfies

$$C_\lambda C_\lambda^\dagger = P_\lambda \quad (14.5)$$

$$\text{tr}(P_\lambda) = d_\lambda \quad (14.6)$$

where  $P_\lambda$  is the projection operator onto the vector space of the rep  $\lambda$  and  $d_\lambda$  is the dimension of that vector space.

Note that if we divide  $C_\lambda$  by  $\sqrt{d_\lambda}$ , then

$$\text{tr} \left( \frac{C_\lambda}{\sqrt{d_\lambda}} \frac{C_\lambda^\dagger}{\sqrt{d_\lambda}} \right) = 1 \quad (14.7)$$

Define

$$P_\lambda = \begin{array}{c} \leftarrow \\ \parallel \\ C_\lambda^\dagger \leftarrow C_\lambda \\ \parallel \\ \leftarrow \end{array} \quad (14.8)$$

$$P_\mu = \frac{d_\mu}{d_\lambda} \quad \begin{array}{c} \text{Diagram: A central node with four lines. Top: green line with arrow pointing left, connected by a green loop. Bottom: red line with arrow pointing right. Left: blue line with arrow pointing left. Right: blue line with arrow pointing left. The central node is labeled with $C_\lambda^\dagger$ on the left and $C_\lambda$ on the right, with double vertical lines connecting them.} \end{array} \quad (14.9)$$

$$P_\nu = \frac{d_\nu}{d_\lambda} \quad \begin{array}{c} \text{Diagram: A central node with four lines. Top: red line with arrow pointing right, connected by a red loop. Bottom: blue line with arrow pointing left, connected by a blue loop. Left: green line with arrow pointing left. Right: green line with arrow pointing left. The central node is labeled with $C_\lambda^\dagger$ on the left and $C_\lambda$ on the right, with double vertical lines connecting them.} \end{array} \quad (14.10)$$

One can check that these operators are projection operators normalized to the dimension of their rep; i.e., for  $\Omega \in \{\lambda, \mu, \nu\}$ ,

$$P_\Omega^2 = P_\Omega \quad (14.11)$$

and

$$\text{tr}(P_\Omega) = d_\Omega \quad (14.12)$$

The normalization of the projectors  $P_\Omega$  can be remembered if one takes the denominator  $d_\lambda$  and splits it into two factors of  $\sqrt{d_\lambda}$  and puts one  $\sqrt{d_\lambda}$  under  $C_\lambda$  and the other under  $C_\lambda^\dagger$ . Then one “trades”  $\frac{C_\lambda}{\sqrt{d_\lambda}}$  by  $\frac{C_\mu}{\sqrt{d_\mu}}$  or  $\frac{C_\nu}{\sqrt{d_\nu}}$ .

Next we define a scaled version of the CG coefficients  $C_\lambda$  as follows

$$\begin{array}{c} \text{Diagram: A central node with four lines. Top: green line with arrow pointing left, labeled $\mu$. Bottom: blue line with arrow pointing left, labeled $\nu$. Left: red line with arrow pointing left, labeled $\lambda$. The central node is labeled with $C_\lambda$ on the left and $C_\lambda$ on the right, with double vertical lines connecting them.} \end{array} = \frac{1}{\sqrt{\kappa_\lambda^{\nu\mu}}} \begin{array}{c} \text{Diagram: A central node with four lines. Top: green line with arrow pointing left, labeled $\mu$. Bottom: blue line with arrow pointing left, labeled $\nu$. Left: red line with arrow pointing left, labeled $\lambda$. The central node is labeled with $\mathfrak{C}_\lambda$ on the left and $\mathfrak{C}_\lambda$ on the right, with double vertical lines connecting them.} \end{array} \quad (14.13)$$

The **scaled CG coefficients**  $\mathfrak{C}_\lambda$  satisfy

$$\begin{array}{c} \text{Diagram: A central node with four lines. Top: green line with arrow pointing left, labeled $\mu$. Bottom: blue line with arrow pointing left, labeled $\nu$. Left: red line with arrow pointing left, labeled $\lambda$. The central node is labeled with $\mathfrak{C}_\lambda$ on the left and $\mathfrak{C}_\sigma^\dagger$ on the right, with double vertical lines connecting them.} \end{array} = \kappa_\lambda^{\nu\mu} \delta(\lambda, \sigma) \begin{array}{c} \text{Diagram: A central node with four lines. Top: green line with arrow pointing left, labeled $\mu$. Bottom: blue line with arrow pointing left, labeled $\nu$. Left: red line with arrow pointing left, labeled $\lambda$. The central node is labeled with $\mathfrak{C}_\lambda$ on the left and $\mathfrak{C}_\sigma^\dagger$ on the right, with double vertical lines connecting them.} \end{array} \quad (14.14)$$

Therefore

$$\mathfrak{e}_\lambda^\dagger \begin{array}{c} \xrightarrow{\mu} \\ \xleftarrow{\lambda} \\ \xrightarrow{\nu} \end{array} \mathfrak{e}_\lambda = \kappa_\lambda^{\nu\mu} d_\lambda \quad (14.15)$$

The projection operators  $P_\Omega$  for  $\Omega \in \{\lambda, \mu, \nu\}$  can be expressed in a more symmetrical form using nodes for the scaled CG coefficients as follows

$$P_\lambda = \frac{1}{\kappa_\lambda^{\nu\mu}} \begin{array}{c} \xleftarrow{\mu} \quad \xleftarrow{\mu} \\ \xleftarrow{\lambda} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \xrightarrow{\lambda} \\ \xrightarrow{\nu} \quad \xrightarrow{\nu} \end{array} \quad (14.16)$$

$$P_\mu = \frac{1}{\kappa_\mu^{\lambda\nu}} \begin{array}{c} \xleftarrow{\nu} \quad \xleftarrow{\nu} \\ \xleftarrow{\mu} \mathfrak{e}_\mu^\dagger \xleftarrow{\mu} \mathfrak{e}_\mu \xrightarrow{\mu} \\ \xrightarrow{\lambda} \quad \xrightarrow{\lambda} \end{array} \quad (14.17)$$

$$P_\nu = \frac{1}{\kappa_\nu^{\mu\lambda}} \begin{array}{c} \xleftarrow{\lambda} \quad \xleftarrow{\lambda} \\ \xleftarrow{\nu} \mathfrak{e}_\nu^\dagger \xleftarrow{\nu} \mathfrak{e}_\nu \xrightarrow{\nu} \\ \xrightarrow{\mu} \quad \xrightarrow{\mu} \end{array} \quad (14.18)$$

The CG series for  $V_\mu \otimes V_\nu = \sum_\lambda V_\lambda$  can be expressed in terms of birdtracks as follows

$$\begin{array}{c} \xleftarrow{\quad} \bullet \xleftarrow{\mu} \\ \xleftarrow{\quad} \bullet \xleftarrow{\nu} \end{array} = \sum_\lambda P_\lambda = \sum_\lambda \frac{d_\lambda}{\mathfrak{e}_\lambda^\dagger \begin{array}{c} \xrightarrow{\mu} \\ \xleftarrow{\lambda} \\ \xrightarrow{\nu} \end{array} \mathfrak{e}_\lambda} \begin{array}{c} \xleftarrow{\mu} \quad \xleftarrow{\mu} \\ \xleftarrow{\lambda} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \xrightarrow{\lambda} \\ \xrightarrow{\nu} \quad \xrightarrow{\nu} \end{array} \quad (14.19)$$

This CG series expresses two **parallel channels** as a sum of s-channels.

The CG series for  $N > 2$  parallel channels  $V_{\mu_1} \otimes V_{\mu_2} \otimes \dots \otimes V_{\mu_N} = \sum_\lambda V_\lambda$  is obtained by combining pairs of vector spaces recursively. The series depends on what vector space pairs are chosen in what order. For example, we can use<sup>2</sup>

<sup>2</sup>For succinctness, we are dropping the rep labels  $\mu, \lambda$  from  $\kappa_\nu^{\mu\lambda}$ , but the  $\kappa_\nu$  still depends on them.



$$\begin{array}{c} \leftarrow \sigma - \mathfrak{e}_\mu^\dagger \leftarrow \mu - \\ \downarrow \omega \\ \leftarrow \rho - \mathfrak{e}_\rho \leftarrow \nu - \end{array} = \sum_\lambda \left[ \begin{array}{c} \overline{\begin{array}{c} \xrightarrow{d_\lambda} \sigma \xrightarrow{\quad} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \xrightarrow{\quad} \rho \end{array}} \quad \overline{\begin{array}{c} \xrightarrow{d_\lambda} \mu \xrightarrow{\quad} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \xrightarrow{\quad} \nu \end{array}} \\ * \quad \begin{array}{c} \begin{array}{c} \swarrow \sigma \quad \nwarrow \mu \\ \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \\ \swarrow \rho \quad \nwarrow \nu \end{array} \quad \begin{array}{c} \mathfrak{e}_\mu^\dagger \\ \downarrow \omega \\ \mathfrak{e}_\rho \end{array} \quad \begin{array}{c} \swarrow \mu \quad \nwarrow \nu \\ \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \end{array} \end{array} \right] \quad (14.22)$$

$$= \sum_\lambda \Phi_\lambda \quad \begin{array}{c} \swarrow \sigma \quad \nwarrow \mu \\ \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \\ \swarrow \rho \quad \nwarrow \nu \end{array} \quad (14.23)$$

where

$$\Phi_\lambda = \frac{\overline{\begin{array}{c} \xrightarrow{d_\lambda} \sigma \xrightarrow{\quad} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \xrightarrow{\quad} \rho \end{array}}}{\begin{array}{c} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \\ \swarrow \sigma \quad \nwarrow \rho \end{array}} \frac{\overline{\begin{array}{c} \xrightarrow{d_\lambda} \mu \xrightarrow{\quad} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \xrightarrow{\quad} \nu \end{array}}}{\begin{array}{c} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \\ \swarrow \mu \quad \nwarrow \nu \end{array}} \frac{\begin{array}{c} \mathfrak{e}_\mu^\dagger \\ \downarrow \omega \\ \mathfrak{e}_\rho \end{array}}{\begin{array}{c} \mathfrak{e}_\lambda \xrightarrow{\lambda} \mathfrak{e}_\lambda^\dagger \end{array}} \quad (14.24)$$

$$= d_\lambda \frac{\begin{array}{c} \mathfrak{e}_\mu^\dagger \\ \downarrow \omega \\ \mathfrak{e}_\rho \end{array}}{\overline{\begin{array}{c} \xrightarrow{d_\lambda} \sigma \xrightarrow{\quad} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \xrightarrow{\quad} \rho \end{array}}} \frac{\begin{array}{c} \mathfrak{e}_\lambda \xrightarrow{\lambda} \mathfrak{e}_\lambda^\dagger \end{array}}{\overline{\begin{array}{c} \xrightarrow{d_\lambda} \mu \xrightarrow{\quad} \mathfrak{e}_\lambda^\dagger \xleftarrow{\lambda} \mathfrak{e}_\lambda \xrightarrow{\quad} \nu \end{array}}} \quad (14.25)$$



### 14.3 Wigner $3n - j$ Coefficients/DAGs

A DAG with no incoming or outgoing arrows is called an **isolated DAG**. Physicists sometimes call it a **vacuum bubble** also. On the right hand side of Eq.(14.25), the isolated DAG with two  $\mathfrak{C}$  is called a  $3j$  **coefficient/DAG**, and the one with 4  $\mathfrak{C}$  is called a  $6j$  **coefficient/DAG**. So far we seen  $3j$  and  $6j$  coefficients/DAGs. Atomic physicists define **Wigner  $3n - j$  coefficients/DAGs**, for  $n = 1, 2, 3, \dots$ . They are called that because they describe how to “add”  $3n$  angular momenta  $j$ . There is only one topological distinct  $3j$  DAG but two  $6j$  DAGs, five  $9j$  DAGs, and so on.

In Chapter 1, we discussed Casimir suns. Next we show that they can always be expressed in terms of  $3j$  and  $6j$  coefficients and CG coefficients. We proceed as we did in Eq.(14.20) but here we use the most general t-channel to sum of s-channels conversion Eq.(14.23).

$$\begin{array}{c} \leftarrow T^{i_1} \\ \uparrow \\ \leftarrow T^{i_2} \\ \uparrow \\ \leftarrow T^{i_3} \\ \uparrow \\ \leftarrow T^{i_4} \end{array} = \sum_{\lambda, \mu} \Phi_\lambda \Phi_\mu \begin{array}{c} \leftarrow \parallel \mathfrak{C}_\lambda^\dagger \leftarrow \mathfrak{C}_\lambda \leftarrow \\ \parallel \\ \leftarrow T^{i_3} \\ \uparrow \\ \leftarrow T^{i_4} \end{array} \quad (14.26)$$

$$= \sum_{\lambda, \mu} \Phi_\lambda \Phi_\mu \begin{array}{c} \leftarrow \parallel \mathfrak{C}_\lambda^\dagger \leftarrow \parallel \mathfrak{C}_\mu^\dagger \leftarrow \mathfrak{C}_\mu \leftarrow \\ \parallel \\ \leftarrow T^{i_4} \end{array} \quad (14.27)$$

$$= \sum_{\lambda, \mu, \nu} \Phi_\lambda \Phi_\mu \Phi_\nu \begin{array}{c} \leftarrow \parallel \mathfrak{C}_\lambda^\dagger \leftarrow \parallel \mathfrak{C}_\mu^\dagger \leftarrow \parallel \mathfrak{C}_\nu^\dagger \leftarrow \mathfrak{C}_\nu \leftarrow \\ \parallel \\ \leftarrow \end{array} \quad (14.28)$$

# Chapter 15

## Recoupling Identities for $U(n)$

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

For  $U(n)$  (as opposed to  $SU(n)$ ), there are no antiparticles (i.e., one can use only lowered indices). A consequence of this is that for proper operators in  $U(n)$ , the total particle number is conserved.

Young Tableaux are discussed in Chapter 26.

Clebsh-Gordan series for  $U(n)$  can be written in terms of Standard Young Tableaux (SYT). For example, the tensor decomposition of  $V^{\otimes 5}$  is:<sup>1</sup>

$$\begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} = \sum_{\alpha, \beta, \gamma, \delta} \left\{ \begin{array}{c} \leftarrow \mathcal{Y}_\alpha \leftarrow \mathcal{Y}_\beta \leftarrow \\ \parallel \\ \leftarrow \parallel \leftarrow \\ \leftarrow \mathcal{Y}_\gamma \leftarrow \mathcal{Y}_\delta \leftarrow \\ \parallel \parallel \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} \right\} \{h.c.\} \quad (15.1)$$

$$= \sum_{\alpha, \beta, \gamma, \delta} \left\{ \begin{array}{c} \leftarrow \mathcal{Y}_\alpha \quad \mathcal{Y}_\beta \\ \parallel \leftarrow 2 \parallel \\ \leftarrow \mathcal{Y}_\gamma \quad \mathcal{Y}_\delta \\ \parallel \leftarrow 2 \parallel \\ \leftarrow \parallel \leftarrow 3 \parallel \leftarrow 5 \end{array} \right\} \{h.c.\} \quad (15.2)$$

where  $\leftarrow p \leftarrow$  means  $p$  parallel arrows superimposed on each other.

---

<sup>1</sup> $x(h.c.) = xx^\dagger$ .

It's always true that

$$\begin{array}{c} \leftarrow \mathcal{Y}_\beta \leftarrow \sigma \leftarrow \mathcal{Y}_\beta \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} = K_\sigma \begin{array}{c} \leftarrow \mathcal{Y}_\beta \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} \quad (15.3)$$

for some  $K_\sigma \in \{-1, 0, 1\}$ . More generally,

$$\begin{array}{c} \leftarrow \mathcal{Y}_\beta \leftarrow \mathcal{Y}_\alpha \leftarrow \mathcal{Y}_\beta \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \leftarrow \mathcal{Y}_\delta \leftarrow \mathcal{Y}_\gamma \leftarrow \mathcal{Y}_\delta \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} = K \begin{array}{c} \leftarrow \mathcal{Y}_\beta \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} \quad (15.4)$$

for some  $K \in \mathbb{R}$  that is independent of  $n$ .

## 15.1 $3j$ Coefficients

Recall that  $|\mathcal{Y}_\alpha|$  or  $|\alpha|$  is the number of boxes (or number of outgoing legs in its birdtrack) in the YT  $\mathcal{Y}_\alpha$ .

Clebsch-Gordan (CG) coefficients are discussed in Chapter 3. One can define a CG coefficient  $\mathfrak{C}_\beta$  in terms of Young Tableaux as follows:

$$\mathfrak{C}_\beta = \begin{array}{c} \leftarrow \beta \leftarrow \mathfrak{C}_\beta \\ \swarrow \alpha \\ \nwarrow \gamma \end{array} = \begin{array}{c} \mathcal{Y}_\beta \leftarrow |\alpha| \leftarrow \mathcal{Y}_\alpha \leftarrow \\ \parallel \\ \leftarrow |\beta| \leftarrow \leftarrow |\gamma| \leftarrow \mathcal{Y}_\gamma \leftarrow \end{array} \quad (15.5)$$

where  $|\beta| = |\alpha| + |\gamma|$

**Claim 35** ( $3j$  coefficient for  $U(n)$  in terms of  $YT$ )<sup>2</sup>

$$\text{tr}(\mathfrak{e}_\beta^\dagger \mathfrak{e}_\beta) = \begin{array}{c} \xrightarrow{\alpha} \\ \mathfrak{e}_\beta^\dagger \xleftarrow{\beta} \mathfrak{e}_\beta \\ \xleftarrow{\gamma} \end{array} = \begin{array}{c} \xleftarrow{\mathcal{Y}_\beta} \xleftarrow{\mathcal{Y}_\alpha} \xleftarrow{\quad} \\ \parallel \\ \xleftarrow{\quad} \xleftarrow{\mathcal{Y}_\gamma} \xleftarrow{\quad} \end{array} \quad (15.6)$$

$$= \dim(\mathcal{Y}_\beta) \quad (15.7)$$

**proof:**

$$\begin{array}{c} \xleftarrow{\mathcal{Y}_\beta} \xleftarrow{\mathcal{Y}_\alpha} \xleftarrow{\quad} \\ \parallel \\ \xleftarrow{\quad} \xleftarrow{\mathcal{Y}_\gamma} \xleftarrow{\quad} \end{array} = \begin{array}{c} \xleftarrow{\mathcal{Y}_\beta} \xleftarrow{\mathcal{Y}_\alpha} \xleftarrow{\mathcal{Y}_\beta} \xleftarrow{\quad} \\ \parallel \qquad \qquad \parallel \\ \xleftarrow{\quad} \xleftarrow{\mathcal{Y}_\gamma} \xleftarrow{\quad} \end{array} \quad (15.8)$$

$$= K \dim(\mathcal{Y}_\beta) \quad (15.9)$$

Ref. [1] shows that for this example,  $K = 1$ .

**QED**

## 15.2 $6j$ Coefficients

**Claim 36** ( $6j$  coefficient for  $U(n)$  in terms of  $YT$ )<sup>3</sup>

$$\begin{array}{c} \mathfrak{e}_\rho^\dagger \\ \downarrow \lambda \\ \mathfrak{e}_\mu \\ \swarrow \mu \quad \searrow \nu \\ \mathfrak{e}_\omega \quad \xrightarrow{\omega} \quad \mathfrak{e}_\omega^\dagger \end{array} = \begin{array}{c} \xleftarrow{\mathcal{Y}_\nu} \xleftarrow{\mathcal{Y}_\mu} \xleftarrow{\mathcal{Y}_\omega} \xleftarrow{\quad} \\ \parallel \qquad \qquad \parallel \\ \xleftarrow{\mathcal{Y}_\rho} \xleftarrow{\mathcal{Y}_\lambda} \xleftarrow{\quad} \xleftarrow{\quad} \\ \parallel \qquad \qquad \parallel \\ \xleftarrow{\quad} \xleftarrow{\mathcal{Y}_\sigma} \xleftarrow{\quad} \xleftarrow{\quad} \end{array} \quad (15.10)$$

$$= K \dim(\mathcal{Y}_\omega) \quad (15.11)$$

where  $K$  is independent of  $n$

<sup>2</sup>Ref.[1], on which this claim is based, uses different labels for the Young projectors. Their labels and ours are related as follows:

$\mathcal{Y}_\alpha \rightarrow X, \mathcal{Y}_\beta \rightarrow Y, \mathcal{Y}_\gamma \rightarrow Z$ .

<sup>3</sup>Ref.[1], on which this claim is based, uses different labels for the Young projectors. Their labels and ours are related as follows:

$\mathcal{Y}_\sigma \rightarrow X, \mathcal{Y}_\omega \rightarrow Y, \mathcal{Y}_\rho \rightarrow U, \mathcal{Y}_\lambda \rightarrow W, \mathcal{Y}_\nu \rightarrow V, \mathcal{Y}_\mu \rightarrow Z$ .

**proof:** Replace each of the 6 Young projectors  $\mathcal{Y}_\alpha$  of the right hand side (RHS) by its square. That gives 12 Young projectors on the RHS. Each of the 4 generators  $\mathfrak{C}_\lambda$  on the left hand side (LHS) is composed of 3 Young projectors so there are 12 Young projectors on the LHS too.

$$\begin{array}{c}
 \leftarrow \mathcal{Y}_\nu \leftarrow \mathcal{Y}_\mu \leftarrow \mathcal{Y}_\omega \leftarrow \\
 \leftarrow \mathcal{Y}_\rho \leftarrow \mathcal{Y}_\lambda \leftarrow \parallel \leftarrow \parallel \leftarrow \\
 \leftarrow \parallel \leftarrow \mathcal{Y}_\sigma \leftarrow \parallel \leftarrow \parallel \leftarrow
 \end{array}
 =
 \begin{array}{c}
 \leftarrow \mathcal{Y}_\omega \leftarrow \mathcal{Y}_\nu \leftarrow \mathcal{Y}_\mu \leftarrow \mathcal{Y}_\omega \leftarrow \\
 \leftarrow \parallel \leftarrow \mathcal{Y}_\rho \leftarrow \mathcal{Y}_\lambda \leftarrow \parallel \leftarrow \parallel \leftarrow \\
 \leftarrow \parallel \leftarrow \mathcal{Y}_\sigma \leftarrow \parallel \leftarrow \parallel \leftarrow
 \end{array}
 \quad (15.12)$$

$$= K \dim(\mathcal{Y}_\omega) \quad (15.13)$$

**QED**

For example, Ref.[1] shows that if

$$\mathcal{Y}_\rho = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 4 & \\ \hline \end{array}, \quad \mathcal{Y}_\nu = \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \quad \mathcal{Y}_\lambda = \begin{array}{|c|} \hline 2 \\ \hline \end{array} \quad (15.14)$$

$$\mathcal{Y}_\sigma = \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array}, \quad \mathcal{Y}_\omega = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array}, \quad \mathcal{Y}_\mu = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \quad (15.15)$$

then

$$K = \frac{1}{3}, \quad \dim \mathcal{Y}_\omega = \frac{n(n^2-1)(n^2-2)}{8} \quad (15.16)$$

## 15.3 Sum Rules

Let

$SYT(n_b)$  = set of SYT with  $n_b$  boxes

$SYT = \bigcup_{n_b=1}^{\infty} SYT(n_b)$

**Claim 37**

$$\sum_{\alpha', \gamma' \in SYT} \mathbb{1}(|\alpha'| + |\gamma'| = |\beta|) \mathfrak{C}_\beta^\dagger \begin{array}{c} \nearrow \alpha' \\ \leftarrow \beta \\ \searrow \gamma' \end{array} \mathfrak{C}_\beta = (|\beta| - 1) \dim(\mathcal{Y}_\beta) \quad (15.17)$$

**proof:**

$$\sum_{\alpha' \in SYT(|\alpha|)} \mathcal{Y}_{\alpha'} = 1, \quad \sum_{\gamma' \in SYT(|\gamma|)} \mathcal{Y}_{\gamma'} = 1 \quad (15.18)$$

$$\sum_{\alpha', \gamma' \in SYT} \mathbb{1}(|\alpha'| + |\gamma'| = |\beta|) \mathfrak{C}_{\beta}^{\dagger} \begin{array}{c} \nearrow \alpha' \\ \leftarrow \beta \\ \searrow \gamma' \end{array} \mathfrak{C}_{\beta} = \sum_{|\alpha'|=1}^{|\beta|-1} \sum_{\substack{\alpha' \in SYT(|\alpha'|) \\ \gamma' \in SYT(|\beta|-|\alpha'|)}} \begin{array}{c} \leftarrow \mathcal{Y}_{\beta} \leftarrow \mathcal{Y}_{\alpha'} \leftarrow \\ \parallel \\ \leftarrow \mathcal{Y}_{\gamma'} \leftarrow \end{array} \quad (15.19)$$

$$= \sum_{|\alpha'|=1}^{|\beta|-1} \begin{array}{c} \leftarrow \mathcal{Y}_{\beta} \leftarrow \\ \parallel \\ \leftarrow \mathcal{Y}_{\gamma'} \leftarrow \end{array} \quad (15.20)$$

$$= (|\beta| - 1) \dim(\mathcal{Y}_{\beta}) \quad (15.21)$$

**QED**

Let

$$A = \{\rho, \nu, \lambda, \sigma, \omega, \mu\}, \quad B = A - \{\omega\} \quad (15.22)$$

$$A' = \{\rho', \nu', \lambda', \sigma', \omega', \mu'\}, \quad B' = A' - \{\omega'\} \quad (15.23)$$

$$J(A) = \mathbb{1} \left( \begin{array}{l} |\sigma| + |\mu| = |\omega|, \\ |\nu| + |\rho| = |\omega|, \\ |\sigma| + |\lambda| = |\rho|, \\ |\lambda| + |\nu| = |\mu| \end{array} \right) \quad (15.24)$$

**Claim 38**

$$\prod_{\alpha' \in B'} \left[ \sum_{\alpha' \in SYT} \right] J(B', \omega) = \frac{1}{2} (|\omega| - 1)(|\omega| - 2) \dim \mathcal{Y}_{\omega} \quad (15.25)$$

**proof:**

See Ref.[1] for proof.

**QED**

# Chapter 16

## Reducibility of Representations

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

### 16.1 Eigenvalue Projectors

Suppose  $M \in \mathbb{C}^{d \times d}$  has eigenvalues  $\lambda_i$  with corresponding eigenvectors  $|\lambda_i\rangle$

$$M|\lambda_i\rangle = \lambda_i|\lambda_i\rangle \quad (16.1)$$

for  $i \in \mathbb{Z}_{[1,r]}$ . The characteristic polynomial of  $M$  is defined as

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M - \lambda) = \prod_{i=1}^r (\lambda - \lambda_i)^{d_i} \quad (16.2)$$

It satisfies

$$cp(\lambda) = 0 \quad (16.3)$$

for  $\lambda = \lambda_i$ .

Note that if  $M$  is Hermitian ( $M^\dagger = M$ ), then all its eigenvalues are real. (because  $\lambda_i = \langle \lambda_i | M | \lambda_i \rangle \in \mathbb{R}$ )

If  $M$  is Hermitian, then there exists a matrix  $C$  that is unitary ( $CC^\dagger = C^\dagger C = 1$ ) and diagonalizes  $M$

$$CMC^\dagger = \begin{bmatrix} \Lambda_{\lambda_1} & 0 & 0 & 0 \\ 0 & \Lambda_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \Lambda_{\lambda_r} \end{bmatrix} \quad (16.4)$$

where

$$\Lambda_{\lambda_i} = \lambda_i \underbrace{\text{diag}(1, 1, \dots, 1)}_{d_i \text{ times}} = \lambda_i I^{d_i \times d_i} \quad (16.5)$$

and

$$d = \sum_{i=1}^r d_i \quad (16.6)$$

As in Chapter 3, let us set

$$\pi_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_i \times d_i} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} \quad (16.7)$$

and

$$P_i = C^\dagger \pi_i C \quad (16.8)$$

For example, when  $d = 2$ ,

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (16.9)$$

so

$$\pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2}, \quad P_1 = \frac{M - \lambda_2}{\lambda_1 - \lambda_2} \quad (16.10)$$

$$\pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1}, \quad P_2 = \frac{M - \lambda_1}{\lambda_2 - \lambda_1} \quad (16.11)$$

$\{\pi_1, \pi_2\}$  is a complete orthogonal set of projection operators, and so is  $\{P_1, P_2\}$ .

Similarly, for  $d > 2$ , we have

$$\pi_i = \prod_{j \neq i} \frac{CMC^\dagger - \lambda_j}{\lambda_i - \lambda_j}, \quad P_i = \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (16.12)$$

$\{\pi_i\}_{i=1}^r$  is a complete set of orthogonal projection operators and  $\{P_i\}_{i=1}^r$  is too.

Note that

$$d_i = \text{tr}(\pi_i) = \text{tr}(P_i) \quad (16.13)$$

## 16.2 $[P_i, M] = 0$ Consequences

From Eq.(16.12), it is clear that  $P_i$  and  $M$  commute

$$[P_i, M] = P_i M - M P_i = 0 \quad (16.14)$$

From the  $P_i$ 's completeness and commutativity with  $M$ , we get



$$M = \sum_{i=1}^r \sum_{j=1}^r P_i M P_j \quad (16.15)$$

$$= \sum_{i=1}^r P_i M P_i \quad (16.16)$$

**Claim 39** For all  $i$ ,

$$M P_i = \lambda_i P_i \text{ (no } i \text{ sum)} \quad (16.17)$$

**proof:**

$$M P_i = [C^\dagger \Lambda C] [C^\dagger \pi_i C] \quad (16.18)$$

$$= \lambda_i [C^\dagger \pi_i C] \quad (16.19)$$

$$= \lambda_i P_i \quad (16.20)$$

**QED**

From the last claim, it immediately follows that if  $f(x)$  can be expressed as a power series in  $x$ , then <sup>1</sup>

$$f(M) P_i = f(\lambda_i) P_i \text{ (no } i \text{ sum)} \quad (16.21)$$

## 16.3 Multiple Invariant Matrices

Suppose  $M^{(1)}, M^{(2)} \in \mathbb{C}^{d \times d}$  are Hermitian matrices that commute

$$[M^{(1)}, M^{(2)}] = 0 \quad (16.22)$$

Use  $M^{(1)}$  to decompose  $V = \mathbb{C}^{d \times d}$  into a direct sum of vector spaces  $\bigoplus_i V_i$ . Then we can use  $M^{(2)}$  to decompose  $V_i$  into  $\bigoplus_j V_{i,j}$ . If  $M^{(1)}$  and  $M^{(2)}$  don't commute, let  $P_i^{(1)}$  be an eigenvalue projection operator of  $M^{(1)}$ . Then replace  $M^{(2)}$  by  $P_i^{(1)} M^{(2)} P_i^{(1)}$ . Now

$$[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] = \sum_j \lambda_j^{(1)} [P_j^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] \quad (16.23)$$

$$= 0 \quad (16.24)$$

---

<sup>1</sup> $M$  must also satisfy some convergence conditions that we won't get into.

## 16.4 $[G, M] = 0$ Consequences

An invariant matrix (see Ch.7) commutes with all the elements  $G$  of a group  $\mathcal{G}$

$$[G, M] = 0 \quad (16.25)$$

If  $P_i$  are the projection operators of  $M$ , then  $P_i = f_i(M)$  so

$$[G, P_i] = 0 \quad (16.26)$$

for all  $G \in \mathcal{G}$  and  $i$ . Hence,

$$G = 1G1 = \sum_i \sum_j P_i G P_j = \sum_j \underbrace{P_j G P_j}_{\stackrel{\text{def}}{=} G'_j} \quad (16.27)$$

Since  $P_i = C^\dagger \pi_i C$ ,

$$[CGC^\dagger, \pi_i] = 0 \quad (16.28)$$

Hence

$$CGC^\dagger = 1G1 = \sum_i \sum_j \pi_i CGC^\dagger \pi_j = \sum_j \underbrace{\pi_j CGC^\dagger \pi_j}_{\stackrel{\text{def}}{=} G_j} = \text{diag}(G_1, G_2, \dots, G_r) \quad (16.29)$$

Note that

$$C^\dagger G_j C = G'_j \quad (16.30)$$

A rep-matrix  $G'_i$  acts only on a  $d_i$  dimensional vector space  $V^{d_i} = P_i V^d$ . In this way, an invariant matrix  $M \in \mathbb{C}^{d \times d}$  with  $r$  distinct eigenvalues, induces a decomposition of  $V^d$  into a direct sum of vector spaces

$$V^d \xrightarrow[M]{} V_1^{d_1} \oplus V_2^{d_2} \oplus \dots \oplus V_r^{d_r} \quad (16.31)$$

If a rep-matrix  $G'_i$  cannot itself be reduced further, it is said to be an **irreducible representation (irrep)**.

Note that sometimes the term representation is used to refer to the vector space  $V_i^{d_i}$  instead of the matrix  $G_i$ .

# Chapter 17

## Spinors

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

The reader should read Chapter 12 before reading this chapter. There he will learn about the group  $Spin(n)_{\mathbb{R}}$ .

Let

$$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_{\mu}, \gamma_{\nu}] \quad (17.1)$$

If  $\gamma_{\mu}^{\dagger} = \gamma_{\mu}$  for  $\mu = 1, 2, \dots, n$  (this assumes the Euclidean metric which is  $g_{\mu\nu} = \text{diag}(1, 1, 1, 1)$  for  $n = 4$ )

$$Spin(n)_{\mathbb{R}} = \{e^{\omega^{\mu\nu}\gamma_{\mu\nu}} | \omega_{\mu\nu} = -\omega_{\nu\mu} \in \mathbb{R}\} \quad (17.2)$$

If  $\gamma_0^{\dagger} = \gamma_0$ , and  $\gamma_i^{\dagger} = -\gamma_i$  (this assumes the mostly-minus metric which is  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  for  $n = 4$ )

$$Spin(1, 3)_{\mathbb{R}} = \{e^{-i\omega^{\mu\nu}\gamma_{\mu\nu}} | \omega_{\mu\nu} = -\omega_{\nu\mu}, \omega_{0,i} \in \mathbb{R}, \omega_{i,j} \in i\mathbb{R}\} \quad (17.3)$$

**Spinors** are the vectors in the vector space upon which the group  $Spin(n)_{\mathbb{R}}$  acts.

$n$	1	2	3	4	5	6	7	8	9	10
$\text{int}(n/2)$	0	1	1	2	2	3	3	4	4	5
$d = 2^{\text{int}(n/2)}$	1	2	2	4	4	8	8	16	16	32

Table 17.1:  $\gamma_{\mu} \in \mathbb{C}^{d \times d}$

Let  $\mu \in \{1, 2, \dots, n\}$

$\text{int}(x) = \text{integer part of } x \in \mathbb{R}$

$a, b \in \{1, 2, \dots, 2^{\text{int}(n/2)}\}$

$\gamma_{\mu} \in \mathbb{C}^{d \times d}$  where  $d = 2^{\text{int}(n/2)}$  (See Table 17.1)

Define the following birdtracks

$$\gamma_{ab}^\mu = \begin{array}{c} \mu \\ \downarrow \\ a \leftarrow \bar{\gamma} \leftarrow b \end{array}, \quad (\gamma_\mu)_{ab} = \begin{array}{c} \mu \\ \uparrow \\ a \leftarrow \underline{\gamma} \leftarrow b \end{array}, \quad \delta_a^b = a \leftarrow \bullet \leftarrow b \quad (17.4)$$

$$g_{\mu\nu} = \mu \longleftarrow \underline{g} \longrightarrow \nu, \quad g^{\mu\nu} = \mu \longrightarrow \bar{g} \longleftarrow \nu \quad (17.5)$$

Note that

$$g_\mu^\mu = \text{red arc} = 2^{int(n/2)} \quad (17.6)$$

Clifford Algebra anticommutator:

$$g_{\mu,\nu} \mathbb{1} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]_+$$

$$\begin{array}{c} \text{cup} \\ \underline{g} \\ \leftarrow \text{dashed line} \end{array} = \begin{array}{c} \uparrow \uparrow \\ \mathcal{S}_2 \\ \uparrow \uparrow \\ \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \end{array} = \frac{1}{2} \left[ \begin{array}{c} \uparrow \uparrow \\ \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \end{array} + \begin{array}{c} \text{cross} \\ \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \end{array} \right] \quad (17.7)$$

By virtue of Eq.(17.7), it is possible to replace all  $g$ 's by swaps, and vice versa, in a linear combination of birdtracks.

**Claim 40**

$$\begin{array}{c} \text{cap} \\ \bar{g} \\ \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \end{array} = n \leftarrow \text{dashed line} \quad (17.8)$$

**proof:**

$$\begin{array}{c} \text{cap} \\ \bar{g} \\ \uparrow \uparrow \\ \mathcal{S}_2 \\ \uparrow \uparrow \\ \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \end{array} = \begin{array}{c} \text{cup} \\ \bar{g} \\ \text{cup} \\ \underline{g} \\ \leftarrow \text{dashed line} \end{array} \quad (17.9)$$

### Claim 41

where we define

$$6 \mathcal{A}_3 = \left[ \begin{array}{c} \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \\ \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \end{array} \right] = \left[ \begin{array}{c} \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \\ - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} + \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} + \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \end{array} \right] \quad (17.15)$$

Now let us express each of the 6 birdtracks on the right hand side of Eq.(17.15) in terms of  $g$ 's instead of swaps.

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \begin{array}{c} \longleftrightarrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} = 2 \begin{array}{c} \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \quad (17.16)$$

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \begin{array}{c} \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \end{array} = 2 \begin{array}{c} \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \quad (17.17)$$

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \begin{array}{c} \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \end{array} = 2 \begin{array}{c} \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \gamma \leftarrow \gamma \leftarrow \gamma \end{array} \quad (17.18)$$

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} = 2 \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} \quad (17.19)$$

$$= 2 \begin{array}{c} \uparrow \\ \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} \quad (17.20)$$

$$= 2 \begin{array}{c} \uparrow \\ \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} - 2 \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad (17.21)$$

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} = 2 \begin{array}{c} \uparrow \\ \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} - 2 \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad \begin{array}{c} \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} \quad (17.22)$$

**QED**

**Claim 42**

$$\frac{1}{2} [\gamma_\nu \gamma_{\mu_1} \dots \gamma_{\mu_a} - (-1)^a \gamma_{\mu_1} \dots \gamma_{\mu_a} \gamma_\nu] = \sum_{k=1}^a (-1)^{k-1} g_{\nu \mu_k} \gamma_{\mu_1} \dots \widehat{\gamma_{\mu_k}} \dots \gamma_{\mu_a} \quad (17.23)$$

where a hat means omission. For  $a = 2$ ,

$$\frac{1}{2} \left[ \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \quad \gamma \end{array} - (-1)^2 \begin{array}{c} \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \end{array} \right] = \begin{array}{c} \uparrow \\ \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \end{array} - \begin{array}{c} \uparrow \\ \leftarrow \quad \leftarrow \\ \gamma \quad \gamma \end{array} \quad (17.24)$$

**proof:**

Note that

$$\begin{array}{c} \nearrow \\ \nwarrow \end{array} = \begin{array}{c} \uparrow \\ \longleftrightarrow \\ \uparrow \\ \longleftrightarrow \\ \uparrow \end{array} \quad (17.25)$$

$\underline{\gamma} \leftarrow - \underline{\gamma} \leftarrow - \underline{\gamma}$

so this claim has been proven before for  $a = 2$  in Eq.(17.22)

**QED**

**Claim 43** *If  $n \neq a$  and  $a$  is odd*

$$\text{tr}(\gamma_{\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_a}) = 0 \quad (17.26)$$

**proof:**

If we left-multiply both sides of Eq.(17.24) by  $\gamma_{\mu_1}$  where  $\mu_1$  is the first index, and then we trace over the spinor indices, we get

$$\frac{1}{2} \left[ \begin{array}{c} \nearrow \\ \nwarrow \end{array} - (-1)^a \begin{array}{c} \nwarrow \\ \nearrow \end{array} \right] = \begin{array}{c} \uparrow \\ \uparrow \end{array} - \begin{array}{c} \nearrow \\ \nwarrow \end{array} \quad (17.27)$$

$\underline{\gamma} \leftarrow - \underline{\gamma} \leftarrow - \underline{\gamma}$

Assuming  $a$  is odd, we get

$$n \mathcal{A}_a = a \mathcal{A}_a \quad (17.28)$$

$\underline{\gamma} \leftarrow - \underline{\gamma} \leftarrow - \underline{\gamma}$

Hence, if  $n \neq a$  and  $a$  is odd, the trace of  $a$  gammas is zero.

The above proof is now complete and it works for odd or even  $n$ . However, let us mention that for even  $n$ , a chirality operator  $\Gamma_5$ , exists, and this claim can be proven using  $\Gamma_5$ . Indeed, for  $n$  even, let

$$\Gamma_5 = K \gamma_1 \gamma_2 \cdots \gamma_n \quad (17.29)$$

$\Gamma_5$  anti-commutes with all  $\gamma_\mu$ :

$$\Gamma_5 \gamma_\mu \Gamma_5^{-1} = -\gamma_\mu \quad (17.30)$$

Hence



$$\text{tr}(\gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_a}) = \text{tr}(\Gamma_5 \gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_a} \Gamma_5^{-1}) \quad (17.31)$$

$$= (-1)^a \text{tr}(\gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_a}) \quad (17.32)$$

So for odd  $a$ ,  $\text{tr}(\gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_a}) = 0$ .

**QED**

**Claim 44**

$$\begin{aligned} \underbrace{\frac{1}{2} \text{tr}([\gamma_\mu, \gamma_\nu]_+)}_{\text{tr}(\gamma_\mu \gamma_\nu)} &= \text{diagram with two vertices connected by a red arc, labeled } g_{\mu\nu} \\ \underbrace{\mu \leftarrow \underline{\gamma} \quad \underline{\gamma} \rightarrow \nu}_{\text{tr}(\gamma_\mu \gamma_\nu)} &= \text{diagram with two vertices connected by a red arc, labeled } \mu \leftarrow \underline{g} \rightarrow \nu \\ &= \text{diagram with two vertices connected by a red arc, labeled } \mu \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \rightarrow \nu \end{aligned} \quad (17.33)$$

$$\begin{aligned} \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) &= \text{diagram with four vertices connected by a red arc, labeled } (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\nu} g_{\sigma\rho}) \\ &= \text{diagram with four vertices connected by a red arc, labeled } \mu \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \rightarrow \sigma \\ &= \text{diagram with four vertices connected by a red arc, labeled } \mu \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \leftarrow \underline{\gamma} \rightarrow \sigma \end{aligned} \quad (17.34)$$

**proof:**

**QED**

Define the antisymmetrized gamma matrix products  $\Gamma^{(k)}$  by:

$$\begin{aligned}
\Gamma^{(0)} &= 1 = \begin{array}{c} \leftarrow \text{---} \end{array} = \begin{array}{c} \uparrow \\ \leftarrow \text{---} \underline{\Gamma}^{(0)} \leftarrow \text{---} \end{array} \\
\Gamma_{\mu}^{(1)} &= \gamma_{\mu} = \begin{array}{c} \uparrow \\ \leftarrow \text{---} \underline{\gamma} \leftarrow \text{---} \end{array} = \begin{array}{c} \uparrow \\ \leftarrow \text{---} \underline{\Gamma}^{(1)} \leftarrow \text{---} \end{array} \\
\Gamma_{\mu\nu}^{(2)} &= \frac{1}{2}[\gamma_{\mu}, \gamma_{\nu}] = \begin{array}{c} \uparrow \quad \uparrow \\ \mathcal{A}_2 = \\ \uparrow \quad \uparrow \\ \leftarrow \text{---} \underline{\gamma} \leftarrow \text{---} \underline{\gamma} \leftarrow \text{---} \end{array} = \begin{array}{c} \uparrow \\ \leftarrow \text{---} \underline{\Gamma}^{(2)} \leftarrow \text{---} \end{array} \\
\Gamma_{\lambda\mu\nu}^{(3)} &= \frac{1}{6}[\gamma_{\lambda}, \gamma_{\mu}, \gamma_{\nu}] = \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \mathcal{A}_3 = \\ \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \text{---} \underline{\gamma} \leftarrow \text{---} \underline{\gamma} \leftarrow \text{---} \underline{\gamma} \leftarrow \text{---} \end{array} = \begin{array}{c} \uparrow \\ \leftarrow \text{---} \underline{\Gamma}^{(3)} \leftarrow \text{---} \end{array}
\end{aligned} \tag{17.35}$$

Henceforth, let

$$\underline{g}^{(t)} = \prod_{i=1}^t g_{\mu_i \nu_i} \tag{17.36}$$

**Claim 45**

$$\underbrace{\mu^{:a} \leftarrow \underline{\Gamma}^{(a)} \overset{\curvearrowright}{\longrightarrow} \underline{\Gamma}^{(b)} \longrightarrow \nu^{:b}} = \delta_a^b \begin{array}{c} \leftarrow \text{---} \end{array} \quad \mu^{:a} \leftarrow \underline{g}^{(a)} \rightarrow \mathcal{A}_a \rightarrow \nu^{:a} \tag{17.37}$$

$$\begin{array}{c} \uparrow \quad \uparrow \\ \leftarrow \text{---} \underline{\Gamma}^{(a)} \leftarrow \text{---} \underline{\Gamma}^{(b)} \leftarrow \text{---} \end{array}$$

**proof:**

**QED**

Define  $\chi_c$  by

$$\chi_c = \begin{array}{c} \leftarrow \mathcal{A}_b \xleftarrow{\underline{g}^{(t)}} \mathcal{A}_a \longrightarrow \\ \quad \nwarrow \underline{g}^{(s)} \quad \nearrow \underline{g}^{(u)} \\ \quad \mathcal{A}_c \\ \quad \downarrow \end{array} \tag{17.38}$$

where

$$\begin{aligned}
s &= (b + c - a)/2 \\
t &= (b + a - c)/2 \\
u &= (a + c - b)/2
\end{aligned} \tag{17.39}$$

**Claim 46**

$$\begin{array}{ccc}
\begin{array}{c} \leftarrow \underline{\Gamma}^{(b)} \leftarrow \cdots \leftarrow \underline{\Gamma}^{(a)} \rightarrow \\ \searrow \quad \nearrow \\ \underline{\Gamma}^{(c)} \\ \downarrow \\ \underbrace{\hspace{10em}} \\ \leftarrow \underline{\Gamma}^{(c)} \leftarrow \underline{\Gamma}^{(b)} \leftarrow \underline{\Gamma}^{(a)} \leftarrow \end{array} & = \frac{a!b!c!}{s!t!u!} \begin{array}{c} \leftarrow \mathcal{A}_b \leftarrow \underline{g}^{(t)} \rightarrow \mathcal{A}_a \rightarrow \\ \searrow \underline{g}^{(s)} \quad \nearrow \underline{g}^{(u)} \\ \mathcal{A}_c \\ \downarrow \\ \underbrace{\hspace{10em}}_{\chi_c} \end{array} & \begin{array}{c} \leftarrow \cdots \\ \text{red arc} \end{array}
\end{array} \tag{17.40}$$

**proof:**  
**QED**

**Claim 47**

$$\begin{array}{ccc}
\begin{array}{c} \uparrow \quad \uparrow \\ \leftarrow \cdots \underline{\Gamma}^{(b)} \leftarrow \cdots \underline{\Gamma}^{(a)} \leftarrow \cdots \end{array} & = \sum_{c=1}^{a+b} K_c \begin{array}{c} \uparrow \quad \uparrow \\ \mathcal{A}_b \leftarrow \underline{g}^{(t)} \rightarrow \mathcal{A}_a \\ \swarrow \quad \searrow \\ \underline{\Gamma}^{(c)} \\ \leftarrow \cdots \end{array} & \tag{17.41}
\end{array}$$

where  $t = a - u = b - s$  and

$$K_c = \frac{a!b!c!}{s!t!u!} \tag{17.42}$$

**proof:**

Eq.(17.41) follows because the  $\Gamma^{(c)}$  span the vector space of  $\Gamma^{(c)}$  products.

If we left-multiply both sides of Eq.(17.41) by  $\Gamma^{(c')}$  and then we trace over the spinor indices, we get

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \Gamma^{(c')} \leftarrow \Gamma^{(b)} \leftarrow \Gamma^{(a)} \leftarrow \\ \text{---} \end{array} = \sum_{c=1}^{a+b} K_c \begin{array}{c} \uparrow \quad \uparrow \\ \mathcal{A}_b \xleftarrow{\underline{g}^{(t)}} \mathcal{A}_a \\ \swarrow \quad \searrow \\ \Gamma^{(c')} \quad \Gamma^{(c)} \\ \text{---} \end{array} \quad (17.43)$$

so  $K_c$  satisfies

$$\frac{a!b!c!}{s!t!u!} \begin{array}{c} \leftarrow \text{---} \\ \text{---} \end{array} \chi_c = K_c \begin{array}{c} \leftarrow \text{---} \\ \text{---} \end{array} \chi_c \quad (17.44)$$

Hence,

$$K_c = \frac{a!b!c!}{s!t!u!} \quad (17.45)$$

**QED**

Define the projector  $P^{(a)}$  by

$$P^{(a)} = \frac{1}{\begin{array}{c} \leftarrow \text{---} \\ \text{---} \end{array}} \begin{array}{c} \text{---} \\ \downarrow \\ \Gamma^{(a)} \end{array} \xleftarrow{\bar{g}^{(a)}} \begin{array}{c} \text{---} \\ \uparrow \\ \Gamma^{(a)} \end{array} = \frac{1}{\begin{array}{c} \leftarrow \text{---} \\ \text{---} \end{array}} \begin{array}{c} \text{---} \\ \downarrow \\ \Gamma^{(a)} \end{array} \rightarrow \begin{array}{c} \text{---} \\ \uparrow \\ \bar{\Gamma}^{(a)} \end{array} \quad (17.46)$$

and the 3 leg vertex  $\begin{smallmatrix} b \wedge a \\ c \end{smallmatrix}$  by

$$\begin{array}{c} \swarrow \quad \searrow \\ b \wedge a \\ c \\ \downarrow \end{array} = \frac{1}{\begin{array}{c} \leftarrow \text{---} \\ \text{---} \end{array}} \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \Gamma^{(c)} \leftarrow \Gamma^{(b)} \leftarrow \Gamma^{(a)} \leftarrow \\ \text{---} \end{array} = K_c \chi_c \quad (17.47)$$

Note that the vertex  $\begin{smallmatrix} b \wedge a \\ c \end{smallmatrix}$  is nonzero iff  $a + b + c$  is even, and if  $a, b$ , and  $c$  satisfy the triangle inequalities  $|a - b| \leq c \leq |a + b|$

**Claim 48**

$$\begin{array}{c} \swarrow \quad \searrow \\ a \wedge b \\ c \\ \downarrow \end{array} = (-1)^{st+tu+us} \begin{array}{c} \swarrow \quad \searrow \\ b \wedge a \\ c \\ \downarrow \end{array} \quad (17.48)$$

**proof:**  
**QED**

Note that

$$d^{(a)} = \text{tr}(P^{(a)}) = \frac{1}{\begin{array}{c} \leftarrow \text{---} \Gamma^{(a)} \xrightarrow{\text{---}} \bar{\Gamma}^{(a)} \\ \leftarrow \text{---} \end{array}} \quad (17.49)$$

$$= \begin{array}{c} \leftarrow \mathcal{A}_a \leftarrow \\ \leftarrow \text{---} \end{array} \quad (17.50)$$

$$= \begin{cases} \binom{n}{a} & \text{if } a \leq n \text{ (see Eq.(22.43))} \\ 0 & \text{otherwise} \end{cases} \quad (17.51)$$

$$\sum_{a=1}^n d^{(a)} = \sum_{a=1}^n \binom{n}{a} = (1+1)^2 = 2^a \quad (17.52)$$

**Claim 49**

$$\begin{array}{c} \uparrow \quad \uparrow \\ \leftarrow \text{---} \Gamma^{(b)} \leftarrow \text{---} \Gamma^{(a)} \leftarrow \text{---} \end{array} = \sum_{c=1}^{a+b} K'_c \begin{array}{c} \nwarrow \quad \nearrow \\ b \wedge a \\ \downarrow C \\ \leftarrow \text{---} \bar{\Gamma}^{(c)} \leftarrow \text{---} \end{array} \quad (17.53)$$

where

$$K'_c = \frac{a!b!c!}{s!t!u!} = K_c \quad (17.54)$$

**proof:**

Eq.(17.53) follows because the  $\Gamma^{(c)}$  span the vector space of  $\Gamma^{(c)}$  products.

If we left-multiply both sides of Eq.(17.53) by  $\Gamma^{(c')}$  and then we trace over the spinor indices, we get

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \text{---} \Gamma^{(c')} \leftarrow \text{---} \Gamma^{(b)} \leftarrow \text{---} \Gamma^{(a)} \leftarrow \text{---} \end{array} = \sum_{c=1}^{a+b} K'_c \begin{array}{c} \nwarrow \quad \nearrow \\ b \wedge a \\ \downarrow C \\ \leftarrow \text{---} \bar{\Gamma}^{(c)} \leftarrow \text{---} \end{array} \quad (17.55)$$

So  $K'_c$  satisfies

$$\frac{a!b!c!}{s!t!u!} \begin{array}{c} \leftarrow \text{---} \end{array} \chi_c = K'_c \begin{array}{c} \leftarrow \text{---} \end{array} \chi_c \quad (17.56)$$

Hence,

$$K'_c = \frac{a!b!c!}{s!t!u!} = K_c \quad (17.57)$$

**QED**

**Claim 50** (*t-channel to sum of s-channels*)

$$\begin{array}{c} \leftarrow \text{---} \underline{\Gamma}^{(a)} \text{---} \leftarrow \\ \downarrow \\ \text{---} \rightarrow \bar{\Gamma}^{(a)} \text{---} \rightarrow \end{array} = \sum_b \Phi_b \begin{array}{c} \leftarrow \text{---} \underline{\Gamma}^{(b)} \text{---} \leftarrow \\ \uparrow \quad \downarrow \\ \text{---} \rightarrow \bar{\Gamma}^{(b)} \text{---} \rightarrow \end{array} \quad (17.58)$$

where

$$\Phi_b = \frac{d_b}{\leftarrow \text{---} \underline{\Gamma}^{(a)} \text{---} \leftarrow} \frac{d_b}{\text{---} \rightarrow \bar{\Gamma}^{(a)} \text{---} \rightarrow} \begin{array}{c} \underline{\Gamma}^{(a)} \\ \downarrow \\ \bar{\Gamma}^{(a)} \\ \uparrow \quad \downarrow \\ \bar{\Gamma}^{(b)} \leftarrow \underline{\Gamma}^{(b)} \end{array} \quad (17.59)$$

$$= d_b \frac{\begin{array}{c} \underline{\Gamma}^{(a)} \\ \downarrow \\ \bar{\Gamma}^{(a)} \\ \uparrow \quad \downarrow \\ \bar{\Gamma}^{(b)} \leftarrow \underline{\Gamma}^{(b)} \end{array}}{\leftarrow \text{---} \underline{\Gamma}^{(a)} \text{---} \leftarrow \quad \text{---} \rightarrow \bar{\Gamma}^{(a)} \text{---} \rightarrow} \quad (17.60)$$

**proof:**

$$\begin{array}{c} \leftarrow \text{---} \underline{\Gamma}^{(a)} \text{---} \leftarrow \\ \downarrow \\ \text{---} \rightarrow \bar{\Gamma}^{(a)} \text{---} \rightarrow \end{array} = \sum_b \left[ * \begin{array}{c} \frac{d_b}{\leftarrow \text{---} \underline{\Gamma}^{(a)} \text{---} \leftarrow} \frac{d_b}{\text{---} \rightarrow \bar{\Gamma}^{(a)} \text{---} \rightarrow} \\ \uparrow \quad \downarrow \\ \underline{\Gamma}^{(b)} \leftarrow \bar{\Gamma}^{(b)} \\ \downarrow \\ \bar{\Gamma}^{(a)} \end{array} \right] \quad (17.61)$$

$$= \sum_b \Phi_b \begin{array}{c} \leftarrow \text{---} \underline{\Gamma}^{(b)} \text{---} \leftarrow \\ \uparrow \quad \downarrow \\ \text{---} \rightarrow \bar{\Gamma}^{(b)} \text{---} \rightarrow \end{array} \quad (17.62)$$

**QED**

# Chapter 18

## Spinors, Their Handedness

This chapter is based on an AI output and Ref.[1].

### 18.1 In 1+3 dim

Consider  $SO(1, 3)$  so  $n = 1 + 3 = 4$

$$\begin{aligned} &\gamma_{ab}^\mu \\ &a, b \in \{1, 2, 3, 4\} \\ &\mu = 1, 2, \dots, n \end{aligned}$$

**Mostly-plus metric (M+M) (a.k.a. East Coast metric)** is  $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$

**Mostly-minus metric (M-M) (a.k.a. West Coast metric)** is  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$

$$[\gamma_\mu, \gamma_\nu]_+ = 2g^{\mu\nu} \quad \text{for } \mu, \nu = 0, 1, 2, 3 \quad (18.1)$$

$$\begin{aligned} \text{M-M: } &\gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i \quad \text{for } i = 1, 2, 3 \\ \text{M+M: } &\gamma_0^\dagger = -\gamma_0, \quad \gamma_i^\dagger = \gamma_i \quad \text{for } i = 1, 2, 3 \end{aligned} \quad (18.2)$$

The last equation is true iff

$$\begin{aligned} \text{M-M: } &\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \\ \text{M+M: } &\gamma_\mu^\dagger = -\gamma_0 \gamma_\mu \gamma_0 \end{aligned} \quad (18.3)$$

In 1+3 dimensional relativistic quantum field theory, the **chirality operator**  $\gamma_5$  is defined in terms of the four Dirac gamma matrices  $\gamma_\mu$ , where  $\mu = 0, 1, 2, 3$ , as follows

$$\begin{aligned} \text{M-M: } &\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \\ \text{M+M: } &\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \end{aligned} \quad (18.4)$$

In the “Dirac representation”,



$$\gamma_5 = \pm \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \quad (18.5)$$

The properties of  $\gamma_5$  might change by a sign depending on which metric we are using. Henceforth, we will use the M-M  $\gamma_5$  properties:

- Anticommutates with  $\gamma_\mu$

$$\gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5 \quad \text{for } \mu = 0, 1, 2, 3 \quad (18.6)$$

This follows because  $g_{\mu\nu}$  is diagonal so, by the Clifford algebra definition Eq.(18.1) (anti-commutator of gammas), the  $\gamma_\mu$  anticommutates with 3 of the 4 gamma matrices in  $\gamma_5$

- Hermitian

$$\gamma_5^\dagger = \gamma_5 \quad (18.7)$$

This follows because

$$\gamma_5^\dagger = -i(-\gamma_3)(-\gamma_2)(-\gamma_1)(+\gamma_0) = i\gamma_3\gamma_2\gamma_1\gamma_0 = (-1)^6 i(\gamma_0\gamma_1\gamma_2\gamma_3) = \gamma_5 \quad (18.8)$$

- Square is 1

$$\gamma_5^2 = I. \quad (18.9)$$

Therefore its eigenvalues are +1 (positive chirality) and -1 (negative chirality). This follows because

$$(\gamma_5)^2 = -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 = -(-1)^6(\gamma_0)^2(\gamma_1)^2(\gamma_2)^2(\gamma_3)^2 = -g_{00}g_{11}g_{22}g_{33} = 1 \quad (18.10)$$

- Traceless

$$\text{tr}(\gamma_5) = 0 \quad (18.11)$$

This follows immediately from the expression for  $\gamma_5$  in the Dirac representation. Alternatively, note that if spinor  $\psi$  satisfies

$$\gamma_5 \psi = +\psi \quad (18.12)$$

(has positive chirality), then spinor  $\gamma_\mu \psi$  has negative chirality because

$$\gamma_5(\gamma_\mu \psi) = -\gamma_\mu \gamma_5 \psi = -\gamma_\mu \psi \quad (18.13)$$

This means that the eigenvalues of  $\gamma_5$  come in  $\pm 1$  pairs. Hence the trace of  $\gamma_5$  must be zero.

•

$$\text{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}, \quad (18.14)$$

with  $\epsilon^{0123} = +1$

• If

$$P_L = P_- = \frac{1}{2}(1 - \gamma_5), \quad P_R = P_+ = \frac{1}{2}(1 + \gamma_5). \quad (18.15)$$

where  $R$ = right handed,  $L$  = left handed then

$$\psi = \underbrace{\psi_L}_{P_L \psi} + \underbrace{\psi_R}_{P_R \psi} \quad (18.16)$$

• If  $\mathbb{P} = \gamma_0$  (Parity Operator) and  $\sigma^{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$  (Lorentz generators)

$$[\gamma_5, \sigma^{\mu\nu}] = 0, \quad \gamma_5 \mathbb{P} = -\mathbb{P} \gamma_5 \quad (18.17)$$

Thus  $\gamma_5$  is **pseudoscalar** under Lorentz transformations (i.e., invariant under proper Lorentz transformations but flips sign under parity).

## 18.2 In $p + q$ dim

Consider  $SO(p, q)$  so  $n = p + q$

$\gamma_{ab}^\mu$

$\text{int}(x)$  = integer part of  $x \in \mathbb{R}$

$a, b \in \{1, 2, \dots, 2^{\text{int}(n/2)}\}$

$\mu = 1, 2, \dots, n$

$n$	1	2	3	4	5	6	7	8	9	10
$\text{int}(n/2)$	0	1	1	2	2	3	3	4	4	5
$d = 2^{\text{int}(n/2)}$	1	2	2	4	4	8	8	16	16	32

Table 18.1:  $\gamma_\mu \in \mathbb{C}^{d \times d}$

$$[\gamma_\mu, \gamma_\nu]_+ = 2g^{\mu\nu} \quad \text{for } \mu, \nu = 0, 1, 2, \dots, n \quad (18.18)$$

$$\Gamma_{\lambda\mu\nu}^{(3)} = \begin{array}{c} \uparrow \\ \mathcal{A}_n \\ \uparrow \uparrow \uparrow \\ \leq -\underline{\gamma} \leq -\underline{\gamma} \leq -\underline{\gamma} \leq - \end{array} = \begin{array}{c} \uparrow \uparrow \uparrow \\ \mathcal{A}_n^{\frac{1}{2}} \\ \uparrow \uparrow \uparrow \\ \leq -\underline{\gamma} \leq -\underline{\gamma} \leq -\underline{\gamma} \leq - \end{array} \quad (18.19)$$

Generalize  $\gamma_5 \rightarrow \Gamma_5$ , from  $SO(1, 3)$  to  $SO(p, q)$ .

$$\Gamma_5 = \frac{1}{\sqrt{n!}} \begin{array}{c} \mathcal{A}_n^{\frac{1}{2}} \\ \hline \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow - \underline{\gamma} \leftarrow - \underline{\gamma} \leftarrow - \underline{\gamma} \leftarrow - \end{array} \quad (18.20)$$

$$= \frac{e^{i\phi}}{n!} \epsilon^{\mu_1 \mu_2 \dots \mu_n} \gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_n} = e^{i\phi} \gamma_1 \gamma_2 \dots \gamma_n \quad (18.21)$$

$\Gamma_5$  properties

- For  $n$  odd,  $\Gamma_5$  is a constant.

The Lorentz group for odd  $n$  has only one irreducible spinor representation. For even  $n$ , it has two inequivalent spinor irreps (positive and negative chirality).

Thus chirality does not exist in odd dimensions because the representation theory of the Lorentz group does not support it.

- $\Gamma_5$  anticommutes with  $\gamma_\mu$  if  $n$  is even. Commutes with  $\gamma_\mu$  if  $n$  is odd.

$$\begin{cases} \gamma_\mu \Gamma_5 = -\Gamma_5 \gamma_\mu & (n \text{ even}) \\ \gamma_\mu \Gamma_5 = \Gamma_5 \gamma_\mu & (n \text{ odd}) \end{cases} \quad (18.22)$$

This follows because  $g_{\mu\nu}$  is diagonal so, by the Clifford algebra definition Eq.(18.18) (anti-cummutator of gammas),

- for  $n$  even, the  $\gamma_\mu$  anticommutes with  $n - 1$  of the  $n$  gamma matrices in  $\Gamma_5$
- for  $n$  odd,  $\Gamma_5$  is a constant

- Square is one

If

$$e^{i\phi} = i^{\frac{n(n-1)}{2}} \sqrt{\prod_{\mu=1}^n g_{\mu\mu}} \quad (18.23)$$

with  $g_{\mu\mu} \in \{1, -1\}$ , then

$$\Gamma_5^2 = 1 \quad (18.24)$$

This follows because

$$\Gamma_5^2 = e^{i2\phi} \gamma_1 \gamma_2 \dots \gamma_n \gamma_1 \gamma_2 \dots \gamma_n \quad (18.25)$$

$$= e^{i2\phi} (-1)^{\frac{n(n-1)}{2}} (\gamma_1)^2 (\gamma_2)^2 \dots (\gamma_n)^2 \quad (18.26)$$

$$= e^{i2\phi} (-1)^{\frac{n(n-1)}{2}} \prod_{\mu=1}^n g_{\mu\mu} \quad (18.27)$$

- If  $n$  is even and

$$P_{\pm} = \frac{1}{2}(1 \pm \Gamma_5), \quad (18.28)$$

then

$$P_+^2 = P_-^2 = 1, \quad P_+ P_- = 0, \quad P_+ + P_- = 1 \quad (18.29)$$

$$\gamma_{\mu} P_+ = P_- \gamma_{\mu} \quad (18.30)$$

$$\gamma_{\mu} = P_+ \gamma_{\mu} P_- + P_- \gamma_{\mu} P_+ \quad (18.31)$$

### 18.3 Weyl and Majorana Spinors

Spinor Type	Condition	# of dofs in 4D	Exists in 4D?
Dirac	none	8	yes
Weyl	$\Gamma_5 \psi = \pm \psi$	4	yes
Majorana	$\psi^c = \psi$	4	yes
Majorana-Weyl	both	0	no

Table 18.2: Different types of spinors, their number of dofs (real degrees of freedom) in 4D and whether they exist in 4D.

- Weyl spinors

$$P_L = \frac{1}{2}(1 - \Gamma_5), \quad P_R = \frac{1}{2}(1 + \Gamma_5) \quad (18.32)$$

A **left-handed Weyl spinor** satisfies

$$\psi_L = P_L \psi, \quad \Gamma_5 \psi_L = -\psi_L \quad (18.33)$$

A **right-handed Weyl spinor** satisfies

$$\psi_R = P_R \psi, \quad \Gamma_5 \psi_R = +\psi_R. \quad (18.34)$$

- Majorana Spinors

The **charge conjugation matrix**  $C$  is defined by

$$C \gamma^{\mu} C^{-1} = -(\gamma^{\mu})^T, \quad (18.35)$$

and exists in any dimension.

Given a Dirac spinor  $\psi$ , its **charge-conjugate** is

$$\psi^c = C\bar{\psi}^T \quad \text{where } \bar{\psi} = \psi^\dagger \gamma^0 \quad (18.36)$$

A **Majorana spinor** is one satisfying

$$\psi = \psi^c \quad (18.37)$$

Whether Majorana spinors exist depends on the spacetime dimension and signature. In 1+3 dimensions, Majorana spinors do exist because one can choose a representation where all gamma matrices are purely real or imaginary so that  $\psi$  can be chosen real.

A Dirac spinor has 4 complex components. A Majorana spinor has 4 real components.

The Weyl spinor and Majorana spinor constraints cannot both be imposed simultaneously in 1+3 dim. But in some dimensions, Majorana–Weyl spinors **do** exist. (see Table 18.3)

$n$	Dirac spinor	Weyl spinor	Majorana spinor	Majorana-Weyl spinor
1	Yes, $\dim_{\mathbb{R}} = 2$	No	No	No
2	Yes, $\dim_{\mathbb{R}} = 4$	Yes, $\dim_{\mathbb{R}} = 2$	Yes, $\dim_{\mathbb{R}} = 2$	Yes, $\dim_{\mathbb{R}} = 1$
3	Yes, $\dim_{\mathbb{R}} = 4$	No	Yes, $\dim_{\mathbb{R}} = 2$	No
4	Yes, $\dim_{\mathbb{R}} = 8$	Yes, $\dim_{\mathbb{R}} = 4$	Yes, $\dim_{\mathbb{R}} = 4$	No
5	Yes, $\dim_{\mathbb{R}} = 8$	No	No	No
6	Yes, $\dim_{\mathbb{R}} = 16$	Yes, $\dim_{\mathbb{R}} = 8$	No	No
7	Yes, $\dim_{\mathbb{R}} = 16$	No	No	No
8	Yes, $\dim_{\mathbb{R}} = 32$	Yes, $\dim_{\mathbb{R}} = 16$	No	No

Table 18.3: Spinor types in Lorentzian signature  $(1, n-1)$  (mostly minus). Note that Majorana-Weyl spinors are possible for  $n = 2, 10$  ( $10 \% 8 = 2$ ). This is why superstring theories are 10 dimensional.

## Chapter 19

# Squashed Entanglement: COMING SOON

## Chapter 20

**Symmetry Breaking: COMING  
SOON**

# Chapter 21

## Symplectic Groups

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Throughout this chapter, we will assume that  $n$  is even. The **symplectic group**  $Sp(n)$  is defined as

$$Sp(n) = \{G \in GL(n, \mathbb{C}) : G^\dagger G = 1, G^T f G = f\} \quad (21.1)$$

where  $f$  is the anti-symmetric matrix

$$f = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \quad (21.2)$$

Note that

$$f^T = -f, \quad f^T f = I_n, \quad f^2 = -I_n \quad (21.3)$$

$f^2$  is a primitive invariant matrix because  $f$  is, so  $f^2$  must be proportional to the identity.

**Claim 51** *If  $G \in Sp(n)$ , then  $\det(G) = 1$ .*

**proof:**

Note that

$$\underbrace{\det(G^T f G)}_{\det^2(G)\det(f)} = \det(f) \quad (21.4)$$

$$\det(f) = \det(I_{n/2})\det(-I_{n/2}) = (-1)^{n/2} \quad (21.5)$$

Hence,

$$\det^2(G) = 1 \implies \det(G) = \pm 1 \quad (21.6)$$

$Sp(n)$  is connected,  $I_n \in Sp(n)$  and  $\det(I_n) = 1$ . Hence  $\det(G) = 1$ .

**QED**



Define the **pseudo metric tensor**  $f_{ab}$  to be a antisymmetric matrix that satisfies:

$$f_{ab} = -f_{ba} = [f]_{ab}, \quad f^{ab} = -f^{ba} = [f]^{ab}, \quad f_a^b = f_b^a = \delta_a^b \quad (21.7)$$

$$f_{ba}x^a = x_b, \quad (f^T)^{cb}x_b = x^c \quad (\text{so } f_{ba}(f^T)^{cb} = \delta_a^c) \quad (21.8)$$

where  $a, b, c \in \{1, 2, \dots, n\}$  and  $x_a$  is any tensor.

$Sp(n)$  leaves invariant the following skew symmetric quadratic form:

$$h(x) = f_{ab}x^ax^b \quad (21.9)$$

where  $a, b \in \{1, \dots, n\}$ . Thus

$$h(Gx) = h(x) \quad (21.10)$$

$$f_{ab}G_{a'}^aG_{b'}^bx^{a'}x^{b'} = f_{a'b'}x^{a'}x^{b'} \implies f_{ab}G_{a'}^aG_{b'}^b = f_{a'b'} \implies G^TfG = f \quad (21.11)$$

In this chapter (and in this book), we will point the arrows in a birdtrack so that the birdtrack is a DAG. Cycles that make the birdtrack not acyclic will have a segment in red. Without that red segment, the birdtrack becomes acyclic. The reason we follow this arrow convention is that it promotes acyclic birdtracks which are more akin to bnets. We will eschew undirected birdtracks for the same reason: bnets are directed.

Let

$$f_a^b = \delta_a^b, \quad \leftarrow f \leftarrow = \leftarrow \quad (21.12)$$

$$f_b^a = \delta_b^a, \quad \rightarrow f \rightarrow = \rightarrow \quad (21.13)$$

$$f_{ac}(f^T)^{cb} = \delta_a^b, \quad \leftarrow \underline{f} \longrightarrow \overline{f}^T \leftarrow = \leftarrow \quad (21.14)$$

Note that we used

$$\underline{f} = [f_{ab}], \quad \overline{f} = [f^{ab}], \quad f^T = -f \quad (21.15)$$

We could write Eq.(21.14) without the overline and underline on  $f$ . Those f-decorations are redundant as omitting them would not introduce any ambiguity. However, we will use them because they make spotting errors in the arrow directions easier.

The generators of symplectic groups will be represented by:

$$(T_i)_a{}^b = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} T_i \text{---} \end{array} \quad (21.16)$$

We will also use

$$(T_i)_a{}^b = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \bar{f}^T T_i \underline{f} \text{---} \end{array} \quad (T_i)_{ab} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} T_i \underline{f} \text{---} \end{array} \quad (T_i)^{ab} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \bar{f}^T T_i \text{---} \end{array} \quad (21.17)$$

For  $G \in Sp(n)$ ,  $f^T G^T f G = 1$  with  $G = e^{iT_i \epsilon_i}$  where  $\epsilon_i \in \mathbb{R}$ . Hence, the generators  $T_i$  must satisfy

$$\underbrace{f^T T_i^T f}_{(f^T T_i f)^T} = -T_i \implies T_i^T = f T_i f \implies (T_i f)^T = T_i f \quad (21.18)$$

$$\boxed{(T_i f)_{ab} = (T_i f)_{ba}} \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} T_i \underline{f} \text{---} \end{array} = \begin{array}{c} a \text{---} T_i \underline{f} \text{---} b \\ \text{---} \end{array} \quad (21.19)$$

$f_a^b = \delta_a^b$  is obviously an invariant matrix.  $f_{ab}$  must be invariant too, so

$$\boxed{\underbrace{(T_i)_a{}^c f_{cb} + (T_i)_b{}^c f_{ac}}_{(T_i f)_{ab} = (T_i f)_{ba}} = 0} \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} T_i \text{---} \end{array} + \begin{array}{c} a \text{---} \underline{f} \text{---} T_i \text{---} b \\ \text{---} \end{array} = 0 \quad (21.20)$$

Hence, the invariance condition Eq.(21.20) reduces to to the statement that  $(T_i f)_{ab}$  is symmetric.

The anti-symmetrizer  $\mathcal{A}_2$  is an invariant tensor (see Section 22.3). Other projectors of the  $V \otimes V$  are not invariant tensors. Therefore, we must have

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} T_i \underline{f} \text{---} \end{array} \sim \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \bar{f}^T T_i \text{---} \end{array} = \begin{array}{c} \text{---} \mathcal{A}_2 \text{---} \\ \text{---} \parallel \text{---} \end{array} \quad (21.21)$$

For  $SO(n)$  and  $O(n)$ , the dimension  $N$  of the adjoint rep (= number of generators) is

$$N = \frac{n(n-1)}{2} = \text{[diagram: a red curved arrow above a wavy line]} \quad (21.22)$$

If you take an  $n \times n$  matrix and remove its diagonal, this  $N$  is the number of entries in the upper (or lower) triangular sector of the matrix. Recall that for  $U(n)$ ,  $N = n^2$ , and for  $SU(n)$ ,  $N = n^2 - 1$ . So for  $U(n)$  (or  $SU(n)$ ), there is a generator for each entry (or each entry minus one) of an  $n \times n$  matrix.

**Claim 52**

$$\Gamma_{fun} \delta_a^b = \sum_i (T_i T_i)_a^b = \frac{n-1}{2} \delta_a^b \quad (21.23)$$

$$\sum_i \text{[diagram: } a \leftarrow T_i \text{ (wavy line with } i \text{ above it)} \rightarrow T_i \leftarrow b \text{]} = \left(\frac{n-1}{2}\right) a \leftarrow \bullet \rightarrow b$$

**proof:**

$$(T_i T_i)_a^b = \text{[diagram: } a \leftarrow T_i \text{ (wavy line with } i \text{ above it)} \rightarrow \bar{f}^T T_i \leftarrow b \text{]} \quad (21.24)$$

$$= \text{[diagram: } a \leftarrow T_i \text{ (wavy line with } i \text{ above it)} \rightarrow \bar{f}^T T_i \leftarrow b \text{]} \quad (21.25)$$

$$= \frac{1}{2} \left[ \text{[diagram: } a \leftarrow \bullet \rightarrow b \text{]} - \text{[diagram: } a \leftarrow \bullet \rightarrow b \text{]} \right] \quad (21.26)$$

$$= \left(\frac{n-1}{2}\right) a \leftarrow \bullet \rightarrow b \quad (21.27)$$

**QED**

## 21.1 $V_{def} \otimes V_{def}$ Decomposition

Define

$$M_{ab}^{cd} = \text{[diagram: } a \leftarrow f \text{ (green arrow)} \rightarrow \bar{f}^T \leftarrow c \text{ (green arrow)} \text{]} \quad (21.28)$$

Note that  $M$  is antisymmetric:

$$\mathcal{A}_2 M = \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} \begin{array}{c} \text{---} f \text{---} \\ \text{---} \overline{f}^T \text{---} \end{array} \quad (21.29)$$

$$= \frac{1}{2} \left[ \begin{array}{c} \leftarrow \leftarrow \\ \text{---} f \text{---} \\ \leftarrow \leftarrow \end{array} \begin{array}{c} \text{---} \overline{f}^T \text{---} \\ \text{---} \end{array} - \begin{array}{c} \leftarrow \leftarrow \\ \text{---} f \text{---} \\ \leftarrow \leftarrow \end{array} \begin{array}{c} \text{---} \overline{f}^T \text{---} \\ \text{---} \end{array} \right] \quad (21.30)$$

$$= M \quad (21.31)$$

Since  $M$  is ant-symmetric, only the anti-symmetric space decomposes.

Note also that

$$M^2 = nM \quad (21.32)$$

Hence,  $(M - n)M = 0$  so  $M$  has two eigenvalues  $\lambda = 0, n$ .

Next we will use the following equation from Chapter 16<sup>1</sup> to obtain a projection (PO) operator for each eigenvalue

$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (21.33)$$

Below, we use the following traces to evaluate the traces of our projection operators

$$\text{tr}(M) = \begin{array}{c} a \text{---} \text{---} d \\ \text{---} f \text{---} \text{---} \overline{f}^T \text{---} \\ b \text{---} \text{---} c \end{array} = \text{tr}(f \overline{f}^T) = n \quad (21.34)$$

$$\text{tr}(\mathcal{A}_2) = \frac{1}{2} \left[ \begin{array}{c} \leftarrow \leftarrow \\ \text{---} \text{---} \\ \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \leftarrow \\ \text{---} \text{---} \\ \leftarrow \leftarrow \end{array} \right] \quad (21.35)$$

$$= \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n - 1) \quad (21.36)$$

$$\text{tr}(\mathcal{S}_2) = \frac{1}{2} \left[ \begin{array}{c} \leftarrow \leftarrow \\ \text{---} \text{---} \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \leftarrow \\ \text{---} \text{---} \\ \leftarrow \leftarrow \end{array} \right] \quad (21.37)$$

$$= \frac{1}{2}(n^2 + n) = \frac{1}{2}n(n + 1) \quad (21.38)$$

---

<sup>1</sup>Note that this equation projects to zero all eigenvalues except one.



Note that

$$P_{TA}P_S = (\mathcal{A}_2 - P_S)P_S = P_S - P_S = 0 \quad (21.48)$$

Hence

$$P_{TA}^2 = (\mathcal{A}_2 - P_S)^2 = P_{TA} \quad (21.49)$$

$P_{SYM}$  is the only of the 3 POs ( $P_{SYM}$ ,  $P_S$ ,  $P_{TA}$ ) that is an invariant tensor so

$$T_i \underline{f} \sim \bar{f}^T T_i = \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} \quad (21.50)$$

**Claim 53** *The Clebsch-Gordan series for  $V \otimes V$  (i.e., decomposition of  $V \otimes V$ ) is*

$$\begin{aligned} \overbrace{V \otimes V}^{\mathcal{V}} &= P_S \mathcal{V} \oplus P_{SYM} \mathcal{V} \oplus P_{TA} \mathcal{V} \\ \square \otimes \square &= \bullet \oplus \square \oplus \begin{array}{c} \square \\ \square \end{array} \\ n^2 &= 1 + \frac{1}{2}n(n+1) + \frac{1}{2}(n+1)(n-2) \end{aligned} \quad (21.51)$$

*The projection operator tree is*

$$\begin{array}{c} \begin{array}{c} \leftarrow P_{ANTI} \\ \leftarrow P_S \\ \leftarrow P_{TA} \\ \leftarrow P_{SYM} \end{array} \end{array}$$

**proof:**  
**QED**

# Chapter 22

## Symmetrization and Antisymmetrization

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

As preparation for this chapter, read Sec.A.9.

### 22.1 Symmetrizer

The set of permutations of 2 elements can be represented by the following  $2! = 2$  birdtracks.

$$\mathbb{1}_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} = \begin{array}{c} a_1 \xleftarrow{\text{green}} \bullet \leftarrow b_1 \\ a_2 \leftarrow \bullet \leftarrow b_2 \end{array} \quad (22.1)$$

$$(\sigma_{(1,2)})_{a_1, a_2}^{b_2, b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} = \begin{array}{c} a_1 \xleftarrow{\text{green}} \bullet \leftarrow b_1 \\ \updownarrow \\ a_2 \leftarrow \bullet \leftarrow b_2 \end{array} \quad (22.2)$$

The vertical double headed arrow is called a **swap**. It moves an upstairs particle downstairs and a downstairs particle upstairs.

The set of values that  $a_i$  and  $b_i$  can assume can be anything, as long as, for some set  $V$ ,  $val(\underline{a}_i) = val(\underline{b}_i) = V$  for all  $i$  and  $|V| = n$ .

The set of permutations of 3 elements can be represented by the following  $3! = 6$  birdtracks:

$$\begin{array}{c} a_1 \xleftarrow{\text{green}} \bullet \leftarrow b_1 \\ \mathbb{1} = \begin{array}{c} a_2 \leftarrow \bullet \leftarrow b_2 \\ a_3 \leftarrow \bullet \leftarrow b_3 \end{array} \end{array} \quad (22.3)$$

$$\sigma_{(1,2)} = \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \leftarrow \bullet \\ \leftarrow \end{array} \quad \sigma_{(2,3)} = \begin{array}{c} \leftarrow \\ \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad \sigma_{(1,3)} = \begin{array}{c} \leftarrow \bullet \\ \leftarrow \updownarrow \leftarrow \\ \leftarrow \bullet \end{array} \quad (22.4)$$

$$\sigma_{(1,2,3)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \updownarrow \leftarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \end{array} \quad (22.5)$$

$$\sigma_{(1,3,2)} = \begin{array}{c} \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} = \begin{array}{c} \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \updownarrow \leftarrow \\ \leftarrow \bullet \leftarrow \end{array} \quad (22.6)$$

Note that

$$(c, a) = (b, c)(b, a)(b, c) \quad \begin{array}{c} \leftarrow a \\ \updownarrow \\ \leftarrow b \\ \downarrow \\ \leftarrow c \end{array} = \begin{array}{c} \leftarrow a \\ \leftarrow \updownarrow \leftarrow b \\ \leftarrow \updownarrow \leftarrow c \end{array} \quad (22.7)$$

The  $p$ -element symmetrizer  $\mathcal{S}_p$  is defined as the birdtrack

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p!} \left\{ \begin{array}{cc} \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \\ \vdots & \vdots \\ \leftarrow & \leftarrow \end{array} + \begin{array}{cc} \leftarrow \bullet \leftarrow & \\ \leftarrow \bullet \leftarrow & \\ \leftarrow & \\ \vdots & \\ \leftarrow & \end{array} + \dots \right\} \quad (22.8)$$

Note that  $\mathcal{S}_p$  satisfies the following identities:

$$\mathcal{S}_p^2 = \mathcal{S}_p \quad \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} \quad (22.9)$$



$$\begin{array}{c} \mathcal{S}_p \mathcal{S}_{[1,q]} = \mathcal{S}_p \end{array}
\begin{array}{ccc}
\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} &
\begin{array}{c} \leftarrow \mathcal{S}_{[1,q]} \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} &
= \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array}
\end{array} \quad (22.10)$$

$$\begin{array}{c} \mathcal{S}_p \sigma_{(1,2)} = \mathcal{S}_p \end{array}
\begin{array}{ccc}
\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} &
\begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} &
= \begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array}
\end{array} \quad (22.11)$$

**Claim 54**

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} = \frac{1}{p} \left( \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \leftarrow \end{array} + (p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{S}_{p-1} \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \leftarrow \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \leftarrow \parallel \leftarrow \end{array} \right) \quad (22.12)$$

**proof:**

We only prove it for  $p = 3$ .

$$3! \begin{array}{c} \leftarrow \mathcal{S}_3 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} = \left( \begin{array}{c} \leftarrow \leftarrow + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \end{array} + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow \end{array} \\ + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \end{array} \right) \quad (22.13)$$

$$2! \begin{array}{c} \leftarrow \\ \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \end{array} = \left( \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} + \begin{array}{c} \leftarrow \\ \bullet \\ \updownarrow \\ \bullet \\ \leftarrow \end{array} \right) \quad (22.14)$$

$$3! \begin{array}{c} \leftarrow \mathcal{S}_3 \leftarrow \\ \parallel \\ \leftarrow \\ \parallel \\ \leftarrow \end{array} - 2! \begin{array}{c} \leftarrow \\ \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \end{array} = \left( \begin{array}{c} \begin{array}{c} \bullet \\ \updownarrow \\ \bullet \end{array} \leftarrow \begin{array}{c} \leftarrow \\ \bullet \\ \updownarrow \\ \bullet \\ \leftarrow \end{array} \\ \leftarrow \begin{array}{c} \bullet \\ \updownarrow \\ \bullet \end{array} \leftarrow \begin{array}{c} \leftarrow \\ \bullet \\ \updownarrow \\ \bullet \\ \leftarrow \end{array} \\ \leftarrow \begin{array}{c} \bullet \\ \updownarrow \\ \bullet \end{array} \leftarrow \begin{array}{c} \leftarrow \\ \bullet \\ \updownarrow \\ \bullet \\ \leftarrow \end{array} \\ + \\ \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \bullet \end{array} \leftarrow \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \bullet \end{array} \\ + \\ \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \bullet \end{array} \leftarrow \begin{array}{c} \leftarrow \bullet \\ \updownarrow \\ \bullet \end{array} \end{array} \right) \quad (22.15)$$

$$= 2!2! \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \bullet \\ \parallel \\ \leftarrow \mathcal{S}_2 \leftarrow \bullet \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \end{array} \quad (22.16)$$

**QED**

Tracing over the identity of Claim 54, we get

$$\begin{array}{c} \leftarrow \mathcal{S}_p \leftarrow \\ \parallel \\ \leftarrow \\ \parallel \\ \leftarrow \\ \vdots \\ \parallel \\ \leftarrow \end{array} = \frac{1}{p} \left( \begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \\ \parallel \\ \leftarrow \\ \vdots \\ \parallel \\ \leftarrow \end{array} + (p-1) \begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \bullet \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \\ \parallel \\ \leftarrow \\ \vdots \\ \parallel \\ \leftarrow \end{array} \right) \quad (22.17)$$

$$= \frac{n+p-1}{p} \left( \begin{array}{c} \leftarrow \mathcal{S}_{p-1} \leftarrow \\ \parallel \\ \leftarrow \\ \parallel \\ \leftarrow \\ \vdots \\ \parallel \\ \leftarrow \end{array} \right) \quad (22.18)$$

Hence

$$\text{tr}_{\mathcal{A}_1} \mathcal{S}_p = \frac{n+p-1}{p} \mathcal{S}_{p-1} \quad (22.19)$$

$$\text{tr}_{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k} \mathcal{S}_p = \frac{(n+p-1)(n+p-2) \dots (n+p-k)}{p(p-1) \dots (p-k+1)} \mathcal{S}_{p-k} \quad (22.20)$$

$$d_{\mathcal{S}_p} = \text{tr}_{\underline{a}^p} \mathcal{S}_p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \quad (22.21)$$

For  $p = 2$ ,

$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \quad (22.22)$$

## 22.2 Antisymmetrizer

The  $p$ -element antisymmetrizer  $\mathcal{A}_p$  is defined as the birdtrack

$$\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} = \frac{1}{p!} \left\{ \begin{array}{cc} \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \\ \vdots & \vdots \\ \leftarrow & \leftarrow \end{array} - \begin{array}{cc} \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \\ \vdots & \vdots \\ \leftarrow & \leftarrow \end{array} + \dots \right\} \quad (22.23)$$

Note that

$$\mathcal{A}_p = 0 \text{ if } n < p \quad (22.24)$$

because when  $n < p$ , there must be two lines with the same value emerging from  $\mathcal{A}_p x$ , so  $-\mathcal{A}_p x = \mathcal{A}_p x = 0$ . For example, for  $n = 2$  and  $p = 3$

$$\begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \mathcal{A}_3 = \parallel \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \end{array} = \frac{1}{6} \left( \begin{array}{ccc} \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \end{array} & + & \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \rightleftharpoons \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} & + & \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \rightleftharpoons \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \\ - & \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \rightleftharpoons \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} & - & \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \rightleftharpoons \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} & - & \begin{array}{c} |a\rangle \quad |a\rangle \quad |b\rangle \\ \downarrow \quad \downarrow \quad \downarrow \\ \bullet \leftarrow \bullet \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \end{array} \right) \quad (22.25)$$

$$\mathcal{A}_3|a, a, b\rangle = \frac{1}{6} \begin{pmatrix} |a, a, b\rangle + |a, b, a\rangle + |b, a, a\rangle \\ -|a, b, a\rangle - |a, a, b\rangle - |b, a, a\rangle \end{pmatrix} \quad (22.26)$$

$$= 0 \quad (22.27)$$

Note that  $\mathcal{A}_p$  satisfies the following identities:

$$\mathcal{A}_p^2 = \mathcal{A}_p \quad (22.28)$$

$$\mathcal{A}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p \quad (22.29)$$

$$\mathcal{A}_p \sigma_{(1,2)} = -\mathcal{A}_p \quad (22.30)$$

$$\mathcal{S}_p \mathcal{A}_q = \mathcal{A}_p \mathcal{S}_q = 0 \quad (22.31)$$

$$\mathcal{S}_p \mathcal{A}_{[1,q]} = \mathcal{A}_p \mathcal{S}_{[1,q]} = 0$$

$$\begin{array}{ccc} \leftarrow \mathcal{S}_p \leftarrow & \leftarrow \mathcal{A}_{[1,q]} \leftarrow & \leftarrow \mathcal{A}_p \leftarrow \quad \leftarrow \mathcal{S}_{[1,q]} \leftarrow \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \quad \leftarrow \parallel \leftarrow \\ \leftarrow \leftarrow & \leftarrow \leftarrow & \leftarrow \leftarrow \quad \leftarrow \leftarrow \\ \vdots & \vdots & \vdots \\ \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow & \leftarrow \parallel \leftarrow \quad \leftarrow \parallel \leftarrow \end{array} = 0 \quad (22.32)$$

**Claim 55**

$$\leftarrow \mathcal{A}_p \leftarrow = \frac{1}{p} \left( \begin{array}{ccc} \leftarrow \mathcal{A}_{p-1} \leftarrow & & \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow & & \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \leftarrow & & \leftarrow \parallel \leftarrow \quad \leftarrow \parallel \leftarrow \\ \vdots & & \vdots \\ \leftarrow \parallel \leftarrow & & \leftarrow \parallel \leftarrow \end{array} + (-1)(p-1) \begin{array}{ccc} \leftarrow \mathcal{A}_{p-1} \leftarrow & & \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow & & \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow & & \leftarrow \parallel \leftarrow \\ \vdots & & \vdots \\ \leftarrow \parallel \leftarrow & & \leftarrow \parallel \leftarrow \end{array} \right) \quad (22.33)$$

**proof:**

We only prove it for  $p = 3$ .

$$3! \leftarrow \mathcal{A}_3 \leftarrow = \left( \begin{array}{ccc} \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \\ \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \\ - \leftarrow \bullet \leftarrow & - \leftarrow \bullet \leftarrow & - \leftarrow \bullet \leftarrow \\ \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \end{array} \right) \quad (22.34)$$

$$2! \leftarrow \mathcal{A}_2 \leftarrow = \left( \begin{array}{ccc} \leftarrow & & \leftarrow \\ \leftarrow & - \leftarrow \bullet \leftarrow & \\ \leftarrow \parallel \leftarrow & \leftarrow \bullet \leftarrow & \leftarrow \bullet \leftarrow \end{array} \right) \quad (22.35)$$

$$\begin{aligned}
3! \begin{array}{c} \leftarrow \mathcal{A}_3 \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} - 2! \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \end{array} &= \left( \begin{array}{cc} \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} & + \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \\ - \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} & - \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \bullet \leftarrow \end{array} \end{array} \right) \\
& \quad (22.36)
\end{aligned}$$

$$\begin{aligned}
&= (-1)2!2! \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{A}_2 \leftarrow \bullet \leftarrow \mathcal{A}_2 \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} \\
& \quad (22.37)
\end{aligned}$$

**QED**

Tracing over the identity of Claim 55, we get

$$\begin{aligned}
\begin{array}{c} \leftarrow \mathcal{A}_p \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} &= \frac{1}{p} \left( \begin{array}{cc} \begin{array}{c} \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} & + (-1)(p-1) \begin{array}{c} \leftarrow \bullet \leftarrow \\ \updownarrow \\ \leftarrow \mathcal{A}_{p-1} \leftarrow \bullet \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \parallel \leftarrow \end{array} \end{array} \right) \\
& \quad (22.38)
\end{aligned}$$

$$\begin{aligned}
&= \frac{n + (-1)(p-1)}{p} \left( \begin{array}{c} \leftarrow \mathcal{A}_{p-1} \leftarrow \\ \leftarrow \parallel \leftarrow \\ \vdots \\ \leftarrow \parallel \leftarrow \end{array} \right) \\
& \quad (22.39)
\end{aligned}$$

Hence,

$$\text{tr}_{\mathcal{A}_1} \mathcal{A}_p = \frac{n-p+1}{p} \mathcal{A}_{p-1} \quad (22.40)$$

$$\text{tr}_{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k} \mathcal{A}_p = \frac{(n-p+1)(n-p+2) \dots (n-p+k)}{p(p-1) \dots (p-k+1)} \mathcal{A}_{p-k} \quad (22.41)$$

$$d_{\mathcal{A}_p} = \text{tr}_{\underline{a}^p} \mathcal{A}_p = \frac{\prod_{i=n}^{n-p+1} i}{p!} \quad (22.42)$$

$$= \begin{cases} \frac{n!}{p!(n-p)!} = \binom{n}{p} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} \quad (22.43)$$

For  $p = 2 \leq n$ ,

$$d_{\mathcal{A}_2} = \binom{n}{2} \quad (22.44)$$

### 22.3 Invariance of $\mathcal{S}_p$ and $\mathcal{A}_p$

The Kronecker delta is obviously always an invariant matrix because

$$\begin{array}{c} \} \\ \leftarrow T_i \leftarrow \delta \leftarrow \end{array} - \begin{array}{c} \} \\ \leftarrow \delta \leftarrow T_i \leftarrow \end{array} = 0 \quad (22.45)$$

An immediate consequence of the the invariance of the Kronecker delta is that the symmetrizer  $\mathcal{S}_p$  and anti-symmetrizer  $\mathcal{A}_p$  are tensor invariants too. Indeed,

$$\begin{array}{c} \} \\ \leftarrow T_i \leftarrow \mathcal{S}_p \\ \leftarrow \parallel \\ \leftarrow \end{array} - \begin{array}{c} \} \\ \leftarrow \mathcal{S}_p \leftarrow T_i \leftarrow \\ \leftarrow \parallel \\ \leftarrow \end{array} = 0 \quad (22.46)$$

$$\begin{array}{c} \} \\ \leftarrow T_i \leftarrow \mathcal{A}_p \\ \leftarrow \parallel \\ \leftarrow \end{array} - \begin{array}{c} \} \\ \leftarrow \mathcal{A}_p \leftarrow T_i \leftarrow \\ \leftarrow \parallel \\ \leftarrow \end{array} = 0 \quad (22.47)$$

### 22.4 Levi-Civita Tensor

The **Levi-Civita tensor**  $\epsilon_{a:p}$  where  $a_i \in \{1, 2, \dots, p\}$  equals  $+1$  (resp.,  $-1$ ) if  $a^:p$  is an even (resp., odd ) permutation of  $(1, 2, \dots, p)$ . Thus

$$\epsilon_{123} = \epsilon^{123} = 1 \quad (22.48)$$

$$\epsilon_{213} = \epsilon^{213} = -1 \quad (22.49)$$

More generally,

$$\epsilon^{123\dots p} = \epsilon_{123\dots p} = 1 \quad (22.50)$$

and

$$\epsilon_{rev(a:p)} = (-1)^{\binom{p}{2}} \epsilon_{a:p} \quad (22.51)$$

Define

$$(C_{\mathcal{A}_p})^1_{a:p} = \frac{\epsilon_{a:p}}{\sqrt{p!}} = \begin{array}{c} a_1 \xleftarrow{\text{green}} \mathcal{A}_p^{\frac{1}{2}} \\ \parallel \\ a_2 \xleftarrow{\quad} \\ \vdots \\ a_p \xleftarrow{\quad} \end{array} \quad (22.52)$$

and

$$(C_{\mathcal{A}_p}^\dagger)_1^{rev(a:p)} = \frac{\epsilon^{rev(a:p)}}{\sqrt{p!}} = \begin{array}{c} \mathcal{A}_p^{\frac{1}{2}} \xleftarrow{\quad} a_1 \\ \parallel \\ \xleftarrow{\quad} a_2 \\ \vdots \\ \xleftarrow{\text{green}} a_p \end{array} \quad (22.53)$$

Then

$$\frac{1}{p!} \epsilon_{a:p} \epsilon^{rev(b:p)} = (\mathcal{A}_p)_{a:p}^{rev(b:p)} = \begin{array}{c} a_1 \xleftarrow{\quad} \mathcal{A}_p^{\frac{1}{2}} \\ \parallel \\ a_2 \xleftarrow{\quad} \\ \vdots \\ a_p \xleftarrow{\quad} \end{array} \mathcal{A}_p^{\frac{1}{2}} \xleftarrow{\quad} b_1 \begin{array}{c} \xleftarrow{\quad} b_2 \\ \vdots \\ \xleftarrow{\quad} b_p \end{array} = \begin{array}{c} a_1 \xleftarrow{\quad} \mathcal{A}_p \xleftarrow{\quad} b_1 \\ \parallel \\ a_2 \xleftarrow{\quad} \xleftarrow{\quad} b_2 \\ \vdots \\ a_p \xleftarrow{\quad} \xleftarrow{\quad} b_p \end{array} \quad (22.54)$$

and



$$e^{i2\phi} \frac{1}{p!} \epsilon^{rev(a:n)} \epsilon_{a:n} = 1 \quad e^{i2\phi} \begin{array}{c} \mathcal{A}_p^{\frac{1}{2}} \longleftarrow \mathcal{A}_p^{\frac{1}{2}} \\ \parallel \longleftarrow \parallel \\ \vdots \\ \parallel \longleftarrow \parallel \end{array} = 1 \quad (22.55)$$

For the FL Convention, we will use  $\phi = 0$ .

For the CC Convention, we must choose

$$e^{i2\phi} = (-1)^{\binom{p}{2}} = e^{i\pi \frac{p(p-1)}{2}} \quad (22.56)$$

so

$$\phi = \frac{\pi}{4} p(p-1) \quad (22.57)$$

## 22.5 Fully-symmetric and Fully-antisymmetric Tensors

A fully symmetric (FS) tensor  $d$

$$d_{a_1 a_2 \dots a_p} = \begin{array}{c} d \\ | \\ a_1 \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} a_2 \\ \dots \\ a_p \end{array} \quad (22.58)$$

is a tensor that satisfies

$$\mathcal{S}_p d = d \quad \begin{array}{c} d \\ | \\ \mathcal{S}_p \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \vdots \\ | \end{array} = \begin{array}{c} d \\ | \\ \vdots \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \vdots \\ | \end{array} \quad (22.59)$$

If  $d$  is a tensor invariant (see Chapter 7), it must satisfy

$$0 = \begin{array}{c} d \\ | \\ T_i \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \vdots \\ | \end{array} + \begin{array}{c} d \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} T_i \\ | \end{array} + \begin{array}{c} d \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \vdots \\ | \end{array} \quad (22.60)$$

$$(22.61)$$

A fully antisymmetric(FA) tensor  $f$

$$(22.62)$$

is a tensor that satisfies

$$(22.63)$$

If  $f$  is a tensor invariant (see Chapter 7), it must satisfy

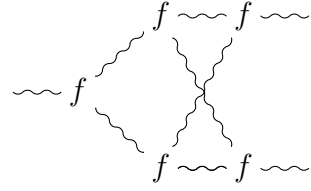
$$(22.64)$$

$$(22.65)$$

## 22.6 Identically Vanishing Birdtracks

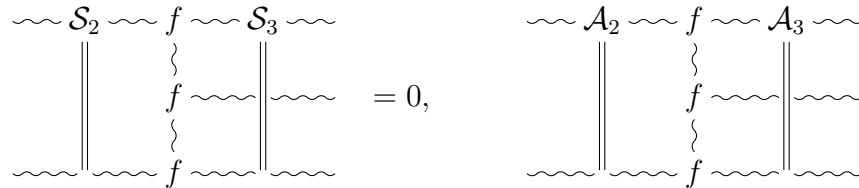
**Identically vanishing (IV) birdtracks** are birdtracks that vanish by virtue of their symmetrized or antisymmetrized components.

- Example of birdtrack that vanishes for any FA tensor  $f$



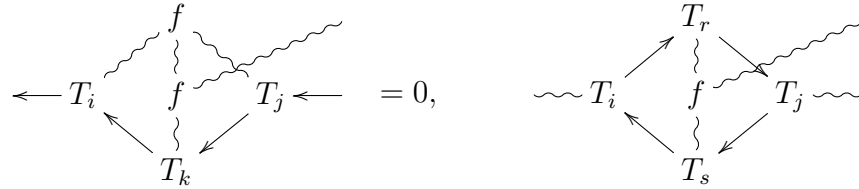
$$= 0 \quad (\text{Kuratowski graph}) \quad (22.66)$$

- Example of birdtrack that vanishes for any  $f$  that is a structure constant of a Lie algebra



$$= 0, \quad = 0 \quad (22.67)$$

- Birdtrack that is zero for an irrep



$$= 0, \quad = 0 \quad (22.68)$$

# Chapter 23

## Unitary Groups

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

Let

**n-dim General Linear group**  $GL(n; \mathbb{C}) = \{G \in \mathbb{C}^{n \times n} : \det(G) \neq 0\}$

**n-dim Special Linear group**  $SL(n; \mathbb{C}) = \{G \in GL(n; \mathbb{C}) : \det(G) = 1\}$

**n-dim Unitary group**,  $U(n) = \{G \in GL(n; \mathbb{C}) : GG^\dagger = G^\dagger G = 1\}$

**n-dim Special Unitary group**  $SU(n) = \{G \in U(n) : \det(G) = 1\}$

Chapter 26 on Young Tableaux is closely connected to this chapter.

### 23.1 $SU(n)$

In  $SU(n) \subset \mathbb{C}^{n \times n}$  in the defining rep, we have the quadratic form

$$m(p, q) = (p_b)^* \delta_b^a q_a \quad (23.1)$$

Let

$$\mathbb{1}_d^a \mathbb{1}_b^c = \delta_b^a \delta_d^c = \begin{array}{c} d \leftarrow \bullet - c \\ a \longrightarrow \bullet b \end{array} \quad (23.2)$$

and

$$M_{a \ c}^d \mathbb{1}_b^c = \delta_d^a \delta_b^c = \begin{array}{c} d \quad c \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \searrow \quad \swarrow \\ a \quad b \end{array} \quad (23.3)$$

Note that

$$M^2 = nM \quad \begin{array}{c} d \quad c \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \searrow \quad \swarrow \\ a \quad b \end{array} \begin{array}{c} d \quad c \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \searrow \quad \swarrow \\ a \quad b \end{array} = n \begin{array}{c} d \quad c \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \searrow \quad \swarrow \\ a \quad b \end{array} \quad (23.4)$$

Hence,  $(M - n)M = 0$  so  $M$  has two eigenvalues  $\lambda = 0, n$ .

Next we will use the following equation from Chapter 16<sup>1</sup> to obtain a projection operator (PO) for each eigenvalue

$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \quad (23.5)$$

1. Singlet PO ( $\lambda_S = 0$ )

$$P_S = \frac{M - 0}{n - 0} = \frac{1}{n}M \quad \begin{array}{c} a \quad b \\ \swarrow \quad \searrow \\ P_S \\ \nearrow \quad \nwarrow \\ c \quad d \end{array} = \frac{1}{n} \begin{array}{c} a \quad b \\ \curvearrowright \quad \curvearrowleft \\ c \quad d \end{array} \quad (23.6)$$

The singlet projection operator  $P_S$  projects the singlet part of a tensor  $x$ :

$$P_S x = \frac{1}{n} x^b_b \delta^c_a \quad (23.7)$$

$P_S$  has dimension 1:

$$\dim(P_S) = \text{tr} P_S = \frac{1}{n} \begin{array}{c} \curvearrowright \quad \curvearrowleft \\ \bullet \quad \bullet \\ \curvearrowleft \quad \curvearrowright \end{array} \quad (23.8)$$

$$= 1 \quad (23.9)$$

2. Adjoint PO

$$P_{adj} = \frac{M - n}{0 - n} = 1 - \frac{1}{n}M \quad \begin{array}{c} a \quad b \\ \swarrow \quad \searrow \\ P_{adj} \\ \nearrow \quad \nwarrow \\ c \quad d \end{array} = \begin{array}{c} a \leftarrow \bullet \rightarrow b \\ c \rightarrow \bullet \leftarrow d \end{array} - \frac{1}{n} \begin{array}{c} a \quad b \\ \curvearrowright \quad \curvearrowleft \\ c \quad d \end{array} = \begin{array}{c} \curvearrowleft \quad \curvearrowright \\ T_i \sim T_i \\ \curvearrowright \quad \curvearrowleft \end{array} \quad (23.10)$$

The adjoint projection operator  $P_{adj}$  projects the traceless part of a tensor  $x$

$$P_{adj} x = x^a_c - \left( \frac{1}{n} x^b_b \delta^c_a \right) \quad (23.11)$$

---

<sup>1</sup>Note that this equation projects to zero all eigenvalues except one.

The  $P_{adj}$  has dimension  $n^2 - 1$

$$\dim(P_{adj}) = \text{tr} P_{adj} = \begin{array}{c} \overleftarrow{\bullet} \overrightarrow{\bullet} \\ \overleftarrow{\bullet} \overrightarrow{\bullet} \end{array} - \frac{1}{n} \begin{array}{c} \overleftarrow{\bullet} \overrightarrow{\bullet} \\ \overleftarrow{\bullet} \overrightarrow{\bullet} \end{array} \quad (23.12)$$

$$= n^2 - 1 \quad (23.13)$$

We will denote the generators  $T_i$  of  $SU(n)$  by

$$(T_i)_a^b = \begin{array}{c} i \\ \vdots \\ a \longleftarrow T^i \longrightarrow b \end{array} \quad (23.14)$$

For  $G \in U(n)$ ,  $G^\dagger G = 1$  with  $G = e^{iT_i \epsilon_i}$  where  $\epsilon_i \in \mathbb{R}$ . Hence, the generators  $T_i$  must be Hermitian

$$T_i^\dagger = T_i \quad (23.15)$$

We will assume that they also satisfy

$$\text{tr}(T_i T_j) = \kappa \delta_i^j \quad (23.16)$$

$$i \rightsquigarrow T_i \overleftarrow{\hspace{0.5cm}} T_j \rightsquigarrow j = \kappa i \rightsquigarrow \bullet \rightsquigarrow j$$

Usually, we set  $\kappa = 1$  and, if necessary, restore the  $\kappa$ 's at the end by dimensional analysis. (Replace each  $T_i$  in a  $\kappa$ -less equation by  $T_i/\sqrt{\kappa}$ .)

The adjoint projection operator for  $SU(n)$  is

$$\begin{array}{c} \overleftarrow{\hspace{0.5cm}} T_i \rightsquigarrow T_i \overrightarrow{\hspace{0.5cm}} \\ \overleftarrow{\hspace{0.5cm}} T_i \rightsquigarrow T_i \overrightarrow{\hspace{0.5cm}} \end{array} \stackrel{\text{def}}{=} P_{adj} = \begin{array}{c} \overleftarrow{\bullet} \overrightarrow{\bullet} \\ \overleftarrow{\bullet} \overrightarrow{\bullet} \end{array} - \frac{1}{n} \begin{array}{c} \overleftarrow{\bullet} \overrightarrow{\bullet} \\ \overleftarrow{\bullet} \overrightarrow{\bullet} \end{array} \quad (23.17)$$

The Lie Algebra commutators for  $SU(n)$  are

$$T_i T_j - T_j T_i = i f_{ijk} T_k$$

$$\begin{array}{c} \overleftarrow{\hspace{0.5cm}} T_i \overleftarrow{\hspace{0.5cm}} T_j \overleftarrow{\hspace{0.5cm}} \\ \vdots \\ i \end{array} - \begin{array}{c} \overleftarrow{\hspace{0.5cm}} T_j \overleftarrow{\hspace{0.5cm}} T_i \overleftarrow{\hspace{0.5cm}} \\ \vdots \\ i \end{array} = \begin{array}{c} \overleftarrow{\hspace{0.5cm}} T_k \overleftarrow{\hspace{0.5cm}} \\ \vdots \\ i f_{ijk} \\ \vdots \\ i \end{array} \quad (23.18)$$

The structure constants  $f_{ijk}$  for  $SU(n)$  is a totally antisymmetric tensor. In the CC convention, the first index of  $f_{ijk}$  corresponds to the green leg in the birdtracks.<sup>2</sup>

Multiplying Lie Algebra commutator by  $T_k$  and taking the trace, we get

$$if_{ijk} = \text{tr}([T_i, T_j]T_k)$$

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 2} - \text{Diagram 3} \\
 &= 2 \times \text{Diagram 4}
 \end{aligned}
 \tag{23.19}$$

The diagrams are birdtracks:   
 Diagram 1: A vertex labeled  $f$  with a wavy line (green) entering from the top and two wavy lines exiting from the bottom-left and bottom-right.   
 Diagram 2: A triangle with vertices  $T_k$  (top),  $T_i$  (bottom-left), and  $T_j$  (bottom-right). Arrows point from  $T_i$  and  $T_j$  to  $T_k$ . A horizontal arrow points from  $T_i$  to  $T_j$ . Wavy lines enter from the bottom-left and bottom-right, and a wavy line exits from the top.   
 Diagram 3: A triangle with vertices  $T_k$  (top),  $T_i$  (bottom-left), and  $T_j$  (bottom-right). Arrows point from  $T_i$  and  $T_j$  to  $T_k$ . A horizontal arrow points from  $T_j$  to  $T_i$ . Wavy lines enter from the bottom-left and bottom-right, and a wavy line exits from the top.   
 Diagram 4: A triangle with vertices  $T_k$  (top),  $T_{i'}$  (bottom-left), and  $T_{j'}$  (bottom-right). Arrows point from  $T_{i'}$  and  $T_{j'}$  to  $T_k$ . A horizontal arrow points from  $T_{i'}$  to  $T_{j'}$ . Below the triangle is a double horizontal line labeled  $\mathcal{A}_2$ . Wavy lines enter from the bottom-left and bottom-right, and a wavy line exits from the top.

One can define a totally symmetric tensor  $d_{ijk}$  analogously by

$$d_{ijk} = \text{tr}([T_i, T_j]_+ T_k)$$

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 2} + \text{Diagram 3} \\
 &= 2 \times \text{Diagram 4}
 \end{aligned}
 \tag{23.20}$$

The diagrams are birdtracks:   
 Diagram 1: Same as in (23.19).   
 Diagram 2: Same as in (23.19).   
 Diagram 3: A triangle with vertices  $T_k$  (top),  $T_i$  (bottom-left), and  $T_j$  (bottom-right). Arrows point from  $T_i$  and  $T_j$  to  $T_k$ . A horizontal arrow points from  $T_j$  to  $T_i$ . Wavy lines enter from the bottom-left and bottom-right, and a wavy line exits from the top.   
 Diagram 4: A triangle with vertices  $T_k$  (top),  $T_{i'}$  (bottom-left), and  $T_{j'}$  (bottom-right). Arrows point from  $T_{i'}$  and  $T_{j'}$  to  $T_k$ . A horizontal arrow points from  $T_{i'}$  to  $T_{j'}$ . Below the triangle is a double horizontal line labeled  $\mathcal{S}_2$ . Wavy lines enter from the bottom-left and bottom-right, and a wavy line exits from the top.

**Claim 56 .**

- $\text{tr}([T_i, T_j]T_k)$  is totally anti-symmetric

---

<sup>2</sup>Actually, it doesn't matter which index is taken first. This is explained in Chapter B

- $\text{tr}([T_i, T_j]_+ T_k)$  is totally symmetric

in the indices  $i, j, k$

**proof:**

$$\text{tr}([T_i, T_j] T_k) = -\text{tr}([T_k, T_j] T_i) \quad (23.21)$$

$$\text{tr}([T_i, T_j] T_k) = +\text{tr}([T_k, T_j]_+ T_i) \quad (23.22)$$

**QED**

**Claim 57**

$$\text{tr}(T_i) = 0 \quad \text{~~~~~} T_i \begin{array}{c} \curvearrowright \end{array} = 0 \quad (23.23)$$

**proof:**

$$0 = P_{adj} P_S = \begin{array}{c} \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} T_i \sim T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \end{array} \quad (23.24)$$

**QED**

**Claim 58**

$$\Gamma_{fun} \delta_a^b = \sum_i (T_i T_i)_a^b = \frac{n^2 - 1}{n} \delta_a^b \quad (23.25)$$

$$\sum_i \begin{array}{c} \text{~~~~~} i \\ \text{~~~~~} \curvearrowright \\ a \longleftarrow T_i \longleftarrow T_i \longleftarrow b \end{array} = \left( \frac{n^2 - 1}{n} \right) a \longleftarrow \bullet \longleftarrow b$$

**proof:**

$$(T_i T_i)_a^b = \begin{array}{c} \text{~~~~~} i \\ \text{~~~~~} \curvearrowright \\ a \longleftarrow T_i \longleftarrow T_i \longleftarrow b \end{array} \quad (23.26)$$

$$= \begin{array}{c} a \longleftarrow \quad b \\ \quad \quad \quad \curvearrowright \\ T_i \sim T_i \\ \quad \quad \quad \curvearrowleft \end{array} \quad (23.27)$$

$$= \begin{array}{c} \longleftarrow \bullet \\ \quad \quad \quad - \frac{1}{n} \quad \begin{array}{c} \curvearrowright \bullet \bullet \curvearrowleft \\ \curvearrowleft \bullet \bullet \curvearrowright \end{array} \end{array} \quad (23.28)$$

$$= \left( n - \frac{1}{n} \right) a \longleftarrow \bullet \longleftarrow b \quad (23.29)$$



QED

Claim 59

$$\begin{array}{c} T_k \\ \swarrow \quad \searrow \\ \sim i \sim T_i \quad T_j \sim j \sim \\ \searrow \quad \swarrow \\ T_k \end{array} = -\frac{1}{n} i \sim \bullet \sim j \quad (23.30)$$

proof:

$$\begin{array}{c} T_k \\ \swarrow \quad \searrow \\ \sim i \sim T_i \quad T_j \sim j \sim \\ \searrow \quad \swarrow \\ T_k \end{array} = \underbrace{\begin{array}{c} \downarrow \quad \uparrow \\ \sim T_i \quad T_j \sim \\ \uparrow \quad \downarrow \end{array}}_{=0} - \frac{1}{n} \begin{array}{c} \leftarrow \quad \rightarrow \\ \sim \sim T_i \quad T_i \sim \sim \end{array} \quad (23.31)$$

QED

Claim 60

$$\delta(i, j) \Gamma_{adj} = -f_{imn} f_{jnm} = 2n \delta(i, j) \quad (23.32)$$

$$(-1) \begin{array}{c} \sim i \sim \\ \sim \sim f \quad f \sim j \sim \\ \sim \sim m \sim \end{array} = 2n i \sim \bullet \sim j$$

proof:

$$A = \begin{array}{c} \sim i \sim \\ \sim \sim f \quad f \sim j \sim \\ \sim \sim m \sim \end{array} = 2 \begin{array}{c} \sim i \sim T_i \leftarrow T_n \sim \sim f \sim j \sim \\ \searrow \quad \uparrow \\ T_m \end{array} \quad (23.33)$$

$$\frac{1}{2} A = \underbrace{\begin{array}{c} T_k \\ \swarrow \quad \searrow \\ \sim i \sim T_i \quad T_n \sim j \sim \\ \searrow \quad \swarrow \\ T_m \end{array}}_{A_1} - \underbrace{\begin{array}{c} T_k \sim j \sim \\ \swarrow \quad \searrow \\ \sim i \sim T_i \quad T_n \\ \searrow \quad \swarrow \\ T_m \end{array}}_{A_2} \quad (23.34)$$

$$A_1 = \frac{n^2 - 1}{n} \delta(i, j) \quad (23.35)$$

$$A_2 = -\frac{1}{n} \delta(i, j) \quad (23.36)$$

$$A = 2(A_1 - A_2) = 2n\delta(i, j) \quad (23.37)$$

QED

## 23.2 Differences Between $U(n)$ and $SU(n)$

### 1. $SU(n)$

primitive invariants: Kronecker delta, Levi-Civita tensor

$$\begin{array}{c} \leftarrow \\ \text{ } \\ \leftarrow \\ \text{ } \\ \leftarrow \end{array} T_i \sim T_i \begin{array}{c} \leftarrow \\ \text{ } \\ \leftarrow \end{array} \stackrel{\text{def}}{=} P_{adj} = \begin{array}{c} \leftarrow \bullet \\ \text{ } \\ \leftarrow \bullet \end{array} - \frac{1}{n} \begin{array}{c} \leftarrow \bullet \\ \text{ } \\ \leftarrow \bullet \end{array} \begin{array}{c} \leftarrow \bullet \\ \text{ } \\ \leftarrow \bullet \end{array} \quad (23.38)$$

$$\dim(P_{adj}) = \text{tr} P_{adj} = \begin{array}{c} \leftarrow \bullet \\ \text{ } \\ \leftarrow \bullet \end{array} - \frac{1}{n} \begin{array}{c} \leftarrow \bullet \\ \text{ } \\ \leftarrow \bullet \end{array} \begin{array}{c} \leftarrow \bullet \\ \text{ } \\ \leftarrow \bullet \end{array} \quad (23.39)$$

$$= n^2 - 1 \quad (23.40)$$

Since the Levi-Civita tensor is an invariant matrix for  $SU(n)$ , we must have

$$0 = \begin{array}{c} \leftarrow T_i \rightarrow \\ \text{ } \\ \leftarrow \mathcal{A}_p \rightarrow T_i \rightarrow \mathcal{A}_p^{\frac{1}{2}} \end{array} = \begin{array}{c} \leftarrow \mathcal{A}_p \rightarrow \mathcal{A}_p^{\frac{1}{2}} \end{array} - \frac{1}{n} \begin{array}{c} \leftarrow \mathcal{A}_p \rightarrow \mathcal{A}_p^{\frac{1}{2}} \end{array} \quad (23.41)$$

### 2. $U(n)$

primitive invariants: Kronecker delta

$$\begin{array}{c} \curvearrowleft \\ \uparrow \\ T_i \sim T_i \\ \downarrow \\ \curvearrowright \end{array} \stackrel{\text{def}}{=} P_{adj} = \begin{array}{c} \leftarrow \bullet \rightarrow \\ \leftarrow \bullet \rightarrow \end{array} \quad (23.42)$$

$$\dim(P_{adj}) = \text{tr} P_{adj} = \begin{array}{c} \leftarrow \bullet \rightarrow \\ \leftarrow \bullet \rightarrow \end{array} \quad (23.43)$$

$$= n^2 \quad (23.44)$$

### 23.3 $V_{def} \otimes V_{def}$ Decomposition

Let

$V_{def} = V =$  vector space in defining representation  $\{|a\rangle\}_{a=1}^n$ .

$$\begin{array}{c} \leftarrow \\ \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} \quad (23.45)$$

$$\begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} = \frac{1}{2} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \end{array} \right\} \quad (23.46)$$

$$\begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} = \frac{1}{2} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \end{array} \right\} \quad (23.47)$$

$$\dim(\mathcal{S}_2) = \frac{1}{2} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \end{array} \right\} \quad (23.48)$$

$$= \frac{n(n+1)}{2} \quad (23.49)$$

$$\dim(\mathcal{A}_2) = \frac{1}{2} \left\{ \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \leftarrow \\ \updownarrow \\ \leftarrow \leftarrow \end{array} \right\} \quad (23.50)$$

$$= \frac{n(n-1)}{2} \quad (23.51)$$

The projection operator tree is

$$\begin{array}{c} \mathcal{A}_2 \\ \perp \\ \mathcal{S}_2 \end{array}$$

## 23.4 $V_{adj} \otimes V_{def}$ Decomposition

Let

$V_{def} = V =$  vector space in defining representation  $\{|a\rangle\}_{a=1}^n$ .

$V_{adj} =$  vector space in adjoint representation  $\{|i\rangle\}_{i=1}^N$ .

$V_{adj} \otimes V \cong (V \otimes V^\dagger) \otimes V$

$$e = \begin{array}{c} \text{~~~~~} \\ \longleftarrow \end{array} \cong \begin{array}{c} \text{~~~~~} T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} T_j \text{~~~~~} \\ \longleftarrow \end{array} \quad (23.52)$$

$$R = \begin{array}{c} \text{~~~~~} T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} T_j \text{~~~~~} \\ \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \end{array} = \begin{array}{c} \text{~~~~~} \\ \begin{array}{c} \diagdown \quad \longleftarrow \quad \diagup \\ T_i \longleftarrow T_j \end{array} \end{array} \quad (23.53)$$

$$Q = \begin{array}{c} \text{~~~~~} T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} T_j \text{~~~~~} \\ \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \end{array} = \begin{array}{c} \text{~~~~~} \\ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \longleftarrow T_j \longleftarrow T_i \longleftarrow \end{array} \end{array} \quad (23.54)$$

Recall that for  $SU(n)$ , the dimension  $N$  of the adjoint rep is

$$N = n^2 - 1 = \begin{array}{c} \text{~~~~~} \\ \text{~~~~~} \end{array} \quad (23.55)$$

For example, for  $SU(2)$ ,  $N = 3$  and for  $SU(3)$ ,  $N = 8$ .

Note that

$$\text{tr}(e) = \begin{array}{c} \text{~~~~~} \\ \longleftarrow \end{array} = Nn \quad (23.56)$$

$$\text{tr}(R) = \begin{array}{c} \text{~~~~~} T_i \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} T_j \text{~~~~~} \\ \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \end{array} = N \quad (23.57)$$

$$\text{tr}(Q) = \begin{array}{c} \text{~~~~~} \\ \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \longleftarrow T_j \longleftarrow T_i \longleftarrow \end{array} \end{array} = N \quad (23.58)$$

**Claim 61**

$$R^2 = \frac{n^2 - 1}{n} R \quad (23.59)$$

$$QR = RQ = -\frac{1}{n} R \quad (23.60)$$

$$Q^2 - e = -\frac{1}{n}R \quad (23.61)$$

**proof:**

$$R^2 = \begin{array}{c} \text{Diagram: A horizontal chain of nodes } T_i \leftarrow T_k \leftarrow T_k \leftarrow T_j. \text{ Above } T_i \text{ and } T_j \text{ are wavy lines. Below } T_i \text{ and } T_j \text{ are diagonal lines pointing down-left and down-right respectively. A curved arrow connects the two } T_k \text{ nodes.} \end{array} \quad (23.62)$$

$$= \frac{n^2 - 1}{n} R \quad (\text{by Eq.(23.25)}) \quad (23.63)$$

$$QR = \begin{array}{c} \text{Diagram: A horizontal chain of nodes } \leftarrow T_k \leftarrow T_i \leftarrow T_k \leftarrow T_j \leftarrow. \text{ Above } T_i \text{ is a wavy line. Above } T_k \text{ and } T_j \text{ are diagonal lines pointing up-left and up-right respectively.} \end{array} \quad (23.64)$$

Define

$$X = \begin{array}{c} \text{Diagram: A horizontal chain of nodes } \leftarrow T_k \leftarrow T_i \leftarrow T_k \leftarrow. \text{ Above } T_i \text{ is a wavy line.} \end{array} \quad (23.65)$$

$$X = \begin{array}{c} \text{Diagram: A central node } T_i \text{ with two nodes } T_k \text{ above it. Wavy lines connect } T_i \text{ to } T_k \text{ and } T_k \text{ to } T_k. \text{ Arrows point from } T_k \text{ to } T_i \text{ and from } T_i \text{ to } T_k. \text{ Diagonal lines point from } T_k \text{ to } T_i \text{ and from } T_i \text{ to } T_k. \end{array} \quad (23.66)$$

$$= \underbrace{\begin{array}{c} \text{Diagram: A central node } T_i \text{ with two horizontal lines above it. Arrows point from the lines to } T_i \text{ and from } T_i \text{ to the lines.} \end{array}}_{=0} - \frac{1}{n} \begin{array}{c} \text{Diagram: A central node } T_i \text{ with two wavy lines above it. Arrows point from the lines to } T_i \text{ and from } T_i \text{ to the lines.} \end{array} \quad (23.67)$$

$$= -\frac{1}{n} \begin{array}{c} \text{Diagram: A central node } T_i \text{ with a wavy line above it. An arrow points from the line to } T_i. \end{array} \quad (23.68)$$

so

$$QR = RQ = -\frac{1}{n}R \quad (23.69)$$

$$Q^2 = \begin{array}{c} \text{Diagram: A horizontal wavy line with two vertices. From each vertex, a wavy line goes up and a wavy line goes down. The left wavy line has an arrow pointing left, labeled $T_k$. The right wavy line has an arrow pointing right, labeled $T_k$. The two vertical wavy lines are connected by a horizontal wavy line. The left vertical wavy line has an arrow pointing left, labeled $T_i$. The right vertical wavy line has an arrow pointing right, labeled $T_j$. \end{array} \quad (23.70)$$

$$= \begin{array}{c} \text{Diagram: A horizontal wavy line with two vertices. From each vertex, a wavy line goes up and a wavy line goes down. The left wavy line has an arrow pointing left, labeled $T_k$. The right wavy line has an arrow pointing right, labeled $T_k$. The two vertical wavy lines are connected by a horizontal wavy line. The left vertical wavy line has an arrow pointing left, labeled $T_i$. The right vertical wavy line has an arrow pointing right, labeled $T_j$. \end{array} \quad (23.71)$$

$$= \begin{array}{c} \text{Diagram: A horizontal wavy line with two vertices. From each vertex, a wavy line goes up and a wavy line goes down. The left wavy line has an arrow pointing left, labeled $T_k$. The right wavy line has an arrow pointing right, labeled $T_k$. The two vertical wavy lines are connected by a horizontal wavy line. The left vertical wavy line has an arrow pointing left, labeled $T_i$. The right vertical wavy line has an arrow pointing right, labeled $T_j$. \end{array} - \frac{1}{n} \begin{array}{c} \text{Diagram: A horizontal wavy line with two vertices. From each vertex, a wavy line goes up and a wavy line goes down. The left wavy line has an arrow pointing left, labeled $T_k$. The right wavy line has an arrow pointing right, labeled $T_k$. The two vertical wavy lines are connected by a horizontal wavy line. The left vertical wavy line has an arrow pointing left, labeled $T_i$. The right vertical wavy line has an arrow pointing right, labeled $T_j$. \end{array} \quad (23.72)$$

$$= \begin{array}{c} \text{Diagram: A horizontal wavy line with two vertices. From each vertex, a wavy line goes up and a wavy line goes down. The left wavy line has an arrow pointing left, labeled $T_k$. The right wavy line has an arrow pointing right, labeled $T_k$. The two vertical wavy lines are connected by a horizontal wavy line. The left vertical wavy line has an arrow pointing left, labeled $T_i$. The right vertical wavy line has an arrow pointing right, labeled $T_j$. \end{array} - \frac{1}{n} \begin{array}{c} \text{Diagram: A horizontal wavy line with two vertices. From each vertex, a wavy line goes up and a wavy line goes down. The left wavy line has an arrow pointing left, labeled $T_k$. The right wavy line has an arrow pointing right, labeled $T_k$. The two vertical wavy lines are connected by a horizontal wavy line. The left vertical wavy line has an arrow pointing left, labeled $T_i$. The right vertical wavy line has an arrow pointing right, labeled $T_j$. \end{array} \quad (23.73)$$

$$= e - \frac{1}{n}R \quad (23.74)$$

**QED**

**Claim 62**

$$P_1 = \frac{n}{n^2 - 1}R \quad (23.75)$$

$$P_2 = \frac{1}{2}P_4(1 + Q) = \frac{1}{2} \left[ e + Q - \frac{1}{n+1}R \right] \quad (23.76)$$

$$P_3 = \frac{1}{2}P_4(1 - Q) = \frac{1}{2} \left[ e - Q - \frac{1}{n-1}R \right] \quad (23.77)$$

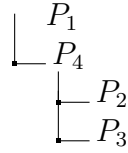
$$P_4 = 1 - P_1 \quad (23.78)$$

are projectors for  $SU(n)$ . The  $V_{adj} \otimes V = \sum_{\lambda} V_{\lambda}$  Clebsch-Gordan series is given by

$$\begin{array}{c}
\overbrace{V_{adj} \otimes V}^{\mathcal{V}} \\
\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \vdots & \\ \hline \square & \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} \\
= \begin{array}{|c|} \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \vdots & & \\ \hline \square & & \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \vdots & \\ \hline \square & \end{array}
\end{array} \quad (23.79)$$

$$\begin{array}{rcl}
(n^2 - 1)n & = & n + \frac{n(n-1)(n+2)}{2} + \frac{n(n+1)(n-2)}{2} \\
SU(3) : 8(3) & = & 3 + 15 + 6
\end{array}$$

The projection operator tree is



**proof:**

$$\text{tr}(P_1) = \frac{n}{n^2 - 1} N = n \quad (23.80)$$

$$\text{tr}(P_2) = \frac{N}{2} \left( n + 1 - \frac{1}{n + 1} \right) \quad (23.81)$$

$$= \frac{N}{2} \frac{n^2 + 2n}{n + 1} \quad (23.82)$$

$$= \frac{N}{2} \frac{n(n + 2)}{n + 1} \quad (23.83)$$

$$= \frac{(n - 1)n(n + 2)}{2} \quad (23.84)$$

$$\text{tr}(P_3) = \frac{N}{2} \left( n - 1 - \frac{1}{n - 1} \right) \quad (23.85)$$

$$= \frac{N}{2} \frac{n^2 - 2n}{n - 1} \quad (23.86)$$

$$= \frac{N}{2} \frac{n(n - 2)}{n - 1} \quad (23.87)$$

$$= \frac{(n + 1)n(n - 2)}{2} \quad (23.88)$$

From  $R^2 = \frac{n^2 - 1}{n} R$ ,

$$P_1 = \frac{n}{n^2 - 1}R \quad (23.89)$$

Define

$$P_4 = e - P_1 \quad (23.90)$$

From  $Q^2 - e = -\frac{1}{n}R$ , we get

$$P_4(Q^2 - 1) = 0 \quad (23.91)$$

Let

$$P_2 = \frac{1}{2}P_4(1 + Q), \quad P_3 = \frac{1}{2}P_4(1 - Q) \quad (23.92)$$

and

$$a = \frac{n}{n^2 - 1} \quad (23.93)$$

Then

$$P_2 = \frac{1}{2}P_4(1 + Q) \quad (23.94)$$

$$= \frac{1}{2}(e - aR)(1 + Q) \quad (23.95)$$

$$= \frac{1}{2}(e - aR + Q - aRQ) \quad (23.96)$$

$$= \frac{1}{2} \left( e + \left( \frac{1}{n} - 1 \right) aR + Q \right) \quad (\text{use } QR = -\frac{1}{n}R) \quad (23.97)$$

where

$$\left( \frac{1}{n} - 1 \right) a = \frac{1 - n}{n} \frac{n}{n^2 - 1} \quad (23.98)$$

$$= -\frac{1}{n + 1} \quad (23.99)$$

Furthermore

$$P_3 = \frac{1}{2}P_4(1 - Q) \quad (23.100)$$

$$= \frac{1}{2}(e - aR)(1 - Q) \quad (23.101)$$

$$= \frac{1}{2}(e - aR - Q + aRQ) \quad (23.102)$$

$$= \frac{1}{2} \left( e - \left( \frac{1}{n} + 1 \right) aR - Q \right) \quad (\text{use } QR = -\frac{1}{n}R) \quad (23.103)$$



where

$$\left(\frac{1}{n} + 1\right) a = \frac{1}{n-1} \quad (23.104)$$

**QED**

Let  $Q_1, Q_2, Q_3 = e, R, Q$

$$Q_\lambda |Q_j\rangle = |Q_\lambda Q_j\rangle = \sum_i A_{ij}^\lambda |Q_i\rangle \quad (23.105)$$

$$\langle Q_i | Q_\lambda | Q_j \rangle = A_{ij}^\lambda \quad (23.106)$$

If  $A^\lambda$  are diagonalized and divided by their eigenvalues, and they have a single non-zero eigenvalue, then they become a complete set of projectors with 1 or 0 along their diagonals.

# Chapter 24

## Weight Diagrams

This chapter is based on Refs. [2] and [3].

In this chapter, we will study irreps of semi-simple Lie algebras, paying special attention to  $\mathfrak{su}(2)$  and  $\mathfrak{su}(3)$ . Suppose  $\mathfrak{g}$  is the Lie algebra of a group  $G$ . Note that an irrep of  $G$  always gives an irrep of  $\mathfrak{g}$ , but an irrep of  $\mathfrak{g}$  might not give an irrep of  $G$ . Suppose groups  $G_1$  and  $G_2$  with different global topology both have Lie algebra  $\mathfrak{g}$ . Some irreps of  $\mathfrak{g}$  might lead to irreps of  $G_1$  but not of  $G_2$ . For example,  $SO(3)$  and  $SU(2)$  have the same Lie algebra  $\mathfrak{su}(2)$ .  $\mathfrak{su}(2)$  has irreps with integral and half integral spin. But only the ones with integral spin are irreps of  $SO(3)$ . Luckily, for  $SU(n)$ , which is the group that most concerns us in this chapter, all irreps of  $\mathfrak{su}(n)$  are irreps of  $SU(n)$  too.

In Chapter 4, we classified semi-simple Lie algebras in terms of their Dynkin diagrams. The Dynkin diagram of a Lie algebra  $\mathfrak{g}$  describes its set of root vectors. An irrep of  $\mathfrak{g}$  is described by a set of weight vectors. Both **root vectors** and **weight vectors** satisfy an eigenvalue equation, namely

$$\begin{aligned} \text{root } \vec{\alpha}: & \quad \overbrace{[\vec{H}, \cdot]^{E_{\vec{\alpha}}}} \\ & [\vec{H}, E_{\vec{\alpha}}] = \vec{\alpha} E_{\vec{\alpha}} \\ \text{weight } \vec{m}: & \quad \vec{H}|j, \vec{m}\rangle = \vec{m}|j, \vec{m}\rangle \end{aligned} \tag{24.1}$$

$\vec{\alpha}, \vec{m} \in \mathbb{R}^r$  where  $r$  is the rank of the Lie algebra.

$|1\rangle, |2\rangle, \dots, |n\rangle \in \mathbb{R}^n$  and  $H_i, E_{\vec{\alpha}} \in \mathbb{R}^{n \times n}$  where  $n$  is the dimension of the fundamental rep of the Lie algebra  $\mathfrak{su}(n)$

$|j, \vec{m}\rangle \in \mathbb{C}^d$  where  $d$  is the dimension of the irrep.

The set of weight vectors of an irrep of a Lie algebra will be called the **Weight Diagram (WD)** of the irrep.

**Claim 63** *If*

$$\vec{H}|\vec{m}\rangle = \vec{m}|\vec{m}\rangle \tag{24.2}$$

*then*

$$\vec{H}E_{\vec{\alpha}}|\vec{m}\rangle = (\vec{m} + \vec{\alpha})E_{\vec{\alpha}}|\vec{m}\rangle \tag{24.3}$$

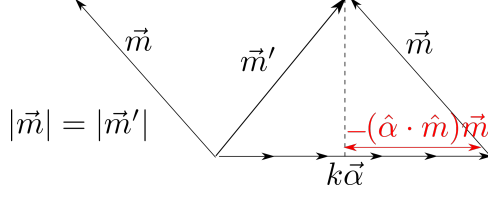


Figure 24.1: Relationship between 2 weights  $\vec{m}$  and  $\vec{m}'$ .

**proof:**

$$[\vec{H}, E_{\vec{\alpha}}] = \vec{\alpha} E_{\vec{\alpha}} \quad (24.4)$$

so

$$\vec{H} E_{\vec{\alpha}} |\vec{m}\rangle = E_{\vec{\alpha}} \vec{H} |\vec{m}\rangle + \vec{\alpha} E_{\vec{\alpha}} |\vec{m}\rangle \quad (24.5)$$

$$= (\vec{m} + \vec{\alpha}) E_{\vec{\alpha}} |\vec{m}\rangle \quad (24.6)$$

**QED**

**Claim 64** See Fig.24.1. For any weight  $\vec{m}$  and root  $\vec{\alpha}$ , if  $k$  is an integer and

$$k = - \frac{2\vec{m} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} \quad (24.7)$$

then

$$\vec{m}' = \vec{m} + k\vec{\alpha} \quad (24.8)$$

is a weight with the same eigenvalue multiplicity as  $\vec{m}$ .

**proof:**

**QED**

## 24.1 $SU(2)$ Weight Diagrams

In this section, we describe the WDs for  $SU(2)$ . These follow from the theory of angular momentum in Quantum Mechanics.

Define

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (24.9)$$

$$\sqrt{2}E_{12} = J_+ = |1\rangle\langle 2|, \quad \sqrt{2}E_{-12} = E_{21} = J_- = |2\rangle\langle 1| \quad (24.10)$$

$$J_z = \frac{1}{2} (|1\rangle\langle 1| - |2\rangle\langle 2|) \quad (24.11)$$

Then

$$\begin{array}{ccc}
 & J_z & \\
 -J_- \swarrow & & \searrow J_+ \\
 J_- & \xleftarrow{2J_z} & J_+
 \end{array} \quad \left\{ \begin{array}{l} [J_+, J_-] = 2J_z \\ [J_z, J_+] = J_+ \\ [J_z, J_-] = -J_- \end{array} \right. \quad (24.12)$$

The WD for irrep  $j$ , where<sup>1</sup>

$$J = j \in \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \right\} \quad (24.13)$$

has weights  $m$ , where

$$J_z = j_z = m = \{-j, -j+1, \dots, j-1, j\} \quad (24.14)$$

$|j, m\rangle$  is the basis vector of irrep  $j$  with weight  $m$ .

## 24.2 $SU(3)$ Weight Diagrams

In this section, we describe the WDs for  $SU(3)$ . These occur in particle physics, when describing particles by their quark constituents.

Define

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (24.15)$$

$$\sqrt{3}E_{12} = T_+ = |1\rangle\langle 2|, \quad \sqrt{3}E_{-12} = E_{21} = T_- = |2\rangle\langle 1| \quad (24.16)$$

$$\sqrt{3}E_{23} = U_+ = |2\rangle\langle 3|, \quad \sqrt{3}E_{-23} = E_{32} = U_- = |3\rangle\langle 2| \quad (24.17)$$

$$\sqrt{3}E_{31} = V_+ = |3\rangle\langle 1|, \quad \sqrt{3}E_{-31} = E_{13} = V_- = |1\rangle\langle 3| \quad (24.18)$$

$$T_z = \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|), \quad U_z = \frac{1}{2}(|2\rangle\langle 2| - |3\rangle\langle 3|), \quad V_z = \frac{1}{2}(|3\rangle\langle 3| - |1\rangle\langle 1|) \quad (24.19)$$

$$\sqrt{\frac{3}{2}}H_1 = T_z, \quad \sqrt{2}H_2 = Y = \frac{2}{3}(U_z - V_z) \quad (24.20)$$

---

<sup>1</sup>Whenever possible, we will follow the convention of denoting operators by upper case letters and their eigenvalues by the same letter in lower case. For example, the eigenvalues of  $J, J_z$  will be denoted by  $j, j_z$ .

**isospin**  $T_z \in \mathbb{Z}/2 = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \dots$

**hypercharge**  $Y \in \mathbb{Z}/3$

$H_1 \in \frac{1}{\sqrt{6}}\mathbb{Z}$

$H_2 \in \frac{1}{3\sqrt{2}}\mathbb{Z}$

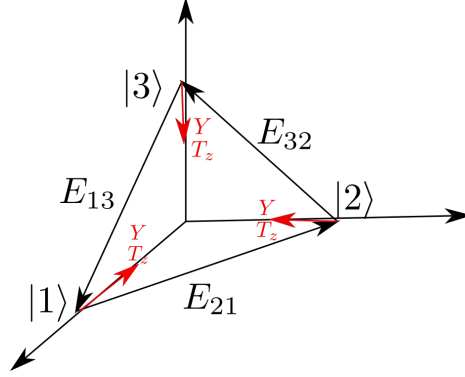


Figure 24.2:  $|1\rangle, |2\rangle, |3\rangle$  vectors, and operators that act on them.  $E_{-ij} = E_{ji}$  is in opposite direction as  $E_{ij}$  for  $i \neq j$  and  $i, j = 1, 2, 3$ .

Fig.24.2 illustrates the states  $|1\rangle, |2\rangle, |3\rangle$ , and operators  $E_{ij}$  that act on them. We can derive the weights of the fundamental rep of  $\mathfrak{su}(3)$  as follows.

$$\begin{pmatrix} \sqrt{\frac{3}{2}}H_1 \\ \sqrt{2}H_2 \end{pmatrix} |1\rangle = \begin{pmatrix} T_z \\ Y \end{pmatrix} |1\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} |1\rangle = \vec{m}(1)|1\rangle \quad (24.21)$$

$$\begin{pmatrix} \sqrt{\frac{3}{2}}H_1 \\ \sqrt{2}H_2 \end{pmatrix} |2\rangle = \begin{pmatrix} T_z \\ Y \end{pmatrix} |2\rangle = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} |2\rangle = \vec{m}(2)|2\rangle \quad (24.22)$$

$$\begin{pmatrix} \sqrt{\frac{3}{2}}H_1 \\ \sqrt{2}H_2 \end{pmatrix} |3\rangle = \begin{pmatrix} T_z \\ Y \end{pmatrix} |3\rangle = \begin{pmatrix} 0 \\ -\frac{2}{3} \end{pmatrix} |3\rangle = \vec{m}(3)|3\rangle \quad (24.23)$$

Fig.24.3 shows the root vectors for  $\mathfrak{su}(3)$  (derived in Chapter 4) and the weight vectors of the fundamental rep of  $\mathfrak{su}(3)$  (just derived). As expected, the difference between any two weight vectors, is  $k$  times a root vector.

As shown in Fig.24.4, each of the six root vectors of  $\mathfrak{su}(3)$  is associated with a raising operator  $T_+, U_+, V_+$  or a lowering operator  $T_-, U_-, V_-$ . In  $\mathfrak{su}(2)$   $J_+$  raises the  $J_z = j_z = m$  of a quantum state by  $1/2$  and  $J_-$  lowers it by a  $1/2$ .  $T_{\pm}, U_{\pm}, V_{\pm}$  change  $T_z, U_z, V_z$  analogously.

### 24.2.1 Examples

In this section, we give examples of  $SU(3)$  WDs.

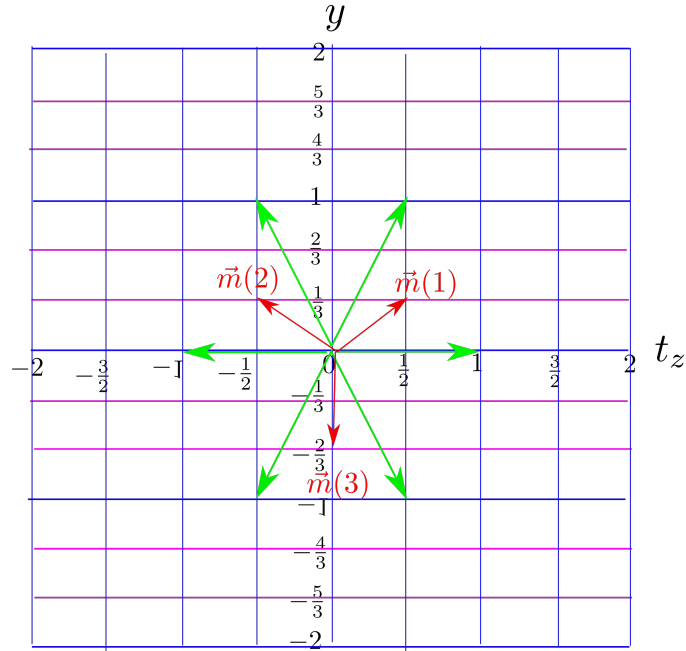


Figure 24.3: The roots vectors of  $\mathfrak{su}(3)$  in green and the weight vectors of the fundamental rep in red.

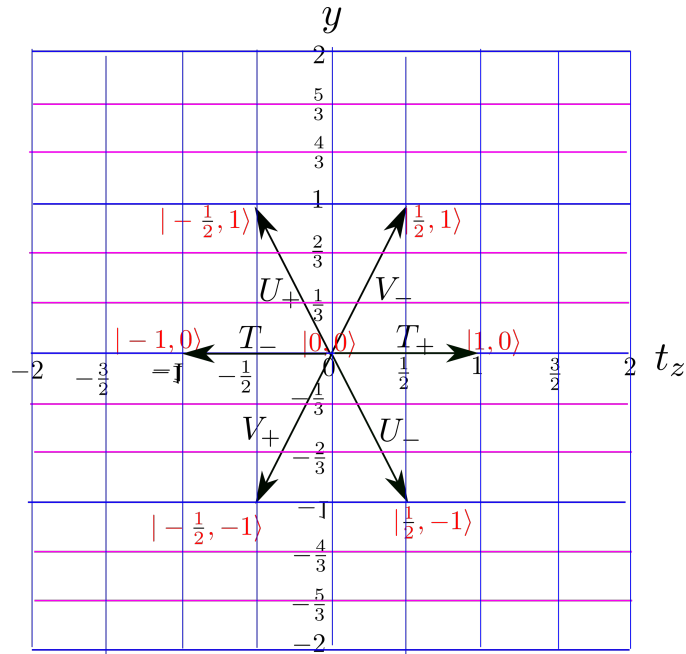


Figure 24.4: Each of the six root vectors of  $\mathfrak{su}(3)$  is associated with a raising operator  $T_+, U_+, V_+$  or a lowering operator  $T_-, U_-, V_-$ .

There is a one-to-one map between the the irreps of  $\mathfrak{su}(3)$  and the WDs of  $\mathfrak{su}(3)$ .

The weights on the **boundary of WD** are called **dominant weights**

In a WD, nondegenerate weights are represented by a single dot.  $k$ -fold degenerate weights (i.e., eigenvalues with **multiplicity**  $k$ ) are represented by a dot with  $k - 1$  rings.

WD are labelled either by

- the dimension  $d_{rep}$  of the irrep, with extra labels in case there are multiple irreps with the same dimension,
- the  $(\lambda, \mu)$  WD-boundary dimensions. See Fig.24.8

One can show that

$$d_{rep} = \frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2) \quad (24.24)$$

For example, see Fig.24.5.  $\mathfrak{su}(3)$  has 2 fundamental irreps,  $\underline{3}$  and  $\underline{3}^*$ . Both are 3 dimensional.  $(\lambda, \mu) = (1, 0)$  for  $\underline{3}$ , and  $(\lambda, \mu) = (0, 1)$  for  $\underline{3}^*$ . The formula Eq.(24.24) for  $d_{rep}$  gives 3 for both.

See the following figures for examples of WDs of  $\mathfrak{su}(3)$ .

- Fig.24.5 shows  $\underline{3}$  and  $\underline{3}^*$
- Fig.24.6 shows  $\underline{10}$  (decuplet)
- Fig.24.7 shows  $\underline{8}$  (octet)
- Fig.24.8 shows  $\underline{15}$

---

### An aside about hypercharge

Several different quantum numbers are called hypercharge in particle physics. For example, in the Gell-mann-Nishijima relation

$$Q = T_z + \frac{1}{2}Y' \quad (24.25)$$

hypercharge=  $Y' \in \mathbb{Z}$ , isospin=  $T_z \in \mathbb{Z}/2$ , so charge=  $Q \in \mathbb{Z}/2$ . For example, for nucleons,

$$\text{proton (p)} : \quad T_z = \frac{1}{2}, \quad Y' = 1 \implies Q = 1 \quad (24.26)$$

$$\text{neutron (n)} : \quad T_z = -\frac{1}{2}, \quad Y' = 1 \implies Q = 0 \quad (24.27)$$

As a mnemonic, remember that a nucleon has 3 quarks with  $Y = 1/3$ , and  $Y'$  for the nucleon is the sum of  $Y$  for each quark.  $Y = 1/3$  for up or down quarks, and  $Y = -2/3$  for the strange quark.  $u, d$  constitute an  $SU(2)$  doublet and  $s$  an  $SU(2)$  singlet.

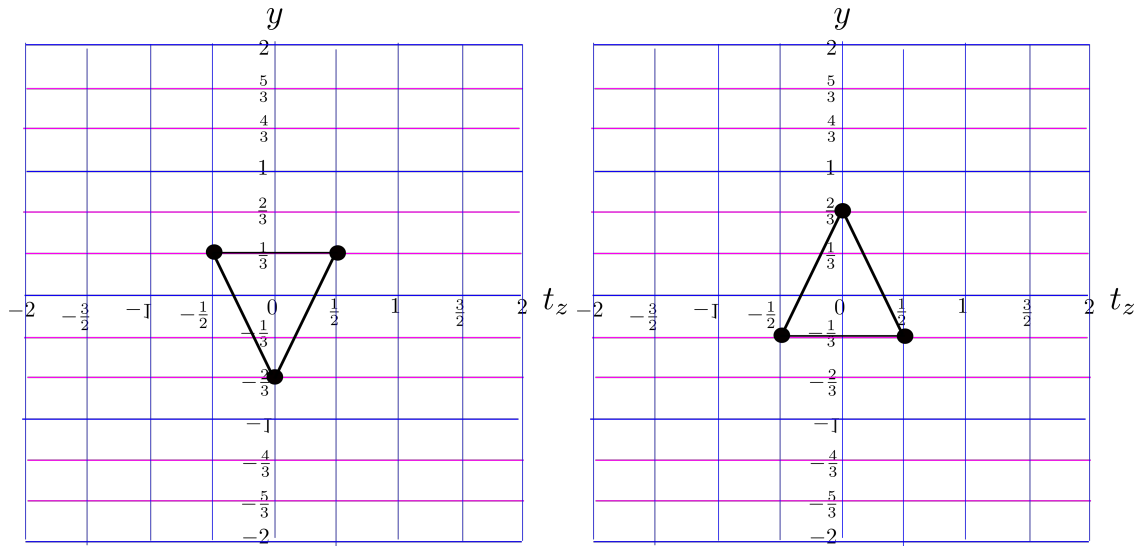


Figure 24.5: WDs for the two fundamental irreps of  $\mathfrak{su}(3)$ ,  $\underline{3}$  and  $\underline{3}^*$ .  $(\lambda, \mu) = (1, 0)$  for  $\underline{3}$ , and  $(\lambda, \mu) = (0, 1)$  for  $\underline{3}^*$ .

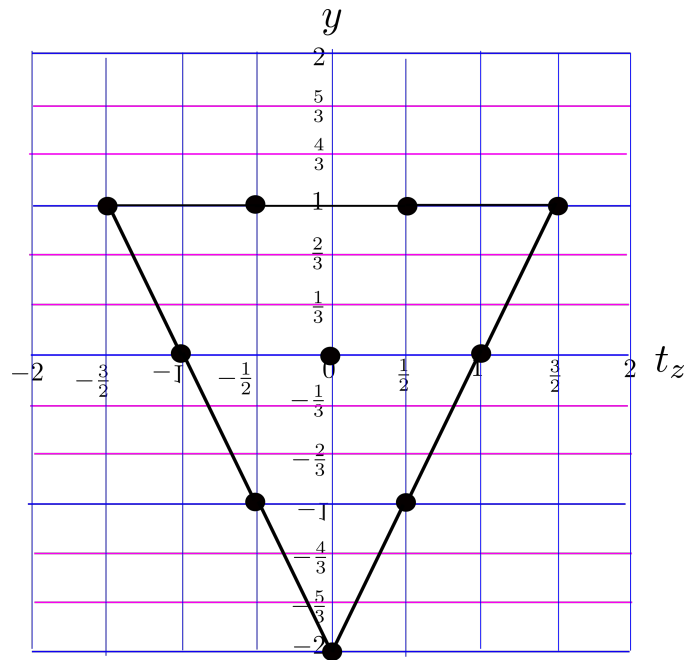


Figure 24.6: WD for the  $\mathfrak{su}(3)$  irrep  $\underline{10}$ .  $(\lambda, \mu) = (3, 0)$ .

## 24.3 Relation between WD and Semi-Standard Young Tableaux

Young Diagrams (YD) and Young Tableaux (YT) are discussed in Chapter 26



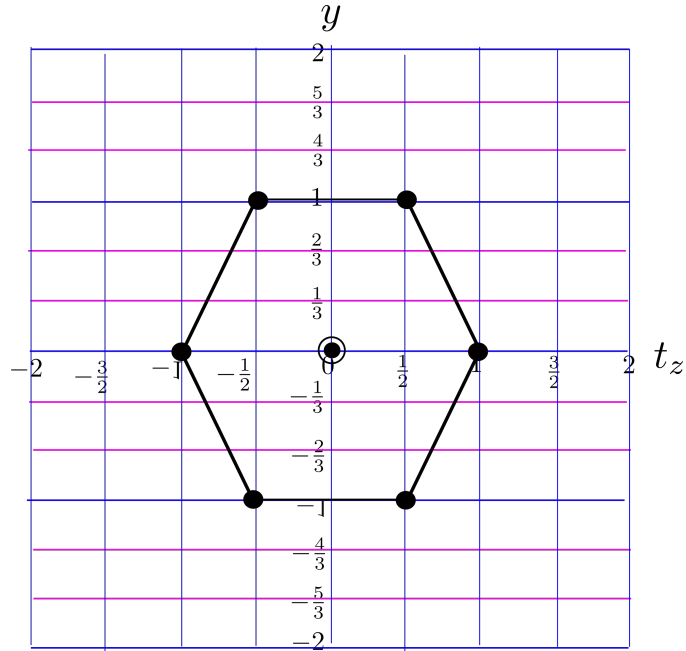


Figure 24.7: WD for the  $\mathfrak{su}(3)$  irrep 8.  $(\lambda, \mu) = (1, 1)$ .

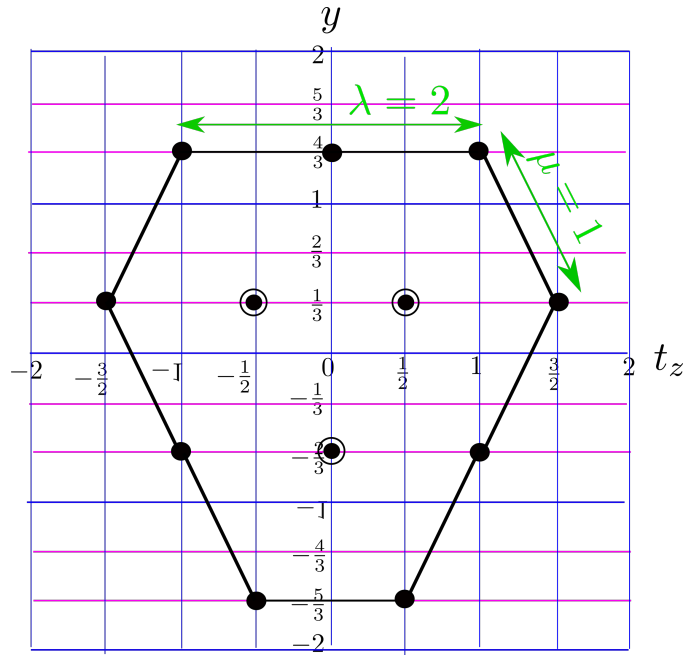


Figure 24.8: WD for the  $\mathfrak{su}(3)$  irrep 15.  $(\lambda, \mu) = (2, 1)$ .

Given a YD, its SYT/SSYT (Standard or Semi-Standard YT) satisfies

- Rows strictly/weakly increasing

- Columns strictly increasing
- Entries in  $\{1, 2, \dots, n\}$ .  $n = 3$  for  $SU(3)$

Thus, SYT/SSYT disallow/allow repetitions along a row.

SYT are used, for example, to label the irreps of  $S_n$ . Hence, there is a 1-1 relation between Young Symmetrizer operators and SYT.

SSYT are used below to label the basis weight vectors of an irrep of  $\mathfrak{su}(n)$ . There is a 1-1 relationship between YD and WD, and a 1-1 relationship between the SSYT of a YD and the weights of the corresponding WD. Degenerate weights have different SSYT.

To summarize,

$$\text{SYTs} \xrightarrow{1 \text{ to } 1} \text{Young operators, SSYTs} \xrightarrow{1 \text{ to } 1} \text{weights of WD}$$

To calculate the isospin, hypercharge  $(t_z, y)$  of a SSYT, use the following equation

$$(t_z, y) = \sum_{\text{boxes}} \begin{cases} (\frac{1}{2}, \frac{1}{3}), & \text{for each 1 box} \\ (-\frac{1}{2}, \frac{1}{3}), & \text{for each 2 box} \\ (0, -\frac{2}{3}), & \text{for each 3 box} \end{cases} \quad (24.28)$$

Equivalently, if  $n_i$  is the number of  $i = 1, 2, 3$  boxes in the SSYT, then

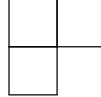
$$t_z = \frac{1}{2}(n_1 - n_2) \quad (24.29)$$

$$y = \frac{1}{3}(n_1 + n_2 - 2n_3) \quad (24.30)$$

---

$SU(3)$  octet example.

An octet  $(\underline{8}, (1,1))$ , Fig.24.7) has the YD



Let

$$\mathcal{Y} = S_{12}A_{13} = [1 + (12)][1 - (13)] \quad (24.31)$$

Then

$$|\psi\rangle = \mathcal{Y}|a\rangle|b\rangle|c\rangle = S_{12}(|a\rangle|b\rangle|c\rangle - |c\rangle|b\rangle|a\rangle) \quad (24.32)$$

$$= [|a\rangle|b\rangle|c\rangle + |b\rangle|a\rangle|c\rangle] - [|c\rangle|b\rangle|a\rangle + |b\rangle|c\rangle|a\rangle] \quad (24.33)$$

The following table gives the SSYT, its  $(t_z, y)$  content, and its wavefunction, for each weight of an  $SU(3)$  octet. This table's information is conveyed pictorially in Fig.24.9.

$$1. \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} \left| (t_z, y) = \left(+\frac{1}{2}, 1\right) \right| |\psi\rangle \text{ with } abc \rightarrow u_1 u_1 u_2 \quad (24.34)$$

$$2. \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \left| (t_z, y) = \left(-\frac{1}{2}, 1\right) \right| |\psi\rangle \text{ with } abc \rightarrow u_1 u_2 u_2 \quad (24.35)$$

$$3. \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array} \left| (t_z, y) = (+1, 0) \right| |\psi\rangle \text{ with } abc \rightarrow u_1 u_1 u_3 \quad (24.36)$$

$$4. \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \left| (t_z, y) = (0, 0) \right| |\psi\rangle \text{ with } abc \rightarrow u_1 u_2 u_3 \quad (24.37)$$

$$5. \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} \left| (t_z, y) = (-1, 0) \right| |\psi\rangle \text{ with } abc \rightarrow u_2 u_2 u_3 \quad (24.38)$$

$$6. \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \left| (t_z, y) = (0, 0) \right| |\psi\rangle \text{ with } abc \rightarrow u_1 u_3 u_2 \quad (24.39)$$

$$7. \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array} \left| (t_z, y) = \left(+\frac{1}{2}, -1\right) \right| |\psi\rangle \text{ with } abc \rightarrow u_1 u_3 u_3 \quad (24.40)$$

$$8. \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array} \left| (t_z, y) = \left(-\frac{1}{2}, -1\right) \right| |\psi\rangle \text{ with } abc \rightarrow u_2 u_3 u_3 \quad (24.41)$$

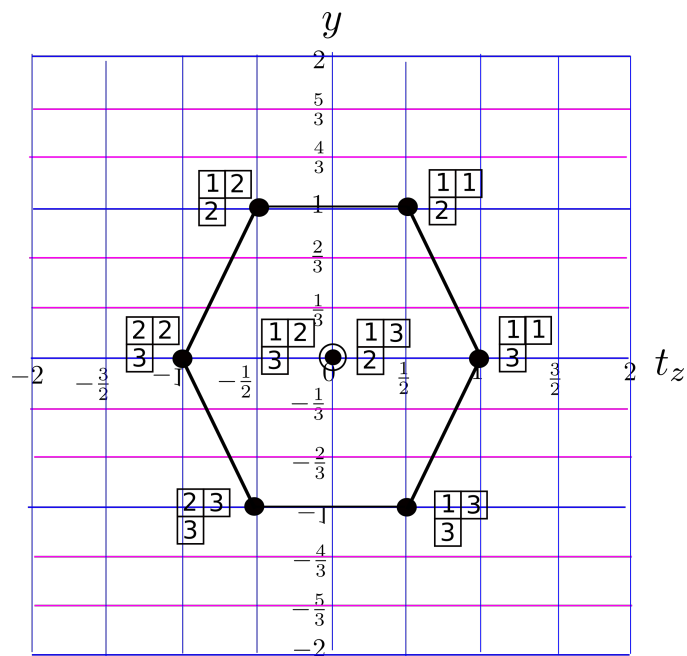


Figure 24.9: WD for the  $SU(3)$  octet with the SSYT for each weight.

# Chapter 25

## Wigner-Ekart Theorem

This chapter on the Wigner-Ekart (WE) Theorem is based on Cvitanovic's Birdtracks book Ref. [1].

### 25.1 WE in General

The birdtracks with no incomming or outgoing arrows are know as **reduced matrix elements, isolated DAGs and vacuum bubbles**

The following 3 claims are related. They reduce a tensor with 1, 2 and 3 indices

**Claim 65** (*one index*)

*If  $M$  is an invariant vector (i.e.,  $G_\lambda(g)M = M$  for all  $g \in \mathcal{G}$ ), then*

$$M_a = \sum_{\lambda} a \leftarrow \lambda - P_{\lambda} \leftarrow \lambda - M \quad (25.1)$$

$$= \sum_{\lambda \in SR} a \leftarrow \lambda - P_{\lambda} \leftarrow \lambda - M \quad (25.2)$$

where  $P_{\lambda} = |\lambda\rangle\langle\lambda| = \mathfrak{C}_{\lambda}^{\dagger}\mathfrak{C}_{\lambda}$  and  $SR =$  set of singlet representations.

**proof:**

**QED**

**Claim 66** (*Schur's Lemma*) (*2 indices*)

*If  $\mu$  and  $\lambda$  are irreps, and  $M$  is an invariant matrix, then*

$$M_{\lambda a}^{\mu b} = a \leftarrow \lambda - M \leftarrow \mu - b \quad (25.3)$$

$$= \frac{1}{d_{\mu}} \left( \begin{array}{c} M \\ \curvearrowright \\ \mu \end{array} \right) \delta(\mu, \lambda) \leftarrow \lambda - \quad (25.4)$$

**proof:**

**QED**

**Claim 67** (*Wigner-Ekart (WE) Theorem*) (*3 indices*)  
 If  $M$  is an invariant 3 index tensor,

$$(M^{\lambda i})_{\lambda_2 a}{}^{\lambda_1 b} = \begin{array}{c} \lambda - i \\ \downarrow \\ a \leftarrow \lambda_2 - M^\lambda \leftarrow \lambda_1 - b \end{array} \quad (25.5)$$

$$= \sum_{\lambda_2} \frac{d_{\lambda_2}}{\mathfrak{e}_{\lambda_2}^\dagger \begin{array}{c} \nearrow \lambda \\ \leftarrow \lambda_2 \\ \searrow \lambda_1 \end{array} \mathfrak{e}_{\lambda_2}} \begin{array}{c} \lambda \\ \downarrow \\ \leftarrow \lambda_2 - M^\lambda \leftarrow \lambda_1 - \end{array} \begin{array}{c} \parallel \\ \mathfrak{e}_{\lambda_2}^\dagger \leftarrow \mathfrak{e}_{\lambda_2} \parallel \\ \parallel \end{array} \quad (25.6)$$

$$= \frac{\begin{array}{c} \lambda \\ \downarrow \\ \text{loop} \\ \lambda_2 \end{array}}{\mathfrak{e}_{\lambda_2}^\dagger \begin{array}{c} \nearrow \lambda \\ \leftarrow \lambda_2 \\ \searrow \lambda_1 \end{array} \mathfrak{e}_{\lambda_2}} \begin{array}{c} \lambda \\ \downarrow \\ \leftarrow \lambda_2 - \mathfrak{e}_{\lambda_2} \leftarrow \lambda_1 - \end{array} \quad (25.7)$$

**proof:**  
**QED**

What about 4 indices and beyond? Consider

$$M = \begin{array}{c} \leftarrow \mu - M \\ \parallel \\ \leftarrow \nu - \\ \rightarrow \rho - \\ \rightarrow \omega - \end{array} \quad (25.8)$$

Then

$$\begin{array}{c} \leftarrow \mu - M \\ \leftarrow \nu - \\ \rightarrow \rho - \\ \rightarrow \omega - \end{array} \parallel \parallel = \sum_{\alpha, \beta} \frac{1}{\kappa_\alpha \kappa_\beta} \begin{array}{c} \leftarrow \parallel \mathfrak{e}_\alpha^\dagger \leftarrow \mathfrak{e}_\alpha \parallel \leftarrow M \\ \leftarrow \parallel \mathfrak{e}_\beta \rightarrow \mathfrak{e}_\beta^\dagger \parallel \rightarrow \end{array} \quad (25.9)$$

$$= \sum_{\alpha} \frac{1}{\kappa_\alpha^2 d_\alpha} \begin{array}{c} \leftarrow \parallel \mathfrak{e}_\alpha^\dagger \leftarrow \mathfrak{e}_\alpha \parallel \leftarrow M \\ \leftarrow \parallel \mathfrak{e}_\alpha \rightarrow \mathfrak{e}_\alpha^\dagger \parallel \rightarrow \end{array} \quad (25.10)$$

Above, we used

$$\begin{array}{c} \leftarrow \parallel \mathfrak{e}_\alpha \parallel \leftarrow M \\ \rightarrow \parallel \mathfrak{e}_\beta^\dagger \parallel \rightarrow \end{array} \parallel \parallel = \frac{\delta(\alpha, \beta)}{d_\alpha} \begin{array}{c} \leftarrow \parallel \mathfrak{e}_\alpha \parallel \leftarrow M \\ \rightarrow \parallel \mathfrak{e}_\alpha^\dagger \parallel \rightarrow \end{array} \quad (25.11)$$

## 25.2 WE for Angular Momentum

Let

$\lambda = J$ ,  $\lambda_i = J_i$  for  $i = 1, 2$ . We will use Greek letters instead of  $J$  so as to keep convention of using Greek letters for rep labels.

$m, m' = -\lambda, -\lambda + 1, \dots, \lambda - 1, \lambda$ . Note that  $d_\lambda = 2\lambda + 1$

for  $i = 1, 2$ ,  $m_i = -\lambda_i, -\lambda_i + 1, \dots, \lambda_i - 1, \lambda_i$ . Note that  $d_{\lambda_i} = 2\lambda_i + 1$

Define

$$\langle (\lambda_1 \lambda_2) \lambda m | \lambda_1 m_1 \lambda_2 m_2 \rangle = \lambda m \leftarrow \begin{array}{c} \leftarrow \lambda_1 m_1 \\ \parallel \\ \mathfrak{C}_\lambda \\ \parallel \\ \leftarrow \lambda_2 m_2 \end{array} \quad (25.12)$$

$$D_{mm'}^\lambda(g) = m \leftarrow D^\lambda \leftarrow m' \quad (25.13)$$

Then the Clebsch-Gordan decomposition of  $D^{\lambda_1} \otimes D^{\lambda_2}$  is

$$D_{m_1 m'_1}^{\lambda_1}(g) D_{m_2 m'_2}^{\lambda_2}(g) = \sum_{\lambda, m, m'} \langle \lambda_1 m_1 \lambda_2 m_2 | \lambda_1 \lambda_2 \lambda m \rangle D_{mm'}^\lambda(g) \langle \lambda_1 \lambda_2 \lambda m' | \lambda_1 m'_1 \lambda_2 m'_2 \rangle$$

$$\begin{array}{c} \leftarrow D^{\lambda_1} \leftarrow \\ \leftarrow D^{\lambda_2} \leftarrow \end{array} = \sum_\lambda \begin{array}{c} \leftarrow \\ \parallel \\ \mathfrak{C}_\lambda^\dagger \leftarrow D^\lambda \leftarrow \mathfrak{C}_\lambda \\ \parallel \\ \leftarrow \end{array} \quad (25.14)$$

We will denote a **tensor operator**  $M_m^\lambda$  by the birdtrack

$$\langle \lambda_2 m_2 | M_m^\lambda | \lambda_1 m_1 \rangle = \begin{array}{c} \lambda m \\ \downarrow \\ \lambda_2 m_2 \leftarrow M_m^\lambda \leftarrow \lambda_1 m_1 \end{array} \quad (25.15)$$

**Claim 68** (*Wigner-Ekart for angular momentum*)

$$\langle \lambda_2 m_2 | M_m^\lambda | \lambda_1 m_1 \rangle = \langle (\lambda \lambda_1) \lambda_2 m_2 | \lambda m \lambda_1 m_1 \rangle Q R$$

$$\begin{array}{c} \lambda m \\ \downarrow \\ \lambda_2 m_2 \leftarrow M_m^\lambda \leftarrow \lambda_1 m_1 \end{array} = Q R \begin{array}{c} \leftarrow \lambda \\ \parallel \\ \leftarrow \lambda_2 - \mathfrak{C}_\lambda \\ \parallel \\ \leftarrow \lambda_1 \end{array} \quad (25.16)$$

where

$$Q(\lambda, \lambda_1, \lambda_2) = \frac{1}{d_{\lambda_2}} \sum_{m_1, m_2, m} \langle \lambda m \lambda_1 m_1 | (\lambda \lambda_1) \lambda_2 m_2 \rangle \langle \lambda_2 m_2 | M_m^\lambda | \lambda_1 m_1 \rangle \quad (25.17)$$

$$= \frac{1}{d_{\lambda_2}} \begin{array}{c} \lambda \\ \downarrow \\ M_m^\lambda \leftarrow \lambda_1 \\ \parallel \\ \mathfrak{C}_{\lambda_2}^\dagger \\ \parallel \\ \lambda_2 \end{array} \quad (25.18)$$



and

$$R(\lambda,\lambda_1,\lambda_2)=\frac{d_{\lambda_2}}{\mathfrak{C}_{\lambda_2}^\dagger\overset{\curvearrowright\lambda\curvearrowleft}{\leftarrow\lambda_2\rightarrow}\mathfrak{C}_{\lambda_2}\overset{\curvearrowright\lambda_1\curvearrowleft}{\rightarrow}}$$
(25.19)

**proof:**  
**QED**

# Chapter 26

## Young Tableau

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

We recommend that the read Chapter 22 on symmetrizers and antisymetrizers before reading this one.

A **Young Diagram** (YD)  $\mathcal{Y} = [\lambda_1, \lambda_2, \dots, \lambda_D]$  consists of  $\lambda_1$  left-aligned empty boxes (LAEB) over  $\lambda_2$  LAEB, over  $\lambda_3$  LAEB, up to  $\lambda_{NR}$  LAEB, where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{NR} \geq 1$ .  $NR$  = number of rows. For example,

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} = [4, 2, 1, 1] \quad (26.1)$$

We will call  $[4, 2, 1, 1]$  the **row lengths** (RL) method of labeling YD.

A alternative method of labelling YD is called the **Dynkin (D) labels** or **row changes (RC)**. These labels list the change in number of columns as we go down the YD. For example,

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} = [2, 1, 0, 1, 0 \dots]_D \quad (26.2)$$

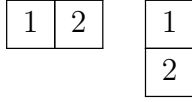
A **Young Tableau** (YT)  $\mathcal{Y}_\alpha$  is a YD in which integers from 1 to  $n$  where  $n \leq n_b$  and  $n_b$  is the number boxes, are inserted according to some rules. The rules for insertion are that integers must increase when reading a row left to right and when reading a column from top to bottom. Obviously, for  $n < n_b$ , some integers are repeated.

A **Standard Young Tableau** (SYT)  $\mathcal{Y}_\alpha$  is a YT such that  $n = n_b$  and no integer is repeated. Fig.26.1 shows all SYT for  $n_b = 1, 2, 3, 4$

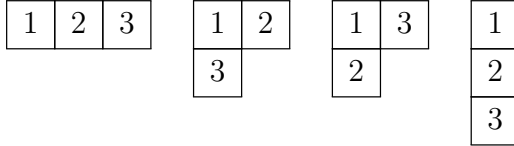
- $n_b = 1$



- $n_b = 2$



- $n_b = 3$



- $n_b = 4$

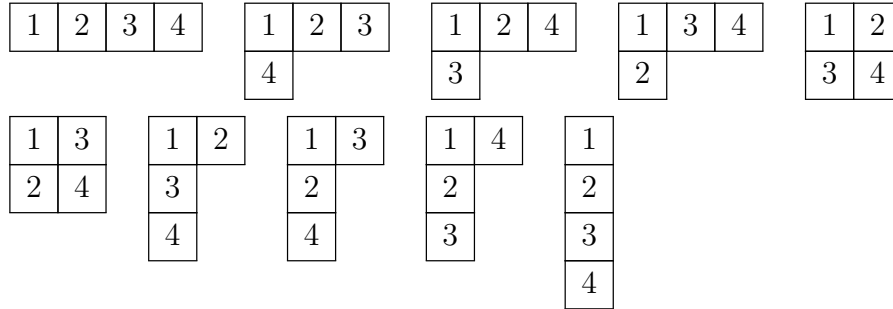


Figure 26.1: All SYT for  $n_b = 1, 2, 3, 4$ .

We will use  $|\mathcal{Y}|$ , or  $|\mathcal{Y}_\alpha|$  or  $|\alpha|$  to denote the number of boxes in a YD or YT.<sup>1</sup>

## 26.1 Symmetric Group $S_{n_b}$

Let

$S_{n_b}$  = the symmetric group in  $n_b$  letters (or  $n_b$  boxes)

$\text{irreps}(S_{n_b})$  = the set of all irreps of  $S_{n_b}$ .

The **transpose of a YT** is defined as if it were a matrix. For example

$$\text{transpose} \left( \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & \\ \hline 4 & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \hline 5 & & \\ \hline \end{array} \quad (26.3)$$

---

<sup>1</sup>For many authors and for us too,  $|S|$  stands for the number of elements in a finite set  $S$ . This should not lead to confusion as a YD or YT are not sets.

**n-dim General Linear group**  $GL(n; \mathbb{C}) = \{G \in \mathbb{C}^{n \times n} : \det(G) \neq 0\}$   
**n-dim Special Linear group**  $SL(n; \mathbb{C}) = \{G \in GL(n; \mathbb{C}) : \det(G) = 1\}$   
**n-dim Unitary group**,  $U(n) = \{G \in GL(n; \mathbb{C}) : GG^\dagger = G^\dagger G = 1\}$   
**n-dim Special Unitary group**  $SU(n) = \{G \in U(n) : \det(G) = 1\}$   
 $YD(n_b)$  = set of YD with  $n_b$  boxes.  $YD = \cup_{n_b=1}^{\infty} YD(n_b)$ .  
 $SYD(n_b)$  = set of SYD with  $n_b$  boxes.  $YT = \cup_{n_b=1}^{\infty} YT(n_b)$ .  
 $SYT(n_b, NR)$  = set of STY with  $n_b$  boxes and  $NR$  rows.  
 $YT(\mathcal{Y})$  = set of YT with a YD  $\mathcal{Y}$ .  
 $SYT(\mathcal{Y})$  = set of SYT with a YD  $\mathcal{Y}$ .  
 $\dim(\mathcal{Y}|S_{n_b})$  = dimension of irrep  $\mathcal{Y}$  of  $S_{n_b}$   
 $\dim(\mathcal{Y}_\alpha|U(n))$  = dimension of irrep  $\mathcal{Y}_\alpha$  of  $U(n)$  or  $SU(n)$ .

### Claim 69

1. The YD with  $n_b$  boxes label all irreps of the symmetric group  $S_{n_b}$ .

$$\text{irreps}(S_{n_b}) = YD(n_b) \quad (26.4)$$

2. The SYT with  $n_b$  boxes and no more than  $n$  rows ( $NR \leq n$ ), label the irreps of  $GL(n)$  and of  $U(n)$

$$\text{irreps}(U(n)) = \bigcup_{n_b \leq n, NR \leq n} SYT(n_b, NR) \quad (26.5)$$

3. The SYT with  $n_b$  boxes and no more than  $n - 1$  rows ( $NR \leq n - 1$ ), label the irreps of  $SL(n)$  and  $SU(n)$ .

$$\text{irreps}(SU(n)) = \bigcup_{n_b \leq n, NR \leq n-1} SYT(n_b, NR) \quad (26.6)$$

**proof:**

**QED**

#### 26.1.1 $\dim(\mathcal{Y}|S_{n_b})$

### Claim 70

$$\dim(\mathcal{Y}|S_{n_b}) = |SYT(\mathcal{Y})| \quad (26.7)$$

**proof:**

**QED**

For example, there are 3 irreps of  $S_4$  associated with the YD

$$\mathcal{Y} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \quad (26.8)$$

And each of those 3 irreps has dimension 3. Why? Because there are 3 possible SYT for this YD:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \implies \dim(\mathcal{Y}|S_4) = 3 \quad (26.9)$$

Thus, we can denote the basis vectors of one of these 3 degenerate irreps by

$$\left| \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \right\rangle, \left| \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array} \right\rangle, \left| \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \right\rangle \quad (26.10)$$

To compute  $hook(\mathcal{Y})$ :

1. Fill each box of the YD with the number of boxes below and to the right of the box, including the box itself.
2. Multiply the numbers in all the boxes.

For example,

$$\mathcal{Y} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \implies hook(\mathcal{Y}) = \begin{array}{|c|c|c|c|} \hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 & \\ \hline 2 & 1 & & \\ \hline \end{array} = 6!3 \quad (26.11)$$

**Claim 71** (*hook rule for computing  $\dim(\mathcal{Y}|S_{n_b})$* )

$$\dim(\mathcal{Y}|S_{n_b}) = \frac{n_b!}{hook(\mathcal{Y})} \quad (26.12)$$

**proof:**

**QED**

For example

$$\mathcal{Y} = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \implies hook(\mathcal{Y}) = \begin{array}{|c|c|} \hline 4 & 1 \\ \hline 2 & \\ \hline 1 & \\ \hline \end{array} = 8 \quad (26.13)$$

so

$$\dim(\mathcal{Y}|S_4) = \frac{4!}{4(2)} = 3 \quad (26.14)$$

### 26.1.2 Regular Representation

The **regular representation** of the symmetric group  $S_{n_b}$  is defined as follows. For each permutation  $\sigma \in S_{n_b}$ , define an independent vector  $|\sigma\rangle$  in a vector space  $\mathcal{V} = \{|\sigma\rangle : \sigma \in S_{n_b}\} = \{|\sigma_i\rangle : i = 1, 2, \dots, n_b!\}$ . Let

$$|x\rangle = \sum_i x_i |\sigma_i\rangle \quad (26.15)$$

For any  $\tau \in S_{n_b}$ , suppose

$$\langle \sigma_j | \tau | \sigma_i \rangle = \langle \sigma_j \tau | \sigma_i \rangle \quad (26.16)$$

$$\langle \sigma_j | \tau | x \rangle = \langle \sigma_j \tau | x \rangle = \sum_i x_i \langle \sigma_j \tau | \sigma_i \rangle \quad (26.17)$$

**Claim 72** *The regular rep is  $n_b!$  dimensional and reducible. In the decomposition of the regular rep of  $S_{n_b}$ , each  $\lambda \in \text{irreps}(S_{n_b})$  appears  $\dim(\lambda|S_{n_b})$  times.*

**proof:**

**QED**

From the last claim, it follows that

$$n_b! = |S_{n_b}| = \sum_{\lambda \in \text{irreps}(S_{n_b})} [\dim(\lambda|S_{n_b})]^2 \quad (26.18)$$

$$= [n_b!]^2 \sum_{\mathcal{Y} \in YD(n_b)} \frac{1}{[\text{hook}(\mathcal{Y})]^2} \quad (\text{Because } |\text{irreps}(S_{n_b})| = |YD(n_b)|) \quad (26.19)$$

Hence,

$$1 = n_b! \sum_{\mathcal{Y} \in YD(n_b)} \frac{1}{[\text{hook}(\mathcal{Y})]^2} \quad (26.20)$$

The Clebsch-Gordan series for the regular rep of  $S_{n_b}$  is

$$1 = \sum_{\mathcal{Y} \in YD(n_b)} P_{\mathcal{Y}} \quad (26.21)$$

where each  $P_{\mathcal{Y}}$  can be further decomposed into

$$P_{\mathcal{Y}} = \sum_{\mathcal{Y}_\alpha \in SYT(\mathcal{Y})} \underbrace{|\mathcal{Y}_\alpha\rangle \langle \mathcal{Y}_\alpha|}_{P_{\mathcal{Y}_\alpha}} \quad (26.22)$$

The projection operators

$$\{P_{\mathcal{Y}_\alpha} : \mathcal{Y}_\alpha \in SYT(\mathcal{Y}), \mathcal{Y} \in YD(n_b)\} = \{P_{\mathcal{Y}_\alpha} : \mathcal{Y}_\alpha \in SYT(n_b)\} \quad (26.23)$$

are complete and orthogonal.

### 26.1.3 Tensor Product Decompositions

$$\begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \quad (26.24)$$

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \quad (26.25)$$

## 26.2 Unitary group $U(n)$

Let

$STY(n_b, NR < n') =$  set of STY with  $n_b$  boxes and number of rows  $NR < n'$   
Recall that<sup>2</sup>

$$irreps(U(n)) = \bigcup_{n_b \leq n, NR \leq n} STY(n_b, NR) = \bigcup_{n_b=1}^n STY(n_b, NR < n) \quad (26.26a)$$

$$irreps(SU(n)) = \bigcup_{n_b \leq n, NR \leq n-1} STY(n_b, NR) = \bigcup_{n_b=1}^n STY(n_b, NR < n-1) \quad (26.26b)$$

A SYT with  $n_b$  boxes represents a tensor with  $n_b$  indices ( $n_b$ -particles state). Each index ranges from 1 to  $n$ .

$n_b = 1$ : A 1-index, 1-box tensor is a 1-particle with  $n$  states. This corresponds to the fundamental representation.

$n_b = 2$ : A 2-index, 2-box tensor is a 2-particle with  $n^2$  states. These  $n^2$  states break into two sets, symmetric and anti-symmetric.

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad (26.27)$$

$$\begin{array}{c} \leftarrow \\ \leftarrow \end{array} = \begin{array}{c} \leftarrow \mathcal{A}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} \quad (26.28)$$

<sup>2</sup>Note that  $STY(n_b)$  only contains STY with  $n_b \leq n$  boxes, so the  $n_b \leq n$  constraint might seem redundant in Eqs.(26.26). It isn't redundant because by  $\cup_{n_b \leq n}$  we mean  $\cup_{n_b=1}^n$ .

The SYT of an irrep describes the symmetry of the indices of a tensor in that irrep. A single column SYT indicates a totally anti-symmetric tensor, a single row SYT indicates a totally symmetric tensor, and a SYT with more than one row or column indicates a mixed symmetry tensor. This is why we can't have more than  $n$  rows, because there are only  $n$  integers to fill all boxes so more than  $n$  rows would require a repetition of an integer in a column, and such a column, after antisymmetrizing, would lead to zero.

### 26.2.1 Young Projection Operators

For each SYT  $\mathcal{Y}_\alpha \in \text{irreps}(U(n))$ , define the **Young projection operator**

$$P_{\mathcal{Y}_\alpha} = \mathcal{N} \left( \prod_i S_i \right) \left( \prod_j A_j \right) \quad (26.29)$$

for some normalization constant  $\mathcal{N}$  yet to be determined. These projection operators are not unique.

**Claim 73**

$$\mathcal{N} = \frac{(\prod_i |S_i|!) (\prod_j |A_j|!)}{\text{hook}(\mathcal{Y})} \quad (26.30)$$

where  $|S_i|$  and  $|A_j|$  are the number of arrows entering the symmetrizer or anti-symmetrizer. Note that the normalization constant  $\mathcal{N}$  depends only on the YD  $\mathcal{Y}$ . Furthermore, the operators  $P_{\mathcal{Y}_\alpha}$  are idempotent (i.e., their square equals themselves) mutually orthogonal and complete:

$$P_{\mathcal{Y}_\alpha} P_{\mathcal{Y}_\beta} = P_{\mathcal{Y}_\alpha} \delta(\alpha, \beta) \quad (26.31)$$

$$1 = \sum_{\mathcal{Y}_\alpha \in \text{SYT}(n_b, NR < n')} P_{\mathcal{Y}_\alpha} \quad (26.32)$$

where

$$n' = \begin{cases} n & \text{for } U(n) \\ n-1 & \text{for } SU(n) \end{cases} \quad (26.33)$$

**proof:**

$$P_{\mathcal{Y}_\alpha} = \mathcal{N} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \left( \begin{array}{c} \leftarrow \\ \leftarrow \\ \vdots \\ \underbrace{\leftarrow}_1 \end{array} + \dots \right) \quad (26.34)$$

From Eq.(26.32)



$$\mathbb{1} = \sum_{\mathcal{Y}_\alpha \in SYT(n_b, NR < n')} \mathcal{N} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \mathbb{1} \quad (26.35)$$

$$= \sum_{\mathcal{Y} \in YD(n_b)} \frac{n_b!}{hook(\mathcal{Y})} \mathcal{N} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \mathbb{1} \quad (26.36)$$

$$= \sum_{\mathcal{Y} \in YD(n_b)} \frac{n_b!}{[hook(\mathcal{Y})]^2} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \mathbb{1} \quad (\text{if assume Eq.(26.30)}) \quad (26.37)$$

$$= \mathbb{1} \quad (\text{by Eq.(26.20)}) \quad (26.38)$$

**QED**

### 26.2.2 $dim(\mathcal{Y}_\alpha|U(n))$

Let  $dim(\mathcal{Y}_\alpha|U(n))$  be the dimension of an irrep of  $U(n)$  with STY given by  $\mathcal{Y}_\alpha \in SYT(n_b, NR < n)$ .

**Claim 74**

$$dim(\mathcal{Y}_\alpha|U(n)) = |YT(\mathcal{Y})| \quad (26.39)$$

*Note that the right hand side is independent of  $\alpha$ , so this dimension is the same for all irreps  $\alpha$  with the same YD  $\mathcal{Y}$ .*

**proof:**

**QED**

Hence,  $\{|\alpha\rangle : \alpha \in YT(\mathcal{Y})\}$  are a basis for the irrep  $\mathcal{Y}_\alpha$  of  $U(n)$ . Note that the irreps of  $U(n)$  are given by SYT  $\mathcal{Y}_\alpha$ , whereas the basis vectors of an irrep are given by YT (not just STY). For example, for

$$\mathcal{Y}_\alpha = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad (26.40)$$

the basis vectors are

$$|\begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array}\rangle, \quad |\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}\rangle, \quad |\begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array}\rangle \quad (26.41)$$

so

$$dim(\mathcal{Y}_\alpha|U(2)) = 3 \quad (26.42)$$

In Eq.(26.39) we gave a way of finding  $dim(\mathcal{Y}_\alpha|U(n))$  A second way is by taking the trace of the corresponding projection operator

$$dim(\mathcal{Y}_\alpha|U(n)) = \text{tr}(P_{\mathcal{Y}_\alpha}) \quad (26.43)$$

For example, if

$$\mathcal{Y}_\alpha = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad (26.44)$$

then

$$\dim(\mathcal{Y}_\alpha | U(n)) = \begin{array}{c} \text{Diagram with two boxes, each containing a double arrow pointing left, with a red arc above them labeled } \mathcal{S}_2 \\ \parallel \\ \text{Diagram with two boxes, each containing a double arrow pointing left, with a red arc above them} \end{array} \quad (26.45)$$

$$= \frac{1}{2} \left( \begin{array}{c} \text{Diagram with two boxes, each containing a double arrow pointing left, with a red arc above them} \\ \text{Diagram with two boxes, each containing a double arrow pointing left, with a red arc above them} \end{array} + \begin{array}{c} \text{Diagram with two boxes, each containing a double arrow pointing left, with a red arc above them} \\ \text{Diagram with two boxes, each containing a double arrow pointing left, with a red arc above them} \end{array} \right) \quad (26.46)$$

$$= \frac{1}{2}(n^2 + n) \quad (26.47)$$

$$= 3 \text{ for } n = 2 \quad (26.48)$$

A third way of computing  $\dim(\mathcal{Y}_\alpha | U(n))$  is by computing the hook and coat functions and using the formula

$$\dim(\mathcal{Y}_\alpha | U(n)) = \frac{\text{coat}(\mathcal{Y})}{\text{hook}(\mathcal{Y})} \quad (26.49)$$

Note that right hand side is independent of  $\alpha$ ; it depends only on the YD. We've already discussed how to compute  $\text{hook}(\mathcal{Y})$ .  $\text{coat}(\mathcal{Y})$  is calculated as follows.<sup>3</sup>

1. Fill  $\mathcal{Y}$  with
  - $n$  at the diagonal blocks
  - $n$  increasing by 1 per block when reading from left to right
  - $n$  decreasing by 1 per block when reading from top to bottom
2. multiply all the boxes

Examples

$$\dim\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, U(2)\right) = \frac{\begin{array}{|c|c|} \hline n & n+1 \\ \hline \end{array}}{\begin{array}{|c|c|} \hline 2 & 1 \\ \hline \end{array}} = \frac{n(n+1)}{2} \quad (26.50)$$

$$\dim\left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, U(2)\right) = \frac{\begin{array}{|c|} \hline n \\ \hline n-1 \\ \hline \end{array}}{\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}} = \frac{n(n-1)}{2} \quad (26.51)$$

---

<sup>3</sup>I invented the name  $\text{coat}(\mathcal{Y})$ . I don't know if it has a name.

$$dim\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & & \\ \hline 7 & & & \\ \hline \end{array}, U(7)\right) = \begin{array}{|c|c|c|c|} \hline n & n+1 & n+2 & n+3 \\ \hline n-1 & n & & \\ \hline n-2 & & & \\ \hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & & \\ \hline 1 & & & \\ \hline \end{array} = \frac{n^2(n^2-1)(n^2-4)(n+3)}{144} \quad (26.52)$$

### 26.2.3 Young Projection Operators for $n_b = 1, 2, 3, 4$

Symmetrizers  $\mathcal{S}_p$  and antisymmetrizers  $\mathcal{A}_p$  are discussed in Chapter 22.

In this section, we will use symmetrizers and antisymmetrizers with “holes” A hole, denoted by an empty square, will signify a particle that the symmetrizer or antisymmetrizer does not touch. For example

$$\begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow 1 \\ \parallel \\ \leftarrow \square \leftarrow 2 \\ \parallel \\ \leftarrow \leftarrow 3 \end{array} \quad (26.53)$$

denotes a symmetrizer of the particles 1 and 3 but not 2.

Note that

$$(c, a) = (b, c)(b, a)(b, c) \quad \begin{array}{c} \leftarrow a \\ \uparrow \\ \leftarrow b \\ \downarrow \\ \leftarrow c \end{array} = \begin{array}{c} \leftarrow a \\ \downarrow \\ \leftarrow b \\ \uparrow \\ \leftarrow c \end{array} \quad (26.54)$$

Similarly

$$\begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \\ \parallel \\ \leftarrow \square \leftarrow \\ \parallel \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \leftarrow \mathcal{S}_2 \leftarrow \leftarrow \\ \parallel \\ \leftarrow \leftarrow \leftarrow \leftarrow \\ \uparrow \quad \downarrow \end{array} \quad (26.55)$$

Hence, one can avoid using symmetrizers and antisymmetrizers with holes, if one is willing to use swaps instead of holes.

Below, we use holes, but keep in mind that those holes can be replaced by swaps.

Below, we give the Clebsch-Gordan decomposition of

$$\begin{array}{c} \square^{\otimes n_b} \\ ( \leftarrow )^{\otimes n_b} \end{array} \quad (26.56)$$

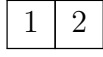
for  $n_b = 1, 2, 3, 4$ .

- $n_b = 1$



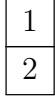
←

- $n_b = 2$



←  $\mathcal{S}_2$  ←

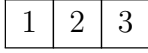
← || ←



←  $\mathcal{A}_2$  ←

← || ←

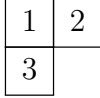
- $n_b = 3$



←  $\mathcal{S}_3$  ←

← || ←

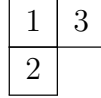
← || ←



←  $\mathcal{S}_2$  ←  $\mathcal{A}_2$  ←

$4/3$  ← || ← □ ←

← || ←



←  $\mathcal{S}_2$  ←  $\mathcal{A}_2$  ←

$4/3$  ← □ ← || ←

← || ←

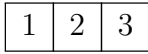


←  $\mathcal{A}_3$  ←

← || ←

← || ←

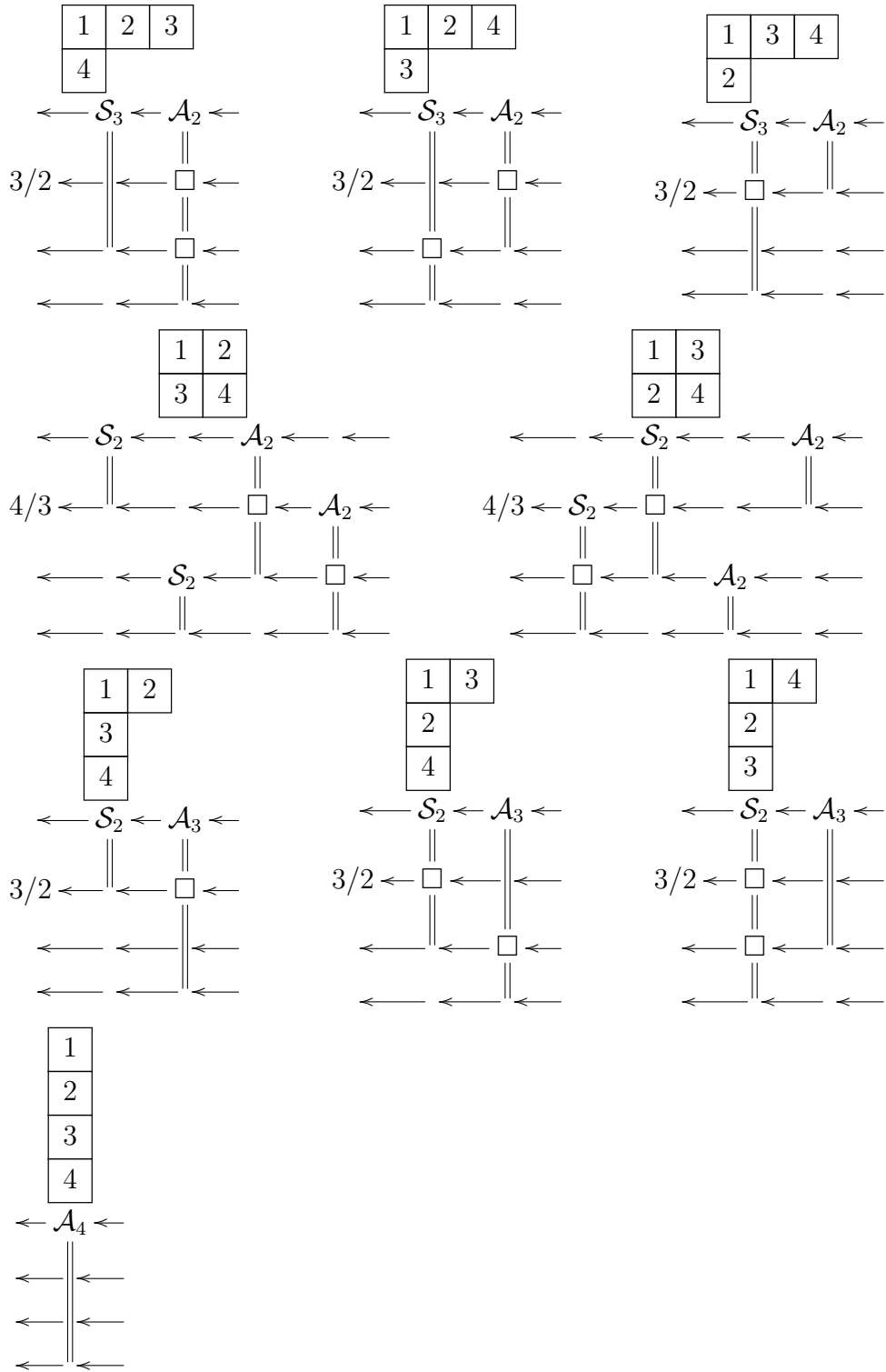
- $n_b = 4$



←  $\mathcal{S}_3$  ←

← || ←

← || ←



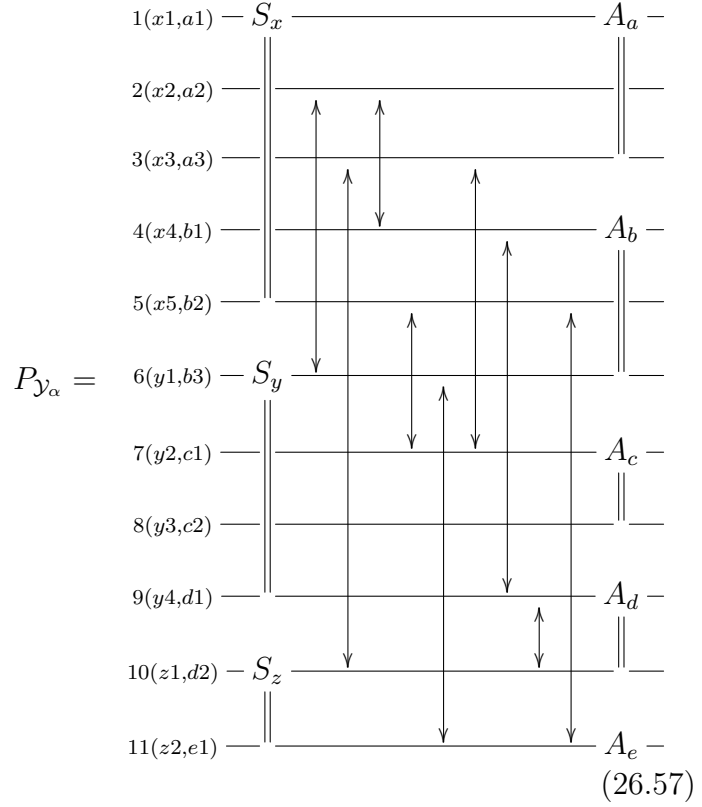
### 26.2.4 Young Projection Operator with Swaps

Eq.(26.57) gives a particular STY  $\mathcal{Y}_\alpha$ , and its projector  $P_{\mathcal{Y}_\alpha}$ . the projector is expressed using swaps instead of holes.

$\mathcal{Y}_\alpha =$

	$A_a$	$A_b$	$A_c$	$A_d$	$A_e$
$S_x$	1	2	3	4	5
$S_y$	6	7	8	9	
$S_z$	10	11			

$a1 \longleftrightarrow 1$
$a2 \longleftrightarrow 6$
$a3 \longleftrightarrow 10$
$b1 \longleftrightarrow 2$
$b2 \longleftrightarrow 7$
$b3 \longleftrightarrow 11$
$c1 \longleftrightarrow 3$
$c2 \longleftrightarrow 8$
$d1 \longleftrightarrow 4$
$d2 \longleftrightarrow 9$
$e1 \longleftrightarrow 5$



### 26.2.5 Tensor Product Decompositions

$$\boxed{1} \otimes \boxed{2} \otimes \boxed{3} = \left( \boxed{1 \ 2} \oplus \boxed{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}} \right) \otimes \boxed{3} \quad (26.58)$$

$$= \boxed{1 \ 2 \ 3} \oplus \boxed{\begin{smallmatrix} 1 \ 2 \\ 3 \end{smallmatrix}} \oplus \boxed{\begin{smallmatrix} 1 \ 3 \\ 2 \end{smallmatrix}} \oplus \boxed{\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}} \quad (26.59)$$

$$\begin{array}{c}
\leftarrow \\
\leftarrow = \\
\leftarrow
\end{array}$$

$$\begin{array}{c}
\leftarrow \mathcal{S}_3 \leftarrow \\
\leftarrow \parallel \leftarrow \\
\leftarrow \parallel \leftarrow
\end{array}
+ \frac{4}{3}
\begin{array}{c}
\leftarrow \mathcal{S}_2 \leftarrow \mathcal{A}_2 \leftarrow \\
\parallel \parallel \\
\leftarrow \parallel \leftarrow \square \leftarrow \\
\leftarrow \parallel \leftarrow
\end{array}
+ \frac{4}{3}
\begin{array}{c}
\leftarrow \mathcal{S}_2 \leftarrow \mathcal{A}_2 \leftarrow \\
\parallel \parallel \\
\leftarrow \parallel \leftarrow \square \leftarrow \\
\leftarrow \parallel \leftarrow
\end{array}
+ \begin{array}{c}
\leftarrow \mathcal{A}_3 \leftarrow \\
\leftarrow \parallel \leftarrow \\
\leftarrow \parallel \leftarrow
\end{array}
\quad (26.60)$$

$$n^3 = \frac{n(n+1)(n+2)}{6} + \frac{n(n^2-1)}{3} + \frac{n(n^2-1)}{3} = \frac{(n-2)(n-1)n}{6} \quad (26.61)$$

$$\begin{array}{c}
\square \square \square \square \otimes \begin{array}{c} \square \square \\ \square \end{array} = \left\{ \begin{array}{c} \begin{array}{c} \square \square \square \square \square \oplus \begin{array}{c} \square \square \square \square \end{array} \\ \square \end{array} \\ \oplus \begin{array}{c} \text{yellow YD} \\ \oplus \begin{array}{c} \text{yellow YD} \end{array} \end{array} \right\} \quad (26.62)
\end{array}$$

For  $SU(n)$ , the yellow YDs are zero for  $n = 2$ , and non-zero for  $n \geq 2$ .

### 26.2.6 $SU(n)$

For  $U(n)$  (as opposed to  $SU(n)$ ), there are no antiparticles (i.e., one can use only lowered indices). A consequence of this is that for proper operators in  $U(n)$ , the total particle number is conserved.

The elements  $G$  of  $SU(n)$  satisfy

$$\underbrace{\epsilon_{12\dots n}}_1 = \underbrace{G_1^{a'_1} G_2^{a'_2} \dots G_n^{a'_n}}_{\det G} \epsilon_{a'_1 a'_2 \dots a'_n} \quad (26.63)$$

$$\epsilon_{a_1 a_2 \dots a_n} = G_{a_1}^{a'_1} G_{a_2}^{a'_2} \dots G_{a_n}^{a'_n} \epsilon_{a'_1 a'_2 \dots a'_n} \quad (26.64)$$

so the Levi-Civita tensor is a primitive invariant of  $SU(n)$  (but not of  $U(n)$ )

This leads to 2 consequences.

1. YD for  $SU(n)$  has a maximum of  $n - 1$  rows.

For an example of this, see Fig.26.2. The yellow columns in that figure are singlets obtained by fully contracting Levi-Civita tensors. Hence, those yellow columns can be removed.

## 2. Conjugate YD

Given a YD  $\mathcal{Y}$ , its **conjugate YD**  $\text{conj}(\mathcal{Y})$  is obtained as follows:

- add yellow colored boxes to the original YD so that the resulting YD is rectangular and has  $n$  rows for each column.
- keep only the yellow boxes, and rotate those clockwise by 180 degrees.

See Fig.26.3 for an example of constructing a conjugate YD.

This is possible because in the intermediate rectangular YD, the columns with  $n$  white and yellow boxes represent a fully contracted Levi-Civita tensor.

**Claim 75** *The reps corresponding to YDs  $\mathcal{Y}$  and  $\text{conj}(\mathcal{Y})$  have the same dimension.*

**proof:**

**QED**

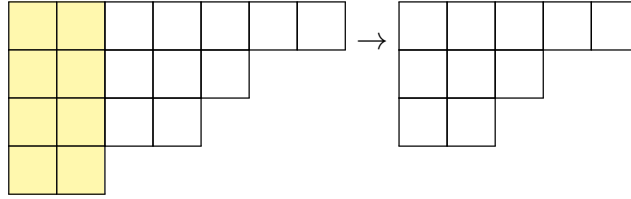


Figure 26.2: Illustration of removal of columns 4 boxes long when dealing with  $SU(4)$ . In this case, the YD in Dynkin notation goes from  $[2, 1, 2, 2, 0, \dots]_D$  to  $[2, 1, 2, 0, \dots]_D$

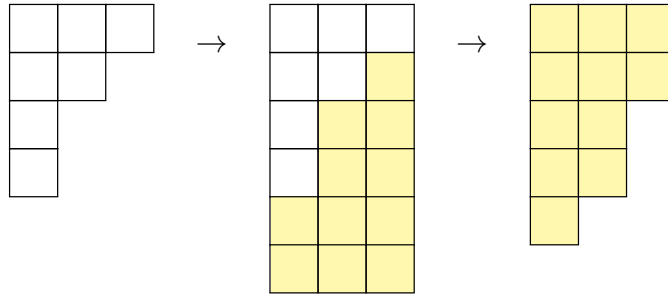


Figure 26.3: Construction of a conjugate YD for  $SU(6)$

Besides the RL (row lengths) and RC/D (row change/Dynkin) methods discussed previously, a third method commonly used to label YDs for  $SU(n)$  is as follows. Label them by their dimension, and then add a subscript or overline if there are more than one reps with a different YD but the same dimension. This method is



used mostly by physicists for  $SU(3)$  (The Eightfold Way). Note that all SYT with the same YD have the same dimension, so this really labels YD instead of YT. For example, for  $SU(3)$  we have

$$\begin{array}{ll}
\begin{array}{|c|} \hline \square \\ \hline \end{array} = [1, 0]_D = 3 & \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} = [0, 1]_D = \bar{3} \\
\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} = [2, 0]_D = 6 & \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = [0, 2]_D = \bar{6} \\
\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} = [1, 1]_D = 8 & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} = [2, 1]_D = 15
\end{array} \tag{26.65}$$

Using this notation, we have for  $SU(n)$ ,

$$\begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \left. \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \vdots \\ \hline \square \\ \hline \end{array} \right\} n-1 \text{ rows} = 1 \oplus \left. \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \vdots & \\ \hline \square & \\ \hline \end{array} \right\} n-1 \text{ rows} \tag{26.66}$$

$$n \otimes \bar{n} = 1 \oplus (n^2 - 1) \tag{26.67}$$

$$\text{fun rep} \otimes \text{conjugate rep} = \text{singlet rep} \oplus \text{adjoint rep} \tag{26.68}$$

### Adjoint representation

$$P_{adj} = \frac{2(n-1)}{n} \quad \begin{array}{c} \leftarrow \mathcal{S}_2 \leftarrow \mathcal{A}_{n-1} \leftarrow \\ \parallel \quad \parallel \\ \leftarrow \leftarrow \square \leftarrow \\ \leftarrow \quad \parallel \quad \leftarrow \\ \leftarrow \quad \parallel \quad \leftarrow \\ \leftarrow \quad \parallel \quad \leftarrow \end{array} \tag{26.69}$$

# Bibliography

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