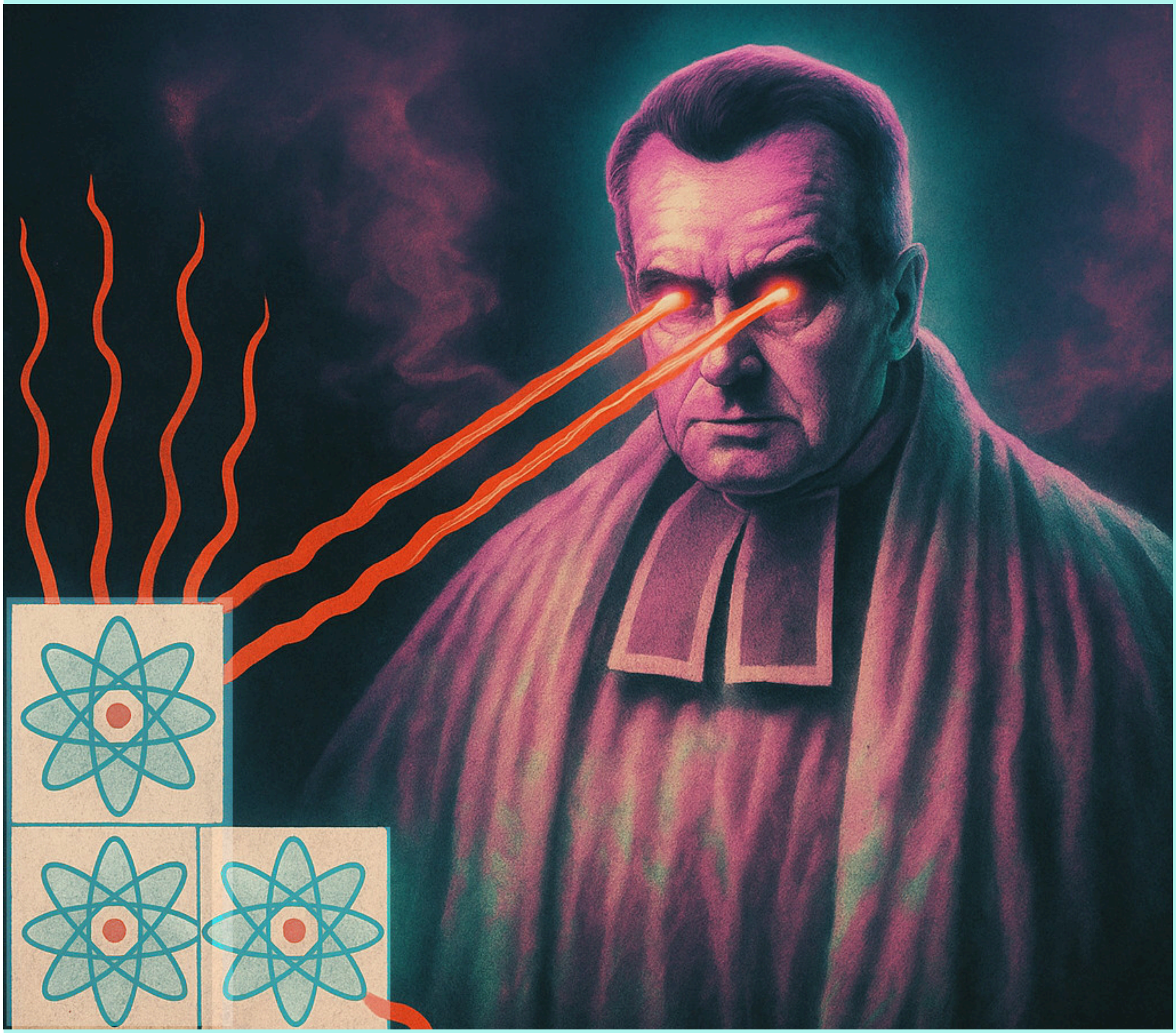
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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August 2, 2025

This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayes Quantico
by Robert R. Tucci

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Appendix A

Notational Conventions and Preliminaries

A.1 Group

A **group** $\mathcal{G}$ is a set of elements with a multiplication map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ such that

1. the multiplication is **associative** ; i.e.,

$$(ab)c = a(bc) \tag{A.1}$$

   for $a, b, c \in \mathcal{G}$.

2. there exists an **identity element** $e \in \mathcal{G}$ such that

$$ea = ae = a \tag{A.2}$$

   for all $a \in \mathcal{G}$

3. for any $g \in \mathcal{G}$, there exists an **inverse** $a^{-1} \in \mathcal{G}$ such that

$$aa^{-1} = a^{-1}a = e \tag{A.3}$$

The number of elements in any set S is denoted by $|S|$. $|\mathcal{G}|$ is called the **order** of the group.

If multiplication is **commutative** (i.e., $ab = ba$ for all $a, b \in \mathcal{G}$, the group is said to be **abelian**.

A **subgroup** $\mathcal{H}$ of $\mathcal{G}$ is a subset of $\mathcal{G}$ ($\mathcal{H} \subset \mathcal{G}$) which is also a group. It's easy to show that any $\mathcal{H} \subset \mathcal{G}$ is a group if it contains the identity and is **closed under multiplication** (i.e., $ab \in \mathcal{H}$ for all $a, b \in \mathcal{H}$)

A.2 Group Representation

A **group representation** of a group $\mathcal{G}$ is a map $\phi : \mathcal{G} \to \mathbb{C}^{n \times n}$[1] such that

$$\phi(a)\phi(b) = \phi(ab) \tag{A.4}$$

Such a map is called a **homomorphism**. When a group is defined using matrices, those matrices are called the **defining representation**. The map ϕ partitions $\mathcal{G}$ into disjoints subsets (equivalence classes), such that all elements of $\mathcal{G}$ in a disjoint set are represented by the same matrix.

For example, the group of **General Linear Transformations** is defined by

$$GL(n, \mathbb{C}) = \{M \in \mathbb{C}^{n \times n} : \det M \neq 0\} \tag{A.5}$$

A.3 Vector Space and Algebra over a field $\mathbb{F}$

A vector (or linear) space $\mathcal{V}$ is defined as a set endowed with two operations: vector addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and scalar multiplication $\mathbb{F} \times \mathcal{V} \to \mathcal{V}$, such that

- $\mathcal{V}$ is an abelian group under $+$ with identity 0 and inverse of $x \in \mathcal{V}$ equal to $-x \in \mathcal{V}$
- For $\alpha, \beta \in \mathbb{F}$ and $x, y \in \mathcal{V}$

$$\begin{aligned} \alpha(x+y) &= \alpha x + \alpha y && \text{(A.6)} \\ (\alpha+\beta)x &= \alpha x + \beta y && \text{(A.7)} \\ \alpha(\beta x) &= (\alpha\beta)x && \text{(A.8)} \\ 1x &= x && \text{(A.9)} \\ 0x &= 0 && \text{(A.10)} \end{aligned}$$

In this book, we will always use either $\mathbb{C}$ or $\mathbb{R}$ for $\mathbb{F}$. Both of these fields are infinite but some fields are finite.

An **algebra** $\mathcal{A}$ is a vector space which, besides being endowed with vector addition and scalar multiplication with which all vector spaces are, it has a **bilinear vector product**. A bilinear vector product is a product that is linear on both sides; i.e.,

$$(\alpha x + \beta y) \cdot z = \alpha x \cdot z + \beta y \cdot z \tag{A.11}$$

and

$$z \cdot (\alpha x + \beta y) = \alpha z \cdot x + \beta z \cdot y \tag{A.12}$$

[1]More generally, the $\mathbb{C}^{n \times n}$ can be replaced by $\mathbb{R}^{n \times n}$ or by $\mathbb{F}^{n \times n}$ for any field $\mathbb{F}$

for $x, y, z \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. The cross product (but not the dot product) for vectors in $\mathbb{R}^3$, the multiplication of 2 complex numbers, and the commutator for square matrices, are all good examples of bilinear vector products.

Let $B = \{\tau_i : i = 1, 2, \ldots, r\}$ be a basis for the vector space $\mathcal{A}$. Then note that B is closed under vector multiplication.

$$\tau_i \cdot \tau_j = \sum_k c_{ij}{}^k \tau_k \tag{A.13}$$

where $c_{ij}{}^k \in \mathbb{C}$. The $c_{ij}{}^k$ are called **structure constants** of B.

An **associative algebra** satisfies $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for $x, y, z \in \mathcal{A}$.

- Not associative: cross product for vectors in $\mathbb{R}^3$.
- Associative: the commutator for square matrices and product of complex numbers

A.4 Tensors

$(x_1, x_2, \ldots, x_n) = x^{:n} \in V^n = \mathbb{C}^{n\times 1}$

Reverse of vector $rev(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_1)$

$y^b = \sum_b g^{ba} x^{:n}$

$(y^1, y^2, \ldots, y^n) = \bar{y}^{:n} \in \bar{V}^n = \mathbb{C}^{n\times 1}$. V^n and $\bar{V}^n$ are **dual vector spaces**.

$M_a{}^b \in \mathbb{C}^{n\times n}$, $a, b \in \mathbb{Z}_{[1,n]}$

Implicit Summation Convention

$$M_a{}^b x_b = \sum_{b=1}^{n} M_a{}^b x_b \tag{A.14}$$

$$\begin{aligned} (M^\dagger)_b{}^a &= (M^*)_a{}^b && \text{(A.15)} \\ &= M_b{}^a \quad \text{(only if } M \text{ is a unitary matrix)} && \text{(A.16)} \end{aligned}$$

For $x_a \in V^n$,

$$(x')_a = M_a{}^b x_b \tag{A.17}$$

For $x^a \in \bar{V}^n$,

$$\begin{aligned} (x'^*)^a &= x^{*b} (M^*)^a{}_b && \text{(A.18)} \\ &= x^{*b} (M^\dagger)_b{}^a && \text{(A.19)} \end{aligned}$$

so

$$(M^\dagger)_b{}^a = (M^*)^a{}_b \tag{A.20}$$

If the Hermitian conjugate $\dagger$ equals $*T$ where $*$ is complex conjugation and T is transpose,

$$(M^T)_b{}^a = M^a{}_b \tag{A.21}$$

This corresponds to flipping M along its horizontal.

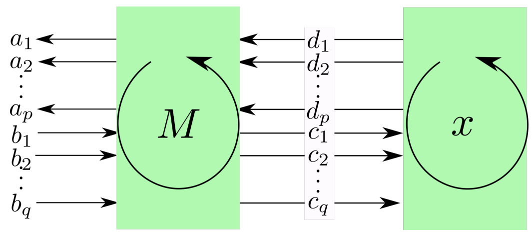


Figure A.1: Index labels for Mx where $M \in \mathbb{C}^{n^{p+q} \times n^{p+q}}$ and $x \in V^{n^p} \otimes \bar{V}^{n^q}$. Note that we list indices in counterclockwise (CC) direction, starting at the top.

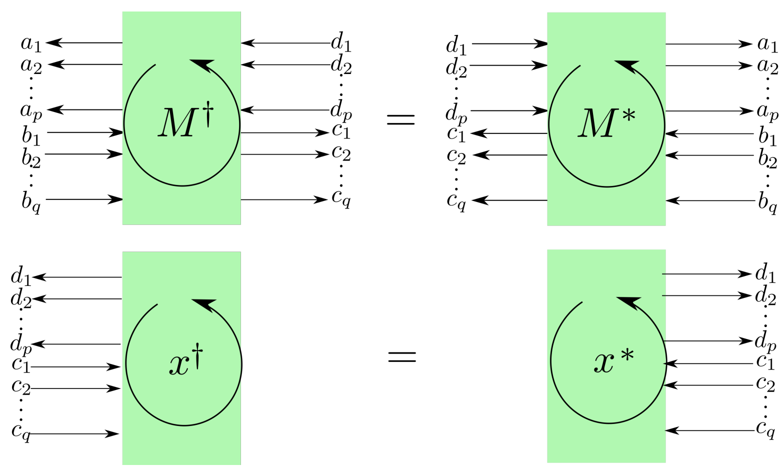


Figure A.2: Index labels for $M^\dagger$ where $M \in \mathbb{C}^{n^{p+q} \times n^{p+q}}$. Note that we list indices in counterclockwise (CC) direction, starting at the top.

Suppose $a_i, b_i, c_i, d_i \in \mathbb{Z}_{[1,n]}$. From Fig.A.1

$$y_{a^{:p}}{}^{b^{:q}} = M_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}} \tag{A.22}$$

$$X_\alpha = X_{a^{:p}}{}^{b^{:q}}, \quad X^\alpha = X_{rev(b^{:q})}{}^{rev(a^{:p})} \tag{A.23}$$

$$x_\alpha = M_\alpha{}^\beta x_\beta \tag{A.24}$$

Hermitian conjugation (see Fig.A.2)

$$\begin{cases} (M^\dagger)_a{}^d = (M^d{}_a)^* \\ (M^\dagger)_\alpha{}^\delta = (M^{rev(\delta)}{}_{rev(\alpha)})^* \end{cases} \tag{A.25}$$

Hermitian matrix

$$M^\dagger = M, \quad \begin{cases} (M^d{}_a)^* = M_a{}^d \\ (M^{rev(\delta)}{}_{rev(\alpha)})^* = M_\alpha{}^\delta \end{cases} \tag{A.26}$$

Note that for $x \in V^n$, $y \in \bar{V}^n$, and $G \in \mathcal{G} \subset GL(n, \mathbb{C})$,

$$(x')_a (y')^b = G^b{}_c G_a{}^d x_d y^c \tag{A.27}$$

If $x \in V^{n^p} \otimes \bar{V}^{n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a^{\cdot p}}{}^{b^{\cdot q}} = \mathbb{G}_{a^{\cdot p}}{}^{b^{\cdot q}}{}_{rev(c^{\cdot q})}{}^{rev(d^{\cdot p})} x_{d^{\cdot p}}{}^{c^{\cdot q}}, \quad (x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta) \tag{A.28}$$

where we define

$$\mathbb{G}_{a^{\cdot p}}{}^{b^{\cdot q}}{}_{rev(c^{\cdot q})}{}^{rev(d^{\cdot p})} \stackrel{\text{def}}{=} \prod_{i=1}^{q} G^{b_i}{}_{c_i} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \tag{A.29}$$

An issue that arises with tensors is this: When is it permissible to represent a tensor by T^{cd}_{ab}? If we define T^{cd}_{ab} by

$$T^{cd}_{ab} = T_{ab}{}^{cd} \tag{A.30}$$

then it's always permissible. Then one can define tensors like $T_a{}^{bcd}$ as

$$T_a{}^{bcd} = g^{bb'} T_{ab'}{}^{cd} = g^{bb'} T^{cd}_{ab'} \tag{A.31}$$

Hence, one drawback of using the notation T^{cd}_{ab} is that if one is interested in using versions of T^{cd}_{ab} with some indices raised or lowered, one has to write down explicitly the metric tensors that do the lowering and raising. Instead of writing $T_a{}^{bcd}$, you'll have to write $g^{bb'} T^{cd}_{ab'}$. This is not very onerous when explaining a topic in which not much lowering and raising of indices is done. But in topics like General Relativity that do use a lot of raising and lowering of indices, it might not be too elegantly concise.

Appendix B

Birdtracks

Cvitanovic Birdtracks book [1]
    Elliott-Dawber book [2]
    My paper “Quantum Bayesian Nets” [3]

B.1 Classical Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $P(y|x) \in [0,1]$ where $x \in val(\underline{x})$ and $y \in val(\underline{y})$

$$\sum_{y \in val(\underline{y})} P(y|x) = 1 \tag{B.1}$$

$$\mathcal{C} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ul] \arrow[ll] \end{tikzcd} \tag{B.2}$$

$$\mathcal{C}(a,b,c) = P(c|b,a)P(b|a)P(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ul] \arrow[ll] \end{tikzcd} \quad P(a) \tag{B.3}$$

$a^{:2} = (a_1, a_2)$

$$\mathcal{C}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^{:2} \arrow[ul, "\underline{a}_1"'] \arrow[ll, "\underline{a}_2"] \end{tikzcd} \tag{B.4}$$

$$\mathcal{C}'(a^{:2},b,c) = P(c|b,a_2)P(a_2|a^{:2})P(b|a_1)P(a_1|a^{:2})P(a^{:2}) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^{:2} \arrow[ul, "a_1"'] \arrow[ll, "a_2"] \end{tikzcd} \quad P(a^{:2}) \tag{B.5}$$

Marginalizer nodes $\underline{a}_1$ and $\underline{a}_2$ have the TPMs

$$P(a'_i|\underline{a}^{:2} = (a_1, a_2)) = \delta(a'_i, a_i) \tag{B.6}$$

for $i = 1, 2$

B.2 Quantum Bayesian Networks and their Instantiations

TPM (Transition Probability Matrix) $A(y|x) \in \mathbb{C}$ where $x \in val(\underline{x})$ and $y \in val(\underline{y})$

$$\sum_{y \in val(\underline{y})} |A(y|x)|^2 = 1 \tag{B.7}$$

$$\mathcal{Q} = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a} \arrow[ll] \arrow[ul] \end{tikzcd} \tag{B.8}$$

$$\mathcal{Q}(a, b, c) = A(c|b, a)A(b|a)A(a) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a \arrow[ll] \arrow[ul] \end{tikzcd} \quad A(a) \tag{B.9}$$

$a^{:2} = (a_1, a_2)$

$$\mathcal{Q}' = \begin{tikzcd} & \underline{b} \arrow[dl] & \\ \underline{c} & & \underline{a}^{:2} \arrow[ll, "\underline{a}_2" description] \arrow[ul, "\underline{a}_1" description] \end{tikzcd} \tag{B.10}$$

$$\mathcal{Q}'(a^{:2}, b, c) = A(c|b, a_2)A(a_2|a^{:2})A(b|a_1)A(a_1|a^{:2})A(a^{:2}) = \begin{tikzcd} & b \arrow[dl] & \\ c & & a^{:2} \arrow[ll, "a_2" description] \arrow[ul, "a_1" description] \end{tikzcd} \quad A(a^{:2}) \tag{B.11}$$

Marginalizer nodes $\underline{a}_1$ and $\underline{a}_2$ have the TAMs

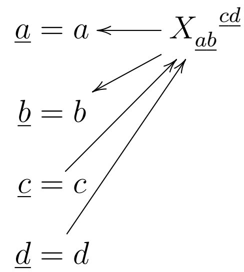
$$A(a'_i|\underline{a}^{:2} = (a_1, a_2)) = \delta(a'_i, a_i) \tag{B.12}$$

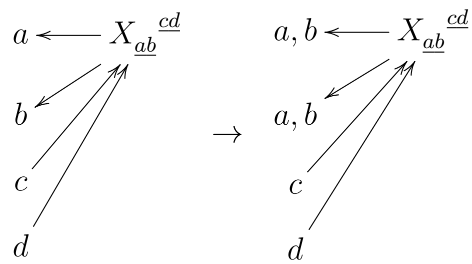
for $i = 1, 2$

B.3 Birdtracks

$$\delta(b,a) = \mathbb{1}(a=b) = \delta_a^b = \; a \leftarrow\!\bullet\!\!- \; b \tag{B.13}$$

$$\langle a,b|X_{\underline{ab}}{}^{\underline{cd}}|c,d\rangle = X_{ab}{}^{cd} = \tag{B.14}$$

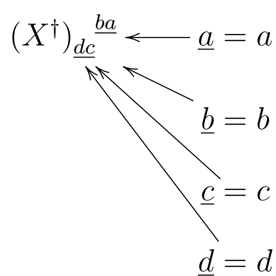




(B.15)

$X_{\underline{ab}}{}^{\underline{cd}} \in V^2 \otimes V_2$. Sometimes, we will omit denote this node simply by X. This if okay as long as we are not using, X to also denote a different version of $X_{\underline{ab}}{}^{\underline{cd}}$ with some of the indices raised or lowered or their order has been changed. [1]

$$(X^\dagger)_{dc}{}^{ba} = \tag{B.16}$$



[1]For matrices, $(A^\dagger)_{i,j} = (A_{j,i})^*$ so taking a Hermitian conjugate involves both taking the complex conjugate of the matrix element and reversing the left-to-right (L2R) order of its indices. This generalizes to $(X^\dagger)_{dc}{}^{ba} = (X_{ab}{}^{cd})^*$. Besides raising and lowering indices, we reverse their L2R order.

$$
(X^\dagger)_{dc}{}^{ba} X_{ab}{}^{cd} = \begin{tikzcd}
(X^\dagger)_{\underline{dc}}{}^{\underline{ba}} \arrow[ddr] \arrow[dddr] & \sum a \arrow[l] & X_{\underline{ab}}{}^{\underline{cd}} \arrow[l] \arrow[dl] \\
& \sum b \arrow[ul] & \\
& \sum c \arrow[uur] & \\
& \sum d \arrow[uuur] &
\end{tikzcd} \tag{B.17}
$$

$$
= \begin{tikzcd}
X^\dagger \arrow[ddr] \arrow[dddr] & {} \arrow[l] & X \arrow[l] \arrow[dl] \\
& {} \arrow[ul] & \\
& {} \arrow[uur] & \\
& {} \arrow[uuur] &
\end{tikzcd} \tag{B.18}
$$

Birdtracks originated as a graphical way to represent the tensors in General Relativity (Gravitation). In General Relativity, one deals with tensors such as $T_a{}^b{}_c$ which have some indices raised and some lowered. One can use the metric $g^{a,b}$ to raise all the lowered indices to get T^{abc}. If we represent this graphically as a node with incoming arrows a, b, c, we need to follow one of the following 2 conventions: either

1. label the arrows as $\underline{a}$, $\underline{b}, \underline{c}$, and define the node as $T^{\underline{abc}}$, or

2. instead of labelling the arrows explicitly $\underline{a}, \underline{b}, \underline{c}$, indicate in the node where is the first arrow $\underline{a}$, and draw the arrows $\underline{a}, \underline{b}, \underline{c}$ so that they enter the node in **counterclockwise** (CC) order. The **left-to-right** (L2R) order of the indices on T corresponds the CC order of the arrows.

If we don't do either 1 or 2, we won't be able to distinguish between the graphical representations of $T^{1,2,3}$ and $T^{2,1,3}$, for example. Cvitanovic's Birdtracks book Ref.[1] follows Convention 2, but most of the time, in this book, we will follow Convention 1 [2] The reason I chose to do so is for the sake of consistency: Convention 2 is closer to the quantum bnet conventions.

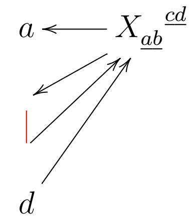
$a^{:m} \in \mathbb{Z}_+^m$

$$
R^{a_3^{:m_3}, b_2^{:n_2}}_{b_3^{:n_3}, a_2^{:m_2}} S^{a_2^{:m_2}, b_1^{:n_1}}_{b_2^{:n_2}, a_1^{:m_1}} = \begin{tikzcd}
b_3^{:n_3} & R \arrow[l] \arrow[dr] & \sum b_2^{:n_2} \arrow[l] & S \arrow[l] \arrow[dr] & b_1^{:n_1} \arrow[l] \\
a_3^{:m_3} \arrow[ur] & & \sum a_2^{:m_2} \arrow[ur] & & a_1^{:m_1}
\end{tikzcd} \tag{B.19}
$$

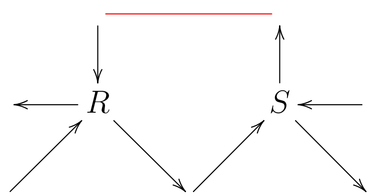
---

[2] If we follow Convention 1, we don't need to reverse the L2R order of the indices when taking a Hermitian conjugate. Thus, $(X^\dagger)^{\underline{ab}}{}_{\underline{cd}} = X_{\underline{ab}}{}^{\underline{cd}} = X^{\underline{dc}}{}_{\underline{ba}}$. As long as $\underline{a}, \underline{b}$ are lower indices and $\underline{c}, \underline{d}$ are upper indices of X, any L2R order of $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ is equivalent under Convention 1.

$$\mathrm{tr}_{\underline{b}} X_{a\underline{b}}{}^{\underline{b}d} = \sum_b X_{ab}{}^{bd} = \tag{B.20}$$



(B.21)



Chapter 1

Casimir Operators: COMING SOON

Chapter 2

Clebsch-Gordan Coefficients

$$\begin{bmatrix} 0 \\ C_\lambda^{d_\lambda \times d} \\ 0 \end{bmatrix}^{d\times d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d \times d} C^{d\times d} \tag{2.1}$$

Let $b^{:nb} = (b_1, b_2, \ldots, b_{nb})$ where $b_i \in Z_{[0, db_i]}$ and $a \in Z_{[1,d_\lambda]}$. Hence,

$$d_\lambda = \prod_{i=1}^{:nb} db_i \tag{2.2}$$

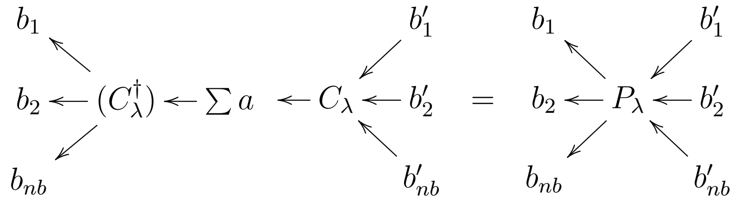
$$(C_\lambda)_a^{rev(b^{:nb})} = \begin{tikzcd} & & b_1 \arrow[dl] \\ a & C_\lambda \arrow[l] & b_2 \arrow[l] \\ & & b_{nb} \arrow[ul] \end{tikzcd} \tag{2.3}$$

$$\begin{bmatrix} 0 & (C^\dagger)_\lambda^{d \times d_\lambda} & 0 \end{bmatrix}^{d\times d} = (C^\dagger)^{d\times d} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I^{d_\lambda \times d_\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{d\times d} \tag{2.4}$$

$$(C_\lambda^\dagger)^a_{b^{:nb}} = \begin{tikzcd} b_1 & & \\ b_2 & (C_\lambda^\dagger) \arrow[l] \arrow[ul] \arrow[dl] & a \arrow[l] \\ b_{nb} & & \end{tikzcd} \tag{2.5}$$

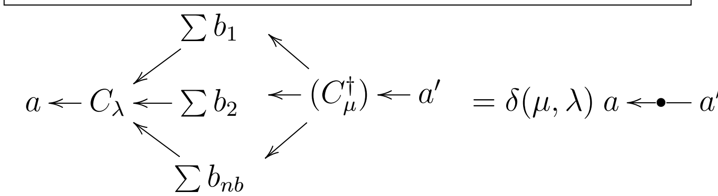
More generally, some of the b_i indices may lowered and their arrows changed to outgoing instead of ingoing. Each b_i represents a different rep (or irrep)

$$\boxed{(C_\lambda^\dagger)^a_{b^{:nb}}(C_\lambda)_a^{rev((b')^{:nb})} = (P_\lambda)^{rev((b')^{:nb})}_{b^{:nb}}, \quad C_\lambda^\dagger C_\lambda = P_\lambda}$$



(2.6)

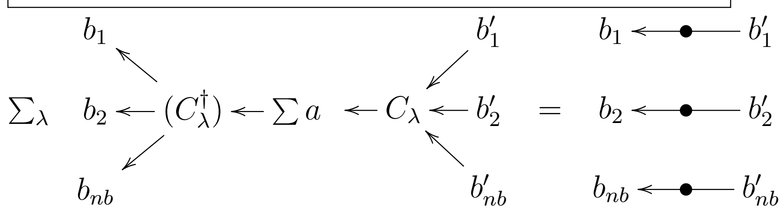
$$\boxed{(C_\lambda)_a^{rev(b^{:nb})}(C_\mu^\dagger)^{a'}_{b^{:nb}} = \delta(\lambda,\mu)\delta_a^{a'}, \quad C_\lambda C_\mu^\dagger = \delta(\mu,\lambda)}$$



(2.7)

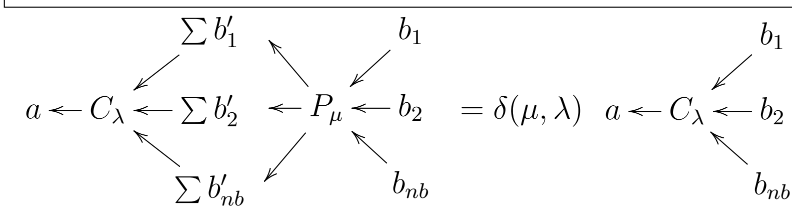
Completeness relation

$$\boxed{\sum_\lambda (C_\lambda^\dagger)^a_{b^{:nb}}(C_\lambda)_a^{rev((b')^{:nb})} = \delta^{rev((b')^{:nb})}_{b^{:nb}}, \quad \sum_\lambda C_\lambda^\dagger C_\lambda = 1}$$



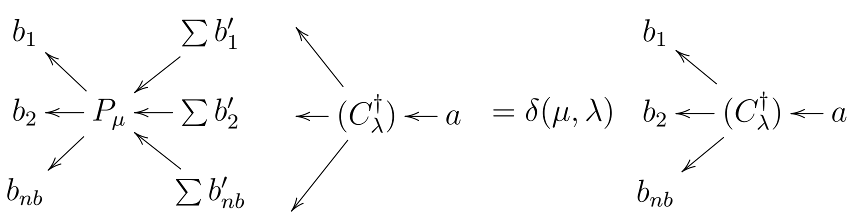
(2.8)

$$\boxed{(C_\lambda)_a^{rev((b')^{:nb})}(P_\mu)^{rev(b^{:nb})}_{(b')^{:nb}} = \delta(\mu,\lambda)(C_\mu)_a^{rev(b^{:nb})}, \quad C_\lambda P_\mu = \delta(\mu,\lambda)C_\mu}$$



(2.9)

$$\boxed{(P_\mu)^{rev((b')^{:nb})}_{b^{:nb}}(C_\lambda^\dagger)^a_{(b')^{:nb}} = \delta(\mu,\lambda)(C_\mu^\dagger)^a_{b^{:nb}}, \quad P_\mu C_\lambda^\dagger = \delta(\mu,\lambda)C_\mu^\dagger}$$



(2.10)

Chapter 3

Determinants: COMING SOON

Chapter 4

Dynkin Diagrams: COMING SOON

Chapter 5

General Relativity Nets: COMING SOON

Chapter 6

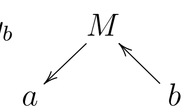
Group Integrals: COMING SOON

Chapter 7

Invariants

Given a **bilinear form**

$$m(\bar{x}^{:n}, y^{:n}) = x^a M_a{}^b y_b \tag{7.1}$$

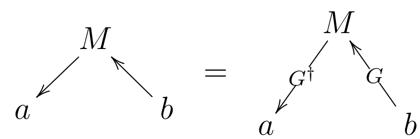


is invariant if

$$m(\bar{x}^{:n}, y^{:n}) = m(\bar{x}^{:n} G^\dagger, G y^{:n}) \tag{7.2}$$

matrix invariant

$$\boxed{M_a{}^b = (G^\dagger)_a{}^{a'} G_{b'}{}^b M_{a'}{}^{b'}} \tag{7.3}$$

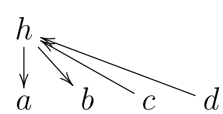


$$M = G^\dagger M G \tag{7.4}$$

$$GM = MG, \quad [G, M] = 0 \tag{7.5}$$

multilinear form

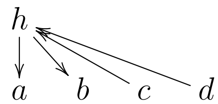
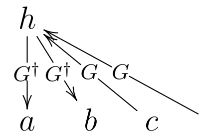
$$h(\bar{w}, \bar{x}, y, z) = h_{ab}{}^{cd} w^a x^b y_c z_d \tag{7.6}$$



is invariant if

$$h(\bar{w}, \bar{x}, y, z) = h(\bar{w} G^\dagger, \bar{x} G^\dagger, G y, G z) \tag{7.7}$$

tensor invariant (TI)

$$\boxed{h_{ab}{}^{cd} = (G^\dagger)_a{}^{a'}(G^\dagger)_b{}^{b'}h_{a'b'}{}^{c'd'}G_{c'}{}^{c}G_{d'}{}^{d}}$$  $=$  (7.8)

A **composed TI** is a TI that can be written as a product or contraction of TIs.

A **tree TI** is a composed TIs without any loops.

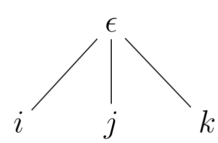
A **primitive TI** is a TI that can be expressed as a linear combination of a finite number of tree TIs.

The **primitiveness assumption**: All TI are primitive.

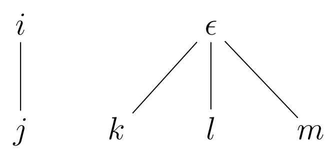
Examples. Consider $\mathbb{R}^3$ vector space.

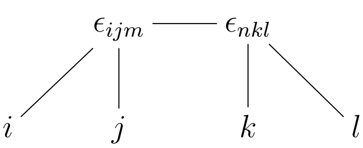
- Primitive TIs

$$length(x) = \delta_{ij}x_i x_i \qquad volume(x,y,z) = \epsilon_{ijk}x_i y_j z_k \tag{7.9}$$

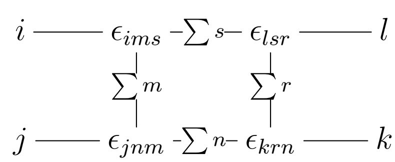
$\delta_{ij} = i$ —— j , $\epsilon_{ijk} =$  (7.10)

- Tree TIs

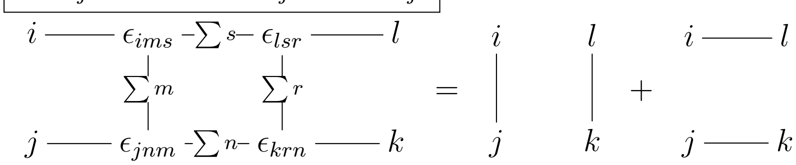
$\delta_{ij}\epsilon_{klm} =$  (7.11)

$\epsilon_{ijm}\delta_{mn}\epsilon_{nkl} =$  (7.12)

- Non-tree TI

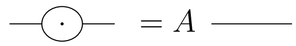
$\epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{lsr} =$  (7.13)

$$\boxed{\epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{lsr} = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}}$$

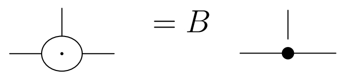
 (7.14)

- Primitiveness Assumption

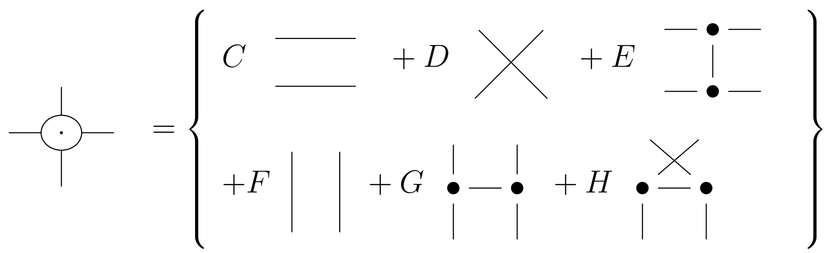
  Suppose $\mathcal{P} = \{\delta_{ij}, f_{ijk}\}$ where f_{ijk} is not ϵ_{ijk}



(7.15)



(7.16)



(7.17)

An **algebra of invariants**

Let $\mathcal{P} = (p_1, p_2, \ldots, p_k)$ be a full set of primitive TIs. By "full", we mean no others exist.

An **invariance group** $\mathcal{G}$ is the set of all linear transformation $G \in \mathcal{G}$ such that

$$p_1(x, \bar{y}) = p_1(Gx, \bar{y}G^\dagger) \tag{7.18}$$

$$p_2(w, x, \bar{y}, \bar{z}) = p_2(Gw, Gx, \bar{y}G^\dagger, \bar{z}G^\dagger) \tag{7.19}$$

$$\text{etc.} \tag{7.20}$$

Example

$$p(\bar{x}, y) = \delta_a^b x^a y_b = x^b y_b \tag{7.21}$$

$$(x')^a (y')_a = x^b (G^\dagger G)_b{}^c y_c = x^b y_b \tag{7.22}$$

So G must be unitary

$$G^\dagger G = 1 \tag{7.23}$$

The group of n dimensional unitary matrices is called $U(n)$

Chapter 8

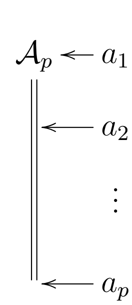
Levi-Civita Tensor

$$\epsilon^{123\ldots p} = \epsilon_{123\ldots p} = 1 \tag{8.1}$$

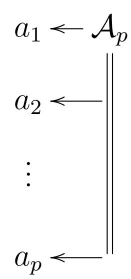
$$\epsilon_{rev(a^{:p})} = (-1)^{\binom{p}{2}} \epsilon_{a^{:p}} \tag{8.2}$$

where $rev(a^{:p})$ is the reverse of $a^{:p}$. $rev(a_1, a_2, \ldots, a_p) = (a_p, a_{p-1}, \ldots, a_1)$

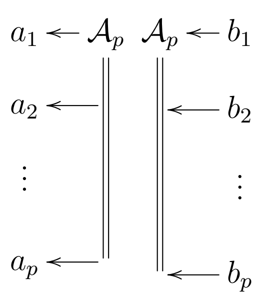
$$(C_{\mathcal{A}_p})_1^{a^{:p}} = e^{i\phi} \frac{\epsilon^{a^{:p}}}{\sqrt{p!}} = \tag{8.3}$$



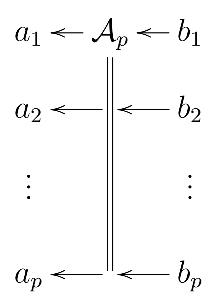
$$(C^\dagger_{\mathcal{A}_p})^1_{a^{:p}} = e^{-i\phi} \frac{\epsilon_{a^{:p}}}{\sqrt{p!}} = \tag{8.4}$$



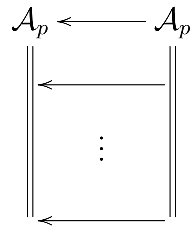
$$\boxed{\frac{1}{p!} \epsilon_{a^{:p}} \epsilon^{b^{:p}} = (\mathcal{A}_p)^{b^{:p}}_{a^{:p}}} \tag{8.5}$$



=



$$\boxed{e^{i2\phi}\frac{1}{p!}\epsilon^{a\colon n}\epsilon_{a\colon n} = \delta^1_1 = 1}$$



$$= 1 \tag{8.6}$$

For Convention 1, we will use $\phi = 0$.
For Convention 2, we must choose

$$e^{i2\phi} = (-1)^{\binom{p}{2}} = e^{i\pi\frac{p(p-1)}{2}} \tag{8.7}$$

so

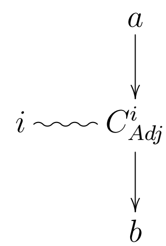
$$\phi = \frac{\pi}{4}p(p-1) \tag{8.8}$$

Chapter 9

Lie Algebras

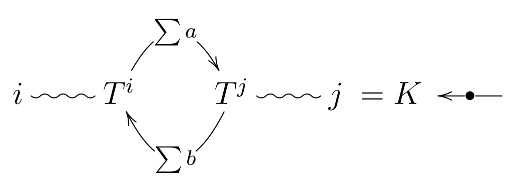
$i \in \mathbb{Z}_{[1,N]}$, $a, b \in \mathbb{Z}_{[1,n]}$

$$(C^i_{Adj})_b{}^a = \frac{1}{\sqrt{K}}(T^i)_b{}^a = \tag{9.1}$$

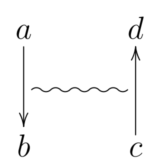


Note that we list the indices of T^i in the counter-clockwise (CC) direction, starting at the i leg. The matrices T^i are called the generators. It's customary to choose them so that they are Hermitian and $K = \frac{1}{2}$. [1]

$$\boxed{(T^i)_b{}^a (T^j)_a{}^b = \mathrm{tr}(T^i T^j) = K\delta(i,j)} \tag{9.2}$$



$$(P_{Adj})^{a,c}_{b,d} = \sum_i \frac{1}{K}(T^i)_b{}^a (T^i)_d{}^c = \frac{1}{K} \tag{9.3}$$



$H \in V^n \otimes \bar{V}^n$

$$(P_{Adj})^{ac}_{bd} H_c{}^d = \sum_i (T^i)_b{}^a \underbrace{\left[\frac{1}{K}(T^i)_d{}^c H_c{}^d\right]}_{h_i \in \mathbb{R}} \tag{9.4}$$

[1] For $SU(2)$, it is customary to use $T^i = \frac{1}{2}\sigma_i$, where σ_i for $i = 1, 2, 3$ are the Pauli matrices. For $SU(3)$, it is customary to choose $T^i = \frac{1}{2}\lambda_i$ where λ_i for $i = 1, 2, \ldots, 8$ are the Gelll-Mann matrices.

$G = 1 + iD \in \mathcal{G}$
$\epsilon_i \in \mathbb{R}$, $|\epsilon_i| << 1$
$D = \sum_i \epsilon_i T^i = V^n \otimes \bar{V}^n$
Recall Eq.(A.28). If $x \in V^{n^p} \otimes \bar{V}^{n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a^{:p}}{}^{b^{:q}} = \mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}}, \quad (x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta) \tag{9.5}$$

where we define

$$\mathbb{G}_\alpha{}^\beta \stackrel{\text{def}}{=} \prod_{i=1}^{q} G^{b_i}{}_{c_i} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \tag{9.6}$$

$$\mathbb{G}_\alpha{}^\beta = 1 + i \sum_j \epsilon_j (\mathbb{T}^j)_\alpha{}^\beta \tag{9.7}$$

$$G^{b_i}{}_{c_i} = 1 + i \sum_j \epsilon_j (T^j)^{b_i}{}_{c_i} \tag{9.8}$$

$$G_{a_i}{}^{d_i} = (G^*)^{d_i}{}_{a_i} = 1 - i \sum_j \epsilon_j (T^{j*})^{d_i}{}_{a_i} = 1 - i \sum_j \epsilon_j (T^j)_{a_i}{}^{d_i} \tag{9.9}$$

When $x'_\alpha = x_\alpha$, to first order in ϵ_i,

$$0 = (\mathbb{T}^j)_\alpha{}^\beta x_\beta = \left[(T^j)^{b_i}{}_{c_i} \frac{1}{\delta^{b_i}_{c_i}} - (T^j)_{a_i}{}^{d_i} \frac{1}{\delta^{d_i}_{a_i}} \right] \delta^{b^{:q}}_{c^{:q}} \delta^{d^{:p}}_{a^{:p}} x_{d^{:p}}{}^{c^{:q}} \tag{9.10}$$

$$\boxed{(\mathbb{T}^j)_\alpha{}^\beta = \left[(T^j)^{b_i}{}_{c_i} \frac{1}{\delta^{b_i}_{c_i}} - (T^j)_{a_i}{}^{d_i} \frac{1}{\delta^{d_i}_{a_i}} \right] \delta^{b^{:q}}_{c^{:q}} \delta^{d^{:p}}_{a^{:p}}} \tag{9.11}$$

Chapter 10

Orthogonal Groups: COMING SOON

Chapter 11

Quantum Shannon Information Theory: COMING SOON

Chapter 12

Recoupling Equations: COMING SOON

Chapter 13

Reducibility

$M \in \mathbb{C}^{d\times d}$

$$M|v\rangle = \lambda|v\rangle \tag{13.1}$$

If M is Hermitian ($H^\dagger = H$), its eigenvalues are real. ($\lambda = \langle\lambda|M\lambda\rangle \in \mathbb{R}$)

$$cp(\lambda) \stackrel{\text{def}}{=} \det(M-\lambda) = 0 \tag{13.2}$$

If M is a Hermitian matrix, then there exists a unitary matric ($CC^\dagger = C^\dagger C = 1$) such that

$$CMC^\dagger = \begin{bmatrix} D_{\lambda_1} & 0 & 0 & 0 \\ 0 & D_{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_{\lambda_r} \end{bmatrix} \tag{13.3}$$

where

$$D_{\lambda_i} = \operatorname{diag}\underbrace{(\lambda_i, \lambda_i, \ldots, \lambda_i)}_{d_i \text{ times}} \tag{13.4}$$

$$d = \sum_{i=1}^{r} d_i \tag{13.5}$$

$$CMC^\dagger = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{13.6}$$

$$CP_1C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{CMC^\dagger - \lambda_2}{\lambda_1 - \lambda_2} \tag{13.7}$$

$$CP_2C^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{CMC^\dagger - \lambda_1}{\lambda_2 - \lambda_1} \tag{13.8}$$

If $I^{d_i \times d_i}$ is the d_i dimensional unit matrix,

$$\begin{aligned} P_i &= C^\dagger diag(0, \ldots, 0, I^{d_i \times d_i}, 0, \ldots, 0)C && (13.9) \\ &= \prod_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} && (13.10) \end{aligned}$$

Note that P_i are Hermitian ($P_i^\dagger = P_i$) because M is Hermitian and its eigenvalues are real.)

Note that P_i and M commute

$$[P_i, M] = P_i M - M P_i = 0 \tag{13.11}$$

orthogonal

$$P_i P_j = \delta(i, j) P_j \tag{13.12}$$

complete

$$\sum_i P_i = 1 \tag{13.13}$$

$$M = \sum_{i=1}^{r} P_i M P_i \tag{13.14}$$

$$d_i = \mathrm{tr} P_i \tag{13.15}$$

$$\begin{aligned} CMP_1C^\dagger &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} && (13.16) \\ &= \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} && (13.17) \end{aligned}$$

$$MP_i = \lambda_i P_i \text{ (no } i \text{ sum)} \tag{13.18}$$

$$f(M) P_i = f(\lambda_i) P_i \text{ (no } i \text{ sum)} \tag{13.19}$$

$M^{(1)}, M^{(2)}$

$$[M^{(1)}, M^{(2)}] = 0 \tag{13.20}$$

Use $M^{(1)}$ to decompose V into $\bigoplus_i V_i$. Use $M^{(2)}$ to decompose V_i into $\bigoplus_j V_{i,j}$. If $M^{(1)}$ and $M^{(2)}$ don't commute, let $P_i^{(1)}$ be the eigenvalue projection operators of $M^{(1)}$. The replace $M^{(2)}$ by $P_i^{(1)} M^{(2)} P_i^{(1)}$

$$[M^{(1)}, P_i^{(1)} M^{(2)} P_i^{(1)}] = 0 \tag{13.21}$$

Chapter 14

Spinors: COMING SOON

Chapter 15

Squashed Entanglement: COMING SOON

Chapter 16

Symplectic Groups: COMING SOON

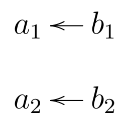
Chapter 17

Symmetrization and Antisymmetrization

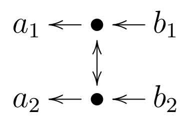
$(1,2)$ transposition, swaps 1 and 2, $1 \to 2 \to 1$. $(3,2,1)$ means $3 \to 2 \to 1 \to 3$. A reordering of $(1,2,3,\ldots,p)$ is a permutation on p letters. A permutation can be expressed as a product of transpositions $(3,2,1) = (3,2)(2,1)$ is an even permutation because it can be expressed as a product of an even number of transpositions. An odd permutation can be expressed as a product of an odd number of permutations.

17.1 Symmetrization

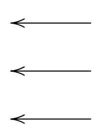
$$\mathbb{1}^{b_2,b_1}_{a_1,a_2} = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2} = \tag{17.1}$$



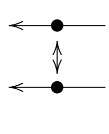
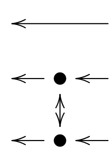
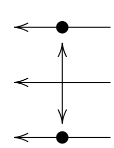
$$(\sigma_{(1,2)})^{b_2,b_1}_{a_1,a_2} = \delta^{b_2}_{a_1}\delta^{b_1}_{a_2} = \tag{17.2}$$



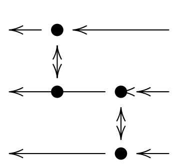
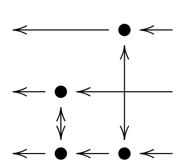
$$\mathbb{1} = \tag{17.3}$$

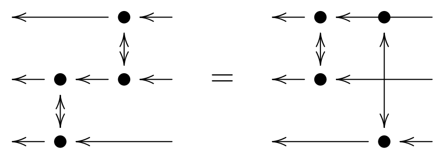


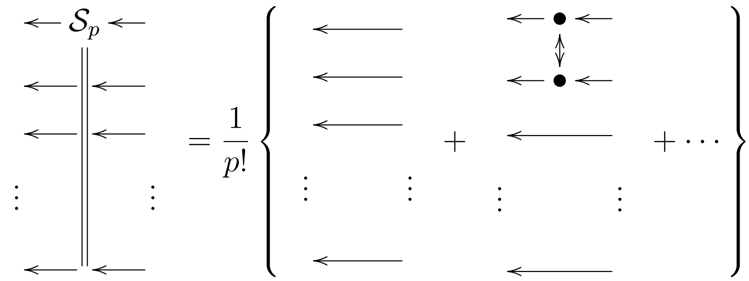
$$\sigma_{(1,2)} = \quad \sigma_{(2,3)} = \quad \sigma_{(1,3)} = \tag{17.4}$$

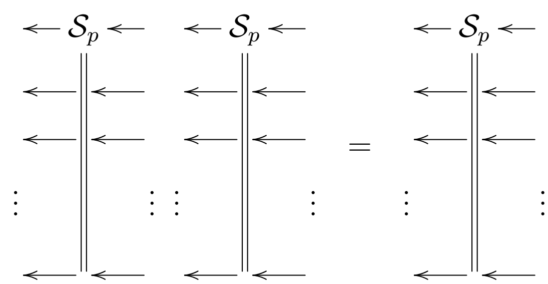




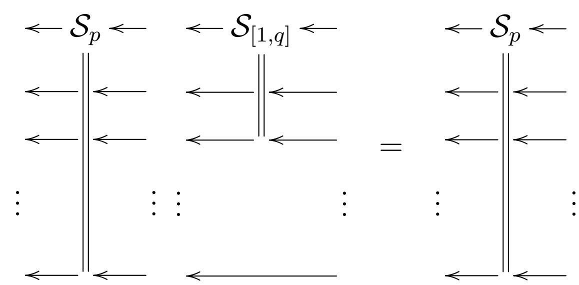
$$\sigma_{(1,2,3)} = \quad = \tag{17.5}$$

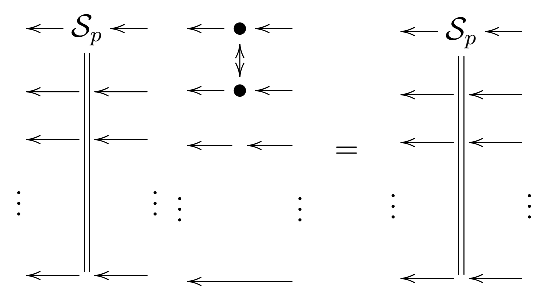



$\sigma_{(1,3,2)} =$  (17.6)

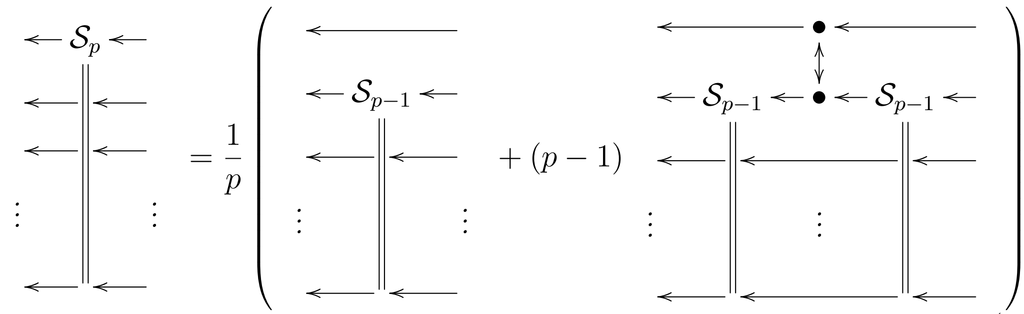
 (17.7)

$\boxed{\mathcal{S}_p^2 = \mathcal{S}_p}$  (17.8)

$\boxed{\mathcal{S}_p\mathcal{S}_{[1,q]} = \mathcal{S}_p}$  (17.9)

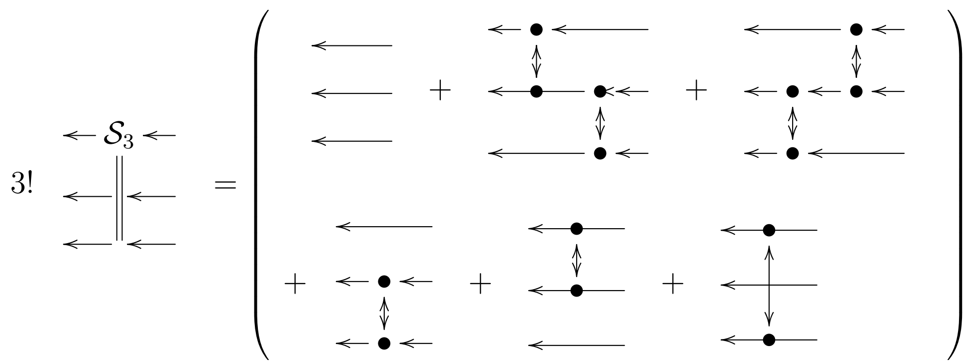
$\boxed{\mathcal{S}_p\sigma_{(1,2)} = \mathcal{S}_p}$  (17.10)

Claim 1



(17.11)

proof: We only prove it for $p = 3$.



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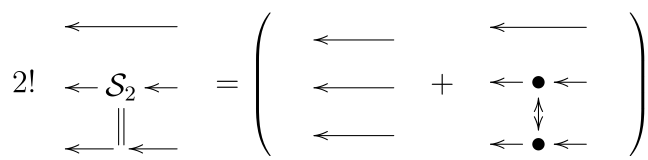

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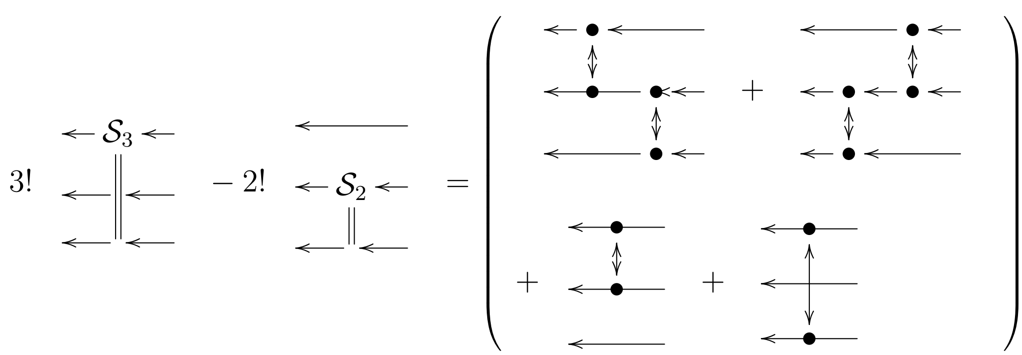
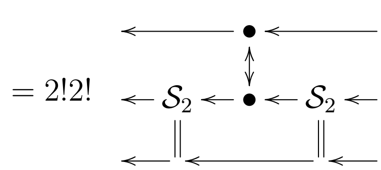
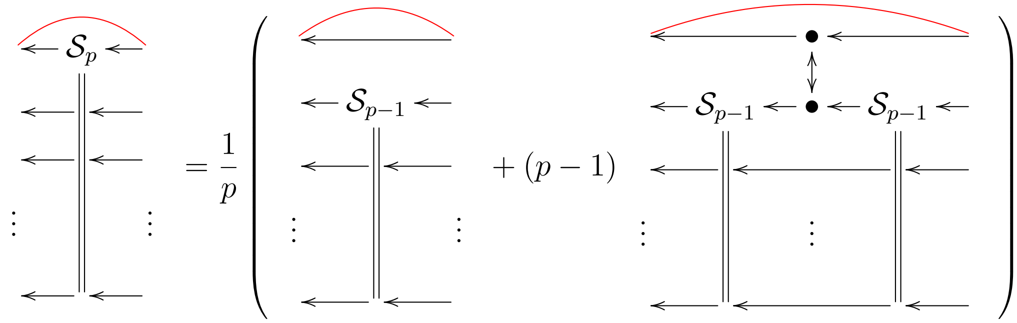


(17.14)

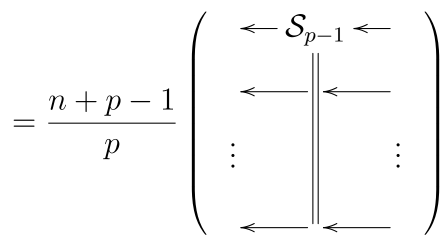


(17.15)

QED



(17.16)



(17.17)

$$\mathrm{tr}_{\underline{a}_1}\mathcal{S}_p = \frac{n+p-1}{p}\mathcal{S}_{p-1} \tag{17.18}$$

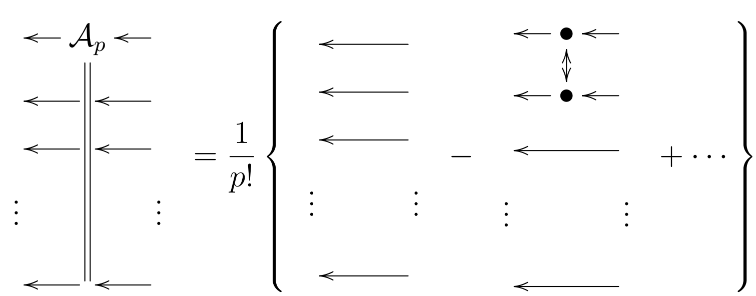
$$\mathrm{tr}_{\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_k}\mathcal{S}_p = \frac{(n+p-1)(n+p-2)\ldots(n=p-k)}{p(p-1)\ldots(p-k+1)}\mathcal{S}_{p-k} \tag{17.19}$$

$$d_{\mathcal{S}_p} = \mathrm{tr}_{\underline{a}^p}\mathcal{S}_p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p} \tag{17.20}$$

For $p = 2$,

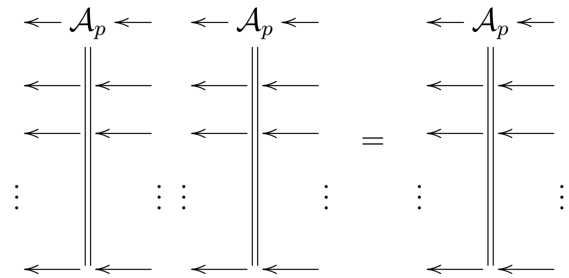
$$d_{\mathcal{S}_2} = \frac{(n+1)n}{2} \tag{17.21}$$

17.2 Antisymmetrization



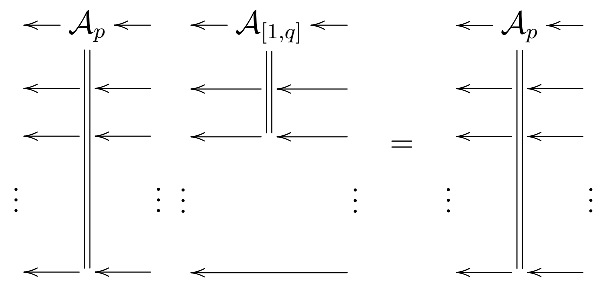
(17.22)

$$\boxed{\mathcal{A}_p^2 = \mathcal{A}_p}$$



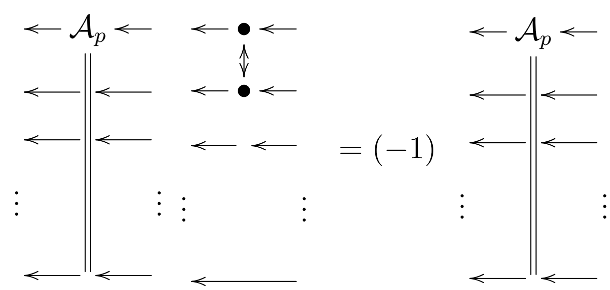
(17.23)

$$\boxed{\mathcal{A}_p\mathcal{A}_{[1,q]} = \mathcal{A}_p}$$



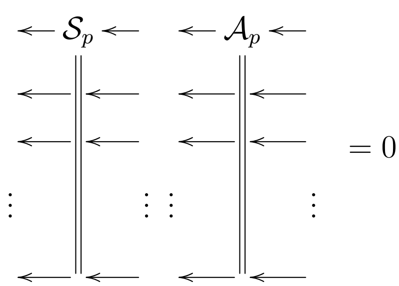
(17.24)

$$\boxed{\mathcal{A}_p\sigma_{(1,2)} = -\mathcal{A}_p}$$



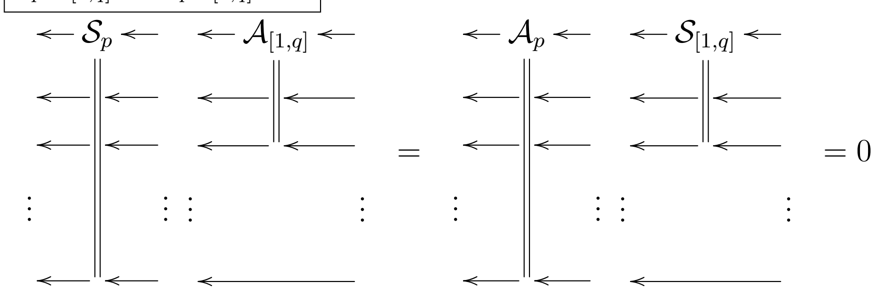
(17.25)

$$\boxed{\mathcal{S}_p\mathcal{A}_q = \mathcal{A}_p\mathcal{S}_q = 0}$$



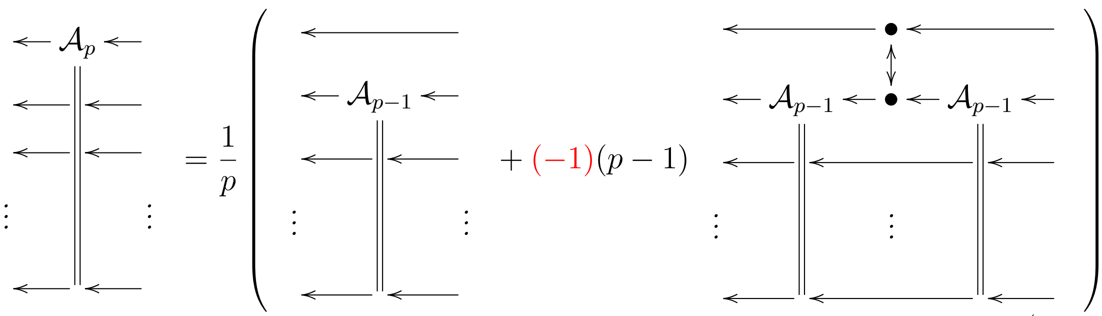
(17.26)

$$\boxed{\mathcal{S}_p\mathcal{A}_{[1,q]} = \mathcal{A}_p\mathcal{S}_{[1,q]} = 0}$$



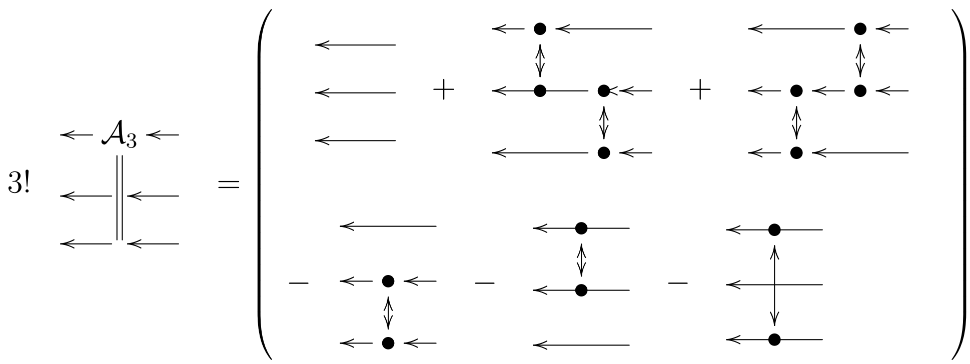
(17.27)

Claim 2

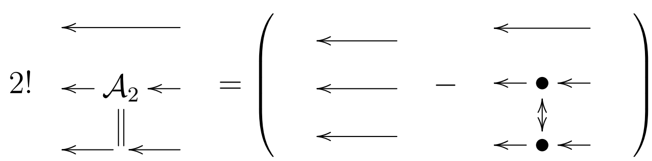


(17.28)

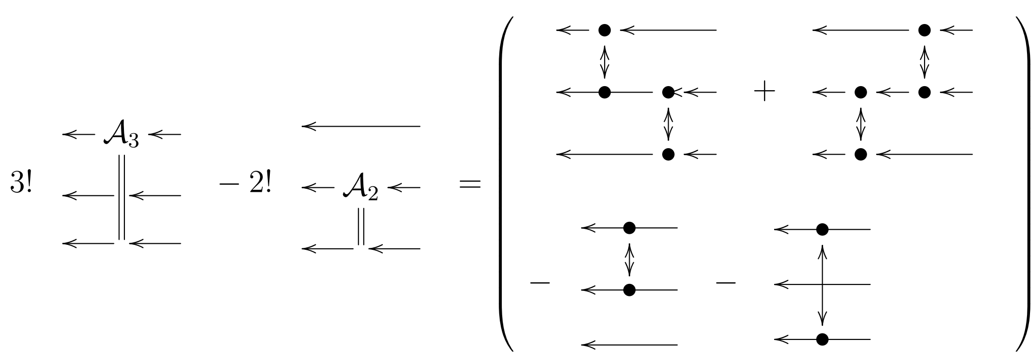
proof: We only prove it for $p = 3$.



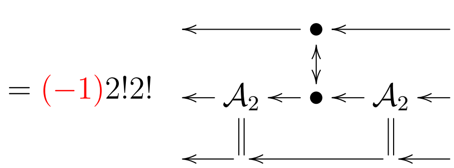
(17.29)



(17.30)

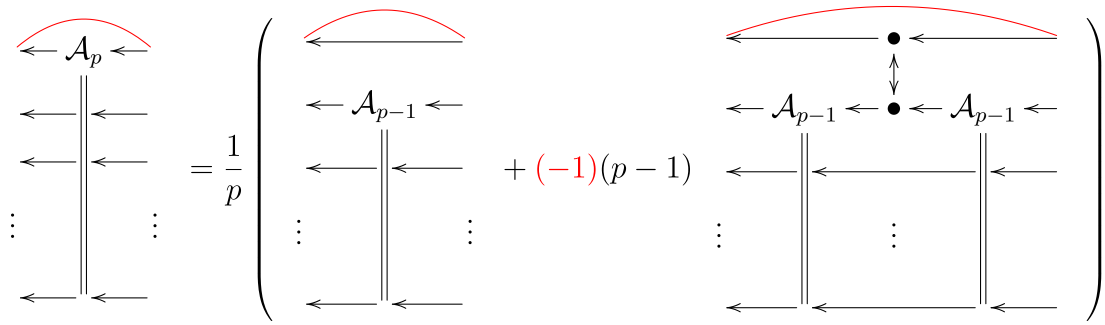


(17.31)



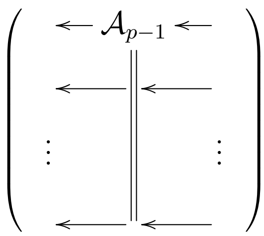
(17.32)

QED



(17.33)

$$= \frac{n + (-1)(p-1)}{p} \tag{17.34}$$



$$\mathrm{tr}_{\underline{a}_1} \mathcal{A}_p = \frac{n-p+1}{p} \mathcal{A}_{p-1} \tag{17.35}$$

$$\mathrm{tr}_{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k} \mathcal{A}_p = \frac{(n-p+1)(n-p+2)\ldots(n-p+k)}{p(p-1)\ldots(p-k+1)} \mathcal{A}_{p-k} \tag{17.36}$$

$$d_{\mathcal{A}_p} = \mathrm{tr}_{\underline{a}^p} \mathcal{A}_p = \frac{\prod_{i=n-p+1}^{n} i}{p!} \tag{17.37}$$

$$= \frac{\prod_{i=n}^{n-p+1} i}{p!} \tag{17.38}$$

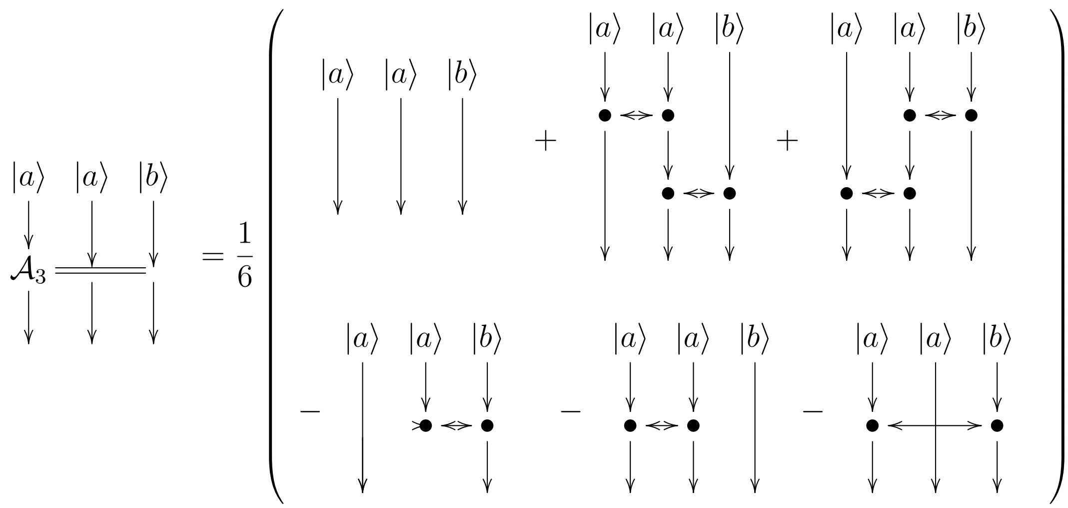
$$= \begin{cases} \frac{n!}{p!(n-p)!} = \binom{n}{p} & \text{if } p \leq n \\ 0 & \text{otherwise} \end{cases} \tag{17.39}$$

For $p = 2 \leq n$,

$$d_{\mathcal{A}_2} = \binom{n}{2} \tag{17.40}$$

$$\mathcal{A}_p = 0 \text{ if } n < p \tag{17.41}$$

For example, for $n = 2$ and $p = 3$



(17.42)

$$
\begin{aligned}
\mathcal{A}_3|a,a,b\rangle &= \frac{1}{6}\begin{pmatrix} |a,a,b\rangle + |a,b,a\rangle + |b,a,a\rangle \\ -|a,b,a\rangle - |a,a,b\rangle - |b,a,a\rangle \end{pmatrix} && (17.43)\\
&= 0 && (17.44)
\end{aligned}
$$

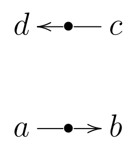
Chapter 18

Unitary Groups: COMING SOON

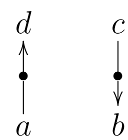
18.1 SU(n)

$$m(p,q) = \delta^a_b \sum_{a=1}^{n} (p_a)^* q_a \tag{18.1}$$

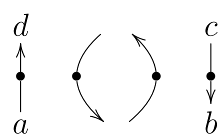
$$\mathbb{1}^{a,c}_{d,b} = \delta^a_b \delta^c_d = \tag{18.2}$$



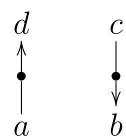
$$\uparrow\downarrow^{a,c}_{d,b} = \delta^a_d \delta^c_b = \tag{18.3}$$



$$\boxed{\uparrow\downarrow^2 = n \uparrow\downarrow} \tag{18.4}$$



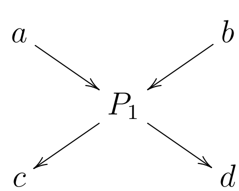
$$= n$$



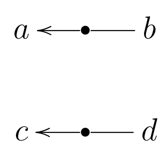
$$P_i = \sum_{j \neq i} \frac{M - \lambda_j}{\lambda_i - \lambda_j} \tag{18.5}$$

$\lambda_1 = n$

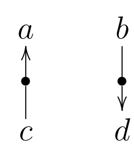
$$\boxed{P_1 = \frac{\uparrow\downarrow - n}{0 - n} = 1 - \frac{1}{n} \uparrow\downarrow} \tag{18.6}$$



$$=$$

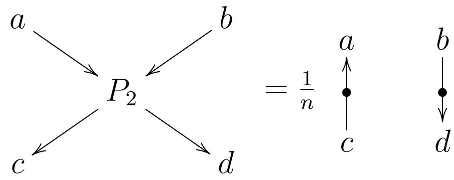


$$- \frac{1}{n}$$



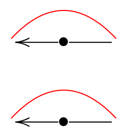
$\lambda_2 = 0$

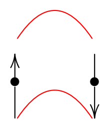
$$\boxed{P_2 = \frac{\uparrow\downarrow - 0}{n - 0} = \frac{1}{n}\uparrow\downarrow}$$



(18.7)

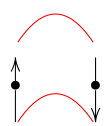
$$\mathrm{tr}P_1 \;=\; \text{[diagram]} \; - \frac{1}{n} \; \text{[diagram]} \tag{18.8}$$





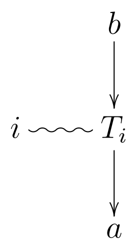
$$= \; n^2 - 1 \tag{18.9}$$

$$\mathrm{tr}P_2 \;=\; \frac{1}{n} \; \text{[diagram]} \tag{18.10}$$



$$= \; 1 \tag{18.11}$$

$$(T_i)_a^b = \tag{18.12}$$



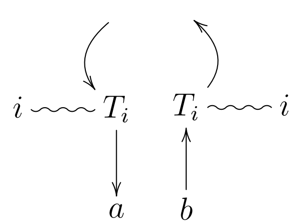
$$T_i^\dagger = T_i \tag{18.13}$$

Claim 3

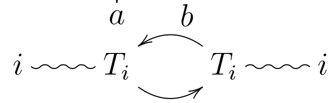
$$C_F \delta_a^b = (T_i T_i)_a^b = \frac{n^2 - 1}{n}\delta_a^b \tag{18.14}$$

proof:

$$(T_i T_i)_a^b \;=\; \sum_i \tag{18.15}$$



$$= \; \sum_i \tag{18.16}$$



QED

Chapter 19

Wigner Coefficients: COMING SOON

Chapter 20

Wigner-Ekart Theorem: COMING SOON

Chapter 21

Young Tableau: COMING SOON

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