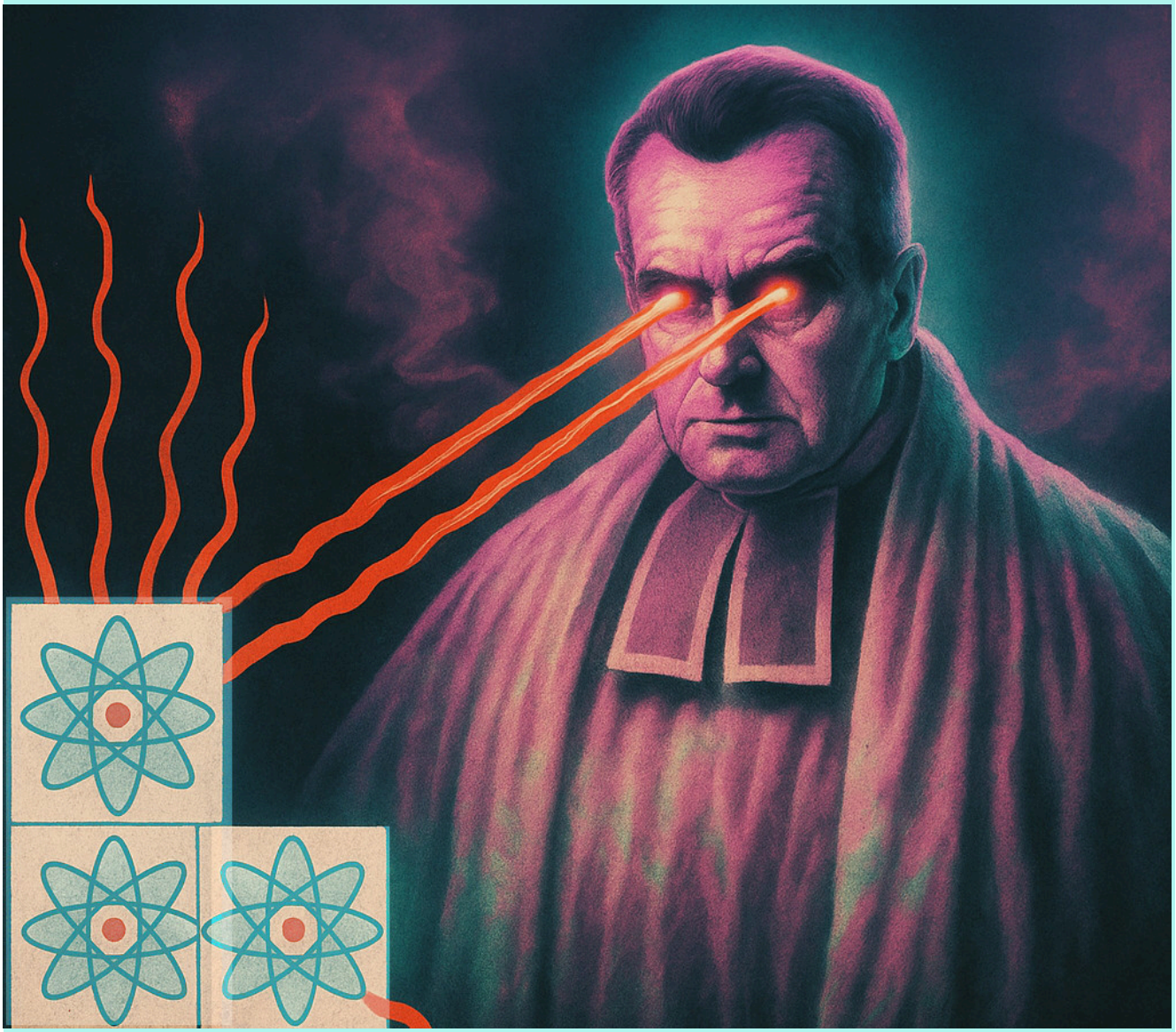
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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December 5, 2025
This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayesuvius Quantico
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Appendices

Chapter 4

Dynkin Diagrams

This chapter is based on Ref.[2], section 20.4.

Lie algebra over reals (real vector space over generators X_r for $r = 1, 2, \ldots \mathcal{D}$)
$\mathcal{D}$ = number of real degrees of freedom, real dimension $\dim_{\mathbb{R}}$ of Lie Algebra

$$[X_q, X_p] = \sum_t f_{qp}{}^t X_t \tag{4.1}$$

$$g_{qs} = \sum_{p,t} f_{qp}{}^t f_{st}{}^p = \quad q \sim\sim f \overset{\frown}{\underset{\smile}{\bigcirc}} f \sim\sim s \tag{4.2}$$

If $\det g = 0$,

$$[X_a, X_b] = 0, \quad [X_q, X_p] = \sum_t f_{qp}{}^t X_t \tag{4.3}$$

Can assume $\det g \neq 0$, Cartan criterion (CC) for group to be semi-simple. If the CC is satisfied, can assume g_{st} is diagonal

$$g_{st} = \delta(s,t) = \quad \sim\sim \tag{4.4}$$

$$f_{qp}{}^t = f_{qpt} \tag{4.5}$$

Will not choose f_{qpt} to be totally antisymmetric

$q_- = 1, 2, \ldots, \mathcal{R}$

$\vec{\alpha} = 1, 2, \ldots, \mathcal{D} - \mathcal{R}$

q = either q_- or $\vec{\alpha}$ but not both.

Let $\{H_{i_-}\}_{i_-=1}^{\mathcal{R}}$ be the largest possible set of mutually commuting X_p. $\mathcal{R}$ is called the **rank** of the group.

$$\boxed{[H_{i_-}, H_{j_-}] = 0} \tag{4.6}$$

Choose $E_{\vec{\alpha}}$ to be eigenvectors of H_{i_-} in the commutator "product"

$$\boxed{[H_{i_-}, E_{\vec{\alpha}}] = \underbrace{\alpha_{i_-}}_{f_{i_-,\vec{\alpha},\vec{\alpha}}} E_{\vec{\alpha}}} \tag{4.7}$$

Then[1]

$$[H_i,[E_{\vec{\alpha}},E_{\vec{\beta}}]] = [[H_i,E_{\vec{\alpha}}],E_{\vec{\beta}}]+[E_{\vec{\alpha}},[H_i,E_{\vec{\beta}}]] \tag{4.8}$$
$$= (\alpha_i+\beta_i)[E_{\vec{\alpha}},E_{\vec{\beta}}] \tag{4.9}$$

If $\vec{\alpha}+\vec{\beta}=0$, $[H_i,[E_{\vec{\alpha}},E_{\vec{\beta}}]]=0$ so

$$\boxed{[E_{\vec{\alpha}},E_{-\vec{\alpha}}]=\sum_{i_-}\underbrace{\alpha^{i_-}}_{f_{\vec{\alpha},-\vec{\alpha},i_-}}H_{i_-}} \tag{4.10}$$

If $\vec{\alpha}+\vec{\beta}\neq 0$,

$$\boxed{[E_{\vec{\alpha}},E_{\vec{\beta}}]=N_{\vec{\alpha},\vec{\beta}}E_{\vec{\alpha}+\vec{\beta}} \quad \text{if } \vec{\alpha}+\vec{\beta}\neq 0} \tag{4.11}$$

$$\alpha^{i_-}=f_{\vec{\alpha},-\vec{\alpha},i_-} \tag{4.12}$$

Dynking Diagram (DD)

$$n=\frac{-2\vec{\alpha}\cdot\vec{\beta}}{\vec{\alpha}\cdot\vec{\alpha}} \tag{4.13}$$

$$p=\frac{-2\vec{\alpha}\cdot\vec{\beta}}{\vec{\beta}\cdot\vec{\beta}} \tag{4.14}$$

$$-\sqrt{\frac{np}{4}}=\hat{\alpha}\cdot\hat{\beta}\in[-1,0] \tag{4.15}$$

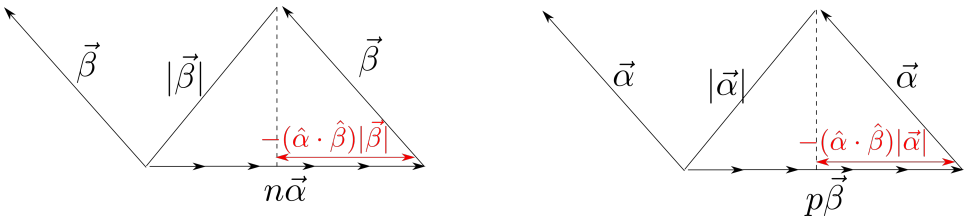

Figure 4.1: Pictorial representation of Eqs.(4.13) and (4.14).

[1]The commutator $[x,y]=xy-yx$ acts like a derivative operator: $[x[a,b]]=[[x,a],b]+[a,[x,b]]$

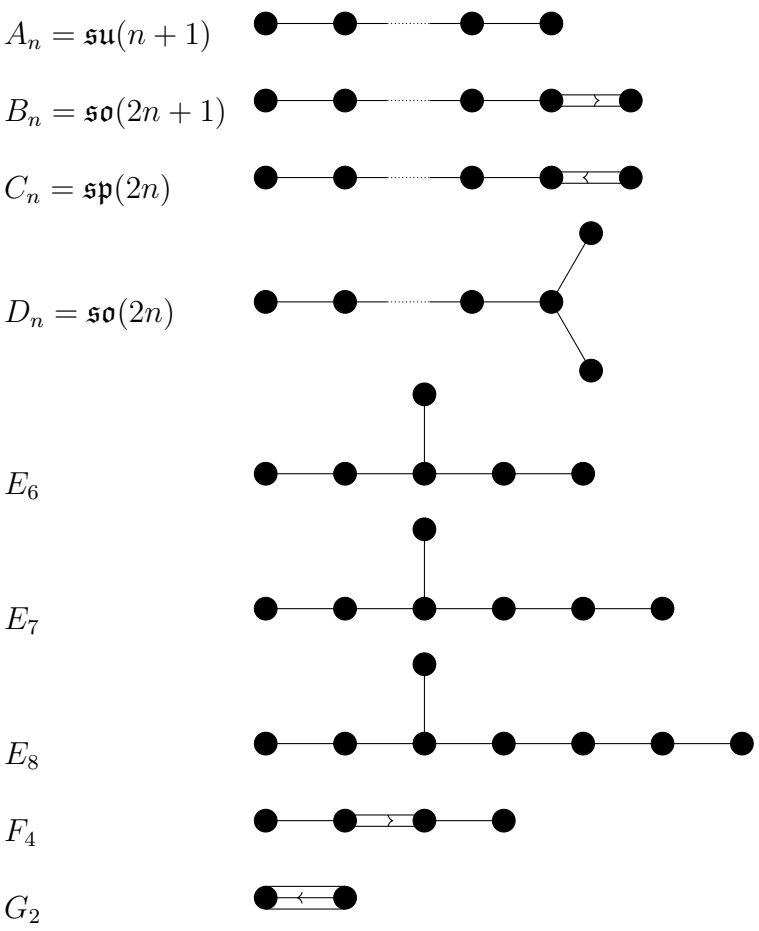

Figure 4.2: Dynkin diagrams for the simple Lie groups

np	$\sqrt{np/4}$	$\arccos\left(-\sqrt{np/4}\right)$
0	0	$\frac{\pi}{2} = 90^o$
1	$\frac{1}{2}$	$\frac{2\pi}{3} = 120^o$
2	$\frac{1}{\sqrt{2}}$	$\frac{3\pi}{4} = 135^o$
3	$\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6} = 150^o$

Table 4.1: Possible root vector angles from Eq.(4.15).

4.1 Examples

- $SO(3)$ and $SU(2)$ have a single dot DD
- $SO(4) \cong SO(3) \times SO(3)$ not a simple Lie algebra, its DD is two disconnected dots
- For $SU(3)$

$$H_1 = T_z, \quad H_2 = \frac{\sqrt{3}}{2} Y \tag{4.16}$$

$$E_{\vec{\alpha}} = \frac{1}{\sqrt{2}} T_+, \quad E_{\vec{\beta}} = \frac{1}{\sqrt{2}} U_+, \quad E_{\vec{\alpha}+\vec{\beta}} = \frac{1}{\sqrt{2}} V_- \tag{4.17}$$

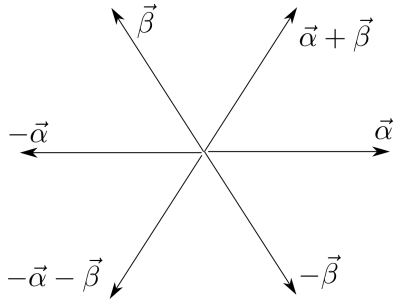

Figure 4.3: Root system for $SU(3)$

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