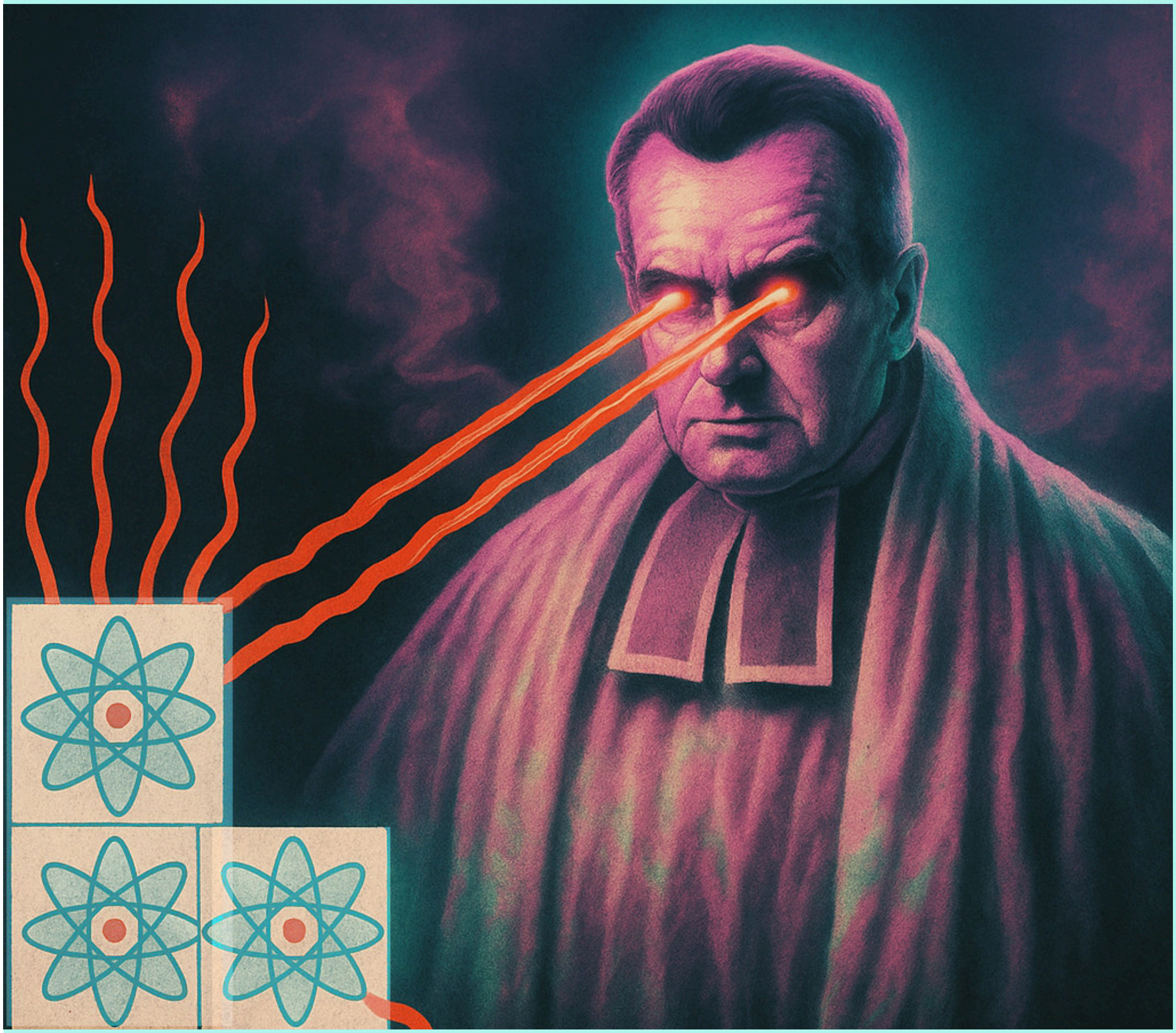
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayesuvius Quantico
by Robert R. Tucci

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Appendices

Chapter 17

Symplectic Groups

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

n even

$$f = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \tag{17.1}$$

$$f^T f = I_n, \quad f^2 = -I_n \tag{17.2}$$

f^2 is a primitive invariant matrix so it must be proportional to the identity.

$$Sp(n) = \{G \in GL(n, \mathbb{C}) : G^T f G = f\} \tag{17.3}$$

Claim 21 *If $G \in Sp(n)$, then $det(G) = 1$.*

proof:

$$det(G^T f G) = det(f) \tag{17.4}$$

$$det() = det(I_{n/2}) det(-I_{n/2}) = (-1)^{n/2} \tag{17.5}$$

$$det(G^T f G) = det^2(G)(-1)^{n/2} \tag{17.6}$$

$$det^2(G) = 1 \tag{17.7}$$

$det(G) = \pm 1$. $Sp(n)$ is connected, $I_n \in Sp(n)$ and $det(I_n) = 1$. Hence $det(G) = 1$.

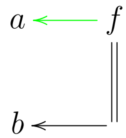
QED

$a, , b \in \{1, 2, \ldots, n\}$. n even

Indices may be raised or lowered without changing tensors. In particular

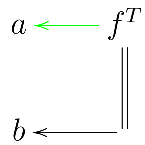
$$f_{ab} = f^a{}_b = f_a{}^b = f^{ab} \tag{17.8}$$

$$f_{ab} =$$

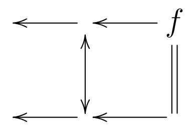


$$= \; a \longleftarrow f \longrightarrow b \tag{17.9}$$

$$f_{ba} = f^T_{ab} =$$

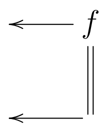


$$=$$



(17.10)

$$\boxed{f_{ab} = -f_{ba}} \, ,$$



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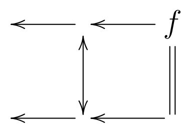

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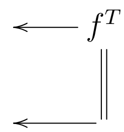


(17.11)

$Sp(n)$ leaves invariant the skew symmetric form

$$h(p,q) = f_{ab}p^a q^b \tag{17.12}$$

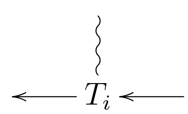
$$h(Gp, Gq) = h(p,q) \implies G^{b'}{}_b G^{a'}{}_a f_{a'b'} = f_{ab} \implies G^T f G = f \tag{17.13}$$

$$\boxed{f^T_{ac} f^{cb} = \delta^b_a} \, , \qquad \longleftarrow f^T \longrightarrow f \longleftarrow \; = \; \longleftarrow \tag{17.14}$$

$$\boxed{f_{ac} f^{cb} = -\delta^b_a} \, , \qquad \longleftarrow f \longrightarrow f \longleftarrow \; = \; - \longleftarrow \tag{17.15}$$

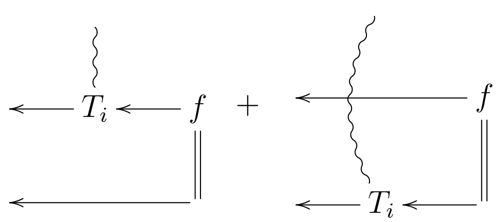
Generator $(T_i)_{ab}$:

$$(T_i)_a{}^b =$$



(17.16)

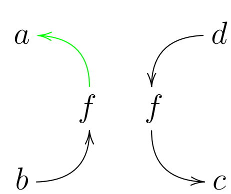
$$\boxed{(T_i)_a{}^c f_{cb} + \underbrace{(T_i)_b{}^c f_{ac}}_{f_{ac}(T_i^T)^c{}_b} = 0}$$



$$= 0$$

(17.17)

$$M_a{}^b{}_c{}^d =$$



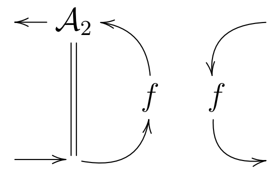
(17.18)

17.1 $V_{def} \otimes V_{def}$ Decomposition

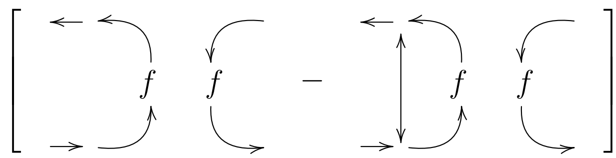
$$M^2 = nM \implies (M-0)(M-n) = 0 \tag{17.19}$$

W has two eigenvalues $0, n$.

$$\mathcal{A}_2 M \quad = \tag{17.20}$$



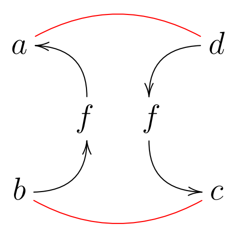
$$= \frac{1}{2} \tag{17.21}$$



$$= \quad M \tag{17.22}$$

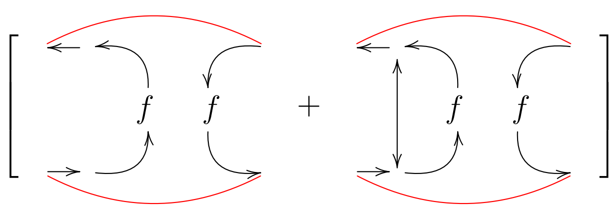
Hence M is ant-symmetric so only anti-symmetric ($\lambda = 0$) space decomposes.

$$\mathrm{tr}(M) = \quad = \mathrm{tr}(f^2) = -n \tag{17.23}$$



$$\mathrm{tr}(\mathcal{A}_2 M) = \mathrm{tr}(M) = -n \tag{17.24}$$

$$\mathrm{tr}(\mathcal{S}_2 M) \quad = \quad \frac{1}{2} \tag{17.25}$$

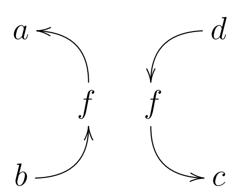


$$= \quad \mathrm{tr}(f^2) + \mathrm{tr}(f^T f) = -n + n = 0 \tag{17.26}$$

Projection operators(PO)

- Singlet PO

$$P_S = \frac{1}{n} f_a{}^b f_c{}^d = \frac{1}{n} \tag{17.27}$$



$$\mathrm{tr}(P_S) = \frac{1}{n}\mathrm{tr}(M) = 1 \tag{17.28}$$

- Traceless Anti-symmetric PO[1]

$$
\begin{aligned}
P_{TA} &= \frac{1}{2}(f_a{}^d f_b{}^c - f_b{}^d f_a{}^c) - \frac{1}{n} f_a{}^b f_c{}^d && (17.29) \\
&= \begin{tikzcd} a & \mathcal{A}_2 \arrow[l] \arrow[d, equal] & f \arrow[l] & d \arrow[l] \\ b & {} \arrow[l] & f \arrow[l] & c \arrow[l] \end{tikzcd} - \frac{1}{n} \begin{tikzcd} a & & & d \arrow[dl, bend right] \\ & f \arrow[ul, bend right] & f \arrow[dr, bend right] & \\ b \arrow[ur, bend right] & & & c \end{tikzcd} && (17.30)
\end{aligned}
$$

- Symmetric PO

$$
\begin{aligned}
P_{SYM} &= \frac{1}{2}(f_a{}^d f_b{}^c + f_b{}^d f_a{}^c) && (17.31) \\
&= \begin{tikzcd} a & \mathcal{S}_2 \arrow[l] \arrow[d, equal] & f \arrow[l] & d \arrow[l] \\ b & {} \arrow[l] & f \arrow[l] & c \arrow[l] \end{tikzcd} && (17.32)
\end{aligned}
$$

$$
\mathrm{tr} P_{SYM} = \mathrm{tr}(\mathcal{S}_2 M) = 0 \qquad (17.33)
$$

[1]Traceless here refers to $P_a{}^a{}_c{}^d V_d{}^c = (PV)_a{}^a = 0$ for any vector $V_d{}^c$. It does not refer to $P_a{}^b{}_b{}^a = 0$

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