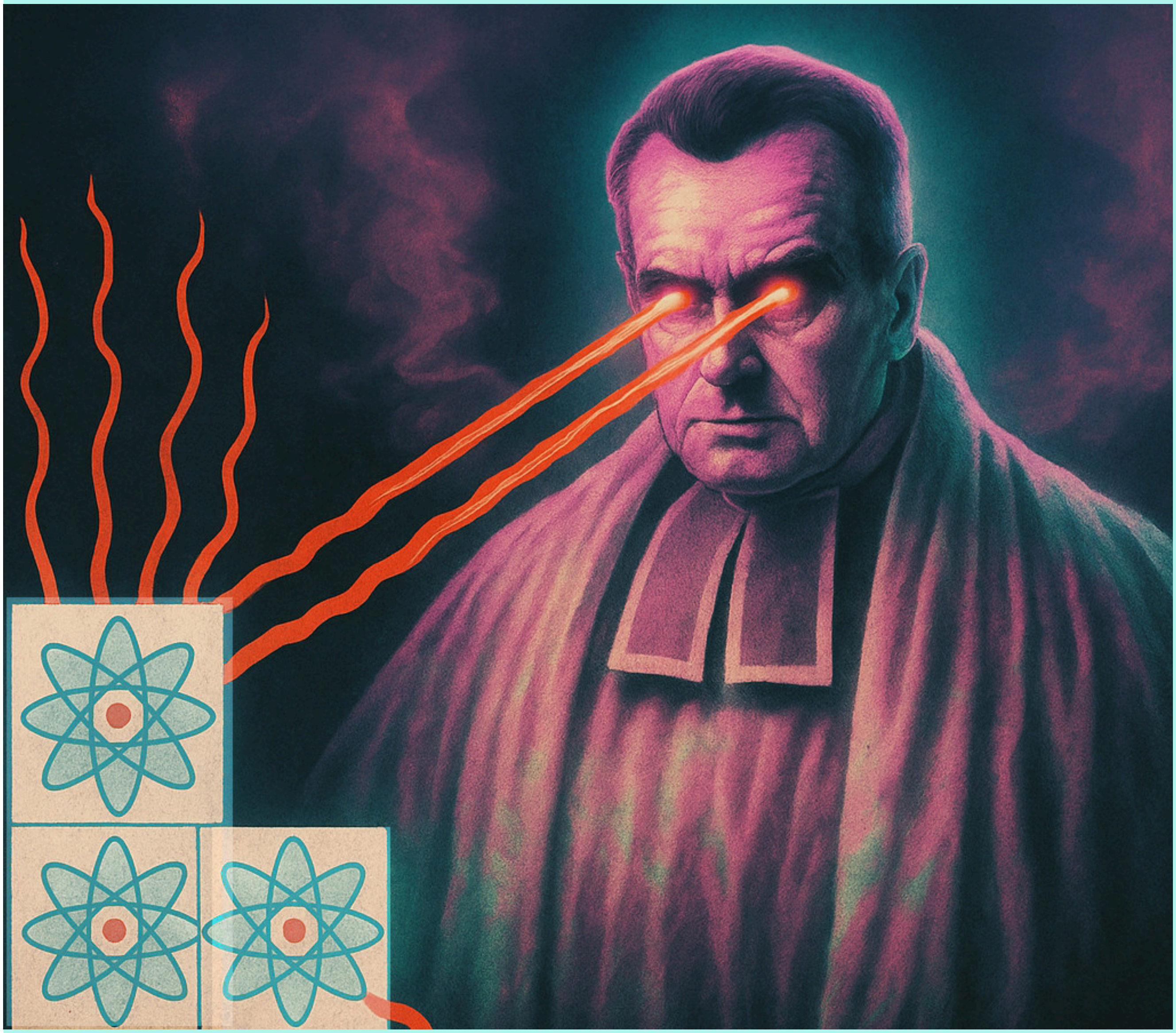
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayesuvius Quantico
by Robert R. Tucci

Contents

Appendices

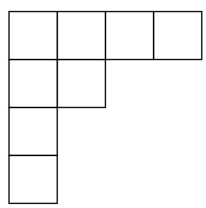
Chapter 20

Young Tableau

This chapter is based on Cvitanovic's Birdtracks book Ref. [1].

We recommend that the read Chapter 17 on symmetrizers and antisymetrizers before reading this one.

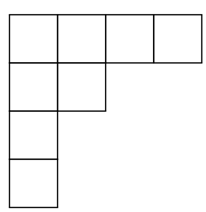
A **Young Diagram** (YD) $\mathcal{Y} = [\lambda_1, \lambda_2, \ldots, \lambda_D]$ consists of λ_1 left-aligned empty boxes (LAEB) over λ_2 LAEB, over λ_3 LAEB, up to λ_{NR} LAEB, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{NR} \geq 1$. $NR =$ number of rows. For example,



$$= [4, 2, 1, 1] \tag{20.1}$$

We will call $[4, 2, 1, 1]$ the **row lengths** (RL) method of labeling YD.

A alternative method of labelling YD is called the **Dynkin (D) labels** or **row changes (RC)**. These labels list the change in number of columns as we go down the YD. For example,



$$= [2, 1, 0, 1, 0 \ldots]_D \tag{20.2}$$

A **Young Tableau** (YT) $\mathcal{Y}_\alpha$ is a YD in which integers from 1 to n where $n \leq n_b$ and n_b is the number boxes, are inserted according to some rules. The rules for insertion are that integers must increase when reading a row left to right and when reading a column from top to bottom. Obviously, for $n < n_b$, some integers are repeated.

A **Standard Young Tableau** (SYT) $\mathcal{Y}_\alpha$ is a YT such that $n = n_b$ and no integer is repeated. Fig.20.1 shows all SYT for $n_b = 1, 2, 3, 4$

- $n_b = 1$

$$\begin{ytableau} 1 \end{ytableau}$$

- $n_b = 2$

$$\begin{ytableau} 1 & 2 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \end{ytableau}$$

- $n_b = 3$

$$\begin{ytableau} 1 & 2 & 3 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$$

- $n_b = 4$

$$\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}$$

Figure 20.1: All SYT for $n_b = 1, 2, 3, 4$.

We will use $|\mathcal{Y}|$, or $|\mathcal{Y}_\alpha|$ or $|\alpha|$ to denote the number of boxes in a YD or YT.[1]

20.1 Symmetric group S_{n_b}

Let

S_{n_b} = the symmetric group in n_b letters (or n_b boxes)
$irreps(S_{n_b})$ = the set of all irreps of S_{n_b}.
The **transpose of a YT** is defined as if it were a matrix. For example

$$transpose\left(\begin{ytableau} 1 & 3 & 5 \\ 2 & 6 \\ 4 \end{ytableau}\right) = \begin{ytableau} 1 & 2 & 4 \\ 3 & 6 \\ 5 \end{ytableau} \tag{20.3}$$

[1]For many authors and for us too, $|S|$ stands for the number of elements in a finite set S. This should not lead to confusion as a YD or YT are not sets.

n-dim General Linear group $GL(n) = \{M \in \mathbb{C}^{n\times n} : det(M) \neq 0\}$
n-dim Special Linear group $SL(n) = \{M \in GL(n) : det(M) = 1\}$
n-dim Unitary group, $U(n) = \{M \in GL(n) : MM^\dagger = M^\dagger M = 1\}$
n-dim Special Unitary group $SU(n) = \{M \in U(n) : det(M) = 1\}$
$YD(n_b)$ = set of YD with n_b boxes. $YD = \cup_{n_n=1}^{\infty} YD(n_b)$.
$SYD(n_b)$ = set of SYD with n_b boxes. $YT = \cup_{n_b=1}^{\infty} YT(n_b)$.
$SYT(n_b, NR)$ = set of STY with n_b boxes and NR rows.
$YT(\mathcal{Y})$ = set of YT with a YD $\mathcal{Y}$.
$SYT(\mathcal{Y})$ = set of SYT with a YD $\mathcal{Y}$.
$dim(\mathcal{Y}|S_{n_b})$ = dimension of irrep $\mathcal{Y}$ of S_{n_b}
$dim(\mathcal{Y}_\alpha|U(n))$ = dimension of irrep $\mathcal{Y}_\alpha$ of $U(n)$ or $SU(n)$.

Claim 30

1. *The YD with n_b boxes label all irreps of the symmetric group S_{n_b}.*

$$irreps(S_{n_b}) = YD(n_b) \tag{20.4}$$

2. *The SYT with n_b boxes and no more than n rows ($NR \leq n$), label the irreps of $GL(n)$ and of $U(n)$*

$$irreps(U(n)) = \bigcup_{n_b \leq n,\ NR \leq n} STY(n_b, NR) \tag{20.5}$$

3. *The SYT with n_b boxes and no more than $n-1$ rows ($NR \leq n-1$), label the irreps of $SL(n)$ and $SU(n)$.*

$$irreps(SU(n)) = \bigcup_{n_b \leq n,\ NR \leq n-1} STY(n_b, NR) \tag{20.6}$$

proof:
QED

20.1.1 $dim(\mathcal{Y}|S_{n_b})$

Claim 31

$$dim(\mathcal{Y}|S_{n_b}) = |SYT(\mathcal{Y})| \tag{20.7}$$

proof:
QED

For example, there are 3 irreps of S_4 associated withe the YD

$$\mathcal{Y} = \begin{array}{|c|c|}\hline & \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array} \tag{20.8}$$

And each of those 3 irreps has dimension 3. Why? Because there are 3 possible SYT for this YD:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array} \implies dim(\mathcal{Y}|S_4) = 3 \tag{20.9}$$

Thus, we can denote the basis vectors of one of these 3 degenerate irreps by

$$|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array}\rangle, \quad |\begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array}\rangle, \quad |\begin{array}{|c|c|}\hline 1 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}\rangle \tag{20.10}$$

To compute $hook(\mathcal{Y})$:

1. Fill each box of the YD with the number of boxes below and to the right of the box, including the box itself.

2. Multiply the numbers in all the boxes.

For example,

$$\mathcal{Y} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \multicolumn{1}{c|}{} \\ \cline{1-3} & \multicolumn{1}{c|}{} \\ \cline{1-2}\end{array} \implies hook(\mathcal{Y}) = \begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & \multicolumn{1}{c|}{1} \\ \cline{1-3} 2 & \multicolumn{1}{c|}{1} \\ \cline{1-2}\end{array} = 6!3 \tag{20.11}$$

Claim 32 (**hook rule** *for computing* $dim(\mathcal{Y}|S_{n_b})$)

$$dim(\mathcal{Y}|S_{n_b}) = \frac{n_b!}{hook(\mathcal{Y})} \tag{20.12}$$

proof:
QED

For example

$$\mathcal{Y} = \begin{array}{|c|c|}\hline & \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array} \implies hook(\mathcal{Y}) = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{1} \\ \cline{1-1}\end{array} = 8 \tag{20.13}$$

so

$$dim(\mathcal{Y}|S_4) = \frac{4!}{4(2)} = 3 \tag{20.14}$$

20.1.2 Regular Representation

The **regular representation** of the symmetric group S_{n_b} is defined as follows. For each permutation $\sigma \in S_{n_b}$, define an independent vector $|\sigma\rangle$ in a vector space $\mathcal{V} = \{|\sigma\rangle : \sigma \in S_{n_b}\} = \{|\sigma_i\rangle : i = 1, 2, \ldots, n_b!\}$. Let

$$|x\rangle = \sum_i x_i |\sigma_i\rangle \tag{20.15}$$

For any $\tau \in S_{n_b}$, suppose

$$\langle \sigma_j|\tau|\sigma_i\rangle = \langle \sigma_j\tau|\sigma_i\rangle \tag{20.16}$$

$$\langle \sigma_j|\tau|x\rangle = \langle \sigma_j\tau|x\rangle = \sum_i x_i \langle \sigma_j\tau|\sigma_i\rangle \tag{20.17}$$

Claim 33 *The regular rep is $n_b!$ dimensional and reducible. In the decomposition of the regular rep of S_{n_b}, each $\lambda \in irreps(S_{n_b})$ appears $dim(\lambda|S_{n_b})$ times.*

proof:
QED

From the last claim, it follows that

$$\begin{aligned} n_b! = |S_{n_b}| &= \sum_{\lambda \in irreps(S_{n_b})} [dim(\lambda|S_{n_b})]^2 \qquad (20.18)\\ &= [n_b!]^2 \sum_{\mathcal{Y} \in YD(n_b)} \frac{1}{[hook(\mathcal{Y})]^2} \quad (\text{Because } |irreps(S_{n_b})| = |YD(n_b)|) \qquad (20.19) \end{aligned}$$

Hence,

$$1 = n_b! \sum_{\mathcal{Y} \in YD(n_b)} \frac{1}{[hook(\mathcal{Y})]^2} \tag{20.20}$$

The Clebsch-Gordan series for the regular rep of S_{n_b} is

$$1 = \sum_{\mathcal{Y} \in YD(n_b)} P_{\mathcal{Y}} \tag{20.21}$$

where each $P_{\mathcal{Y}}$ can be further decomposed into

$$P_{\mathcal{Y}} = \sum_{\mathcal{Y}_\alpha \in SYT(\mathcal{Y})} \underbrace{|\mathcal{Y}_\alpha\rangle\langle\mathcal{Y}_\alpha|}_{P_{\mathcal{Y}_\alpha}} \tag{20.22}$$

The projection operators

$$\{P_{\mathcal{Y}_\alpha} : \mathcal{Y}_\alpha \in STY(\mathcal{Y}), \mathcal{Y} \in YD(n_b)\} = \{P_{\mathcal{Y}_\alpha} : \mathcal{Y}_\alpha \in SYT(n_b)\} \tag{20.23}$$

are complete and orthogonal.

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \tag{20.24}$$

$$\begin{gathered}\ydiagram{3} \otimes \ydiagram{2,1} = \\ \ydiagram{5,1} \oplus \ydiagram{4,2} \oplus \ydiagram{4,1,1} \oplus \ydiagram{3,2,1}\end{gathered} \tag{20.25}$$

20.2 Unitary group $U(n)$

Let

$STY(n_b, NR < n')$ = set of STY with n_b boxes and number of rows $NR < n'$

Recall that[2]

$$irreps(U(n)) = \bigcup_{n_b \leq n,\ NR \leq n} STY(n_b, NR) = \bigcup_{n_b=1}^{n} STY(n_b, NR < n) \tag{20.26a}$$

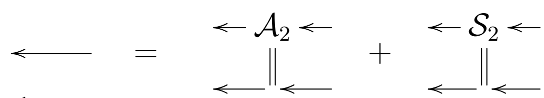
$$irreps(SU(n)) = \bigcup_{n_b \leq n,\ NR \leq n-1} STY(n_b, NR) = \bigcup_{n_b=1}^{n} STY(n_b, NR < n-1) \tag{20.26b}$$

A SYT with n_b boxes represents a tensor with n_b indices (n_b-particles state). Each index ranges from 1 to n.

$n_b = 1$: A 1-index, 1-box tensor is a 1-particle with n states. This corresponds to the fundamental representation.

$n_b = 2$: A 2-index, 2-box tensor is a 2-particle with n^2 states. These n^2 states break into two sets, symmetric and anti-symmetric.

$$\ytableaushort{1} \otimes \ytableaushort{1} = \begin{ytableau} 1 \\ 2 \end{ytableau} \oplus \ytableaushort{12} \tag{20.27}$$



(20.28)

[2]Note that $STY(n_b)$ only contains STY with $n_b \leq n$ boxes, so the $n_b \leq n$ constraint might seem redundant in Eqs.(20.26). It isn't redundant because by $\cup_{n_b \leq n}$ we mean $\cup_{n_b=1}^{n}$.

The SYT of an irrep describes the symmetry of the indices of a tensor in that irrep. A single column SYT indicates a totally anti-symmetric tensor, a single row SYT indicates a totally symmetric tensor, and a SYT with more than one row or column indicates a mixed symmetry tensor. This is why we can't have more than n rows, because there are only n integers to fill all boxes so more than n rows would require a repetition of an integer in a column, and such a column, after antisymetrizing, would lead to zero.

20.2.1 Young Projection operators

For each SYT $\mathcal{Y}_\alpha \in irreps(U(n))$, define the **Young projection operator**

$$P_{\mathcal{Y}_\alpha} = \mathcal{N}\left(\prod_i S_i\right)\left(\prod_j A_j\right) \tag{20.29}$$

for some normalization constant $\mathcal{N}$ yet to bee determined. These projection operators are not unique.

Claim 34

$$\mathcal{N} = \frac{\left(\prod_i |S_i|!\right)\left(\prod_j |A_j|!\right)}{hook(\mathcal{Y})} \tag{20.30}$$

where $|S_i|$ and $|A_j|$ are the number of arrows entering the symmetrizer or anti-symmetrizer. Note that the normalization constant $\mathcal{N}$ depends only on the YD $\mathcal{Y}$. Furthermore, the operators $P_{\mathcal{Y}_\alpha}$ are idempotent (i.e., their square equals themselves) mutually orthogonal and complete:

$$P_{\mathcal{Y}_\alpha} P_{\mathcal{Y}_\beta} = P_{\mathcal{Y}_\alpha}\delta(\alpha,\beta) \tag{20.31}$$

$$1 = \sum_{\mathcal{Y}_\alpha \in SYT(n_b,\, NR<n')} P_{\mathcal{Y}_\alpha} \tag{20.32}$$

where

$$n' = \begin{cases} n & for\ U(n) \\ n-1 & for\ SU(n) \end{cases} \tag{20.33}$$

proof:

$$P_{\mathcal{Y}_\alpha} = \mathcal{N}\frac{1}{\prod_i |S_i|!\prod_j |A_j|!}\left(\underbrace{\begin{array}{l}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{array}}_{\mathbb{1}} + \cdots\right) \tag{20.34}$$

From Eq.(20.32)

$$
\begin{aligned}
\mathbb{1} &= \sum_{\mathcal{Y}_\alpha \in SYT(n_b, NR<n')} \mathcal{N} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} && (20.35) \\
&= \sum_{\mathcal{Y} \in YD(n_b)} \frac{n_b!}{hook(\mathcal{Y})} \mathcal{N} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} && (20.36) \\
&= \sum_{\mathcal{Y} \in YD(n_b)} \frac{n_b!}{[hook(\mathcal{Y})]^2} \frac{1}{\prod_i |S_i|! \prod_j |A_j|!} \quad \mathbb{1} \quad \text{(if assume Eq.(20.30))} && (20.37) \\
&= \mathbb{1} \quad \text{(by Eq.(20.20))} && (20.38)
\end{aligned}
$$

QED

20.2.2 $dim(\mathcal{Y}_\alpha | U(n))$

Let $dim(\mathcal{Y}_\alpha | U(n))$ be the dimension of an irrep of $U(n)$ with STY given by $\mathcal{Y}_\alpha \in STY(n_b, NR < n)$.

Claim 35

$$dim(\mathcal{Y}_\alpha | U(n)) = |YT(\mathcal{Y})| \tag{20.39}$$

Note that the right hand side is independent of α, so this dimension is the same for all irreps α with the same YD $\mathcal{Y}$.

proof:
QED

Hence, $\{|\alpha\rangle : \alpha \in YT(\mathcal{Y})\}$ are a basis for the irrep $\mathcal{Y}_\alpha$ of $U(n)$. Note that the irreps of $U(n)$ are given by SYT $\mathcal{Y}_\alpha$, whereas the basis vectors of an irrep are given by YT (not just STY). For example, for

$$\mathcal{Y}_\alpha = \boxed{1}\boxed{2} \tag{20.40}$$

the basis vectors are

$$|\boxed{1}\boxed{1}\rangle, \quad |\boxed{1}\boxed{2}\rangle, \quad |\boxed{2}\boxed{2}\rangle \tag{20.41}$$

so

$$dim(\mathcal{Y}_\alpha | U(2)) = 3 \tag{20.42}$$

In Eq.(20.39) we gave a way of finding $dim(\mathcal{Y}_\alpha | U(n))$ A second way is by taking the trace of the corresponding projection operator

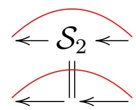
$$dim(\mathcal{Y}_\alpha | U(n)) = \mathrm{tr}(P_{\mathcal{Y}_\alpha}) \tag{20.43}$$

For example, if

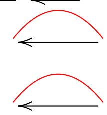
$$\mathcal{Y}_\alpha = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \tag{20.44}$$

then

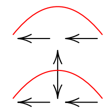
$$dim(\mathcal{Y}_\alpha|U(n)) = \tag{20.45}$$



$$= \frac{1}{2}\Bigg(\tag{20.46}$$



$$+$$



$$\Bigg)$$

$$= \frac{1}{2}(n^2+n) \tag{20.47}$$

$$= 3 \text{ for } n = 2 \tag{20.48}$$

A third way of computing $dim(\mathcal{Y}_\alpha|U(n))$ is by computing the hook and coat functions and using the formula

$$dim(\mathcal{Y}_\alpha|U(n)) = \frac{coat(\mathcal{Y})}{hook(\mathcal{Y})} \tag{20.49}$$

Note that right hand side is independent of α; it depends only on the YD. We've already discussed how to compute $hook(\mathcal{Y})$. $coat(\mathcal{Y})$ is calculated as follows.[3]

1. Fill $\mathcal{Y}$ with
   - n at the diagonal blocks
   - n increments increasing by 1 when reading from left to right
   - n increments decreasing by 1 when reading from top to bottom
2. multiply all the boxes

   Examples

$$dim(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, U(2)) = \frac{\begin{array}{|c|c|}\hline \text{n} & \text{n+1} \\ \hline\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}} = \frac{n(n+1)}{2} \tag{20.50}$$

$$dim(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, U(2)) = \frac{\begin{array}{|c|}\hline \text{n} \\ \hline \text{n-1} \\ \hline\end{array}}{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}} = \frac{n(n-1)}{2} \tag{20.51}$$

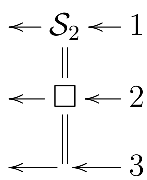
[3]I invented the name $coat(\mathcal{Y})$. I don't know if it has a name.

$$dim\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&6\\ \cline{1-2} 7\\ \cline{1-1}\end{array}, U(7)\right) = \frac{\begin{array}{|c|c|c|c|}\hline n&n+1&n+2&n+3\\ \hline n-1&n\\ \cline{1-2} n-2\\ \cline{1-1}\end{array}}{\begin{array}{|c|c|c|c|}\hline 6&4&2&1\\ \hline 3&1\\ \cline{1-2} 1\\ \cline{1-1}\end{array}} = \frac{n^2(n^2-1)(n^2-4)(n+3)}{144} \tag{20.52}$$

20.3 Young Projection operators for $n_b = 1, 2, 3, 4$

Symmetrizers $\mathcal{S}_p$ and antisymmetriers $\mathcal{A}_p$ are discussed in Chapter 17.

In this section, we will use symmetrizers and antisymmetrizers with "holes" Holes, denoted by empty square, will signify particles the symmetrizer or antisymmetrizer does not touch. For example

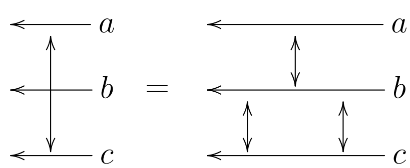


(20.53)

denotes a symmetrizer of the particles 1 and 3 but not 2.

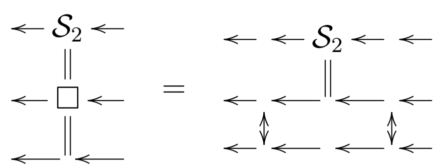
Note that

$(c, a) = (b, c)(b, a)(b, c)$



(20.54)

Similarly

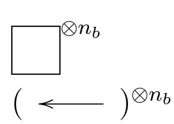


(20.55)

Hence, one can avoid using symmetrizers and antisymmetrizers with holes, if one is willing to use swaps instead of holes.

Below, we use holes, but keep in mind that those holes can we replaced by swaps.

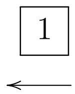
Below, we give the Clebsch decomposition of



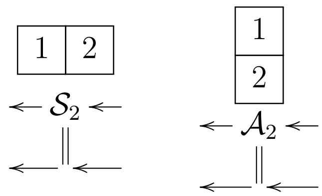
(20.56)

for $n_b = 1, 2, 3, 4$.

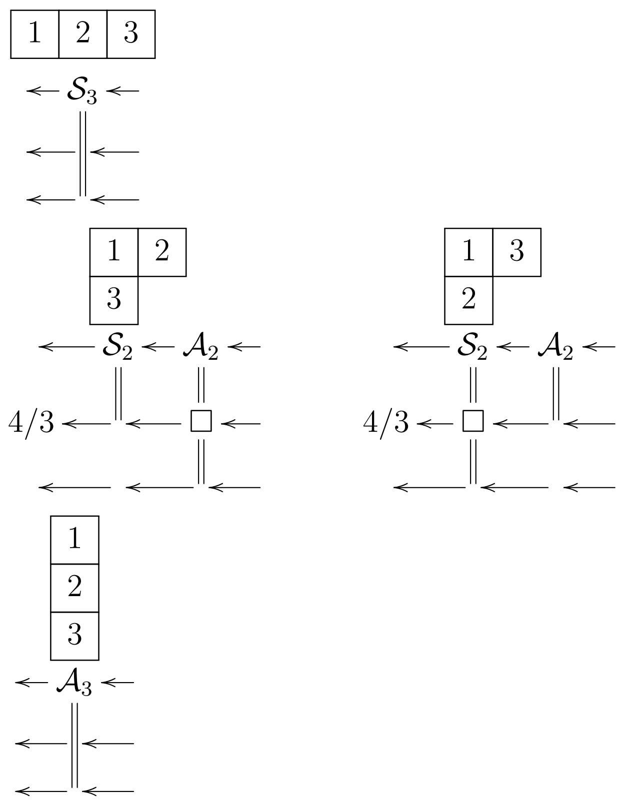
- $n_b = 1$



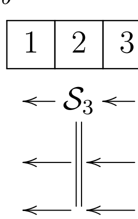
- $n_b = 2$

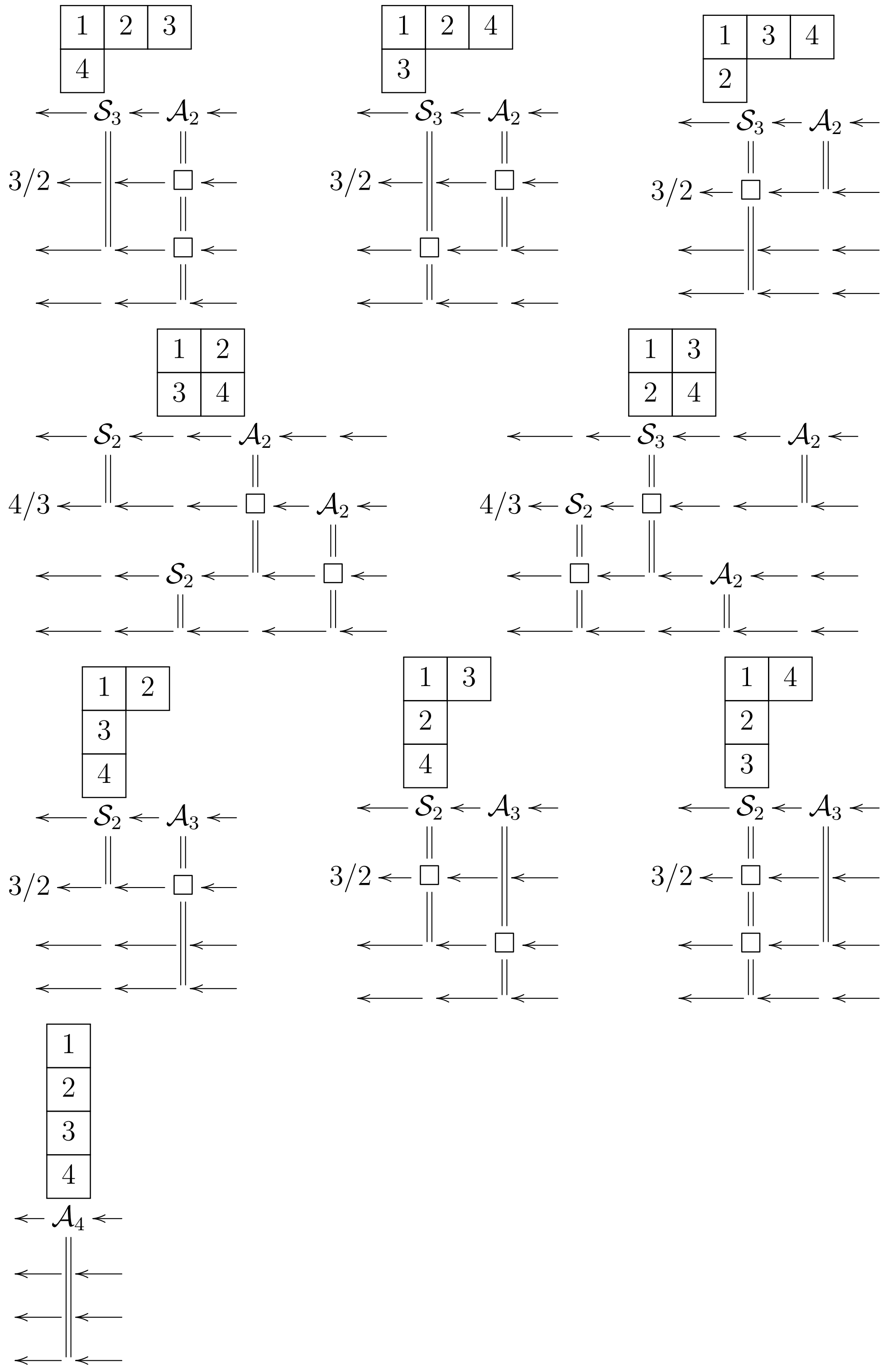


- $n_b = 3$



- $n_b = 4$




1 2 3
4
$\mathcal{S}_3$ $\mathcal{A}_2$
3/2
1 2 4
3
$\mathcal{S}_3$ $\mathcal{A}_2$
3/2
1 3 4
2
$\mathcal{S}_3$ $\mathcal{A}_2$
3/2
1 2
3 4
$\mathcal{S}_2$ $\mathcal{A}_2$
4/3
$\mathcal{A}_2$
$\mathcal{S}_2$
1 3
2 4
$\mathcal{S}_3$ $\mathcal{A}_2$
4/3 $\mathcal{S}_2$
$\mathcal{A}_2$
1 2
3
4
$\mathcal{S}_2$ $\mathcal{A}_3$
3/2
1 3
2
4
$\mathcal{S}_2$ $\mathcal{A}_3$
3/2
1 4
2
3
$\mathcal{S}_2$ $\mathcal{A}_3$
3/2
1
2
3
4
$\mathcal{A}_4$

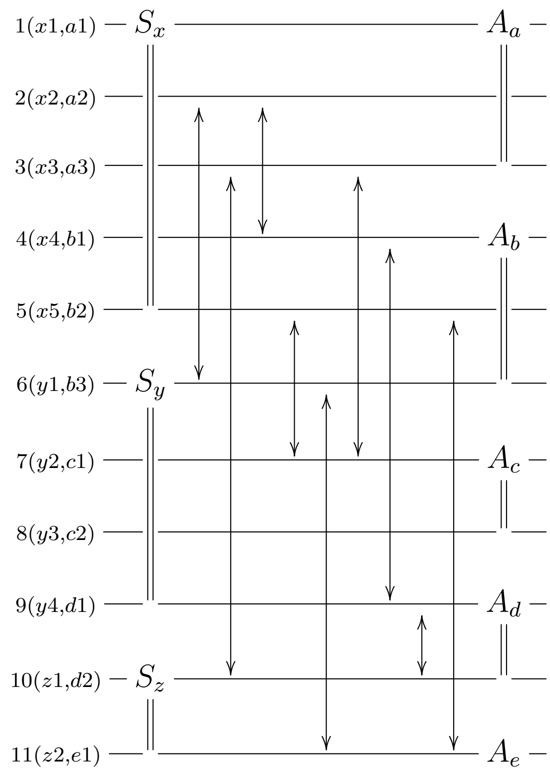
20.4 Young Projection Operator with swaps

Same info as in Section 20.3 but using swaps instead of holes.

$$
\mathcal{Y}_\alpha =
\begin{array}{c|c|c|c|c|c|}
 & A_a & A_b & A_c & A_d & \multicolumn{1}{c}{A_e} \\ \cline{2-6}
S_x & 1 & 2 & 3 & 4 & 5 \\ \cline{2-6}
S_y & 6 & 7 & 8 & 9 & \multicolumn{1}{c}{} \\ \cline{2-5}
S_z & 10 & 11 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-3}
\end{array}
$$

$a1 \longleftrightarrow 1$	
$a2 \longleftrightarrow 6$	
$a3 \longleftrightarrow 10$	
$b1 \longleftrightarrow 2$	
$b2 \longleftrightarrow 7$	
$b3 \longleftrightarrow 11$	
$c1 \longleftrightarrow 3$	
$c2 \longleftrightarrow 8$	
$d1 \longleftrightarrow 4$	
$d2 \longleftrightarrow 9$	
$e1 \longleftrightarrow 5$	

$P_{\mathcal{Y}_\alpha} =$

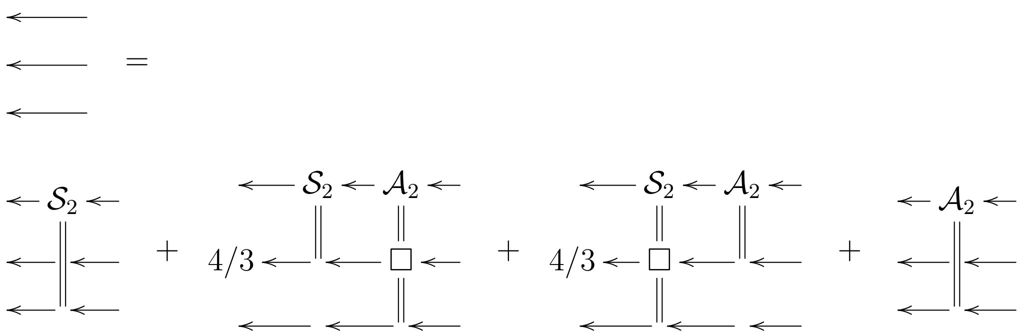


(20.57)

20.5 Tensor product decompositions

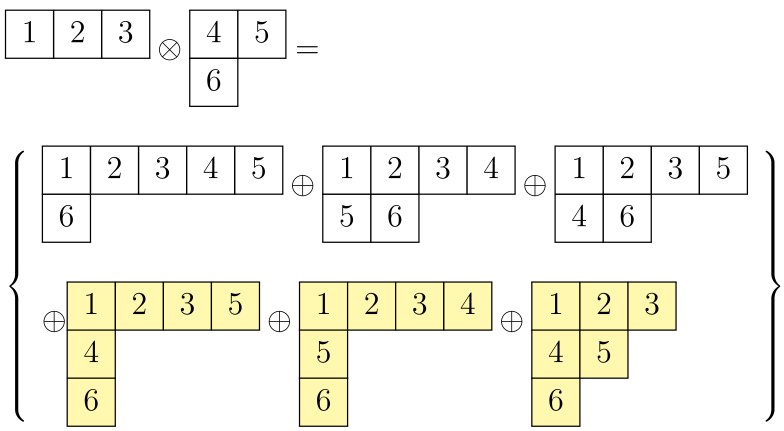
$$
\boxed{1} \otimes \boxed{2} \otimes \boxed{3} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right) \otimes \boxed{3} \tag{20.58}
$$

$$
= \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \tag{20.59}
$$



(20.60)

$$n^3 = \frac{n(n+1)(n+2)}{6} + \frac{n(n^2-1)}{3} + \frac{n(n^2-1)}{3} = \frac{(n-2)(n-1)n}{6} \tag{20.61}$$



(20.62)

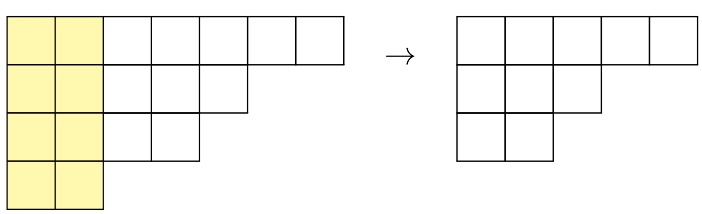
For $U(n)$, the blue YT are zero for $n_b = 2$, and non-zero otherwise.

20.6 $SU(n)$

$\mathfrak{su}(\mathfrak{n})$ A_n

$$\epsilon_{a_1 a_2 \dots a_n} = G_{a_1}{}^{a'_1} G_{a_2}{}^{a'_2} \cdots G_{a_n}{}^{a'_n}\, \epsilon_{a'_1 a'_2 \dots a'_n} \tag{20.63}$$

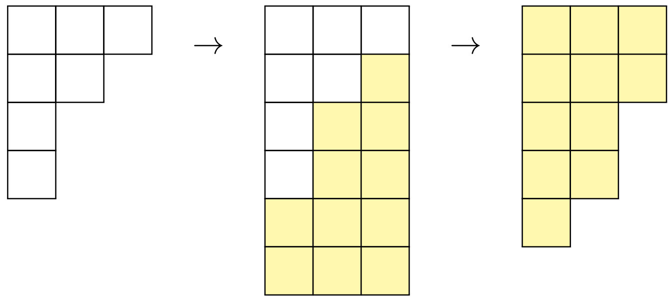
YD for $SU(n)$ has a maximum of $n-1$ rows. For $SU(4)$



(20.64)

$[5,3,2]_{RL}$ row lengths, $[2,1,2,0,\dots]_D$ row changes (Dynkin labels).

$[b_1, b_2, \ldots, b_{n-1}]$
For $SU(6)$,



(20.65)

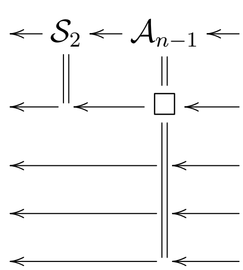
Besides RL and RC, a third way of labelling YD (reps of $SU(n)$) is by their dimension, and then adding a subscript or overline if there are more than one reps with a different YD but the same dimension. This method is used mostly by physicists for $SU(3)$ (The Eightfold Way). Note that all YT with the same YD have the same dimension, so this really labels YD instead of YT. For example, for $SU(3)$ we have

$$
\begin{array}{ll}
\ydiagram{1} = [1,0]_D = 3 & \ydiagram{2} = [0,1]_D = \overline{3} \\
\ydiagram{2} = [2,0]_D = 6 & \ydiagram{2,2} = [0,2]_D = \overline{6} \\
\ydiagram{2,1} = [1,1]_D = 8 & \ydiagram{3,1} = [2,1]_D = 15
\end{array}
\tag{20.66}
$$

$$
\begin{aligned}
\ydiagram{1} \otimes \left.\begin{array}{c} \ydiagram{1} \\ \ydiagram{1} \\ \vdots \\ \ydiagram{1} \end{array}\right\} n-1 \quad &= \quad 1 \oplus \left.\begin{array}{l} \ydiagram{2} \\ \ydiagram{1} \\ \vdots \\ \ydiagram{1} \end{array}\right\} n-1 && (20.67) \\
n \otimes \overline{n} \quad &= \quad 1 \oplus (n^2 - 1) && (20.68) \\
\text{fun rep} \otimes \text{conjugate rep} \quad &= \quad \text{singlet rep} \oplus \ \text{adjoint rep} && (20.69)
\end{aligned}
$$

Adjoint representation

$$P_{adj} = \frac{2(n-1)}{n}$$



(20.70)

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