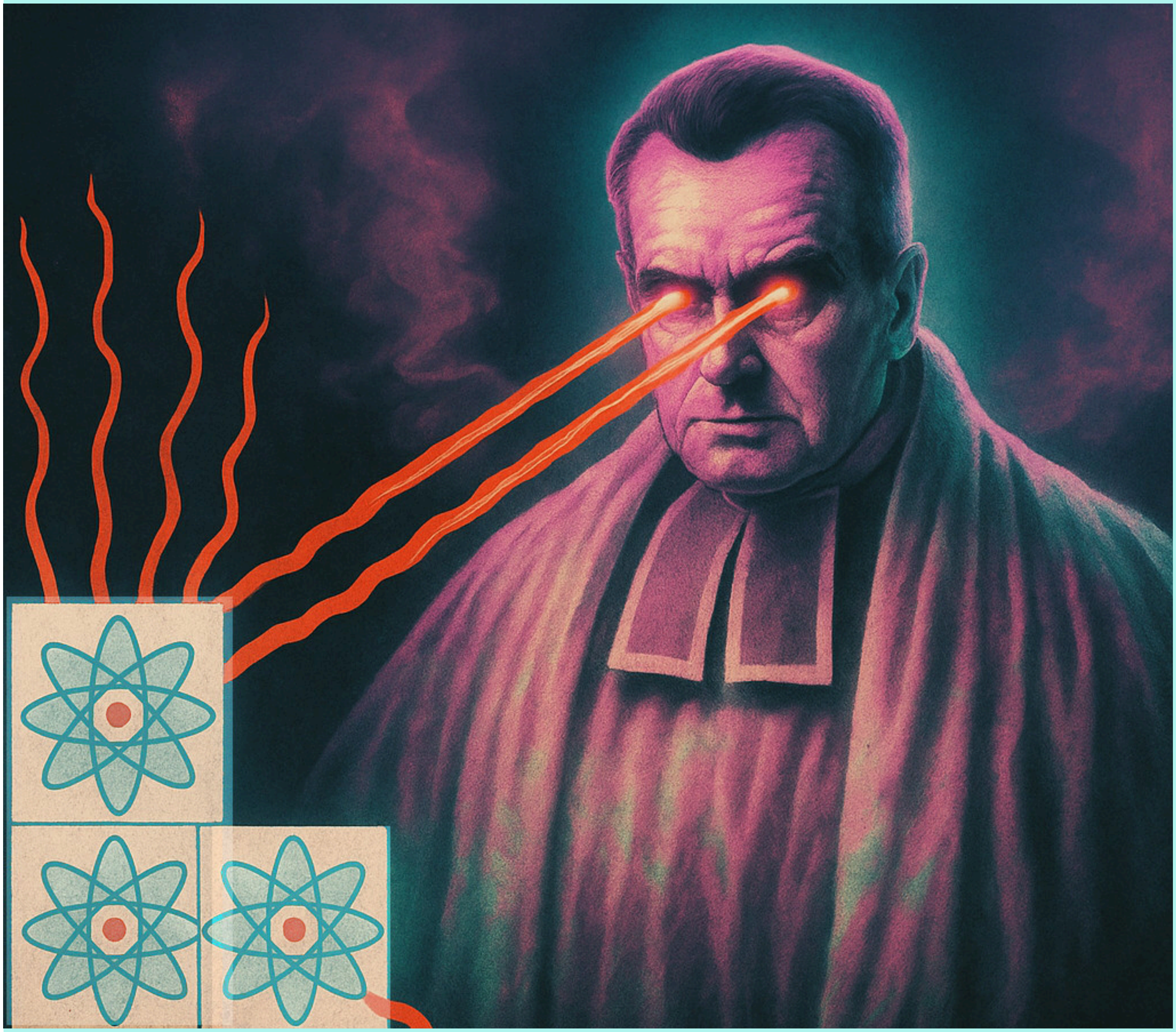
BAYESUVIUS QUANTICO

A VISUAL DICTIONARY OF QUANTUM BAYESIAN NETWORKS



ROBERT R. TUCCI

Bayesuvius Quantico,
a visual dictionary of Quantum Bayesian Networks

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This book is constantly being expanded and improved. To download the latest version, go to
`https://github.com/rrtucci/bayes-quantico`

Bayes Quantico
by Robert R. Tucci

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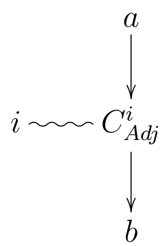
Lie Algebras

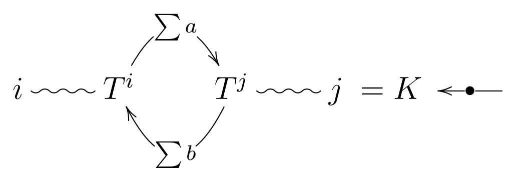
$i \in \mathbb{Z}_{[1,N]}$, $a, b \in \mathbb{Z}_{[1,n]}$

$$(C^i_{Adj})_b{}^a = \frac{1}{\sqrt{K}} (T^i)_b{}^a = \tag{9.1}$$

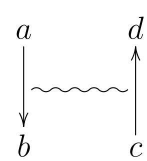


Note that we list the indices of T^i in the counter-clockwise (CC) direction, starting at the i leg. The matrices T^i are called the generators. It's customary to choose them so that they are Hermitian and $K = \frac{1}{2}$. [1]

$$\boxed{(T^i)_b{}^a (T^j)_a{}^b = \mathrm{tr}(T^i T^j) = K\delta(i,j)} \tag{9.2}$$



$$(P_{Adj})^{a,c}_{b,d} = \sum_i \frac{1}{K} (T^i)_b{}^a (T^i)_d{}^c = \frac{1}{K} \tag{9.3}$$



$H \in V^n \otimes \bar{V}^n$

$$(P_{Adj})^{ac}_{bd} H_c{}^d = \sum_i (T^i)_b{}^a \underbrace{\left[\frac{1}{K} (T^i)_d{}^c H_c{}^d \right]}_{h_i \in \mathbb{R}} \tag{9.4}$$

[1] For $SU(2)$, it is customary to use $T^i = \frac{1}{2}\sigma_i$, where σ_i for $i = 1, 2, 3$ are the Pauli matrices. For $SU(3)$, it is customary to choose $T^i = \frac{1}{2}\lambda_i$ where λ_i for $i = 1, 2, \ldots, 8$ are the Gelll-Mann matrices.

$G = 1 + iD \in \mathcal{G}$
$\epsilon_i \in \mathbb{R}$, $|\epsilon_i| << 1$
$D = \sum_i \epsilon_i T^i = V^n \otimes \bar{V}^n$
Recall Eq.(A.28). If $x \in V^{n^p} \otimes \bar{V}^{n^q}$, $\mathbb{G} \in \mathcal{G} \subset GL(n^{p+q}, \mathbb{C})$,

$$(x')_{a^{:p}}{}^{b^{:q}} = \mathbb{G}_{a^{:p}}{}^{b^{:q}}{}_{rev(c^{:q})}{}^{rev(d^{:p})} x_{d^{:p}}{}^{c^{:q}}, \quad (x'_\alpha = \mathbb{G}_\alpha{}^\beta x_\beta) \tag{9.5}$$

where we define

$$\mathbb{G}_\alpha{}^\beta \stackrel{\text{def}}{=} \prod_{i=1}^{p} G_{a_i}{}^{d_i} \prod_{i=1}^{q} G^{b_i}{}_{c_i} \tag{9.6}$$

$$\mathbb{G}_\alpha{}^\beta = 1 + i \sum_j \epsilon_j (\mathbb{T}^j)_\alpha{}^\beta \tag{9.7}$$

$$G_{a_i}{}^{d_i} = 1 + i \sum_j \epsilon_j (T^j)_{a_i}{}^{d_i} \tag{9.8}$$

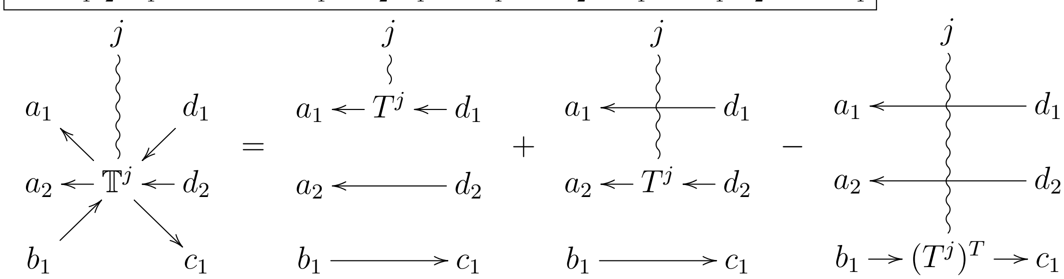
$$G^{b_i}{}_{c_i} = (G^*)_{b_i}{}^{c_i} = 1 - i \sum_j \epsilon_j (T^{j*})_{b_i}{}^{c_i} = 1 - i \sum_j \epsilon_j (T^j)^{b_i}{}_{c_i} \tag{9.9}$$

When $x'_\alpha = x_\alpha$, to first order in ϵ_i,

$$0 = (\mathbb{T}^j)_\alpha{}^\beta x_\beta = \left[(T^j)_{a_i}{}^{d_i} \frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}{}_{c_i} \frac{1}{\delta_{c_i}^{b_i}} \right] \delta_{a^{:p}}^{d^{:p}} \delta_{c^{:q}}^{b^{:q}} x_{d^{:p}}{}^{c^{:q}} \tag{9.10}$$

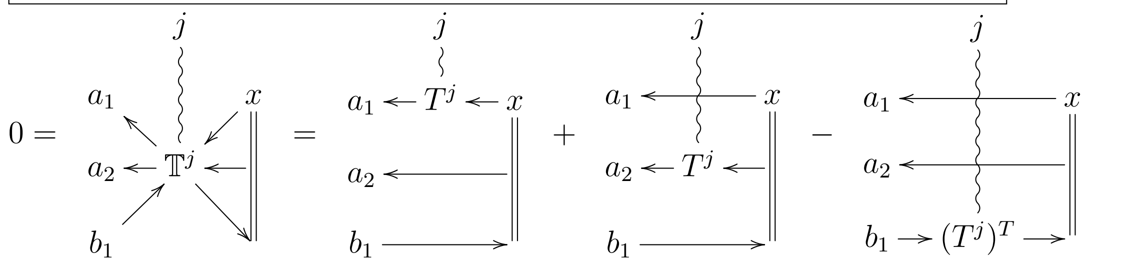
$$(\mathbb{T}^j)_\alpha{}^\beta = \left[(T^j)_{a_i}{}^{d_i} \frac{1}{\delta_{a_i}^{d_i}} - (T^j)^{b_i}{}_{c_i} \frac{1}{\delta_{c_i}^{b_i}} \right] \delta_{a^{:p}}^{d^{:p}} \delta_{c^{:q}}^{b^{:q}} \tag{9.11}$$

$$\boxed{(\mathbb{T}^j)_{a_1 a_2}{}^{b_1}{}_{c_1}{}^{d_2 d_1} = (T^j)_{a_1}{}^{d_1} \delta_{a_2}^{d_2} \delta_{c_1}^{b_1} + \delta_{a_1}^{d_1} (T^j)_{a_2}{}^{d_2} \delta_{c_1}^{b_1} - \delta_{a_1}^{d_1} \delta_{a_2}^{d_2} (T^j)^{b_1}{}_{c_1}}$$



(9.12)

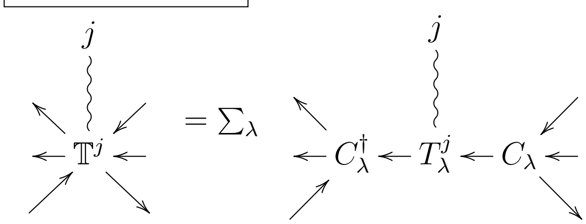
$$\boxed{0=(\mathbb{T}^j x)_{a_1a_2}{}^{b_1} = \left[(T^j)_{a_1}{}^{d_1}\delta_{a_2}^{d_2}\delta_{c_1}^{b_1} + \delta_{a_1}^{d_1}(T^j)_{a_2}{}^{d_2}\delta^{b_1}_{c_1} - \delta^{d_1}_{a_1}\delta^{d_2}_{a_2}(T^j)^{b_1}{}_{c_1}\right] x_{d_1d_2}{}^{c_1}}$$



(9.13)

Since $\mathbb{G} = 1 + i\sum_j \epsilon_j \mathbb{T}^j$, generators decompose in the same way as the group elements

$$\boxed{\mathbb{T}^j = \sum_\lambda C_\lambda^\dagger T_\lambda^j C_\lambda}$$



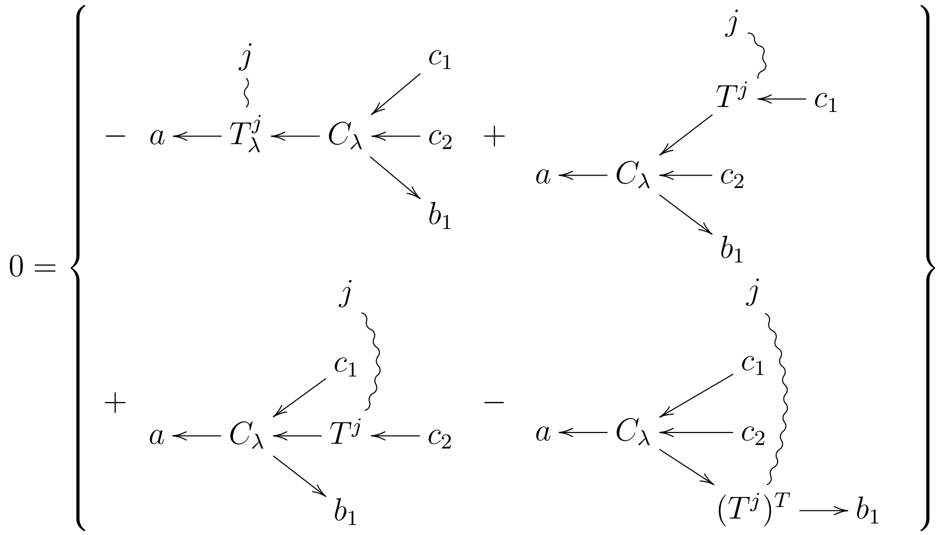
(9.14)

Clebsch-Gordan matrices are invariant matrices.

$$[G, C_\lambda] = GC_\lambda - C_\lambda G = 0 \tag{9.15}$$

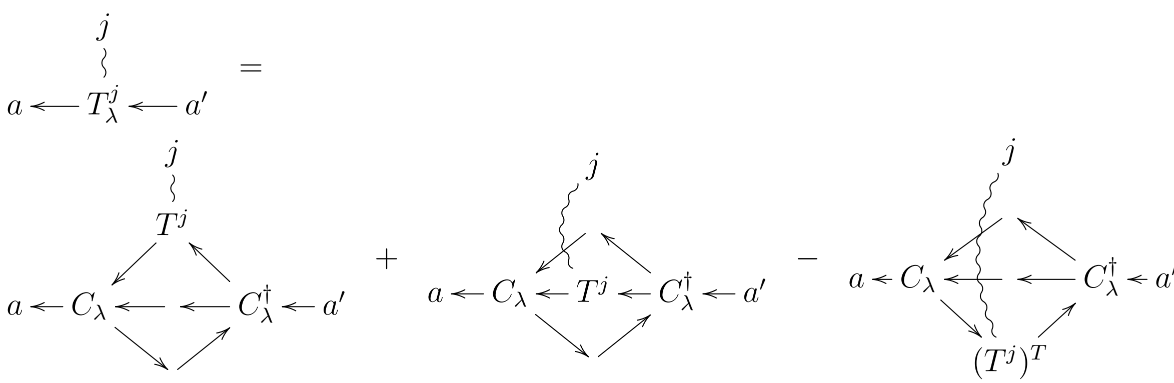
$$C_\lambda = G^\dagger C_\lambda G = G_\lambda^\dagger C_\lambda G \tag{9.16}$$

$$0 = -T_\lambda^j C_\lambda + C_\lambda T^j \tag{9.17}$$



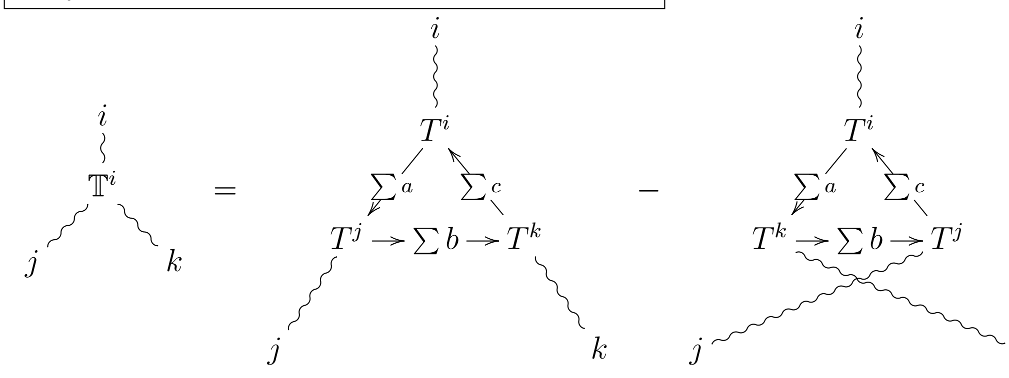
(9.18)

Multiplying on the left $C_\lambda^\dagger$, we get



(9.19)

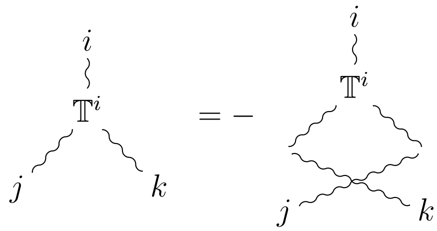
$$\boxed{(\mathbb{T}^i)_{jk} = (T^i)_a{}^c (T^k)_c{}^b (T^j)_b{}^a - (T^i)_a{}^c (T^j)_c{}^b (T^k)_b{}^a}$$



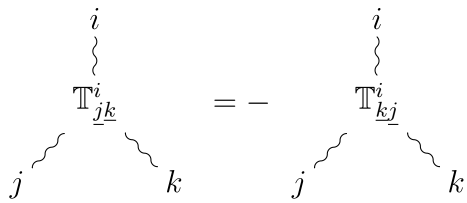
(9.20)

Note that

$$\boxed{(\mathbb{T}^i)_{jk} = -(\mathbb{T}^i)_{kj}}$$



(Convention CC)



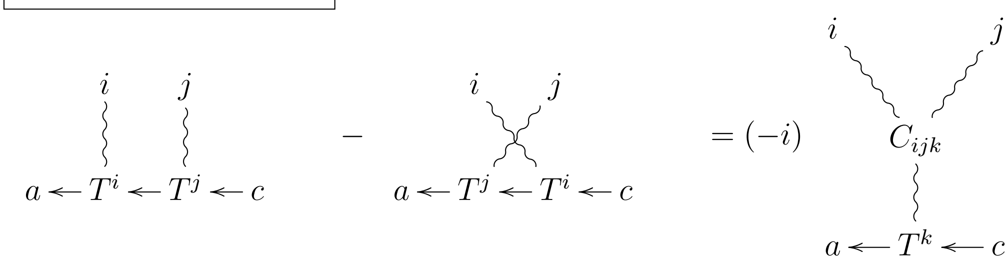
(Convention L)

(9.21)

$$(\mathbb{T}^i)_{jk} \stackrel{\text{def}}{=} -iC_{ijk} = -\frac{1}{K}\mathrm{tr}([T^i, T^j]T^k) \tag{9.22}$$

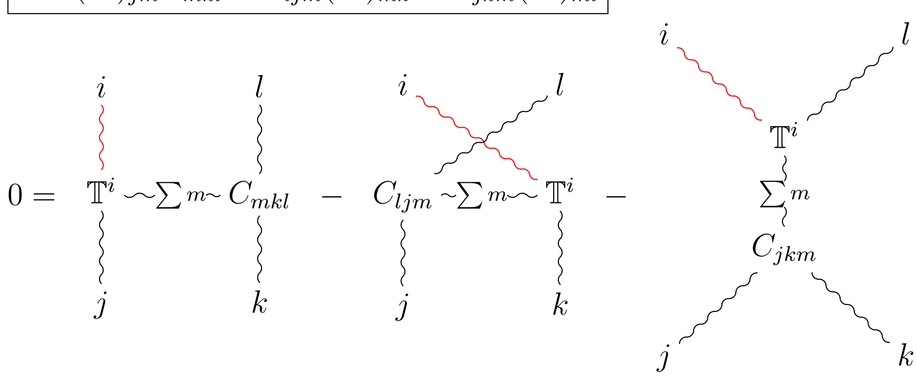
C_{ijk}, called the bf structure constants, is totally antisymmetric

$$\boxed{\underbrace{T^iT^j - T^jT^i}_{[T^i,T^j]} = iC_{ijk}T^k}$$ (**Lie Algebra** commutation relations)



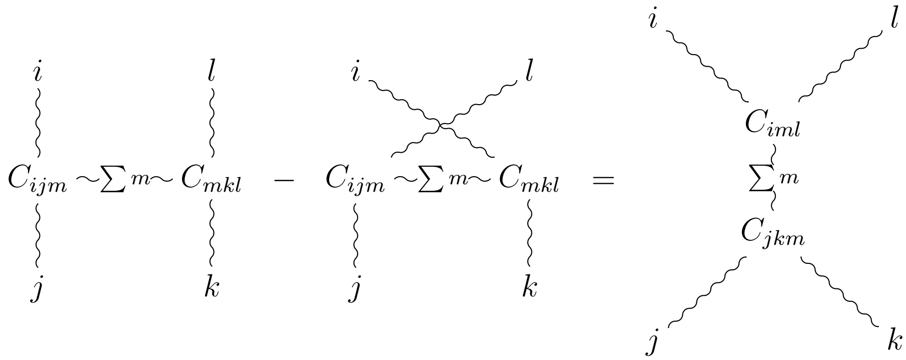
(9.23)

$$\boxed{0 = (\mathbb{T}^i)_{jm}C_{mkl} - C_{ljm}(\mathbb{T}^i)_{mk} - C_{jkm}(\mathbb{T}^i)_{ml}}$$



(9.24)

$$\boxed{C_{ijm}C_{mkl} - C_{ljm}C_{mki} = C_{iml}C_{jkm}}$$



(9.25)

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