

Chapter 1

Do Calculus proofs

In Chapter ?? of Bayesuvius, we explained Do Calculus but referred to this chapter for proofs of claims that use Do Calculus. In this chapter, we've aggregated all proofs, from throughout the book, of claims that use Do Calculus.

Note that even though the 3 rules of Do Calculus are great for proving adjustment formulae for general classes of DAGs, they are sometimes overkill for proving adjustment formulae for a single specific DAG. After all, the 3 rules of Do Calculus are a consequence of the d-separation theorem. Hence, all adjustment formulae should be provable from first principles, assuming only the d-separation theorem and the standard rules of probability theory.

We use the following conventions. Random variables are underlined and their values are not. For example, $\underline{a} = a$ means the random variable \underline{a} takes the value a . Diagrams with nodes that are underlined represent Bayesian Networks (bnets) and the same diagram with the letters not underlined represents a specific instantiation of that bnet. For example $\underline{a} \rightarrow \underline{b}$ represents the bnet with conditional probability distribution $P(b|a)$, whereas $a \rightarrow b$ represents $P(b|a)$ itself.

If \underline{a} is a root node, then $\sum \underline{a}$ signifies a weighted sum $\sum_a P(a)$. For example, $\sum \underline{a} \rightarrow \underline{b} = \sum_a P(a)P(b|a)$. If \underline{a} is not a root node as in $x \rightarrow \sum \underline{a} \rightarrow y = \sum_a P(y|a)P(a|x)$, then $\sum \underline{a}$ signifies a simple unweighted sum \sum_a .

Unobserved nodes are indicated by enclosing them in a dashed circle. For example, (\underline{u}) .

Selection diagrams with selection nodes are discussed in Chapter ??. In a selection diagram with a selection node $\underline{s} \in \{0, 1\}$, if a node \underline{x} has parents $pa(\underline{x})$ where $\underline{s} \notin pa(\underline{x})$, then the TPM of \underline{x} is $P(x|pa(x))$. If, on the other hand, \underline{x} has parents $pa(\underline{x}) \cup \underline{s}$, then the TPM of \underline{x} is $P(x|pa(x), \underline{s})$, where $P(x|pa(x), \underline{s} = 0) = P(x|pa(x))$ and $P(x|pa(x), \underline{s} = 1) = P^*(x|pa(x))$.

Some identities that are used in this chapter:

1.

$$P(y|x_1, x_2) = \sum_a P(y|a, x_1, x_2)P(a|x_1, x_2) . \quad (1.1)$$

$$\begin{array}{ccc}
x_1 & & x_1 \\
\searrow & & \searrow \\
& y & \sum a \\
\nearrow & & \nearrow \\
x_2 & & x_2
\end{array}
=
\begin{array}{ccc}
x_1 & & x_1 \\
\searrow & & \searrow \\
& \sum a & \longrightarrow y \\
\nearrow & & \nearrow \\
x_2 & & x_2
\end{array}
. \tag{1.2}$$

One can describe this identity as “giving \underline{y} a universal backdoor”, because $\sum a$ is a backdoor (i.e., input) to y , and $\sum a$ is universal in the sense that it is entered by every arrow that enters y except $\sum a$ itself.

2.

$$\sum_a \underbrace{P(a)P(a|x_1)P(a|x_2)}_{P(a|x_1, x_2)} = 1 \tag{1.3}$$

$$\begin{array}{ccc}
x_1 & & \\
\searrow & & \\
& \sum a & \xrightarrow{0} \\
\nearrow & & \\
x_2 & &
\end{array}
= 1 \tag{1.4}$$

One can describe this identity as “summing over the values of a collider node which has no emerging arrows”¹. Eq.(1.4) can be understood as an edge case (when $\underline{y} = \emptyset$) of Eq.(1.2).

3.

$$\sum_a P(x_2|a)P(a|x_1) = P(x_2|x_1) \tag{1.5}$$

$$x_1 \longrightarrow \sum a \longrightarrow x_2 = x_1 \longrightarrow x_2 \tag{1.6}$$

One can describe this identity as “summing over the values of a mediator node”.

4.

$$P(x) = \sum_a P(x|a)P(a) = \sum_b P(x|b) \tag{1.7}$$

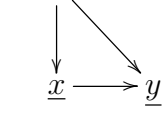
$$\longrightarrow_0 \sum a \longrightarrow x = \longrightarrow_0 \sum b \longrightarrow x \tag{1.8}$$

One can describe this identity as “averaging over different priors”. Eq.(1.8) can be understood as an edge case of Eq.(1.6).

¹A zeroed arrow means the same as no arrow.

Claim 1 (*Backdoor Adjustment Formula*)

If \underline{z} then



$$P(y|\mathcal{D}\underline{x} = x) = \sum_z P(y|x, z)P(z) \quad (1.9)$$

$$\begin{array}{c} \mathcal{D}\underline{x} = x \longrightarrow y \quad \sum z \\ = \\ x \longrightarrow y \end{array} \quad (1.10)$$

proof:

* **proof 1:**

$$\begin{array}{c} P(y|\mathcal{D}\underline{x} = x) = \sum_z P(y|x, z)P(z) \\ \sum z \quad \quad \quad \sum z \\ \mathcal{D}\underline{x} = x \longrightarrow y = x \longrightarrow y \end{array}$$

* **proof 2:**

$$\begin{array}{l} P(y|\mathcal{D}\underline{x} = x) = \sum_z P(y|\mathcal{D}\underline{x} = x, z)P(z|\mathcal{D}\underline{x} = x) \\ \text{by Probability Axioms} \\ = \sum_z P(y|x, z)P(z|\mathcal{D}\underline{x} = x) \\ P(y|\mathcal{D}\underline{x} = x, z) \rightarrow P(y|x, z) \\ \text{by Rule 2: } \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{L}_{\underline{a}} \mathcal{D}_{\underline{r}} G, \text{ then} \\ \mathcal{D}\underline{a} = a. \leftrightarrow \underline{a} = a. \\ \underline{y} \perp \underline{x} | \underline{z} \text{ in } \mathcal{L}_{\underline{x}} \mathcal{D}_{\emptyset} G : \begin{array}{c} \underline{z} \\ \downarrow \quad \searrow \\ \underline{x} \quad \underline{y} \end{array} \\ = \sum_z P(y|x, z)P(z) \\ P(z|\mathcal{D}\underline{x} = x) \rightarrow P(z) \\ \text{by Rule 3: } \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{D}_{\underline{a} - an(\underline{s})} \mathcal{D}_{\underline{r}} G, \text{ then} \\ \mathcal{D}\underline{a} = a. \leftrightarrow 1 \\ \underline{z} \perp \underline{x} \text{ in } \mathcal{D}_{\underline{x}} \mathcal{D}_{\emptyset} G : \begin{array}{c} \underline{z} \\ \searrow \\ \underline{x} \longrightarrow \underline{y} \end{array} \end{array}$$

QED

Claim 2 (*Frontdoor Adjustment Formula*)

If $\begin{array}{c} (\underline{c}) \\ \swarrow \quad \searrow \\ \underline{x} \longrightarrow \underline{m} \longrightarrow \underline{y} \end{array}$ then

$$P(y|\mathcal{D}\underline{x} = x) = \sum_m \left[\sum_{x'} P(y|x', m) P(x') \right] P(m|x) \quad (1.11)$$

$$\begin{array}{ccc} \mathcal{D}\underline{x} = x & \longrightarrow & y \\ & & \nwarrow \quad \nearrow \\ & & \sum x' \\ & & \searrow \\ & & y \end{array} \quad (1.12)$$

$$= \begin{array}{ccc} x & \longrightarrow & \sum m \longrightarrow y \end{array}$$

proof:

* **proof 1:**

$$P(y|\mathcal{D}\underline{x} = x) = \sum_{m,c,x'} P(y|m, c) P(c|x') P(x') P(m|\mathcal{D}\underline{x} = x)$$

$$\begin{array}{ccc} & (\sum c) & \sum x' \longrightarrow (\sum c) \\ & \searrow & \searrow \\ \mathcal{D}\underline{x} = x & \longrightarrow \sum m \longrightarrow y & = \mathcal{D}\underline{x} = x \longrightarrow \sum m \longrightarrow y \\ = \sum_{m,x'} P(y|m, x') P(x') P(m|x) & & \\ & \sum x' & \\ = & \searrow & \\ & x \longrightarrow \sum m \longrightarrow y & \end{array}$$

* **proof 2:**

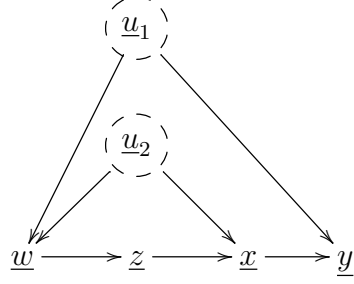
$$\begin{array}{l} P(y|\mathcal{D}\underline{x} = x) = \sum_m P(y|\mathcal{D}\underline{x} = x, m) P(m|\mathcal{D}\underline{x} = x) \\ \text{by Probability Axioms} \\ = \sum_m P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m) P(m|\mathcal{D}\underline{x} = x) \\ P(y|\mathcal{D}\underline{x} = x, m) \rightarrow P(y|\mathcal{D}\underline{x} = x, \mathcal{D}m = m) \\ \text{by Rule 2: If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{L}_{\underline{a}} \mathcal{D}_{\underline{r}} G, \text{ then} \\ \mathcal{D}\underline{a} = \underline{a} \leftrightarrow \underline{a} = \underline{a}. \end{array}$$

$$\begin{aligned}
& \underline{y} \perp \underline{m} | \underline{x} \text{ in } \mathcal{L}_{\underline{m}} \mathcal{D}_{\underline{x}} G : \quad \begin{array}{c} (\underline{c}) \\ \swarrow \quad \searrow \\ \underline{x} \longrightarrow \underline{m} \quad \underline{y} \end{array} \\
&= \sum_m P(y | \mathcal{D}_{\underline{x}} = x, \mathcal{D}_{\underline{m}} = m) P(m | x) \\
&\quad P(m | \mathcal{D}_{\underline{x}} = x) \rightarrow P(m | x) \\
&\text{by Rule 2:} \quad \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{L}_{\underline{a}} \mathcal{D}_{\underline{r}} G, \text{ then} \\
&\quad \mathcal{D}_{\underline{a}} = a. \leftrightarrow \underline{a} = a. \\
&\underline{m} \perp \underline{x} \text{ in } \mathcal{L}_{\underline{x}} \mathcal{D}_{\emptyset} G : \quad \begin{array}{c} (\underline{c}) \\ \swarrow \quad \searrow \\ \underline{x} \quad \underline{m} \longrightarrow \underline{y} \end{array} \\
&= \sum_m P(y | \mathcal{D}_{\underline{m}} = m) P(m | x) \\
&\quad P(y | \mathcal{D}_{\underline{x}} = x, \mathcal{D}_{\underline{m}} = m) \rightarrow P(y | \mathcal{D}_{\underline{m}} = m) \\
&\text{by Rule 3:} \quad \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{D}_{\underline{a} - an(\underline{s})} \mathcal{D}_{\underline{r}} G, \text{ then} \\
&\quad \mathcal{D}_{\underline{a}} = a. \leftrightarrow 1 \\
&\underline{y} \perp \underline{x} | \underline{m} \text{ in } \mathcal{D}_{\underline{x}} \mathcal{D}_{\underline{m}} G : \quad \begin{array}{c} (\underline{c}) \\ \swarrow \quad \searrow \\ \underline{x} \quad \underline{m} \longrightarrow \underline{y} \end{array} \\
&= \sum_{x'} \sum_m P(y | \mathcal{D}_{\underline{m}} = m, x') P(x' | \mathcal{D}_{\underline{m}} = m) P(m | x) \\
&\quad \text{by Probability Axioms} \\
&= \sum_{x'} \sum_m P(y | m, x') P(x' | \mathcal{D}_{\underline{m}} = m) P(m | x) \\
&\quad P(y | \mathcal{D}_{\underline{m}} = m, x') \rightarrow P(y | m, x') \\
&\text{by Rule 2:} \quad \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{L}_{\underline{a}} \mathcal{D}_{\underline{r}} G, \text{ then} \\
&\quad \mathcal{D}_{\underline{a}} = a. \leftrightarrow \underline{a} = a. \\
&\underline{y} \perp \underline{m} | \underline{x} \text{ in } \mathcal{L}_{\underline{m}} \mathcal{D}_{\emptyset} G : \quad \begin{array}{c} (\underline{c}) \\ \swarrow \quad \searrow \\ \underline{x} \longrightarrow \underline{m} \quad \underline{y} \end{array} \\
&= \sum_{x'} \sum_m P(y | m, x') P(x') P(m | x) \\
&\quad P(x' | \mathcal{D}_{\underline{m}} = m) \rightarrow P(x') \\
&\text{by Rule 3:} \quad \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{D}_{\underline{a} - an(\underline{s})} \mathcal{D}_{\underline{r}} G, \text{ then} \\
&\quad \mathcal{D}_{\underline{a}} = a. \leftrightarrow 1 \\
&\underline{x} \perp \underline{m} \text{ in } \mathcal{D}_{\underline{m}} \mathcal{D}_{\emptyset} G : \quad \begin{array}{c} (\underline{c}) \\ \swarrow \quad \searrow \\ \underline{x} \quad \underline{m} \longrightarrow \underline{y} \end{array}
\end{aligned}$$

QED

Claim 3 (*Napkin problem from Ref.[3]*)

If then



$$P(y|\mathcal{D}\underline{x} = x) = \sum_{w,z} P(y|x, w, z)P(w, z) \quad (1.13)$$

$$\mathcal{D}\underline{x} = x \longrightarrow y \quad = \quad \begin{array}{c} w, z \\ \searrow \\ x \longrightarrow y \end{array} \quad (1.14)$$

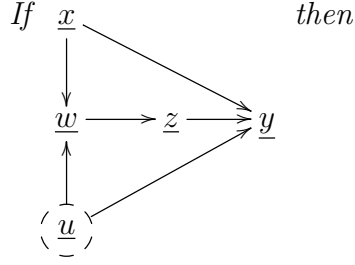
proof:

$$P(y|\mathcal{D}\underline{x} = x) = \sum_{u_1} P(y|x, u_1)P(u_1)$$

$$\begin{aligned} & \begin{array}{c} \text{Bayesian network with } \sum u_1, \sum u_2, \sum w, \sum z, \mathcal{D}\underline{x} = x, y \end{array} = \begin{array}{c} \text{Bayesian network with } \sum u_1, x, y \end{array} \\ & = \sum_{w,z} \sum_{u_1} P(y|x, u_1)P(u_1|w, z)P(w, z) \\ & = \sum_{w,z} P(y|x, w, z)P(w, z) \\ & = \begin{array}{c} w, z \longrightarrow y \end{array} \end{aligned}$$

QED

Claim 4 (from Ref.[3])



$$P(y|\mathcal{D}\underline{z} = z, x) = \sum_w P(y|z, x, w)P(w) \quad (1.15)$$

$$\begin{array}{c} x \\ \searrow \\ \mathcal{D}\underline{z} = z \longrightarrow y \end{array} = \sum_w \begin{array}{c} x \\ \searrow \\ z \longrightarrow y \end{array} \quad (1.16)$$

proof:

$$\begin{aligned} & P(y|\mathcal{D}\underline{z} = z, x) = \sum_u P(y|z, x, u)P(u) \\ & \begin{array}{c} x \\ \downarrow \\ \sum w \\ \uparrow \\ (\sum u) \end{array} \begin{array}{c} \mathcal{D}\underline{z} = z \\ \longrightarrow \end{array} y = \begin{array}{c} x \\ \searrow \\ z \longrightarrow y \\ (\sum u) \end{array} \\ & = \sum_w \sum_u P(y|z, x, u)P(u|w)P(w) \\ & \begin{array}{c} x \\ \searrow \\ \sum w \\ \downarrow \\ (\sum u) \end{array} \begin{array}{c} z \\ \longrightarrow \end{array} y \\ & = \sum_w P(y|z, x, w)P(w) \\ & \begin{array}{c} x \\ \searrow \\ \sum w \\ \searrow \\ z \longrightarrow y \end{array} \end{aligned}$$

QED

Claim 5 (*Trivial Transportability, from Ref.[2]*)

If $\underline{s} \in \{0, 1\}$ is a selection node, then

$$\begin{array}{c}
 \begin{array}{ccc}
 & \underline{z} & \underline{s} \\
 & \swarrow \quad \searrow & \downarrow \\
 \underline{x} & \xrightarrow{\quad} & \underline{y} \\
 & \curvearrowright &
 \end{array}
 \end{array}
 \quad P^*(y|\mathcal{D}\underline{x} = x, z) = P^*(y|x, z) \quad (1.17)$$

proof:

$$\begin{array}{ccc}
 P(y|\mathcal{D}\underline{x} = x, z, \underline{s} = 1) = P(y|x, z, \underline{s} = 1) \\
 \begin{array}{ccc}
 z & \underline{s} = 1 & z \\
 & \downarrow & \downarrow \\
 \mathcal{D}\underline{x} = x & \xrightarrow{\quad} & y = x \xrightarrow{\quad} y
 \end{array}
 \end{array}$$

QED

Claim 6 (*Direct Transportability, a.k.a. External Validity, from Ref.[2]*)

If $\underline{s} \longrightarrow \underline{z}$ where $\underline{s} \in \{0, 1\}$ is a selection node, then

$$\begin{array}{c}
 \begin{array}{ccc}
 \underline{s} & \longrightarrow & \underline{z} \\
 & \swarrow \quad \searrow & \\
 \underline{x} & \xrightarrow{\quad} & \underline{y} \\
 & \curvearrowright &
 \end{array}
 \end{array}
 \quad P^*(y|\mathcal{D}\underline{x} = x, z) = P(y|\mathcal{D}\underline{x} = x, z) \quad (1.18)$$

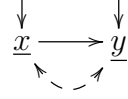
proof:

$$\begin{array}{ccc}
 P(y|\mathcal{D}\underline{x} = x, z, \underline{s} = 1) = P(y|\mathcal{D}\underline{x} = x, z) \\
 \begin{array}{ccc}
 \underline{s} = 1 & \longrightarrow & z \\
 & \searrow & \\
 \mathcal{D}\underline{x} = x & \xrightarrow{\quad} & y = \mathcal{D}\underline{x} = x \xrightarrow{\quad} y
 \end{array}
 \end{array}
 \quad \text{Because } \underline{s} \perp \underline{y} | \underline{z}$$

QED

Claim 7 (*S-Admissable Transportability, from Ref.[2]*)

If $\underline{s} \longrightarrow \underline{z} \longrightarrow \underline{a}$ where $\underline{s} \in \{0, 1\}$ is a selection node, then



$$P^*(y|\mathcal{D}\underline{x} = x) = \sum_a P(y|\mathcal{D}\underline{x} = x, a)P^*(a) \quad (1.19)$$

$$\begin{array}{ccc} \underline{s} = 1 & & \underline{s} = 1 \longrightarrow \sum a \\ & \searrow & \downarrow \\ \mathcal{D}\underline{x} = x \longrightarrow y & = & \mathcal{D}\underline{x} = x \longrightarrow y \end{array} \quad (1.20)$$

proof:

$$P(y|\mathcal{D}\underline{x} = x, \underline{s} = 1) = \sum_a P(y|\mathcal{D}\underline{x} = x, a)P(a|\underline{s} = 1)$$

$$\begin{array}{ccc} \underline{s} = 1 \longrightarrow \sum z \longrightarrow \sum a & & \underline{s} = 1 \longrightarrow \sum a \\ & \downarrow & \downarrow \\ \mathcal{D}\underline{x} = x \longrightarrow y & = & \mathcal{D}\underline{x} = x \longrightarrow y \end{array}$$

QED

Claim 8 (from Ref.[2])

If $\underline{s} \longrightarrow \underline{z}$ where $\underline{s} \in \{0, 1\}$ is a selection node, then



$$P^*(y|\mathcal{D}\underline{x} = x) = \sum_z P(y|\mathcal{D}\underline{x} = x, z)P^*(z) \quad (1.21)$$

$$\begin{array}{ccc} \underline{s} = 1 & & \underline{s} = 1 \longrightarrow \sum z \\ & \searrow & \searrow \\ \mathcal{D}\underline{x} = x \longrightarrow y & = & \mathcal{D}\underline{x} = x \longrightarrow y \end{array} \quad (1.22)$$

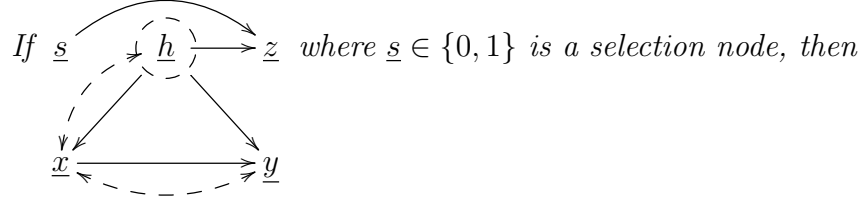
proof:

$$P(y|\mathcal{D}\underline{x} = x, \underline{s} = 1) = \sum_z P(y|\mathcal{D}\underline{x} = x, z)P(z|\underline{s} = 1)$$

$$\begin{array}{ccc} \underline{s} = 1 \longrightarrow \sum z & & \\ & \searrow & \\ \mathcal{D}\underline{x} = x \longrightarrow y & & \end{array}$$

QED

Claim 9 (from Ref.[2])



$$P^*(y|\mathcal{D}\underline{x} = x) = P(y|\mathcal{D}\underline{x} = x) \quad (1.23)$$

$$\begin{array}{c} \underline{s} = 1 \\ \searrow \\ \mathcal{D}\underline{x} = x \longrightarrow y = \mathcal{D}\underline{x} = x \longrightarrow y \end{array} \quad (1.24)$$

proof:

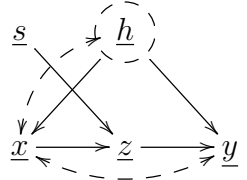
$$P(y|\mathcal{D}\underline{x} = x, \underline{s} = 1) = \sum_h P(y|\mathcal{D}\underline{x} = x, h)P(h)$$

$$\begin{array}{c} \underline{s} = 1 \xrightarrow{\quad} (\sum h) \xrightarrow{\quad} \sum z \\ \quad \quad \quad \searrow \\ \mathcal{D}\underline{x} = x \longrightarrow y = \mathcal{D}\underline{x} = x \longrightarrow y \\ = \sum_h P(y|\mathcal{D}\underline{x} = x, h)P(h|\mathcal{D}\underline{x} = x) \end{array}$$

$$\begin{array}{c} \quad \quad \quad (\sum h) \\ \quad \quad \quad \nearrow \quad \searrow \\ = \mathcal{D}\underline{x} = x \longrightarrow y \\ = P(y|\mathcal{D}\underline{x} = x) \\ = \mathcal{D}\underline{x} = x \longrightarrow y \end{array}$$

QED

Claim 10 (from Ref.[2])

If \underline{s}  where $\underline{s} \in \{0, 1\}$ is a selection node, then

$$P^*(y|\mathcal{D}\underline{x} = x) = \sum_z P(y|\mathcal{D}\underline{x} = x, z)P^*(z|x) \quad (1.25)$$

$$\begin{array}{c} \underline{s} = 1 \quad \mathcal{D}\underline{x} = x \quad \underline{s} = 1 \quad \mathcal{D}\underline{x} = x \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ y \quad = \quad x \longrightarrow \sum z \longrightarrow y \end{array} \quad (1.26)$$

proof:

$$P(y|\mathcal{D}\underline{x} = x, \underline{s} = 1) = \sum_h \sum_z P(y|h, z)P(h)P(z|\mathcal{D}\underline{x} = x, \underline{s} = 1)$$

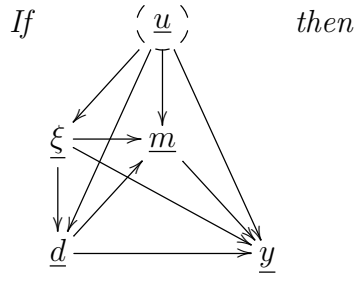
$$\begin{array}{c} \underline{s} = 1 \quad (\sum h) \\ \downarrow \quad \downarrow \\ \mathcal{D}\underline{x} = x \longrightarrow \sum z \longrightarrow y \end{array}$$

$$= \sum_h \sum_z P(y|h, z)P(h|\mathcal{D}\underline{x} = x)P(z|x, \underline{s} = 1)$$

$$\begin{array}{c} \underline{s} = 1 \quad (\sum h) \longleftarrow \mathcal{D}\underline{x} = x \\ \downarrow \quad \downarrow \\ = \quad x \longrightarrow \sum z \longrightarrow y \\ = \sum_z P(y|\mathcal{D}\underline{x} = x, z)P(z|x, \underline{s} = 1) \\ \underline{s} = 1 \quad \mathcal{D}\underline{x} = x \\ \downarrow \quad \downarrow \\ = \quad x \longrightarrow \sum z \longrightarrow y \end{array}$$

QED

Claim 11 (from Ref.[1])



$$P(y|\mathcal{D}\underline{d} = d, \mathcal{I}_m \underline{d} = d') = \sum_{\xi} \sum_m P(y|d, m, \xi) P(m|d', \xi) P(\xi) \quad (1.27)$$

$$\begin{array}{ccc} \mathcal{I}\underline{d} = d' & \mathcal{I}\underline{d} = d' & \sum \xi \longrightarrow \sum m \\ \searrow & \searrow & \searrow \\ \mathcal{D}\underline{d} = d \longrightarrow y & = & \mathcal{D}\underline{d} = d \longrightarrow y \end{array} \quad (1.28)$$

proof:

$$P(y|\mathcal{D}\underline{d} = d, \mathcal{I}\underline{d} = d') = \sum_{\xi, u} \sum_m P(y|d, m, \xi, u) P(m|d', \xi, u) \underbrace{P(\xi|u)P(u)}_{P(\xi, u)}$$

$$\begin{array}{ccc} & (\sum u) & \\ & \downarrow & \\ \mathcal{I}\underline{d} = d' & \sum \xi \longrightarrow \sum m & \\ \searrow & \searrow & \searrow \\ \mathcal{D}\underline{d} = d \longrightarrow y & & y \\ = \sum_{\xi} \sum_m P(y|d, m, \xi) P(m|d', \xi) P(\xi) & & \end{array}$$

$$\begin{array}{ccc} \mathcal{I}\underline{d} = d' & \sum \xi \longrightarrow \sum m & \\ \searrow & \searrow & \searrow \\ = \mathcal{D}\underline{d} = d \longrightarrow y & & y \end{array}$$

We switch from averaging over the prior of ξ, u to averaging over the prior of ξ .

QED

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