

# Chapter 1

## Do Calculus proofs

In Chapter ?? of Bayesuvius, we explained Do Calculus but referred to this chapter for proofs of claims that use Do Calculus. In this chapter, we've aggregated all proofs, from throughout the book, of claims that use Do Calculus.

Note that even though the 3 rules of Do Calculus are great for proving adjustment formulas for general classes of DAGs, they are sometimes overkill for proving adjustment formulas for a single specific DAG. After all, the 3 rules of Do Calculus are a consequence of the d-separation theorem. Hence, all adjustment formulas should be provable from first principles, assuming only the d-separation theorem and the standard rules of probability theory.

We use the following conventions. Random values are underlined and their values are not. For example,  $\underline{a} = a$  means the random variable  $\underline{a}$  takes the value  $a$ . Diagrams with nodes that are underlined represent Bayesian Networks (bnets) and the same diagram with the letters not underlined represents a specific instantiation of that bnet. For example  $\underline{a} \rightarrow \underline{b}$  represents the bnet with conditional probability distribution  $P(b|a)$ , whereas  $a \rightarrow b$  represents  $P(b|a)$  itself. Thus, the identity

$$P(y|x) = \sum_a P(y|a, x)P(a|x) \quad (1.1)$$

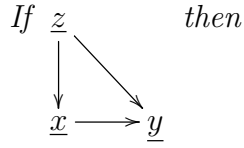
can be represented graphically by

$$x \longrightarrow y = \sum_a \begin{array}{c} \phantom{x} \nearrow a \\ \uparrow \\ x \longrightarrow y \end{array} \quad (1.2)$$

where  $\sum a$  means the random variable  $\underline{a}$  is summed over.

Unobserved nodes are indicated by enclosing them in a dashed circle.

**Claim 1** (*Backdoor Adjustment Formula*)



$$P(y|\mathcal{D}\underline{x} = x) = \sum_z P(y|x, z)P(z) \quad (1.3)$$

$$\mathcal{D}\underline{x} = x \longrightarrow y = \sum_z \begin{array}{c} z \\ \searrow \\ x \longrightarrow y \end{array} \quad (1.4)$$

**proof:**

**\* proof 1:**

$$\begin{aligned} P(y|\mathcal{D}\underline{x} = x) &= \sum_z P(y|\mathcal{D}\underline{x} = x, z)P(z|\mathcal{D}\underline{x} = x) \\ &= \sum_z \begin{array}{c} z \\ \searrow \\ \mathcal{D}\underline{x} = x \longrightarrow y \end{array} = \sum_z \begin{array}{c} \begin{array}{c} \uparrow 0 \\ z \end{array} \\ \searrow \\ \mathcal{D}\underline{x} = x \longrightarrow y \end{array} \\ &= \sum_z P(y|x, z)P(z|\mathcal{D}\underline{x} = x) \end{aligned}$$

$$\begin{aligned} &= \begin{array}{c} \sum z \\ \uparrow 0 \\ \mathcal{D}\underline{x} = x \end{array} \begin{array}{c} \searrow \\ y \end{array} \quad \text{by Rule 2} \\ &= \sum_z \begin{array}{c} z \\ \searrow \\ x \longrightarrow y \end{array} \\ &= \sum_z P(y|x, z)P(z) \end{aligned}$$

$$\begin{aligned} &= \begin{array}{c} \sum z \\ \searrow \\ x \longrightarrow y \end{array} \quad \begin{array}{l} \text{by Rule 3.} \\ \text{No info transmission} \\ \text{between } \mathcal{D}\underline{x} \text{ and } \underline{z}. \end{array} \end{aligned}$$

**\* proof 2:**

$$\begin{aligned} P(y|\mathcal{D}\underline{x} = x) &= \sum_z P(y|\mathcal{D}\underline{x} = x, z)P(z|\mathcal{D}\underline{x} = x) \\ &\quad \text{by Probability Axioms} \\ &= \sum_z P(y|x, z)P(z|\mathcal{D}\underline{x} = x) \\ &\quad P(y|\mathcal{D}\underline{x} = x, z) \rightarrow P(y|x, z) \end{aligned}$$

by Rule 2: If  $(\underline{b}. \perp \underline{a} | \underline{r}., \underline{s}.)$  in  $\mathcal{L}_{\underline{a}} \mathcal{D}_{\underline{r}} G$ , then  
 $\mathcal{D}_{\underline{a}}. = \underline{a}. \leftrightarrow \underline{a}. = \underline{a}.$

$$\begin{aligned}
& \mathcal{L}_{\underline{x}} \mathcal{D}_{\emptyset} G : \underline{z} \\
& \quad \downarrow \quad \searrow \\
& \quad \underline{x} \quad \underline{y} \\
& = \sum_z P(y|x, z) P(z) \\
& \quad P(z | \mathcal{D}\underline{x} = x) \rightarrow P(z) \\
& \text{by Rule 3: If } (\underline{b}. \perp \underline{a} | \underline{r}., \underline{s}.) \text{ in } \mathcal{D}_{\underline{a}. - an(\underline{s}.)} \mathcal{D}_{\underline{r}} G, \text{ then} \\
& \quad \mathcal{D}_{\underline{a}}. = \underline{a}. \leftrightarrow 1 \\
& \mathcal{D}_{\underline{x}} \mathcal{D}_{\emptyset} G : \underline{z} \\
& \quad \searrow \\
& \quad \underline{x} \longrightarrow \underline{y}
\end{aligned}$$

**QED**

**Claim 2** (*Frontdoor Adjustment Formula*)

If  $\begin{array}{ccc} & \underline{c} & \\ \swarrow & & \searrow \\ \underline{x} & \longrightarrow \underline{m} & \longrightarrow \underline{y} \end{array}$  then

$$P(y | \mathcal{D}\underline{x} = x) = \sum_m \left[ \sum_c P(y | c, m) P(c) \right] P(m | x) \quad (1.5)$$

$$\begin{aligned}
\mathcal{D}\underline{x} = x \longrightarrow y &= \begin{array}{ccc} & \sum c & \\ & \searrow & \\ x \longrightarrow \sum m & \longrightarrow & y \end{array} \quad (1.6)
\end{aligned}$$

**proof:**

\* **proof 1:**

$$P(y | \mathcal{D}\underline{x} = x) = \sum_{m, c} P(y | \mathcal{D}\underline{x} = x, m, c) P(m | \mathcal{D}\underline{x} = x)$$

$$\begin{aligned}
& \begin{array}{ccc} & \sum c & \\ & \searrow & \\ \mathcal{D}\underline{x} = x \longrightarrow \sum m & \longrightarrow & y \end{array} = \begin{array}{ccc} & \sum c & \\ & \searrow & \\ \mathcal{D}\underline{x} = x \longrightarrow \sum m & \longrightarrow & y \\ & \underbrace{\hspace{1.5cm}}_0 & \end{array}
\end{aligned}$$

$$= \sum_{m,c} P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m, c) P(m|\mathcal{D}\underline{x} = x)$$

$$= \sum_m \mathcal{D}\underline{x} = x \begin{array}{c} \xrightarrow{\quad} m \\ \quad \quad \quad \searrow \quad \quad \quad \nearrow \\ \quad \quad \quad 0 \quad \quad \quad \end{array} \begin{array}{c} \sum c \\ \searrow \\ y \end{array} \quad \text{by Rule 2}$$

$\mathcal{D}\underline{m} = m$

$$= \sum_{m,c} P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m, c) P(m|x)$$

$$= \sum_m \mathcal{D}\underline{x} = x \begin{array}{c} \xrightarrow{\quad} m \\ \quad \quad \quad \searrow \quad \quad \quad \nearrow \\ \quad \quad \quad 0 \quad \quad \quad \end{array} \begin{array}{c} \sum c \\ \searrow \\ y \end{array} \quad \text{by Rule 2}$$

$\mathcal{D}\underline{m} = m$

$x$

$$= \sum_{m,c} P(y|\mathcal{D}\underline{m} = m, c) P(m|x)$$

by Rule 3.  
No info transmission  
between  $\mathcal{D}\underline{x}$  and  $\underline{y}$   
if condition on  $\mathcal{D}\underline{m}$ .

$$= \sum_m \begin{array}{c} \sum c \\ \searrow \\ m \\ \nearrow \\ x \end{array} \begin{array}{c} \nearrow \\ \mathcal{D}\underline{m} = m \\ \nearrow \\ y \end{array}$$

$$= \sum_{m,c} P(y|\mathcal{D}\underline{m} = m, c) P(c|\mathcal{D}\underline{m} = m) P(m|x)$$

$$= \sum_m \begin{array}{c} \sum c \\ \searrow \quad \quad \quad \nearrow \\ \quad \quad \quad 0 \quad \quad \quad \end{array} \begin{array}{c} \nearrow \\ m \\ \nearrow \\ y \end{array} \quad \text{by Rule 3.}$$

$\mathcal{D}\underline{m} = m$

$x$

$$= \sum_{m,c} P(y|m, c) P(c|\mathcal{D}\underline{m} = m) P(m|x)$$

$$\begin{aligned}
& \sum c \xleftarrow{0} \mathcal{D}\underline{m} = m \quad \text{by Rule 2} \\
& \begin{array}{ccc} & & \searrow \\ = \sum_m & & \\ & \nearrow m & \longrightarrow y \\ x & & \end{array} \\
& = \sum_{m,c} P(y|m,c)P(c)P(m|x) \\
& \begin{array}{ccc} & \sum c & \\ & \searrow & \\ = & & \\ x \longrightarrow & \sum m \longrightarrow & y \end{array} \quad \begin{array}{l} \text{by Rule 3.} \\ \text{No info transmission} \\ \text{between } \mathcal{D}\underline{m} \text{ and } \underline{c}. \end{array}
\end{aligned}$$

\* **proof 2:**

$$\begin{aligned}
& P(y|\mathcal{D}\underline{x} = x) = \sum_m P(y|\mathcal{D}\underline{x} = x, m)P(m|\mathcal{D}\underline{x} = x) \\
& \quad \text{by Probability Axioms} \\
& = \sum_m P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m)P(m|\mathcal{D}\underline{x} = x) \\
& \quad P(y|\mathcal{D}\underline{x} = x, m) \rightarrow P(y|\mathcal{D}\underline{x} = x, \mathcal{D}m = m) \\
& \quad \text{by Rule 2: } \begin{array}{l} \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{L}_{\underline{a}}.\mathcal{D}_{\underline{r}}.G, \text{ then} \\ \mathcal{D}\underline{a} = a. \leftrightarrow \underline{a} = a. \end{array} \\
& \quad \mathcal{L}_{\underline{m}}\mathcal{D}_{\underline{x}}G : \quad \begin{array}{ccc} & \underline{c} & \\ & \searrow & \\ \underline{x} \longrightarrow \underline{m} & & \underline{y} \end{array} \\
& = \sum_m P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m)P(m|x) \\
& \quad P(m|\mathcal{D}\underline{x} = x) \rightarrow P(m|x) \\
& \quad \text{by Rule 2: } \begin{array}{l} \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{L}_{\underline{a}}.\mathcal{D}_{\underline{r}}.G, \text{ then} \\ \mathcal{D}\underline{a} = a. \leftrightarrow \underline{a} = a. \end{array} \\
& \quad \mathcal{L}_{\underline{x}}\mathcal{D}_{\emptyset}G : \quad \begin{array}{ccc} & \underline{c} & \\ \swarrow & & \searrow \\ \underline{x} & & \underline{m} \longrightarrow \underline{y} \end{array} \\
& = \sum_m P(y|\mathcal{D}\underline{m} = m)P(m|x) \\
& \quad P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m) \rightarrow P(y|\mathcal{D}\underline{m} = m) \\
& \quad \text{by Rule 3: } \begin{array}{l} \text{If } (\underline{b} \perp \underline{a} | \underline{r}, \underline{s}) \text{ in } \mathcal{D}_{\underline{a} - an(\underline{s})}.\mathcal{D}_{\underline{r}}.G, \text{ then} \\ \mathcal{D}\underline{a} = a. \leftrightarrow 1 \end{array} \\
& \quad \mathcal{D}_{\underline{x}}\mathcal{D}_{\underline{m}}G : \quad \begin{array}{ccc} & \underline{c} & \\ & \searrow & \\ \underline{x} & & \underline{m} \longrightarrow \underline{y} \end{array} \\
& = \sum_c \sum_m P(y|\mathcal{D}\underline{m} = m, c)P(c|\mathcal{D}\underline{m} = m)P(m|x)
\end{aligned}$$

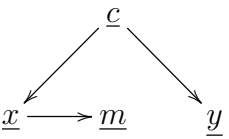
by Probability Axioms

$$= \sum_c \sum_m P(y|m, c) P(c|\mathcal{D}\underline{m} = m) P(m|x)$$

$$P(y|\mathcal{D}\underline{m} = m, c) \rightarrow P(y|m, c)$$

by Rule 2: If  $(\underline{b} \perp \underline{a} | \underline{r}, \underline{s})$  in  $\mathcal{L}_{\underline{a}} \mathcal{D}_{\underline{r}} G$ , then  
 $\mathcal{D}\underline{a} = a. \leftrightarrow \underline{a} = a.$

$\mathcal{L}_{\underline{m}} \mathcal{D}_{\emptyset} G :$

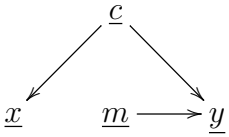


$$= \sum_c \sum_m P(y|m, c) P(c) P(m|x)$$

$$P(c|\mathcal{D}\underline{m} = m) \rightarrow P(c)$$

by Rule 3: If  $(\underline{b} \perp \underline{a} | \underline{r}, \underline{s})$  in  $\mathcal{D}_{\underline{a} - an(\underline{s})} \mathcal{D}_{\underline{r}} G$ , then  
 $\mathcal{D}\underline{a} = a. \leftrightarrow 1$

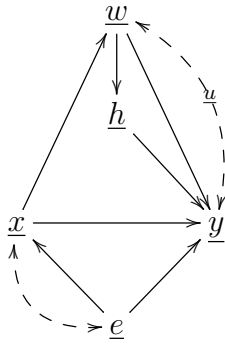
$\mathcal{D}_{\underline{m}} \mathcal{D}_{\emptyset} G :$



**QED**

**Claim 3** (from Ref.[1])

If  $\quad \quad \quad$  then



$$P(y|\mathcal{D}\underline{x} = x) = \sum_e P(y|x, e) P(e) \quad (1.7)$$

$$\mathcal{D}\underline{x} = x \longrightarrow y \quad = \quad x \longrightarrow y \quad (1.8)$$

$\sum e$  (with an arrow pointing from the text to the arrow between x and y)

**proof:**

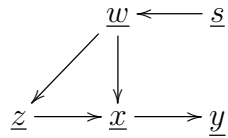
$$P(y|\mathcal{D}\underline{x} = x) =$$

$$\begin{array}{c}
\begin{array}{ccc}
& \sum w & \\
& \downarrow & \swarrow \text{dashed } \sum u \\
\mathcal{D}\underline{x} = x & \xrightarrow{\quad} & y \\
& \uparrow \sum h & \\
& & \sum e
\end{array}
= 
\begin{array}{ccc}
& \sum w & \\
& \downarrow & \swarrow \text{dashed } \sum u \\
\mathcal{D}\underline{x} = x & \xrightarrow{\quad} & y \\
& \uparrow \sum h & \\
& & \sum e
\end{array}
\\
= \sum_e P(y|\mathcal{D}\underline{x} = x, e) P(e|\mathcal{D}\underline{x} = x)
\\
\begin{array}{ccc}
& \sum w & \\
& \downarrow & \swarrow \text{dashed } \sum u \\
\mathcal{D}\underline{x} = x & \xrightarrow{\quad} & y \\
& \uparrow \sum h & \\
& & \sum e
\end{array}
= 
\begin{array}{ccc}
& \sum w & \\
& \downarrow & \swarrow \text{dashed } \sum u \\
\mathcal{D}\underline{x} = x & \xrightarrow{\quad} & y \\
& \uparrow \sum h & \\
& & \sum e
\end{array}
\\
= \sum_e P(y|x, e) P(e|\mathcal{D}\underline{x} = x)
\\
\begin{array}{ccc}
x & \xrightarrow{\quad} & y \\
& \searrow & \\
& & \sum e
\end{array}
\quad \text{by Rule 2}
\\
= \begin{array}{ccc}
\mathcal{D}\underline{x} = x & \xrightarrow{\quad} & y \\
& \searrow & \\
& & \sum e
\end{array}
\\
= \sum_e P(y|x, e) P(e)
\\
\begin{array}{ccc}
x & \xrightarrow{\quad} & y \\
& \searrow & \\
& & \sum e
\end{array}
\quad \begin{array}{l} \text{by Rule 3.} \\ \text{No information transmission} \\ \text{between } \mathcal{D}\underline{x} \text{ and } \underline{e}. \end{array}
\\
= \begin{array}{ccc}
& \sum w & \\
& \downarrow & \swarrow \text{dashed } \sum u \\
x & \xrightarrow{\quad} & y \\
& \uparrow \sum h & \\
& & \sum e
\end{array}
\end{array}$$

QED

**Claim 4** (from Ref.[1])

If  $\underline{w} \leftarrow \underline{s}$  then



$$P(y|\mathcal{D}\underline{x} = x) = \sum_z P(y|x, z, w, \underline{s} = 1) P(z|w, \underline{s} = 1) \quad (1.9)$$

$$\mathcal{D}\underline{x} = x \longrightarrow y = \sum z \begin{array}{c} \underline{w} = w \quad \underline{s} = 1 \\ \swarrow \quad \searrow \quad \downarrow \\ \underline{x} = x \longrightarrow y \\ \text{---} \end{array} \quad (1.10)$$

**proof:**

$$P(y|\mathcal{D}\underline{x} = x) = P(y|\mathcal{D}\underline{x} = x, w, \underline{s} = 1)$$

$$\sum z \begin{array}{c} \sum w \longleftarrow \sum s \\ \swarrow \quad \searrow \\ \mathcal{D}\underline{x} = x \longrightarrow y \end{array} = \sum z \begin{array}{c} \underline{w} = w \longleftarrow \underline{s} = 1 \\ \swarrow \quad \searrow \\ \mathcal{D}\underline{x} = x \longrightarrow y \end{array} \quad \text{by Rule 1}$$

$$= \sum_z P(y|\mathcal{D}\underline{x} = x, z, w, \underline{s} = 1) P(z|\mathcal{D}\underline{x} = x, w, \underline{s} = 1)$$

$$= \sum z \begin{array}{c} \underline{w} = w \longleftarrow \underline{s} = 1 \\ \swarrow \quad \searrow \quad \downarrow \\ \mathcal{D}\underline{x} = x \longrightarrow y \\ \text{---} \end{array}$$

$$= \sum_z P(y|\underline{x} = x, z, w, \underline{s} = 1) P(z|\mathcal{D}\underline{x} = x, w, \underline{s} = 1)$$

$$= \sum z \begin{array}{c} \underline{w} = w \longleftarrow \underline{s} = 1 \\ \swarrow \quad \searrow \quad \downarrow \\ \mathcal{D}\underline{x} = x \longrightarrow y \\ \text{---} \end{array} \quad \text{by Rule 2}$$

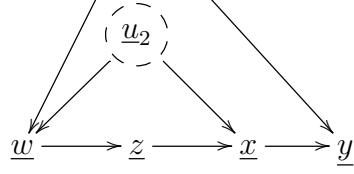
$$= \sum_z P(y|\underline{x} = x, z, w, \underline{s} = 1) P(z|w, \underline{s} = 1)$$

$$= \sum z \begin{array}{c} \underline{w} = w \quad \underline{s} = 1 \\ \swarrow \quad \searrow \quad \downarrow \\ x \longrightarrow y \\ \text{---} \end{array} \quad \begin{array}{l} \text{by Rule 3.} \\ \text{No info transmission} \\ \text{between } \mathcal{D}\underline{x} \text{ and } \underline{z}. \end{array}$$

**Claim 5** (*Napkin problem from Ref.[2]*)



If  $(\underline{u}_1)$  then



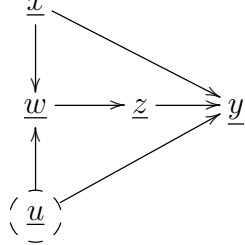
$$P(y|\mathcal{D}\underline{x} = x) = \tag{1.11}$$

**proof:** coming soon

**QED**

**Claim 6** (from Ref.[2])

If  $\underline{x}$  then



$$P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{z} = z) = \tag{1.12}$$

**proof:** coming soon

**QED**

# Bibliography

- [1] Paul Hunermund and Elias Bareinboim. Causal inference and data fusion in econometrics. <https://arxiv.org/abs/1912.09104>, 2021.
- [2] Judea Pearl and Dana Mackenzie. *The book of why: the new science of cause and effect*. Basic Books, 2018.