Chapter 1

Do Calculus proofs

In Chapter **??** of Bayesuvius, we explained Do Calculus but referred to this chapter for proofs of claims that use Do Calculus. In this chapter, we've aggregated all proofs, from throughout the book, of claims that use Do Calculus.

Note that even though the 3 rules of Do Calculus are great for proving adjustment formulae for general classes of DAGs, they are sometimes overkill for proving adjustment formulae for a single specific DAG. After all, the 3 rules of Do Calculus are a consequence of the d-separation theorem. Hence, all adjustment formulae should be provable from first principles, assuming only the d-separation theorem and the standard rules of probability theory.

We use the following conventions. Random variables are underlined and their values are not. For example, $\underline{a} = a$ means the random variable $\underline{a}$ takes the value a. Diagrams with nodes that are underlined represent Bayesian Networks (bnets) and the same diagram with the letters not underlined represents a specific instantiation of that bnet. For example $\underline{a} \rightarrow \underline{b}$ represents the bnet with conditional probability distribution $P(b|a)$, whereas $a \rightarrow b$ represents $P(b|a)$ itself.

If $\underline{a}$ is a root node, then $\sum a$ signifies a weighted sum $\sum_a P(a)$. For example, $\sum a \rightarrow b = \sum_a P(a)P(b|a)$. If $\underline{a}$ is not a root node as in $x \rightarrow \sum a \rightarrow y = \sum_a P(y|a)P(a|x)$, then $\sum a$ signifies a simple unweighted sum $\sum_a$.

Unobserved nodes are indicated by enclosing them in a dashed circle. For example, u.

Selection diagrams with selection nodes are discussed in Chapter **??**. In a selection diagram with a selection node $\underline{s} \in \{0, 1\}$, if a node $\underline{x}$ has parents $pa(\underline{x})$ where $\underline{s} \notin pa(\underline{x})$, then the TPM of $\underline{x}$ is $P(x|pa(x))$. If, on the other hand, $\underline{x}$ has parents $pa(\underline{x}) \cup \underline{s}$, then the TPM of $\underline{x}$ is $P(x|pa(x), s)$, where $P(x|pa(x), \underline{s} = 0) = P(x|pa(x))$ and $P(x|pa(x), \underline{s} = 1) = P^*(x|pa(x))$.

Some identities that are used in this chapter:

1.
$$P(y|x_1, x_2) = \sum_a P(y|a, x_1, x_2)P(a|x_1, x_2) \; . \tag{1.1}$$

$$
\begin{tikzcd}[column sep=small]
x_1 \arrow[dr] & & & x_1 \arrow[d] \arrow[drr] & & \\
& y & = & \sum a \arrow[rr] & & y \\
x_2 \arrow[ur] & & & x_2 \arrow[u] \arrow[urr] & &
\end{tikzcd}
\quad . \tag{1.2}
$$

One can describe this identity as “giving $\underline{y}$ a universal backdoor”, because $\sum a$ is a backdoor (i.e., input) to y, and $\sum a$ is universal in the sense that it is entered by every arrow that enters y except $\sum a$ itself.

2.

$$\sum_a P(a|x_1, x_2) = 1 \tag{1.3}$$

$$
\begin{tikzcd}
x_1 \arrow[dr] & & \\
& \sum a \arrow[r, "0"'] & {} \\
x_2 \arrow[ur] & &
\end{tikzcd}
= 1 \tag{1.4}
$$

One can describe this identity as “summing over the values of a collider node which has no emerging arrows”[1]. Eq.(1.4) can be understood as an edge case (when $\underline{y} = \emptyset$) of Eq.(1.2).

3.

$$\sum_a P(x_2|a)P(a|x_1) = P(x_2|x_1) \tag{1.5}$$

$$x_1 \longrightarrow \sum a \longrightarrow x_2 \qquad = \qquad x_1 \longrightarrow x_2 \tag{1.6}$$

One can describe this identity as “summing over the values of a mediator node”.

4.

$$P(x) = \sum_a P(x|a)P(a) = \sum_b P(x|b)P(b) \tag{1.7}$$

$$P(x) \qquad = \qquad \xrightarrow[0]{} \sum a \longrightarrow x \qquad = \qquad \xrightarrow[0]{} \sum b \longrightarrow x \tag{1.8}$$

One can describe this identity as “averaging over different priors”. Eq.(1.8) can be understood as an edge case of Eq.(1.6).

[1]A zeroed arrow means the same as no arrow.

Claim 1 *(Backdoor Adjustment Formula)*

If
$$\begin{tikzcd} \underline{z} \arrow[d] \arrow[dr] & \\ \underline{x} \arrow[r] & \underline{y} \end{tikzcd}$$
then

$$P(y|\mathcal{D}\underline{x}=x) = \sum_z P(y|x,z)P(z) \tag{1.9}$$

$$\mathcal{D}\underline{x}=x \longrightarrow y \quad = \quad \begin{tikzcd} \sum z \arrow[dr] & \\ x \arrow[r] & y \end{tikzcd} \tag{1.10}$$

proof:

*** proof 1:**

$$P(y|\mathcal{D}\underline{x}=x) = \textstyle\sum_z P(y|x,z)P(z)$$

$$\begin{tikzcd} \sum z \arrow[drr] & & \\ \mathcal{D}\underline{x}=x \arrow[rr] & & y \end{tikzcd} \quad = \quad \begin{tikzcd} \sum z \arrow[dr] & \\ x \arrow[r] & y \end{tikzcd}$$

*** proof 2:**

$P(y|\mathcal{D}\underline{x}=x) = \sum_z P(y|\mathcal{D}\underline{x}=x,z)P(z|\mathcal{D}\underline{x}=x)$

by Probability Axioms

$= \sum_z P(y|x,z)P(z|\mathcal{D}\underline{x}=x)$

$P(y|\mathcal{D}\underline{x}=x,z) \to P(y|x,z)$

by Rule 2: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\mathcal{L}_{\underline{a}.}\mathcal{D}_{\underline{r}.}G$, then $\mathcal{D}\underline{a}. = a. \leftrightarrow \underline{a}. = a.$

$\underline{y} \perp \underline{x}|\underline{z}$ in $\mathcal{L}_{\underline{x}}\mathcal{D}_{\emptyset}G$:
$$\begin{tikzcd} \underline{z} \arrow[d] \arrow[dr] & \\ \underline{x} & \underline{y} \end{tikzcd}$$

$= \sum_z P(y|x,z)P(z)$

$P(z|\mathcal{D}\underline{x}=x) \to P(z)$

by Rule 3: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\mathcal{D}_{\underline{a}.-an(\underline{s}.)}\mathcal{D}_{\underline{r}.}G$, then $\mathcal{D}\underline{a}. = a. \leftrightarrow 1$

$\underline{z} \perp \underline{x}$ in $\mathcal{D}_{\underline{x}}\mathcal{D}_{\emptyset}G$:
$$\begin{tikzcd} \underline{z} \arrow[dr] & \\ \underline{x} \arrow[r] & \underline{y} \end{tikzcd}$$

QED

Claim 2 *(Frontdoor Adjustment Formula)*

If

$$\begin{tikzcd} & \underline{c} \arrow[dl] \arrow[dr] & \\ \underline{x} \arrow[r] & \underline{m} \arrow[r] & \underline{y} \end{tikzcd}$$

then

$$P(y|\mathcal{D}\underline{x} = x) = \sum_m \left[\sum_{x'} P(y|x', m)P(x') \right] P(m|x) \tag{1.11}$$

$$\begin{tikzcd} \mathcal{D}\underline{x} = x \arrow[r] & y \end{tikzcd} = \begin{tikzcd} & \sum x' \arrow[dr] & \\ x \arrow[r] & \sum m \arrow[r] & y \end{tikzcd} \tag{1.12}$$

proof:

*** proof 1:**

$P(y|\mathcal{D}\underline{x} = x) = \sum_{m,c,x'} P(y|m, c)P(c|x')P(x')P(m|\mathcal{D}\underline{x} = x)$

$$\begin{tikzcd} & & \sum c \arrow[dr] & \\ \mathcal{D}\underline{x} = x \arrow[r] & \sum m \arrow[rr] & & y \end{tikzcd} = \begin{tikzcd} & \sum x' \arrow[r] & \sum c \arrow[dr] & \\ \mathcal{D}\underline{x} = x \arrow[r] & \sum m \arrow[rr] & & y \end{tikzcd}$$

$= \sum_{m,x'} P(y|m, x')P(x')P(m|x)$

$$= \begin{tikzcd} & \sum x' \arrow[dr] & \\ x \arrow[r] & \sum m \arrow[r] & y \end{tikzcd}$$

*** proof 2:**

$P(y|\mathcal{D}\underline{x} = x) = \sum_m P(y|\mathcal{D}\underline{x} = x, m)P(m|\mathcal{D}\underline{x} = x)$

by Probability Axioms

$= \sum_m P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m)P(m|\mathcal{D}\underline{x} = x)$

$P(y|\mathcal{D}\underline{x} = x, m) \to P(y|\mathcal{D}\underline{x} = x, \mathcal{D}m = m)$

by Rule 2: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\mathcal{L}_{\underline{a}.}\mathcal{D}_{\underline{r}.}G$, then
$\mathcal{D}\underline{a}. = a. \leftrightarrow \underline{a}. = a.$

$\underline{y} \perp \underline{m}|\underline{x}$ in $\mathcal{L}_{\underline{m}}\mathcal{D}_{\underline{x}}G$:

$$\begin{tikzcd}
 & \underline{c} \arrow[dr] & \\
\underline{x} \arrow[r] & \underline{m} & \underline{y}
\end{tikzcd}$$

$= \sum_m P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m)P(m|x)$

$P(m|\mathcal{D}\underline{x} = x) \to P(m|x)$

by Rule 2: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\mathcal{L}_{\underline{a}.}\mathcal{D}_{\underline{r}.}G$, then $\mathcal{D}\underline{a}. = a. \leftrightarrow \underline{a}. = a.$

$\underline{m} \perp \underline{x}$ in $\mathcal{L}_{\underline{x}}\mathcal{D}_{\emptyset}G$:

$$\begin{tikzcd}
 & \underline{c} \arrow[dl] \arrow[dr] & \\
\underline{x} & \underline{m} \arrow[r] & \underline{y}
\end{tikzcd}$$

$= \sum_m P(y|\mathcal{D}\underline{m} = m)P(m|x)$

$P(y|\mathcal{D}\underline{x} = x, \mathcal{D}\underline{m} = m) \to P(y|\mathcal{D}\underline{m} = m)$

by Rule 3: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\mathcal{D}_{\underline{a}.-an(\underline{s}.)}\mathcal{D}_{\underline{r}.}G$, then $\mathcal{D}\underline{a}. = a. \leftrightarrow 1$

$\underline{y} \perp \underline{x}|\underline{m}$ in $\mathcal{D}_{\underline{x}}\mathcal{D}_{\underline{m}}G$:

$$\begin{tikzcd}
 & \underline{c} \arrow[dr] & \\
\underline{x} & \underline{m} \arrow[r] & \underline{y}
\end{tikzcd}$$

$= \sum_{x'} \sum_m P(y|\mathcal{D}\underline{m} = m, x')P(x'|\mathcal{D}\underline{m} = m)P(m|x)$

by Probability Axioms

$= \sum_{x'} \sum_m P(y|m, x')P(x'|\mathcal{D}\underline{m} = m)P(m|x)$

$P(y|\mathcal{D}\underline{m} = m, x') \to \quad P(y|m, x')$

by Rule 2: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\mathcal{L}_{\underline{a}.}\mathcal{D}_{\underline{r}.}G$, then $\mathcal{D}\underline{a}. = a. \leftrightarrow \underline{a}. = a.$

$\underline{y} \perp \underline{m}|\underline{x}$ in $\mathcal{L}_{\underline{m}}\mathcal{D}_{\emptyset}G$:

$$\begin{tikzcd}
 & \underline{c} \arrow[dl] \arrow[dr] & \\
\underline{x} \arrow[r] & \underline{m} & \underline{y}
\end{tikzcd}$$

$= \sum_{x'} \sum_m P(y|m, x')P(x')P(m|x)$

$P(x'|\mathcal{D}\underline{m} = m) \to P(x')$

by Rule 3: If $(\underline{b}. \perp \underline{a}.|\underline{r}., \underline{s}.)$ in $\mathcal{D}_{\underline{a}.-an(\underline{s}.)}\mathcal{D}_{\underline{r}.}G$, then $\mathcal{D}\underline{a}. = a. \leftrightarrow 1$

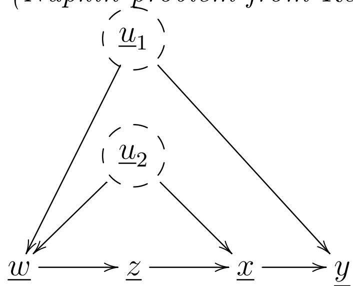
$\underline{x} \perp \underline{m}$ in $\mathcal{D}_{\underline{m}}\mathcal{D}_{\emptyset}G$:

$$\begin{tikzcd}
 & \underline{c} \arrow[dl] \arrow[dr] & \\
\underline{x} & \underline{m} \arrow[r] & \underline{y}
\end{tikzcd}$$

QED

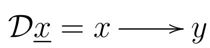
Claim 3 *(Napkin problem from Ref.[3])*

If

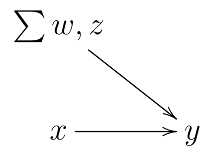


then

$$P(y|\mathcal{D}\underline{x} = x) = \sum_{w,z} P(y|x, w, z)P(w, z) \tag{1.13}$$



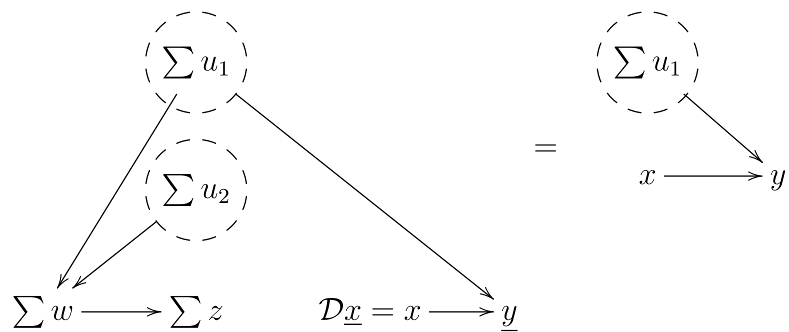
$=$



(1.14)

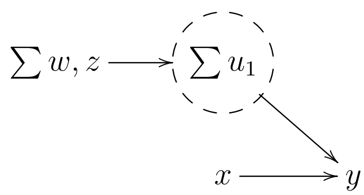
proof:

$P(y|\mathcal{D}\underline{x} = x) = \sum_{u_1} P(y|x, u_1)P(u_1)$



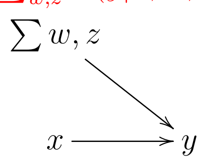
$= \sum_{w,z} \sum_{u_1} P(y|x, u_1)P(u_1|w, z)P(w, z)$

$=$



$= \sum_{w,z} P(y|x, w, z)P(w, z)$

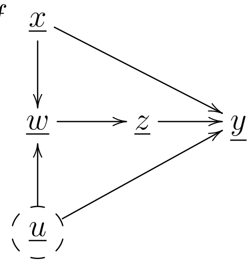
$=$



QED

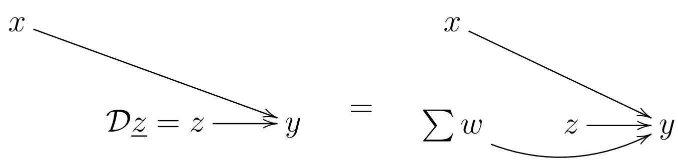
Claim 4 *(from Ref.[3])*

If



then

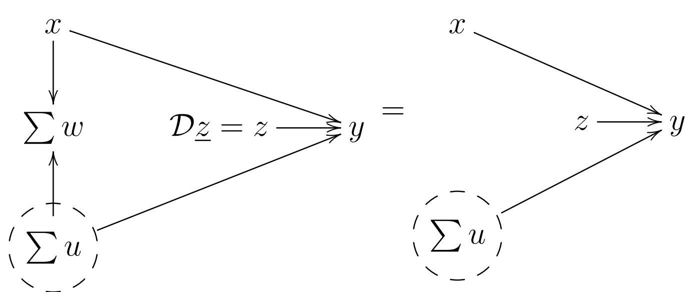
$$P(y|\mathcal{D}\underline{z} = z, x) = \sum_w P(y|z, x, w)P(w) \tag{1.15}$$



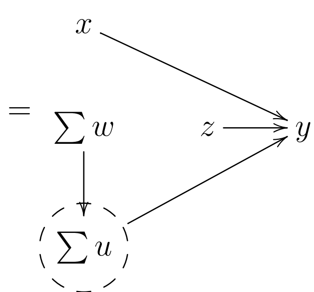
(1.16)

proof:

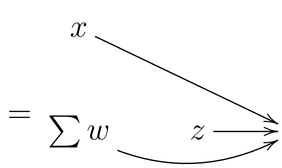
$P(y|\mathcal{D}\underline{z} = z, x) = \sum_u P(y|z, x, u)P(u)$



$= \sum_w \sum_u P(y|z, x, u)P(u|w)P(w)$



$= \sum_w P(y|z, x, w)P(w)$



QED

Claim 5 *(Trivial Memoryless Transportability, from Ref.[2])*
If

$$\begin{tikzcd}
 & \underline{z} \arrow[dl] \arrow[dr] & \underline{s} \arrow[d] \\
\underline{x} \arrow[rr] \arrow[ur, dashed, bend left, <->] \arrow[rr, dashed, bend right, <->] & & \underline{y}
\end{tikzcd}$$

where $\underline{s} \in \{0,1\}$ *is a selection node, then*

$$P^*(y|\mathcal{D}\underline{x}=x,z) = P^*(y|x,z) \quad \textit{(replace } \mathcal{D} \textit{ by 1, keep } P^*\textit{)} \tag{1.17}$$

$$\begin{tikzcd}
 & z \arrow[dr] & \underline{s}=1 \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[rr] & & y
\end{tikzcd}
=
\begin{tikzcd}
 & z \arrow[dr] & \underline{s}=1 \arrow[d] \\
x \arrow[rr] & & y
\end{tikzcd} \tag{1.18}$$

proof:

$$P(y|\mathcal{D}\underline{x}=x,z,\underline{s}=1) = P(y|x,z,\underline{s}=1)$$

$$\begin{tikzcd}
 & z \arrow[dr] & \underline{s}=1 \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[rr] & & y
\end{tikzcd}
=
\begin{tikzcd}
 & z \arrow[dr] & \underline{s}=1 \arrow[d] \\
x \arrow[rr] & & y
\end{tikzcd}$$

QED

Claim 6 *(Direct Transportability, a.k.a. External Validity, from Ref.[2])*
If

$$\begin{tikzcd}
\underline{s} \arrow[r] & \underline{z} \arrow[dl] \arrow[dr] & \\
\underline{x} \arrow[rr] \arrow[ur, dashed, bend left, <->] \arrow[rr, dashed, bend right, <->] & & \underline{y}
\end{tikzcd}$$

where $\underline{s} \in \{0,1\}$ *is a selection node, then*

$$P^*(y|\mathcal{D}\underline{x}=x,z) = P(y|\mathcal{D}\underline{x}=x,z) \quad \textit{(replace } P^* \textit{ by } P\textit{, keep } \mathcal{D}\textit{)} \tag{1.19}$$

$$\begin{tikzcd}
\underline{s}=1 \arrow[r] & z \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[r] & y
\end{tikzcd}
=
\begin{tikzcd}
 & z \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[r] & y
\end{tikzcd} \tag{1.20}$$

Furthermore,

$$P^*(y|\mathcal{D}\underline{x}=x) = \sum_z P(y|\mathcal{D}\underline{x}=x,z)P^*(z) \tag{1.21}$$

$$
\begin{tikzcd}
\underline{s}=1 \arrow[drr] & & \\
\mathcal{D}\underline{x}=x \arrow[rr] & & y
\end{tikzcd}
=
\begin{tikzcd}
\underline{s}=1 \arrow[r] & \sum z \arrow[dr] & \\
\mathcal{D}\underline{x}=x \arrow[rr] & & y
\end{tikzcd}
\tag{1.22}
$$

proof:

$$
\color{red}{P(y|\mathcal{D}\underline{x}=x,z,\underline{s}=1)=P(y|\mathcal{D}\underline{x}=x,z)}
$$

$$
\begin{tikzcd}
\underline{s}=1 \arrow[r] & z \arrow[dr] & \\
\mathcal{D}\underline{x}=x \arrow[rr] & & y
\end{tikzcd}
=
\begin{tikzcd}
 & z \arrow[dr] & \\
\mathcal{D}\underline{x}=x \arrow[rr] & & y
\end{tikzcd}
\qquad \text{Because } \underline{s}\perp\underline{y}|\underline{z}
$$

Furthermore,

$$
\color{red}{P(y|\mathcal{D}\underline{x}=x,\underline{s}=1)=\textstyle\sum_z P(y|\mathcal{D}\underline{x}=x,z)P(z|\underline{s}=1)}
$$

$$
\begin{tikzcd}
\underline{s}=1 \arrow[r] & \sum z \arrow[dr] & \\
\mathcal{D}\underline{x}=x \arrow[rr] & & y
\end{tikzcd}
$$

QED

Claim 7 *(S-Admisssible Transportability, from Ref.[2])*

If
$$
\begin{tikzcd}
\underline{s} \arrow[r] & \underline{z} \arrow[r] \arrow[dl] \arrow[dl, dashed, bend right, leftrightarrow] & \underline{a} \arrow[d] \\
\underline{x} \arrow[rr] \arrow[rr, dashed, bend right, leftrightarrow] & & \underline{y}
\end{tikzcd}
$$
where $\underline{s}\in\{0,1\}$ *is a selection node, then*

$$
P^*(y|\mathcal{D}\underline{x}=x)=\sum_a P(y|\mathcal{D}\underline{x}=x,a)P^*(a) \tag{1.23}
$$

$$
\begin{tikzcd}
\underline{s}=1 \arrow[dr] & \\
\mathcal{D}\underline{x}=x \arrow[r] & y
\end{tikzcd}
=
\begin{tikzcd}
\underline{s}=1 \arrow[r] & \sum a \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[r] & y
\end{tikzcd}
\tag{1.24}
$$

proof:

$$
\color{red}{P(y|\mathcal{D}\underline{x}=x,\underline{s}=1)=\textstyle\sum_a P(y|\mathcal{D}\underline{x}=x,a)P(a|\underline{s}=1)}
$$

$$
\begin{tikzcd}
\underline{s}=1 \arrow[r] & \sum z \arrow[r] & \sum a \arrow[d] \\
& \mathcal{D}\underline{x}=x \arrow[r] & y
\end{tikzcd}
=
\begin{tikzcd}
\underline{s}=1 \arrow[r] & \sum a \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[r] & y
\end{tikzcd}
$$

QED

Claim 8 *(Non-transportability, from Ref.[2])*

If
$$
\begin{tikzcd}
& \underline{h} \arrow[dl] \arrow[dr] & \underline{s} \arrow[d] \\
\underline{x} \arrow[rr] & & \underline{y}
\end{tikzcd}
$$
where $\underline{s} \in \{0,1\}$ is a selection node, then

$$P^*(y|\mathcal{D}\underline{x}=x) = P^*(y|\mathcal{D}\underline{x}=x) \tag{1.25}$$

$$
\begin{tikzcd}
& \underline{s}=1 \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[r] & y
\end{tikzcd}
= \textit{same} \tag{1.26}
$$

proof:

$$P^*(y|\mathcal{D}\underline{x}=x) = P^*(y|\mathcal{D}\underline{x}=x)$$

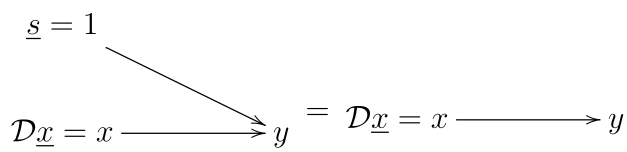
$$
\begin{tikzcd}
& \sum h \arrow[dr] & \underline{s}=1 \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[rr] & & y
\end{tikzcd}
=
\begin{tikzcd}
& \underline{s}=1 \arrow[d] \\
\mathcal{D}\underline{x}=x \arrow[r] & y
\end{tikzcd}
$$

Can't replace $\mathcal{D}\underline{x}=x$ by x because $\underline{y} \not\perp \underline{x}$ in $\mathcal{L}_{\underline{x}}G$. Hence, Rule 2 not satisfied.
QED

Claim 9 *(from Ref.[2])*

If
$$
\begin{tikzcd}
\underline{s} \arrow[rr, bend left] & \underline{h} \arrow[r] \arrow[dl] \arrow[dr] & \underline{z} \\
\underline{x} \arrow[rr] \arrow[ur, dashed, bend left, leftrightarrow] \arrow[rr, dashed, bend right, leftrightarrow] & & \underline{y}
\end{tikzcd}
$$
where $\underline{s} \in \{0,1\}$ is a selection node, then

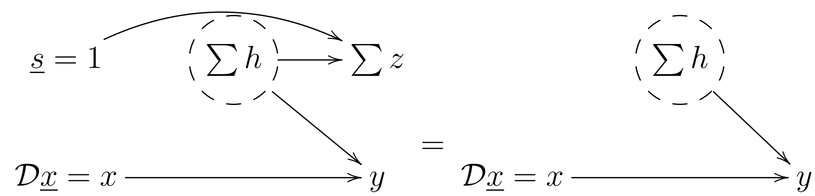
$$P^*(y|\mathcal{D}\underline{x}=x) = P(y|\mathcal{D}\underline{x}=x) \tag{1.27}$$



(1.28)

proof:

$P(y|\mathcal{D}\underline{x} = x, \underline{s} = 1) = \sum_h P(y|\mathcal{D}\underline{x} = x, h)P(h)$



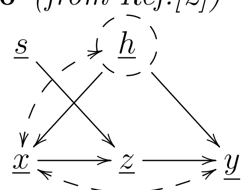
$= P(y|\mathcal{D}\underline{x} = x)$

$= \mathcal{D}\underline{x} = x \longrightarrow y$

QED

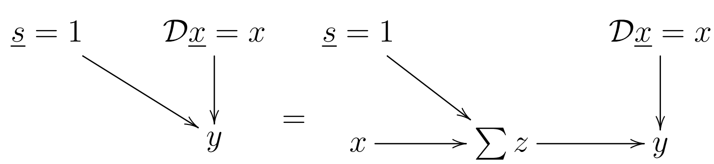
Claim 10 *(from Ref.[2])*

If



where $\underline{s} \in \{0, 1\}$ *is a selection node, then*

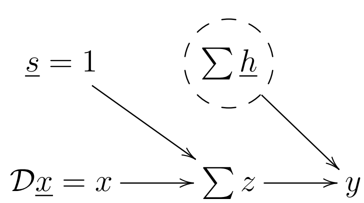
$$P^*(y|\mathcal{D}\underline{x} = x) = \sum_z P(y|\mathcal{D}\underline{x} = x, z)P^*(z|x) \tag{1.29}$$



(1.30)

proof:

$P(y|\mathcal{D}\underline{x} = x, \underline{s} = 1) = \sum_h \sum_z P(y|h, z)P(h)P(z|\mathcal{D}\underline{x} = x, \underline{s} = 1)$



$= \sum_h \sum_z P(y|h, z)P(h|\mathcal{D}\underline{x} = x)P(z|x, \underline{s} = 1)$

$$
= \begin{tikzcd}
\underline{s}=1 \arrow[dr] & \sum \underline{h} \arrow[dr] & \mathcal{D}\underline{x}=x \arrow[l] \\
x \arrow[r] & \sum z \arrow[r] & y
\end{tikzcd}
$$

$$
= \sum_z P(y|\mathcal{D}\underline{x}=x, z)P(z|x, \underline{s}=1)
$$

$$
= \begin{tikzcd}
\underline{s}=1 \arrow[dr] & & \mathcal{D}\underline{x}=x \arrow[d] \\
x \arrow[r] & \sum z \arrow[r] & y
\end{tikzcd}
$$

QED

Claim 11 *(Unconfounded Mediation, from Ref.[1])*

If
$$
\begin{tikzcd}
& \underline{m} \arrow[dr] & \\
\underline{d} \arrow[ur] \arrow[rr] & & \underline{y}
\end{tikzcd}
$$
then

$$
P(y|\mathcal{D}\underline{d}=d, \mathcal{I}_{\underline{m}}\underline{d}=d') = \sum_m P(y|d,m)P(m|d') \tag{1.31}
$$

$$
\begin{tikzcd}
\mathcal{I}\underline{d}=d' \arrow[dr] & \\
\mathcal{D}\underline{d}=d \arrow[r] & y
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{I}\underline{d}=d' \arrow[r] & \sum m \arrow[dr] & \\
\mathcal{D}\underline{d}=d \arrow[rr] & & y
\end{tikzcd} \tag{1.32}
$$

proof:

$$
P(y|\mathcal{D}\underline{d}=d, \mathcal{I}\underline{d}=d') = \sum_m P(y|d,m)P(m|d')
$$

$$
\begin{tikzcd}
\mathcal{I}\underline{d}=d' \arrow[r] & \sum m \arrow[dr] & \\
\mathcal{D}\underline{d}=d \arrow[rr] & & y
\end{tikzcd}
$$

QED

Claim 12 *(Mediation with universal prior $\underline{\xi}$ and universal confouder $\underline{u}$, from Ref.[1])*

If

$$\begin{tikzcd}
 & \underline{u} \arrow[dl] \arrow[d] \arrow[ddr] \arrow[ddl] & \\
\underline{\xi} \arrow[r] \arrow[d] \arrow[ddr, phantom] \arrow[drr] & \underline{m} \arrow[dr] & \\
\underline{d} \arrow[ur] \arrow[rr] & & \underline{y}
\end{tikzcd}$$

then

$$P(y|\mathcal{D}\underline{d}=d,\mathcal{I}_{\underline{m}}\underline{d}=d')=\sum_{\xi}\sum_{m}P(y|d,m,\xi)P(m|d',\xi)P(\xi) \tag{1.33}$$

$$\begin{tikzcd}
\mathcal{I}\underline{d}=d' \arrow[dr] & \\
\mathcal{D}\underline{d}=d \arrow[r] & y
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{I}\underline{d}=d' \arrow[rr, bend right] & \sum\xi \arrow[r] \arrow[dr] & \sum m \arrow[d] \\
\mathcal{D}\underline{d}=d \arrow[rr] & & y
\end{tikzcd} \tag{1.34}$$

proof:

$$P(y|\mathcal{D}\underline{d}=d,\mathcal{I}\underline{d}=d')=\sum_{\xi,u}\sum_{m}P(y|d,m,\xi,u)P(m|d',\xi,u)\underbrace{P(\xi|u)P(u)}_{P(\xi,u)}$$

$$\begin{tikzcd}
 & & \sum u \arrow[dl] \arrow[d] \arrow[ddr] & \\
\mathcal{I}\underline{d}=d' \arrow[rr, bend right] & \sum\xi \arrow[r] \arrow[drr] & \sum m \arrow[dr] & \\
\mathcal{D}\underline{d}=d \arrow[rrr] & & & y
\end{tikzcd}$$

$$=\sum_{\xi}\sum_{m}P(y|d,m,\xi)P(m|d',\xi)P(\xi)$$

$$=\begin{tikzcd}
\mathcal{I}\underline{d}=d' \arrow[rr, bend right] & \sum\xi \arrow[r] \arrow[dr] & \sum m \arrow[d] \\
\mathcal{D}\underline{d}=d \arrow[rr] & & y
\end{tikzcd}$$

We switch from averaging over the prior of ξ, u to averaging over the prior of ξ.

QED

Bibliography

[1] Judea Pearl. Causal and counterfactual inference. *The Handbook of Rationality*, pages 1–41, 2019. `https://ftp.cs.ucla.edu/pub/stat_ser/r485.pdf`.

[2] Judea Pearl and Elias Bareinboim. Transportability of causal and statistical relations: A formal approach. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 25, 2011. `https://ojs.aaai.org/index.php/AAAI/article/view/7861`.

[3] Judea Pearl and Dana Mackenzie. *The book of why: the new science of cause and effect*. Basic Books, 2018.