Aequationes Mathematicae 19 (1979) 194–226
University of Waterloo

Birkhäuser Verlag, Basel

On the simultaneous associativity of $F(x, y)$ and $x+y-F(x, y)$.

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1. Introduction

Consider the set of continuous two-place functions F from the unit square $[0, 1]^2$ to the unit interval $[0, 1]$ which satisfy the boundary conditions,

$$F(0, x) = F(x, 0) = 0, \qquad F(1, x) = F(x, 1) = x, \quad \text{for every } x.$$

Given F, define the function $F^\wedge$ on $[0, 1]^2$ via

$$F^\wedge(x, y) = x + y - F(x, y).$$

The main purpose of this paper is to find all functions F such that *both* F and $F^\wedge$ are associative.

This problem originally arose in connection with the author's work on a certain family of binary operations defined on the space Δ of probability distribution functions. Briefly, these are the operations $\sigma_{C,+}$, one for each copula C (the copulas form a subclass of the functions F), defined via

$$\sigma_{C,+}(D_1, D_2)(x) = \iint\limits_{u+v\leq x} dC(D_1(u), D_2(v)), \qquad D_1, D_2 \in \Delta.$$

In [2] we obtained the complete solution of the associativity equation for this family; and, as one step of the proof (Theorem 9), we found that the associativity of C and $C^\wedge$ are necessary conditions for the associativity of $\sigma_{C,+}$.

The problem merits investigation in its own right, both as an interesting functional equation and as a natural question about semigroups on an interval. Moreover, the family $\sigma_{C,+}$ can be embedded in a much wider class of operations

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AMS (1970) subject classification: Primary 39A30. Secondary 22A15, 60E05.

Manuscript received October 26, 1976; final version received June 30, 1977.

on Δ in which the simultaneous associativity plays the same role, but where the ad hoc methods employed in [2] fail. This generalization is discussed in Section 5.

Clearly the above problem is equivalent to finding all solution pairs (F, G) of the system of equations,

$$F(x, y)+G(x, y)=x+y, \tag{1.1a}$$

$$F(F(x, y), z)=F(x, F(y, z)), \tag{1.1b}$$

$$G(G(x, y), z)=G(x, G(y, z)), \tag{1.1c}$$

where F and G satisfy the following conditions:

$$\operatorname{Dom} F=\operatorname{Dom} G=[0, 1]^2, \quad \operatorname{Ran} F=\operatorname{Ran} G=[0, 1]; \tag{1.2a}$$

$$F \text{ and } G \text{ are continuous;} \tag{1.2b}$$

$$F(0, x)=F(x, 0)=0, \qquad F(1, x)=F(x, 1)=x, \quad \text{for every} \quad x\in[0, 1]; \tag{1.2c}$$

$$G(0, x)=G(x, 0)=x, \qquad G(1, x)=G(x, 1)=1, \quad \text{for every} \quad x\in[0, 1]. \tag{1.2d}$$

Observe that the roles of F and G are interchangeable except for the boundary conditions (1.2c) and (1.2d). Also, the problem is equivalent to finding all continuous functions $G:[0, 1]^2\to[0, 1]$ satisfying the boundary conditions (1.2d) such that both G and $G^\wedge$ are associative. In algebraic notation, writing $F(x, y)=x\circ y$ and $G(x, y)=x * y$, a solution of (1.1) consists of a pair of topological semigroups on $[0, 1]$ that are related by

$$(x\circ y)+(x * y)=x+y.$$

Remark. Once either function of a solution (F, G) of the system (1.1) is specified, the other one is completely determined by equation (1.1a). Thus there is actually just one unknown function in the system (1.1). For instance, putting $G(x, y)=x+y-F(x, y)$ into (1.1c) and simplifying, we obtain:

$$F(x+y-F(x, y), z)+F(x, y)=F(x, y+z-F(y, z))+F(y, z), \tag{1.3}$$

so the system (1.1) is equivalent to the system $\{(1.1b), (1.3)\}$. The apparent complication created by introducing a second (redundant) function will greatly facilitate the complete solution of the problem.

Consider the sets of functions F_s and G_s, $0 \leq s \leq \infty$, defined on $[0, 1]^2$ via

$$F_0(x, y) = \text{minimum}\,(x, y), \qquad G_0(x, y) = \text{maximum}\,(x, y), \tag{1.4}$$

$$F_1(x, y) = x \cdot y, \qquad G_1(x, y) = x + y - x \cdot y, \tag{1.5}$$

$$F_\infty(x, y) = \text{maximum}\,(x + y - 1, 0), \qquad G_\infty(x, y) = \text{minimum}\,(x + y, 1), \tag{1.6}$$

$$\left.\begin{aligned} &F_s(x, y) = \log_s\,[1 + (s^x - 1)(s^y - 1)/(s - 1)], \\ &G_s(x, y) = 1 - \log_s\,[1 + (s^{1-x} - 1)(s^{1-y} - 1)/(s - 1)], \quad \text{for} \quad 0 < s < \infty,\ s \neq 1. \end{aligned}\right\} \tag{1.7}$$

These functions satisfy conditions (1.2), and it is readily established by direct calculations that each pair (F_s, G_s) is a solution of the system (1.1). [In [2] we conjectured that (F_1, G_1) is the only solution of (1.1) such that F and G are strictly increasing in each argument over the open unit square. (See also [11].) While trying to prove this, we discovered the family (1.7) of such solutions.]

The functions F_s, $0 \leq s \leq \infty$, form a *single family* in the sense that F_0, F_1, and F_∞ are the limits of members of the set F_s corresponding to their subscripts. Note that in view of the relation (1.1a), the identical statement then holds for the G_s. To verify this fact for F_0, observe first that for $0 < s < 1$, rearrangement of terms yields

$$F_s(x, y) = x + \log\,[1 + (s^x - 1)(s^{y-x} - s^{1-x})/(s - 1)]/\log s.$$

Hence when $0 \leq x \leq y \leq 1$, $F_s(x, y) \to x$ as $s \to 0$, and so $\lim_{s\to 0} F_s(x, y) = \min\,(x, y)$. The other limits are most easily obtained by appealing to generators of the F_s; this is done at the end of Section 2.

The family (F_s, G_s) and ordinal sums formed from its members constitute the complete solution set of (1.1). [The notion of ordinal sum is discussed in Section 2.] For we shall prove:

THEOREM 1.1. *Let F and G be functions satisfying conditions* (1.2). *Then the pair (F, G) is a solution of the system of equations* (1.1) *if and only if*

(1) *$F = F_s$, $G = G_s$, for some s, $0 \leq s \leq \infty$,*

or

(2) *F is representable as an ordinal sum of functions, each of which is a member of the family F_s, $0 < s \leq \infty$, and G is obtained from F via equation* (1.1*a*).

This paper is divided into six sections, the first of which is this introduction. The second and third sections are devoted to the proof of Theorem 1.1. In Section 2, some well known results on topological semigroups are exploited to reduce the search for solutions of the system (1.1) to those associative F and G which have the so-called Archimedean property. The representation theorem for such functions then makes possible the transformation of (1.1) to equation (2.8), which involves two functions of one variable. In Section 3, this equation is solved via reduction to several differential equations. In the fourth section, we solve some problems closely related to the main one. In the fifth section, Theorem 1.1 is employed to solve a certain distributivity equation. We then explore its consequences for the integral operations mentioned above. All of these results are easily extended to functions F and G similarly defined on any finite interval $[a, b]$. [We have restricted the discussion to $[0, 1]$ primarily for clarity of exposition.] The sixth section is devoted to stating the theorems in this more general form and to indicating what is involved in extending their proofs. Finally, an example is given which shows that the system (1.1) has other solutions when F and G are defined on the entire real line.

2. Reduction to Archimedean F and G; the generator equation

For the rest of this paper, $\mathcal{F}$ will denote the set of continuous and associative functions F from $[0, 1]^2$ to $[0, 1]$ which satisfy the boundary conditions (1.2c); similarly, $\mathcal{G}$ will denote the set of continuous and associative functions G from $[0, 1]^2$ to $[0, 1]$ which satisfy (1.2d). Thus *(F, G) is a solution of the system* (1.1) *if and only if $F \in \mathcal{F}$, $G \in \mathcal{G}$, and (F, G) satisfies equation* (1.1*a*). Observe that $\mathcal{F}$ (resp., $\mathcal{G}$) consists of all topological semigroup operations on $[0, 1]$ with null element 0 and unit 1 (resp., null element 1 and unit 0).

Some important properties of elements of $\mathcal{F}$ and $\mathcal{G}$ are collected below.

LEMMA 2.1. *If $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then:*

(A) *F and G are commutative, i.e.,*

$$F(x, y) = F(y, x), \qquad G(x, y) = G(y, x), \quad \textit{for every } x, y \in [0, 1]; \tag{2.1}$$

F and G are non-decreasing in each argument, i.e.,

$$F(x_2, y_2) - F(x_1, y_1) \geq 0, \tag{2.2a}$$

$$G(x_2, y_2) - G(x_1, y_1) \geq 0, \tag{2.2b}$$

whenever $0 \leq x_1 \leq x_2 \leq 1$ *and* $0 \leq y_1 \leq y_2 \leq 1$.

(B) *If, in addition,* (F, G) *satisfies equation* $(1.1a)$, *we have:*

$$0 \leq F(x_2, y_2) - F(x_1, y_1) \leq (x_2 - x_1) + (y_2 - y_1), \tag{2.3a}$$

$$0 \leq G(x_2, y_2) - G(x_1, y_1) \leq (x_2 - x_1) + (y_2 - y_1), \tag{2.3b}$$

whenever $0 \leq x_1 \leq x_2 \leq 1$ *and* $0 \leq y_1 \leq y_2 \leq 1$.

Proof. Part (A) follows from results in the theory of topological semigroups. There, elements of $\mathscr{F}$ are often called I-semigroups, and it is well known that they are commutative and order-preserving (i.e., non-decreasing as functions). (See [5, Lemmas 5.1.1 and 5.1.2] or [7, pages 84–87].) Now for $G \in \mathscr{G}$, let $F'(x, y) = 1 - G(1-x, 1-y)$. It is easily shown that $F' \in \mathscr{F}$. Hence F' is commutative and non-decreasing, and it follows readily that G inherits these properties.

To establish part (B), substitute $x + y - F(x, y)$ for $G(x, y)$ in (2.2b) and $x + y - G(x, y)$ for $F(x, y)$ in (2.2a).

Note that the boundary conditions (1.2c) and (1.2d) together with (2.2) immediately yield the inequalities:

$$F(x, y) \leq F_0(x, y), \qquad G(x, y) \geq G_0(x, y), \tag{2.4}$$

for every $F \in \mathscr{F}$, $G \in \mathscr{G}$, and $(x, y) \in [0, 1]^2$.

Lemma 2.1 shows that the sets $\mathscr{F}$ and $\mathscr{G}$ are precisely the so-called continuous t-norms and t-conorms, respectively–two families of functions which have been studied extensively [3; 4; 8; 9].

Remark. The continuity assumptions (1.2b) on F and G can be replaced by the monotonicity conditions (2.2) without affecting the set of solutions of the system (1.1); i.e., in the presence of the other assumptions, (1.2b) and (2.2) are interchangeable. In one direction, this statement is an immediate consequence of Lemma 2.1(A), since (1.1) and (1.2) together imply (2.2). To prove the converse, note that the proof of Lemma 2.1(B) requires only (1.1a) and (2.2). Hence non-decreasing functions F and G related by (1.1a) must satisfy the inequalities (2.3), from which the continuity of F and G immediately follow.

The initial step in proving Theorem 1.1 consists of characterizing "non-Archimedean" solutions of the system (1.1) via their "ordinal sum decompositions." In the following paragraphs, we discuss the concepts required to prove this result.

Topological semigroups on an interval $[a, b]$ of the kind under consideration can be characterized by their interior idempotents, i.e., by those $x \in (a, b)$ for which $x \circ x = x$. To facilitate this characterization, we make the following definition: a semigroup on a closed interval is said to be *Archimedean* if it has no interior idempotents.

The set of non-idempotents is a countable union of open intervals whose closures J_n are algebraically closed under the operation $\circ$, i.e., for each n, $(J_n, \circ)$ is an Archimedean subsemigroup [5; 7].

Let $\mathcal{F}_A$ and $\mathcal{G}_A$ denote the sets of Archimedean elements of $\mathcal{F}$ and $\mathcal{G}$, respectively. As a consequence of the inequalities (2.4) and the fact that $F_0(x, x) = G_0(x, x) = x$ for every x, we have:

$$\mathcal{F}_A = \{F \in \mathcal{F} : F(x, x) < x \quad \text{for every} \quad x \in (0, 1)\},$$
$$\mathcal{G}_A = \{G \in \mathcal{G} : G(x, x) > x \quad \text{for every} \quad x \in (0, 1)\}.$$

The facts presented in the two preceding paragraphs lead to the following construction of non-Archimedean elements of $\mathcal{F}$ and $\mathcal{G}$.

Let $\{J_n\}$ be a countable family of non-overlapping, closed, proper subintervals of $[0, 1]$. With each $J_n = [a_n, b_n]$, associate a function $F_n \in \mathcal{F}_A$ and a function $G_n \in \mathcal{G}_A$. Let F and G be the functions defined on $[0, 1]^2$ via

$$F(x, y) = \begin{cases} a_n + (b_n - a_n) F_n\left(\dfrac{x - a_n}{b_n - a_n}, \dfrac{y - a_n}{b_n - a_n}\right), (x, y) \in [a_n, b_n]^2, & n = 1, 2, \dots, \\ \min(x, y), \quad \text{otherwise}, \end{cases} \tag{2.5a}$$

$$G(x, y) = \begin{cases} a_n + (b_n - a_n) G_n\left(\dfrac{x - a_n}{b_n - a_n}, \dfrac{y - a_n}{b_n - a_n}\right), (x, y) \in [a_n, b_n]^2, & n = 1, 2, \dots, \\ \max(x, y), \quad \text{otherwise}. \end{cases} \tag{2.5b}$$

F and G are called the *ordinal sums* of the semigroups (J_n, F_n) and (J_n, G_n), respectively, and each F_n or G_n is called a *summand*. Straightforward calculations yield:

LEMMA 2.2. *Let F and G be the ordinal sums given by* (2.5), *where $F_n \in \mathcal{F}_A$ and $G_n \in \mathcal{G}_A$ for $n = 1, 2, \dots$. Then $F \in \mathcal{F}$ and $G \in \mathcal{G}$.*

The ordinal sums F and G given by (2.5) have idempotents $[0, 1] - \bigcup_n J_n^0$. Since the J_n are assumed to be proper, F and G are definitely not Archimedean. Note that, if $\{J_n\}$ is empty, then $F = F_0$ and $G = G_0$.

The graph of an ordinal sum F can be constructed by starting with the graph of F_0, removing those portions over the squares J_n^2 along the main diagonal, then filling in the holes with appropriately scaled down graphs of the F_n. The process for G is identical, except that one begins with the graph of G_0.

The chief importance of the ordinal sum construction is that the converse of Lemma 2.2 is true. This result is a direct consequence of [5, Theorem B].

LEMMA 2.3. *If* $F \in \mathcal{F} - \mathcal{F}_A$ *(resp., if* $G \in \mathcal{G} - \mathcal{G}_A$*), then* F *(resp.,* G*) is representable as an ordinal sum, each of whose summands is in* $\mathcal{F}_A$ *(resp.,* $\mathcal{G}_A$*).*

We are now ready to show that non-Archimedean solutions of the system (1.1) are ordinal sums of Archimedean solutions. Observe that if (F, G) is a solution of (1.1), then on setting $x = y$ in (1.1a), it is clear that F and G must have identical idempotent sets. Thus $F \in \mathcal{F}_A$ if and only if $G \in \mathcal{G}_A$, and if $F \in \mathcal{F} - \mathcal{F}_A$ and $G \in \mathcal{G} - \mathcal{G}_A$, their ordinal sum decompositions must involve identical interval sets $\{J_n\}$.

THEOREM 2.1. *The pair* (F, G) *is a solution of the system* (1.1) *if and only if one of the following statements holds:*

(1) $F \in \mathcal{F}_A$, $G \in \mathcal{G}_A$, *and* (F, G) *satisfies equation* (1.1*a*);

(2) $F \in \mathcal{F} - \mathcal{F}_A$, $G \in \mathcal{G} - \mathcal{G}_A$, *and, in the ordinal sum decompositions* (2.5) *of* F *and* G, (F_n, G_n) *satisfies equation* (1.1*a*) *for* $n = 1, 2, \ldots$.

Proof. Statement (1) is immediate. To prove (2), observe that for any $(x, y) \in J_n^2$, $n = 1, 2, \ldots$,

$$x + y = 2a_n + (b_n - a_n)\left(\frac{x - a_n}{b_n - a_n} + \frac{y - a_n}{b_n - a_n}\right),$$

$$F(x, y) + G(x, y) = 2a_n + (b_n - a_n)\left[F_n\left(\frac{x - a_n}{b_n - a_n}, \frac{y - a_n}{b_n - a_n}\right) + G_n\left(\frac{x - a_n}{b_n - a_n}, \frac{y - a_n}{b_n - a_n}\right)\right],$$

and so equation (1.1a) holds for $(x, y) \in J_n^2$ if and only if

$$F_n\left(\frac{x - a_n}{b_n - a_n}, \frac{y - a_n}{b_n - a_n}\right) + G_n\left(\frac{x - a_n}{b_n - a_n}, \frac{y - a_n}{b_n - a_n}\right) = \frac{x - a_n}{b_n - a_n} + \frac{y - a_n}{b_n - a_n}. \tag{2.6}$$

Suppose first that (F_n, G_n) is a solution of (1.1a) for each n. Then from (2.6), (2.5), and the fact that $\min(x, y) + \max(x, y) = x + y$, it follows that for any $(x, y) \in [0, 1]^2$, (F, G) satisfies (1.1a). Conversely, suppose that (F, G) satisfies (1.1a). Then (2.6) and the fact that the mapping

$$t \to (t - a_n)/(b_n - a_n)$$

is a one-to-one correspondence between J_n and $[0, 1]$ imply that (1.1a) holds for each (F_n, G_n), and the proof is complete.

Note that when $F=F_0$ and $G=G_0$, statement (2) in Theorem 2.1 holds vacuously.

To prove Theorem 1.1, it now suffices to show that the (F_s, G_s), $0<s\leq\infty$, are the only pairs of Archimedean functions which satisfy equation (1.1a). We shall accomplish this by solving an equivalent equation which involves two one-place functions.

It is well known that any continuous solution of the associativity equation (on an interval) having certain regularity properties admits a representation via a continuous and strictly monotonic one-place function [1, page 253 ff]. This result was extended to include all of $\mathcal{F}_A$ and $\mathcal{G}_A$ in [4], as summarized below.

LEMMA 2.4. (A) *For each* $F\in\mathcal{F}_A$ *there exists a strictly decreasing and continuous function* f *from* $[0, 1]$ *to* $[0, +\infty]$, *with* $f(1)=0$, *such that*

$$F(x, y)=f^{(-1)}(f(x)+f(y)), \quad \textit{for every} \quad (x, y)\in[0, 1]^2, \tag{2.7a}$$

where $f^{(-1)}$ *is given by*

$$f^{(-1)}(u)=\begin{cases} f^{-1}(u), & 0\leq u\leq f(0),\\ 0, & f(0)<u\leq\infty.\end{cases}$$

Similarly, for each $G\in\mathcal{G}_A$ *there exists a strictly increasing and continuous function* g *from* $[0, 1]$ *to* $[0,\infty]$, *with* $g(0)=0$, *such that*

$$G(x, y)=g^{(-1)}(g(x)+g(y)), \quad \textit{for every} \quad (x, y)\in[0, 1]^2, \tag{2.7b}$$

where $g^{(-1)}$ *is given by*

$$g^{(-1)}(u)=\begin{cases} g^{-1}(u), & 0\leq u\leq g(1),\\ 1, & g(1)<u\leq\infty.\end{cases}$$

The functions f *and* g *in* (2.7) *are called* additive generators (*or, briefly,* generators) *of* F *and* G.

(B) *Conversely, given any* f *and* g *as above, the functions* F *and* G *defined by* (2.7) *are elements of* $\mathcal{F}_A$ *and* $\mathcal{G}_A$, *respectively.*

(C) *Two functions are generators of the same* F *or* G *if and only if they differ by a positive multiplicative constant.*

(D) *F and G are strictly increasing in each argument over the open square* $(0,1)^2$ *if and only if their generators satisfy* $f(0)=\infty$ *and* $g(1)=\infty$, *in which case* $f^{(-1)}=f^{-1}$ *and* $g^{(-1)}=g^{-1}$. *We shall say that such F and G are* strict.

Accordingly, given $F \in \mathcal{F}_A$ and $G \in \mathcal{G}_A$, the pair (F, G) satisfies equation (1.1a) if and only if any pair of their generators (f, g) is a solution of the equation

$$f^{(-1)}(f(x)+f(y))+g^{(-1)}(g(x)+g(y))=x+y. \tag{2.8}$$

Solutions (f, g) of (2.8) fall naturally into equivalence classes – one class for each pair of Archimedean functions (F, G) satisfying (1.1a). For consider the equivalence relation defined on generators f via

$$f_1 \sim f_2 \Leftrightarrow f_1 = kf_2, \quad \text{for some} \quad k>0. \tag{2.9}$$

Lemma 2.4(C) asserts that f_1 and f_2 are generators of the same $F \in \mathcal{F}_A$ if and only if $f_1 \sim f_2$, and that a similar statement holds for generators g of elements of $\mathcal{G}_A$. The relation (2.9) thus induces an equivalence relation on solutions of equation (2.8) via

$$(f_1, g_1) \sim (f_2, g_2) \Leftrightarrow f_1 \sim f_2 \text{ and } g_1 \sim g_2 \Leftrightarrow f_1 = k_1 f_2 \text{ and } g_1 = k_2 g_2 \tag{2.10}$$

for some positive constants k_1 and k_2.

Generators of members of the family (F_s, G_s) defined in (1.5), (1.6), and (1.7) are:

$$f_1(x) = -\log x, \qquad g_1(x) = -\log(1-x); \tag{2.11}$$

$$f_\infty(x) = 1-x, \qquad g_\infty(x) = x; \tag{2.12}$$

$$f_s(x) = -\log\left(\frac{s^x-1}{s-1}\right), \qquad g_s(x) = -\log\left(\frac{s^{1-x}-1}{s-1}\right). \tag{2.13}$$

Note that by virtue of Lemma 2.4(D), F_s and G_s are strict for $0<s<\infty$, but F_∞ and G_∞ are not.

We now verify the remaining limiting cases of the family F_s. First, an application of L'Hospital's Theorem gives $f_s(x) \to -\log x$ as $s \to 1$, so that $\lim_{s\to 1} F_s(x, y) = x \cdot y$. Next, on rearranging (2.13) we get $f_s(x)/\log s = 1-x-\log[(1-s^{-x})/(1-s^{-1})]/\log s$, whence $f_s(x)/\log s \to 1-x$ as $s\to\infty$. Since $f_s(x) \sim f_s(x)/\log s$, it follows that $\lim_{s\to\infty} F_s(x, y) = \max(x+y-1, 0)$.

3. Solutions of the generator equation

To complete the proof of Theorem 1.1, it remains to show that the pairs (f_s, g_s), given in (2.11), (2.12), and (2.13), and all pairs equivalent to any of these (via (2.10)) are the only solutions (f, g) of the generator equation (2.8). We begin by showing that f and g must be convex and, unless $(f, g) \sim (f_\infty, g_\infty)$, invertible functions.

LEMMA 3.1. *Let f and g satisfy the conditions given in Lemma* 2.4(A). *If (f, g) is a solution of equation* (2.8), *then both f and g must be convex.*

Proof. Suppose that (f, g) is a solution of (2.8). Let F and G be the elements of $\mathcal{F}_A$ and $\mathcal{G}_A$ given by (2.7). Then (F, G) satisfies (1.1a), and by Lemma 2.1(B) F and G must satisfy the inequalities (2.3).

It is shown in [6, Theorem 3.2] that any $F \in \mathcal{F}_A$ for which (2.3a) holds must satisfy the monotonicity condition,

$$F(x_1, y_1) - F(x_2, y_1) - F(x_1, y_2) + F(x_2, y_2) \geq 0, \tag{3.1}$$

whenever $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$. Now from [8, Theorem 9], it follows that if $F \in \mathcal{F}_A$ satisfies (3.1), then any generator f of F is convex. [This result was proved in [8] with the more restrictive hypothesis that F is strict. However, the arguments given there readily extend to Archimedean F. A function F that satisfies conditions (1.2a), (1.2c), and (3.1) is called a *copula*.]

The arguments used to prove the results quoted in the preceding paragraph can easily be modified to prove that $G \in \mathcal{G}_A$ must satisfy (3.1), and then that g is convex.

LEMMA 3.2. *Let f and g be functions satisfying the conditions given in Lemma* 2.4(A). *If (f, g) is a solution of equation* (2.8), *then either $(f, g) \sim (f_\infty, g_\infty)$ or $f(0) = g(1) = \infty$.*

Proof. Suppose that (f, g) is a solution of (2.8). By Lemma 3.1, f and g are convex, and the lemma then follows from [2, Theorem 10]. For there it is proved, with different notation, that if $F \in \mathcal{F}_A$ has a convex generator f and if F is a solution of equation (1.3) – equivalently, of the system (1.1) – then either $f \sim f_\infty$ (so that $F = F_\infty$) or $f(0) = \infty$ (so that F is strict). Replacing F by G and f by g in that proof, it is easy to obtain the analogous result for g.

Note that Lemma 3.2 asserts that (F_∞, G_∞) is the only solution of the system (1.1) for which F and G are Archimedean but not strict.

For future reference, we now collect the results on Archimedean solutions of (1.1) obtained so far.

THEOREM 3.1. *Let $F \in \mathscr{F}_A$ and $G \in \mathscr{G}_A$ possess generators f and g as in* (2.5). *Then (F, G) satisfies equation* (1.1a) *if and only if either $(F, G) = (F_\infty, G_\infty)$ or (f, g) is a solution of the equation*

$$f^{-1}(f(x)+f(y))+g^{-1}(g(x)+g(y))=x+y, \tag{3.2}$$

where f and g satisfy the following conditions:

$$\text{Dom } f = \text{Dom } g = [0, 1], \qquad \text{Ran } f = \text{Ran } g = [0, \infty]; \tag{3.3a}$$

$$\left.\begin{array}{l} f \text{ is strictly decreasing and convex;} \\ g \text{ is strictly increasing and convex;} \end{array}\right. \tag{3.3b}$$

$$f(1) = g(0) = 0, \qquad f(0) = g(1) = \infty. \tag{3.3c}$$

Note that, if f and g satisfy conditions (3.3), then f^{-1} is strictly decreasing and convex, g^{-1} is strictly increasing and concave, $f^{-1}(0) = g^{-1}(\infty) = 1$, and $f^{-1}(\infty) = g^{-1}(0) = 0$.

Henceforth whenever we assert that (f, g) is a solution of equation (3.2), it will be tacitly assumed that f and g satisfy conditions (3.3) and that F and G are the elements of $\mathscr{F}_A$ and $\mathscr{G}_A$ whose generators are f and g, respectively. Note that in view of (1.3), equation (3.2) is equivalent to the following equation in one unknown function:

$$\begin{aligned} &f^{-1}\{f[x+y-f^{-1}(f(x)+f(y))]+f(z)\}+f^{-1}(f(x)+f(y)) \\ &\qquad = f^{-1}\{f(x)+f[y+z-f^{-1}(f(y)+f(z))]\}+f^{-1}(f(y)+f(z)). \end{aligned} \tag{3.4}$$

Equation (3.2) will ultimately be solved via three second-order differential equations. We begin by showing that f and g possess strictly increasing derivatives.

LEMMA 3.3. *If (f, g) is a solution of equation* (3.2), *then f and g are differentiable on* $(0, 1)$.

Proof. Suppose that (f, g) is a solution of (3.2), and fix an arbitrary x in $(0, 1)$. Since f is convex, the left-hand and right-hand derivatives of f at x, denoted by $f'_-(x)$ and $f'_+(x)$, respectively, exist, and furthermore $f'_-(x) \le f'_+(x)$. To show the existence of $f'(x)$, it thus suffices to establish the reverse inequality,

$$f'_-(x) \geq f'_+(x). \tag{3.5}$$

To this end, let $\{a_n\}$ be any sequence such that $x < a_n < 1$, $a_n \nearrow 1$, and let $\{y_n\}$ and $\{z_n\}$ be the sequences obtained from $\{a_n\}$ via

$$y_n = f^{-1}(f(a_n)+f(x)), \qquad z_n = f^{-1}(f(x)-f(a_n)). \tag{3.6}$$

Since f and f^{-1} are strictly decreasing and $f(1)=0$, it easily follows that $y_n < x < z_n$, $y_n \nearrow x$, $z_n \searrow x$. Moreover, as a picture clearly reveals, the convexity of f^{-1} guarantees:

$$x - y_n \leq z_n - x. \tag{3.7}$$

Now the definition of one-sided derivatives, the listed properties of the above sequences, the decreasing character of f, and its convexity yield:

$$f(a_n)/(x-y_n) \searrow -f'_-(x) > 0, \qquad 0 < f(a_n)/(z_n - x) \nearrow -f'_+(x),$$

from which we easily get the inequality,

$$\frac{z_n - x}{x - y_n} = \frac{f(a_n)/(x-y_n)}{f(a_n)/(z_n - x)} \geq f'_-(x)/f'_+(x). \tag{3.8}$$

(Here, and subsequently, we make use of obvious relationships between the slopes of secant lines on the graph of a convex function.) On combining (3.2) and the relations (3.6), we find that

$$a_n + x - y_n = a_n + x - f^{-1}(f(a_n)+f(x)) = g^{-1}(g(a_n)+g(x)),$$

and

$$a_n + z_n - x = a_n + z_n - f^{-1}(f(a_n)+f(z_n)) = g^{-1}(g(a_n)+g(z_n)),$$

which clearly are equivalent to:

$$g(a_n + x - y_n) - g(a_n) = g(x), \qquad g(a_n + z_n - x) - g(a_n) = g(z_n). \tag{3.9}$$

By (3.7), $a_n + x - y_n \leq a_n + z_n - x$. Using this fact, the convexity of g, and the equations (3.9), we obtain:

$$\begin{aligned} g(x)/(x-y_n) &= [g(a_n+x-y_n)-g(a_n)]/[(a_n+x-y_n)-a_n] \\ &\leq [g(a_n+z_n-x)-g(a_n)]/[a_n+z_n-x)-a_n] \\ &= g(z_n)/(z_n-x), \end{aligned}$$

which, when combined with (3.8), yields,

$$g(z_n)/g(x) \geq f'(x)/f'_+(x). \tag{3.10}$$

But $g(z_n) \searrow g(x)$ because g is continuous and increasing; consequently, letting $n \to \infty$ in (3.10), we have:

$$f'_-(x)/f'_+(x) \leq 1,$$

which, since f is decreasing, is equivalent to (3.5).

To prove the differentiability of g at x, it is necessary only to reverse the roles of f and g in the above argument, take $\{a_n\}$ so that $0 < a_n < x$, $a_n \searrow 0$, and reverse the appropriate inequalities. We omit the details.

COROLLARY 3.3.1. *If* (f, g) *is a solution of equation* (3.2), *then:*

(A) f' *and* g' *are non-decreasing and hence continuous on* $(0, 1)$;
(B) $f'(0+) = -\infty$, $f'(x) < 0$ *for* $x \in (0, 1)$, $f'(1-) \leq 0$;
(C) $g'(0+) \geq 0$, $g'(x) > 0$ *for* $x \in (0, 1)$, $g'(1-) = +\infty$.

Upon differentiating equation (3.2) with respect to x, we immediately obtain the next important result.

LEMMA 3.4. *If* (f, g) *is a solution of equation* (3.2), *then for every* $(x, y) \in (0, 1)^2$,

$$f'(x)/f'F(x, y) + g'(x)/g'G(x, y) = 1. \tag{3.11}$$

Henceforth, juxtaposition of function symbols denotes composition.

LEMMA 3.5. *If* (f, g) *is a solution of equation* (3.2), *then* f' *and* g' *are strictly increasing on* $(0, 1)$.

Proof. By Corollary 3.3.1(A), f' is non-decreasing. Suppose, to the contrary, that $f'(z)=f'(x)$ for some x and z such that $0<z<x<1$. Then by Corollary 3.3.1(B), $f'(z)<0$. Let $y=f^{-1}(f(z)-f(x))$. Then $0<y<1$, $z=F(x, y)$, and by (3.11) we have:

$$g'(x)/g'G(x, y)=1-f'(x)/f'F(x, y)=1-f'(z)/f'(z)=0.$$

Since $G(x, y)<1$, it follows that $g'G(x, y)<\infty$, whence $g'(x)=0$, contradicting Corollary 3.3.1(C). Therefore f' must be strictly increasing. An analogous argument yields the same result for g'.

The second stage of our analysis of equation (3.2) consists of showing that each of the four possible assumptions, (1) $f'(1-)=0$, (2) $g'(0+)=0$, (3) $f'(1-)\neq 0$, and (4) $g'(0+)\neq 0$, implies a further relationship between f and g. We begin with cases (1) and (2).

LEMMA 3.6. *Suppose that* (f, g) *is a solution of equation* (3.2).

(A) *If* $f'(1-)=0$, *then* $f'\cdot g$ *is constant on* $(0, 1)$.
(B) *If* $g'(0+)=0$, *then* $f\cdot g'$ *is constant on* $(0, 1)$.

Proof. We prove part (A) only. As before, the proof of part (B) follows in a similar fashion.

First, observe that for any $z\in(0, 1)$,

$$-f'(z)=\lim_{x\to 1-} f(x)/[z-F(z, x)]. \tag{3.12}$$

For, on setting $a=z-F(z, x)$, it is evident that each side of (3.12) is equal to $-\lim_{a\to 0+}[f(z)-f(z-a)]/a$.

Fix any u and v so that $0<u<v<1$. Then using (3.12) and the fact that $F(z, x)+G(z, x)=z+x$, we get:

$$\frac{f'(u)}{f'(v)}=\lim_{x\to 1-}\frac{v-F(v, x)}{u-F(u, x)}=\lim_{x\to 1-}\frac{G(v, x)-x}{G(u, x)-x}. \tag{3.13}$$

Also, upon considering the slopes of various tangent and secant lines on the graph of g at any point $x>v$, the convexity of g guarantees that

$$0<g'(x)<g(u)/[G(u, x)-x]<g(v)/[G(v, x)-x]<g'G(v, x),$$

whence:

$$\frac{g'(x)}{g'G(v,x)}<\frac{g(u)/[G(u,x)-x]}{g(v)/[G(v,x)-x]}<1, \quad \text{for every} \quad x>v. \tag{3.14}$$

Now suppose that $f'(1-)=0$. Then equation (3.11), the continuity of f' and F, and the boundary conditions (1.2c) yield:

$$\begin{aligned}\lim_{x\to 1-} g'(x)/g'G(v,x) &= 1-\lim_{x\to 1-} f'(x)/f'F(v,x)\\ &= 1-f'(1-)/f'(v)=1.\end{aligned} \tag{3.15}$$

On combining (3.13), (3.14), and (3.15), we obtain:

$$\frac{f'(u)g(u)}{f'(v)g(v)}=\frac{g(u)}{g(v)}\lim_{x\to 1-}\frac{G(v,x)-x}{G(u,x)-x}=\lim_{x\to 1-}\frac{g(u)/[G(u,x)-x]}{g(v)/[G(v,x)-x]}=1,$$

i.e., $f'(u)g(u)=f'(v)g(v)$. Thus $f'\cdot g$ is constant on $(0, 1)$.

The properties listed in Lemma 3.7 follow easily from Corollary 3.3.1 and Lemma 3.5.

LEMMA 3.7. *Suppose that (f, g) is a solution of equation (3.2). Let φ and ψ be the functions defined on $(0, 1)$ via*

$$\varphi(x)=1-f'(1-)/f'(x), \tag{3.16a}$$

$$\psi(x)=1-g'(0+)/g'(x), \tag{3.16b}$$

and extended to $[0, 1]$ to continuity.

(A) *If $f'(1-)\neq 0$, then φ is strictly decreasing and continuous, with $\varphi(0)=1$ and $\varphi(1)=0$.*

(B) *If $g'(0+)\neq 0$, then ψ is strictly increasing and continuous, with $\psi(0)=0$ and $\psi(1)=1$.*

LEMMA 3.8. *Suppose that (f, g) is a solution of equation (3.2). Let φ and ψ be the functions given in (3.16).*

(A) *If $f'(1-)\neq 0$, then $-\log\varphi$ is a generator of G.*

(B) *If $g'(0+)\neq 0$, then $-\log\psi$ is a generator of F.*

Proof. Suppose first that $f'(1-)\neq 0$. Then by Lemma 3.7(A), $-\log\varphi$ is strictly increasing and continuous on $[0, 1]$, with $-\log \varphi(0)=0$ and $-\log \varphi(1)=\infty$. It thus suffices to show that $-\log \varphi$ satisfies (2.7b) (with $-\log \varphi(x)=g$), i.e.,

$$-\log \varphi G(x, y)=-\log \varphi(x)-\log \varphi(y),$$

or, equivalently,

$$\varphi G(x, y)=\varphi(x)\varphi(y), \tag{3.17}$$

and φ is thus a "multiplicative generator" of G[8; 9].

To verify (3.17), observe that by (3.16a), (3.11), the continuity of g' and G, and the boundary conditions (1.2c), we have for any $y\in(0, 1)$,

$$\begin{aligned}\varphi(y)&=1-f'(1-)/f'(y)\\ &=1-\lim_{z\to 1-} f'(z)/f'F(z, y)\\ &=\lim_{z\to 1-} g'(z)/g'G(z, y),\end{aligned}\tag{3.18}$$

and consequently, for any $x, y\in(0, 1)$,

$$\begin{aligned}\varphi(x)&=\lim_{G(z, y)\to 1-} g'G(z, y)/g'G(G(z, y), x) && \text{(substituting } x \text{ for } y \text{ and } G(z, y) \text{ for } z \text{ in (3.18))}\\ &=\lim_{z\to 1-} g'G(z, y)/g'G(G(z, y), x) && \text{(since } G(z, y)\to 1- \text{ if and only if } z\to 1- \text{ for any fixed } y<1)\\ &=\lim_{z\to 1-} g'G(z, y)/g'G(z, G(x, y)) && \text{(since } G \text{ is associative and commutative).}\end{aligned}\tag{3.19}$$

Now, multiplying (3.18) and (3.19), and then using (3.18) with y replaced by $G(x, y)$, we obtain (3.17) as follows:

$$\begin{aligned}\varphi(x)\varphi(y)&=\lim_{z\to 1-}\left[\frac{g'(z)}{g'G(z, y)}\cdot\frac{g'G(z, y)}{g'G(z, G(x, y))}\right]\\ &=\lim_{z\to 1-} g'(z)/g'G(z, G(x, y))\\ &=\varphi G(x, y).\end{aligned}$$

Assuming that $g'(0+)\neq 0$, the fact that $\psi F(x, y)=\psi(x)\psi(y)$ can be established by a similar argument. For it can be shown that, for any $y\in(0,1)$,

$$\psi(y)=\lim_{z\to 0+} f'(z)/f'F(z, y), \tag{3.20}$$

and the remaining steps in the proof are analogous.

Remark. The relation (3.17) can be established in a different manner: differentiate equation (3.4) with respect to z, let $z\to 1-$, and divide both sides of the resulting equation by $f'(1-)$.

Notice that the functions φ and ψ defined in (3.16) are independent of the representatives f and g chosen from the equivalence classes determined by the relation (2.9). Thus Lemma 3.8 can be rewritten in the following more useful form.

LEMMA 3.9. *Suppose that* (f, g) *is a solution of equation* (3.2).
(A) *If* $f'(1-)\neq 0$, *then* $g_0\sim g$, *where* g_0 *is given by*

$$g_0(x)=-\log[1-f'(1-)/f'(x)]. \tag{3.21}$$

(B) *If* $g'(0+)\neq 0$, *then* $f_0\sim f$, *where* f_0 *is given by*

$$f_0(x)=-\log[1-g'(0+)/g'(x)]. \tag{3.22}$$

LEMMA 3.10. *If* (f, g) *is a solution of equation* (3.2), *then* f' *and* g' *are continuously differentiable on* $(0, 1)$.

Proof. By virtue of Lemmas 3.6 and 3.9 we always have (1) either $g(x)=-a/f'(x)$ or $g(x)=-a\log[1-f'(1-)/f'(x)]$ *and* (2) either $f(x)=b/g'(x)$ or $f(x)=-b\log[1-g'(0+)/g'(x)]$, for some positive constants a and b. Since f and g are continuously differentiable on $(0, 1)$, the lemma follows immediately from the above relations.

In the third and final stage of solving equation (3.2), we analyze separately the four possible cases: (1) $f'(1-)=g'(0+)=0$, (2) $f'(1-)=0$, $g'(0+)\neq 0$, (3) $f'(1-)\neq 0$, $g'(0+)=0$, (4) $f'(1-)\neq 0\neq g'(0+)$. In each case we obtain a second-order differential equation in f or g and find those solutions (if any) which satisfy conditions (3.3).

LEMMA 3.11. *Suppose that* f *and* g *satisfy conditions* (3.3) *and that* $f'(1-)=g'(0+)=0$. *Then* (f, g) *is not a solution of equation* (3.2).

Proof. Suppose, to the contrary, that (f, g) is a solution of equation (3.2), with $f'(1-)=g'(0+)=0$. Then by Lemma 3.6,

$$f' \cdot g \equiv c_1 < 0, \tag{3.23}$$

$$f \cdot g' \equiv c_2 > 0, \tag{3.24}$$

on $(0, 1)$. Moreover, by Lemma 3.10, f'' is continuous there. We shall obtain the desired contradiction by showing that no function f which satisfies all of these conditions can satisfy conditions (3.3c).

On differentiating (3.23), we have:

$$f' \cdot g' + f'' \cdot g = 0. \tag{3.25}$$

Noting that $f>0$ and $f'<0$ on $(0, 1)$, using (3.23) and (3.24) to eliminate g in (3.25), we obtain

$$c_2 f'/f + c_1 f''/f' = 0, \qquad c_1 < 0, \qquad c_2 > 0;$$

integrating then yields

$$c_2 \log f + c_1 \log(-f') = -\log \alpha, \qquad \alpha > 0,$$

whence,

$$f' = -af^k, \qquad a > 0, \qquad k > 0.$$

If $k=1$, then

$$f(x) = be^{-ax}, \qquad a>0, \qquad b>0. \tag{3.26}$$

If $k \neq 1$, then

$$f(x) = [(1-k)(c-ax)]^{1/(1-k)}, \qquad a>0, \qquad k>0. \tag{3.27}$$

But no member f of either of the families (3.26) and (3.27) satisfies conditions (3.3c). Clearly this is so for the family (3.26) because $f(0)=b<\infty$. Now for the family (3.27), in order that $0=f(1)=[(1-k)(c-a)]^{1/(1-k)}$, we must have $c=a$ and $0<k<1$, in which case again $f(0)=[(1-k)c]^{1/(1-k)}<\infty$.

LEMMA 3.12. *Suppose that f and g satisfy conditions* (3.3) *and that* $f'(1-)=0$

and $g'(0+)\neq 0$. *Then* (f, g) *is not a solution of equation* (3.2). *The same conclusion holds in case* $f'(1-)\neq 0$ *and* $g'(0+)=0$.

Proof. Suppose, to the contrary, that (f, g) is a solution of equation (3.2), and suppose that $f'(1-)=0$ and $g'(0+)\neq 0$. Let $g_0(x)=g(x)/g'(0+)$, and let f_0 be given by (3.22). Now $g_0\sim g$, $f_0\sim f$ by Lemma 3.9(B), and so $(f_0, g_0)\sim(f, g)$ by (2.10). Thus (f_0, g_0) is a solution of equation (3.2), $f_0'(1-)=0$, $g_0'(0+)=1$, and by Lemmas 3.6(A) and 3.9(B), f_0 and g_0 must satisfy the equations,

$$f_0'(x)=-c/g_0(x), \qquad c>0, \tag{3.28}$$

$$f_0(x)=-\log[1-1/g_0'(x)], \tag{3.29}$$

for every $x\in(0, 1)$.

Upon differentiating (3.29) and equating the result with (3.28), we get (dropping subscripts):

$$c\frac{g'}{g}=\frac{g''}{g'-1}, \qquad c>0.$$

Since g' is increasing and $g'(0+)=1$, integrating yields

$$c\log g+\log a=\log(g'-1), \qquad a>0,$$

whence,

$$g'=1+ag^c, \qquad a>0, \qquad c>0. \tag{3.30}$$

Accordingly, g_0 must be a solution of equation (3.30). For arbitrary c the solutions g of this equation cannot be given explicitly. We therefore show that, with the current assumptions, c must equal 1. First of all, for any y in $[0, 1]$, we have:

$$\begin{aligned}\lim_{x\to 0+} gF(x, y)/g(x) &= \lim_{x\to 0+} f'(x)/f'F(x, y) \quad &&\text{(by (3.28))}\\ &=\psi(y)=1-1/g'(y) &&\text{(by (3.20) and (3.16b))}\\ &=1-1/[1+ag^c(y)] &&\text{(by (3.30))}\\ &=ag^c(y)/[1+ag^c(y)]. &&\end{aligned} \tag{3.31}$$

Next, by (3.29) and (3.30), for any x in $[0, 1]$,

$$f(x)=-\log[1-1/g'(x)]=\log[1+1/ag^c(x)],$$

whence, solving for g,

$$g(x)=[a(e^{f(x)}-1)]^{-1/c}.$$

Therefore,

$$gF(x, y)=gf^{-1}(f(x)+f(y))=[a(e^{f(x)+f(y)}-1]^{-1/c}=\left[\frac{ag^c(x)g^c(y)}{1+ag^c(x)+ag^c(y)}\right]^{1/c},$$

and so, dividing by $g(x)$ and letting $x\to 0+$, we get:

$$\lim_{x\to 0+} gF(x, y)/g(x)=\left[\frac{ag^c(y)}{1+ag^c(y)}\right]^{1/c}. \tag{3.32}$$

From (3.31) and (3.32), it follows that $c=1$, since $ag^c/[1+ag^c]$ cannot be identically 0 or 1.

As a consequence g_0 must be a solution of the equation

$$g'=1+ag, \qquad a>0,$$

which, when integrated, yields

$$g(x)=Ae^{ax}-\frac{1}{a}, \qquad a>0, \qquad A>0.$$

But no member g of this family satisfies conditions (3.3c) since $g(1)<\infty$. This contradiction completes the proof of the first assertion. The second can be verified analogously, with the roles of f and g reversed.

The force of Lemmas 3.11 and 3.12 is that all solutions (f, g) of equation (3.2) must satisfy $f'(1-)\neq 0\neq g'(0+)$. Our analysis of this case will be similar to the preceding ones. We begin with a preliminary result.

LEMMA 3.13. *Suppose that* (f, g) *is a solution of equation* (3.2), *with* $f'(1-)\neq 0\neq g'(0+)$. *Let* φ *and* ψ *be the functions given in* (3.16). *Then*

$$-\varphi'(0+)=\psi'(1-). \tag{3.33}$$

Proof. By Lemma 3.10, φ and ψ are differentiable, and hence,

$$\varphi'(x)=f'(1-)f''(x)/[f'(x)]^2, \tag{3.34a}$$

$$\psi'(x)=g'(0+)g''(x)/[g'(x)]^2, \tag{3.34b}$$

for every $x \in (0, 1)$. Also, since f'' and g'' exist, we may differentiate equation (3.11) with respect to y, thus obtaining the first of the following string of equalities:

$$\begin{aligned}
\frac{-f'(x)f'(y)f''F(x,y)}{[f'F(x,y)]^3} &= \frac{g'(x)g'(y)g''G(x,y)}{[g'G(x,y)]^3} \\
&= \frac{g'(x)g'(y)}{g'G(x,y)} \cdot \frac{\psi'G(x,y)}{g'(0+)} && \text{(substituting } G(x,y) \text{ for } x \text{ in (3.34b))} \\
&= \psi'G(x,y) \cdot \frac{1}{1-\psi(x)} \cdot \frac{g'(y)}{g'G(x,y)} && \text{(by (3.16b))} \\
&= \psi'G(x,y) \cdot \frac{1}{1-\psi(x)} \\
&\quad \times [1 - f'(y)/f'F(x,y)] && \text{(by (3.11)).} \qquad (3.35)
\end{aligned}$$

Consequently,

$$\begin{aligned}
-\varphi'(x) &= -f'(x)f'(1-)f''(x)/[f'(x)]^3 && \text{(by (3.34a))} \\
&= \lim_{y \to 1-} -f'(x)f'(y)f''F(x,y)/[f'F(x,y)]^3 && \text{(by the continuity of } f', f'', \text{ and } F, \text{ and (1.2c))} \\
&= \lim_{y \to 1-} \{\psi'G(x,y) \cdot \frac{1}{1-\psi(x)}[1 - f'(y)/f'F(x,y)]\} && \text{(by (3.35))} \\
&= \psi'(1-) \cdot \frac{1}{1-\psi(x)}[1 - f'(1-)/f'(x)] && \text{(by the continuity of } \psi', G, \text{ and } f').
\end{aligned}$$

Now on letting $x \to 0+$ in the last equation, we have:

$$-\varphi'(0+) = \psi'(1-) \cdot \frac{1}{1-\psi(0)} \cdot \varphi(0) = \psi'(1-).$$

LEMMA 3.14. *Suppose that f and g satisfy conditions* (3.3) *and that $f'(1-) \neq 0 \neq g'(0+)$. If (f, g) is a solution of equation* (3.2), *then $(f, g) \sim (f_s, g_s)$ for some positive s, where f_s and g_s are the functions given in* (2.11) *and* (2.13).

Proof. Suppose that (f, g) is a solution of equation (3.2) and that $f'(1-) \neq 0 \neq g'(0+)$. Let f_0 and g_0 be the functions given in (3.22) and (3.21), respectively. Then $(f_0, g_0) \sim (f, g)$, so (f_0, g_0) is a solution of (3.2) and hence by Lemma 3.9, f_0 and g_0 must satisfy the equations,

$$g_0(x) = -\log[1 - f_0'(1-)/f_0'(x)], \tag{3.36}$$

$$f_0(x) = -\log[1 - g_0'(0+)/g_0'(x)], \tag{3.37}$$

for every $x \in (0, 1)$. Since $f_0 = -\log \varphi$ and $g_0 = -\log \psi$ by definition, Lemma 3.13 gives:

$$g_0'(0+) = -\varphi'(0+)/\varphi(0) = -\varphi'(0+) = \psi'(1-) = \psi'(1-)/\psi(1) = -f_0'(1-). \tag{3.38}$$

Next, differentiating (3.36), solving (3.37) for g_0', equating the resulting expressions, and using (3.38), we find that f_0 must be a solution of the following equation:

$$\frac{e^f \cdot f'}{e^f - 1} = \frac{f''}{f'(1-) - f'},$$

which, when integrated, becomes

$$f' = f'(1-) - b(e^f - 1), \qquad b > 0.$$

Now setting $y = f(x)$ and $a = -f'(1-)$, it is apparent that f^{-1} must be of the following form:

$$x = f^{-1}(y) = \int \frac{-dy}{a + b(e^y - 1)}, \qquad a > 0, \qquad b > 0. \tag{3.39}$$

To complete the proof, it suffices to show that the f_s $(0 < s < \infty)$ are the only solutions of (3.39) that satisfy conditions (3.3c). There are two cases to consider: $a = b$ and $a \neq b$. First, if $a = b$, then

$$x = f^{-1}(y) = \int -dy/ae^{-y} = \frac{1}{a} e^{-y} + k, \qquad a > 0.$$

In order that $f^{-1}(\infty) = 0$, we must have $k = 0$; then in order that $f^{-1}(0) = 1$, $a = 1$. Thus $f^{-1}(y) = e^{-y}$, and so $f = f_1$.

Finally, if $a \neq b$, then substituting $u = e^y - 1$ in (3.39), decomposing by partial fractions, and integrating, we obtain:

$$x=f^{-1}(y)=\frac{-1}{a-b}\log\left[\frac{e^y}{a+b(e^y-1)}\right]+k, \qquad a>0, \qquad b>0.$$

In order that $f^{-1}(0)=1$, we muat have $k=1-\log a/(a-b)$, and so,

$$x=f^{-1}(y)=1-\frac{1}{a-b}\log\left[\frac{ae^y}{a+b(e^y-1)}\right], \qquad a>0, \qquad b>0,$$

which, when solved for y, becomes:

$$f(x)=-\log\left[\frac{ae^{(b-a)(1-x)}-b}{a-b}\right], \qquad a>0, \qquad b>0. \tag{3.40}$$

Now in order that $f(0)=\infty$, we must have:

$$a\neq b, \qquad ae^{b-a}-b=0.$$

Hence, putting $s=a/b=e^{a-b}$ in (3.40) and simplifying yields:

$$f(x)=-\log\left[\frac{a(b/a)^{1-x}-b}{a-b}\right]=-\log\left[\frac{(a/b)^x-1}{a/b-1}\right]=-\log\left(\frac{s^x-1}{s-1}\right),$$

and so $f=f_s$ for some positive $s\neq 1$.

On combining Lemmas 3.3–3.14 (especially Lemmas 3.11, 3.12, and 3.14), we obtain the desired result:

THEOREM 3.2. *The pair* (f, g) *is a solution of equation* (3.2) *if and only if* $(f, g)\sim(f_s, g_s)$, *whence* $(F, G)=(F_s, G_s)$, *for some positive s.*

Finally, Theorem 1.1 follows from Theorems 2.1, 3.1, and 3.2.

4. Some consequences of Theorem 1.1

In this section, by direct and rather elementary applications of Theorem 1.1, we shall solve two outstanding problems and point out a connection with probability.

The first application concerns solutions of the equation

$$x+y-H(x,y)=1-H(1-x,1-y), \tag{4.1}$$

where $H:[0,1]^2 \to [0,1]$ *is assumed to be associative and to satisfy one of the sets of boundary conditions* (1.2*c*) *and* (1.2*d*). For convenience, we denote by $H^\wedge$ and $\bar{H}$ the functions derived from H via

$$H^\wedge(x, y) = x + y - H(x, y), \tag{4.2}$$

$$\bar{H}(x, y) = 1 - H(1-x, 1-y), \tag{4.3}$$

which enables us to write equation (4.1) as: $H^\wedge = \bar{H}$.

It is readily verified that the associativity of H implies the associativity of $\bar{H}$. If H satisfies (1.2c), the clearly $\bar{H}$ satisfies (1.2d), and vice versa. Also, if H is assumed to be either continuous or non-decreasing, then $\bar{H}$ has the same property.

Now suppose that H is a solution of equation (4.1). Then the statements in the previous paragraph must hold when $\bar{H}$ is replaced by $H^\wedge$. Therefore, if H is continuous, either $(H, H^\wedge)$ or $(H^\wedge, H)$ must be a solution of the system (1.1). Moreover, there are no discontinuous, non-decreasing solutions of equation (4.1), for in view of the remark following Lemma 2.1, the same result holds when "continuous" is replaced by "non-decreasing".

Accordingly, the Archimedean case can be quickly resolved. For, by (2.13), $g_s(x) = f_s(1-x)$ and therefore,

$$G_s(x, y) = 1 - F_s(1-x, 1-y), \qquad F_s(x, y) = 1 - G_s(1-x, 1-y), \qquad \text{for} \quad 0 \leqslant s \leqslant \infty;$$

and since $G = F^\wedge$ and $F = G^\wedge$, it follows that all members of the families F_s and G_s are solutions of (4.1). Combining these facts with Theorem 1.1 yields:

THEOREM 4.1. *The functions F_s and G_s, $0 < s \leqslant \infty$, are the only members H of $\mathcal{F}_A \cup \mathcal{G}_A$ which are solutions of equation* (4.1).

This solves the second problem posed in [11].

Non-Archimedean solutions F and G of (1.1) are not, in general, solutions of (4.1). In fact, the idempotent sets of $H^\wedge$ and $\bar{H}$ usually do not coincide, for $H(x, x) = x$ if and only if $H^\wedge(x, x) = x$ if and only if $\bar{H}(1-x, 1-x) = 1-x$. It is thus necessary that the idempotent set of H be symmetric about $\frac{1}{2}$. It is necessary also that, in the ordinal sum decomposition of H, the same summand be associated with the intervals $[a_n, b_n]$ and $[1-b_n, 1-a_n]$ for any n. Simple computations based on the representations (2.5) reveal that for $H \in (\mathcal{F} - \mathcal{F}_A) \cup (\mathcal{G} - \mathcal{G}_A)$, these two conditions are sufficient as well as necessary in order that H be a solution of (4.1).

The second application resolves a conjecture made in [3] concerning so-called invertibility of semigroups on [0, 1]. Given any $H \in \mathcal{F} \cup \mathcal{G}$, define the function H^* on $[0, 1]^2$ via

$$H^*(x, y) = x + y - 1 + H(1 - x, 1 - y). \tag{4.4}$$

Note that by (4.2) and (4.3), $H^* = (\bar{H})^\wedge$.

Clearly H^* is continuous and satisfies the same boundary conditions as does H. Hence if H^* is associative, then H^* is an element of, correspondingly, $\mathcal{F}$ or $\mathcal{G}$. Following [3], we call such an H^* the *invert* of H and say that the semigroup operation H is *invertible.* This terminology is reasonable because, as is easily checked, $(H^*)^* = H$.

Invertible semigroups $H \in \mathcal{F} \cup \mathcal{G}$ are easily characterized by appealing to Theorem 1.1. For since $\bar{H}$ is always associative and since $H^* = (\bar{H})^\wedge$, H is invertible precisely when either $(\bar{H}, (\bar{H})^\wedge)$ or $((\bar{H})^\wedge, \bar{H})$ is a solution of the system (1.1). But it is simple to show that H is one member of a solution of (1.1) if and only if $\bar{H}$ is the opposite member of a (possibly different) solution. Therefore we have:

THEOREM 4.2. *The semigroup operation $H \in \mathcal{F} \cup \mathcal{G}$ is invertible (equivalently, $H^* \in \mathcal{F} \cup \mathcal{G}$) if and only if H is one of the functions F or G listed in Theorem* 1.1.

(To obtain an alternate proof of Theorem 4.2, put (4.4) into the associativity equation $H^*(H^*(1-x, 1-y), 1-z) = H^*(1-x, H^*(1-y, 1-z))$. After simplifying, equation (1.3) emerges.)

It is interesting to note that for invertible, Archimedean H, $H^* = H$. For, by Theorem 4.1, $H^* = (\bar{H})^\wedge = (H^\wedge)^\wedge = H$. Once again, if the continuity assumption on H is replaced by the assumption that both H and H^* are non-decreasing, then the conclusion of Theorem 4.2 remains valid. (Consult the remark following Lemma 2.1.) Therefore, Theorem 1.1 yields the only invertible t-norms (t-conorms) whose inverts are t-norms (t-conorms).

The third application concerns a family of binary operations defined on the space Δ of one-dimensional (cumulative) probability distribution functions (d.f.'s). [Δ consists of all non-decreasing functions D defined on the extended real line and such that $D(-\infty) = 0$ and $D(+\infty) = 1$. It is customary to normalize Δ by requiring its elements to be either left- or right-continuous.]

Let C be a copula, i.e., a two-place function from $[0, 1]^2$ to $[0, 1]$ that satisfies the boundary conditions (1.2c) and the monotonicity condition (3.1). It is readily established that both C and $C^\wedge$ are non-decreasing in each argument and are continuous, and that C satisfies the inequalities,

$$F_\infty(x, y) \leq C(x, y) \leq F_0(x, y), \quad \text{for every} \quad (x, y) \in [0, 1]^2.$$

Hence, for any D_1 and D_2 in Δ, the functions $\pi_C(D_1, D_2)$ and $\pi_{C^\wedge}(D_1, D_2)$, defined on $[-\infty, +\infty]$ via

$$\begin{aligned} &\pi_C(D_1, D_2)(x) = C(D_1(x), D_2(x)),\\ &\pi_{C^\wedge}(D_1, D_2)(x) = C^\wedge(D_1(x), D_2(x)), \end{aligned} \tag{4.5}$$

are elements of Δ, and so the mappings π_C and $\pi_{C^\wedge}$ are binary operations on Δ. These simple "pointwise" operations are special cases of the family $\sigma_{C,B}$ discussed at the end of the next section. It is an obvious consequence of (4.5) that π_C is associative (i.e., (Δ, π_C) is a semigroup) if and only if C is associative, and similarly for $\pi_{C^\wedge}$. Thus, *both* π_C and $\pi_{C^\wedge}$ are associative if and only if $(C, C^\wedge)$ is a solution of the system (1.1), and we have thus proved:

THEOREM 4.3. *Let C be a copula, and let π_C and $\pi_{C^\wedge}$ be the binary operations on Δ defined by* (4.5). *Then both π_C and $\pi_{C^\wedge}$ are associative if and only if C is one of the functions F listed in Theorem* 1.1.

To demonstrate the significance of Theorem 4.3, we shall make use of a result, stated in [8] and proved in [10], which establishes the role played by copulas in probability. This theorem also is a necessary background for the deeper application discussed in Section 5.

LEMMA 4.1. *Let X and Y be extended real-valued random variables defined on a common probability space with d.f.'s D_X and D_Y and joint d.f. K_{XY}. Then there exists a copula C_{XY} (called a* connecting copula *of X and Y) such that for every $(u, v) \in [-\infty, +\infty]^2$,*

$$K_{XY}(u, v) = C_{XY}(D_X(u), D_Y(v)). \tag{4.6}$$

In the other direction, if D_X and D_Y are elements of Δ and if C is a copula, then the two-place function K defined by (4.6) *is a two-dimensional d.f. whose margins are D_X and D_Y.*

The operations π_C and $\pi_{C^\wedge}$ yield the d.f.'s of the maximum and the minimum of two random variables. For it is easy to show that if C_{XY} is a connecting copula of X and Y, then the d.f.'s of Max (X, Y) and Min (X, Y) are given by,

$$D_{\mathrm{Max}\,(X,Y)} = \pi_{C_{XY}}(D_X, D_Y),\ D_{\mathrm{Min}\,(X,Y)} = \pi_{C^{\wedge}_{XY}}(D_X, D_Y). \tag{4.7}$$

Theorem 4.3 asserts that members of the family F_s and their ordinal sums are the only connecting copulas C for which both π_C and $\pi_{C^{\wedge}}$ are associative. Notice that F_1 is a connecting copula of X and Y if and only if they are independent. If D_X and D_Y are continuous, F_0 (resp., F_∞) is a connecting copula of X and Y if and only if one of them is a non-decreasing (resp., non-increasing) function of the other almost everywhere. There is apparently no simple probabilistic interpretation for other members of the family F_s.

5. A distributivity equation; an application to integral operations

We shall use Theorem 1.1 to prove the following result:

THEOREM 5.1. *The function* $F \in \mathscr{F}$ *is a solution of the equation*

$$F(x+y-F(x, y), z) = F(x, z)+F(y, z)-F(F(x, y), z) \tag{5.1}$$

if and only if (1) $F = F_0$, (2) $F = F_1$, *or* (3) F *is representable as an ordinal sum of* F_1*'s.*

Theorem 5.1 thus solves the problem of finding all continuous functions $F:[0, 1]^2 \to [0, 1]$ which satisfy the boundary conditions (1.2c) and such that F is a solution of the system of equations {(1.1b), (5.1)}. This system is discussed in [2] in connection with the operations $\sigma_{C,+}$ mentioned in the introduction.

LEMMA 5.1. *If* $F \in \mathscr{F}$ *is a solution of equation* (5.1), *then* $(F, F^{\wedge})$ *must be a solution of the system of equations* (1.1).

Proof. In view of the remark following conditions (1.2), it suffices to show that any solution F of equation (5.1) must be a solution of equation (1.3). To accomplish this, interchange x and z in (5.1), and subtract this equation from (5.1). The resulting equation can easily be simplified to equation (1.3) via the associativity and commutativity (Lemma 2.1(A)) of F.

Accordingly, members of the family F_s, $0 < s \leq \infty$, are the only candidates for solutions $F \in \mathscr{F}_A$ of (5.1). Clearly this equation holds with $F = F_1$. However, setting $x = y = z = \frac{1}{2}$ in (5.1), with $F = F_\infty$ the left-hand side equals $\frac{1}{2}$ while the right equals 0, and with $F=F_s$ for finite s straightforward calculations yield:

$$\text{left-hand side} = \log_s\left[\frac{3\sqrt{s}+s}{2(\sqrt{s}+1)}\right],$$

$$\text{right-hand side} = \log_s\left[\frac{4\sqrt{s}}{3+\sqrt{s}}\right],$$

and these expressions are unequal when $s \neq 1$. Thus we have proved:

LEMMA 5.2. *F_1 is the only solution $F \in \mathcal{F}_A$ of equation* (5.1).

For $F \in \mathcal{F} - \mathcal{F}_A$ we shall again rely on the ordinal sum representation (2.5a) and Lemma 2.3. Observe that F_0 is a solution of (5.1).

LEMMA 5.3. *If $F \in \mathcal{F} - \mathcal{F}_A$ is a solution of equation* (5.1), *then each summand in its ordinal sum decomposition must be the function F_1.*

Proof. Suppose that F is the ordinal sum of (J_n, F_n). Fix an arbitrary n, and choose any x, y, and z in J_n. Putting (2.5a) into (5.1) and simplifying, it follows that F_n must satisfy (5.1) with $x=(x-a_n)/(b_n-a_n)$, $y=(y-a_n)/(b_n-a_n)$, and $z=(z-a_n)/(b_n-a_n)$, and therefore that F_n must be a solution of (5.1). But $F_n \in \mathcal{F}_A$, whence by Lemma 5.2, F_n must be the function F_1.

To complete the proof of Theorem 5.1, it suffices to demonstrate that any ordinal sum of F_1's is a solution of equation (5.1).

LEMMA 5.4. *If F is representable as an ordinal sum of F_1's, then F is a solution of equation* (5.1).

Proof. The equation may be verified directly. By (2.5a),

$$F(x, y) = \begin{cases} a_n + (x-a_n)(y-a_n)/(b_n-a_n), & (x, y) \in [a_n, b_n]^2, \qquad n = 1, 2, \ldots, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

Splitting into cases according to the positions of x, y, and z relative to $[a_n, b_n]$, elementary but tedious calculations show that (5.1) holds universally.

An alternate, but indirect, proof follows from two results in [2]. For there it is shown that if F is an ordinal sum of F_1's then the operation $\sigma_{F,+}$ is associative (Theorem 15), and also that if $\sigma_{F,+}$ is associative, then $F \in \mathcal{F}$ must be a solution of equation (5.1) (Theorem 8).

Theorem 5.1 has an immediate analogue for $G \in \mathcal{G}$. For it is easily verified that $F \in \mathcal{F}$ satisfies (5.1) if and only if $G(x, y) = x + y - F(x, y)$ satisfies the same equation. Consequently, we have:

COROLLARY 5.1.1. *The function $G \in \mathcal{G}$ is a solution of equation* (5.1) (*replacing F with G*) *if and only if* (1) $G = G_0$, (2) $G = G_1$, *or* (3) *G is representable as an ordinal sum of G_1's.*

We conclude this section with a brief account of the implications of Theorem 5.1 for the important family of binary operations on Δ that motivated the author's interest in solving the system (1.1).

Let X and Y be extended real-valued random variables with d.f.'s D_X and D_Y and joint d.f. K_{XY}. For any Borel-measurable two-place real function B, the d.f. of the random variable $B(X, Y)$ is given by the Lebesgue–Stieltjes integral,

$$D_{B(X,Y)}(x) = \iint\limits_{B(u,v)\leq x} dK_{XY}(u, v). \tag{5.2}$$

If C_{XY} is a connecting copula of X and Y (see Lemma 4.1), then substituting (4.6) into (5.2) yields:

$$D_{B(X,Y)}(x) = \iint\limits_{B(u,v)\leq x} dC_{XY}(D_X(u), D_Y(v)). \tag{5.3}$$

Moreover, by the last part of Lemma 4.1, for any copula C and D_1 and D_2 in Δ, the integrator in (5.3) is a two-dimensional d.f., and thus, for any B as above, the right-hand side of (5.3) is an element of Δ. The above discussion yields the next result.

LEMMA 5.5. *For any Borel-measurable two-place real function B and any copula C, the mapping $\sigma_{C,B}$ defined via*

$$\sigma_{C,B}(D_1, D_2)(x) = \iint\limits_{B(u,v)\leq x} dC(D_1(u), D_2(v)),$$

$$D_1, D_2 \in \Delta, \qquad x \in [-\infty, +\infty], \tag{5.4}$$

is a binary operation on Δ. Moreover, if C_{XY} is a connecting copula of the random variables X and Y, then

$$D_{B(X,Y)} = \sigma_{C_{XY},B}(D_X, D_Y). \tag{5.5}$$

Surprisingly little is known about the family $\sigma_{C,B}$ except in certain very special cases. In particular, it would be of great interest to determine which of these operations are associative.

For the cases $B = \text{Max}$ and $B = \text{Min}$, this question is easily answered: $\sigma_{C,\text{Max}}$ (resp., $\sigma_{C,\text{Min}}$) is associative if and only if C (resp., $C^\wedge$) is associative. This is so because

$$\sigma_{C,\text{Max}} = \pi_C, \qquad \sigma_{C,\text{Min}} = \pi_{C^\wedge},$$

where π_C and $\pi_{C^\wedge}$ are given by (4.5), as can be shown directly from (5.4) (see [10, Theorem 5]) or indirectly by comparing (4.7) with (5.5).

For the case $B = \text{addition}$, the associativity question is the subject of [2], where it is shown that $\sigma_{C,+}$ is associative if and only if (1) $C = F_0$, (2) $C = F_1(\sigma_{F_1,+}$ is ordinary convolution), or (3) C is representable as an ordinal sum of F_1's. One step of the proof (Theorem 8) reduces the set of possible solutions to those operations $\sigma_{C,+}$ for which C is both associative and a solution of equation (5.1). Theorem 11 of [2] is thus an immediate corollary of Lemma 5.2.

For the general case, the associativity question has not yet been completely solved. However, the author has shown that the necessary conditions stated above for $\sigma_{C,+}$ are also necessary for $\sigma_{C,B}$, provided that B is associative and satisfies certain other mild restrictions, and, moreover, with slightly more stringent restrictions on B, that these conditions are also sufficient. Loosely speaking, we thus have:

"THEOREM" 5.2. *If B is associative and satisfies certain other (fairly weak) conditions, the operation $\sigma_{C,B}$ defined by* (5.4) *is associative if and only if* (1) $C = F_0$, (2) $C = F_1$, *or* (3) *C is representable as an ordinal sum of F_1's.*

A precise statement and proof of this result will appear elsewhere.

6. Extension to $[a, b]$

As was mentioned in the introduction, our results can be extended from $[0, 1]$ to any finite, closed interval $[a, b]$. To see that this is so, first observe that all of the results on topological semigroups and on solutions of the associativity equation which we used in Section 2 (Lemmas 2.1–2.4) are special cases of theorems that are valid with the obvious modifications on $[a, b]$ (c.f. [4; 7]). Second, a careful examination of the arguments in Section 3 will reveal that no special properties of $[0, 1]$ were used there. Theorem 1.1 can thus be extended as follows:

THEOREM 6.1. *Let F and G be functions satisfying the following conditions, where $-\infty < a < b < +\infty$:*

$$\operatorname{Dom} F = \operatorname{Dom} G = [a, b]^2, \qquad \operatorname{Ran} F = \operatorname{Ran} G = [a, b]; \tag{6.1a}$$

$$F \text{ and } G \text{ are continuous;} \tag{6.1b}$$

$$F(a, x) = F(x, a) = a, \qquad F(b, x) = F(x, b) = x, \quad \text{for every} \quad x \in [a, b]; \tag{6.1c}$$

$$G(a, x) = G(x, a) = x, \qquad G(b, x) = G(x, b) = b, \quad \text{for every} \quad x \in [a, b]. \tag{6.1d}$$

Then the pair (F, G) is a solution of the system of equation (1.1) *if and only if*

(1) *F is a member of the family F_s, $0 \leq s \leq \infty$, where*

$$F_0(x, y) = \text{minimum}\ (x, y),$$
$$F_1(x, y) = a + (x - a)(y - a)/(b - a),$$
$$F_\infty(x, y) = \text{maximum}\ (x + y - b, a),$$
$$F_s(x, y) = a + \log_s [1 + (s^{x-a} - 1)(s^{y-a} - 1)/(s^{b-a} - 1)], \quad \text{for} \quad 0 < s < \infty,$$

or

(2) *F is representable as an ordinal sum (where the ordinal sum of semigroups on $[a, b]$ is defined exactly as before, except that the J_n are subintervals of $[a, b]$) of functions, each of which is one of the functions F_s, $0 < s \leq \infty$; and G is obtained from F via equation* (1.1a).

The first two results in Section 4 and the solution of equation (5.1) can be extended similarly to functions satisfying conditions (6.1). Here the definition of $H^\wedge$ is given by (4.2), but the definition of $\bar{H}$ must be modified to

$$\bar{H}(x, y) = c - H(c - x, c - y),$$

where $c = a + b$, so that equation (4.1) and the operator $*$ defined by (4.4) become, respectively:

$$x+y-H(x, y)=c-H(c-x, c-y),$$
$$H^*(x, y)=x+y-c+H(c-x, c-y).$$

Theorems 4.1, 4.2, and 5.1 (and Corollary 5.1.1) are then valid when the functions listed in Theorem 1.1 are replaced by those in Theorem 6.1.

We conclude by showing that the situation is entirely different on infinite intervals. In the first place, if either $a=-\infty$ or $b=+\infty$, the definition of the family F_s cannot be extended to $[a, b]$ as before. Second, some of the preliminary lemmas (notably, Lemma 2.1) are no longer valid. Finally, as the following example shows, there are other solutions of the system (1.1) on $[-\infty, +\infty]$.

For any fixed $t>0$, let F_t and G_t be the functions defined on $[-\infty, +\infty]^2$ via

$$F_t(x, y)=-t\log(e^{-x/t}+e^{-y/t}), \qquad G_t(x, y)=t\log(e^{x/t}+e^{y/t}).$$

It is easy to show that F_t and G_t satisfy conditions (6.1) with $a=-\infty$ and $b=+\infty$. Also, F_t and G_t are associative; in fact, they have generators,

$$f_t(x)=e^{-x/t}, \qquad g_t(x)=e^{x/t}.$$

By direct calculation we have $F_t(x, y)+G_t(x, y)=x+y$, so (F_t, G_t) *is a solution of the system* (1.1) *for any* $t>0$. However, F_t is not a solution of the distributivity equation (5.1). For with $x=y=z=0$, simple calculations yield for (5.1): left-hand side $=-t\log\frac{3}{2}$, right-hand side $=-t\log\frac{4}{3}$.

Observe that the functions f_t (and by implication, g_t) emerge as potential solutions of the generator equation (3.2) in Lemma 3.11 (specifically, in (3.26)). Careful analysis of the discussion in Section 3 reveals that the pairs (f_t, g_t) are the only solutions of equation (3.2) when conditions (3.3) are appropriately modified for functions defined on $[-\infty, +\infty]$. However, it is possible that there are solutions (f, g) of equation (2.8) with f and g not convex, because the proof of Lemma 3.1 (and hence that of Lemma 3.2) is not valid on $[-\infty, +\infty]$.

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