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Factor copula models for multivariate data

Pavel Krupskii *, Harry Joe

Department of Statistics, University of British Columbia, Vancouver, BC, V6T 1Z4 Canada

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ABSTRACT

General conditional independence models for d observed variables, in terms of p latent variables, are presented in terms of bivariate copulas that link observed data to latent variables. The representation is called a factor copula model and the classical multivariate normal model with a correlation matrix having a factor structure is a special case. Dependence and tail properties of the model are obtained. The factor copula model can handle multivariate data with tail dependence and tail asymmetry, properties that the multivariate normal copula does not possess. It is a good choice for modeling high-dimensional data as a parametric form can be specified to have $O(d)$ dependence parameters instead of $O(d^2)$ parameters. Data examples show that, based on the Akaike information criterion, the factor copula model provides a good fit to financial return data, in comparison with related truncated vine copula models.

1. Introduction

The multivariate normality assumption is widely used to model the joint distribution of high-dimensional data. The univariate margins are transformed to normality and then the multivariate normal distribution is fitted to the transformed data. In this case, the dependence structure is completely defined by the correlation matrix and different models on the correlation structure can be used to reduce the number of dependence parameters from $O(d^2)$ to $O(d)$, where d is the multivariate dimension or number of variables. When dependence in the observed variables is thought to be explained by a few latent variables, the Gaussian or normal factor model assumes a linear relation on a few unobserved normally distributed factors.

The main contribution of this paper is to propose and study the copula version of the multivariate normal distribution with a correlation matrix that has a factor structure. We name the extension as the *factor copula model*. The classical factor model is a special case but within our framework, the parameterization is different as it involves partial correlations. The factor copula model is useful when the dependence in observed variables is based on a few unobserved variables, and there exists tail asymmetry or tail dependence in the data, so that the multivariate normality assumption is not valid.

The copula is a function linking univariate margins into the joint distribution. Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a random d-dimensional vector with the joint cumulative distribution function (cdf) $F_{\mathbf{X}}$. Let F_{X_j} be the marginal cdf of X_j for $j = 1, \ldots, d$. The copula $C_{\mathbf{X}}$, corresponding to $F_{\mathbf{X}}$, is a multivariate uniform cdf such that $F_{\mathbf{X}}(x_1, \ldots, x_d) = C_{\mathbf{X}}(F_{X_1}(x_1), \ldots, F_{X_d}(x_d))$. By Sklar [26], there exists a unique copula $C_{\mathbf{X}}$ if $F_{\mathbf{X}}$ is continuous. Copulas are suitable for modeling non-normal data such as financial asset returns or insurance data; see [24,21] and others. The superiority of non-normal copulas over the normal copula in modeling financial and insurance data has been discussed in [7].

* Corresponding author.
E-mail address: KrupskiiPV@yandex.ru (P. Krupskii).

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The vine copula or the pair-copula construction has been popular in recent years; see, for example [20,4]. The number of bivariate (conditional or marginal) copulas used in the vine construction is $d(d-1)/2$ for d variables, so typically vine copulas involve $O(d^2)$ number of parameters. Dißmann et al. [6] propose an algorithm which allows the fit of regular vines to data. If d is large (e.g., many asset returns), the conditional independence can be assumed at higher levels of the vine to reduce the number of parameters in a truncated vine model to $O(d)$; see also [5]. The factor copula model is an alternative copula modeling approach to truncated vines that has the order of $O(d)$ dependence parameters.

In many multivariate applications, the dependence in observed variables can be explained through latent variables; in multivariate item response in psychology applications, latent variables are related to the abstract variable being measured through items, and in finance applications, latent variables are related to economic factors. Classical factor analysis assumes (after transforms) that all observed and latent random variables are jointly multivariate normal. Books on multivariate analysis (see for example [16]) often have examples with factor analysis and financial returns. We show for some financial return data that, in terms of the Akaike or Bayesian information criteria, the factor copula model can be a better fit than truncated vines (because of a simpler dependence structure) and the classical factor model (because of tail dependence).

An important advantage of factor models is that they can be nicely interpreted. In case of stocks in a common sector, the current state of this sector can affect all of their change of prices, but the sector index, if measured, might not contain all of the latent information that explains the dependence. Similarly for market data, the state of the economy as a whole can determine the latent dependence structure. The "state variables" are aggregated from many exogenous variables (such as interest rate, refinancing rate, political instabilities, etc.) and cannot be easily measured, therefore factor copula models based on latent variables might be a good choice.

The rest of this paper is organized as follows. In Section 2 we define the factor copula model and give more details for the one-factor and two-factor models. Some dependence and tail properties of bivariate margins of the factor copula models are given in Section 3. The results imply that different types of dependence and tail asymmetry can be modeled with appropriate choices of bivariate linking copulas. Computational details for maximum likelihood estimation of the factor copula model are given in Section 4. Section 5 discusses diagnostics for choices of bivariate linking copulas, reports on some simulation results, and shows applications of the factor copula model to US stock returns. Section 6 concludes with a discussion of future research.

2. Factor copula model

In multivariate models with copulas, a copula or multivariate uniform distribution is combined with a set of univariate margins. This is equivalent to assuming that variables $X_1, \dots, X_d$ have been transformed to uniform random variables. So we assume that $\mathbf{U} = (U_1, \dots, U_d)$ is a random vector with $U_i \sim U(0, 1)$. The joint cdf of the vector $\mathbf{U}$ is then given by $C(u_1, \dots, u_d)$ where C is a d-dimensional copula. In the p-factor copula model, $U_1, \dots, U_d$ are assumed to be conditionally independent given p latent variables $V_1, \dots, V_p$. Without loss of generality, we can assume V_i are independent and identically distributed (i.i.d.) $U(0, 1)$. Let the conditional cdf of U_j given $V_1, \dots, V_p$ be denoted by $F_{j|V_1,\dots,V_p}$. Then,

$$C(u_1, \dots, u_d) = \int_{[0,1]^p} \prod_{j=1}^{d} F_{j|V_1,\dots,V_p}(u_j|v_1, \dots, v_p)\, dv_1 \cdots dv_p. \tag{1}$$

Any conditional independence model, based on p independent latent variables, can be put in this form after transforms to $U(0, 1)$ random variables. Hence, the dependence structure of $\mathbf{U}$ is then defined through conditional distributions $F_{1|V_1,\dots,V_p}, \dots, F_{d|V_1,\dots,V_p}$. We will call (1) a *factor copula model*, with $F_{j|V_1,\dots,V_p}$ expressed appropriately in terms of a sequence of bivariate copulas that link the observed variables U_j to the latent variables V_k. Some of the bivariate copulas are applied to conditional distributions. Details are given in Sections 2.1 and 2.2.

In the finance literature there are several factor copula models (e.g., Section 9.7.2 of McNeil et al. [21], Hull and White [12] and Oh and Patton [23]); these all have a linear latent structure and are not as general as our model. Oh and Patton [23] have overlapping ideas with our research but our approach was developed independently of their approach. With the conditional independence model with 2 or more latent variables, there could be alternative ways to specify a model for $F_{j|V_1,\dots,V_p}$ than we have. Throughout the remainder of this paper, we assume that all copulas are absolutely continuous and have densities, so that the log-likelihood for continuous data will involve the density of the factor copula.

2.1. One- and two-factor copula models

We first study the case of $p = 1$ latent variable in (1). For $j = 1, \dots, d$, denote the joint cdf and density of (U_j, V_1) by C_{j,V_1} and c_{j,V_1} respectively. Since U_1, V_j are $U(0, 1)$ random variables, then $F_{j|V_1}$ is just a partial derivative of the copula C_{j,V_1} with respect to the second argument. That is, $F_{j|V_1}(u_j|v_1) = C_{j|V_1}(u_j|v_1) = \partial C_{j,V_1}(u_j, v_1)/\partial v_1$. With $p = 1$, Eq. (1) becomes:

$$C(u_1, \dots, u_d) = \int_0^1 \prod_{j=1}^{d} F_{j|V_1}(u_j|v_1)\, dv_1 = \int_0^1 \prod_{j=1}^{d} C_{j|V_1}(u_j|v_1)\, dv_1. \tag{2}$$

Note that $\frac{\partial}{\partial u}C_{j|V_1}(u|v_1) = \frac{\partial^2}{\partial u \partial v_1}C_{j,V_1}(u, v_1) = c_{j,V_1}(u, v_1)$. Then (2) implies by differentiation that the density of the 1-factor copula is

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \cdots \partial u_d} = \int_0^1 \prod_{j=1}^d c_{j,V_1}(u_j, v_1)\, dv_1. \tag{3}$$

In the model, dependence is defined by d bivariate linking copulas $C_{1,V_1}, \dots, C_{d,V_1}$; there are no constraints amongst these bivariate copulas. Note that any conditional independence model for absolutely continuous random variables, conditioned on one latent variable, can be written in this form. Below we show that when C_{j,V_1} are all bivariate normal copulas, then (3) becomes the copula of the multivariate normal distribution with a 1-factor correlation matrix.

A main advantage of the model is that it allows for different types of tail dependence structure. As it was shown in [15,14], if all bivariate linking copulas are lower (upper) tail dependent then all bivariate margins of **U** are also lower (upper) tail dependent respectively. Thus, with appropriately chosen linking copulas asymmetric dependence structure as well as tail dependence can be easily modeled.

For the special case of bivariate normal linking copulas, let C_{j,V_1} be the bivariate normal copula with correlation $\alpha_{j1}, j = 1, \dots, d$. Let Φ, ϕ denote the standard normal cdf and density function, and let $\Phi_2(\cdot; \rho)$ be bivariate normal cdf with correlation ρ. Then $C_{j,V_1}(u, v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); \alpha_{j1})$ and

$$F_{j|V_1}(u|v) = \Phi\left(\left[\Phi^{-1}(u) - \alpha_{j1}\Phi^{-1}(v)\right] \Big/ \sqrt{1-\alpha_{j1}^2}\right).$$

For Eq. (2), let $u_j = \Phi(z_j)$ to get a multivariate distribution with $N(0, 1)$ margins. Then

$$\begin{aligned} F(z_1, \dots, z_d) &:= C(\Phi(z_1), \dots, \Phi(z_d)) = \int_0^1 \prod_{j=1}^d \left\{\Phi\left(\left[z_j - \alpha_{j1}\Phi^{-1}(v_1)\right] \Big/ \sqrt{1-\alpha_{j1}^2}\right)\right\} dv_1 \\ &= \int_{-\infty}^{\infty} \prod_{j=1}^d \left\{\Phi\left(\left[z_j - \alpha_{j1}w\right] \Big/ \sqrt{1-\alpha_{j1}^2}\right)\right\} \phi(w)\, dw. \end{aligned}$$

Hence this model is the same as a multivariate normal model with a 1-factor correlation structure because this multivariate cdf comes from the representation:

$$Z_j = \alpha_{j1}W + \sqrt{1-\alpha_{j1}^2}\,\epsilon_j, \quad j = 1, \dots, d,$$

where $W, \epsilon_1, \dots, \epsilon_d$ are i.i.d. $N(0, 1)$ random variables.

If C_{j,V_1} is the Student t copula with correlation α_{j1}, and ν_i degrees of freedom, $j = 1, \dots, d$, $c(u_1, \dots, u_d)$ is no longer the multivariate Student t copula density. When the C_{j,V_1} are chosen from the Student t copula or another copula family, then the simplest representation is a 1-dimensional integral, but this is not a problem for likelihood inference, as shown in Section 4.

We next show details for $p = 2$. Let C_{j,V_1} be the copula of (U_j, V_1) as before. Also let $C_{j,V_2;V_1}$ be the copula for $F_{j|V_1} = F_{U_j|V_1}$ and $F_{V_2|V_1}$, and let $c_{j,V_2;V_1}$ be its density. We make the simplifying assumption that the copula for $F_{U_j|V_1}(\cdot|v_1)$ and $F_{V_2|V_1}(\cdot|v_1)$ does not depend on v_1; this is the same assumption used in vine copulas or the pair-copula construction. For both theory and applications, it is a first step in understanding factor copula models with more than 1 latent factor. Note that $F_{V_2|V_1}$ is the $U(0, 1)$ cdf since we assume that V_2 is independent of V_1. Then the independence of V_1, V_2 implies

$$\begin{aligned} F_{j|V_1,V_2}(u|v_1, v_2) &= \Pr(U_j \le u|V_1 = v_1, V_2 = v_2) = \frac{\partial}{\partial v_2}\Pr(U_j \le u, V_2 \le v_2|V_1 = v_1) \\ &= \frac{\partial}{\partial v_2}C_{jV_2;V_1}\big(C_{j|V_1}(u|v_1), v_2\big) = C_{j|V_2;V_1}\big(C_{j|V_1}(u|v_1)|v_2\big), \end{aligned} \tag{4}$$

where $C_{j|V_2;V_1}(x|v) = \partial C_{j,V_2;V_1}(x, v)/\partial v$. Eq. (1) becomes:

$$C(u_1, \dots, u_d) = \int_0^1 \int_0^1 \prod_{j=1}^d F_{j|V_1,V_2}(u_j|v_1, v_2)\, dv_1 dv_2 = \int_0^1 \int_0^1 \prod_{j=1}^d C_{j|V_2;V_1}(C_{j|V_1}(u_j|v_1)|v_2)\, dv_1 dv_2. \tag{5}$$

By differentiation with respect to $u_1, \dots, u_d$, (5) implies that the 2-factor copula density is

$$c(u_1, \dots, u_d) = \int_0^1 \int_0^1 \prod_{j=1}^d \left\{c_{j,V_2;V_1}(C_{j|V_1}(u_j|v_1), v_2) \cdot c_{j,V_1}(u_j, v_1)\right\} dv_1 dv_2. \tag{6}$$

The dependence structure is defined through $2d$ linking copulas $C_{1,V_1}, \dots, C_{d,V_1}, C_{1,V_2;V_1}, \dots, C_{d,V_2;V_1}$; there are no constraints amongst these $2d$ bivariate copulas. Clearly, this is an extension of the 1-factor copula model and different types of dependence can be modeled. More details on the properties of bivariate margins are provided in Section 3.

This model includes the 2-factor multivariate normal model as a special case. Suppose C_{j,V_1} and $C_{j,V_2;V_1}$ are the bivariate copulas with correlations α_{j1} and $\gamma_j = \alpha_{j2}/(1-\alpha_{j1}^2)^{1/2}$ respectively, $j = 1, \ldots, d$. Here α_{j2} is a correlation of $Z_j = \Phi(U_j)$ and $W_2 = \Phi(V_2)$ so that the independence of V_1, V_2 implies that γ_j is the partial correlation of Z_j and W_2 given $W_1 = \Phi(V_1)$ (in general $\rho_{ZW_2;W_1} = [\rho_{ZW_2} - \rho_{ZW_1}\rho_{W_2W_1}]/[(1-\rho_{ZW_1}^2)(1-\rho_{W_2W_1}^2)]^{1/2}$). Then, using the above conditional distribution of the bivariate normal copula,

$$C_{j|V_2;V_1}(C_{j|V_1}(u|v_1)|v_2) = \Phi\left(\left[\frac{\Phi^{-1}(u)-\alpha_{j1}\Phi^{-1}(v_1)}{(1-\alpha_{j1}^2)^{1/2}} - \gamma_j\Phi^{-1}(v_2)\right]\Big/\sqrt{1-\gamma_j^2}\right)$$
$$= \Phi\left(\frac{\Phi^{-1}(u)-\alpha_{j1}\Phi^{-1}(v_1)-\gamma_j(1-\alpha_{j1}^2)^{1/2}\Phi^{-1}(v_2)}{\sqrt{(1-\alpha_{j1}^2)(1-\gamma_j^2)}}\right).$$

With $z_j = \Phi(u_j), j = 1, \ldots, d$, the cdf for the 2-factor model becomes

$$F(z_1, \ldots, z_d) := C(\Phi(z_1), \ldots, \Phi(z_d))$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{j=1}^{d}\Phi\left(\frac{z_j-\alpha_{j1}w_1-\gamma_j(1-\alpha_{j1}^2)^{1/2}w_2}{\sqrt{(1-\alpha_{j1}^2)(1-\gamma_j^2)}}\right)\cdot\phi(w_1)\phi(w_2)\,dw_1dw_2.$$

Hence this model is the same as a multivariate normal model with a 2-factor correlation structure because this multivariate cdf comes from the representation:

$$Z_j = \alpha_{j1}W_1 + \alpha_{j2}W_2 + \sqrt{(1-\alpha_{j1}^2)(1-\gamma_j^2)}\,\epsilon_j, \quad j = 1, \ldots, d,$$

where $W_1, W_2, \epsilon_1, \ldots, \epsilon_d$ are i.i.d. $N(0, 1)$ random variables.

2.2. Models with $p > 2$ factors

The factor copula model can be straightforwardly extended to $p > 2$ factors and it becomes an extension of the p-factor multivariate normal distribution with a correlation matrix Σ that has the p-factor structure, that is, $\Sigma = AA^T + \Psi$, where A is a $d \times p$ matrix of loadings and Ψ is a diagonal matrix. The main difference is that for the factor copula model, the parameters for the second to pth factors are partial correlations $\rho_{Z_jW_k;W_1\ldots W_{k-1}}$ (the jth observed variable and kth latent variable given latent variables $1, \ldots, k-1$). The advantage of this parameterization is that all of these partial correlations and $\rho_{Z_jW_1}$ are algebraically independent in the interval $(-1, 1)$, and the partial correlation parameterization is the one that can extend to the factor copula, by replacing each correlation with factor 1 by a bivariate copula, and each partial correlation for factors 2 to p with a bivariate copula applied to conditional distributions.

We now provide some details on how the parameters of bivariate normal linking copulas are related to the matrix of loadings A in the classical factor model with p factors. In the model,

$$Z_j = \sum_{i=1}^{p}\alpha_{ji}W_i + \epsilon_j, \tag{7}$$

where $W_1, \ldots, W_p,\ \epsilon_1, \ldots, \epsilon_d$ are i.i.d. $N(0, 1)$ random variables. Let the matrix of loadings be $A = (A_{ij})$; one possibility has $A_{ij} = \alpha_{ji}, i = 1, \ldots, p, j = 1, \ldots, d$. This matrix is unique up to orthogonal transformations. The unconditional distribution of Z_j and W_1 is given by a bivariate normal distribution with correlation α_{j1} as it follows from (7). Similarly, for $k = 2, \ldots, p$ the conditional distribution $F_{Z_j,W_k;W_1,\ldots,W_{k-1}}$ is a bivariate normal distribution with correlation

$$\rho_{Z_j,W_k;W_1,\ldots,W_{k-1}} = \frac{\mathrm{Cov}(Z_j, W_k|W_1, \ldots, W_{k-1})}{[\mathrm{Var}(Z_j|W_1, \ldots, W_{k-1})\mathrm{Var}(W_k|W_1, \ldots, W_{k-1})]^{1/2}}$$
$$= \frac{\alpha_{jk}}{(1-\alpha_{j1}^2-\cdots-\alpha_{j,k-1}^2)^{1/2}}. \tag{8}$$

As a result, C_{Z_j,W_1} is a bivariate copula with correlation α_{j1} and $C_{Z_j,W_k;W_1,\ldots,W_{k-1}}$ is a bivariate normal copula with correlation $\rho_{Z_j,W_k;W_1,\ldots,W_{k-1}}$ as given in (8).

The density for a general $2 \le p < d$ can be obtained if we note that due to independence $V_1, \ldots, V_p$

$$F_{j|V_1,\ldots,V_p}(u|v_1, \ldots, v_p) = \frac{\partial\,\Pr(U_j \le u, V_p \le v_p|V_1 = v_1, \ldots, V_{p-1} = v_{p-1})}{\partial v_p}$$
$$= \frac{\partial C_{j,V_p;V_1,\ldots,V_{p-1}}(F_{j|V_1,\ldots,V_{p-1}}(u|v_1, \ldots, v_{p-1}), v_p)}{\partial v_p}$$
$$= C_{j|V_p;V_1,\ldots,V_{p-1}}(F_{j|V_1,\ldots,V_{p-1}}(u|v_1, \ldots, v_{p-1})|v_p).$$

This recursion formula for $F_{j|V_1,\dots,V_p}(u|v_1,\dots,v_p)$ can be further expanded to express this conditional probability in terms of bivariate linking copulas $C_{j|V_k;V_1,\dots,V_{k-1}}$, $k \le p$ and then used in (1) to get the joint cdf $C(u_1,\dots,u_d)$. The density can be then obtained by differentiating the cdf and applying the chain rule.

In the preceding subsections, we have derived details of the factor copula models with one and two factors as conditional independence models to show that they are very general latent variable models. At the same time, factor copula models are equivalent to truncated C-vines rooted at the latent variables; the view as C-vines means that one can obtain the joint density of the observed and unobserved variables through the copula density representation of Bedford and Cooke [3] and then integrate the latent variables to get the density of the observed variables. In particular, the factor copula model with p latent variables $V_1,\dots,V_p$ can be represented as a C-vine copula model for $(V_1,\dots,V_p,U_1,\dots,U_d)$ rooted at the latent variables and truncated at the pth level (all copulas at higher levels are independence copulas). More details on the C-vine copula construction are given in [2]. By integrating over latent variables, note that the p-factor copula density involves a p-dimensional integral in general when the bivariate linking copulas are not all normal.

3. Properties of 1- and 2-factor copula models

In a later section, we show that the 1-factor and 2-factor copula models are good fits to some multivariate data sets compared with existing copula families. Hence it is important to know more properties about them to help in the choice of the bivariate linking copulas.

In this section we investigate different types of tail behavior and dependence properties of bivariate marginal copulas that can be obtained in factor copula models. Without loss of generality we restrict our attention to the copula $C_{1,2}$ corresponding to the joint distribution of U_1 and U_2.

3.1. Dependence properties

In the factor copula model some positive dependence properties of the linking copulas extend to the bivariate margins under some mild conditions. These properties include positive quadrant dependence (PQD), increasing in the concordance ordering and stochastic increasing (SI). PQD indicates that small (or large) values of a pair of variables distributed with a PQD copula tend to occur more often than in the independence case. It means dependence is stronger in the first and third quadrants. Similarly, negative quadrant dependence indicates that dependence is stronger in the second and fourth quadrants. Increasing in the concordance ordering is a natural property when dependence becomes stronger when the dependence parameter increases (with other dependence parameters held fixed for copula families with several parameters). SI is a property of conditional distribution of a pair of variables when the second variable is more likely to take a larger value as the first variable increases.

The detailed overview of these dependence concepts and some examples are given in Chapter 2 of Joe [13]. Many commonly-used bivariate copulas, including the normal copula, satisfy all or some of these properties in the range of positive dependence. For all results in this section we will assume that all bivariate linking copulas are twice continuously differentiable functions on $(0,1)^2$. It follows that in the factor copula model the three properties – PQD, increasing in concordance ordering and SI – hold under some mild conditions on bivariate linking copulas. Basically, SI linking (conditional) copulas imply all these properties for bivariate margins in a 1-factor copula model as it is shown in Propositions 1 and 2.

Proposition 1. *For $j = 1, 2$, suppose $C_{j|V_1} = C_{U_j|V_1}$ is SI, that is, $\Pr(U_j > u|V_1 = v) = 1 - C_{j|V_1}(u|v)$ is increasing (or non-decreasing) in $v \in (0,1)$ for any $0 < u < 1$. Let $(U_1,U_2) \sim C_{1,2}$, where $C_{1,2}$ is a bivariate margin of the 1-factor copula (2). Then $\mathrm{Cov}(U_1,U_2) \ge 0$ and $C_{1,2}$ is PQD or $C_{1,2}(u_1,u_2) \ge u_1u_2$ for any $0 < u_1 < 1$ and $0 < u_2 < 1$.*

Proof. We have

$$\mathrm{Cov}(U_1,U_2) = \mathrm{E}[\mathrm{Cov}(U_1,U_2|V_1)] + \mathrm{Cov}(\mathrm{E}(U_1|V_1),\mathrm{E}(U_2|V_1)) = \mathrm{Cov}(\mathrm{E}(U_1|V_1),\mathrm{E}(U_2|V_1)) \ge 0$$

since $\mathrm{E}(U_1|V_1 = v)$ and $\mathrm{E}(U_2|V_1 = v)$ are increasing in v from the SI assumption, and the covariance of two increasing functions is non-negative from Chebyshev's inequality for similarly ordered functions (see [9]). Similarly, for any $0 < u_1 < 1$ and $0 < u_2 < 1$,

$$\begin{aligned}\Pr\{U_1 \ge u_1, U_2 \ge u_2\} - (1-u_1)(1-u_2) &= \mathrm{Cov}(I\{U_1 \ge u_1\}, I\{U_2 \ge u_2\}) \\ &= \mathrm{Cov}(\mathrm{E}[I\{U_1 \ge u_1\}|V_1], \mathrm{E}[I\{U_2 \ge u_2\}|V_1]) \ge 0,\end{aligned}$$

because $\mathrm{E}[I\{U_j \ge u_j\}|V_1 = v] = \Pr\{U_j \ge u_j|V_1 = v\}$ for $j = 1, 2$ are increasing functions of v from the SI assumption. With $\overline{C}$ being the survival function of C, $\overline{C}(u_1,u_2) \ge (1-u_1)(1-u_2)$ which is the same thing as $C(u_1,u_2) \ge u_1u_2$. □

Proposition 2. *Consider the bivariate margin $C_{1,2}$ of (2). Assume that C_{2,V_1} is fixed and that $C_{2|V_1}$ is stochastically increasing (respectively decreasing).* (a) *Assume that C_{1,V_1} increases in the concordance ordering. Then $C_{1,2}$ is increasing (respectively decreasing) in concordance.* (b) *Assume that $C_{V_1|1}$ is SI. Then $C_{2|1}$ is stochastically increasing (respectively decreasing).*

Proof. Suppose C_{1,V_1} is parameterized by a parameter θ and C_{2,V_1} is fixed. The increasing in concordance assumption implies that $C_{1,V_1}(\cdot;\theta_2) - C_{1,V_1}(\cdot;\theta_1) \geq 0$ for $\theta_1 < \theta_2$. Using the integration by parts formula we get:

$$C_{1,2}(u_1,u_2;\theta) = \int_0^1 C_{1|V_1}(u_1|v;\theta)C_{2|V_1}(u_2|v)dv = u_1 C_{2|V_1}(u_2|1) - \int_0^1 C_{1,V_1}(u_1,v;\theta)\frac{\partial C_{2|V_1}(u_2|v)}{\partial v}dv. \tag{9}$$

With the assumption of twice continuous differentiability, $\partial C_{2|V_1}(u_2|v)/\partial v$ is a continuous function of v for $v \in (0,1)$ but can be unbounded at 0 or 1. Nevertheless, the integrand is an integrable function since

$$\int_0^1 \left| C_{1,V_1}(u_1,v;\theta)\frac{\partial C_{2|V_1}(u_2|v)}{\partial v}\right| dv \leq \left|\int_0^1 \frac{\partial C_{2|V_1}(u_2|v)}{\partial v}dv\right| = \left|C_{2|V_1}(u_2|0) - C_{2|V_1}(u_2|1)\right|.$$

Therefore the formula (9) is valid. For $\theta_2 > \theta_1$ we have:

$$C_{1,2}(u_1,u_2;\theta_2) - C_{1,2}(u_1,u_2;\theta_1) = \int_0^1 [C_{1,V_1}(u_1,v;\theta_1) - C_{1,V_1}(u_1,v;\theta_2)]\cdot\frac{\partial C_{2|V_1}(u_2|v)}{\partial v}dv. \tag{10}$$

Since $C_{1,V_1}(u_1,v;\theta_2) \geq C_{1,V_1}(u_1,v;\theta_1)$ and $\partial C_{2|V_1}(u_2|v)/\partial v \leq (\geq)\ 0$ by the assumption of stochastic increasing (decreasing), we get $C_{1,2}(u_1,u_2;\theta_2) \geq (\leq)\ C_{1,2}(u_1,u_2;\theta_1)$ respectively, that is $C_{1,2}$ is increasing (decreasing) in concordance.

Similarly, one can show that for $u_1 \in (0,1)$ both parts of (9) can be differentiated with respect to u_1 twice to get

$$\frac{\partial^2 C_{1,2}(u_1,u_2;\theta)}{\partial u_1^2} = \frac{\partial C_{2|1}(u_2|u_1;\theta)}{\partial u_1} = -\int_0^1 \frac{\partial C_{V_1|1}(v|u_1;\theta)}{\partial u_1}\cdot\frac{\partial C_{2|V_1}(u_2|v)}{\partial v}dv.$$

Assuming $C_{V_1|1}$ is SI we get $\partial C_{V_1|1}(v|u_1;\theta)/\partial u_1 \leq 0$; since also $\partial C_{2|V_1}(u_2|v;\theta)/\partial v \leq (\geq)\ 0$ by the assumption of stochastically increasing (decreasing), then $\partial C_{2|1}(u_2|u_1;\theta)/\partial u_1 \leq (\geq$, respectively) 0, that is, $C_{2|1}$ is stochastically increasing (decreasing). □

While the result on SI cannot be readily extended for the 2-factor model, the results for PQD and increasing in the concordance ordering hold in this model under similar assumptions. We will not include the proofs as they are similar to those for the 1-factor model.

Proposition 3. *For the 2-factor copula model in (5), suppose that $C_{j|V_1}(\cdot|v_1)$ and $C_{j|V_2;V_1}(\cdot|v_2)$ are SI for $j=1,2$. Then the margin $C_{1,2}$ of (5) is PQD.*

Proposition 4. *Consider the bivariate margin $C_{1,2}$ of (5). Assume that $C_{1,V_2;V_1}$ increases in the concordance ordering and $C_{2|V_2;V_1}$ is stochastically increasing (decreasing). Then $C_{1,2}$ is increasing (decreasing, respectively) in concordance.*

3.2. Tail properties

In this subsection, we prove some tail properties of the 1-factor and 2-factor copula models. We consider the properties of $C_{1,2}$ in the lower tail, as the properties in the upper tail can be obtained by reflections $U_j \to 1-U_j$. In [11], the concept of tail order is introduced in a multivariate context to study a range of tail behavior. The lower tail order of a bivariate copula $C_{1,2}$ is κ_L if $C_{1,2}(u,u) \sim \ell_L(u)u^{\kappa_L}$ as $u \to 0$ where $\ell_L(u)$ is a slowly varying function (such as a constant or a power of $-\log u$). If $C_{1,2}(u,u) = 0$ for all $0 < u < u_0$ for some positive u_0, then define $\kappa_L = \infty$. Similarly the upper tail order κ_U is such that $C_{1,2;R}(u,u) \sim \ell_U(u)u^{\kappa_U}$ as $u \to 0$, where $C_{1,2;R}(u_1,u_2) = u_1+u_2-1+C_{1,2}(1-u_1,1-u_2)$ is the survival or reflection copula. A property is that $\kappa_L \geq 1$ and $\kappa_U \geq 1$ with a smaller value corresponding to more dependence in the tail (more probability in the corner). Thus, the strongest tail dependence corresponds to $\kappa_L = 1$ or $\kappa_U = 1$. For the comonotonic (perfect positive dependence) tail, $\kappa_L = \kappa_U = 1$ and $\ell_L(u) = \ell_U(u) = 1$ for strongest tail dependence. For the countermonotonic (perfect negative dependence) tail, $\kappa_L = \kappa_U = \infty$ because there is no probability in the upper and lower corners. These tail orders also provide a simple condition to establish the direction of tail asymmetry, namely: if $\kappa_L > \kappa_U$ ($\kappa_L < \kappa_U$) then $C_{1,2}$ has tail asymmetry skewed to the upper (lower) tail; and if $C_{1,2}(u,u) \sim \lambda_L u^\kappa$ and $C_{12,R}(u,u) \sim \lambda_U u^\kappa$ as $u \to 0$ with $\lambda_U > \lambda_L > 0$ ($\lambda_L > \lambda_U > 0$), then $C_{1,2}$ has tail asymmetry skewed to the upper (lower) tail.

Below we consider three types of tail behavior for the bivariate copula $C_{1,2}$ for κ being κ_L or κ_U: tail dependence ($\kappa = 1$), intermediate tail dependence ($1 < \kappa < 2$) and tail quadrant independence ($\kappa = 2$). The proofs are more technical, so are deferred to the Appendix.

Tail dependence

It is shown in [14] that in the 1-factor model, $C_{1,2}$ has lower (upper) tail dependence if both C_{1,V_1} and C_{2,V_1} have lower (upper) tail dependence. Under mild assumptions, tail dependence is inherited by $C_{1,2}$ in the 2-factor model. More formally, the next proposition holds.

Proposition 5. *Let $\lim_{u\to 0} C_{j|V_1}(u|hu) = t_j(h)$ and assume $C_{j|V_2;V_1}(u|v)$ and $C_{j|V_1}(u|v)$ are continuous functions of u and v, $j=1,2$. Assume $\lim_{h\to 0} t_j(h) = t_{j0} > 0$, and $C_{j|V_2;V_1}(t_{j0}|0) \geq k_0 > 0$, $j=1,2$. Then the bivariate margin $C_{1,2}$ of the copula in (6) is lower tail dependent. A parallel result holds for upper tail dependence.*

Proof. See the Appendix. $\square$

The condition for $C_{j|V_1}$ is implied by C_{j,V_1} having lower tail dependence—see [14]. Hence, lower (upper) tail dependence for the observed variables can be modeled by choosing lower (upper) tail dependent linking copulas for the first factor.

Intermediate tail dependence

Intermediate tail dependence can be inherited by $C_{1,2}$ in many cases. One example of a copula with intermediate dependence is a bivariate normal copula with positive correlation $0 < \rho < 1$, for which $\kappa = 2/(1+\rho)$. If the bivariate linking copulas are normal, with correlations α_{jV_1} for C_{j,V_1} and $\gamma_j = \alpha_{jV_2}/(1-\alpha_{jV_1}^2)^{1/2}$ for $C_{j,V_2;V_1}$ respectively, $j = 1, 2$, then the marginal copula $C_{1,2}$ is again a normal copula with parameter $\rho_{12} = \alpha_{1V_1}\alpha_{2V_1} + \alpha_{1V_2}\alpha_{2V_2}$, Therefore it has intermediate tail dependence if all of the $\alpha_{jV_1}, \alpha_{jV_2}$ parameters are positive.

Intermediate lower tail dependence can also be obtained if bivariate linking copulas are extreme value copulas, which have upper tail dependence and intermediate lower tail dependence. Because of the structure of extreme value copulas, this case is amenable to mathematical proofs of tail properties; we expect the behavior to be similar in general for linking copulas with intermediate tail dependence.

We consider the 1-factor model first. Let $C_{i,V_1}(u_1, u_2)$ be a bivariate extreme value copula. From Chapter 6 of Joe [13], there is the following representation:

$$C_{i,V_1}(u_1, u_2) = \exp\{-(w_1 + w_2)A_i(w_2/[w_1 + w_2])\} = (u_1u_2)^{A_i(\ln u_2/\ln(u_1u_2))}, \quad w_j = -\ln u_j,\ j = 1, 2, \tag{11}$$

where $A_i(\cdot) : [0, 1] \mapsto [0.5, 1]$ is a convex function such that $A_i(t) \geq \max\{t, 1-t\}$, $i = 1, 2$.

Proposition 6. *Assume $A_i(t)$ in (11) is a continuously differentiable function and $A_i'(t) > -1$ for $0 < t < 0.5$, $i = 1, 2$. The lower tail order of the bivariate marginal copula $C_{1,2}$ in (2) is equal to $\xi^* = \min_{0<s<\infty}\{\xi(s)\} \in [1, 2]$, where*

$$\xi(s) = (s+1)\left[A_1\left(\frac{s}{s+1}\right) + A_2\left(\frac{s}{s+1}\right)\right] - s. \tag{12}$$

Proof. See the Appendix. $\square$

Under the assumptions of Proposition 6, the marginal copula $C_{1,2}$ of (2) has intermediate lower tail dependence if $1 < \xi^* < 2$. Note that $\xi(s) \geq (s+1)\left(\frac{s}{s+1} + \frac{1}{s+1}\right) - s = 1$ and $\xi(s) = 1$ only if $A_i\left(\frac{s^*}{s^*+1}\right) = \frac{s^*}{s^*+1}$ and $A_{2-i}\left(\frac{s^*}{s^*+1}\right) = \frac{1}{s^*+1}$ for some s^*. Since $A_i(t) \geq \max\{t, 1-t\}$ it implies $s^* = 1$ and $A_i(1/2) = 1/2$ which is possible only if both C_{1,V_1} and C_{2,V_1} are comonotonic copulas. Otherwise, the lower tail order is always larger than 1. Let us consider some examples.

Example 1. Gumbel copula: $A_i(t) = [t^{\theta_i} + (1-t)^{\theta_i}]^{1/\theta_i}$, $\theta_i > 1$, for $i = 1, 2$. It is seen that $\xi(s) = (1+s^{\theta_1})^{1/\theta_1} + (1+s^{\theta_2})^{1/\theta_2} - s$. If $\theta_1 = \theta_2 = \theta_0$, $s^* = \left(2^{\frac{\theta_0}{\theta_0-1}} - 1\right)^{-1/\theta_0}$ and $\xi^* = \left(2^{\frac{\theta_0}{\theta_0-1}} - 1\right)^{(\theta_0-1)/\theta_0}$. It follows that $1 < \xi^* < 2$ for any $\theta_1 > 1$, $\theta_2 > 1$. Hence, $C_{1,2}$ has intermediate lower tail independence. The Gumbel copula has upper tail dependence and therefore $C_{1,2}$ also has upper tail dependence. As a result, $C_{1,2}$ has tail asymmetry skewed to the upper tail.

If $\theta_1 = \theta_2 = \theta_0 > 1$, the tail orders of Gumbel linking copulas are $2^{1/\theta_0}$ and this is less than ξ^*. This demonstrates the general pattern that the conditional independence in the 1-factor model "dampens" the strength of dependence in the tail (or tail order increases). Tail quadrant independence (tail order 2) is possible when the linking copulas with V_1 have intermediate tail dependence, as shown in the next example.

Example 2. Let $A_i(t) = 1 - t(1-t)/\theta_i$, $\theta_i \geq 1$. Then $\xi(s) = 2 + s(1 - \alpha_0/(s+1))$ where $\alpha_0 = 1/\theta_1 + 1/\theta_2$. If $\alpha_0 \leq 1$ we have $\xi(s) \geq 2 + s^2/(s+1)$ therefore $\xi^* = 2$. However, the lower tail order of C_{i,V_1} is equal to $2A_i(1/2) = 2 - 1/(2\theta_i) < 2$. This example shows that intermediate tail dependence of C_{i,V_1} does not necessarily imply intermediate tail dependence of $C_{1,2}$.

Now consider 2-factor copula models. Suppose the first level $C_{1,V_1}(u_1, u_2)$, $C_{2,V_1}(u_1, u_2)$ are extreme value copulas as above, and in addition copulas at the second level are also extreme value copulas:

$$C_{i,V_2;V_1}(u_1, u_2) = (u_1u_2)^{A_{i;1}(\ln u_2/\ln(u_1u_2))}, \tag{13}$$

where $A_{i;1}(\cdot) : [0, 1] \mapsto [0.5, 1]$ is a convex function such that $A_{i;1}(t) \geq \max\{t, 1-t\}$, $i = 1, 2$.

Proposition 7. *Assume $A_i(t)$, $A_{i;1}(t)$ in (11) and (13) are continuously differentiable functions and $A_i'(t) > -1$, $A_{i;1}'(t) > -1$ for $0 < t < 0.5$. The lower tail order of the bivariate marginal copula $C_{1,2}$ in (5) is equal to $\xi_2^* = \min_{0<s_1,s_2<\infty}\{\tilde{\xi}_2(s_1, s_2)\} \in [1, 2]$, where*

$$\tilde{\xi}_2(s_1, s_2) = (s_1 + \tilde{m}_1(s_2))A_{1;1}\left(\frac{s_1}{s_1 + \tilde{m}_1(s_2)}\right) + (s_1 + \tilde{m}_2(s_2))A_{2;1}\left(\frac{s_1}{s_1 + \tilde{m}_2(s_2)}\right) - s_1 + s_2, \tag{14}$$

$$\tilde{m}_i(s_2) = (s_2 + 1)A_i\left(\frac{s_2}{s_2 + 1}\right) - s_2, \quad i = 1, 2.$$

Proof. See the Appendix. □

Note that $\tilde{\xi}_2(0, s_2) = \xi(s_2)$ as given in (12) and (14), and therefore $\xi_2^* \le \xi^*$. That is, the lower tail order of $C_{1,2}$ in the 2-factor model is lower than that in the 1-factor model when the same linking copulas C_{i,V_1} are used at the first level. It means the intermediate tail dependence is stronger if the second factor is added to the model. For example, if all linking copulas are Gumbel copulas, the lower tail order of $C_{1,2}$ is always smaller than 2. According to Proposition 5, $C_{1,2}$ is an upper tail dependent copula. So all Gumbel or survival Gumbel linking copulas imply $C_{1,2}$ has tail asymmetry as in the 1-factor model.

This behavior of the tail order is not the same for the 2-factor model as with normal copulas. If C_{i,V_1} and $C_{i,V_2;V_1}$ are all normal copulas with positive correlation ρ, then $C_{1,2}$ is a normal copula with correlation $\rho_{12} = \rho^2 + (1-\rho^2)\rho^2$. Hence the tail order $2/(1+\rho_{12})$ is smaller than $2/(1+\rho)$ only if $(\sqrt{5}-1)/2 < \rho < 1$.

Also, $\tilde{\xi}_2(s_1, s_2) \ge \tilde{m}_1(s_2) + s_2 \ge 1$ and the equality is possible if and only if $A_1\left(\frac{s_2}{1+s_2}\right) = \frac{1}{1+s_2}, A_{1;1}\left(\frac{s_1}{s_1+\tilde{m}_1(s_2)}\right) = \frac{\tilde{m}_1(s_2)}{s_1+\tilde{m}_1(s_2)}, A_{2;1}\left(\frac{s_1}{s_1+\tilde{m}_2(s_2)}\right) = \frac{s_1}{s_1+m_2(s_2)}$. Since $A_{i;1}(t) \ge \max\{t, 1-t\}, A_i(t) \ge \max\{t, 1-t\}$, it implies $s_1 \le \tilde{m}_1(s_2) = 1-s_2 \le \tilde{m}_2(s_2) \le s_1$. Hence $\tilde{m}_1(s_2) = \tilde{m}_2(s_2) = s_1 = 1-s_2$ and $A_{i;1}(1/2) = 1/2$, that is linking copulas at the second level $C_{1,V_2;V_1}, C_{2,V_2;V_1}$ should be comonotonic copulas.

Tail quadrant independence

Many bivariate copulas $C(u, v)$ with (lower) tail order 2 satisfy the condition: $C_{U|V}(u|v) \le v_0 u$ if $u > 0$ is small enough, where $v_0 > 0$ is a constant. If the bivariate linking copulas C_{j,V_1} are positive quadrant dependent and satisfy this condition with positive constants v_{0j}, then $C_{1,2}$ has tail quadrant independence with $\kappa = 2$. For example, in the 1-factor model we have

$$C_{1,2}(u, u) = \int_0^1 C_{1|V_1}(u|v_1)\, C_{2|V_1}(u|v_1)\, dv_1 \le \int_0^1 v_{01} u v_{02} u dv_1 = v_{01} v_{02} u^2$$

and the tail order is not more than 2 with positive dependence for $C_{1,2}$. One can show tail quadrant independence of $C_{1,2}$ in a similar way. Hence, tail quadrant independence can be obtained by choosing such linking copulas with tail order 2. Copula families for which the above condition is satisfied include the independence, Frank and Plackett copulas.

In summary, the choice of linking copulas affects the types of dependence both in the lower tail and the upper tail of $C_{1,2}$, and more generally, the margin $C_{j,k}$ of (2) or (5). Asymmetric tail dependence can be achieved with reflection asymmetric linking copulas.

4. Computational details

In this section, we discuss some computational details of numerical integration and optimization for maximum likelihood.

We consider the estimation of parameters in the factor copula models based on the set of i.i.d. data vectors with margins transformed to $U(0, 1)$ random variables and the joint distribution given by the copula $C(u_1, \ldots, u_d; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector of dependence parameters. For parametric versions of (3) and (6), a parametric copula family is used for each linking copula, and $\boldsymbol{\theta}$ is the vector of all of the dependence parameters. Similar to Nikoloulopoulos et al. [22] for vine copulas, appropriate choices of the bivariate linking copulas can be made based on bivariate normal scores plots for the data and the properties in Section 3.

With parameters of univariate margins estimated first followed by conversion to $U(0, 1)$ data $\{(u_{i1}, \ldots, u_{id})\}_{i=1}^n$, the parameter vector $\boldsymbol{\theta}$ of the joint density $c(\cdot; \boldsymbol{\theta})$ can then be estimated using maximum likelihood. The likelihood can be written as

$$L(u_1, \ldots, u_d; \boldsymbol{\theta}) = \prod_{i=1}^n c(u_{i1}, \ldots, u_{id}; \boldsymbol{\theta}).$$

That is, the parameters of uniform margins are estimated at the first step and dependence parameters at the second step with parameters of the univariate margins fixed at the estimates obtained from the first step. The two-step estimation procedure significantly simplifies the computation process. The second stage optimization of the log-likelihood of the copula model is done via a quasi-Newton or modified Newton–Raphson numerical method and this outputs a Hessian matrix from which standard errors (conditional on the first step) are obtained. To obtain standard errors for the two-step method, appropriate resampling methods can be used.

In the p-factor copula model numerical integration is required to evaluate the p-dimensional integral. This is a minor issue for $p = 1$. Based on the formula (3), with parameter θ_j for copula C_{j,V_1}, and $\boldsymbol{\theta} = (\theta_j)_{1 \le j \le d}$,

$$c(u_{i1}, \ldots, u_{id}; \boldsymbol{\theta}) = \int_0^1 \prod_{j=1}^d c_{j,V_1}(u_{ij}, v_1; \theta_j) dv_1.$$

For many commonly used parametric copula families, the density $c_{j,V_1}(u_{ij}, v_1; \theta)$ approaches infinity as $(u_{ij}, v_1) \to (0, 0)$ or $(u_{ij}, v_1) \to (1, 1)$, therefore the integrand could be unbounded.

One can transform the variable of integration with $v_1 \mapsto \Phi(t_1)$, where Φ is the standard normal cdf and then use Gauss–Hermite quadrature. However we find that Gauss–Legendre quadrature [27] works fine; this approach approximates the integral as a weighted combination of integrands evaluated at quadrature points:

$$c(u_{i1}, \dots, u_{id}; \boldsymbol{\theta}) \approx \sum_{k=1}^{n_q} w_k \prod_{j=1}^{d} c_{j,V_1}(u_{ij}, x_k; \theta_j)$$

where $\{x_k\}$ are the nodes and $\{w_k\}$ are the quadrature weights, and n_q is the number of quadrature points. n_q between 21 and 25 tends to give a good approximation of these integrals. The nice property of Gauss–Legendre quadrature is that the same nodes $\{x_k\}$ and weights $\{w_k\}$ are used for different functions to compute the integral quickly and with a high precision. The same nodes also help in smooth numerical derivatives for numerical optimization.

The same methods can be used to compute two-dimensional integrals if $p = 2$. Approximation using Gauss–Legendre quadrature is now a double sum. Assuming the parameters are θ_{j1} for C_{j,V_1} and θ_{j2} for $C_{j,V_2|V_1}$, from (6) we get

$$c(u_{i1}, \dots, u_{id}; \boldsymbol{\theta}) \approx \sum_{k_1=1}^{n_q} \sum_{k_2=1}^{n_q} w_{k_1} w_{k_2} \prod_{j=1}^{d} \left\{ c_{j,V_2;V_1}(C_{j|V_1}(u_{ij}|x_{k_1}; \theta_{j1}), x_{k_2}; \theta_{j2}) \cdot c_{j,V_1}(u_{ij}, x_{k_1}; \theta_{j1}) \right\}.$$

Usually n_q between 15 and 21 per dimension is enough to compute the integrals with a good precision. One should try using different starting points in the algorithms to compute maximum likelihood estimates to ensure that a global optimum is obtained.

The fastest implementation that we have is through a modified Newton–Raphson algorithm, with log-likelihood functions in Fortran90 code linked to optimization in R. The analytical derivatives of the integrand function in (3) and (6) can be computed to get the formula for the first and second order derivatives of the log-likelihood function. The step size is controlled to ensure the value of the likelihood increases at each iteration.

Note that in the Gaussian 2-factor models (with bivariate normal linking copulas) and a multivariate normal distribution with correlation matrix having factor structure, the number of independent correlation parameters is $2d - 1$, because the loading matrix is non-unique and can be rotated so that one correlation for the second factor is 0. But, as a rough guideline, the multivariate normal model can be used to choose an appropriate number of factors p in the factor copula model. Different number of factors can be used to get likelihoods for different choices of p; the number of factors can be chosen so that there is not much improvement in terms of the likelihood or Akaike information criterion when adding an additional factor to the model.

However, the two-factor copula model, not based on all bivariate normal copulas, is identifiable if $2d$ bivariate linking copulas are used. To indicate the more general situation, we use the Morgenstern copula: $uv[1 + \theta(1 - u)(1 - v)]$, $-1 \le \theta \le 1$, because the integrals simplify in this case (we are not recommending that this copula be used with factor copulas in applications because it has limited dependence). If $C_{j,V_1}, C_{j,V_2;V_1}, j = 1, \dots, d$, are Morgenstern copulas with respective parameters $\theta_{1,j}, \theta_{2,j}$, then one can show that

$$c(u_{i1}, \dots, u_{id}; \boldsymbol{\theta}) = 1 + \frac{1}{3} \sum_{j_1<j_2} (\theta_{1,j_1}\theta_{1,j_2} + \theta_{2,j_1}\theta_{2,j_2})(1 - 2u_{j_1})(1 - 2u_{j_2}) + R(\boldsymbol{\theta}),$$

where $R(\boldsymbol{\theta})$ is a higher-order polynomial of $\theta_{1,1}, \dots, \theta_{1,d}, \theta_{2,1}, \dots, \theta_{2,d}$. The second order term depends on $\theta_{1,j_1}\theta_{1,j_2} + \theta_{2,j_1}\theta_{2,j_2}$ and as in the case of a normal copula the number of such independent parameters is $2d - 1$. However, with higher-order terms in $R(\boldsymbol{\theta})$ the total number of independent parameters in this model is $2d$.

In cases where the log-likelihood is quite flat (near non-identifiable), we can set one of the $C_{j,V_2;V_1}$ to be an independence copula (this can be done without loss of generality in the case of normal copulas).

5. Empirical results for simulated and financial data sets

The main aims of this section are to report on the accuracy of the MLEs in factor copula models for simulated data sets in Section 5.3 and apply the proposed models to financial data sets of asset returns in Section 5.4. To assess how well different copulas fit the data we use several measures of dependence for bivariate margins, some of which put more weight on one of the joint tails; see Section 5.2. For each margin, we compute the empirical estimate of a dependence measure and compare it to the model-based estimate obtained using MLEs for the linking copula parameters. Before these topics, we discuss choices of suitable bivariate linking copulas in Section 5.1.

5.1. Choice of bivariate linking copulas

The factor copula models are completely defined by the bivariate linking copulas. However, there are many options available for the choice of bivariate parametric copula families, therefore some preliminary analysis of data might be required. For each pair of variables (U_{j_1}, U_{j_2}), diagnostic methods using bivariate normal scores plots and tail-weighted measures of dependence can be applied to better assess the dependence structure of bivariate margins. If there is some evidence of dependence in the lower (upper) tail, the corresponding linking copulas with lower (upper) tail dependence should be used. The reflected Gumbel copula is a good choice at the first level (both for the factor and vine models) in the case of only lower tail dependence. In the case of tail dependence and approximate tail symmetry, the Student t copula might be a better option for linking copulas.

With a copula family chosen at the first level the same copula family can be used at the second level. One may also try different copula families and choose the best one based on Akaike information criterion (AIC) when the number of parameters differ for different choices.

5.2. Dependence measures

We use Spearman's correlation as a measure of bivariate dependence in the middle. To estimate dependence in the tails, small-sample (around $n = 220$ for one year of US stock data) empirical estimates of tail dependence coefficients can be estimated quite poorly. Instead, we use tail-weighted measures of dependence as proposed in [19]. Unlike tail dependence coefficients, the tail-weighted measures of dependence are defined as correlations and not as limiting values. Hence, these measures can estimate dependence in the tails efficiently even if the sample size is not large. With variables transformed to U_1, U_2 that are $U(0, 1)$ random variables, one good choice of tail-weighted dependence measures consists of $\alpha_L = \mathrm{Cor}((1 - 2U_1)^6, (1 - 2U_2)^6 | U_1 < 0.5, U_2 < 0.5)$ and $\alpha_U = \mathrm{Cor}((2U_1 - 1)^6, (2U_2 - 1)^6 | U_1 > 0.5, U_2 > 0.5)$. The empirical version involves sample correlations after transformation and truncation, and the model-based version involves numerical integration.

For a given copula and a bivariate margin (j_1, j_2), $1 \le j_1 < j_2 \le d$, we estimate Spearman's correlation $[\hat{S}_\rho^{\mathrm{emp}}]_{j_1,j_2}$, $[\hat{S}_\rho^{\mathrm{model}}]_{j_1,j_2}$, the lower and upper tail-weighted measure of dependence $[\alpha_L^{\mathrm{emp}}]_{j_1,j_2}$, $[\alpha_L^{\mathrm{model}}]_{j_1,j_2}$ and $[\alpha_U^{\mathrm{emp}}]_{j_1,j_2}$, $[\alpha_U^{\mathrm{model}}]_{j_1,j_2}$ (empirical and the model-based estimates). For M being one of $S_\rho, \alpha_L, \alpha_U$, we define:

$$[\hat{M}^{\mathrm{diff}}]_{\max} = \max_{j_1<j_2} \left| [\hat{M}^{\mathrm{emp}}]_{j_1,j_2} - [\hat{M}^{\mathrm{model}}]_{j_1,j_2} \right|;$$

$$[\hat{M}^{\mathrm{diff}}]_{\mathrm{mean}} = \frac{2}{d(d-1)} \sum_{j_1<j_2} \left| [\hat{M}^{\mathrm{emp}}]_{j_1,j_2} - [\hat{M}^{\mathrm{model}}]_{j_1,j_2} \right|.$$

These are averages and maxima over $d(d - 1)/2$ bivariate margins. We compute these quantities for different choices of bivariate linking copulas both in 1- and 2-factor copula models.

5.3. Simulation results

In this subsection, the accuracy of the maximum likelihood estimates (MLEs) is given in detail for one simulated data set, but the pattern is similar for other choices of copulas and parameters. We use a 2-factor model with Gumbel linking copulas at both levels with $d = 30$ and sample size $n = 500$. The resulting multivariate copula has upper tail dependence and lower intermediate tail dependence. The parameters for the linking copulas for the first and second factors, θ_1 and θ_2 respectively, have been chosen as follows:

$$10\theta_1 = (20, 22, 24, 26, 28, 30, 30, 30, 30, 30 : 40, 40, 39, 38, 37, 36, 35, 35, 35, 35, 35)$$
$$10\theta_2 = (15 : 20, 20, 22, 24, 26, 28, 30[7], 28, 26, 24, 22, 20, 18, 16, 15, 15, 15, 15, 15)$$

where in the above, colons indicate consecutive sequences of integers and 30[7] means eight 30s in a row.

To estimate the simulated data we use 1- and 2-factor copula models with normal, Frank, Gumbel, and reflected Gumbel linking copulas. This is for comparison with misspecification, so we did not completely follow the guidelines in the preceding subsections. The values of $[\hat{S}_\rho^{\mathrm{diff}}]_{\mathrm{mean}}$, $[\hat{S}_\rho^{\mathrm{diff}}]_{\max}$, $[\hat{\alpha}_L^{\mathrm{diff}}]_{\mathrm{mean}}$, $[\hat{\alpha}_L^{\mathrm{diff}}]_{\max}$, $[\hat{\alpha}_U^{\mathrm{diff}}]_{\mathrm{mean}}$, $[\hat{\alpha}_U^{\mathrm{diff}}]_{\max}$ are summarized in Table 1.

It is seen the model with the reflected Gumbel copula is the worst one, both in terms of likelihood and the accuracy of tail-weighted dependence estimates. The model heavily overestimates dependence in the lower tail and underestimates it in the upper tail. The models with the Frank and normal copulas underestimate dependence in the upper tail and normal copulas overestimate dependence in the lower tail. Note that even the 1-factor model with the Gumbel copula provides quite accurate estimates of tail dependence because it correctly matches the asymmetric dependence structure of the simulated data set.

We compare the vectors of estimated dependence parameters, $\hat{\theta}_1$ and $\hat{\theta}_2$, in the 2-factor Gumbel copula model with the vectors of true values, θ_1 and θ_2. The estimated values are very close to the true values with the mean absolute difference about 0.21 resulting in good estimates of tail-weighted dependence. The running time was about 22 min on an Intel core i5-2410M CPU at 2.3 GHz and convergence was achieved in 15 iterations. The time can vary depending on the choice of linking copula. For the reflected Gumbel copula in the 2-factor model the convergence was achieved in 24 iterations whereas 10 iterations were used for the Frank copula. Note that the convergence is very fast in 1-factor models and it usually takes 10–20 s. The value of the final likelihood can be slightly different for different starting points but it does not affect much the dependence characteristics of the estimated distribution.

Table 1

$[\hat{S}_{\rho}^{\text{diff}}]_{\max}$, $[\hat{S}_{\rho}^{\text{diff}}]_{\text{mean}}$, $[\hat{\alpha}_{L}^{\text{diff}}]_{\max}$, $[\hat{\alpha}_{L}^{\text{diff}}]_{\text{mean}}$, $[\hat{\alpha}_{U}^{\text{diff}}]_{\max}$, $[\hat{\alpha}_{U}^{\text{diff}}]_{\text{mean}}$ (averages and maxima over $\binom{d}{2}$ bivariate margins) and the maximum log-likelihood value for different copulas in 1- and 2-factor models; simulated data set with sample size $N = 500$ and dimension $d = 30$.

Copula	$[\hat{S}_{\rho}^{\text{diff}}]_{\text{mean}}$	$[\hat{S}_{\rho}^{\text{diff}}]_{\max}$	$[\hat{\alpha}_{L}^{\text{diff}}]_{\text{mean}}$	$[\hat{\alpha}_{L}^{\text{diff}}]_{\max}$	$[\hat{\alpha}_{U}^{\text{diff}}]_{\text{mean}}$	$[\hat{\alpha}_{U}^{\text{diff}}]_{\max}$	Loglik
1-factor copula model							
Normal	0.01	0.04	0.13	0.32	0.09	0.24	15 318
Frank	0.01	0.06	0.13	0.29	0.34	0.51	14 701
Gumbel	0.01	0.04	0.04	0.17	0.06	0.15	16 258
rGumbel	0.04	0.10	0.30	0.48	0.24	0.50	13 539
2-factor copula model							
Normal	0.01	0.04	0.13	0.32	0.09	0.25	15 417
Frank	0.01	0.06	0.05	0.22	0.19	0.36	15 800
Gumbel	0.01	0.04	0.04	0.16	0.06	0.15	16 818
rGumbel	0.03	0.11	0.13	0.31	0.31	0.62	14 728

Other simulated data sets were used to assess the accuracy of the MLEs. With larger sample size, such as $n = 1000$, usually fewer iterations are required for convergence and the running time does not increase significantly. At the same time the accuracy of the estimates increases.

5.4. Financial return data

We have fitted the 1-factor and 2-factor copula models (as well as various vine copula models) to different financial data sets including returns over several stocks in the same sector and European index returns. The results were similar (except factor copula models have bigger improvements over truncated vine models for stock returns than for market index returns), so we provide details for only one data set in this subsection. Vine copulas have quite flexible dependence and are good approximations to many multivariate copulas, so they are a baseline to compare our factor copula models. In particular, we consider two special cases of truncated vine copula models, C-vines and D-vines. A reference for C-vines and D-vines is Aas et al. [2], and a reference for truncated vines is Brechmann et al. [5].

The example shown here is GARCH-filtered financial return data converted to uniform scores. We consider 8 US stocks in the IT sector: AAPL, ADBE, CSCO, DELL, INTC, MOT, MSFT, and NOVL in the year 2001. The AR(1)-GARCH(1, 1) model with symmetric Student t innovations was used to fit univariate margins in each set.

Consider d financial assets and let the price of the j-th asset at time t be P_{jt}. The copula-GARCH model in [17] and many other subsequent papers is the following. Let

$$r_{jt} = \mu_j + \rho_j r_{j,t-1} + \sigma_{jt}\epsilon_{jt}, \qquad \sigma_{jt}^2 = \omega_j + \alpha_j r_{j,t-1}^2 + \beta_j \sigma_{j,t-1}^2, \quad j = 1, \ldots, d,\ t = 1, \ldots, T,$$

where $r_{jt} = \log(P_{jt}/P_{j,t-1})$ and $\epsilon_{\mathbf{t}} = (\epsilon_{1t}, \ldots, \epsilon_{dt})$ are i.i.d. vectors with distribution

$$F(z_1, \ldots, z_d) = C(F_{\nu_1}(z_1), \ldots, F_{\nu_d}(z_d); \boldsymbol{\theta}).$$

F_{ν} denotes the cdf of Student t distribution, with ν degrees of freedom and standardized to have variance 1, used to model innovations in the GARCH model.

Parameters $\mu_j, \rho_j, \omega_j, \alpha_j, \beta_j, \nu_j, j = 1, \ldots, d$ were estimated at the first step. Then different models (truncated vines and factor copula models) for the d-variate copula $C(u_1, \ldots, u_d; \boldsymbol{\theta})$ were applied to the GARCH filtered data transformed to uniform scores to get estimates for the vector of dependence parameters θ. The maximum likelihood at the second step was computed for 8 different copula models (1- and 2-factor copula models, D-vine and C-vine copula models truncated after 1, 2 and 3 levels), with various choices for the parametric bivariate linking copulas.

In each model, we use 5 different linking copula families: normal, Gumbel, reflected Gumbel, Frank, and BB1 copula (see [13] for definitions) to cover different types of dependence structure. The Gumbel (reflected Gumbel) copula is an example of a copula with asymmetric dependence and upper (lower) tail dependence respectively. The Frank copula is reflection symmetric and has tail order of 2. The dependence in the tails is weaker than the normal copula in the case of positive dependence. Finally, the BB1 copula is reflection asymmetric with lower and upper tail dependence.

Based on preliminary analysis using normal scores plots of the return data, there was not enough tail asymmetry to consider copula families with tail order of 1 in one joint tail and 2 in the other joint tail. For each copula family choice at the first level, the same family was used at the second level and the Frank family was applied at the third level for vine models truncated after 3 levels. In the 2-factor model with the BB1 copula family for the first factor, the Frank family was used for the second factor as estimated parameters pointed to weaker dependence at the second factor in this case, and tail dependence for the first factor is sufficient to get tail dependence for the 2-factor copula. The Frank copula has negative as well as positive dependence, so it is useful at the second factor or third vine level to allow for negative or positive conditional dependence. For factor and vine copula models, all bivariate margins have upper (lower) tail dependence if the linking copulas at factor or level 1 have upper (lower) tail dependence.

Table 2
$[\hat{S}_\rho^{\text{diff}}]_{\max}$, $[\hat{S}_\rho^{\text{diff}}]_{\text{mean}}$, $[\hat{\alpha}_L^{\text{diff}}]_{\max}$, $[\hat{\alpha}_L^{\text{diff}}]_{\text{mean}}$, $[\hat{\alpha}_U^{\text{diff}}]_{\max}$, $[\hat{\alpha}_U^{\text{diff}}]_{\text{mean}}$ (averages or maxima over $\binom{8}{2} = 28$ bivariate margins) and the maximum log-likelihood value for different copulas in 1- and 2-factor models and for MVt distribution; US stock data, year 2001.

Copula	$[\hat{S}_\rho^{\text{diff}}]_{\text{mean}}$	$[\hat{S}_\rho^{\text{diff}}]_{\max}$	$[\hat{\alpha}_L^{\text{diff}}]_{\text{mean}}$	$[\hat{\alpha}_L^{\text{diff}}]_{\max}$	$[\hat{\alpha}_U^{\text{diff}}]_{\text{mean}}$	$[\hat{\alpha}_U^{\text{diff}}]_{\max}$	Loglik
1-factor copula model							
Normal	0.02	0.09	0.10	0.33	0.11	0.30	516.2
Frank	0.04	0.08	0.12	0.44	0.26	0.44	504.8
Gumbel	0.03	0.10	0.11	0.42	0.11	0.27	520.1
rGumbel	0.03	0.13	0.24	0.42	0.27	0.57	488.3
BB1	0.02	0.09	0.10	0.38	0.06	0.18	547.7
MVt(10)	0.02	0.08	0.15	0.40	0.07	0.21	543.9
2-factor copula model							
Normal	0.02	0.07	0.09	0.31	0.11	0.30	525.9
Frank	0.03	0.09	0.09	0.35	0.19	0.37	537.1
Gumbel	0.02	0.07	0.11	0.41	0.11	0.29	534.4
rGumbel	0.02	0.09	0.23	0.43	0.25	0.57	506.4
BB1+Frank	0.03	0.08	0.08	0.35	0.06	0.21	565.1
MVt(9)	0.02	0.07	0.14	0.38	0.07	0.22	552.5

Since the order of indexing the assets is important for vine models, the variables were rearranged. The order for US log-return variables is INTC, CSCO, NOVL, MOT, AAPL, MSFT, DELL, and ADBE both for the D-vine and C-vine. The choice is based on the estimated correlation matrix of GARCH filtered data transformed to the uniform scores, when looking at the correlation matrices of the transformed data. The variables were rearranged to make the correlation matrix of the permuted data closer to the corresponding truncated vine structure, which means that we include strongest pairs of variables for the copulas applied to bivariate margins. For example, for the C-vine order, the first column of the correlation matrix should mostly contain higher correlations than other columns.

Preliminary analysis based on normal plots of each pair of variables and tail-weighted measures of dependence shows that for the US stock data, dependence is stronger in the upper tail. Based on this analysis one can suggest linking copulas with asymmetric tail dependence, such as the BB1 copula, to fit these data. We employ different copula families to see if the log-likelihood based approach confirms these suggestions.

We compute $[\hat{S}_\rho^{\text{diff}}]_{\text{mean}}$, $[\hat{S}_\rho^{\text{diff}}]_{\max}$, $[\hat{\alpha}_L^{\text{diff}}]_{\text{mean}}$, $[\hat{\alpha}_L^{\text{diff}}]_{\max}$, $[\hat{\alpha}_U^{\text{diff}}]_{\text{mean}}$, $[\hat{\alpha}_U^{\text{diff}}]_{\max}$ both in 1- and 2-factor copula models. The results are presented in Table 2. In addition, we use a multivariate Student factor model with 1 and 2 common factors for comparison. This model is popular in finance where it is used to model the joint distribution of financial assets. For example [18] study a model with multivariate t, where the correlation matrix has factor correlation structure. Note that the multivariate Student factor model is not the same as the factor copula model with Student bivariate linking copulas as the joint distribution given by (2) or (5) is no longer a Student distribution. In fact, the Student copula model and multivariate Student factor model produce similar results but the latter is preferable as it has fewer parameters. For this reason, we do not report results on the factor copula model with t bivariate linking copulas.

The factor copula model with reflected Gumbel is the worst one as indicated by the very large difference between the model-based and empirical estimates. Comparing $\hat{\alpha}_L^{\text{emp}}$ and $\hat{\alpha}_U^{\text{emp}}$ with $\hat{\alpha}_L^{\text{model}}$ and $\hat{\alpha}_U^{\text{model}}$ we conclude that the model heavily overestimates dependence in the lower tail and underestimates dependence in the upper tail. This confirms the fact that IT stock data has stronger dependence in the upper tail and the Gumbel copula is more suitable. The Frank copula underestimates dependence in both tails. The model with BB1 copula provides even better estimates of tail dependence as this copula allows for more flexible dependence structure while the Spearman correlation is adequately assessed by all models. At the same time, the multivariate Student distribution tends to overestimate dependence in the lower tail.

In 1-factor models, dependence parameter estimates are always consistent for different copulas, in terms of the Spearman rho values linking the latent variable to each observed variable. For the data, the weakest dependence can be found in the copulas C_{2,V_1}, C_{7,V_1}, C_{8,V_1} linking the latent variable and the second, seventh and eighth indices (CSCO, DELL, ADBE) respectively. The strongest dependence can be found in the copula C_{5,V_1} linking the latent variable and the fifth index (AAPL). This might indicate that AAPL reflects most of the changes in the US IT stock market. However, in 2-factor models, parameter estimates are not always consistent in relative strength. Dependence is stronger in the first factor for the choices of MVN and MVt factor models and it is weaker for Frank, Gumbel, BB1 and reflected Gumbel copulas; see Table 3. By choosing different starting values stronger dependence in the first factor can be obtained for these copulas, however the likelihood will be smaller in this case pointing to local maximum. So depending on the choice of bivariate linking copulas, sometimes the second factor has more dependence than the first factor. For the normal factor model, note that the parameters are partial correlations for the second factor, as mentioned in Section 2. For the fitted BB1 copulas, the parameters are such that there is tail asymmetry based on λ_L, λ_U.

The log-likelihoods for different models including truncated vines are shown in Table 4. Based on likelihood or AIC, the 2-factor copula model is the best one in each row of Table 4 and the BB1 copula is best for most columns. In general, tail

Table 3
MLEs for parameters of linking copula families in 2-factor models, US IT stock data, year 2001; variables 1 = INTC, 2 = CSCO, 3 = NOVL, 4 = MOT, 5 = AAPL, 6 = MSFT, 7 = DELL, 8 = ADBE. First partial correlation is set to 0 for the normal and t factor model because it does not have $2d$ identifiable parameters.

Family	C_{1,V_1}	C_{2,V_1}	C_{3,V_1}	C_{4,V_1}	C_{5,V_1}	C_{6,V_1}	C_{7,V_1}	C_{8,V_1}
Normal	0.75	0.62	0.78	0.83	0.88	0.79	0.63	0.65
MVt(9)	0.75	0.65	0.78	0.82	0.87	0.79	0.62	0.68
Frank	4.8	2.1	4.3	5.2	7.1	3.3	3.1	2.8
Gumbel	1.3	1.1	1.1	1.3	1.3	1.3	1.1	1.2
rGumbel	1.0	1.6	1.1	1.1	1.0	1.1	1.1	1.0
BB1	0.0, 1.8	0.0, 1.4	0.4, 1.4	0.2, 2.0	1.2, 1.5	0.2, 1.6	0.0, 1.5	0.6, 1.2
Copula	$C_{1,V_2;V_1}$	$C_{2,V_2;V_1}$	$C_{3,V_2;V_1}$	$C_{4,V_2;V_1}$	$C_{5,V_2;V_1}$	$C_{6,V_2;V_1}$	$C_{7,V_2;V_1}$	$C_{8,V_2;V_1}$
Normal	0.00	0.22	1.00	−0.01	0.11	−0.13	0.16	−0.01
MVt(9)	0.00	0.21	1.00	−0.04	0.09	−0.14	0.16	0.01
Frank	4.0	5.6	5.9	5.3	7.4	6.1	3.6	4.3
Gumbel	2.0	2.0	2.7	2.4	2.9	2.1	1.8	1.7
rGumbel	2.0	2.2	2.4	2.4	3.4	2.1	1.7	1.9
Frank/BB1	3.7	5.1	6.4	4.4	6.7	4.5	3.6	3.7

Table 4
Log-likelihoods for different models, US data, year 2001; number of dependence parameters for each model is shown in parentheses.

Copula	1-factor	2-factor	1 D-vine	2 D-vine	3 D-vine	1 C-vine	2 C-vine	3 C-vine
Frank	505(8)	537(16)	310(7)	422(13)	468(18)	433(7)	482(13)	499(18)
Gumbel	520(8)	534(16)	323(7)	418(13)	468(18)	439(7)	487(13)	503(18)
rGumbel	488(8)	506(16)	296(7)	388(13)	440(18)	418(7)	454(13)	470(18)
Normal	516(8)	526(15)	316(7)	425(13)	470(18)	446(7)	490(13)	502(18)
t	544(9)	553(16)	339(14)	453(26)	497(31)	464(14)	519(26)	531(31)
BB1	548(16)	565(24)	339(14)	451(26)	498(31)	468(14)	515(26)	530(31)

Table 5
Log-likelihoods for R-vine and factor models: US data, year 2001.

Data	1-factor MVt	1-factor BB1	2-factor MVt	2-factor BB1	1 R-vine	2 R-vine	3 R-vine
US	544	548	553	565	478	537	550

dependent copulas perform better in all cases providing additional evidence of tail dependence for these financial data. The log-likelihoods are quite a bit larger for the factor copula models compared with the truncated vine models.

Smaller values of the log-likelihoods for the truncated vine models might be due to special vine structure (namely, D-vine and C-vine) used to fit the stock return data. To make a better comparison with factor copulas, we fit more general R-vines using the R package by Schepsmeier et al. [25], and the six choices for each linking copula as before. The details of the algorithm for fitting "optimal" R-vines are given in [6]. The likelihoods for fitted R-vine models truncated after the first, second and third level are given in Table 5. The likelihoods for the 1- and 2-factor models with BB1 copulas as well as multivariate Student 1- and 2-factor models are also included for comparison. The 2-factor model with the BB1 copula is better, based on log-likelihood and AIC, than the R-vine truncated after the third level. Note that the 1-factor and 2-factor copula models can achieve even larger log-likelihoods and smaller AICs if we allow for different copula families for different links of observed to latent variables. However for simplicity of interpretation, we have kept the copula family to be the same for each factor level. Thus, the factor copula model with latent variables might be more interpretable than a truncated R-vine where the best-fitted linking copulas and vine structure can change with different years.

6. Concluding remarks

Factor copula models provide a wide range of dependence with $O(d)$ number of dependence parameters and allow for different types of tail behavior. They can be preferred over the classical factor model based on multivariate normality when there is tail dependence or tail asymmetry, and inferences concerning joint tails are important. We have derived factor copula models as conditional independence models and shown that the classical multivariate normal factor models are special cases. Some properties of our version of factor copulas can be obtained from the fact that the truncated C-vines rooted at the latent variables. The link to vines is useful for other models, with and without latent variables.

For factor models that account for sector information or other measured factors, Heinen and Valdesogo [10] and Brechmann and Czado [4] have copula models that use vines. Factor copula models can be further extended to the case of dependent latent variables, with each observed variable linked only to a subset of the latent variables, and these are truncated regular vines. The parametric version of these "structured" factor models might have fewer parameters than the general model in (1), so that log-likelihood might be less flat. These alternative factor models can be used to model dependence in variables that can be grouped into subsets. One example consists of financial data with stocks from two or more common sectors. In each sector, dependence can be modeled via one or more common latent variables and the latent variables for different groups of data can have a special dependence structure.

Using a modified Newton–Raphson algorithm, numerical maximum log-likelihood can be implemented for up to 3 factors and over 50 variables, with reasonable computational time. Our proposed factor copula models are closed under marginalization, whereas truncated vine copula models are not, and this property helps in getting starting values for numerical optimization for large d. These comments also apply to the "structured" factor models mentioned above. The numerical techniques of Gauss–Legendre quadrature and modified Newton–Raphson should be simpler to implement than the simulation-based estimation method of Oh and Patton [23].

In addition to their theoretical plausibility, the examples given or mentioned in this paper show the factor copula models to be good fits to some financial return data sets. Future research includes doing validations via Value-at-Risk measures, such as in [1,23], to assess what the improvement in fits based on AIC means for quantities relevant to financial analysts.

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Appendix

A.1. Proof of Proposition 5, Section 3.2

The (1, 2) margin of (5) is:

$$\begin{aligned}C_{1,2}(u,u) &= \int_0^1\int_0^1 C_{1|V_2;V_1}(C_{1|V_1}(u|v_1)|v_2)\cdot C_{2|V_2;V_1}(C_{2|V_1}(u|v_1)|v_2)\,dv_1dv_2\\ &= u\int_0^1\int_0^{1/u} C_{1|V_2;V_1}(C_{1|V_1}(u|hu)|v_2)\cdot C_{2|V_2;V_1}(C_{2|V_1}(u|hu)|v_2)\,dh\,dv_2.\end{aligned}$$

Let $t_j(h) = \lim_{u\to 0} C_{j|V_1}(u|hu)$. This is continuous for small $h > 0$ from the continuity assumptions on $C_{j|V_1}$. Since $\lim_{h\to 0} t_j(h) = t_{j0}$, for every $\epsilon > 0$ we can find $h = h(\epsilon) > 0$ such that $|t_{j0} - t_j(h)| < \epsilon$ if $h \le h(\epsilon), j = 1, 2$. For a fixed $\epsilon > 0$ there exists $u(\epsilon) > 0$ such that $|C_{j|V_1}(u|h(\epsilon)u) - t_j(h(\epsilon))| < \epsilon$ for $0 < u \le u(\epsilon)$. It implies $|t_{j0} - C_{j|V_1}(u|h(\epsilon)u)| < 2\epsilon$ for $u \le u(\epsilon)$. Due to uniform continuity of $C_{j|V_1}$ on $[0, 1] \times [0, 1]$ we can find $u_0 > 0$, h_1, h_2 such that $h_1 < h(\epsilon) < h_2$ and $|C_{j|V_1}(u|h(\epsilon)u) - C_{j|V_1}(u|hu)| < \epsilon$ for $0 < u \le u_0$, $h_1 \le h \le h_2$. Therefore $|t_{j0} - C_{j|V_1}(u|hu)| < 3\epsilon$ for $0 < u \le \min\{u_0, u(\epsilon)\}$, $h_1 < h \le h(\epsilon)$.

For the copulas at the second level we have $C_{j|V_2;V_1}(t_{j0}|0) \ge k_0 > 0$ and due to uniform continuity of $C_{j|V_2;V_1}, j = 1, 2$, for a fixed $\epsilon_0 \in (0, k_0)$ we get $C_{j|V_2;V_1}(z|v_2) \ge (k_0 - \epsilon_0) > 0$ for some $z(\epsilon_0) \le z < \min\{t_{10}, t_{20}\}$ and $0 < v_2 \le v_2(\epsilon)$. Let $\epsilon = (k_0 - \epsilon_0)/3$ and $u < u(\epsilon)$. It follows that if $0 < u \le \min\{u_0, u(\epsilon)\}$, then

$$\begin{aligned}C_{1,2}(u,u) &\ge u\int_0^{v_2(\epsilon)}\int_{h_1}^{h(\epsilon)} C_{1|V_2;V_1}(C_{1|V_1}(u|hu)|v_2)\cdot C_{2|V_2;V_1}(C_{2|V_1}(u|hu)|v_2)\,dh\,dv_2\\ &\ge uv_2(\epsilon)(h(\epsilon)-h_1)(k_0-\epsilon_0)^2.\end{aligned}$$

Therefore $\lim_{u\to 0} C_{1,2}(u, u)/u \ge v_2(\epsilon)(h(\epsilon) - h_1)(k_0 - \epsilon_0)^2 > 0$ and $C_{1,2}$ has lower tail dependence. $\square$

A.2. Proof of Proposition 6, Section 3.2

The conditional distributions are:

$$C_{i|V_1}(u_1|u_2) = \frac{C_{i,V_1}(u_1,u_2)}{u_2}\left[A_i\left(\frac{\ln u_2}{\ln(u_1u_2)}\right) + \frac{\ln u_1}{\ln(u_1u_2)}A_i'\left(\frac{\ln u_2}{\ln(u_1u_2)}\right)\right],$$

and therefore

$$C_{i|V_1}(u|u^s) = u^{(s+1)A_i\left(\frac{s}{s+1}\right)-s}\left[A_i\left(\frac{s}{s+1}\right) + \frac{1}{s+1}A_i'\left(\frac{s}{s+1}\right)\right],\quad 0 < s < \infty. \tag{15}$$

Let $\tilde u = -\ln u$, $B_i(s) = A_i\left(\frac{s}{s+1}\right) + \frac{1}{s+1}A_i'\left(\frac{s}{s+1}\right)$ for $i = 1, 2$ and $B(s) = B_1(s)B_2(s)$. In the 1-factor model, for the bivariate copula $C_{1,2}$ we have

$$C_{1,2}(u,u) = \int_0^1 C_{1|V_1}(u|v)C_{2|V_1}(u|v)dv = \tilde u\int_0^\infty C_{1|V_1}(u|u^s)C_{2|V_1}(u|u^s)u^s ds = \tilde u\int_0^\infty u^{\xi(s)}B(s)ds, \tag{16}$$

where

$$\xi(s) = (s+1)\left[A_1\left(\frac{s}{s+1}\right) + A_2\left(\frac{s}{s+1}\right)\right] - s.$$

Because $A_i(s/[s+1]) \geq s/[s+1]$, then $\xi(s) \geq s$; also $\xi(0) = 2$. Hence the minimum $\xi^* \leq 2$ exists and $C_{1,2}(u,u) = \tilde{u}u^{\xi^*}\int_0^\infty u^{\xi(s)-\xi^*}B(s)ds$. Since $A_i' \leq 1$ and $A_i(s/[s+1]) \geq 1/(s+1)$,

$$B_i(s) = A_i\left(\frac{s}{s+1}\right) + \frac{1}{s+1}A_i'\left(\frac{s}{s+1}\right) \geq \frac{1}{s+1}\left[1 + A_i'\left(\frac{s}{s+1}\right)\right] \geq 0$$

and since A_i is convex,

$$B_i(s) = A_i\left(\frac{s}{s+1}\right) + \left(1 - \frac{s}{s+1}\right)A_i'\left(\frac{s}{s+1}\right) \leq A_i(1) = 1. \tag{17}$$

Therefore, $0 \leq B(s) \leq 1$. The equality $B(s) = 0$ is only possible if $s = 0$ or $s = \infty$, because $A_i'(t) > -1$ for $0 < t < 0.5$ by assumption and $A_i'(t) > -1$ for $0.5 \leq t < 1$ by the convexity and lower bound on A_i'.

The lower tail order of $C_{1,2}$ cannot be larger than 2 because by Garralda and Guillem [8] bivariate extreme value copulas are SI and then by Proposition 1, $C_{1,2}$ is PQD and satisfies $C_{1,2}(u,u) \geq u^2$. If $\xi^* = 2$, $C_{1,2}(u,u) \leq u^2\tilde{u}\{\int_0^2 B(s)ds + \int_2^\infty u^{s-2}ds\} = u^2\{\tilde{u}\int_0^2 B(s)ds + 1\}$. Hence the lower tail order of $C_{1,2}$ equals 2. Next assume $\xi^* < 2$. Let $s^* = \operatorname{argmin}\{\xi(s)\}$. Then $0 < s^* < 2$. Due to continuity and differentiability of $\xi(s)$ there exist an interval (s_l^*, s_u^*) containing s^* and a constant $d_0 > 0$ such that $\xi(s) \leq \xi^* + d_0|s - s^*|$ if $0 < s_l^* < s < s_u^* \leq 2$. Due to continuity of $B(s)$, there exists a constant $B_0 > 0$ such that $B(s) \geq B_0$ if $s_l^* < s < s_u^*$. Hence, in bounding (16),

$$C_{1,2}(u,u) \leq u^{\xi^*}\tilde{u}\left(\int_0^{\xi^*} B(s)ds + \int_{\xi^*}^\infty u^{s-\xi^*}ds\right) = u^{\xi^*}\left(\tilde{u}\int_0^{\xi^*} B(s)ds + 1\right),$$

where the rightmost term with $\tilde{u}$ is slowly varying in u as $u \to 0$, and

$$C_{1,2}(u,u) \geq B_0 u^{\xi^*}\tilde{u}\int_{s_l^*}^{s_u^*} u^{d_0|s-s^*|}ds = \frac{B_0 u^{\xi^*}}{d_0}\left(2 - u^{d_0(s^*-s_l^*)} - u^{d_0(s_u^*-s^*)}\right).$$

Therefore, the lower tail order of $C_{1,2}$ equals ξ^*. □

A.3. Proof of Proposition 7, Section 3.2

For $0 < s_1 < \infty$, $0 < s < \infty$ we have:

$$C_{i|V_2;V_1}(u^s|u^{s_1}) = u^{(s_1+s)A_{i;1}\left(\frac{s_1}{s_1+s}\right)-s_1}\left[A_{i;1}\left(\frac{s_1}{s_1+s}\right) + \frac{s}{s_1+s}A_{i;1}'\left(\frac{s_1}{s_1+s}\right)\right].$$

For $i = 1, 2$, let $B_{i;1}(s_1,s) = A_{i;1}\left(\frac{s_1}{s_1+s}\right) + \frac{s}{s_1+s}A_{i;1}'\left(\frac{s_1}{s_1+s}\right)$, $g_i(s_1,t) = (s_1+t)A_{i;1}\left(\frac{s_1}{s_1+t}\right)$ and $m_i(s_2) = (s_2+1)A_i\left(\frac{s_2}{s_2+1}\right) - s_2 + \log_u(B_i(s_2))$, and let B_i be as defined in the proof of Proposition 6. From (15), $C_{i|V_1}(u|u^{s_2}) = u^{m_i(s_2)}$ for $0 < s_2 < \infty$, and therefore,

$$C_{i|V_2;V_1}(C_{i|V_1}(u|u^{s_2})|u^{s_1}) = u^{g_i(s_1,m_i(s_2))-s_1}B_{i;1}(s_1, m_i(s_2)).$$

Let $\tilde{u} = -\ln u$. For the marginal copula $C_{1,2}$ of the 2-factor model, we get

$$\begin{aligned}
C_{1,2}(u,u) &= \int_0^1\int_0^1 C_{1|V_2;V_1}(C_{1|V_1}(u|v_1)|v_2)\,C_{2|V_2;V_1}(C_{2|V_1}(u|v_1)|v_2)dv_1dv_2 \\
&= \tilde{u}^2\int_0^\infty\int_0^\infty C_{1|V_2;V_1}(C_{1|V_1}(u_1|u^{s_1})|u^{s_2})\,C_{2|V_2;V_1}(C_{2|V_1}(u|u^{s_1})|u^{s_2})u^{s_1+s_2}ds_1ds_2 \\
&= \tilde{u}^2\int_0^\infty\int_0^\infty u^{\xi_2(s_1,s_2)}B_{1;1}(s_1,m_1(s_2))\,B_{2;1}(s_1,m_2(s_2))ds_1ds_2,
\end{aligned}$$

where $\xi_2(s_1,s_2) = g_1(s_1,m_1(s_2)) + g_2(s_1,m_2(s_2)) - s_1 + s_2$.

Now let $\tilde{m}_i(s_2) = m_i(s_2) - \log_u(B_i(s_2))$ and then $\tilde{\xi}_2(s_1,s_2) = g_1(s_1,\tilde{m}_1(s_2)) + g_2(s_1,\tilde{m}_2(s_2)) - s_1 + s_2$. For a fixed s_1, the function $g_i(s_1,t)$ is an increasing function of t:

$$\partial g_i(s_1,t)/\partial t = A_{i;1}\left(\frac{s_1}{s_1+t}\right) - \frac{s_1}{s_1+t}A_{i;1}'\left(\frac{s_1}{s_1+t}\right) \geq \frac{s_1}{s_1+t}\left(1 - A_{i;1}'\left(\frac{s_1}{s_1+t}\right)\right) \geq 0.$$

Since $\log_u(B_i(s_2)) = \ln(B_i(s_2))/\ln u, 0 < u < 1$, and $0 < B_i(s_2) \le 1$ from (17), we have $\log_u(B_i(s_2)) \ge 0$. It implies $g_i(s_1, \tilde{m}_i(s_2)) \le g_i(s_1, m_i(s_2))$ and $\tilde{\xi}_2(s_1, s_2) \le \xi_2(s_1, s_2)$. Note that $\xi_2(s_1, s_2) \ge \tilde{\xi}_2(s_1, s_2) \ge s_1 + s_2$ and $B_{i;1}(s_1, m_i(s_2)) \le 1$. It follows that

$$C_{1,2}(u, u) \le u_1^{\xi_2^*} \tilde{u}^2 \int_0^\infty \int_0^\infty u^{\tilde{\xi}_2(s_1,s_2)-\xi_2^*} ds_1 ds_2$$

$$\le u^{\xi_2^*} \tilde{u}^2 \left(\iint_{\substack{s_1 \ge 0, s_2 \ge 0: \\ s_1+s_2 \le \xi_2^*}} ds_1 ds_2 + \iint_{\substack{s_1 \ge 0, s_2 \ge 0: \\ s_1+s_2 > \xi_2^*}} u^{s_1+s_2-\xi_2^*} ds_1 ds_2 \right) = u^{\xi_2^*} \left(\frac{1}{2}(\xi_2^*)^2 \tilde{u}^2 + \xi_2^* \tilde{u} + 1 \right),$$

where the rightmost term involving $\tilde{u}$ is slowly varying in u as $u \to 0$.

Also, due to convexity of $A_{i;1}$, we have

$$g_i(s_1, m_i(s_2)) \le (s_1 + m_i(s_2)) \left[\frac{s_1 + \tilde{m}_i(s_2)}{s_1 + m_i(s_2)} A_{i;1} \left(\frac{s_1}{s_1 + \tilde{m}_i(s_2)} \right) + \frac{m_i(s_2) - \tilde{m}_i(s_2)}{s_1 + m_i(s_2)} A_{i;1}(0) \right]$$

$$= g_i(s_1, \tilde{m}_i(s_2)) + (m_i(s_2) - \tilde{m}_i(s_2)) = g_i(s_1, \tilde{m}_i(s_2)) + \log_u(B_i(s_2)).$$

It implies $\tilde{\xi}_2(s_1, s_2) \ge \xi_2(s_1, s_2) - \log_u(B_1(s_2)B_2(s_2))$.

Let $(s_1^*, s_2^*) = \operatorname{argmin} \tilde{\xi}_2^*(s_1, s_2)$. Consider the case when $s_1^* > 0, s_2^* > 0$. Due to continuity and differentiability of $\tilde{\xi}_2(s_1, s_2)$, there exist $s_{l,i}^* < s_i^* < s_{u,i}^*, i = 1, 2$, and $d_0 > 0$ such that $\tilde{\xi}_2(s_1, s_2) \le \xi_2^* + d_0(|s_1 - s_1^*| + |s_2 - s_2^*|)$ if $0 < s_{l,i}^* < s_i < s_{u,i}^*$. Since $B_i(s_2) > 0$ and $B_{i;1}(s_1, m_i(s_2)) > 0$ for $0 < s_1 < \infty, 0 < s_2 < \infty$, due to continuity of $B_i, B_{i;1}$, there exists a constant $B_0 > 0$ such that $B_{12}(s_1, s_2) = \prod_{i=1}^2 \{B_i(s_2)B_{i;1}(s_1, m_i(s_2))\} \ge B_0$ if $s_{l,i}^* < s_i < s_{u,i}^*$. It implies

$$C_{1,2}(u, u) \ge u^{\xi_2^*} \tilde{u}^2 \int_{s_{l,1}^*}^{s_{u,1}^*} \int_{s_{l,2}^*}^{s_{u,2}^*} u^{\tilde{\xi}_2(s_1,s_2)} B_{12}(s_1, s_2) ds_1 ds_2$$

$$\ge B_0 u^{\xi_2^*} \tilde{u}^2 \int_{s_{l,1}^*}^{s_{u,1}^*} \int_{s_{l,2}^*}^{s_{u,2}^*} u^{d_0(|s_1-s_1^*|+|s_2-s_2^*|)} ds_1 ds_2 = \frac{B_0 u^{\xi_2^*}}{d_0^2} \prod_{i=1}^2 \left\{ 2 - u^{d_0(s_i^* - s_{l,i}^*)} - u^{d_0(s_{u,i}^* - s_i^*)} \right\}.$$

Hence, the lower tail order of $C_{1,2}$ equals ξ_2^*.

It is left to consider the case when $\xi_2^* < 2$ and $s_1^* s_2^* = 0$. We still can find $s_{l,i}^*$ and $s_{u,i}^*, i = 1, 2$, such that $\tilde{\xi}_2(s_1, s_2) \le \xi_2^* + d_0(|s_1 - s_1^*| + |s_2 - s_2^*|)$ if $s_{l,i}^* < s_i < s_{u,i}^*$. For every $0 < \epsilon < \min_i(s_{u,i}^* - s_i^*)$, there exists a constant $B_0 > 0$ such that $B_{12}(s_1, s_2) \ge B_0$ for $s_i^* + \epsilon < s_i < s_{u,i}^*$. It follows that

$$C_{1,2}(u, u) \ge B_0 u^{\xi_2^*} \tilde{u}^2 \int_{s_1^*+\epsilon}^{s_{u,1}^*} \int_{s_2^*+\epsilon}^{s_{u,2}^*} u^{d_0(|s_1-s_1^*|+|s_2-s_2^*|)} ds_1 ds_2 = \frac{B_0 u^{\xi_2^*+d_0\epsilon}}{d_0^2} \prod_{i=1}^2 \{1 - u^{d_0(s_{i,u} - s_i^* - \epsilon)}\},$$

and the lower tail order of $C_{1,2}$ is not greater than $\xi_2^* + d_0\epsilon$. Also, the lower tail order of $C_{1,2}$ is not smaller than ξ_2^* as shown above. Hence, the lower tail order of $C_{1,2}$ equals ξ_2^*. □

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