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Flexible copula models with dynamic dependence and application to financial data

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ABSTRACT

A new class of copula models with dynamic dependence is introduced; it can be used when one can assume that there exist a common latent factor that affects all of the observed variables. Conditional on this factor, the distribution of these variables is given by the Gaussian copula with a time-varying correlation matrix, and some observed driving variables can be used to model dynamic correlations. This structure allows one to build flexible and parsimonious models for multivariate data with non-Gaussian dependence that changes over time. The model is computationally tractable in high dimensions and the numerical maximum likelihood estimation is feasible. The proposed class of models is applied to analyze three financial data sets of bond yields, CDS spreads and stock returns. The estimated model is used to construct projected distributions and, for the bond yield and CDS spread datasets, compute the expected maximum number of investments in distress under different scenarios.

1. Introduction

The main goal of this article is to develop a dependence model that (a) can handle arbitrary continuous univariate time series models, (b) has dependence that can mostly be explained by one latent variable, (c) has dynamic or time-varying dependence, (d) has asymmetric tail dependence, and (e) is computationally tractable for maximum likelihood estimation.

Because of (a), (b) and (d), we consider factor copula models; and because of (b) and (e), we allow for conditional dependence given the latent variable. By choosing copulas linking observed variables to the latent variable having asymmetric tail dependence, and specifying the conditional distribution (i.e., conditional dependence) of the observed variables given the latent variable having a Gaussian copula, we can get the joint distribution of all observed variables with tail dependence. By having one latent variable and a Gaussian copula for the conditional dependence, the copula density of the observed variable involves only one-dimensional integration and the dependence structure can be quite flexible to allow for between group and within group dependence when the variables are in non-overlapping groups. The use of the Gaussian copula for conditional dependence also means that we can specify different parsimonious structural time-varying correlation matrices.

Our main application is for time series over many years of multiple financial assets such as stock returns, bond yields or CDS spreads. Our models are specified to match empirical and dependence behavior seen in initial data analysis of asset

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return data. Financial asset log returns typically show some positive serial correlations in the absolute log returns, and this is typically handled with GARCH filtering.

Suppose the data are denoted as $(y_{i1}, \dots, y_{id})$, $i = 1, \dots, n$. To see some dependence properties, it is useful to obtain summaries based on the rank transforms to normal scores; the jth variable vector $(y_{m_a j}, \dots, y_{m_b j})$ in the period with $m_a < i \le m_b$, is ranked in the increasing order to get ranks $R_{m_a j}, \dots, R_{m_b j}$ and then $\hat{z}_{ij} = \Phi^{-1}([R_{ij} - 0.5]/[m_b - m_a])$ is obtained using the normal scores transform of the jth variable in this time interval. Different intervals $(m_a, m_b]$ can be used to assess time-varying patterns when the data correspond to many years (say over 3 years) so that stationarity over time might not be a good assumption.

The following patterns tend to hold with or without GARCH filtering.

1. Consider moving window averages and standard deviations (SD) of $(y_{m_a j}, \dots, y_{m_b j})$ separately by asset j, and moving window Spearman's rank correlations, lower/upper semi-correlations (Section 2.17 of Joe (2014)) and lower/upper tail-weighted dependence measures (Krupskii and Joe (2015b)) for pairs from $(\hat{z}_{ij}, \hat{z}_{ik})$, $m_a < i \le m_b$, $j \ne k$. The quantities with the most variability are the SDs of returns. In contrast, Spearman's rank correlations, semi-correlations and tail-weighted dependence measures change slowly over time.
2. Bivariate normal scores plots of $(\hat{z}_{ij}, \hat{z}_{ik})$, $m_a < i \le m_b$, $j \ne k$, often show lower and upper tail dependence (with clouds of points that are sharper than ellipses in the lower and upper quadrant).
3. The empirical correlation matrix of normal scores $(\{\hat{z}_{ij}\})$ often has close to a low-dimensional factor dependence structure.

For high-dimensional financial asset return data, the best fitting models for the short term have been vine copulas (Dißmann et al. (2013)) and factor copulas based on vines (Krupskii and Joe (2013), Krupskii and Joe (2015a)). Advantages of factor copulas over vine copulas include closure under margins, and possibly simpler interpretation with time-varying dependence.

An assumption of unchanging dependence can be too restrictive when modeling data over a long period of time. One example of such data consists of financial returns that have stronger dependence during market downturns. In this case, one can fit a static model to data for several non-overlapping time intervals and analyze the change in dependence in these intervals (Tamakoshi and Hamori, 2014) or use a normal or Student-t copula with time varying correlations proposed by C. Ausin and F. Lopes (2010). Silva et al. (2014) used this model to analyze the co-movement of credit default swaps (CDS) spreads and stock prices and Atil et al. (2016) analyzed the pairwise dependence in the sovereign CDS spreads between the US and Europe. This model cannot handle data with asymmetric dependence, however, and it can have many parameters to estimate if the dimension is very high.

Examples of copula models with time-varying dependence include bivariate observation-driven models in Patton (2006) and Creal et al. (2013), bivariate stochastic autoregressive copula models in Hafner and Manner (2012), semiparametric dynamic bivariate copula model in Hafner and Reznikova (2010), regime switching copulas in Okimoto (2008) (bivariate copulas), Chollette et al. (2009) (multivariate Gaussian and canonical vine copulas), time-varying high-dimensional D-vine copulas in Almeida et al. (2016).

Recently, H. Oh and Patton (2017) introduced a flexible linear factor model for modeling multivariate financial data and H. Oh and Patton (2018) proposed an extension of this model with time-varying coefficients which can be used for modeling non-Gaussian data with dynamic dependence. The authors adopted the generalized autoregressive score (GAS) model of Creal et al. (2011) to model time-varying coefficients in their model. A similar approach was used by Lucas et al. (2014) to model dynamics in coefficients of the skew-t distribution. The driving variable in these models is hard to interpret and some simplifying assumptions about the coefficients in these models are required to make them computationally tractable in high dimensions.

In this paper, we propose a new class of dynamic copula models which is an extension of factor copula models proposed by Krupskii and Joe (2013, 2015a). The variables have a multivariate Gaussian distribution with time-varying correlations after conditioning on an unobserved latent factor. The copulas linking observed variables and the unobserved factor can be selected to handle tail dependence and asymmetry, and some exogenous variables can be included to explain time-varying correlations for the conditional Gaussian distribution. Flexible and computationally tractable models can be obtained with some parsimonious parameterizations of the correlation matrix.

The rest of this paper is organized as follows. In Section 2 we introduce a new class of one-factor conditional Gaussian copulas and discuss their dependence properties; we also consider different parsimonious parameterizations for the correlation matrix in this section. We provide more details on parameter estimation in Section 3. Simulations studies are given in Section 4. The proposed models are applied to analyze government bond yield data, CDS data and several sectors of stock return data in Section 5. The estimated models can be used for stress testing and we estimate the expected maximum number of companies or governments in distress under different scenarios for the first two data sets. Section 6 concludes with a discussion.

2. One latent factor combined with Gaussian copula for conditional dependence

In this section, we derive a copula model with one latent variable and conditional dependence given the latent variable specified with a Gaussian copula. In Section 2.1 we show through dependence properties that this model can handle joint upper and joint lower tail dependence with tail asymmetry, as well as having flexible dependence structure. In Section 2.2 we consider different parsimonious parameterizations of the correlation matrix of the conditional Gaussian copula.

Let d be the number of variables. To define a copula model, we can start with d dependent $U(0, 1)$ random variables $U_1, \ldots, U_d$. let $\mathbf{U} = (U_1, \ldots, U_d)^\top$. We assume that $V \sim U(0, 1)$ is an unobserved latent variable and that the joint cumulative distribution function (cdf) of $\mathbf{U}$ conditional on V is given by the Gaussian or normal copula, $C_N(\cdot\,; \boldsymbol{\Sigma})$, with correlation matrix $\boldsymbol{\Sigma}$. Specifically, let $C_{j,0}$ be the copula cdf linking U_j and V, let $C_{j|0}(u_j|v) := \partial C_{j,0}(u_j, v)/\partial v$ be the conditional cdf of the copula, and let $c_{j,0}$ be the corresponding copula probability density function (pdf). We assume that all these bivariate copulas and their densities are strictly positive and continuous functions on $(0, 1)^2$.

Let $\mathbf{u}_t = (u_{1t}, \ldots, u_{dt})^\top$ be a vector of values at time t. The joint copula cdf and pdf of $\mathbf{U}$ at time $t = 1, \ldots, T$ can then be written as

$$C^t_{\mathbf{U}}(\mathbf{u}_t) = \int_0^1 C_N(C_{1|0}(u_{1t}|v), \ldots, C_{d|0}(u_{dt}|v); \boldsymbol{\Sigma}_t)\,\mathrm{d}v, \quad \text{and}$$
$$c^t_{\mathbf{U}}(\mathbf{u}_t) = \int_0^1 c_N(C_{1|0}(u_{1t}|v), \ldots, C_{d|0}(u_{dt}|v); \boldsymbol{\Sigma}_t) \prod_j c_{j,0}(u_{jt}, v)\,\mathrm{d}v. \tag{1}$$

We assume that copulas $C_{j,0}$ do not depend on t and the matrix $\boldsymbol{\Sigma}_t$ depends on t and does not depend on v. For this model to be more useful, we would like $\boldsymbol{\Sigma}_t$ to be parsimonious (say, structural form with $O(d)$ parameters) and change slowly with t.

For simplicity, we will omit the superscript t for $C^t_{\mathbf{U}}$ and $c^t_{\mathbf{u}}$ in most cases in subsequent sections.

2.1. Dependence and tail asymmetry properties

In this subsection, we obtain some dependence properties of the copula $C_{\mathbf{U}}$ in (1), assuming that there is not time-varying dependence.

Properties that are straightfoward to prove are the following.

1. If $\boldsymbol{\Sigma}_t = I_d$ is the identity matrix, then $C_{\mathbf{U}}$ simplifies to the one-factor copula cdf (Krupskii and Joe, 2013).
2. $C_{\mathbf{U}}$ increases in concordance as $\boldsymbol{\Sigma}_t$ increases (while being positive definite), with $C_{1,0}, \ldots, C_{d,0}$ fixed.
3. If all linking copulas $C_{1,0}, \ldots, C_{d,0}$ are independence copulas then $C_{\mathbf{U}}$ is the Gaussian copula with the correlation matrix $\boldsymbol{\Sigma}_t$.

Tail properties of the copula $C_{\mathbf{U}}$ will depend on tail properties of the linking copulas, $C_{j,0},\ j = 1, \ldots, d,$ similar to the one-factor copula models. The next proposition has a result for the lower tail; analogous properties in the upper tail can be obtained in a similar way.

Proposition 1. *Let $\boldsymbol{w} = (w_1, \ldots, w_d)^\top$ be a vector of positive values. Assume that $\lim_{u\to 0} C_{j|0}(uw_j|uw_0) = b_{j|0}(w_j|w_0)$ and $\lim_{u\to 0} C_{0|j}(uw_0|uw_j) = b_{0|j}(w_0|w_j)$, where $b_{j|0}(\cdot|w_0)$ is a proper distribution function for any $w_0 > 0$ and $b_{0|j}(\cdot|w_j)$ is a proper distribution function for any $w_j > 0,\ j = 1, \ldots, d$. Then the lower tail function is $b_{1:d}(\boldsymbol{w}) := \lim_{u\to 0} C_{\mathbf{U}}(u\boldsymbol{w})/u > 0$; that is, $C_{\mathbf{U}}$ has joint lower tail dependence.*

Proof. By extending Theorem 8.76 of Joe (2014), we get

$$C_{\mathbf{U}}(\mathbf{u}) = u \int_0^{1/u} C_N\{C_{1|0}(uw_1|uw_0), \ldots, C_{d|0}(uw_d|uw_0); \boldsymbol{\Sigma}\}\mathrm{d}w_0$$
$$= u \int_0^{\infty} C_N\{b_{1|0}(w_1|w_0), \ldots, b_{d|0}(w_d|w_0); \boldsymbol{\Sigma}\}\mathrm{d}w_0 + o(u),$$

and hence

$$b(\boldsymbol{w}) = \int_0^{\infty} C_N\{b_{1|0}(w_1|w_0), \ldots, b_{d|0}(w_d|w_0); \boldsymbol{\Sigma}\}\mathrm{d}w_0 > 0. \qquad \square$$

The assumptions of Proposition 1 are satisfied for many bivariate copula families with lower tail dependence, such as the reflected Gumbel and BB1; see Chapter 4 of Joe (2014) for more details on bivariate parametric copula families and their dependence properties.

The next result concerns tail behavior when there may not necessarily be tail dependence. We use the concept of tail order of a copula, as defined in Hua and Joe, 2011. If $\Pr(U_j \le u, j = 1, \ldots, d) = C_{\mathbf{U}}(u\mathbf{1}_d) \sim u^{\kappa_L}\ell_L(u)$ as $u \to 0$, where ℓ_L is a slowly varying function, then the lower tail order is κ_L. Similarly, the upper tail order is defined via $\Pr(U_j \ge 1 - u, j = 1, \ldots, d) \sim u^{\kappa_U}\ell_U(u)$ as $u \to 0$.

If $C_{j,0}$ has lower tail quadrant independence with $\kappa_L = 2$ or conditional distributions that behave close to independence as $u \to 0$, then $C_{\mathbf{U}}$ is a copula with no lower tail dependence as the next proposition shows.

Proposition 2. *Assume that $\boldsymbol{\Sigma}$ is non-degenerate and $m_j^- u \le C_{j|0}(u|v) \le m_j^+ u$ for small enough $u > 0$ and some constants $m_j^-, m_j^+ > 0$, $j = 1, \dots, d$. Then $\kappa_L = \kappa_{\boldsymbol{\Sigma}}$ where $\kappa_{\boldsymbol{\Sigma}} = \mathrm{tr}(\boldsymbol{\Sigma}^{-1})$ is the tail order of the normal copula with the correlation matrix $\boldsymbol{\Sigma}$.*

Proof. For small enough $u > 0$, $m^- = \min_j(m_j^-)$ and $m^+ = \max_j(m_j^+)$ we have:

$$C_{\mathbf{U}}(\mathbf{u}) \le \int_0^1 C_N(m_1^+ u, \dots, m_d^+ u; \boldsymbol{\Sigma}) \mathrm{d}v \le C_N(m^+ u, \dots, m^+ u; \boldsymbol{\Sigma}) \sim_{u \to 0} (m^+ u)^{\kappa_{\Sigma}} \ell(u),$$

and

$$C_{\mathbf{U}}(\mathbf{u}) \ge \int_0^1 C_N(m_1^- u, \dots, m_d^- u; \boldsymbol{\Sigma}) \mathrm{d}v \ge C_N(m^- u, \dots, m^- u; \boldsymbol{\Sigma}) \sim_{u \to 0} (m^- u)^{\kappa_{\Sigma}} \ell(u),$$

where $\ell(u)$ is a slowly varying function. This implies that $\kappa_L = \kappa_{\boldsymbol{\Sigma}}$, if it exists. □

It is easy to check that the assumption of Proposition 2 is satisfied for bivariate parametric copula families with lower tail quadrant independence, including the Frank and Plackett copulas.

The two propositions show that the strength of dependence in the tails for the linking copulas $C_{j,0}$ affect the multivariate tail properties of $C_{\mathbf{U}}$.

If each of the bivariate copulas $C_{j,0}$ is reflection symmetric (copula density $c_{j,0}(u, v) = c_{j,0}(1-u, 1-v)$ for $0 < u, v < 1$), then it can be shown using the technique of the proof Theorem 8.67 of Joe (2014) that the copula density in (1) is reflection symmetric with $c_U^t(u_1, \dots, u_d) = c_U^t(1-u_1, \dots, 1-u_d)$ for $0 < u_j < 1$, $j = 1, \dots, d$.

If all of the bivariate copulas $C_{j,0}$ are reflection asymmetric in the same direction, then one would expect (1) to be reflection asymmetric in the same direction. The proof of this is not tractable in the general case, but the reflection and tail asymmetry can be shown in plots such as in Fig. 1 with tail asymmetric BB1 and Gumbel copula families.

2.2. Parsimonious parameterization of $\boldsymbol{\Sigma}_t$

In this section, we consider different parsimonious parameterizations of $\boldsymbol{\Sigma}_t$ using $O(d)$ parameters. We assume that, for $i \ne j$, $(\boldsymbol{\Sigma}_t)_{i,j} = \rho_{i,j}^* \rho_{i,j}(t)$ where $-1 \le \rho_{i,j}^* \le 1$ is a fixed parameter, and $\rho_{ij}(t)$ is a dynamic component. Further assumptions are needed in order that $(\boldsymbol{\Sigma}_t)$ is positive definite for all t. Three examples are given below. In these examples, the matrix inverse and determinant have closed form so that numerical computation of the copula density $c_{\mathbf{U}}$ is fast even for large d.

1. *Equicorrelation matrix*. We assume that $\rho_{i,j}(t) = \rho(t)$ for $0 \le \rho(t) \le 1$, and $\rho_{i,j}^* = 1$ for $i \ne j$. The correlation matrix inverse and determinant can be easily calculated as follows:

$$(\boldsymbol{\Sigma}_t^{-1})_{i,j} = \frac{\{(d-1)\rho(t)+1\}1\{i=j\} - \rho(t)}{-(d-1)\rho^2(t) + (d-2)\rho(t) + 1}, \quad \det(\boldsymbol{\Sigma}_t) = \{1-\rho(t)\}^{d-1}\{1+(d-1)\rho(t)\}.$$

   We get an exchangeable dependence structure in $\mathbf{U}$ after conditioning on V, where $\boldsymbol{\Sigma}$ is a correlation matrix of $\boldsymbol{Z} = (W_1, \dots, W_d)$ where

$$W_i = \rho^{1/2}(t) Z_0 + \{1-\rho(t)\}^{1/2} Z_i, \quad Z_0, Z_1, \dots, Z_d \sim_{\text{i.i.d.}} N(0,1),$$

   and $Z_0 = \Phi^{-1}(V)$, $W_i = \Phi^{-1}(U_i)$ and Φ is the standard normal cdf. This exchangeability assumption can be too restrictive when modeling data with heterogeneous dependence.

2. *One-factor structure*. We assume that $\rho_{i,j}(t) = \rho(t)$ for $0 \le \rho(t) \le 1$, and $\rho_{i,j}^* = \alpha_i \alpha_j$ where $-1 \le \alpha_1, \dots, \alpha_d \le 1$ are some constants. The correlation matrix inverse and determinant can be calculated as follows:

$$(\boldsymbol{\Sigma}_t^{-1})_{i,j} = \frac{1\{i=j\}}{1-\rho(t)\alpha_i^2} - \frac{\alpha_i \alpha_j \rho(t)}{\{1-\rho(t)\alpha_i^2\}\{1-\rho(t)\alpha_j^2\}} \cdot \left\{1 + \sum_{i=1}^d \frac{\rho(t)\alpha_i^2}{1-\rho(t)\alpha_i^2}\right\}^{-1},$$

   and

$$\det(\boldsymbol{\Sigma}) = \prod_{i=1}^d \{1-\rho(t)\alpha_i^2\} \cdot \left\{\sum_{i=1}^d \frac{1}{1-\rho(t)\alpha_i^2} + 1 - d\right\}.$$

   $\boldsymbol{\Sigma}_t$ is a correlation matrix of $\boldsymbol{W} = (W_1, \dots, W_d)$ where

$$W_i = \alpha_i \rho^{1/2}(t) Z_0 + \{1-\alpha_i^2 \rho(t)\}^{1/2} Z_i, \quad Z_0, Z_1, \dots, Z_d \sim_{\text{i.i.d.}} N(0,1),$$

   and $Z_0 = \Phi^{-1}(V)$, $W_i = \Phi^{-1}(U_i)$. The model is more flexible as it allows one to obtain nonexchangeable residual dependence. The copula $C_{\mathbf{U}}$ is a special case of a two-factor copula (Krupskii and Joe, 2015a) with time varying coefficients. An exchangeable structure can be obtained when $\alpha_i = 1$, $i = 1, \dots, d$.

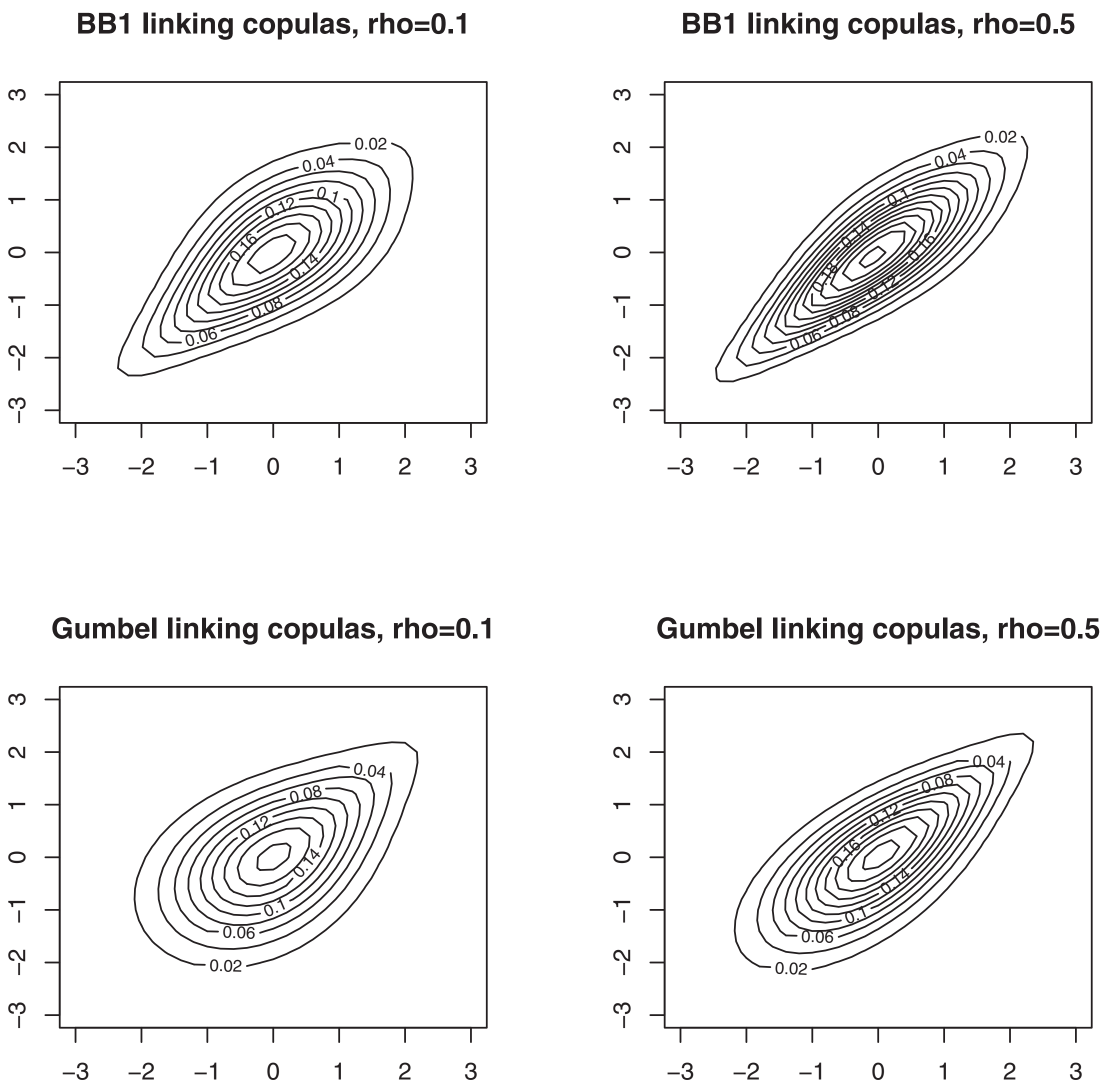

Fig. 1. Bivariate (1,2) marginal copula densities with univariate $N(0, 1)$ margins. Top row: BB1 linking copulas ($\lambda_L = 0.6, \lambda_U = 0.4$ for variable 1; $\lambda_L = 0.7, \lambda_U = 0.5$ for variable 2). Bottom row: Gumbel linking copulas ($\lambda_U = 0.6$ for variable 1; $\lambda_U = 0.5$ for variable 2). Left side: $\rho_{12} = 0.1$ for conditional dependence given latent variable. Right side: $\rho_{12} = 0.5$.

3. *Nested structure*. The model with a one-factor correlation matrix $\sum_t$ and positive α_i's assumes that dependence changes in the same direction for all observed variables. If $\rho(t)$ increases/decreases, then all correlations $(\boldsymbol{\Sigma}_t)_{i,j}$ increase/decrease, respectively. The one-factor structure might therefore be not suitable when dependence gets stronger for some variables and weaker for some other variables.
   We assume that the data can be split into $G > 1$ non-overlapping groups, with n_g variables in the gth group, such that dependence among variables in each group changes in the same direction. We assume that the correlation between variables i and j in the gth group is $(\boldsymbol{\Sigma}_t)^{(g)}_{i,j} = \rho_g(t)\alpha_{ig}\alpha_{jg}$ and the correlation between variable i in the g_1th group and variable j in the g_2th group is $(\boldsymbol{\Sigma}_t)^{(g_1,g_2)}_{i,j} = \rho^*_{g_1}\rho^*_{g_2}\alpha_{ig_1}\alpha_{jg_2} \cdot \{\rho_{g_1}(t)\rho_{g_2}(t)\}^{1/2}$ where $0 \le \rho_g(t) \le 1$ and $-1 \le \alpha_{ig}, \rho^*_g \le 1,\ i = 1, \ldots, n_g$ and $g = 1, \ldots, G$. Here, $\boldsymbol{\Sigma}$ is positive definite because it is the correlation matrix of $\boldsymbol{Z} = (W_{11}, \ldots, W_{n_{11}}, \cdots, W_{1G}, \ldots, W_{n_{GG}})$ where

$$W_{ig} = \{\rho_g(t)\}^{1/2}\rho^*_g\alpha_{ig}Z_0 + \{\rho_g(t)(1-\rho^{*2}_g)\}^{1/2}\alpha_{ig}Z_g + \{1-\rho_g(t)\alpha^2_{ig}\}^{1/2}Z_{ig}, \tag{2}$$

   and $Z_0, Z_1, \ldots, Z_G, Z_{11}, \ldots, Z_{n_GG} \sim_{\text{i.i.d.}} N(0, 1)$. The dynamic component $\rho_g(t)$ controls the change in dependence in the gth group, and the correlation matrix inverse and determinant can be obtained in a simple form (Krupskii, 2014).

We next consider a possible choice for the dynamic component $\rho(t)$, discuss some identifiability issues and possible extensions of the model for the correlation matrix $\boldsymbol{\Sigma}_t$.

Modeling dynamic components $\rho_g(t)$

We use the logit transformation $\eta_g(t)$ to ensure that $0 \le \rho_g(t) \le 1$:

$$\rho_g(t) = \frac{1}{1+\exp\{-\eta_g(t)\}}, \quad \eta_g(t) = \psi_{0g} + \sum_{m=1}^{M} \psi_{mg}\eta_g(t-m) + \sum_{k=1}^{K} \gamma_{kg}\mathcal{V}_{kg}(t), \tag{3}$$

where $\mathcal{V}_{11}, \dots, \mathcal{V}_{KG}$ are some observed driving variables. In (3), if $\gamma_{kg} > 0$, then larger values of $\mathcal{V}_{kg}(t)$ lead to larger values of $\rho_g(t)$. For financial data, driving variables can include sector indexes or macroeconomic parameters such as the gross domestic product and unemployment rate. Different driving variables can be used for $\rho_g(t)$, $g = 1, \dots, G$. The index g can be omitted if $G = 1$.

Identifiability issues

With $G = 2$ groups, two parameters ρ_1^* and ρ_2^* cannot be identified as $\boldsymbol{\Sigma}(t)$ depends on their product $\rho_1^*\rho_2^*$ only. In this case, one can assume $\rho_1^* = \rho_2^* = \rho^*$ or $\rho_1^* = -\rho_2^* = \rho^*$ for positive or negative dependence between the two groups, respectively, for identifiability. If $G \ge 3$, these parameters are identifiable; however, for more parsimony, one can assume $\rho_g^* = \rho^*$ for $g = 1, \dots, G$ if G is not large.

Coefficients ψ_m, $m = 1, \dots, M$, in (3) are nearly non-identifiable in many cases when driving variables are slowly varying over time t and hence these coefficients cannot be estimated with good accuracy. They can therefore be set equal to zero without changing the model fit much. To illustrate these ideas, consider the process $\eta(t)$ with $M = K = 1$ and $\mathcal{V}_1(t) = t$:

$$\eta(t) = \psi_0 + \psi_1\eta(t-1) + \gamma_1 t. \tag{4}$$

It is straightforward to check that non-identifiability occurs if $\eta(t)$ is linear because in this case,

$$\eta(t) = \frac{\psi_0}{1-\psi_1} - \frac{\psi_1\gamma_1}{(1-\psi_1)^2} + \frac{\gamma_1 t}{1-\psi_1}$$

satisfies (4) and therefore one can redefine

$$\psi_0 \leftarrow \frac{\psi_0}{1-\psi_1} - \frac{\psi_1\gamma_1}{(1-\psi_1)^2}, \quad \psi_1 \leftarrow 0, \quad \gamma_1 \leftarrow \frac{\gamma_1 t}{1-\psi_1}.$$

Possible extension

If variables naturally fit into non-overlapping groups, then we might like to consider other covariance matrices $\boldsymbol{\Sigma}_t$ that have stronger within-group dependence than between-group dependence. Future research will consider parsimonious dependence structures of this form with the time-varying parameter so that the positive definite constraint is always satisfied.

3. Parameter estimation

In this section, some details are given in order that maximum likelihood estimation of model (1) can proceed efficiently when there is a parametric family for each $C_{j,0}$.

The details are provided for the model with $\boldsymbol{\Sigma}_t$ which has a nested structure, as the other cases in Section 2 are simpler. We assume that there are $G > 1$ non-overlapping groups of variables of size $n_1, \dots, n_G$. Let the observed data $\{\boldsymbol{z}_t = (z_{11}^t, \dots, z_{n_{11}}^t, \cdots, z_{1G}^t, \dots, z_{n_{GG}}^t)\}_{t=1}^T$. We assume that $\boldsymbol{z}_1, \dots, \boldsymbol{z}_T$ are realizations of independent random vectors $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_T$ that are not identically distributed.

Let $F_{ig}^t(\cdot; \boldsymbol{\theta}_{ig}^F)$ be the univariate marginal cdf of Z_{ig}^t for $i = 1, \dots, n_g$ and $g = 1, \dots, G$. Here $\boldsymbol{\theta}_{ig}^F$ is a vector of unknown parameters for F_{ig}^t; this distribution can also depend on some external variables that change in time. Parameters of the univariate distributions can be estimated in the first step and the data can be converted to the uniform variables using $u_{ig}^t = (F_{ig}^t)(z_{ig}^t; \widehat{\boldsymbol{\theta}}_{ig}^F)$.

Parameters of (1), including parameters of the correlation matrix $\boldsymbol{\Sigma}_t$, can then be estimated using the copula pseudo-log-likelihood $\ell(\mathbf{u}; \boldsymbol{\theta}) = \sum_{t=1}^T \ln \int_0^1 L(\mathbf{u}_t, v; \boldsymbol{\theta}) \mathrm{d}v$, where

$$L(\mathbf{u}_t, v; \boldsymbol{\theta}) = c_N(C_{11|0}(u_{11}^t|v; \boldsymbol{\theta}_{11}), \dots, C_{n_GG|0}(u_{n_GG}^t|v; \boldsymbol{\theta}_{n_GG}); \boldsymbol{\Sigma}_t) \prod_{i,g} c_{ig,0}(u_{ig}^t, v; \boldsymbol{\theta}_{ig}),$$

with $\mathbf{u}_t = (u_{11}^t, \dots, u_{n_{11}}^t, \cdots, u_{1G}^t, \dots, u_{n_{GG}}^t)$. Here $C_{ig,0}$ is the copula linking u_{ij}^t and v, and $\boldsymbol{\theta}_{ig}$ is a vector of parameters of $C_{ig,0}$, and $\boldsymbol{\theta}$ is a vector of all unknown parameters.

The results in Section 2 can be used to select suitable parametric copula families $C_{ig,0}$. Some tail-weighted measures of dependence or normal scores scatterplots can be used to check for some departures from normality, such as tail dependence or asymmetry. For data with upper/lower tail dependence, one can select linking copulas with upper/lower tail dependence, respectively. Assuming homogeneous dependence in each group, the same parametric family of copulas can be used for the gth group.

A quasi-Newton method with analytical derivatives can be used to estimate parameters in (1); it is because we can code the gradient of the log-likelihood that we can handle maximum likelihood estimation for a large number of variables. In particular,

$$\frac{\partial \ell(\mathbf{u};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{ig}} = \sum_{t=1}^{T} \frac{\int_0^1 L(\mathbf{u}_t, v;\boldsymbol{\theta}) \cdot h_{ig}(\mathbf{u}_t, v;\boldsymbol{\theta}) \mathrm{d}v}{\int_0^1 L(\mathbf{u}_t, v;\boldsymbol{\theta}) \mathrm{d}v},$$

$$h_{ig}(\mathbf{u}_t, v;\boldsymbol{\theta}) = \frac{\partial c_{ig,0}(u_{ig}^t, v;\boldsymbol{\theta}_{ig})/\partial \boldsymbol{\theta}_{ig}}{c_{ig,0}(u_{ig}^t, v;\boldsymbol{\theta}_{ig})} - \frac{\partial C_{ig|0}(u_{ig}^t|v;\boldsymbol{\theta}_{ig})}{\partial \boldsymbol{\theta}_{ig}} \cdot \frac{\boldsymbol{\Sigma}_t^{-1} S(\mathbf{u}_t, v;\boldsymbol{\theta})}{\phi(\varphi_{ig}(u_{ig}|v;\boldsymbol{\theta}_{ig}))}$$
$$+ \frac{\partial C_{ig|0}(u_{ig}^t|v;\boldsymbol{\theta}_{ig})}{\partial \boldsymbol{\theta}_{ig}} \cdot \frac{\varphi_{ig}(u_{ig}|v;\boldsymbol{\theta}_{ig})}{\phi(\varphi_{ig}(u_{ig}|v;\boldsymbol{\theta}_{ig}))}, \text{ and}$$

$$S(\mathbf{u}_t, v;\boldsymbol{\theta}) = (\varphi_{11}(u_{11}|v;\boldsymbol{\theta}_{11}), \ldots, \varphi_{n_G G}(u_{n_G G}|v;\boldsymbol{\theta}_{n_G G}))^\top,$$

where $\phi(\cdot)$ is the standard normal pdf and

$$\varphi_{ig}(u_{ig}|v;\boldsymbol{\theta}_{ig}) = \Phi^{-1}(C_{ig|0}(u_{ig}^t|v;\boldsymbol{\theta}_{ig})), \quad i = 1, \ldots, n_g, \; g = 1, \ldots, G.$$

To obtain partial derivatives with respect to correlation parameters, $\alpha_{ig}, \rho_g^*, \eta_m, \gamma_k, i = 1, \ldots, n_g, g = 1, \ldots, G, m = 0, \ldots, M, k = 1, \ldots, K$, one needs to compute derivatives of $\boldsymbol{\Sigma}_t$ with respect to these parameters. In particular, we use (3) to compute $\partial \boldsymbol{\Sigma}_t/\psi_{mg}$ and $\partial \boldsymbol{\Sigma}_t/\gamma_{kg}$ iteratively as follows:

$$\frac{\partial \boldsymbol{\Sigma}_t}{\partial \gamma_{kg}} = \rho_g(t)(1-\rho_g(t))\frac{\partial \boldsymbol{\Sigma}_t}{\partial \rho_g(t)} \cdot \frac{\partial \eta_g(t)}{\partial \gamma_{kg}}, \quad \frac{\partial \eta_g(t)}{\partial \gamma_{kg}} = \mathcal{V}_{kg}(t) + \sum_{m=1}^{M} \frac{\partial \eta_g(t-m)}{\partial \gamma_{kg}},$$

and

$$\frac{\partial \boldsymbol{\Sigma}_t}{\partial \psi_{mg}} = \rho_g(t)(1-\rho_g(t))\frac{\partial \boldsymbol{\Sigma}_t}{\partial \rho_g(t)} \cdot \frac{\partial \eta_g(t)}{\partial \psi_{mg}}, \quad \frac{\partial \eta_g(t)}{\partial \psi_{mg}} = \eta_g(t-m) + \sum_{m'=1}^{M} \psi_{m'g}\frac{\partial \eta_g(t-m')}{\partial \psi_{mg}}.$$

One-dimensional numerical integration is required to compute the log-likelihood and its partial derivatives. Gaussian quadrature can be used (Stroud and Secrest, 1966); we found that these integrals can be computed with good accuracy when $n_q = 25$ quadrature points are used.

Asymptotic properties of estimators

If the driving variables are correctly specified, then maximum likelihood estimators are consistent and asymptotically normal under standard regularity conditions (Domowitz and White, 1982; Bates and White, 1985). In practice, the true driving variables are not known and some proxy variables can be used to model the dynamic dependence structure. In this case, the quasi-maximum likelihood estimators obtained by maximizing the copula pdf in (1) minimize the Kullback-Leibler divergence

$$KL(\boldsymbol{\theta}) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} KL\{C_{\mathbf{U}}^t(\cdot;\boldsymbol{\theta}), \tilde{C}_{\mathbf{U}}^t\},$$

where $KL\{C_{\mathbf{U}}^t(\cdot;\boldsymbol{\theta}), \tilde{C}_{\mathbf{U}}^t\}$ is the Kullback-Leibler divergence between the true copula cdf of $\mathbf{U}_t$, $\tilde{C}_{\mathbf{U}}^t$, and the copula $C_{\mathbf{U}}^t$ parameterized using a vector of parameters $\boldsymbol{\theta}$, $t = 1, \ldots, T$. These results hold if the driving variables are bounded.

4. Simulation results

In this section we show that the models in Section 2 can be used to handle heterogeneous dependence for data with group structures and check the performance of the estimation algorithm for the two simulated data sets generated from the model (1) with parameters corresponding to slowly and quickly changing dependence structure. We obtained similar results for other sets of copula parameters.

Dependence properties for bivariate marginals

We consider the nested structure for $\boldsymbol{\Sigma}_t$ as defined in Section 2.2. For different linking copulas and correlation parameters, we compute the Spearman's correlation, S_ρ, and lower and upper tail dependence coefficients, λ_L and λ_U, for bivariate copulas corresponding to the distribution of two variables U_{11}, U_{21} from the first group and two variables U_{11}, U_{12} from two different groups. For a given common factor V, we denote the conditional correlation between U_{11} and U_{21} (U_{11} and U_{12}) by $\rho_{11,21}$ ($\rho_{11,12}$, respectively). Table 1 shows the results. It is seen that a wide range of dependence structures can be obtained with different linking copulas and correlation parameters (including tail dependence and tail asymmetry), with weaker dependence between variables from different groups.

First simulation

We generate a data set from the model (1) with BB1 linking copulas and $\boldsymbol{\Sigma}(t)$ with the nested correlation structure with 3 groups of size 5 each (15 variables in total). We use the following parameters: $\boldsymbol{\theta}_{i1} = (0.5, 2)^\top, \boldsymbol{\theta}_{i2} = (1, 1.5)^\top, \boldsymbol{\theta}_{i3} = (0.5, 2)^\top$ for the BB1 linking copulas and $\alpha_{ig} = 0.8$, $\rho_g^* = 0.4$ for the $\boldsymbol{\Sigma}(t)$ matrix, $i = 1, \ldots, 5$ and $g = 1, \ldots, 3$. To model dynamic dependence, we use the logit transformation (3) with $M = 0, K = 1, \mathcal{V}_1(t) = 3\sin\{\pi(t-0.5)\}$, $\mathcal{V}_2(t) = 3\cos(4t-1)$, $\mathcal{V}_3(t) = 7|t-0.5|$ and $\eta_0 = 2, \beta_1 = 1$ for each of the three groups.

Table 1
S_ρ, λ_L and λ_U for copulas $C_{11,21}$, $C_{11,12}$ linking U_{11} and U_{21}, U_{11} and U_{12} for different correlation parameters $\rho_{11,21}, \rho_{11,12}$ and copulas $C_{11,0}$, $C_{12,0}$, $C_{21,0}$ linking U_{11}, U_{12}, U_{21} and unobserved factor V (copula parameters are shown in parentheses).

$C_{11,0}$, $C_{12,0}$, $C_{21,0}$	$\rho_{11,21}, \rho_{11,12}$	same group S_ρ, λ_L, λ_U for $C_{11,21}$	different groups S_ρ, λ_L, λ_U for $C_{11,12}$
all normal(0.3)	0.8, 0.2	0.80, 0.00, 0.00	0.26, 0.00, 0.00
all $t(0.3, 4)$	0.8, 0.2	0.80, 0.34, 0.34	0.25, 0.07, 0.07
all Gumbel(1.2)	0.8, 0.2	0.80, 0.00, 0.37	0.25, 0.00, 0.10
all BB1(0.15, 1.1)	0.8, 0.2	0.80, 0.12, 0.24	0.24, 0.00, 0.04

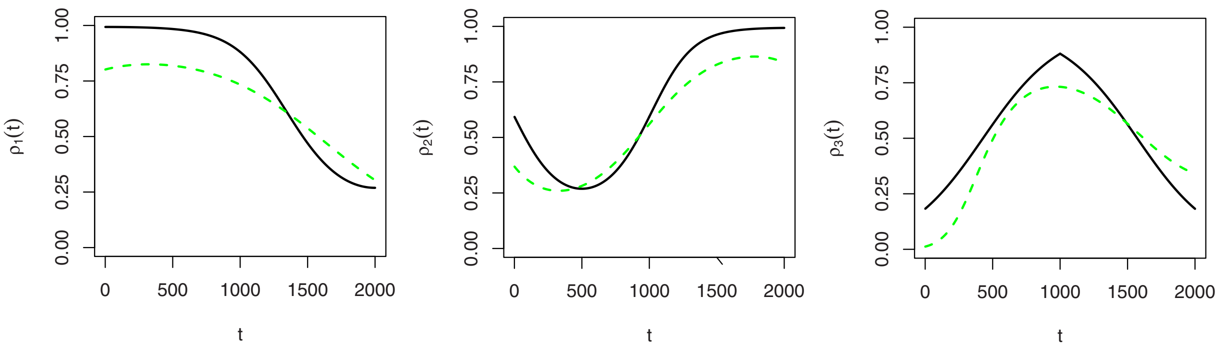

Fig. 2. True and estimated dynamic components $\rho_g(t)$, $g = 1, 2, 3$ (black and dashed green lines, respectively).

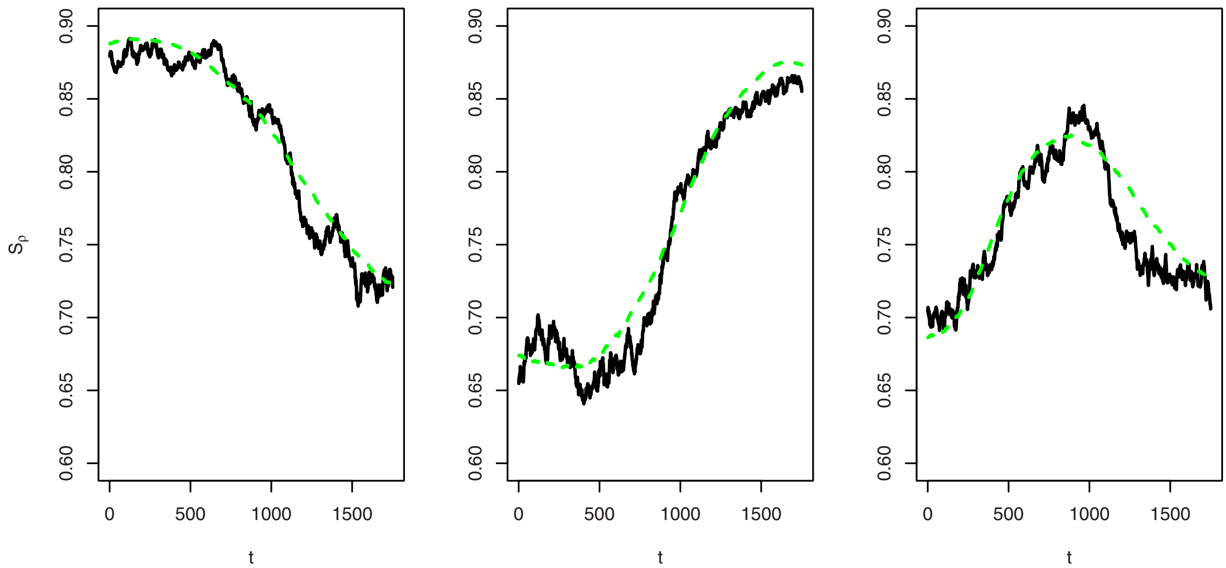

Fig. 3. Correlation averages, S_ρ, for the three groups of variables for the original data set and data set generated from the estimated model (black and dashed green lines, respectively).

With $t = (1:2000)/2000$, we fit a misspecified model with $\mathcal{V}_k^*(t) = t^k$, $k = 1, 2, 3$, as three observed driving variables for each group. We estimate parameters in the model using the likelihood approach; Fig. 2 shows the true and estimated dynamic components, $\rho_g(t)$, for $g = 1, 2, 3$.

The dynamic dependence is correctly estimated despite misspecified driving variables $\mathcal{V}_k^*(t)$, $k = 1, 2, 3$, were used. We simulate data from the estimated model and compute within group Spearman's correlation averages for the three groups of variables using these simulated data and rolling window of size 250. Fig. 3 shows the correlation averages, S_ρ, for the original simulated data set (black lines) and the data set generated from the estimated model (dashed green lines). The correlation structure is estimated very well for the original data set.

Second simulation

Again, we generate a data set from the model (1) with BB1 linking copulas and $\mathbf{\Sigma}(t)$ with the nested correlation structure with 3 groups of size 5 each. We use the same parameters for the BB1 linking copulas and the logit transformation (3) with $M = 0, K = 1$ to model the dynamic dependence, but with different driving variables: $\mathcal{V}_1(t) = 20\sin(5t)$, $\mathcal{V}_2(t) =$

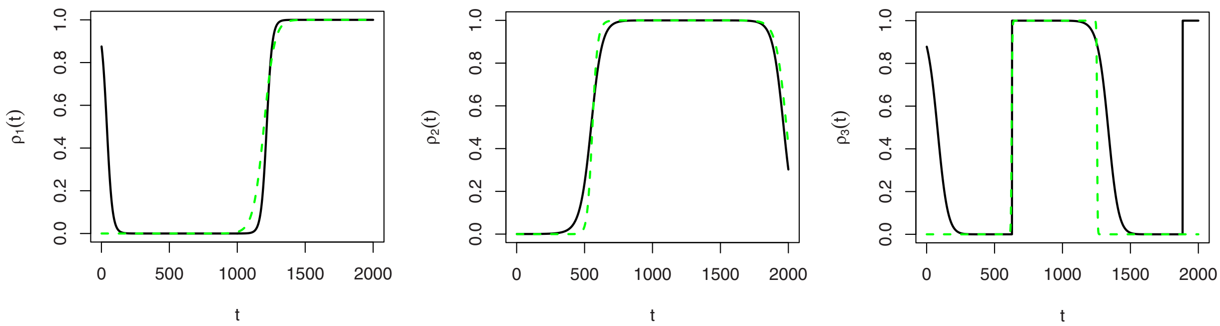

Fig. 4. True and estimated dynamic components $\rho_g(t)$, $g = 1, 2, 3$ (black and dashed green lines, respectively).

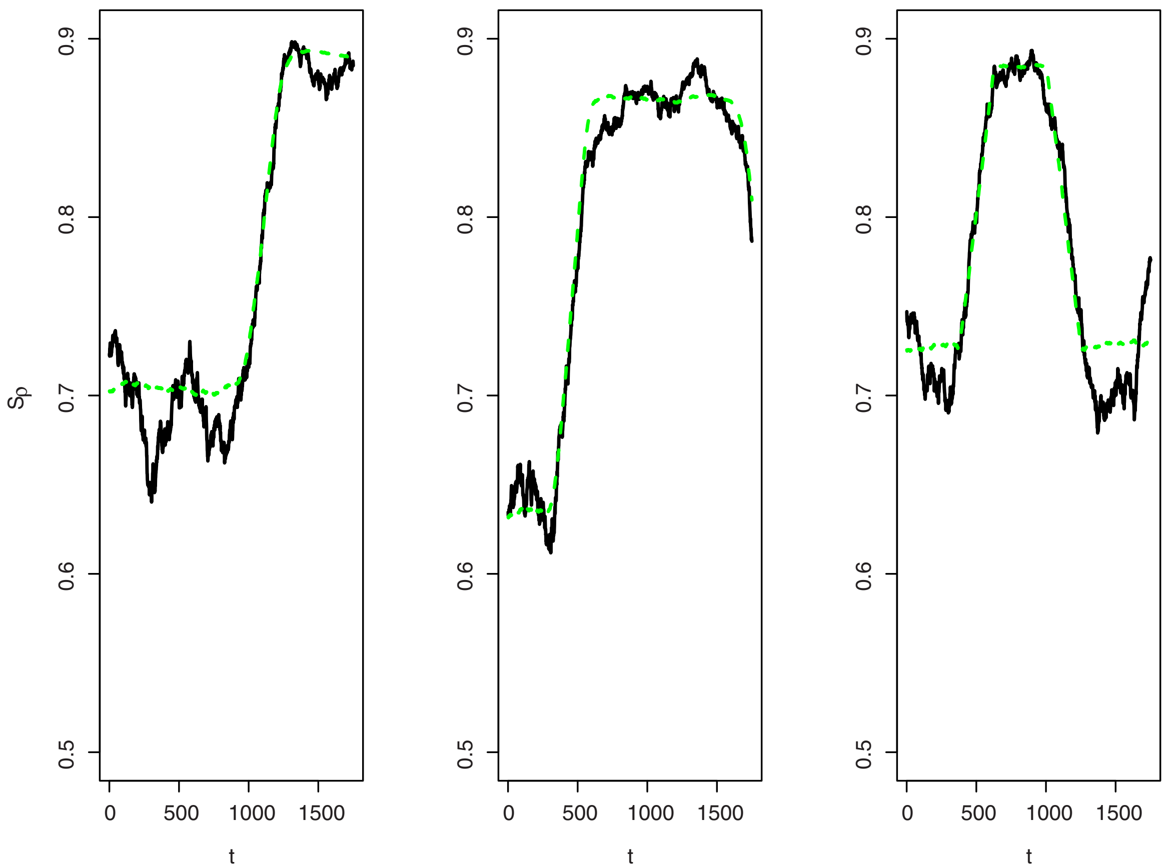

Fig. 5. Correlation averages, S_ρ, for the three groups of variables for the original data set and data set generated from the estimated model (black and dashed green lines, respectively).

$10\cos(5t)$, $\mathcal{V}_3(t) = 10\tan(5t)$ and $\eta_0 = 2, \beta_1 = 1$ for each of the three groups, corresponding to quickly changing dependence.

We fit a model with $\mathcal{V}_k^*(t) = t^k$, $k = 1, 2, 3$, as three observed driving variables for each group, as before. Fig. 4 shows the true and estimated dynamic components, $\rho_g(t)$, for $g = 1, 2, 3$.

The dynamic dependence is recovered quite well despite slowly varying driving variables were used to model quickly changing dependence. The estimated dynamic component is a bit less accurate for the third group with very sharp changes in the dependence structure. Fig. 5 shows the correlation averages, S_ρ, for the original simulated data set (black lines) and the data set generated from the estimated model (dashed green lines) obtained using a rolling window of size 250. Similar to the first data set, the correlation structure is estimated very well.

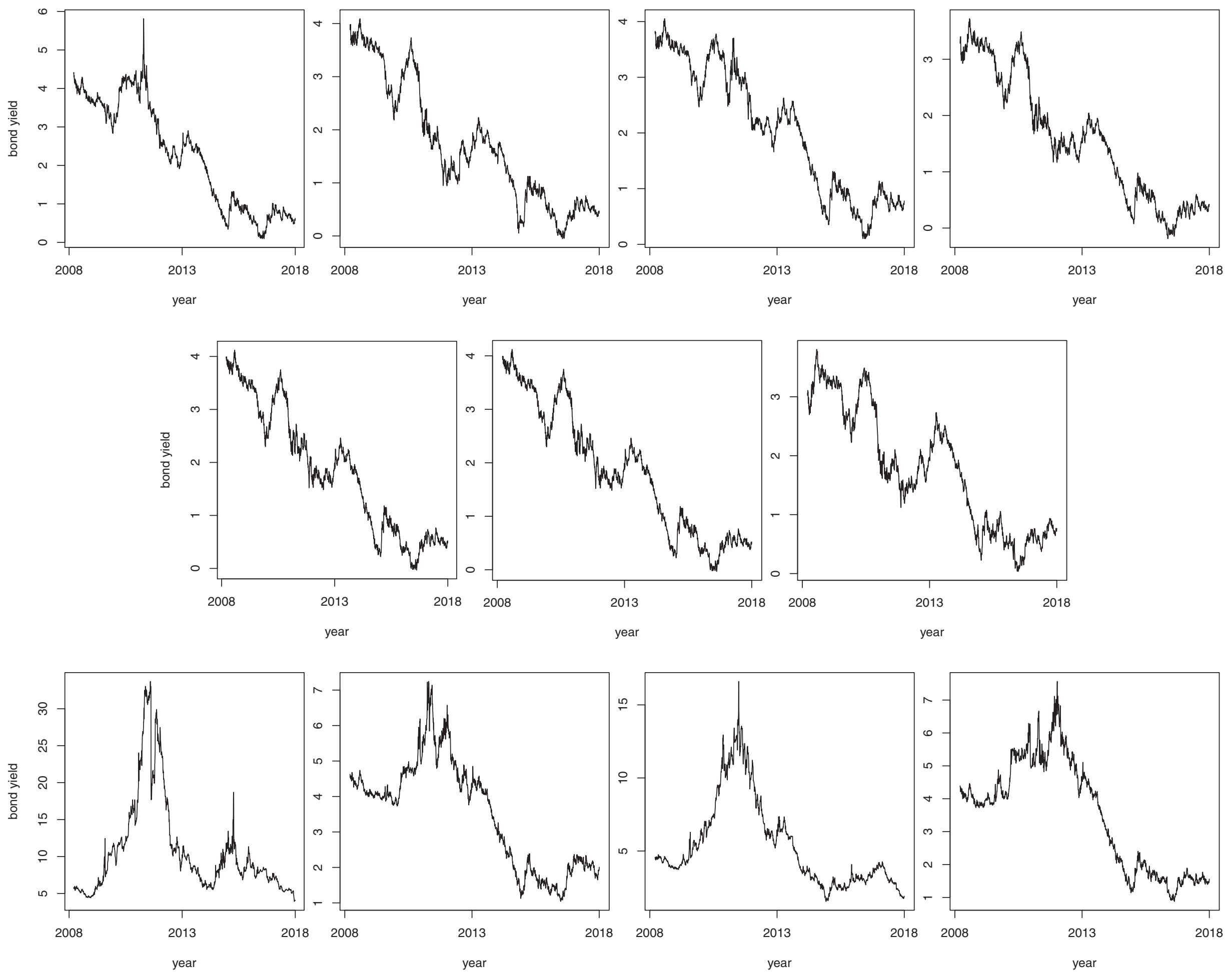

Fig. 6. 10-year government bond daily yields for 11 countries, left to right: Belgium, Denmark, France, Germany (first row), Netherlands, Sweden, United Kingdom (second row) and Greece, Italy, Spain, Portugal (third row).

5. Empirical studies

In this section, we apply the model (1) with a nested structure of the correlation matrix $\mathbf{\Sigma}_t$ to analyze three financial data sets. The data sets used in the first two subsections were downloaded via a UBC Bloomberg terminal.

In Section 5.1 the first data set includes 10-year government bond yields for 11 European countries: Belgium, Denmark, France, Germany, Greece, Italy, Netherlands, Portugal, Spain and Sweden and United Kingdom. We use daily yields from March 2008 to December 2017, with 2265 business days in total. The countries are divided into $G = 2$ groups based on strength of economy.

In Section 5.2, the second data set includes CDS spreads of 24 US companies in $G = 4$ sectors or groups: 5 companies from the technology sector (tickers CSCO, DXC, IBM, MSI, XRX), 4 companies from the energy sector (tickers APC, APA, COP, VLO), 11 companies from the industrials sector (tickers CAT, CSX, HON, JCI, LMT, NSC, NOC, RTN, LUV, UNP, UPS), and 4 companies from the consumer staples sector (tickers CAG,KR,WMT,YUM). We use daily data from January 2010 to December 2017, with 2011 business days in total.

Lastly, to show that the proposed approach can handle a larger number of variables, we use stock returns of 81 US companies in Section 5.4. We include stocks from $G = 4$ sectors: 17 companies from the consumer staples sector, 24 companies from the energy sector, 21 companies from the financials sector and 19 companies from the health care sector. We use daily data from January 2010 to December 2018, with 2263 business days in total.

5.1. European bonds data

Fig. 6 shows plots of daily bond yields for the eleven countries. The first group of countries with strong economies (Belgium, Denmark, France, Germany, Netherlands, Sweden and United Kingdom) have their government bond yields declined,

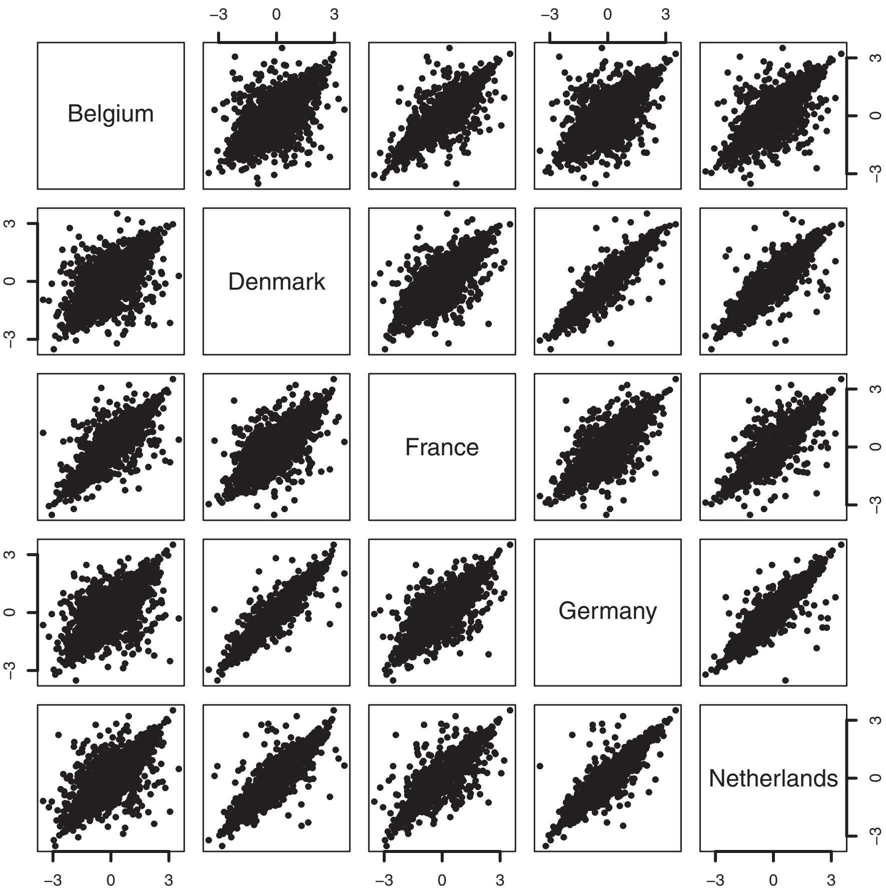

Fig. 7. Normal scores scatterplots of the residuals obtained for bond yields of Belgium, Denmark, France, Germany and Netherlands.

with some bonds reaching negative yields. Four European countries (Greece, Italy, Spain and Portugal) experienced financial crisis in 2010-2013 reflected by very high bond yields. It is therefore natural to consider a group structure when modeling dynamic dependence of the eleven bond yields with seven countries with stronger economies in the first group and four countries in the Southern Europe in the second group.

We start with fitting univariate marginal models to the data. Let $w_{j,t}$ be the yield of the j-th government bond at time t and $r_{j,t} = w_{j,t} - w_{j,t-1}$. To handle serial dependence, we use `AR(2)-GARCH(1,1)` model with the t_ν innovations:

$$r_{j,t} = \phi_{j,0} + \sum_{k=1}^{2} \phi_{j,k} r_{j,t-k} + \sigma_{j,t}\epsilon_{j,t}, \quad \epsilon_{j,t} \sim \mathrm{t}_\nu, \text{ and}$$
$$\sigma_{j,t}^2 = \beta_{j,0} + \beta_{j,1}\sigma_{j,t-1}^2 + \beta_{j,2} r_{j,t-1}^2. \tag{5}$$

We use differences $r_{j,t}$ rather than log-returns because of the negative values of $w_{j,t}$ for some countries in 2017.

We transform residuals from the estimated model (5) for each time series to the uniform data $\{\mathbf{u}_t\}_{t=1}^T$ using non-parametric ranks:

$$\mathbf{u}_t = (u_{1,t}, \ldots, u_{11,t}), \quad u_{j,t} = \frac{\mathrm{rank}(u_{j,t}) - 0.5}{T},$$

where $T = 2265$ is the number of observations. Alternatively, an integral transform can be applied using the estimated marginal distributions to obtain the uniform data. Both methods give very similar results. Fig. 7 shows normal scores scatterplots of the residuals obtained for bond yields of five countries from the first group. Sharp tails of the scatterplots point to strong tail dependence and therefore suggest a model that allows for tail dependence.

We now estimate the joint distribution of $\mathbf{u}_t$ using the copula $C_{\mathbf{U}}$ as defined in (1) and we assume the nested structure (2) for the correlation matrix $\boldsymbol{\Sigma}_t$, with $\rho_1^* = \rho_2^* = \rho^*$ as explained in Section 2.2. We select BB1 linking copulas $C_{j,0}$ for each group as this class of copulas allows for asymmetric tail dependence. We use daily values of the CAC 40, Euronext 100 and DAX 30 indexes as driving variables in (3) to model the dynamic dependence. We use these indexes as indicators of performance of the European markets, with smaller values pointing to declining markets.

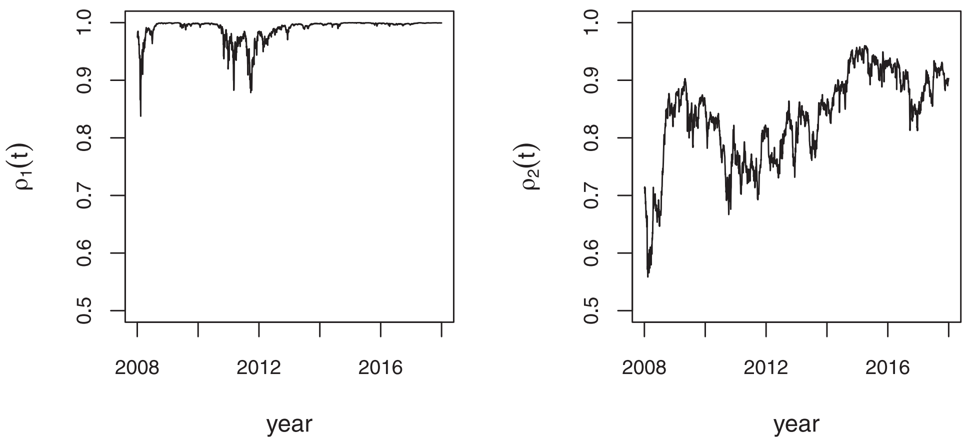

Fig. 8. The estimated dynamic components, $\rho_1(t)$ and $\rho_2(t)$, for bond yields for two groups of countries: Belgium, Denmark, France, Germany, Netherlands, Sweden (left) and Greece, Italy, Portugal, Spain (right).

Fig. 9. Spearman correlation moving averages obtained using a rolling window of size 250, for bond yields for two groups of countries: Belgium, Denmark, France, Gemrany, Netherlands, Sweden (left) and Greece, Italy, Portugal, Spain (right) and for the data simulated from the fitted copula model (1) (black and dashed green lines, respectively).

We find that ρ^* is negative which implies quite weak dependence between the two groups of countries. For comparison, we consider a restricted model (2) with (a) $\rho^* = 0$ (no dynamic dependence between the two groups) and (b) $\rho(t) = 1$ (with no dynamic dependence among variables between the two groups as well as within each group). We obtain BIC values of -26102, -26073 and -26015 for the full and restricted models (a) and (b), respectively, and therefore dynamic dependence within each group of variables cannot be ignored. Fig. 8 shows the estimated dynamic component $\rho_1(t)$ and $\rho_2(t)$ for the two groups of countries.

Copula parameter estimates are reported in Table A.6 in the Appendix. Estimates indicate stronger dependence in the first group with both lower and upper tail dependence and no significant tail asymmetry. Dependence in the upper tail is a bit stronger in the second group.

For both groups of variables, dependence was weaker in 2011-2012 years, and didn't change much otherwise for the first group. We simulate data from the estimated full model and compute within group correlation moving averages for the two groups of countries using the simulated data and rolling window of size 250. Fig. 9 shows the correlation averages, S_ρ, for the original data set (black lines) and the simulated data set (dashed green lines).

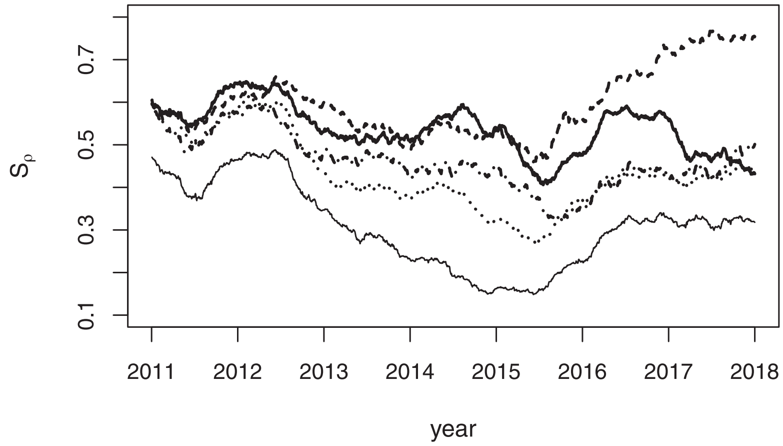

Fig. 10. Spearman correlation averages for the technology, energy, industrials, consumer staples sectors (thick solid, dashed, dotted and dotdash lines, respectively) and the overall average (thin solid line).

It is seen that the estimated model fits the data quite well except for a period of weaker dependence in 2011-2012 years. One reason is that quick changes in dependence over a short period of time cannot be estimated well as we showed in Section 4. Another reason could be a weak predictive power of the three European indexes we selected as external variables.

5.2. CDS spreads data

Let $w_{j,t}$ be the CDS spread of the jth company at time t. We define log-returns $r_{j,t} = \ln(w_{j,t}/w_{j,t-1})$. We use `AR(3)-GARCH(1,1)` model with the skew-t innovations and the VIX volatility index used to model the conditional variance:

$$
\begin{aligned}
r_{j,t} &= \phi_{j,0} + \sum_{k=1}^{3} \phi_{j,k} r_{j,t-k} + \sigma_{j,t}\epsilon_{j,t}, \quad \epsilon_{j,t} \sim \text{skew-}t(\nu,\delta), \text{ and} \\
\sigma^2_{j,t} &= \beta_{j,0} + \beta_{j,1}\sigma^2_{j,t-1} + \beta_{j,2} r^2_{j,t-1} + \beta_{j,3}\mathrm{VIX}_t.
\end{aligned} \tag{6}
$$

We find that this model captures serial dependence well for the CDS data.

We transform residuals from the estimated model (6) for each time series to the uniform data $\{\mathbf{u}_t\}_{t=1}^{2011}$ using non-parametric ranks and use a rolling window of size 250 to compute within group correlation averages for the four groups corresponding to different sectors and overall average. Fig. 10 shows the correlation averages, S_ρ.

It is seen that within group and overall dependence, as measured by S_ρ, is stronger in 2011-2012 and it is weaker in subsequent years except for the energy sector where it becomes stronger in 2017-2018. The assumption that the distribution of $\mathbf{u}_t$ does not change in time is therefore not appropriate for these data. The overall average correlation is smaller than the group averages pointing to stronger dependence among CDS spreads within each group.

We estimate the joint distribution of $\mathbf{u}_t$ using the copula model $C_{\mathbf{U}}$ as defined in (1). Similar to the bond yields data, we assume the nested structure (2) for the correlation matrix $\boldsymbol{\Sigma}_t$. For simplicity we assume that $\rho^*_g = \rho^*$ for $g = 1, \ldots, 4$ and $\eta_m = 0$ for $m = 1, \ldots, M$ to avoid near non-identifiability in this model. We select BB1 linking copulas $C_{j,0}$ for each group and we use daily values of the volatility index VIX, and S&P 500 index as driving variables in (3) to model the dynamic dependence. Larger values of VIX can indicate stronger dependence among different CDS swaps and S&P500 index can be a good predictor of the state of economy as a whole.

For comparison, we also consider two special cases of model (2) with $\rho^* = 0$ and with $\rho(t) = 1$. The first condition implies conditional independence of variables from different groups with no dynamic dependence for these variables. The second condition implies no dynamic dependence for all the variables, both within each group and between different groups. Table 2 shows the results.

Model 1 has the smallest AIC and BIC values which implies dynamic dependence for variables within each group and between different groups can not be ignored for the CDS data set. Fig. 11 shows the estimated dynamic component $\rho_g(t)$ for the four sectors. For CDS swaps from the technology, industrials and consumer staples sector, dependence decreased over the period of 9 years while for the energy sector, dependence increased over the same period.

Copula parameter estimates are reported in Table A.7 in the Appendix. Estimates indicate tail asymmetry with very weak or no lower tail dependence and upper tail dependence for all variables. The estimates of γ_{1g} are positive for all groups of variables which implies that dependence among CDS spreads in these groups is stronger if the volatility is higher, as expected. The estimates of γ_{2g} are negative for all but the second group so that the estimated dependence for the first,

Table 2
AIC and BIC for model (1) with the correlation matrix $\boldsymbol{\Sigma}_t$ with the nested structure (2) and with $\eta_m = 0$ and $\rho_g^* = \rho^*$, $m = 1, \dots, M$, $g = 1, \dots, 4$ (Model 1, first row), special case of Model 1 with $\rho^* = 0$ (Model 2, conditional independence given latent variable, second row) and special case of Model 1 with $\rho(t) = 1$ (Model 3, no dynamic dependence, third row). There are 48 parameters for the 24 BB1 copulas linking to the latent factor and 24 individual correlation parameters for the conditional dependence with the normal copula. For models 1 and 2, there are $4 \times 3 = 12$ regression parameters for $\rho_1(t), \dots, \rho_4(t)$. For models 1 and 3, there is an extra ρ^* parameter for between group dependence.

Model	#parameters	AIC	BIC
Model 1	85	−16840	−16363
Model 2	84	−16418	−15947
Model 3	73	−16250	−15841

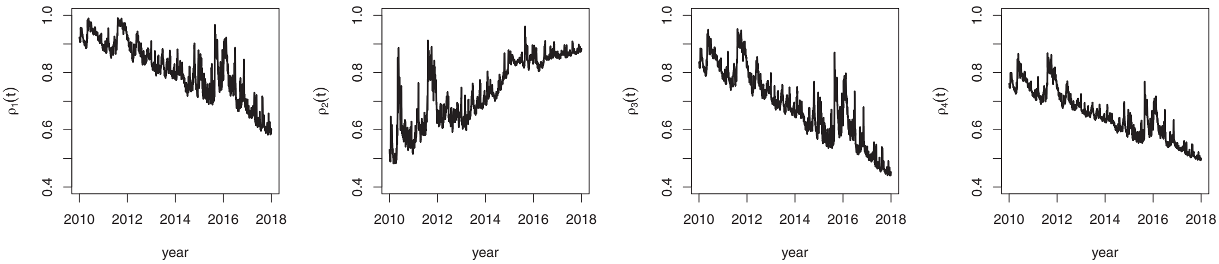

Fig. 11. The estimated dynamic component $\rho_g(t)$ for technology sector, energy sector, industrials sector and consumer staples sector (left to right).

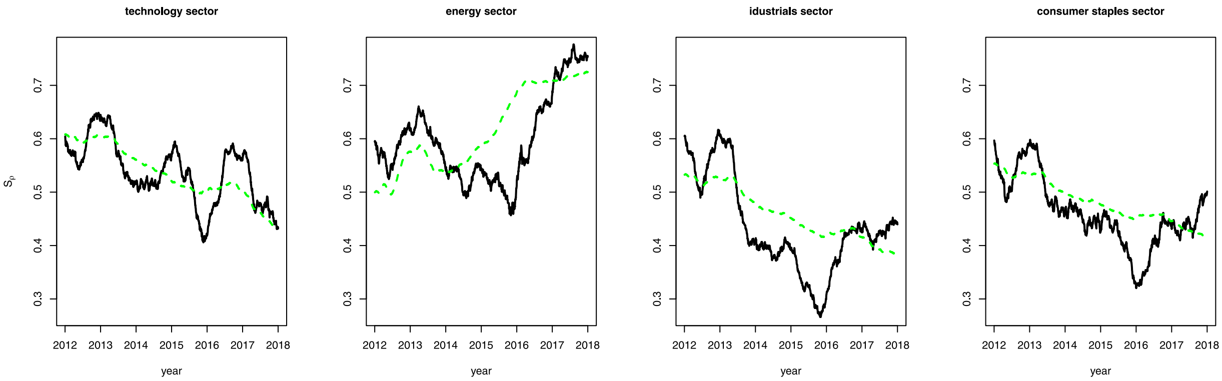

Fig. 12. Left to right: Spearman correlation moving averages obtained using a rolling window of size 250, for the technology, energy, industrials, consumer staples sectors (black lines) and for the data simulated from the estimated Model 1 (dashed green lines).

third and fourth groups is weaker if the value of the S&P 500 index is larger. Lastly, positive estimate of ρ^* indicates positive dynamic dependence between each pair of groups.

Finally, we simulate data from the estimated Model 1 and compute within group correlation moving averages for the four sectors using the simulated data and rolling window of size 250 as we did for the bond yields data; see Fig. 12.

It is seen that Model 1 captures the general trend of dynamic dependence for the four sectors quite well and S&P 500 and VIX indexes can be good predictors for the performance of the CDS spreads of the US companies.

5.3. Stress testing under different scenarios

The estimated models for the first two data sets can be used to construct the projected distribution of bond yields and CDS spreads and estimate different quantities of interest for investors under different scenarios. In this section we analyze how many investments could be in distress simultaneously for a period $0 < T_1 < t \leq T_2 < T_0$ in the past where the data used to fit the model (1) are observed for $0 < t \leq T_0$ with $T_0 = 2265$ for the bond yield data and $T_0 = 2011$ for the CDS spreads data.

Table 3
Estimates of M_f for the bond yields data of 11 countries under different scenarios (assuming model (1) with BB1/ Gaussian linking copulas/ independence). Estimates are reported for different shifts, S_1, S_2 and S_3, of the rescaled observed driving variables $\mathcal{V}_1^* = 10^{-3}\mathcal{V}_1, \mathcal{V}_2^* = 10^{-2}\mathcal{V}_2$ and $\mathcal{V}_3^* = 10^{-3}\mathcal{V}_3$, where $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$ are daily prices of the CAC 40, Euronext 100 and DAX 30 indexes.

T_f	S_1	S_2	S_3	M_f	T_f	S_1	S_2	S_3	M_f
250	−1	−2	−2.5	6.7/6.7/3.6	1250	−1	−2	−2.5	7.2/7.0/3.7
250	−1	−2	+2.5	6.3/6.1/3.6	1250	−1	−2	+2.5	6.7/6.4/3.7
250	+1	+2	−2.5	7.0/7.0/3.6	1250	+1	+2	−2.5	7.4/7.2/3.7
250	+1	+2	+2.5	7.1/6.9/3.6	1250	+1	+2	+2.5	7.3/7.2/3.7

Table 4
Estimates of M_f for the CDS spreads data of 24 US companies under different scenarios (assuming model (1) with BB1/ Gaussian linking copulas/ independence). Estimates are reported for different rescaling factors, S_1, and shifts, S_2, of the observed driving variables $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1, \mathcal{V}_2$ are daily values of the VIX volatility index and S&P 500 index.

T_f	S_1	S_2	M_f	T_f	S_1	S_2	M_f
250	3.50	−1000	11.2/10.7/6.1	1250	3.50	−1000	11.8/11.2/7.8
250	3.50	+1000	10.5/9.9/6.1	1250	3.50	+1000	11.2/10.6/7.8
250	0.28	−1000	9.4/8.9/6.4	1250	0.28	−1000	10.5/10.3/8.7
250	0.28	+1000	8.8/8.4/6.4	1250	0.28	+1000	10.4/9.9/8.7

As larger values of bond yields and CDS spreads indicate higher probability of default, we define a distress event of the j-th variable for both data sets as

$$\{T_1 < t \le T_2 : w_{j,t} > w_j^{95}\}, \quad w_j^{95} \text{ is the 95\% quantile of } \{w_{j,t}, T_1 < t \le T_2\}.$$

We want to estimate $M_f = \max_{T_1 < t \le T_2} M_t$ where M_t is the number of investments in distress at time t. A similar approach is used in H. Oh and Patton (2018) to estimate the probability that a certain number of variables is in distress at a given time t and in Lucas et al. (2014) where the joint probability of default is estimated with the default defined as a CDS spread exceeding some threshold.

We select year 2013 with no significant shocks on the market as a starting point and consider possible alternative outcomes in this and subsequent years under different scenarios. To compare the outcomes for the bond data for positive or negative market shifts relative to year 2013, we consider observed values of the CAC 40, Euronext 100 and DAX 30 indexes and shift them by a constant. For the CDS data, we shift the observed values of the S&P500 and rescale the observed values of the VIX index.

We then use the estimated parameters of $C_{\mathbf{U}}^t$ in (1) to generate $T_f = T_2 - T_1$ vectors $\mathbf{u}_t = (u_{1t}, \dots, u_{dt})^\top$, $t = T_1 + 1, \dots, T_2$. We use marginal models (5) and (6) to generate sequences $w_{j,t}$ of bond yields (with $j = 1, \dots, 11$) and CDS spreads (with $j = 1, \dots, 24$), respectively. For the generated sequences, M_f can be computed. We repeat this procedure 1000 times and then average M_f values. For comparison, we repeat these procedure and compute M_f for the Gaussian model with dynamic correlations which is a special case of (1) with the Gaussian bivariate linking copulas and also in the independence model assuming all variables are independent.

Table 3 shows results for the bond yield data set for different shifts corresponding to weaker or stronger markets. It is seen that M_f is quite large (around 7) in all scenarios and it does not change much for different values of the driving variables. The Gaussian copula with dynamic correlations yields similar results especially for $T_f = 250$, while the independence model significantly underestimates M_f. We obtained similar results for other choices of $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$. Larger values of M_f are unlikely to occur due to very weak dependence between the two groups of countries: distress event in the first group does not affect much the second group and vice versa.

Table 4 shows results for the CDS spreads data set for different shifts of the S&P500 index and rescaled values of the VIX index. The estimated M_f is larger for model (1) with BB1 linking copulas than for the Gaussian copula pointing to stronger dependence in the joint upper tail. Higher volatility or smaller values of the S&P500 index result in larger values of M_f which is reasonable as these two parameters can be good predictors of the US economy. This difference is less pronounced for a longer horizon, $T_f = 1250$, with larger values of M_f under different scenarios. One can conclude that high volatility and/or large negative S&P500 returns have a bigger effect on the probability of default for the US companies over a shorter horizon.

5.4. US stock returns data

In this section we show that our copula model can be used to analyze large number of financial variables over a period of many years, with dependence explainable through a few sector-based latent variables as well as auxiliary variables. We also

Table 5
AIC and BIC for model (1) with the correlation matrix $\boldsymbol{\Sigma}_t$ with the nested structure (2) and with $\eta_m = 0$ and $\rho_g^* = \rho^*$, $m = 1, \ldots, M$, $g = 1, \ldots, 4$ (Model 1, first row) and special case of Model 1 with $\rho(t) = 1$ (Model 2, no dynamic dependence, second row). There are 162 parameters for the 81 BB1 copulas linking to the latent factor and 81 individual correlation parameters for the conditional dependence with the normal copula. For models 1 and 2, there are $4 \times 3 = 12$ regression parameters for $\rho_1(t), \ldots, \rho_4(t)$ and an extra ρ^* parameter for between group dependence.

Model	#parameters	AIC	BIC
Model 1	256	-1.416×10^5	-1.336×10^5
Model 2	244	-1.400×10^5	-1.321×10^5

make some comparisons with the dynamic factor model (DFM) studied by Forni et al. (2005). For stock data, these models assume that the log-returns are stationary and can be expressed as a linear combination of dynamic common factors.

Let $w_{j,t}$ be the stock price of the jth company at time t. Similar to Section 5.2, we define log-returns $r_{j,t} = \ln(w_{j,t}/w_{j,t-1})$. The DFM with q factors assumes the decomposition

$$r_{j,t} = \chi_{j,t} + \epsilon_{j,t}, \quad j = 1, \ldots, d, \quad t = 1, 2, \ldots$$

where $\epsilon_{j,t}$ are idiosyncratic components and $\chi_{j,t} = b_{j1}(L)u_{1t} + \cdots + b_{jq}(L)u_{qt}$, $u_{1t}, \ldots, u_{qt}$ are q common orthonormal dynamic factors, and $b_{j1}(L), \ldots, b_{jq}(L)$ are lag operators. Forni et al. (2015, 2017) proposed and studied estimators of the common factors and Barigozzi and Hallin (2017) constructed volatility estimators based on the DFM.

The DFM is observation driven (autoregressive in the latent variables) and can be used for forecasting and non-tail-based inferences from predictive distributions. In contrast, our model is aimed at explaining features such as time-varying standard deviations, central dependence and tail dependence, with auxiliary or external variables to account for the non-stationarity over time. Inferences regarding systemic risks, such as the number of investments under distress depending on the strength of tail dependence, are therefore not readily available for the DFM.

Dynamic dependence could be introduced into the DFM by allowing the coefficients in the polynomial lag operators to depend on external variables. Tail dependence of $(r_{1t}, \ldots, r_{dt})$ can occur if some of $u_{1t}, \ldots, u_{qt}$ have densities with regularly varying tails and ϵ_{it} have densities with lighter tails. More detailed comparison of DFM and copula-based approach requires further investigation and we leave it as a topic for future research.

Hallin and Liška (2007) proposed an algorithm to determine the number of factors, q, in the DFM. We applied this method to the US stock returns data and extract some results on the factor structure to compare with our approach. Without using sector information, the algorithm found that $q = 4$ factors are selected for the 81 stock returns. Let $\mathbf{B}(L)$ be the $d \times q \times n_{lag}$ array with the $b_{jk}(L)$ elements, where n_{lag} is the numbers of lags that are used. For our data set, $\mathbf{B}(L)$ contains small values for all lags greater than one, so we next summarize the (lag 1) loading matrix of dimension $d \times q$. Factor 1 has loadings around 0.6 for stocks in the consumer staples sector and mostly between 0.2 to 0.4 for the other groups; factor 2 has loadings mostly between 0.1 and 0.2 for the consumer staples sector and mostly between 0.45 and 0.65 for other groups; factor 3 loads on the energy and financials sectors; factor 4 can be interpreted as a contrast of the energy and financials sectors because the loadings that are quite different from 0 have opposite signs for these two groups. The loading matrix can often be rotated to a matrix with more entries closer to 0 while preserving the interpretations of group information. So it appears that the DFM can help in dividing variables into groups if the variables depend on group-based latent variables with unknown group membership.

Barigozzi and Hallin (2017) constructed volatility estimators for DFM; their approach performed favorably compared with other methods. We use VIX as an external variable to explain the time-varying standard deviations so cannot compare with their analysis; our main inference concerns the effect of external variables on strength of tail dependence.

For the proposed copula-based approach, we use `AR(2)-GARCH(1,1)` model with the skew-t innovations and the VIX volatility index to model the conditional variance:

$$\begin{aligned} r_{j,t} &= \phi_{j,0} + \sum_{k=1}^{2} \phi_{j,k} r_{j,t-k} + \sigma_{j,t}\epsilon_{j,t}, \quad \epsilon_{j,t} \sim \text{skew-}t(\nu, \delta), \text{ and} \\ \sigma_{j,t}^2 &= \beta_{j,0} + \beta_{j,1}\sigma_{j,t-1}^2 + \beta_{j,2} r_{j,t-1}^2 + \beta_{j,3}\mathrm{VIX}_t. \end{aligned} \tag{7}$$

We find that this model captures serial dependence well for the US stock data.

Similar to the CDS data, we transform residuals from the estimated model (7) to the uniform data $\{\mathbf{u}_t\}_{t=1}^{2263}$ and estimate the joint distribution of $\mathbf{u}_t$ using the copula model (1). Again, we assume the nested structure (2) for the correlation matrix $\boldsymbol{\Sigma}_t$ and $\rho_g^* = \rho^*$ for $g = 1, \ldots, 4$ and $\eta_m = 0$ for $m = 1, \ldots, M$. We select BB1 linking copulas $C_{j,0}$ for each group and we use daily values of the volatility index VIX as a driving variable for all four groups. We also use daily values of 4 sector indexes (consumer staples, energy, financials, health care sectors) as group-specific driving variables.

For comparison, we consider the special case of (1) with no dynamic dependence, i.e., $\rho(t) = 1$. Table 5 shows the results for Model 1 with dynamic dependence and for Model 2 with no dynamic dependence.

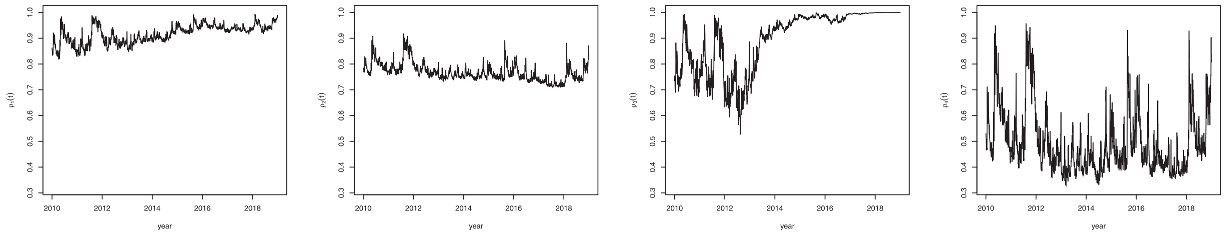

Fig. 13. The estimated dynamic components $\rho_g(t)$ for stock returns from consumer staples sector, energy sector, financials sector and health care sector (left to right).

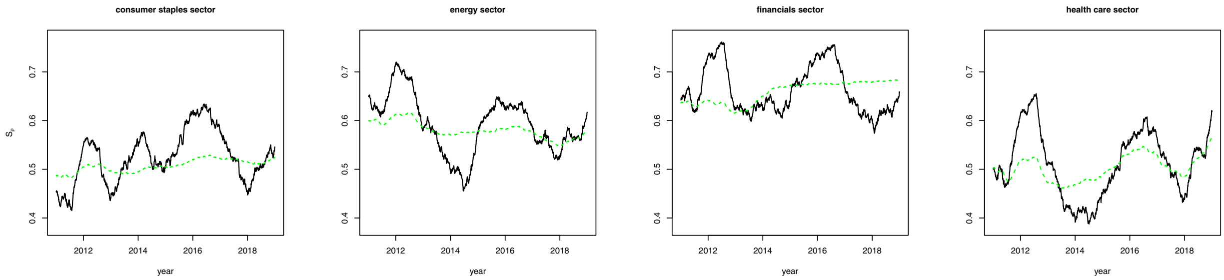

Fig. 14. Left to right: Spearman correlation moving averages obtained using a rolling window of size 250 for the consumer staples, energy, financials, health care sectors (black lines) and for the data simulated from the estimated Model 1 (dashed green lines).

Again, the model with dynamic dependence has the smallest AIC and BIC values. The estimated coefficient ρ^* is close to zero indicating very weak to no dynamic dependence for variables from different groups. Fig. 13 shows estimated dynamic component $\rho_g(t)$ for the four sectors. The strength of dependence changed for variables from all sectors, with no clear trend over time.

Copula parameter estimates are reported in Table A.8 in the Appendix. Estimates indicate both lower and upper tail dependence for all variables, with stronger dependence among variables from the financials sector and with stronger lower tail dependence among variables from the health care sector. Positive γ coefficients imply stronger dependence if the volatility or value of the sector index is higher.

Using the estimated parameters, we simulate data from Model 1 and compute within group correlation moving averages for the four sectors using the simulated data and rolling window of size 250; see Fig. 14.

Model 1 fits the dynamic correlation structure quite well, except for the financials sector where the range of dependence is narrower. Sector information can therefore be useful when estimating dynamic dependence among US stock returns.

6. Discussion

We proposed a copula-based model for modeling non-Gaussian data with dynamic dependence. The model allows greater flexibility when modeling data with heterogeneous dynamic dependence, in particular when dependence changes differently for different groups of variables. Parameters in the model can be estimated using the maximum likelihood approach even for high-dimensional data sets. The driving observed variables are used to model dynamic dependence and this allows interpretability. One example is financial data where market indexes or different macroeconomic parameters can be good predictors of the market. We discussed possible extensions of the proposed model and applied it to analyze two financial data sets. We found that S&P500, VIX and sector indexes can be good predictors for CDS spreads and stock returns of the US companies and that European market indexes might not be able to capture well quickly changing dependence among European government bonds.

Declaration of Competing Interest

We have no conflicts of interest.

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Appendix A. Parameters estimates of C_U^t for the three data sets

Tables A.6–A.8 show BB1 linking copula parameter estimates $\boldsymbol{\theta}_{ig} = (\theta_{ig}^1, \theta_{ig}^2)$, correlation matrix parameters α_{ig}, ψ_{0g}, γ_{ig} and ρ^* for the three data sets from Section 5. Here, θ_{ig}^1 and θ_{ig}^2 are the first and second parameters of the BB1 copula, respectively, α_{ig} are correlation parameters from the nested model in Section 2.2 and ψ_{0g}, γ_{ig} and ρ^* are parameters of the dynamic component $\rho_g(t)$ in (3). We also report lower and upper tail dependence coefficients for the estimated BB1 linking copulas, λ_L and λ_U.

Table A.6
Parameter estimates of C_U^t for the government bond yield data set and lower and upper tail dependence coefficients for BB1 linking copulas. Estimates are reported for the rescaled driving variables $\mathcal{V}_1^* = 10^{-3}\mathcal{V}_1$, $\mathcal{V}_2^* = 10^{-2}\mathcal{V}_2$ and $\mathcal{V}_3^* = 10^{-3}\mathcal{V}_3$, where $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$ are daily prices of the CAC 40, Euronext 100 and DAX 30 indexes. Standard errors are shown in parentheses.

	group 2						
θ_{ig}^1	0.51(0.14)	0.52(0.08)	0.81(0.34)	0.63(0.14)			
θ_{ig}^2	2.53(0.70)	2.86(0.40)	3.42(0.69)	3.39(0.64)			
α_{ig}	−0.64(0.16)	0.74(0.09)	−0.45(0.32)	0.89(0.08)			
λ_L	0.58	0.63	0.78	0.72			
λ_U	0.68	0.73	0.78	0.77			
	ψ_{01}	γ_{11}	γ_{21}	γ_{31}			
	−2.27(0.04)	−3.72(0.06)	0.68(0.07)	2.30(0.07)			
	group 1						
θ_{ig}^1	0.89(0.12)	0.36(0.04)	0.38(0.06)	0.03(0.03)	0.33(0.09)	0.12(0.08)	0.28(0.09)
θ_{ig}^2	3.62(0.35)	1.72(0.07)	1.76(0.10)	1.06(0.03)	1.34(0.12)	1.19(0.07)	1.29(0.12)
α_{ig}	0.42(0.35)	0.35(0.08)	0.36(0.10)	0.38(0.05)	0.93(0.03)	0.65(0.05)	0.95(0.02)
λ_L	0.81	0.33	0.36	0.00	0.20	0.01	0.14
λ_U	0.79	0.50	0.52	0.07	0.32	0.21	0.29
	ψ_{02}	γ_{12}	γ_{22}	γ_{32}			
	−6.82(0.09)	0.18(0.11)	2.38(0.12)	−1.11(0.13)			
	ρ^*						
	−0.61(0.11)						

Table A.7
Parameter estimates of C_U^t for the US CDS data set and lower and upper tail dependence coefficients for BB1 linking copulas. Estimates are reported for the rescaled driving variables $\mathcal{V}_1^* = 10^{-2}\mathcal{V}_1$, $\mathcal{V}_2^* = 10^{-3}\mathcal{V}_2$, where $\mathcal{V}_1, \mathcal{V}_2$ are daily values of the VIX volatility index and S&P 500 index. The order of magnitude of the standard errors is 0.01 – 0.09 for estimates of $\theta_{ig}^1, \theta_{ig}^2$ and α_{ig} and 0.02 – 0.13 for estimates of ψ_{0g}, γ_{lg} and ρ^*.

	group 1					group 2				group 4			
θ_{ig}^1	0.05	0.07	0.06	0.13	0.17	0.06	0.04	0.07	0.05	0.05	0.05	0.07	0.01
θ_{ig}^2	1.27	1.21	1.23	1.15	1.19	1.18	1.18	1.19	1.24	1.19	1.19	1.19	1.22
α_{ig}	0.67	0.77	0.70	0.78	0.80	0.99	0.92	0.94	0.78	0.77	0.89	0.66	0.79
λ_L	0.00	0.00	0.00	0.01	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
λ_U	0.27	0.23	0.24	0.17	0.21	0.20	0.20	0.21	0.25	0.21	0.21	0.21	0.23
ρ^*	ψ_{01}	γ_{11}	γ_{21}			ψ_{02}	γ_{12}	γ_{22}		ψ_{04}	γ_{14}	γ_{24}	
0.47	1.03	7.72	−0.71			−1.67	8.06	1.43		0.71	2.55	−0.49	
	group 3												
θ_{ig}^1	0.18	0.16	0.27	0.16	0.37	0.15	0.58	0.59	0.02	0.19	0.17		
θ_{ig}^2	1.34	1.18	1.46	1.29	1.67	1.23	1.64	1.58	1.30	1.20	1.26		
α_{ig}	0.58	0.99	0.55	0.50	0.58	0.99	0.58	0.58	0.65	0.99	0.69		
λ_L	0.06	0.02	0.17	0.04	0.33	0.03	0.48	0.48	0.00	0.05	0.04		
λ_U	0.32	0.20	0.39	0.29	0.49	0.25	0.47	0.45	0.30	0.22	0.27		
	ψ_{03}	γ_{13}	γ_{23}										
	0.76	4.80	−0.73										

Table A.8

Parameter estimates of C_U^t for the US stock returns data set and lower and upper tail dependence coefficients for BB1 linking copulas. Estimates are reported for the rescaled driving variables $\mathcal{V}_0^* = 10^{-1}\mathcal{V}_0, \mathcal{V}_g^* = 5 \cdot 10^{-3}\mathcal{V}_g,\ g = 1, \ldots, 4,$ where $\mathcal{V}_0, \mathcal{V}_1, \ldots, V_4$ are daily values of the VIX volatility index and consumer staples, energy, financials and health care sector indexes. The order of magnitude of the standard errors is 0.01 – 0.09 for estimates of $\theta_{ig}^1, \theta_{ig}^2$ and α_{ig} and 0.02 – 0.13 for estimates of ψ_{0g}, γ_{lg} and ρ^*.

group 1																								
θ_{i1}^1	0.33	0.44	0.36	0.32	0.39	0.35	0.41	0.45	0.50	0.36	0.43	0.31	0.41	0.31	0.49	0.41	0.45							
θ_{i1}^2	1.27	1.28	1.31	1.26	1.33	1.31	1.25	1.37	1.33	1.20	1.24	1.29	1.35	1.37	1.31	1.15	1.32							
α_{i1}	0.68	0.59	0.74	0.63	0.61	0.64	0.74	0.69	0.56	0.40	0.59	0.63	0.63	0.63	0.55	0.31	0.62							
λ_L	0.19	0.28	0.23	0.17	0.26	0.22	0.25	0.32	0.35	0.20	0.27	0.17	0.28	0.20	0.33	0.23	0.31							
λ_U	0.27	0.27	0.30	0.26	0.31	0.30	0.25	0.34	0.31	0.21	0.24	0.28	0.32	0.34	0.29	0.17	0.30							
group 2																								
θ_{i2}^1	0.46	0.40	0.35	0.50	0.46	0.39	0.50	0.51	0.47	0.51	0.43	0.41	0.54	0.24	0.48	0.38	0.44	0.47	0.50	0.44	0.44	0.39	0.38	0.50
θ_{i2}^2	1.30	1.29	1.17	1.45	1.23	1.25	1.37	1.26	1.28	1.46	1.33	1.29	1.31	1.23	1.32	1.34	1.30	1.37	1.25	1.27	1.41	1.33	1.29	1.24
α_{i2}	0.96	0.80	0.62	0.83	0.90	0.94	0.94	0.99	0.99	0.75	0.91	0.91	0.97	0.36	0.94	0.84	0.94	0.88	0.63	0.96	0.92	0.81	0.34	0.66
λ_L	0.31	0.26	0.18	0.39	0.30	0.24	0.37	0.34	0.31	0.39	0.30	0.27	0.38	0.09	0.34	0.25	0.30	0.34	0.33	0.29	0.33	0.26	0.25	0.32
λ_U	0.29	0.29	0.19	0.39	0.25	0.26	0.34	0.27	0.28	0.39	0.32	0.29	0.30	0.24	0.31	0.32	0.30	0.34	0.26	0.27	0.36	0.31	0.29	0.25
group 3																								
θ_{i3}^1	0.63	0.58	0.65	0.65	0.62	0.71	0.67	0.73	0.55	0.61	0.67	0.50	0.88	0.67	0.59	0.52	0.76	0.64	0.69	0.72	0.74			
θ_{i3}^2	1.51	1.49	1.45	1.43	1.50	1.53	1.39	1.46	1.41	1.42	1.78	2.16	1.72	1.51	1.67	2.00	2.02	1.87	2.04	2.00	1.82			
α_{i3}	0.83	0.84	0.85	0.86	0.75	0.82	0.87	0.86	0.67	0.83	0.06	0.09	0.27	0.00	−0.03	0.15	0.42	0.54	0.41	0.55	0.47			
λ_L	0.48	0.44	0.48	0.48	0.47	0.53	0.47	0.52	0.41	0.45	0.56	0.52	0.63	0.50	0.49	0.51	0.63	0.56	0.61	0.62	0.60			
λ_U	0.41	0.40	0.39	0.38	0.41	0.43	0.35	0.39	0.36	0.37	0.52	0.62	0.50	0.41	0.48	0.58	0.59	0.55	0.59	0.58	0.53			
group 4																								
θ_{i4}^1	0.60	0.36	0.69	0.52	0.58	0.60	0.75	0.43	0.57	0.51	0.45	0.63	0.64	0.42	0.58	0.58	0.66	0.55	0.56					
θ_{i4}^2	1.29	1.17	1.39	1.29	1.35	1.20	1.47	1.15	1.20	1.23	1.19	1.33	1.40	1.24	1.37	1.30	1.43	1.34	1.34					
α_{i4}	0.72	0.53	0.74	0.74	0.83	0.76	0.65	0.69	0.67	0.70	0.76	0.83	0.75	0.60	0.85	0.72	0.80	0.78	0.77					
λ_L	0.40	0.20	0.48	0.36	0.41	0.38	0.53	0.25	0.36	0.33	0.28	0.43	0.46	0.27	0.42	0.39	0.48	0.39	0.39					
λ_U	0.29	0.19	0.35	0.29	0.33	0.22	0.40	0.18	0.22	0.24	0.21	0.31	0.36	0.25	0.34	0.29	0.37	0.32	0.32					

ρ^*	ψ_{01}	γ_{11}	γ_{21}	ψ_{02}	γ_{12}	γ_{22}	ψ_{03}	γ_{13}	γ_{23}	ψ_{04}	γ_{14}	γ_{24}
0.07	−1.50	0.89	1.12	0.34	0.39	0.07	−7.86	1.61	5.73	−2.48	1.09	0.42

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