Journal of Applied Statistics

ISSN: 0266-4763 (Print) 1360-0532 (Online) Journal homepage: www.tandfonline.com/journals/cjas20

Tail-weighted measures of dependence

Pavel Krupskii & Harry Joe

Journal of Applied Statistics, 2015
Vol. 42, No. 3, 614–629, http://dx.doi.org/10.1080/02664763.2014.980787

Taylor & Francis
Taylor & Francis Group

Tail-weighted measures of dependence

Pavel Krupskii* and Harry Joe

Department of Statistics, University of British Columbia, Vancouver, Canada

(*Received 11 February 2014; accepted 22 October 2014*)

Multivariate copula models are commonly used in place of Gaussian dependence models when plots of the data suggest tail dependence and tail asymmetry. In these cases, it is useful to have simple statistics to summarize the strength of dependence in different joint tails. Measures of monotone association such as Kendall's tau and Spearman's rho are insufficient to distinguish commonly used parametric bivariate families with different tail properties. We propose lower and upper tail-weighted bivariate measures of dependence as additional scalar measures to distinguish bivariate copulas with roughly the same overall monotone dependence. These measures allow the efficient estimation of strength of dependence in the joint tails and can be used as a guide for selection of bivariate linking copulas in vine and factor models as well as for assessing the adequacy of fit of multivariate copula models. We apply the tail-weighted measures of dependence to a financial data set and show that the measures better discriminate models with different tail properties compared to commonly used risk measures – the portfolio value-at-risk and conditional tail expectation.

Keywords: copula; dependence measure; factor model; intermediate tail dependence; tail asymmetry; tail dependence

AMS Subject Classification: 62H20

1. Introduction

When multivariate data show significant departures from Gaussianity, models with flexible dependence structure and tail properties are required. In this regard, copula models are becoming more popular for the modeling dependence structure of multidimensional data such as stock or exchange rate returns; see [14,23] and others. The copula is a multivariate cumulative distribution function (cdf) with $U(0,1)$ margins, and can be used to summarize the dependence in a joint cdf. By Sklar [26], for a continuous d-dimensional cdf F with univariate margins $F_1,\dots,F_d$, there exists a unique copula C such that $F(x_1,\dots,x_d) = C(F_1(x_1),\dots,F_d(x_d))$.

Simple multivariate copula families, such as Archimedean, do not have flexible dependence structure. To get flexibility, a popular approach is to use a sequence of algebraically independent bivariate copulas applied to either marginal or univariate conditional distributions. Approaches that use a sequence of bivariate copulas include vine pair-copula constructions [18] and factor

*Corresponding author. Email: krupskiiPV@yandex.ru

copulas [16]; the latter are vines with latent variables. These classes of copulas have been used for finance and other applications because they can cover a wide range of dependence and tail behavior; see for example, [1,3,21].

Appropriate choices of bivariate copulas used in the vine and factor copula models are very important. Inappropriate choices can cause the model to provide incorrect inference on joint tail probabilities. This is crucial in some applications such as quantitative risk analysis in finance and insurance where good estimates of joint tail probabilities are essential. In this paper, we propose tail-weighted measures of dependence which can be used to distinguish different bivariate copula families through their strength of dependence in the joint tails. The tail-weighted measures provide additional information to commonly used monotone dependence measures such as Kendall's τ and Spearman's ρ_S, because a single dependence measure, which is dominated by the copula density in the center part of the unit square, cannot summarize all of the important dependence and tail properties of the copula.

Standard measures of tail dependence used in the copula literature are tail dependence coefficients. For a bivariate copula C, they are defined as limiting quantities:

$$\lambda_L = \lim_{q\downarrow 0} \frac{C(q,q)}{q} \quad \text{and} \quad \lambda_U = \lim_{q\downarrow 0} \frac{\bar{C}(1-q,1-q)}{q}, \tag{1}$$

where $\bar{C}(u_1,u_2) = 1 - u_1 - u_2 + C(u_1,u_2)$ is the survival function. Because they are defined through limits, the true sample versions of λ_L and λ_U are not available. Frahm *et al.* [6] discuss the difficulty of estimating the tail dependence parameters. Dobrić and Schmid [4] have 'empirical versions' of λ_L and λ_U for data based on values of q in Equation (1) that are small positive numbers, but they do not necessarily work well. Furthermore, tail dependence coefficients cannot distinguish copulas when one or both tail dependence coefficients equal zero.

Hua and Joe [10] assess tail dependence and asymmetry based on the rates that $C(u,u)$ and $\bar{C}(1-u,1-u)$ go to 0 as $u \to 0$. They define the tail order that is the reciprocal of a quantity used in [9,19]. It measures the strength of dependence in the joint lower and upper tails. If $C(u,u) \sim \ell_L(u)u^{\kappa_L}$ as $u \to 0$, where $\ell_L(u)$ is a slowly varying function, then we say that the lower tail order is κ_L. Similarly, if $\bar{C}(1-u,1-u) \sim \ell_U(u)u^{\kappa_U}$ as $u \to 0$, where $\ell_U(u)$ is a slowly varying function, then the upper tail order is κ_U. If $1 < \kappa_L < 2$ (or $1 < \kappa_U < 2$) then $\lambda_L = 0$ ($\lambda_U = 0$, respectively), and this is termed intermediate tail dependence in [10]. A smaller value of the tail order corresponds to more tail dependence (more probability in the joint tail). Then, the strongest dependence in the tail corresponds to $\kappa_L = 1$ or $\kappa_U = 1$. The tail order can be used to assess tail dependence strength, if tail dependence coefficient is zero. Nevertheless, tail orders are also defined through limits and good sample-based estimates have not been proposed for data.

We propose bivariate tail-weighted measures of dependence which are defined as correlations of transformed variables; they have easily computed sample versions. The measures can be applied to each pair of variables in a multidimensional data set. These measures put more weight in the tails and thus can efficiently estimate the strength of dependence in the tail even if the sample size is not very large.

Note that the purpose of these tail-weighted measures of dependence to assess models is quite different in aim than copula goodness-of-fit procedures; Huang and Prokhorov [11], Schepsmeier [24] have developed likelihood-based copula goodness-of-fit tests and Genest *et al.* [8] have an overview of bivariate goodness-of-fit tests for copulas. However, these previously proposed goodness-of-fit procedures are not diagnostic in suggesting improved models when the P-values are small. For preliminary data analysis, the tail-weighted measures of dependence can provide a guide to the choice of bivariate copulas to use in vine and factor models. After fitting competing copula models, the comparison of empirical and model-based estimates is a method to assess whether the best-fitting copula models based on AIC or BIC are adequate for tail inference or whether alternative models should be considered. Different copula models with a similar dependence structure can typically perform similarly in the comparison of empirical versus model-based measure of monotone association such Spearman's rho, but they can differ more for tail inference. This is because while the dependence structure, which is based mostly on the middle of the data, dominates the log-likelihood of parameter copula models, a secondary contribution to the log-likelihood is based on the quality of fit in the tails. Most parametric copula families, including Gaussian, have densities that asymptote to infinity at one of more corners of the hypercube, and hence parametric copula families with different strengths of dependence in the tails (or different rates of asymptotes to infinity) can perform quite differently when assessed by tail-weighted dependence measures.

The rest of the paper is organized as follows. In Section 2, we give the formal definition of the tail-weighted measures of dependence in terms of a weighting function $a(\cdot)$ that satisfies some properties, and discuss some desirable properties for these measures. Empirical versions and asymptotic properties of the measures are presented in Section 3; also some details are shown on numerical computation of the measures for a given bivariate copula and a Monte Carlo simulation study is conducted to compare measures of dependence based on different choices of the function a. In Section 4, we apply the tail-weighted measures of dependence to a data set of financial returns to investigate the dependence structure and compare different copula models using the model-based estimates of the measures. Section 5 concludes with a summary.

2. Tail-weighted measures of dependence

Let $(U_1, U_2) \sim C$ where C is a bivariate copula. Let $a(\cdot) : [0, 1] \to (0, \infty)$ be a continuous function and let p be a truncation level satisfying $0 < p \le 0.5$. Define

$$\varrho_L(a, p) = \mathrm{Cor}\left[a\left(1 - \frac{U_1}{p}\right),\ a\left(1 - \frac{U_2}{p}\right) \middle| U_1 < p,\ U_2 < p \right],$$
$$\varrho_U(a, p) = \mathrm{Cor}\left[a\left(1 - \frac{1 - U_1}{p}\right),\ a\left(1 - \frac{1 - U_2}{p}\right) \middle| 1 - U_1 < p,\ 1 - U_2 < p \right].$$

If $a(0) = 0$ and $a(\cdot)$ is monotonically increasing, then for ϱ_L, $a(1 - U_j/p)$ is decreasing as U_j increases from 0 to p, and for ϱ_L, $a(1 - [1 - U_j]/p)$ is increasing as U_j increases from $1 - p$ to 1; in either case, more weight is given to the joint tail. The symbol C in ϱ_L, ϱ_U will be suppressed unless we are referring to more than one copula. ϱ_L and ϱ_U are bivariate tail-weighted measures of dependence in the joint lower and upper tails, respectively. They are defined as correlations of transformed variables and have sample versions for data. Unlike tail dependence coefficients λ, the measures are not defined as limits so that empirical estimates of $\varrho_L(a, p), \varrho_U(a, p)$ are more reliable than something like the estimate of λ's in [4].

With some choices of the function $a(\cdot)$, the measures $\varrho_L(a, p)$ and $\varrho_U(a, p)$ include some dependence measures that have been previously used in the literature as special cases. Schmid and Schmidt [25] define conditional Spearman's rhos:

$$\rho_L = \mathrm{Cor}(U_1, U_2 | U_1 < 0.5, U_2 < 0.5), \quad \rho_U = \mathrm{Cor}(U_1, U_2 | U_1 > 0.5, U_2 > 0.5), \tag{2}$$

which corresponds to $a(v) = v$ and $p = 0.5$. Another example consists of the semi-correlations of normal scores, defined as

$$\begin{aligned} \rho_N^- &= \mathrm{Cor}(\Phi^{-1}(U_1), \Phi^{-1}(U_2) | U_1 < 0.5, U_2 < 0.5), \\ \rho_N^+ &= \mathrm{Cor}(\Phi^{-1}(U_1), \Phi^{-1}(U_2) | U_1 > 0.5, U_2 > 0.5); \end{aligned} \tag{3}$$

these are obtained with $a(v) = \Phi^{-1}(0.5(1 + v))$ and $p = 0.5$ where Φ denotes the standard normal cdf. That is, ρ_N^- and ρ_N^+ are correlations of truncated data transformed using the inverse standard normal cdf. These measures were employed in some publications to study the comovements of financial assets; see, for example, [2,7]. The semi-correlations of normal scores naturally follow as a diagnostic from [21], where it is advocated to plot pairs of variables after a normal score transform in order to check for deviations from the elliptical-shaped scatterplot expected for the bivariate Gaussian copula.

The conditional Spearman's rho might be not very sensitive to the strength of dependence in the tails of a bivariate copula, and we would like tail-weighted dependence measures that are faster to compute than the semi-correlations of normal scores when there are $\binom{d}{2}$ bivariate margins and d is large. In the next section, we propose better choices for the weighting function $a(\cdot)$ so that the corresponding tail-weighted measures of dependence $\varrho_L(a, p)$, $\varrho_U(a, p)$ can more efficiently differentiate copula families, either analytically or numerically or both. We restrict our attention to monotone weighting functions a to ensure the following properties:

$$
\begin{aligned}
&(1)\quad \varrho_L(a,p) = 1 \quad \text{iff } U_1 \cdot I_L = U_2 \cdot I_L, \quad \varrho_U(a,p) = 1 \quad \text{iff } U_1 \cdot I_U = U_2 \cdot I_U,\\
&(2)\quad \varrho_L(a,p) = -1 \quad \text{iff } U_1 \cdot I_L = (p - U_2) \cdot I_L,\\
&\qquad \varrho_U(a,p) = -1 \quad \text{iff } U_1 \cdot I_U = (2 - p - U_2) \cdot I_U,\\
&(3)\quad \varrho_L(a,p) = 0 \quad \text{if } U_1 \cdot I_L \perp\!\!\!\perp U_2 \cdot I_L, \quad \varrho_U(a,p) = 0 \quad \text{if } U_1 \cdot I_U \perp\!\!\!\perp U_2 \cdot I_U,
\end{aligned}
$$

where $I_L = I\{U_1 < p,\ U_2 < p\}$ and $I_U = I\{U_1 > 1 - p,\ U_2 > 1 - p\}$. Monotonicity of the weighting function is important to guarantee that the maximum (minimum) value of the measure can only be achieved when there is perfect comonotonic (countermonotonic, respectively) dependence in the tail. Then, we expect large (small) values of $\varrho_L(a, p)$ and $\varrho_U(a, p)$ to indicate that dependence in the tails of C is strong (weak, respectively).

Note that these tail-weighted dependence measures (and likewise the tail dependence parameters) do not satisfy some properties of positive dependence measures as given in [15]; for example, they are not defined for some copulas such as the countermonotonic copula that has zero probability on the set $U_1 < p$, $U_2 < p$ with $0 < p < 0.5$. The criterion of tail weighting is to efficiently provide additional information to the commonly used monotone dependence measures, because a single dependence measure cannot adequately summarize dependence and tail properties of a bivariate copula. Some theory on how to choose the weight function $a(\cdot)$ is given in the next section.

3. Empirical version of tail-weighted measures and the choice of the weighting function

Let $\{(X_{i1}, X_{i2})\}_{i=1}^n$ be a sample of size n generated from a bivariate distribution $F = C(F_1, F_2)$. Since the marginal distributions of X_{i1} and X_{i2} may not be uniform, the data can be converted to uniform scores. Define $R_{i1} = [\text{rank}(X_{i1}) - 0.5]/n$ and $R_{i2} = [\text{rank}(X_{i2}) - 0.5]/n$ via increasing ranks. The scores defined in this way provide a better approximation to a uniform random variable as they preserve the first moment and have the second moment close to 1/12. The empirical estimates of $\varrho_L(a, p)$ and $\varrho_U(a, p)$ can be defined as the sample correlations of the ranked data:

$$
\hat{\varrho}_L(a,p) = \widehat{\text{Cor}}\left[a\left(1 - \frac{R_{i1}}{p}\right),\ a\left(1 - \frac{R_{i2}}{p}\right) \middle|\, R_{i1} < p,\ R_{i2} < p \right],
$$

$$
\hat{\varrho}_U(a,p) = \widehat{\text{Cor}}\left[a\left(1 - \frac{1 - R_{i1}}{p}\right),\ a\left(1 - \frac{1 - R_{i2}}{p}\right) \middle|\, 1 - R_{i1} < p,\ 1 - R_{i2} < p \right],
$$

where the notation $\widehat{\mathrm{Cor}}[y_{i1}, y_{i2}|(y_{i1}, y_{i2}) \in B]$ is the shorthand for

$$\frac{\sum_{i\in J_B} y_{i1}y_{i2} - n_B^{-1}\sum_{i\in J_B} y_{i1}\sum_{i\in J_B} y_{i2}}{\left[\sum_{i\in J_B} y_{i1}^2 - n_B^{-1}\left(\sum_{i\in J_B} y_{i1}\right)^2\right]^{1/2}\left[\sum_{i\in J_B} y_{i2}^2 - n_B^{-1}\left(\sum_{i\in J_B} y_{i2}\right)^2\right]^{1/2}},$$

with $J_B = \{i : (y_{i1}, y_{i2}) \in B\}$ and n_B is the cardinality of J_B.

Under mild conditions on the weighting function $a(\cdot)$, these estimates are asymptotically normal as stated next.

PROPOSITION 1 *Let $a(\cdot)$ be a continuously differentiable function on $[0, 1]$ such that $a(0) = 0$ and let C be a bivariate copula cdf with continuous partial derivatives of the first order. Then, both $\hat{\varrho}_L(a, p)$ and $\hat{\varrho}_U(a, p)$ are asymptotically normal.*

Proof See the appendix. ■

Good choices of the truncation level p and the weighting function $a(\cdot)$ are important for the measure to discriminate well copula families with different tail properties. Without loss of generality, we now restrict our attention to the lower tail. More formally, assume that we have two samples from bivariate copulas C_1 and C_2 with the same value of Spearman's ρ_S but with different tail properties. We want the difference

$$\Delta(a, p; C_1, C_2) = \varrho_L(a, p; C_1) - \varrho_L(a, p; C_2) \tag{4}$$

to be large in absolute value and the variance $\sigma^2(\hat{\Delta})$ of the estimate $\hat{\Delta}(a, p; C_1, C_2) = \hat{\varrho}_L(a, p; C_1) - \hat{\varrho}_L(a, p; C_2)$ to be small so that the large absolute value of $\hat{\Delta}(a, p; C_1, C_2)$ does not have a big standard error. This is also important for assessing how different copula models fit data in the tails. When likelihood maximization is used to estimate copula parameters, the model-based estimates of Spearman's ρ_S or other measure of overall monotone dependence can be about the same for different models. The reason is that the overall dependence characteristics such as Spearman's ρ_S are often estimated fairly well by the likelihood regardless of the copula choice but tail characteristics can be estimated very poorly.

It is seen that in the case of copula C with comonotonic dependence in the lower tail, we get $\varrho_L(a, p; C) = 1$ and the variance of the sampling distribution is $\sigma^2(\hat{\varrho}_L) = 0$. With the independence copula $C^{\perp}(u, v) = uv$, we get $\varrho_L(a, p; C^{\perp}) = 0$ and $\sigma^2(\hat{\varrho}_L) = 1/p^2$ as follows from the next result.

PROPOSITION 2 *Let $a(\cdot)$ be a weighting function satisfying the conditions of Proposition 1 and $C^{\perp}$ be bivariate independence copula. It follows that the asymptotic variance $\sigma^2(\hat{\varrho}_L(a, p; C^{\perp})) = 1/p^2$.*

Proof See the appendix. ■

This result implies that with weak to moderate overall dependence as measured by Spearman's ρ_S or Kendall's τ, it is preferable to use larger value of the truncation level p as the standard errors of the estimates for ϱ_L are smaller and the weighting function can be chosen to make the difference $\Delta(a, p; C_1, C_2)$ large in absolute value for C_1, C_2 with different lower tail properties. This is especially important if the sample size is small so that smaller values of p result in a very small truncated sample size to estimate ϱ_L.

To illustrate these ideas, we consider the power weighting functions $a(u) = u^k$ with $k = 1, 2, \ldots$. The power weighting functions satisfy all conditions of Proposition 1 and, with larger

k, more weight is put in the joint tail, whereas with small values of k even those points far from the tail receive quite large weights. To compute $\varrho_L(a,p;C)$ for a given bivariate copula C, one can use the formula:

$$\varrho_L(a,p;C) = \frac{C(p,p)m_{12} - m_1 m_2}{[\{C(p,p)m_{11} - m_1^2\}\{C(p,p)m_{22} - m_2^2\}]^{1/2}},$$

where

$$m_{12} = \frac{1}{p^2}\int_0^p \int_0^p a'\left(1 - \frac{u_1}{p}\right) a'\left(1 - \frac{u_2}{p}\right) C(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2,$$

$$m_1 = \frac{1}{p}\int_0^p a'\left(1 - \frac{u_1}{p}\right) C(u_1,p)\,\mathrm{d}u_1, \quad m_2 = \frac{1}{p}\int_0^p a'\left(1 - \frac{u_2}{p}\right) C(p,u_2)\,\mathrm{d}u_2,$$

$$m_{11} = \frac{1}{p}\int_0^p 2a\left(1 - \frac{u_1}{p}\right) a'\left(1 - \frac{u_1}{p}\right) C(u_1,p)\,\mathrm{d}u_1,$$

$$m_{22} = \frac{1}{p}\int_0^p 2a\left(1 - \frac{u_2}{p}\right) a'\left(1 - \frac{u_2}{p}\right) C(p,u_2)\,\mathrm{d}u_2;$$

see the proof of Proposition 1 for details. All integrands here are bounded functions so that numerical integration should be fast and stable. With some other weighting functions, such as the semi-correlations of normal scores, the integrands are unbounded so that numerical integration can be slower. This is especially important if one wants to compute the model-based estimates of the measures for each pair of variables in a multidimensional data set.

In practice using copulas with intermediate dependence (tail order $1 < \kappa < 2$ based on [10]) or tail quadrant independence ($\kappa = 2$ and the slowly varying function is a constant) can lead to incorrect inferences in the tails if the true copula is tail dependent. It means that a good measure of tail-weighted dependence should discriminate well copulas with tail dependence and copulas that are not tail dependent. We use the following bivariate parametric copula families for comparisons in the tails.

- Student t copula with 4 degrees of freedom: This is a reflection symmetric tail-dependent copula.
- Gaussian copula: This is a reflection symmetric copula with intermediate dependence for $0 < \rho < 1$. The tail order is $\kappa_L = 2/(1+\rho)$ and it gets closer to 1 as ρ increases. Therefore, it is harder to discriminate the Student t and Gaussian copulas if the overall dependence is strong.
- Gumbel copula: This is a tail asymmetric copula with intermediate lower tail dependence; the tail order is $\kappa_L = 2^{1/\theta}$ where $\theta > 1$ is the copula dependence parameter.
- Reflected Gumbel (rGumbel) copula: This is a tail asymmetric copula with lower tail dependence.
- Frank copula: This is a reflection symmetric copula with tail quadrant independence.
- BB1 copula: This is a tail asymmetric and tail-dependent copula.

More details on these copulas can be found in [12,20]. Results are how tail properties of the bivariate linking copulas affect the tail properties of the bivariate margins of vine and factor copulas are given in [13,16].

The various tail-weighted dependence measures can discriminate strength of dependence in the tail somewhat like the (limiting) tail order, since the t_4 and BB1 copulas have lower and upper tail order of 1, the Gaussian copula has lower and upper tail order in $(1,2)$ for $0 < \rho < 1$, the Gumbel copula has $\kappa_U = 1$ and $\kappa_L \in (1,2)$ and the Frank copula has $\kappa_L = \kappa_U = 2$; see

Table 1. Tail-weighted dependence measures: ρ_L and ρ_U (conditional Spearman's rhos), ρ_N^- and ρ_N^+ (semi-correlations), and ϱ_L, ϱ_U with $a(u) = u^6, p = 0.5$, for different bivariate copulas with Spearman's rho equals 0.7.

Copula	(λ_L, λ_U)	(κ_L, κ_U)	ρ_L	ρ_U	ρ_N^-	ρ_N^+	ϱ_L	ϱ_U
Gaussian	(0.000, 0.000)	(1.17, 1.17)	0.40	0.40	0.47	0.47	0.46	0.46
t(4)	(0.397, 0.397)	(1.00, 1.00)	0.48	0.48	0.58	0.58	0.59	0.59
BB1	(0.446, 0.446)	(1.00, 1.00)	0.47	0.44	0.58	0.54	0.60	0.56
BB1	(0.083, 0.548)	(1.00, 1.00)	0.37	0.52	0.45	0.64	0.43	0.66
BB1	(0.655, 0.201)	(1.00, 1.00)	0.59	0.27	0.71	0.34	0.76	0.31
Gumbel	(0.000, 0.586)	(1.41, 1.00)	0.32	0.55	0.36	0.67	0.33	0.70
Frank	(0.000, 0.000)	(2.00, 2.00)	0.35	0.35	0.32	0.32	0.26	0.26

Table 1. Note that the conditional Spearman's rhos in Equation (2) and the normal scores semi-correlations in Equation (3) are less sensitive (especially the former) to stronger dependence in the tails and tail asymmetry. For the tail-dependent BB1 copula, the tail-weighted dependence measures ϱ_L and ϱ_U discriminate well three different cases when dependence in the lower tail is about equal, weaker or stronger than dependence in the upper tail.

The next step is a comparison of $a(\cdot)$ functions that can be used for the tail-weighted dependence measures; we use power functions for $a(\cdot)$ as the resulting measures are fast to compute. In Table 2, we compute $\Delta(a, p = 0.5; C_1, C_2)$ in Equation (4) for two copulas C_1, C_2 with the same value of Spearman's ρ_S such that C_1 is a tail-dependent copula and C_2 is not; also asymptotic standard errors are included. Spearman's ρ_S equals 0.5 in the top part of the table and 0.7 in the bottom part. Standard errors with the sample size equals 400 are computed using Monte-Carlo simulations. An alternative way is to use the delta method to obtain the formula for $\sigma(\hat{\Delta}(a, p; C_1, C_2))$ and then use numerical integration. It is seen, that a higher power k results in a larger absolute difference $|\Delta(a, p; C_1, C_2)|$. The main difference between the tail-dependent copula C_1 and the copula C_2 with intermediate dependence or tail quadrant independence can be found in the tail. The power function with large k puts more weight in the tail and thus makes the difference $\Delta(a, p; C_1, C_2)$ larger. The standard errors of the empirical version increase slowly unless the power k is very large. The ratio $\Delta/\sigma(\hat{\Delta})$ attains its maximum at k in the interval 6–8 for all pairs of copulas considered in Table 2. It implies that $k = 6$ can be a good choice when the absolute difference $|\Delta(a, p; C_1, C_2)|$ is quite large and the asymptotic variance is reasonably small.

The values $|\Delta(a, p; C_1, C_2)|$ can be slightly larger for p smaller than 0.5 but the asymptotic variance increases significantly so that there is no improvement. To illustrate this, we compute $\varrho_L(a, p; C)$ with $a(u) = u^6$ for Gaussian copula and reflected Gumbel copula with different ρ_S and different truncation levels p; see Table 3. Similar results can be obtained for other pairs of copulas. With $\rho_S = 0$ we have $\varrho_L = 0$ and the asymptotic variance equals $1/p^2$. With larger ρ_S the variance can increase slightly but eventually it goes to zero as ρ_S goes to 1. It is seen that with smaller $p < 0.5$, the standard errors are larger whereas the values of the tail-weighted measures do not change much. As a result, we propose to use $p = 0.5$ and $a(u) = u^6$ for the tail-weighted measure of dependence $\varrho_L(a, p)$ in the next section.

4. Empirical study with financial return data

In this section, we illustrate the use of the tail-weighted measures of dependence $\varrho_L(u^6, 0.5)$ and $\varrho_U(u^6, 0.5)$ for (i) choosing bivariate linking copulas in a structured factor model and (ii) comparing adequacy of fit to several copula models after maximum likelihood estimation. We

Table 2. The values $\Delta(a, p = 0.5; C_1, C_2)$ for different pairs of copulas with $\rho_S = 0.5, 0.7$, based on 20,000 samples of size 20,000; asymptotic standard errors for the sample size 400 are shown in brackets.

Pair (C_1, C_2) $\rho_S = 0.5$	t(4)-Gaussian	rGumbel-Gaussian	t(4)-Gumbel	rGumbel-Gumbel	t(4)-Frank	rGumbel Frank
$a(u) = u$	0.11 (0.11)	0.16 (0.11)	0.17 (0.12)	0.21 (0.12)	0.15 (0.11)	0.20 (0.11)
$a(u) = u^2$	0.14 (0.12)	0.20 (0.12)	0.20 (0.12)	**0.26** (0.12)	0.20 (0.12)	**0.26** (0.11)
$a(u) = u^4$	0.17 (0.13)	**0.24** (0.12)	0.24 (0.13)	**0.32** (0.12)	**0.27** (0.13)	**0.34** (0.12)
$a(u) = u^6$	0.18 (0.14)	**0.27** (0.13)	**0.27** (0.14)	**0.35** (0.13)	**0.30** (0.13)	**0.39** (0.13)
$a(u) = u^8$	0.20 (0.15)	**0.28** (0.14)	0.28 (0.15)	**0.37** (0.14)	**0.33** (0.14)	**0.42** (0.14)
Pair (C_1, C_2) $\rho_S = 0.7$	t (4)-Gaussian	rGumbel-Gaussian	t (4)-Gumbel	rGumbel-Gumbel	t (4)-Frank	rGumbel Frank
$a(u) = u$	0.08 (0.10)	0.15 (0.09)	0.16 (0.10)	**0.23** (0.10)	0.13 (0.10)	**0.20** (0.10)
$a(u) = u^2$	0.10 (0.10)	**0.18** (0.09)	0.19 (0.10)	**0.28** (0.10)	0.19 (0.10)	**0.28** (0.10)
$a(u) = u^4$	0.12 (0.11)	**0.22** (0.10)	**0.23** (0.11)	**0.33** (0.10)	**0.28** (0.11)	**0.38** (0.10)
$a(u) = u^6$	0.13 (0.12)	**0.24** (0.11)	**0.26** (0.12)	**0.37** (0.11)	**0.33** (0.12)	**0.44** (0.11)
$a(u) = u^8$	0.14 (0.13)	**0.26** (0.12)	**0.28** (0.13)	**0.39** (0.12)	**0.37** (0.13)	**0.48** (0.12)

Note: The values that are significantly positive at the 5% significance level are shown in bold font.

Table 3. The values $\varrho(a, p; C)$ for Gaussian and reflected Gumbel copula with different Spearman's rho, based on 20,000 samples of size 20,000; asymptotic standard errors for the sample size 400 are shown in brackets.

	Gaussian copula			rGumbel copula		
ρ_S	$p = 0.5$	$p = 0.4$	$p = 0.3$	$p = 0.5$	$p = 0.4$	$p = 0.3$
0.00	0.00 (0.10)	0.00 (0.13)	0.00 (0.17)	0.00 (0.10)	0.00 (0.13)	0.00 (0.17)
0.30	0.13 (0.10)	0.11 (0.12)	0.10 (0.15)	0.36 (0.11)	0.37 (0.13)	0.38 (0.15)
0.55	0.33 (0.10)	0.30 (0.11)	0.27 (0.13)	0.59 (0.08)	0.59 (0.09)	0.59 (0.11)
0.80	0.61 (0.07)	0.57 (0.09)	0.54 (0.11)	0.80 (0.05)	0.79 (0.06)	0.79 (0.07)
0.95	0.89 (0.03)	0.87 (0.03)	0.85 (0.04)	0.95 (0.01)	0.95 (0.02)	0.95 (0.02)
1.00	1.00 (0.00)	1.00 (0.00)	1.00 (0.00)	1.00 (0.00)	1.00 (0.00)	1.00 (0.00)

are not illustrating the full analysis of several classes of copula models, as the latter is the main emphasis of our other research publications.

Parametric copula models have been used a lot for modeling of financial returns. We want to show the main uses of tail-weighted dependence measures without lengthy tabular summaries, so we choose a data set with only nine stocks. We also use a relative small sample size from two years of data because the dependence structure can change in time, and with averaging over longer periods, the short-term tail asymmetry might not be seen. In general use of the proposed tail-weighted measures of dependence, we find that best-fitting copula models vary for different groups of stock returns and/or different periods of time.

We use a data set of nine S&P500 stock returns from Consumer Staples sector, years 2011–2012, and show how they can be used for model selection and assessment of model adequacy. We include four stocks with tickers LO (Lorillard, Inc.), MO (Altria Group Inc.), PM (Philip Morris International, Inc.), RAI (Reynolds American Inc.) from tobacco industry and five stocks with tickers CCE (Coca-Cola Enterprises Inc.), DPS (Dr Pepper Snapple Group), KO (Coca-Cola Company), MNST (Monster Beverage Corporation), PEP (Pepsico) from companies producing soft drinks. We use the autoregressive-generalized autoregressive conditional heteroscedastic (AR(1)-GARCH(1,1)) model (see [1,14]) to fit univariate marginals and GARCH-filtered data are then converted to uniform scores.

Table 4. Empirical Spearman's ρ_S for GARCH-filtered log return data.

	LO	MO	PM	RAI	CCE	DPS	KO	MNST	PEP
LO	1.000	0.554	0.538	0.623	0.375	0.360	0.437	0.309	0.388
MO	0.554	1.000	0.561	0.703	0.409	0.358	0.493	0.320	0.509
PM	0.538	0.561	1.000	0.604	0.490	0.371	0.527	0.361	0.468
RAI	0.623	0.703	0.604	1.000	0.443	0.400	0.511	0.358	0.498
CCE	0.375	0.409	0.490	0.443	1.000	0.477	0.547	0.415	0.520
DPS	0.360	0.358	0.371	0.400	0.477	1.000	0.431	0.331	0.444
KO	0.437	0.493	0.527	0.511	0.547	0.431	1.000	0.409	0.589
MNST	0.309	0.320	0.361	0.358	0.415	0.331	0.409	1.000	0.383
PEP	0.388	0.509	0.468	0.498	0.520	0.444	0.589	0.383	1.000

Table 5. Empirical lower (upper) tail-weighted dependence measures ϱ_L (ϱ_U) for GARCH-filtered log returns in lower (upper) triangle.

	LO	MO	PM	RAI	CCE	DPS	KO	MNST	PEP
LO		0.281	0.300	0.463	0.309	0.142	0.234	0.013	0.255
MO	0.429		0.313	0.515	0.351	0.313	0.304	0.144	0.176
PM	0.375	0.545		0.336	0.362	0.060	0.303	0.041	0.249
RAI	0.436	0.573	0.549		0.403	0.312	0.524	0.162	0.365
CCE	0.418	0.492	0.383	0.462		0.239	0.395	0.132	0.355
DPS	0.380	0.426	0.338	0.340	0.463		0.223	0.150	0.178
KO	0.277	0.589	0.521	0.447	0.501	0.427		0.190	0.481
MNST	0.283	0.188	0.012	0.179	0.319	0.342	0.280		0.057
PEP	0.338	0.475	0.442	0.436	0.391	0.439	0.563	0.186	

Table 6. Overall, tobacco group and soft drinks group averages of $\hat{\rho}_S$, $\hat{\varrho}_L$, $\hat{\varrho}_U$ and $\hat{\varrho}_N$; GARCH-filtered log-returns from S&P500 index, consumer staples sector, years 2011–2012.

	Overall average	Tobacco group average	Soft drinks group average
$\hat{\rho}_S$	0.46	0.60	0.45
$\hat{\varrho}_L$	0.40	0.48	0.39
$\hat{\varrho}_U$	0.27	0.37	0.24
$\hat{\varrho}_N$	0.24	0.36	0.23

We compute Spearman's ρ_S and ϱ_L, ϱ_U for each pair of the GARCH-filter returns. We denote the estimates by $\hat{\rho}_S$, $\hat{\varrho}_L$ and $\hat{\varrho}_U$, respectively; see Tables 4 and 5. Table 5 shows that there is tail asymmetry toward the joint lower tail for most pairs.

In addition, for each pair of returns, the estimated Spearman's ρ_S parameters were converted to a Gaussian copula correlation parameter and the model-based estimates of ϱ_L, ϱ_U are then obtained assuming a bivariate Gaussian distribution of the returns. We denote these estimates by $\hat{\varrho}_N$ (so that $\varrho_L = \varrho_U = \varrho_N$ for a Gaussian copula). This is done to compare the strength of dependence in the tails compared to Gaussian copula. The Gaussian copula with positive dependence is a copula with intermediate dependence so that if the model-based estimate $\hat{\varrho}_N$ for a pair is significantly smaller than empirical estimates $\hat{\varrho}_L$ or $\hat{\varrho}_U$, one might assume lower or upper tail dependence for the pair. The number of all pairs is 36 therefore to summarize the dependence structure of the data set we compute the average of $\hat{\rho}_S$, $\hat{\varrho}_L$, $\hat{\varrho}_U$ and $\hat{\varrho}_N$ for all pairs of uniform scores. We also compute the average of these estimates for all pairs of stock returns from tobacco industry and from soft drink producers. The results are presented in Table 6.

It is seen that the overall dependence, as measured by Spearman's ρ_S, is stronger within the first (tobacco) group. Dependence in the tails, as measured by ϱ_L and ϱ_U is also stronger for the tobacco group. It means models accounting for the group structure might be more appropriate for modeling such data set. In addition, dependence in the lower tail is on average much stronger comparing to that for Gaussian copula and dependence in the upper tail is comparable to that for Gaussian copula. It implies a tail asymmetric dependence structure of the data set with lower tail dependence so that using models with reflection symmetric copulas could be inappropriate.

For modeling the joint dependence of the GARCH-filtered data, in preliminary analyses, we considered and fitted several different classes of parametric copula models. For brevity of space, we provide summaries only for the class of nested copula models proposed in [17], as the dependence structure matches this data set and the models are simple to explain. We fit different nested copula models and compare how well different characteristics of the joint distribution of returns are estimated by these models. Assume that we have G groups of returns and that for a fixed $g = 1, \dots, G$, $U_{1g}, \dots, U_{d_g g}$ are conditionally independent given $V_g \sim U(0,1)$, and the joint cdf of $\boldsymbol{V} = (V_1, \dots, V_G)$ is given by the copula $C_{\boldsymbol{V}}$. We also assume that U_{ig} in group g does not depend on $V_{g'}$ for $g' \neq g$. That is, we have G groups of variables and G latent factors where the gth latent factor defines dependence structure in the gth group. Let C_{U_{ig},V_g} be the copula cdf of (U_{ig}, V_g) and $C_{U_{ig}|V_g}$ be the corresponding conditional distribution. With a vector $\mathbf{u} = (u_{11}, \dots, u_{d_1 1}, \cdots, u_{1G}, \dots, u_{d_G G})$ we get:

$$C_{\boldsymbol{U}}(\mathbf{u}) = \int_{[0,1]^G} \left\{ \prod_{g=1}^{G} \prod_{i=1}^{d_g} C_{U_{ig}|V_g}(u_{ig}|v_g) \right\} c_{\boldsymbol{V}}(v_1, \dots, v_G)\, \mathrm{d}v_1 \cdots \mathrm{d}v_G, \tag{5}$$

where C_{U_{ig},V_g} is the copula linking U_{ig} and V_g. Now we additionally assume that latent factors $V_1, \dots, V_G$ are conditionally independent given another latent variable V_0, that is, the joint distribution of $\boldsymbol{V}$ has one-factor copula structure. Then we get:

$$c_{\boldsymbol{V}}(v_1, \dots, v_G) = \int_0^1 \left\{ \prod_{g=1}^{G} c_{V_g,V_0}(v_g, v_0) \right\} \mathrm{d}v_0,$$

where C_{V_g,V_0} is the copula linking V_g and V_0.

In a nested copula model, stock returns in each of the two groups in our example are assumed to be independent given the group latent variable. The two group latent variables, V_1 and V_2, in turn, are independent given some other latent variable V_0. This model is an extension of a classical nested Gaussian model when different types of dependence structure, including tail dependence and asymmetry, can be modeled for each group. To model dependence within each group, one should choose a bivariate copula linking the group latent variable and stock returns, and to model dependence between groups, the copula linking the two group latent variables should be selected. We use the following models:

(1) Gaussian nested model. This is a model with reflection symmetric dependence structure and intermediate tail dependence between each two stocks from the data set
(2) Student t nested model: multivariate Student t_ν model with the same correlation matrix as in the nested Gaussian model. This is a model with reflection symmetric dependence structure and tail dependence between each pair of stocks.
(3) Frank nested model: all linking copulas are Frank copulas. This is a model with reflection symmetric dependence structure and tail quadrant independence between each pair of stocks.
(4) Reflected Gumbel/Reflected BB1 model: The reflected Gumbel (rGumbel) copula is used to model dependence between the two groups and the reflected BB1 (rBB1) copula is used

Table 7. Negative log-likelihood and AIC values for copula models applied to GARCH-filtered log returns; AIC is 2 times the negative log-likelihood + 2 times #parameters.

Model	-loglik	#parameters	AIC
Gaussian	– 1023.6	10	– 2027.2
Student t_ν $\nu = 15$	– 1069.9	11	– 2117.8
Frank	– 961.6	10	– 1903.2
rGumbel/rBB1	– 1072.2	19	– 2106.4

to model dependence within each of the two groups. This is a model with tail asymmetric dependence structure and lower/upper tail dependence between each pair of stocks.

The BB1 copula is a tail asymmetric copula with the lower and upper tail dependence; see [12] for details. Hence only the last model allows for asymmetric tail dependence, and based on the results presented in Table 6, the first three models might be not appropriate for the data set, but they are included to show how tail-weighted dependence measures are used to assess model adequacy. We next indicate the comparison based on negative log-likelihood and values of Akaike information criterion (AIC) for the uniform-transformed GARCH-filtered log returns. For the nested factor models based on Gaussian, Student t, Frank and rGumbel/rBB1, they are summarized in Table 7; the choice of $\nu = 15$ led to the largest log-likelihood for multivariate Student t. As the preliminary analysis based on tail-weighted dependence measures suggested tail dependence, it is not surprising that the best models based on AIC are the two models with tail dependence.

We could try to find copula models with more parameters that can lead to smaller values of AIC and better fits in the tails, but we next assess whether any of the models in Table 7 is adequate for relevant context-based inferences such as tail inferences. Although the Student t nested factor model is better based on AIC, we show below that the rGumbel/rBB1 nested factor model is better in adequate fit to the joint tails.

To compare the performance of these models, we compute the model-based characteristics of the joint distribution of data and compare them to the corresponding empirical estimates. The first characteristic is the portfolio Value-at-Risk (VaR). VaR is a popular risk measure that is extensively used in financial applications. It is defined as a quantile of a portfolio distribution where portfolio is a weighted sum of stock returns. For the analysis of the data set we assume an equally weighted portfolio of stock returns. We consider 1%, 5%, 95% and 99% VaR which we denote by $\mathrm{VaR}_{0.01}, \mathrm{VaR}_{0.05}, \mathrm{VaR}_{0.95}, \mathrm{VaR}_{0.99}$ respectively. The VaR with upper quantiles represent maximal possible loss for investors, who short sell the portfolio, that can occur with a given probability so that both lower and upper quantiles for VaR are used in applications to quantify risks.

The second characteristic is the portfolio conditional tail expectation (CTE) that is defined as the conditional return of a portfolio given that the return is smaller (greater) than some lower (upper) threshold value. The threshold is usually set equal to VaR so that we use 4 different thresholds: $\mathrm{VaR}_{0.01}, \mathrm{VaR}_{0.05}$ for the lower threshold and $\mathrm{VaR}_{0.95}, \mathrm{VaR}_{0.99}$ for the upper threshold. We denote these quantities by $\mathrm{CTE}_{0.01}, \mathrm{CTE}_{0.05}, \mathrm{CTE}_{0.95}, \mathrm{CTE}_{0.99}$ respectively.

In addition, we compute the model-based estimates of ρ_S, ϱ_L and ϱ_U for each pair of GARCH-filtered returns and then compute the overall average of these estimates together with the tobacco group and soft drinks group average. By condensing to these averages, there are 3 quantities rather than 36 quantities to compare, and the confidence intervals for these averages are shorter than those for a single tail-weighted dependence measure. To account for variability in parameter

Table 8. Empirical estimates and the model-based 95% confidence intervals for VaR_α and CTE_α with $\alpha = 0.01, 0.05, 0.95, 0.99$; GARCH-filtered log-returns from S&P500 index, consumer staples sector, years 2011–2012.

Model	$\mathrm{VaR}_{0.01}$	$\mathrm{VaR}_{0.05}$	$\mathrm{VaR}_{0.95}$	$\mathrm{VaR}_{0.99}$
Empirical estimate	− 0.026	− 0.015	0.016	0.028
Nested Gaussian	(− 0.032, − 0.020)	(− 0.018, − 0.012)	(0.014, 0.020)	(0.022, 0.034)
Nested Student *t*	(− 0.032, − 0.020)	(− 0.018, − 0.012)	(0.014, 0.019)	(0.022, 0.033)
Nested Frank	(− 0.026, − 0.016)	(− 0.017, − 0.011)	(0.013, 0.019)	(0.018, 0.028)
Nested rGumbel/rBB1	(− 0.036, − 0.023)	(− 0.018, − 0.013)	(0.014, 0.019)	(0.021, 0.033)
Model	$\mathrm{CTE}_{0.01}$	$\mathrm{CTE}_{0.05}$	$\mathrm{CTE}_{0.95}$	$\mathrm{CTE}_{0.99}$
Empirical estimate	− 0.035	− 0.023	0.023	0.033
Nested Gaussian	(− 0.041, − 0.031)	(− 0.024, − 0.020)	(0.021, 0.025)	(0.033, 0.042)
Nested Student *t*	(− 0.039, − 0.031)	(− 0.024, − 0.020)	(0.021, 0.025)	(0.033, 0.041)
Nested Frank	(− 0.045, − 0.029)	(− 0.022, − 0.018)	(0.019, 0.022)	(0.031, 0.048)
Nested rGumbel/rBB1	(− 0.043, − 0.034)	(− 0.027, − 0.022)	(0.021, 0.025)	(0.034, 0.044)

Table 9. Estimated averages (overall, tobacco group and soft drinks group) of ρ_S, ϱ_L, ϱ_U and the model-based 95% confidence intervals (intervals that do not contain the empirical value are shown in bold font); GARCH-filtered log-returns from S&P500 index, consumer staples sector, years 2011–2012.

Model	ρ_S(all)	ρ_S(tobacco)	ρ_S(softdrinks)
Empirical estimate	0.46	0.60	0.45
Nested Gaussian	(0.41, 0.49)	(0.55, 0.64)	(0.40, 0.49)
Nested Student *t*	(0.42, 0.50)	(0.56, 0.64)	(0.41, 0.50)
Nested Frank	(0.44, 0.53)	(0.58, 0.67)	(0.43, 0.53)
Nested rGumbel/rBB1	(0.41, 0.49)	(0.54, 0.63)	(0.39, 0.48)
Model	ϱ_L(all)	ϱ_L(tobacco)	ϱ_L(softdrinks)
Empirical estimate	0.40	0.48	0.39
Nested Gaussian	**(0.21, 0.27)**	**(0.31, 0.40)**	**(0.19, 0.26)**
Nested Student *t*	**(0.27, 0.33)**	**(0.37, 0.45)**	**(0.26, 0.32)**
Nested Frank	**(0.08, 0.11)**	**(0.14, 0.19)**	**(0.08, 0.12)**
Nested rGumbel/rBB1	(0.36, 0.47)	(0.44, 0.57)	(0.32, 0.45)
Model	ϱ_U(all)	ϱ_U(tobacco)	ϱ_U(softdrinks)
Empirical estimate	0.27	0.37	0.24
Nested Gaussian	(0.21, 0.27)	(0.31, 0.40)	(0.19, 0.26)
Nested Student t	(0.27, 0.33)	(0.37, 0.45)	**(0.26, 0.32)**
Nested Frank	**(0.08, 0.11)**	**(0.14, 0.19)**	**(0.08, 0.12)**
Nested rGumbel/rBB1	(0.19, 0.29)	(0.32, 0.46)	(0.20, 0.34)

estimates, we use bootstrap to construct 95% confidence intervals for the aforementioned model-based characteristics of the joint distribution. To do this, we resample GARCH-filtered residuals, and use the resampled residuals plus the estimated GARCH parameters to get a bootstrap sample of returns; see [22] for more details. The results are presented in Tables 8 and 9.

One can see that the model-based estimates for VaR are quite close to the empirical estimates for all four of the models under consideration. The Frank nested model slightly underestimates $\mathrm{CTE}_{0.05}$ and $\mathrm{CTE}_{0.95}$, and the rGumbel/rBB1 model slightly overestimates $\mathrm{CTE}_{0.99}$. Both the Gaussian and Student t models do well in terms of the two risk measures, with all confidence intervals containing the corresponding empirical estimates. Furthermore, the model-based estimates for ρ_S are very close for all 4 models. However, the confidence intervals for the tail-weighted measures of dependence clearly indicate that all models but the last one are not appropriate for modeling dependence in the tails. For the nested Gaussian and Student t models, dependence in the lower tail is significantly underestimated. Dependence in the lower tail is stronger for the Student t nested model but it is still underestimated by the model. The Frank nested model grossly underestimates the strength of dependence in both tails. And only the last model does reasonably well in the tails as it accounts for tail asymmetric dependence structure of the data set. It confirms our findings based on the preliminary analysis that tail asymmetric models with the lower tail dependence could fit the data well.

As a result, the two risk measures, portfolio VaR and CTE, failed to discriminate well models with different tail properties. The possible reason is that smaller quantiles for VaR and smaller thresholds for CTE are needed to discriminate different models. Otherwise the inferences on VaR and CTE are mainly dominated by the fit in the middle. However, with small sample size, the resulting confidence intervals for these quantities can be very large so that efficient discrimination is not possible. At the same time, the tail-weighted measures of dependence are more sensitive to models with different types of dependence structure and therefore they can be used for model validation in applications where tail properties of the joint distribution are extremely important.

Other data sets of stock returns have been analyzed using the tail-weighted measures of dependence. While the best fitted models may be different, the proposed measures of dependence were always more sensitive to the strength of dependence in the tails, compared with VaR and CTE, thus allowing to select appropriate models for these data sets.

5. Discussion

The tail-weighted measures proposed in this paper can be used to assess the strength of dependence in the tails in pairs of variables. Even with a small sample size the measures can efficiently discriminate copulas with tail dependence and copulas that are not tail dependent. The measures can be used as summaries in addition to general concordance measures such as Kendall's τ and Spearman's ρ_S to facilitate the choice of a model that could fit data well in the tails. Furthermore, the measures can be used for model validation to assess the model adequacy of fit in the tails. Quantile-based measures, such as portfolio VaR, might not discriminate dependence models, and the overall dependence can be estimated fairly well by models with different tail properties. The tail-weighted measures of dependence can be estimated using data in the joint tails, by putting time more weight on the extremes, and thus they efficiently discriminate models with different tail orders. This is especially important for many applications in finance, insurance and other areas when model with misspecified tail characteristics can lead to incorrect inferences.

Model adequacy assessment for the strength of dependence in the tails is similar to that for monotone dependence, by comparing empirical versus model-based quantities. If discrepancies are found, then there are indications on how to improve the model in either the tail properties and/or dependence structure. Note also that conclusions about model comparisons based on assessment of adequacy of fit with tail-weighted dependence measures could be different from conclusions based on AIC because log-likelihood values are dominated by the fit in the 'middle' of the data.

Future research includes the study of multivariate extensions of the tail-weighted measures when the measure is defined for a d-dimensional random vector with $d > 2$. Another research direction is the development of measures of asymmetry and dependence based on the empirical copula function. These measures can be useful in detecting different types of dependence in the middle of a multivariate distribution when the use of tail-weighted measures of dependence is not very efficient.

Acknowledgments

This research has been supported by an NSERC Discovery Grant. Thanks to the referees for their careful reading and comments.

References

[1] K. Aas, C. Czado, A. Frigessi, and H. Bakken, *Pair-copula constructions of multiple dependence*, Insur. Math. Econ. 44(2) (2009), pp. 182–198.

[2] A. Ang and J. Chen, *Asymmetric correlations of equity portfolios*, J. Financ. Econ. 63(3) (2002), pp. 443–494.

[3] E.C. Brechmann, C. Czado, and K. Aas, *Truncated regular vines in high dimensions with applications to financial data*, Can. J. Stat. 40 (2012), pp. 68–85.

[4] J. Dobrić and F. Schmid, *Nonparametric estimation of the lower tail dependence λ_l in bivariate copulas*, J. Appl. Stat. 32(4) (2005), pp. 387–407.

[5] J. Fermanian, D. Radulovic, and M. Wegkamp, *Weak convergence of empirical copula processes*, Bernoulli 10(5) (2004), pp. 847–860.

[6] G. Frahm, M. Junker, and R. Schmidt, *Estimating the tail dependence coefficient: Properties and pitfalls*, Insur. Math. Econ. 37 (2005), pp. 80–100.

[7] G. Gabbi, *Semi-correlations as a tool of geographical and sector asset allocation*, Eur. J. Financ. 11(3) (2005), pp. 271–281.

[8] C. Genest, B. Rémillard, and D. Beaudoin, *Goodness-of-fit tests for copulas: A review and a power study*, Insur. Math. Econ. 44(2) (2009), pp. 199–213.

[9] J.E. Heffernan, *A directory of coefficients of tail dependence*, Extremes 3 (2000), pp. 279–290.

[10] L. Hua and H. Joe, *Tail order and intermediate tail dependence of multivariate copulas*, J. Multivariate Anal. 102 (2011), pp. 1454–1471.

[11] W. Huang and A. Prokhorov, *A goodness-of-fit test for copulas*, Econom. Rev. 33(7) (2014), pp. 751–771.

[12] H. Joe, *Multivariate Models and Dependence Concepts*, Chapman and Hall, London, 1997.

[13] H. Joe, H. Li, and A. Nikoloulopoulos, *Tail dependence functions and vine copulas*, J. Multivariate Anal. 101 (2010), pp. 252–270.

[14] E. Jondeau and M. Rockinger, *The copula-GARCH model of conditional dependencies: An international stock market application*, J. Int. Money Financ. 25 (2006), pp. 827–853.

[15] G. Kimeldorf and A.R. Sampson, *A framework for positive dependence*, Ann. Inst. Statist. Math. 41 (1989), pp. 31–45.

[16] P. Krupskii and H. Joe, *Factor copula models for multivariate data*, J. Multivariate Anal. 120 (2013), pp. 85–101.

[17] P. Krupskii and H Joe, *Structured factor copula models: Theory, inference and computation*, J. Multivariate Anal. (2015), accepted.

[18] D. Kurowicka and H. Joe, *Dependence Modeling: Vine Copula Handbook*, World Scientific, Singapore, 2011.

[19] A. Ledford and J. Tawn, *Statistic for near independence in multivariate extreme values*, Biometrica 83 (1996), pp. 168–187.

[20] R.B. Nelsen, *An Introduction to Copulas*, 2nd ed., Springer, New York, 2006.

[21] A.K. Nikoloulopoulos, H. Joe, and H. Li, *Vine copulas with asymmetric tail dependence and applications to financial return data*, Comput. Stat. Data Anal. 56 (2012), pp. 3659–3673.

[22] L. Pascual, J. Romo, and E. Ruiz, *Bootstrap prediction for returns and volatilities in GARCH models*, Comput. Stat. Data Anal. 50(9) (2006), pp. 2293–2312.

[23] A. Patton, *Modeling asymmetric exchange rate dependence*, Int. Econ. Rev. 47(2) (2006), pp. 527–556.

[24] U. Schepsmeier, *Estimating standard errors and efficient goodness-of-fit tests for regular vine copula models*, PhD thesis, Technische Universität München, 2014.

[25] F. Schmid and R. Schmidt, *Multivariate conditional versions of Spearman's rho and related measures of tail dependence*, J. Multivariate Anal. 98 (2007), pp. 1123–1140.

[26] A. Sklar, *Fonctions de répartition à n dimensions et leurs marges*, Publ. Inst. Statist. Univ. Paris 8 (1959), pp. 229–231.

Appendix

Proof of Proposition 1. We prove the result for $\hat{\varrho}_L$ as the proof for $\hat{\varrho}_U$ is similar.

To simplify the notation, we redefine $b(u) := a(1 - u/p)$. In new notation, $b(p) = 0$. If b_j is differentiable with $b_j(p) = 0$, we can write $b_j(v) = -\int_v^p b_j'(u)\,\mathrm{d}u$ for $0 \le v \le p$ and $j = 1, 2$. Then for a bivariate copula cdf C, after substitution of this identity and interchanging the order of integration.

$$\int_0^p \int_0^p b_1(v_1)b_2(v_2)\,\mathrm{d}C(v_1,v_2) = \int_0^p \int_0^p b_1'(u_1)b_2'(u_2)C(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2,$$
$$\int_0^p \int_0^p b_1(v_1)\,\mathrm{d}C(v_1,v_2) = -\int_0^p b_1'(u_1)C(u_1,p)\,\mathrm{d}u_1,$$
$$\int_0^p \int_0^p b_2(v_2)\,\mathrm{d}C(v_1,v_2) = -\int_0^p b_2'(u_2)C(p,u_2)\,\mathrm{d}u_2,$$

Let $C_n(u_1,u_2) = (1/n)I\{R_{i1} < u_1, R_{i2} < u_2\}$ be the empirical copula for $0 < u_1 < 1$ and $0 < u_2 < 1$. The above equations also hold for C_n so that we have:

$$\hat{m}_{12} = \frac{1}{n}\sum_{i=1}^n b(R_{i1})b(R_{i2})I\{R_{i1} < p, R_{i2} < p\} = \int_0^p \int_0^p b'(u_1)b'(u_2)\hat{C}_n(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2,$$
$$\hat{m}_1 = \frac{1}{n}\sum_{i=1}^n b(R_{i1})I\{R_{i1} < p, R_{i2} < p\} = -\int_0^p b'(u_1)\hat{C}_n(u_1,p)\,\mathrm{d}u_1,$$
$$\hat{m}_2 = \frac{1}{n}\sum_{i=1}^n b(R_{i2})I\{R_{i1} < p, R_{i2} < p\} = -\int_0^p b'(u_2)\hat{C}_n(p,u_2)\,\mathrm{d}u_2.$$

It implies that $m_{12} = \mathrm{E}[\hat{m}_{12}] = \int_0^p \int_0^p b'(u_1)b'(u_2)C(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2$, $m_1 = \mathrm{E}[\hat{m}_1] = -\int_0^p b'(u_1)C(u_1,p)\,\mathrm{d}u_1$, $m_2 = \mathrm{E}[\hat{m}_2] = -\int_0^p b'(u_2)C(p,u_2)\,\mathrm{d}u_2$. Denote $\mathcal{G}_n(u_1,u_2) = \sqrt{n}(\hat{C}_n(u_1,u_2) - C(u_1,u_2))$. As it was shown by Fermanian *et al.* [5], $\mathcal{G}_n(u_1,u_2) \xrightarrow{d} \mathcal{G}(u_1,u_2)$, where $\mathcal{G}$ is a Gaussian process. As a result,

$$\begin{aligned}\sqrt{n}(\hat{m}_{12} - m_{12}) &= \int_0^p \int_0^p b'(u_1)b'(u_2)\mathcal{G}_n(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 \\ &\xrightarrow{d} \int_0^p \int_0^p b'(u_1)b'(u_2)\mathcal{G}(u_1,u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2,\end{aligned} \tag{A1}$$

$$\sqrt{n}(\hat{m}_1 - m_1) = -\int_0^p b'(u_1)\mathcal{G}_n(u_1,p)\,\mathrm{d}u_1 \xrightarrow{d} -\int_0^p b'(u_1)\mathcal{G}(u_1,p)\,\mathrm{d}u_1, \tag{A2}$$

$$\sqrt{n}(\hat{m}_2 - m_2) = -\int_0^p b'(u_2)\mathcal{G}_n(p,u_2)\,\mathrm{d}u_2 \xrightarrow{d} -\int_0^p b'(u_2)\mathcal{G}(p,u_2)\,\mathrm{d}u_2. \tag{A3}$$

Similarly, we get

$$\begin{aligned}\sqrt{n}(\hat{m}_{11} - m_{11}) &\xrightarrow{d} -\int_0^p 2b(u_1)b'(u_1)\mathcal{G}(u_1,p)\,\mathrm{d}u_1, \\ \sqrt{n}(\hat{m}_{22} - m_{22}) &\xrightarrow{d} -\int_0^p 2b(u_2)b'(u_2)\mathcal{G}(p,u_2)\,\mathrm{d}u_2,\end{aligned} \tag{A4}$$

where $\hat{m}_{jj} = (1/n)\sum_{i=1}^n [b(R_{ij})]^2 I\{R_{i1} < p, R_{i2} < p\}$ and $m_{jj} = \mathrm{E}[\hat{m}_{jj}]$, $j = 1,2$.

In addition, $\hat{n}_p = (1/n)\sum_{i=1}^n I\{R_{i1} < p, R_{i2} < p\} = \hat{C}_n(p,p)$ and $\sqrt{n}(\hat{n}_p - n_p) \xrightarrow{d} \mathcal{G}(p,p)$, with $n_p = C(p,p)$. Hence, from Equations (A1)–(A4) we get the joint asymptotic normality of $\hat{m}_{12}, \hat{m}_1, \hat{m}_2, \hat{m}_{11}, \hat{m}_{22}, \hat{n}_p$ and the asymptotic normality of $\hat{\varrho}_L(a,p) = (\hat{n}_p\hat{m}_{12} - \hat{m}_1\hat{m}_2)/[(\hat{n}_p\hat{m}_{11} - \hat{m}_1^2)(\hat{n}_p\hat{m}_{22} - \hat{m}_2^2)]^{1/2}$ follows from the delta method and Cramer–Wold theorem. ∎

Proof of Proposition 2. To prove the result, we use the delta method and the result of Theorem 1. Denote $m_{j0} = (n_p m_{jj} - m_j^2)$. In case of independence copula, we get $n_p m_{12} = m_1 m_2$, $m_1 = m_2$, $m_{11} = m_{22}$, $m_{10} = m_{20}$ and $n_p = p^2$. If we write $\varrho_L(a,p)$ as a function of $m_{12}, m_1, m_2, m_{10}, m_{20}, n_p$, we find

$$\varrho_L(a,p) = \frac{n_p m_{12} - m_1 m_2}{[m_{10}m_{20}]^{1/2}},$$

Note that $\partial\varrho_L/\partial m_{j0} = 0$ for $j = 1,2$ so it suffices to find the asymptotic variance of the covariance $\hat{W}_{12} = \hat{n}_p\hat{m}_{12} - \hat{m}_1\hat{m}_2$. The gradient for $W_{12} = n_p m_{12} - m_1 m_2$ with respect to m_{12}, m_1, m_2, n_p is as follows:

$$\nabla W_{12} = \left(p^2, \quad -m_1, \quad -m_1, \quad \frac{m_1^2}{p^2}\right)^{\mathrm{T}}.$$

Redefine $b(u) := a(1 - u/p)$. Now we find the covariance function for the Gaussian process $\mathcal{G}(u_1, u_2)$ for independence copula. We have

$$\mathcal{G}(u_1, u_2) = \mathcal{B}(u_1, u_2) - \frac{\partial C(u_1, u_2)}{\partial u_1}\mathcal{B}(u_1, 1) - \frac{\partial C(u_1, u_2)}{\partial u_2}\mathcal{B}(1, u_2),$$

where $\mathcal{B}(u_1, u_2)$ is a two-dimensional Brownian bridge and $\mathrm{E}[\mathcal{B}(u_1, u_2)\mathcal{B}(u_3, u_4)] = C(u_1 \wedge u_3, u_2 \wedge u_4) - C(u_1, u_2)C(u_3, u_4)$. If $C(u_1, u_2) = u_1u_2$, we get

$$\mathcal{D}(u_1, u_2, u_3, u_4) = \mathrm{E}[\mathcal{G}(u_1, u_2)\mathcal{G}(u_3, u_4)] = (u_1 \wedge u_3 - u_1u_3)(u_2 \wedge u_4 - u_2u_4).$$

Let

$$\Sigma = \begin{pmatrix} \sigma_{12}^2 & v_{12,1} & v_{12,2} & v_{12,p} \\ v_{12,1} & \sigma_1^2 & v_{1,2} & v_{1,p} \\ v_{12,2} & v_{1,2} & \sigma_2^2 & v_{2,p} \\ v_{12,p} & v_{1,p} & v_{2,p} & \sigma_p^2 \end{pmatrix}$$

be the asymptotic covariance matrix of the vector $\mathcal{M} = n^{-1/2}(\hat{m}_{12}, \hat{m}_1, \hat{m}_2, \hat{n}_p)$. From Equations (A1)–(A3), we find

$$\begin{aligned}
\sigma_{12}^2 &= \int_0^p\int_0^p\int_0^p\int_0^p b'(u_1)b'(u_2)b'(u_3)b'(u_4)\mathcal{D}(u_1, u_2, u_3, u_4)\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3\,\mathrm{d}u_4 = \xi^2,\\
\sigma_1^2 = \sigma_2^2 &= \int_0^p\int_0^p b'(u_1)b'(u_3)\mathcal{D}(u_1, p, u_3, p)\,\mathrm{d}u_1\,\mathrm{d}u_3 = p(1-p)\xi,\\
\sigma_p^2 &= \mathcal{D}(p, p, p, p) = p^2(1-p)^2,\\
v_{12,1} = v_{12,2} &= -\int_0^p\int_0^p\int_0^p b'(u_1)b'(u_2)b'(u_3)\mathcal{D}(u_1, u_2, u_3, p)\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 = (1-p)m_1\xi/p,\\
v_{1,2} = v_{12,p} &= \int_0^p\int_0^p b'(u_1)b'(u_4)\mathcal{D}(u_1, p, p, u_4)\,\mathrm{d}u_1\,\mathrm{d}u_4 = (1-p)^2m_1^2/p^2,\\
v_{1,p} = v_{2,p} &= -\int_0^p b'(u_1)\mathcal{D}(u_1, p, p, p)\,\mathrm{d}u_1 = (1-p)^2m_1,
\end{aligned}$$

where

$$\xi = \int_0^p\int_0^p b'(u_1)b'(u_2)(u_1 \wedge u_2 - u_1u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 = \frac{m_{10}}{p^3} + \frac{(1-p)m_1^2}{p^3}.$$

Using the delta method and straightforward algebraic calculations, we obtain the formula for the asymptotic variance $\sigma^2(\hat{W}_{12})$:

$$\sigma^2(\hat{W}_{12}) = (\nabla W_{12})^{\mathrm{T}}\Sigma(\nabla W_{12}) = \frac{m_{10}^2}{p^2}$$

and finally, the variance of the sampling distribution of $\hat{\varrho}_L$ is $\sigma^2(\hat{\varrho}_L) = 1/p^2$. ■