RADI603: Medical Statistics and Programming for Data Science and Clinical Informatics
Mahidol University
Faculty of Medicine - Ramathibodi Hospital
Department of Clinical Epidemiology and Biostatistics

Assignment II: Vector Assignments

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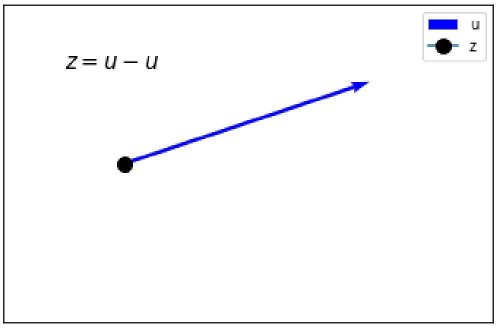
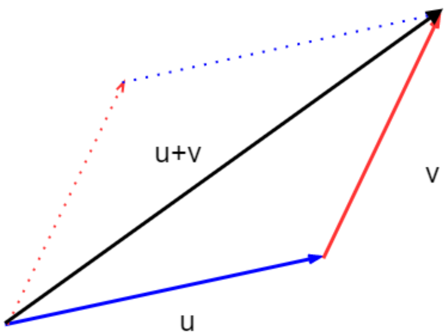
1. Suppose vectors $\vec{u}, \vec{v}$, and $\vec{w}$ have the same initial point (i.e., origin (0, 0)). Let $\vec{z}$ be the resulting vector to every vector operations, such that every vector illustrated in this solution are approximated and scaled with respect to the given vector and solution vector $\vec{z}$. We calculated each problem using the Parallelogram Law (column 1) or using assumed values of vectors (column 2) to check and compare its graph. For column 2, we assumed the following values: $\vec{u} = (3, 1), \vec{v} = (1, 3)$, and $\vec{w} = (-2, 3)$ and draw its resulting graph via Python.

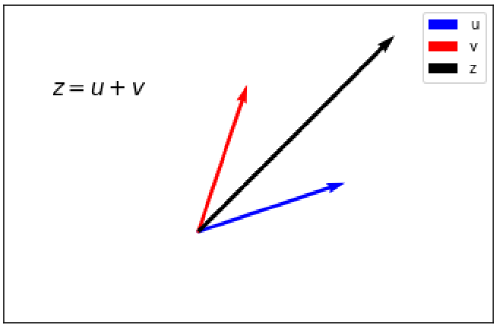
(a) (1.1) $\vec{u} - \vec{u}$ i.e., (zero vector)



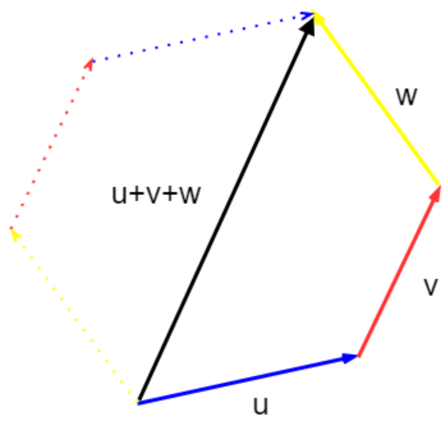
(b) (1.1) $\vec{u} - \vec{u}$



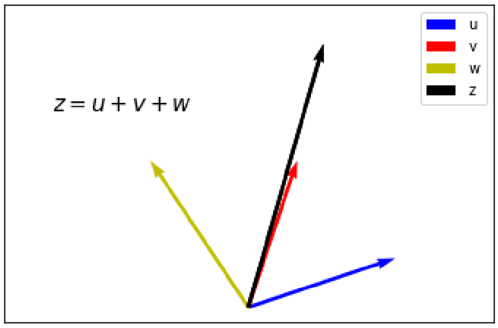
(a) (1.2) $\vec{u} + \vec{v}$ by Parallelogram Law



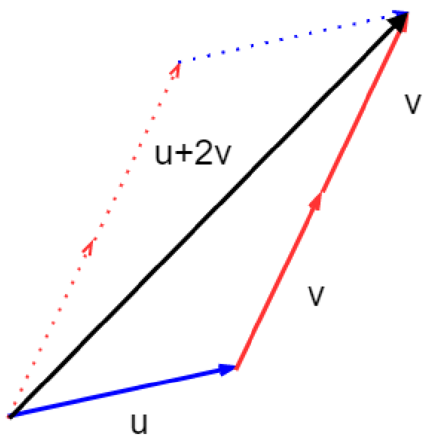
(b) (1.2) $\vec{u} + \vec{v}$



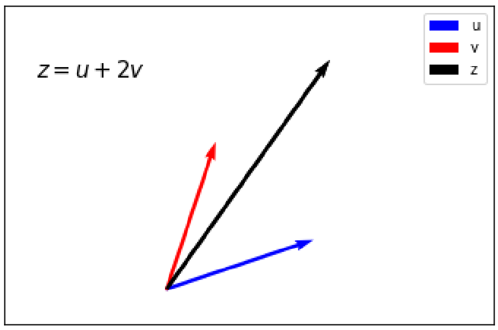
(a) (1.3) $\vec{u} + \vec{v} + \vec{w}$ by Parallelogram Law



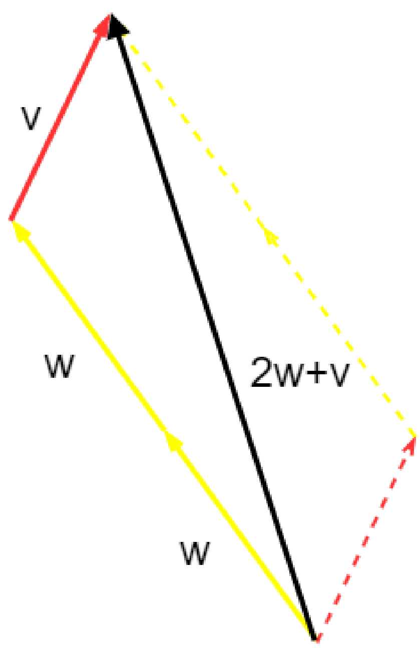
(b) (1.3) $\vec{u} + \vec{v} + \vec{w}$



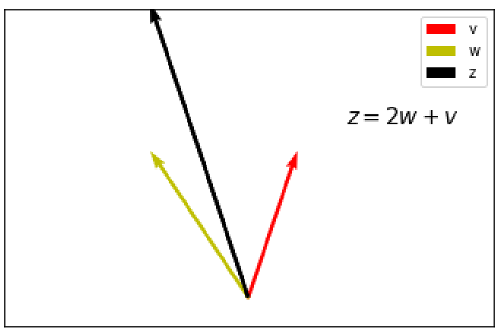
(a) (1.4) $\vec{u} + 2\vec{v}$ by Parallelogram Law



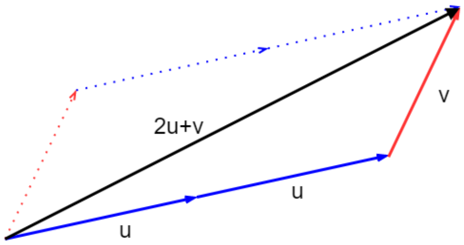
(b) (1.4) $\vec{u} + 2\vec{v}$



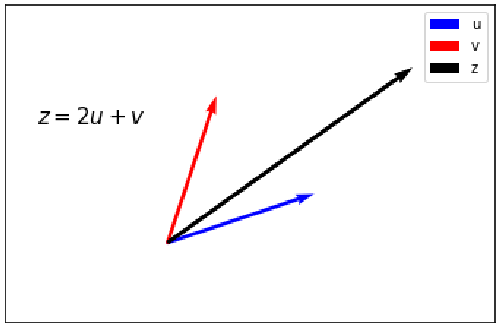
(a) (1.5) $2\vec{w} + \vec{v}$ by Parallelogram Law



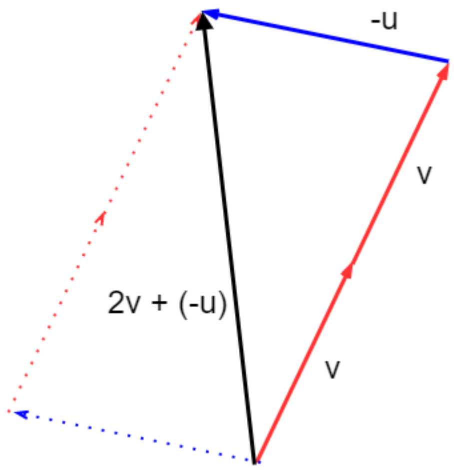
(b) (1.5) $2\vec{w} + \vec{v}$



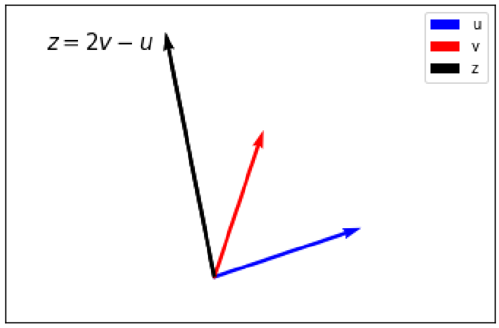
(a) (1.6) $2\vec{u} + \vec{v}$ by Parallelogram Law



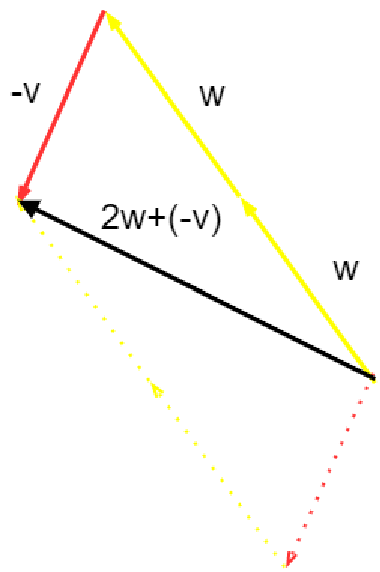
(b) (1.6) $2\vec{u} + \vec{v}$



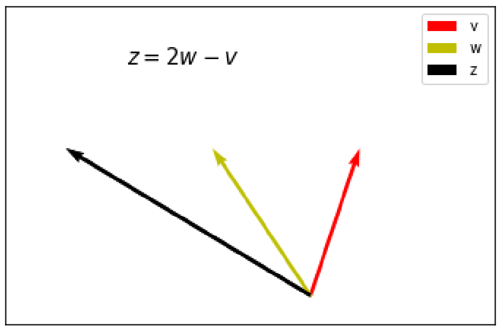
(a) (1.7) $2\vec{v} + \vec{u}$ by Parallelogram Law



(b) (1.7) $2\vec{v} + \vec{u}$



(a) (1.8) $2\vec{w} - \vec{v}$ by Parallelogram Law



(b) (1.8) $2\vec{w} - \vec{v}$

2. Given the vectors: $\vec{A} = (5, 10, 12, 9)$, $\vec{B} = (1, 2, 3, 4)$, $\vec{C} = (3, 1, 9, 3)$, and $\vec{D} = (4, 2, 3, 1)$, find the following vectors.

(a) $\vec{A}\cdot\vec{B} = (5,10,12,9)\cdot(1,2,3,4)$
$= (5)(1)+(10)(2)+(12)(3)+(9)(4)$
$= 5+20+36+36$
$\vec{A}\cdot\vec{B} = 97$

(b) $\vec{C}\cdot\vec{D} = (3,1,9,3)\cdot(4,2,3,1)$
$= (3)(4)+(1)(2)+(9)(3)+(3)(1)$
$= 12+2+27+3$
$\vec{C}\cdot\vec{D} = 44$

(c) $\vec{A}\cdot(\vec{B}+\vec{C}) = (5,10,12,9)\cdot((1,2,3,4)+(3,1,9,3))$
$= (5,10,12,9)\cdot(1+3,2+1,3+9,4+3)$
$= (5,10,12,9)\cdot(4,3,12,7)$
$= (5)(4)+(10)(3)+(12)(12)+(9)(7) = 20+30+144+63$
$\vec{A}\cdot(\vec{B}+\vec{C}) = 257$

(d) $(\vec{A}\cdot\vec{B})+(\vec{C}\cdot\vec{D}) = ((5,10,12,9)\cdot(1,2,3,4))+((3,1,9,3)\cdot(4,2,3,1))$
$= ((5)(1)+(10)(2)+(12)(3)+(9)(4))+((3)(4)+(1)(2)+(9)(3)+(3)(1))$
$= 5+20+36+36+12+2+27+3$
$(\vec{A}\cdot\vec{B})+(\vec{C}\cdot\vec{D}) = 141$

(e) $2\vec{A}\cdot(\vec{D}+\vec{C}) = 2\cdot(5,10,12,9)\cdot((4,2,3,1)+(3,1,9,3))$
$= ((2)(5),(2)(10),(2)(12),(2)(9))\cdot(4+3,2+1,3+9,1+3)$
$= (10,20,24,18)\cdot(7,3,12,4)$
$= (10)(7)+(20)(3)+(24)(12)+(18)(4)$
$= 70+60+288+72$
$2\vec{A}\cdot(\vec{D}+\vec{C}) = 490$

3. Are the vectors linearly independent?

Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbf{R}^n$. Let $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$. Then the vectors are linearly independent if and only if the determinant of the matrix $A = [v_{ij}]_{n\times n} \neq 0$. Otherwise, the vectors are linearly dependent if the determinant of the matrix $A = [v_{ij}]_{n\times n} = 0$.

(a) $\vec{A}_1 = \begin{bmatrix}3\\5\\1\end{bmatrix}, \vec{A}_2 = \begin{bmatrix}4\\2\\4\end{bmatrix}, \vec{A}_3 = \begin{bmatrix}2\\3\\1\end{bmatrix}$

$$\det \vec{A} = \begin{bmatrix} 3 & 4 & 2 \\ 5 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 & 3 & 4 \\ 5 & 2 & 3 & 5 & 2 \\ -1 & 4 & 1 & -1 & 4 \end{bmatrix}$$
$$\det \vec{A} = (3)(2)(1) + (4)(3)(-1) + (2)(5)(4) - [(-1)(2)(2) + (4)(3)(3) + (1)(5)(4)]$$
$$\det \vec{A} = 6 - 12 + 40 - (-4 + 36 + 20)$$
$$\det \vec{A} = -18.00$$

Since $\det \vec{A} \neq 0$, the vectors $A_1, \vec{A_2}, A_3$ are **linearly independent**.

(b) $\vec{A_1} = \begin{bmatrix} -5 \\ -10 \\ -70 \end{bmatrix}, \vec{A_2} = \begin{bmatrix} -50 \\ -2 \\ -30 \end{bmatrix}, \vec{A_3} = \begin{bmatrix} -20 \\ -30 \\ -50 \end{bmatrix}$

$$\det \vec{A} = \begin{bmatrix} -5 & -50 & -20 \\ -10 & -2 & -30 \\ -70 & -30 & -50 \end{bmatrix} = \begin{bmatrix} -5 & -50 & -20 & -5 & -50 \\ -10 & -2 & -30 & -10 & -2 \\ -70 & -30 & -50 & -70 & -30 \end{bmatrix}$$
$$\det \vec{A} = -500 - 105000 - 600 + 2800 + 4500 + 25000$$
$$\det \vec{A} = -79200.0$$

Since $\det \vec{A} \neq 0$, the vectors $A_1, \vec{A_2}, A_3$ are **linearly independent**.

(c) $\vec{A_1} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \vec{A_2} = \begin{bmatrix} -6 \\ 3 \\ -9 \end{bmatrix}, \vec{A_3} = \begin{bmatrix} 10 \\ -5 \\ 15 \end{bmatrix}$

$$\det \vec{A} = \begin{bmatrix} 2 & -6 & 10 \\ -1 & 3 & -5 \\ 3 & -9 & 15 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 10 & 2 & -6 \\ -1 & 3 & -5 & -1 & 3 \\ 3 & -9 & 15 & 3 & -9 \end{bmatrix}$$
$$\det \vec{A} = 90 + 90 + 90 - (90 + 90 + 90)$$
$$\det \vec{A} = 0$$

Since $\det \vec{A} = 0$, the vectors $A_1, \vec{A_2}, A_3$ are **linearly dependent**.

4. What is the rank of the following vectors?

(a) $\vec{A_1} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \vec{A_2} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}, \vec{A_3} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$$\mathbf{rank}(A) = \begin{bmatrix} 3 & 4 & 2 \\ 5 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix} R_2 - \tfrac{5}{3}R_1 \to R_2 \begin{bmatrix} 3 & 4 & 2 \\ 0 & \frac{-14}{3} & \frac{-1}{3} \\ -1 & 4 & 1 \end{bmatrix} R_3 - \tfrac{-1}{3}R_1 \to R_3$$
$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & \frac{-14}{3} & \frac{-1}{3} \\ 0 & \frac{16}{3} & \frac{5}{3} \end{bmatrix} R_3 + \tfrac{8}{7}R_2 \to R_3 \begin{bmatrix} 3 & 4 & 2 \\ 0 & \frac{-14}{3} & \frac{-1}{3} \\ 0 & 0 & \frac{9}{7} \end{bmatrix}$$

Since there are three non-zero rows in the row-echelon form, hence, rank(A) = 3.

(b) $\vec{A_1} = \begin{bmatrix} -5 \\ -10 \\ -70 \end{bmatrix}, \vec{A_2} = \begin{bmatrix} -50 \\ -2 \\ -30 \end{bmatrix}, \vec{A_3} = \begin{bmatrix} -20 \\ -30 \\ -50 \end{bmatrix}$

$$\textbf{rank}(A) = \begin{bmatrix} -5 & -50 & -20 \\ -10 & -2 & -30 \\ -70 & -30 & -50 \end{bmatrix} R_2 - 2R_1 \to R_2 \begin{bmatrix} -5 & -50 & -20 \\ 0 & 98 & 10 \\ -70 & -30 & -50 \end{bmatrix} R_3 - 14R_1 \to R_3$$
$$\begin{bmatrix} -5 & -50 & -20 \\ 0 & 98 & 10 \\ 0 & 670 & 230 \end{bmatrix} R_3 - \tfrac{335}{49} R_2 \to R_3 \begin{bmatrix} -5 & -50 & -20 \\ 0 & 98 & 10 \\ 0 & 0 & \frac{7920}{49} \end{bmatrix}$$

Since there are three non-zero rows in the row-echelon form, hence, rank(A) = 3.

(c) $\vec{A_1} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \vec{A_2} = \begin{bmatrix} -6 \\ 3 \\ -9 \end{bmatrix}, \vec{A_3} = \begin{bmatrix} 10 \\ -5 \\ 15 \end{bmatrix}$

$$\textbf{rank}(A) = \begin{bmatrix} 2 & -6 & 10 \\ -1 & 3 & -5 \\ 3 & -9 & 15 \end{bmatrix} R_2 + \tfrac{1}{2} R_1 \to R_2 \begin{bmatrix} 2 & -6 & 10 \\ 0 & 0 & 0 \\ 3 & -9 & 15 \end{bmatrix} R_3 - \tfrac{3}{2} R_1 \to R_3$$
$$\begin{bmatrix} 2 & -6 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are only one non-zero row in the row-echelon form, hence, rank(A) = 1.

5. Determine the angle between the following vectors.

(a) $\vec{a} = (3, -4, -1)$ and $\vec{b} = (0, 5, 2)$

Since a or b are non-zero vectors, then $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta \\ (3, -4, -1) \cdot (0, 5, 2) &= \|(3, -4, -1)\| \cdot \|(0, 5, 2)\| \cdot \cos\theta \\ (3)(0) + (-4)(5) + (-1)(2) &= \left[\sqrt{3^2 + (-4)^2 + (-1)^2} \cdot \sqrt{0^2 + (5)^2 + (2)^2}\right] \cdot \cos\theta \\ -22 &= 27.45906043549196 \cdot \cos\theta \\ \frac{-22}{27.45906043549196} &= \cos\theta \\ \theta &= \cos^{-1} \frac{-22}{27.45906043549196} \\ \theta &= 2.5 \quad \text{(radians)} \\ \theta &= 2.5 \cdot \frac{180}{\pi} \\ \theta &= 143.24 \quad \text{(degrees)} \end{aligned}$$

(b) $\vec{a} = (1,1,1)$ and $\vec{b} = (3,3,3)$
Since a or b are non-zero vectors, then $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta \\
(1,1,1) \cdot (3,3,3) &= \|(1,1,1)\| \cdot \|(3,3,3)\| \cdot \cos\theta \\
(1)(3) + (1)(3) + (1)(3) &= \left[\sqrt{1^2 + 1^2 + 1^2} \cdot \sqrt{3^2 + 3^2 + 3^2}\right] \cdot \cos\theta \\
9 &= 9 \cdot \cos\theta \\
\frac{9}{9} &= \cos\theta \\
\theta &= \cos^{-1} 1 \\
\theta &= 0 \quad \text{(radians)} \\
\theta &= 0 \cdot \frac{180}{\pi} \\
\theta &= 0 \quad \text{(degrees)}
\end{aligned}$$

(c) $\vec{a} = (-3,4,1)$ and $\vec{b} = (0,-5,-2)$
Since a or b are non-zero vectors, then $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta \\
(-3,4,1) \cdot (0,-5,-2) &= \|(-3,4,1)\| \cdot \|(0,-5,-2)\| \cdot \cos\theta \\
(-3)(0) + (4)(-5) + (1)(-2) &= \left[\sqrt{(-3)^2 + (4)^2 + (1)^2} \cdot \sqrt{(0)^2 + (-5)^2 + (-2)^2}\right] \cdot \cos\theta \\
-22 &= 27.45906043549196 \cdot \cos\theta \\
\frac{-22}{27.45906043549196} &= \cos\theta \\
\theta &= \cos^{-1} \frac{-22}{27.45906043549196} \\
\theta &= 2.5 \quad \text{(radians)} \\
\theta &= 2.5 \cdot \frac{180}{\pi} \\
\theta &= 143.24 \quad \text{(degrees)}
\end{aligned}$$