CALCULUS

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OUTLINES

- Function
- Limit and Continuity
- Derivative
- Integral

LIMIT AND CONTINUITY

WHY LIMIT

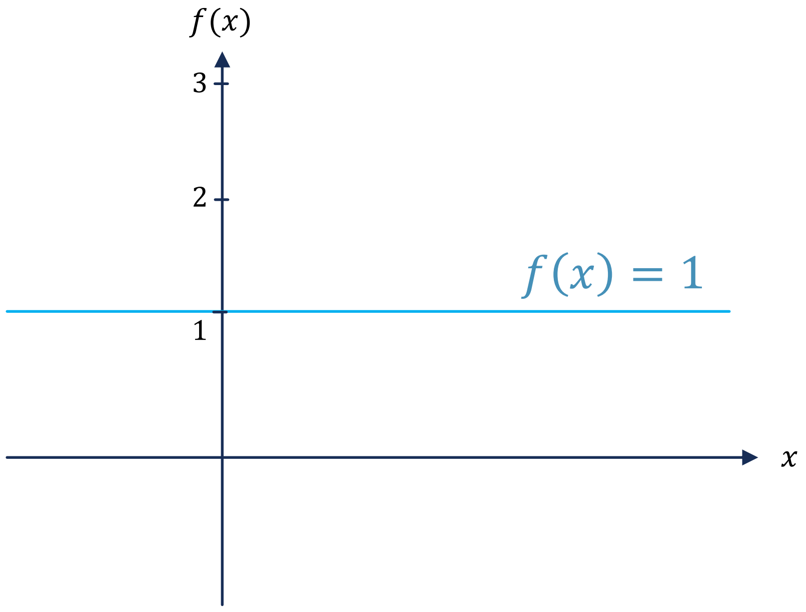
- Given a function

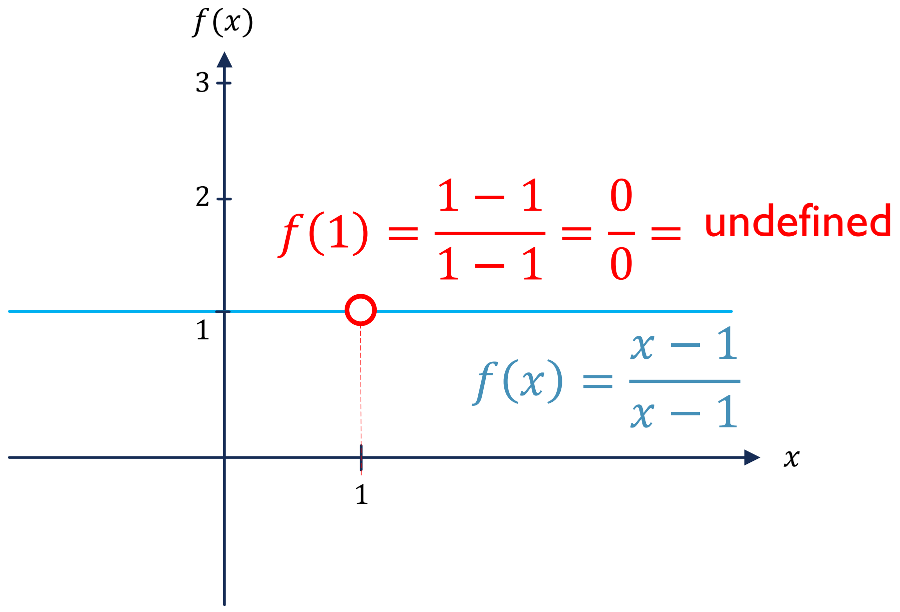
$$f(x) = \frac{x-1}{x-1}$$

Q: Can $f(x)$ be simplified as $f(x) = 1$?

WHY LIMIT

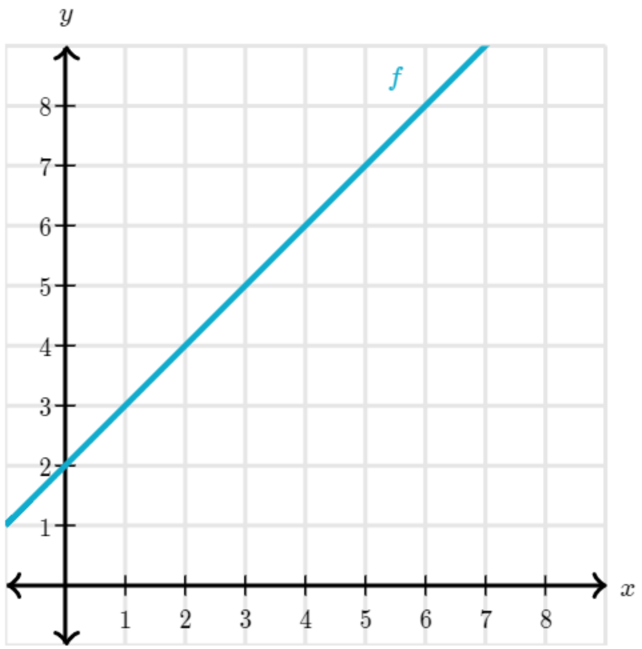
- Plotting





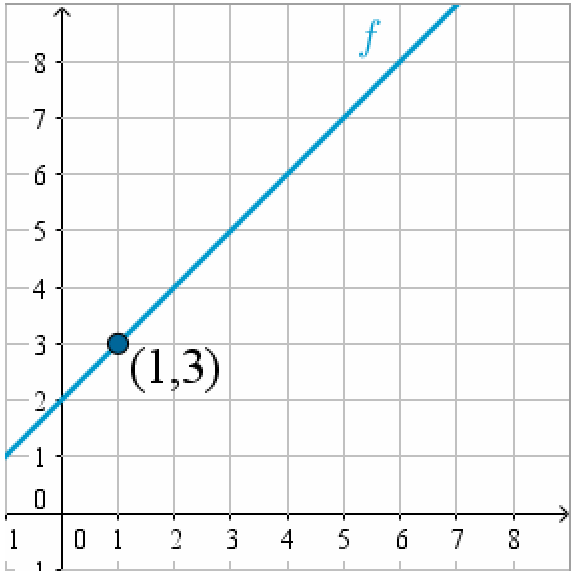
APPROACHING

- Given a function $f(x) = x + 2$



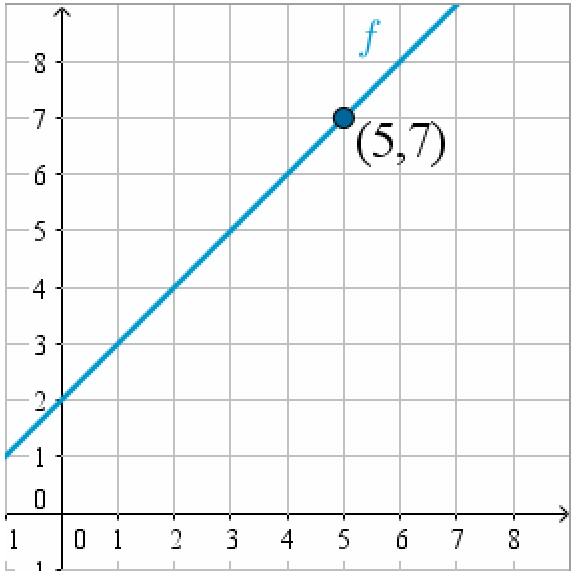
APPROACHING

- Approaching from the left from $x = 1$ to $x = 3$



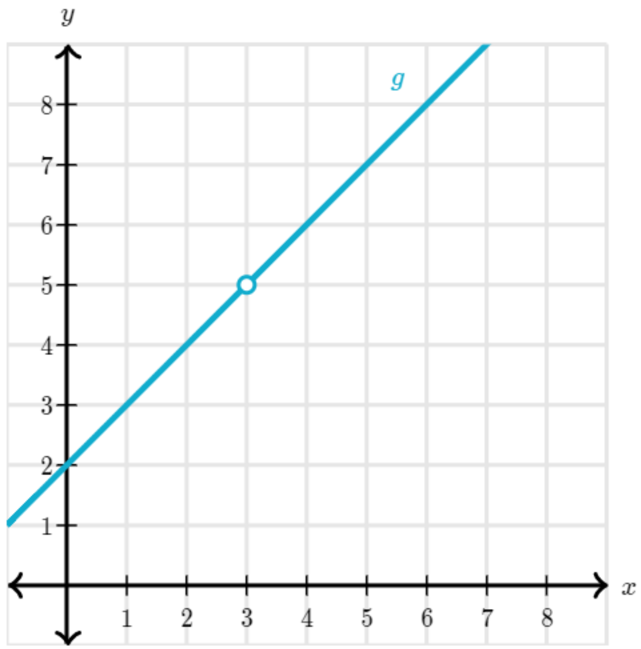
APPROACHING

- Approaching from the right from $x = 5$ to $x =$



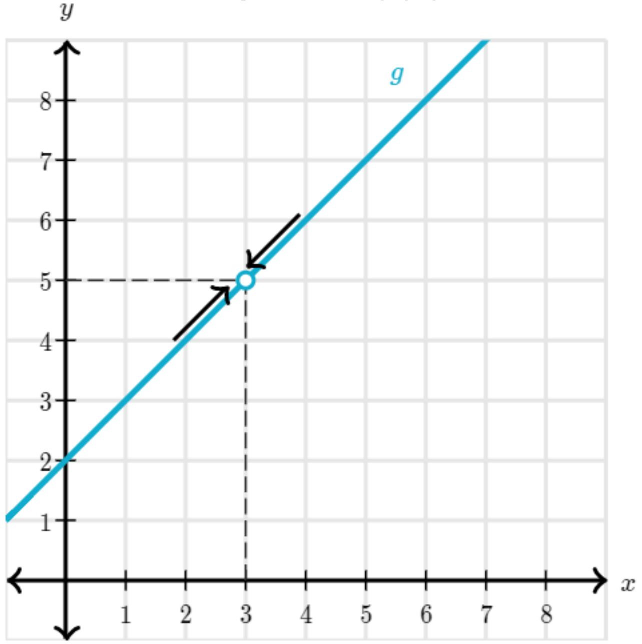
LIMIT AND FUNCTION VALUE

- Given $g(x) = x + 2, \forall_x | (x \in \mathbb{R}) \cap (x \neq 3)$



LIMIT AND FUNCTION VALUE

- The limit of g(x) when x is approach to 3 is NOT equal to g(3)



NOTATION

- Formal

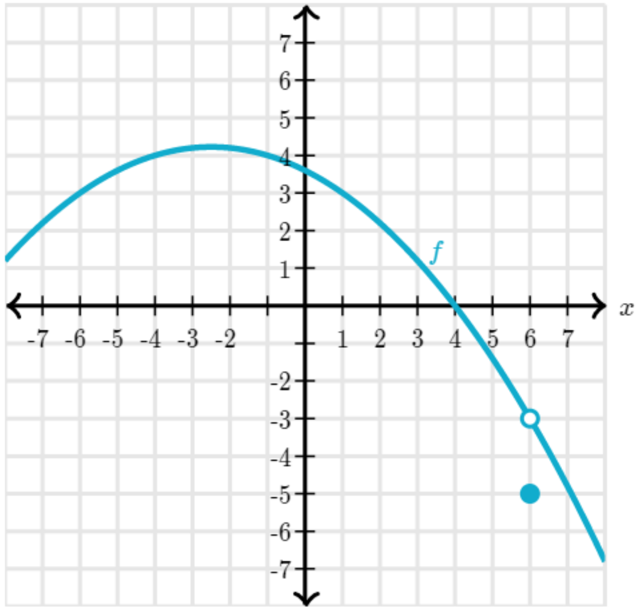
"The limit of..." "...the function f..."

$$\lim_{x \to 3} f(x)$$

"...as x approaches 3."

EXERCISE 1

- What is a reasonable estimate for $\lim_{x \to 6} f(x)$



(a) 5
(b) -3
(c) 6
(d) Limit does not exist

INFINITY CLOSE

- Consider $f = x + 2$ when x is approaching to 3

x	$f(x)$
2.9	4.9
2.99	4.99
$\underbrace{2.999}_{\text{close to } 3}$	$\underbrace{4.999}_{\text{close to } 5}$

Approaching from the left

INFINITY CLOSE

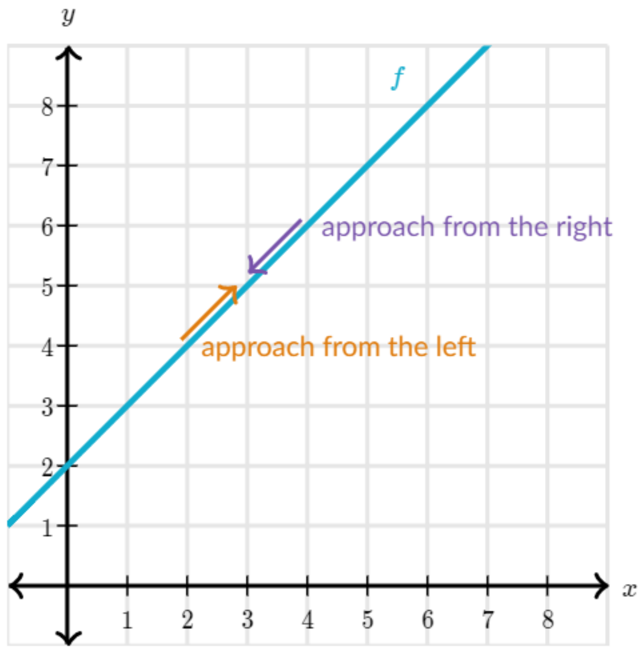
- Consider $f = x + 2$ when x is approaching to 3

x	$f(x)$
3.1	5.1
3.01	5.01
$\underbrace{3.001}_{\text{close to } 3}$	$\underbrace{5.001}_{\text{close to } 5}$

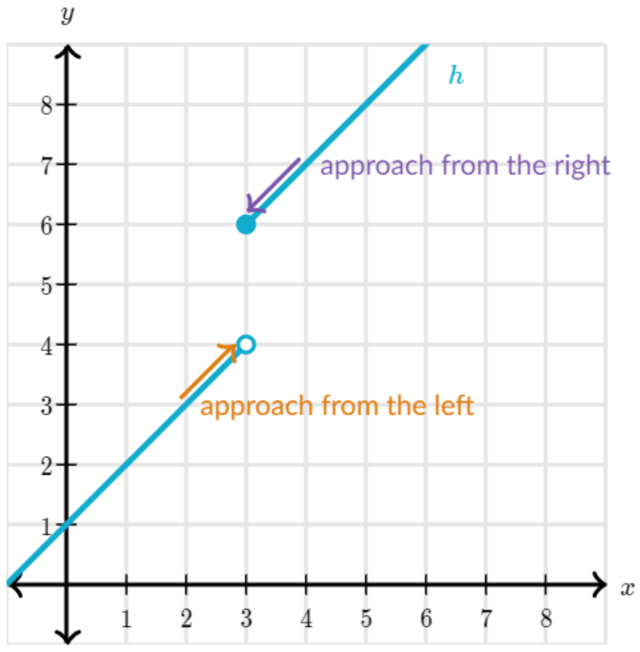
Approaching from the right

CONTINUITY

- A function is continuous if and only if the **limit does exist for all x**



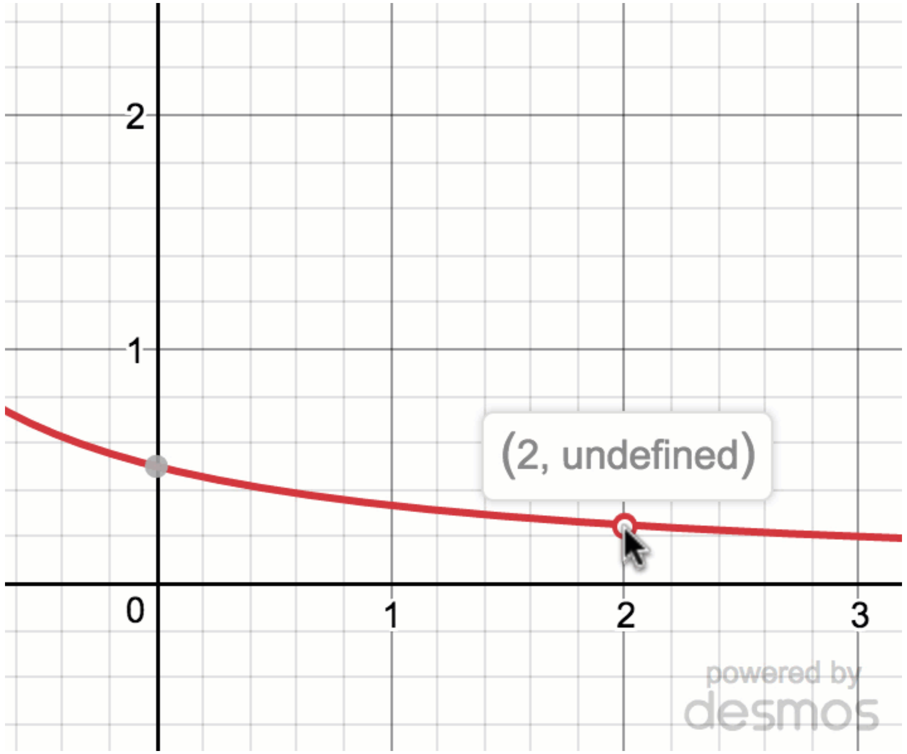
Limit does exist



Limit does not exist

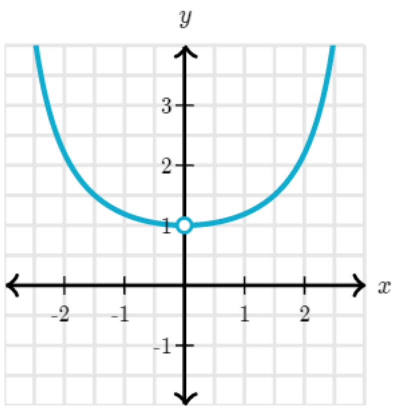
ESTIMATING LIMIT FROM GRAPH

- $\lim_{x \to 2} \frac{x-2}{x^2-4}$



ESTIMATING LIMIT FROM GRAPH

- Consider the following graph



LIMIT PROPERTIES

- Multiplicative factor

$$\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$$

- Distributivity (sum or difference)

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

  - where

$$\lim_{x\to a}(f(x) \pm g(x)) \neq \infty \pm \infty$$

LIMIT PROPERTIES

- Distributivity (Multiplication)

$$\lim_{x \to a}(f(x)g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

  - where

$$\lim_{x \to a} f(x) \lim_{x \to a} g(x) \neq \infty \times \infty$$

- Distributivity (Division)

$$\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

  - where

$$\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \neq \frac{\infty}{\infty}$$

LIMIT PROPERTIES

- Exponential

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n \qquad n \in \mathbb{R}$$

- Constant

$$\lim_{x \to a} c = c \qquad c \in \mathbb{R}$$

COMPUTING LIMITS

- If a function is continuous, limit is computed by using substitution

$$\lim_{x \to a} f(x) = f(a)$$

- Not all cases can be use substitution
  - Example: Zero divided by zero

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

COMPUTING LIMIT

- Handling zero divided by zero using algebra
  - Example: Determine $\lim_{h \to 0} \frac{2(-3+h)^2 - 18}{h}$

$$\begin{aligned}
\lim_{h \to 0} \frac{2(-3+h)^2 - 18}{h} &= \lim_{h \to 0} \frac{2\left(9 - 6h + h^2\right) - 18}{h} \\
&= \lim_{h \to 0} \frac{18 - 12h + 2h^2 - 18}{h} \\
&= \lim_{h \to 0} \frac{-12h + 2h^2}{h}
\end{aligned}$$

If we make substitution now, limit is undefined

COMPUTING LIMIT

- Example (cont.)

Modifying using factorization

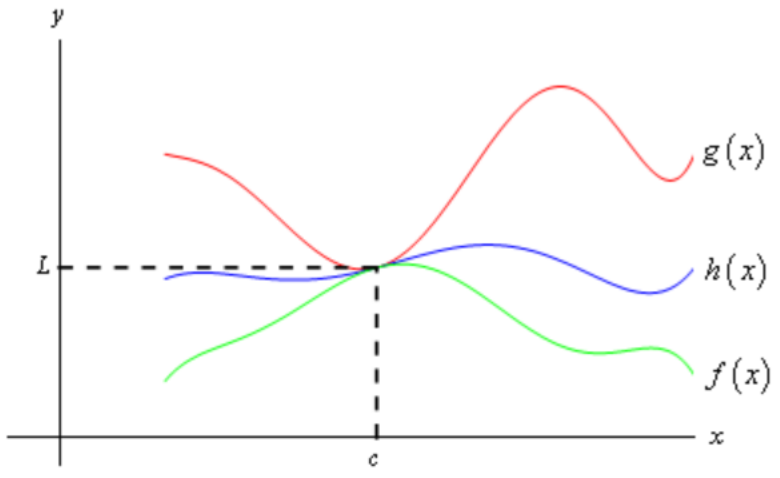
$$
\begin{aligned}
&= \lim_{h \to 0} \frac{-12h + 2h^2}{h} \\
&= \lim_{h \to 0} \frac{h\left(-12 + 2h\right)}{h} \\
&= \lim_{h \to 0} \; -12 + 2h = -12
\end{aligned}
$$

COMPUTING LIMIT

- Exercise (10 minutes): Determine $\lim_{t \to 4} \dfrac{t - \sqrt{3t+4}}{4 - t}$

SQUEEZE THEOREM

- Given f(x), g(x) and h(x)



Suppose that for all x on $[a, b]$ (except possibly at $x = c$),

If $f(x) \leq h(x) \leq g(x)$

And

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$$

Hence

$$\lim_{x \to c} h(x) = L$$

INFINITE LIMITS

- Definition

$$\lim_{x \to a} f(x) = \infty$$

  - If $f(x)$ is arbitrarily large and **positive** for all x sufficiently close to a, from both sides, without letting $x = a$

$$\lim_{x \to a} f(x) = -\infty$$

  - If $f(x)$ is arbitrarily large and **negative** for all x sufficiently close to a, from both sides, without letting $x = a$

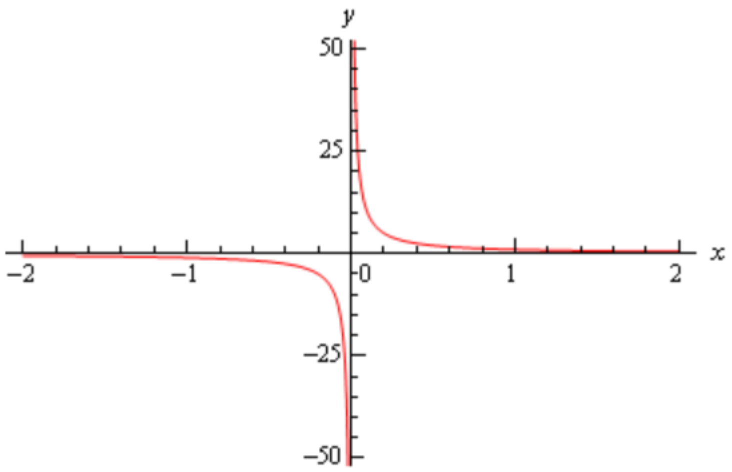
INFINITE LIMITS

- Example: Determine these limits

$$f(x) = \frac{1}{x}$$

$$\lim_{x \to 0^+} \frac{1}{x} \qquad \lim_{x \to 0^-} \frac{1}{x} \qquad \lim_{x \to 0} \frac{1}{x}$$

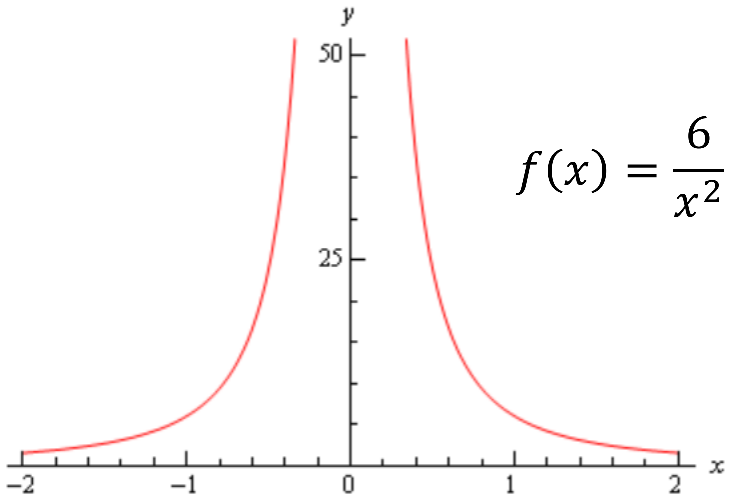
x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10000	0.0001	10000



INFINITE LIMITS

- Example: Determine these limits

$$\lim_{x \to 0^+} \frac{6}{x^2} \qquad \lim_{x \to 0^-} \frac{6}{x^2} \qquad \lim_{x \to 0} \frac{6}{x^2}$$



$$f(x) = \frac{6}{x^2}$$

$$\lim_{x \to 0^+} \frac{6}{x^2} = \infty \qquad \lim_{x \to 0^-} \frac{6}{x^2} = \infty \qquad \lim_{x \to 0} \frac{6}{x^2} = \infty$$

INFINITE LIMITS

- Facts: Given f(x) and g(x) and suppose

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L \qquad L \in \mathbb{R}$$

- Therefore

$$\lim_{x\to c}(f(x) \pm g(x)) = \infty$$

- If $L > 0$

$$\lim_{x\to c}(f(x)g(x)) = \infty$$

- If $L < 0$

$$\lim_{x\to c}(f(x)g(x)) = -\infty$$

- And

$$\lim_{x\to c}\frac{g(x)}{f(x)} = 0$$

LIMIT AT INFINITY

- Definition

$$\lim_{x \to \infty} f(x) \quad \text{or} \quad \lim_{x \to -\infty} f(x)$$

- Facts I
  - If r is a positive rational number and c is any real number, then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0$$

  - If r is a positive rational number, c is any real number and x^r is defined for $x < 0$, then

$$\lim_{x \to -\infty} \frac{c}{x^r} = 0$$

LIMIT AT INFINITY

- Example: Determine $\lim_{x \to \infty} (2x^4 - x^2 - 8x)$
  - If we use substitution

$$\lim_{x \to \infty} (2x^4 - x^2 - 8x) = \infty - \infty - \infty$$

  - Applying factorization

$$\lim_{x \to \infty} (2x^4 - x^2 - 8x) = \lim_{x \to \infty} \left[x^4 \left(2 - \frac{1}{x^2} - \frac{8}{x^3} \right) \right]$$

  - Consider each term

$$\lim_{x \to \infty} x^4 = \infty \qquad \lim_{x \to \infty} \left(2 - \frac{1}{x^2} - \frac{8}{x^3} \right) = 2$$

$$\lim_{x \to \infty} (2x^4 - x^2 - 8x) = \infty$$

LIMIT AT INFINITY

- Exercise (10 minutes): Determine $\lim_{t \to -\infty} \left(\frac{1}{3} t^5 + 2t^3 - t^2 + 8 \right)$

$$\lim_{t \to -\infty} \left(\frac{1}{3} t^5 + 2t^3 - t^2 + 8 \right) = -\infty$$

- HINT: Negative number raised to an odd power is still negative

LIMIT AT INFINITY

- Fact II
  - If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n $(a_n \neq 0)$, then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$$

and

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n$$

LIMIT AT INFINITY

- Example: Determine $\lim_{x\to\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7}$
- Substitution is not applicable

$$\lim_{x\to\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} = \frac{\infty}{-\infty}$$

- Apply factorization

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^4 - x^2 + 8x}{-5x^4 + 7} &= \lim_{x\to\infty} \frac{x^4\left(2 - \frac{1}{x^2} + \frac{8}{x^3}\right)}{x^4\left(-5 + \frac{7}{x^4}\right)} \\ &= \lim_{x\to\infty} \frac{2 - \frac{1}{x^2} + \frac{8}{x^3}}{-5 + \frac{7}{x^4}} \\ &= \frac{2 + 0 + 0}{-5 + 0} \\ &= -\frac{2}{5}\end{aligned}$$

LIMIT AT INFINITY

- Exponential function

$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to \infty} e^{-x} = 0$$

$$\lim_{x \to -\infty} e^{-x} = \infty$$

LIMIT AT INFINITY

- Example: Determine $\lim_{x \to \infty} \mathbf{e}^{2-4x-8x^2}$

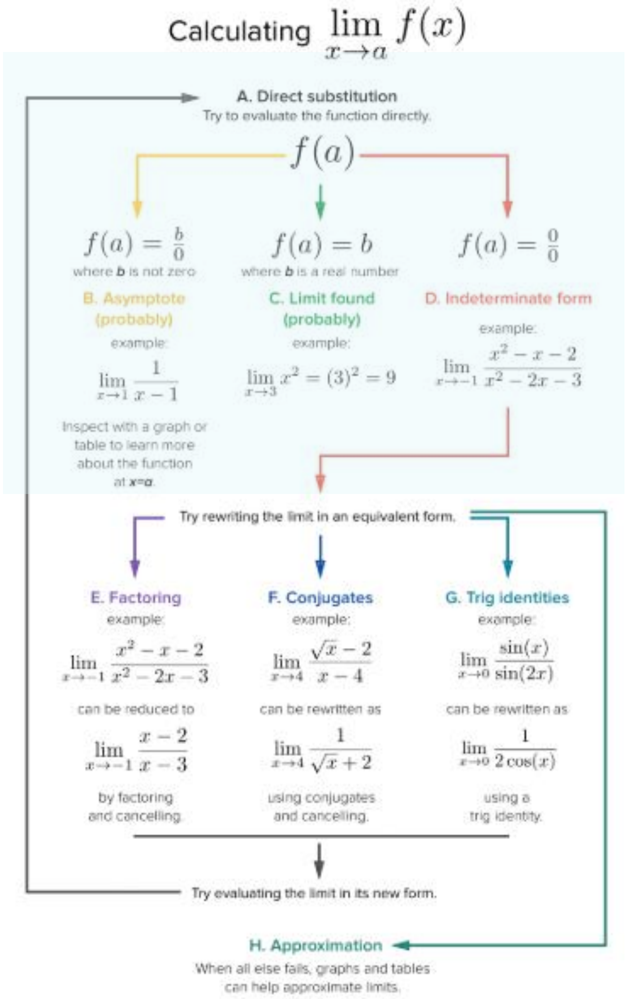
- Consider the exponential part

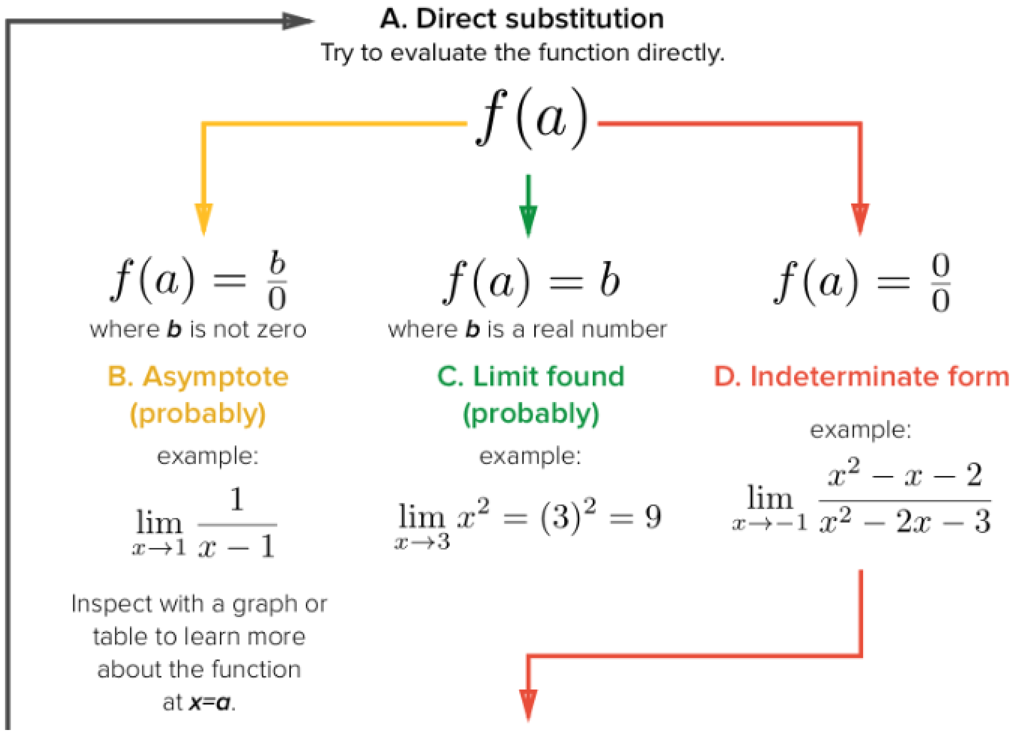
$$\lim_{x \to \infty} \left(2 - 4x - 8x^2\right) = -\infty$$

- Therefore

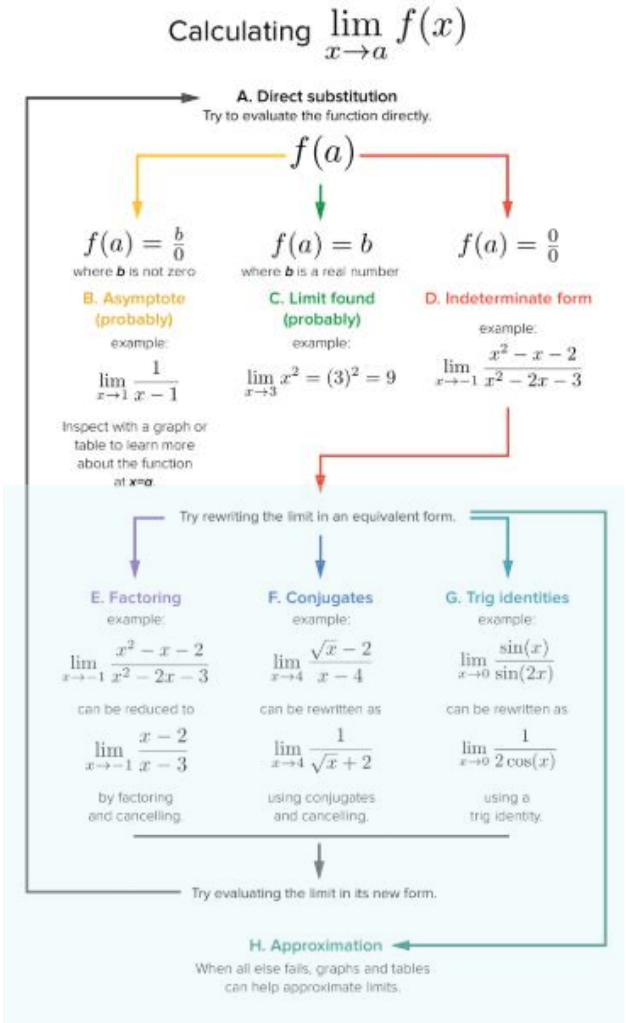
$$\lim_{x \to \infty} \mathbf{e}^{2-4x-8x^2} = 0$$

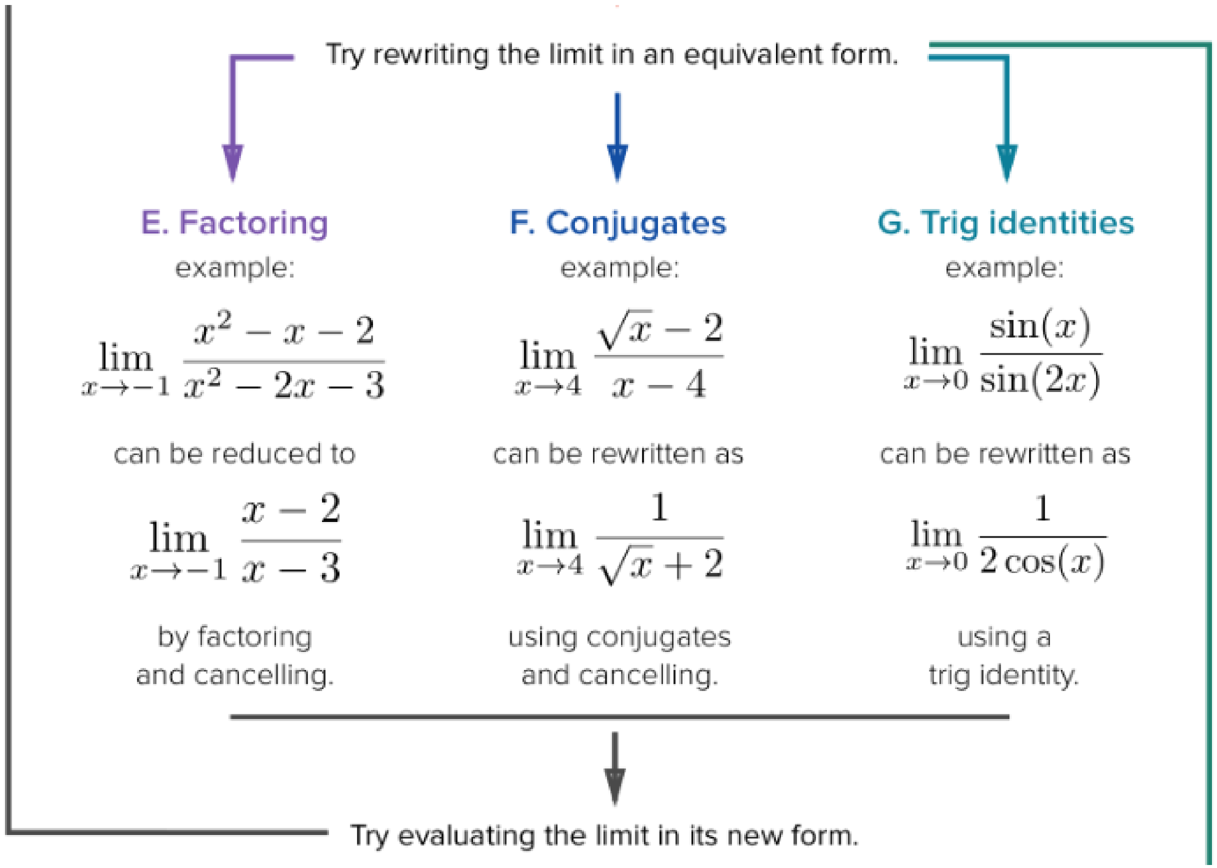
STRATEGY TO FIND LIMIT





STRATEGY TO FIND LIMIT





DERIVATIVE

DEFINITION

- The derivative of $f(x)$ with respect to x is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Notation

$$f'(x) = \frac{d}{dx} f(x)$$

DEFINITION

- Example: Determine derivative of

$$f\left(x\right) = 2x^2 - 16x + 35$$

- Substitution $f(x)$ to definition of derivative

$$\begin{aligned} f'\left(x\right) &= \lim_{h \to 0} \frac{f\left(x+h\right) - f\left(x\right)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 - 16\left(x+h\right) + 35 - \left(2x^2 - 16x + 35\right)}{h} \end{aligned}$$

DEFINITION

- Example (cont.)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - 16h}{h} \\
&= \lim_{h\to 0} \frac{h(4x + 2h - 16)}{h} \\
&= \lim_{h\to 0} 4x + 2h - 16 \\
&= 4x - 16
\end{aligned}
$$

DIFFERENTIABLE

- A function $f(x)$ is called differentiable at $x = a$ if $f'(a)$ exists
- $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval

- Theorem

> If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$

  - This theorem does not work in reverse

DIFFERENTIABLE

- Example $f(x) = |x|$

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{|0+h| - |0|}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \end{aligned}$$

DIFFERENTIABLE

- Example (cont.)

$$|h| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$$

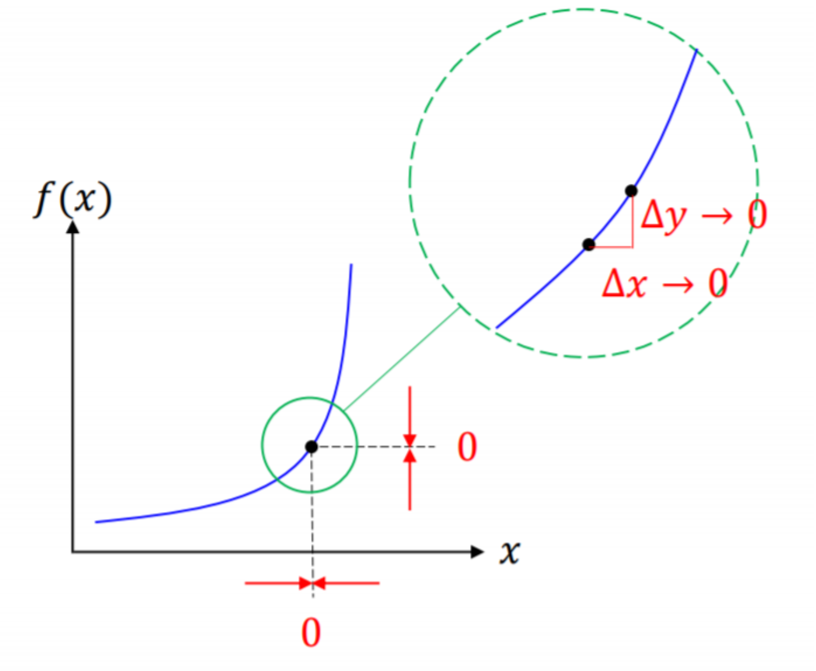
$$\begin{aligned} \lim_{h \to 0^-} \frac{|h|}{h} &= \lim_{h \to 0^-} \frac{-h}{h} \qquad \text{because } h < 0 \text{ in a left-hand limit.} \\ &= \lim_{h \to 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0^+} \frac{|h|}{h} &= \lim_{h \to 0^+} \frac{h}{h} \qquad \text{because } h > 0 \text{ in a right-hand limit.} \\ &= \lim_{h \to 0^+} 1 \\ &= 1 \end{aligned}$$

The two one-sided limits are different and so $\lim_{h \to 0} \frac{|h|}{h}$ does not exist

INTERPRETATION OF DERIVATIVE

- Rate of change
  - If $f(x)$ represents a quantity at and x then $f'(a)$ represents the instantaneous rate of change of $f(x)$ at $x = a$



INTERPRETATION OF DERIVATIVE

- Example: Suppose that the amount of water in a holding tank at t minutes is given by $V(t) = 2t^2 - 16t + 35$. Determine each of the following.
  - Is the volume of water in the tank increasing or decreasing at $t = 1$ minute
  - When the volume of water in the tank is not changing

INTERPRETATION OF DERIVATIVE

- Example (Solution)
  - Derivative

$$V'(t) = 4t - 16 \qquad \text{OR} \qquad \frac{dV}{dt} = 4t - 16$$

  - at $t = 1$

$$V'(1) = -12 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=1} = -12$$

  - Conclusion: water level is decreasing at this time

INTERPRETATION OF DERIVATIVE

- Example (Solution)
  - Derivative is set to zero

$$V'(t) = 0 \qquad \text{OR} \qquad \frac{dV}{dt} = 0$$

  - Solving equation

$$4t - 16 = 0$$
$$t = 4$$

  - Conclusion: Water is not changing at $t = 4$

DIFFERENTIATION FORMULAS

- Properties I

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

- Properties II

$$(cf(x))' = cf'(x)$$

DIFFERENTIATION FORMULAS

- Product rule
  - If $f(x)$ and $g(x)$ are differentiable then the product is differentiable and defined as

$$(fg)' = fg' + gf'$$

- Quotient rule
  - If $f(x)$ and $g(x)$ are differentiable then the quotient is differentiable and defined as

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

DIFFERENTIATION FORMULAS

- Example: Differentiate the function $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

- Conversion

$$y = x^{\frac{2}{3}}\left(2x - x^2\right)$$

- Differentiate

$$y' = \frac{2}{3}x^{-\frac{1}{3}}\left(2x - x^2\right) + x^{\frac{2}{3}}(2 - 2x)$$

- Rearrange format

$$y' = \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{3}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

DIFFERENTIATION FORMULAS

- Exercise: Given differentiable functions $f(x), g(x)$ and $h(x)$,determine $(fgh)'$

DERIVATIVE OF TRIGONOMETRY FUNCTION

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \qquad \frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

DERIVATIVE OF TRIGONOMETRY FUNCTION

- Example: Determine derivative of $g(x) = 3\sec(x) - 10\cot(x)$

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x) \qquad \frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \qquad \frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

$$\begin{aligned} g'(x) &= 3\sec(x)\tan(x) - 10\left(-\csc^2(x)\right) \\ &= 3\sec(x)\tan(x) + 10\csc^2(x) \end{aligned}$$

DERIVATIVE OF EXPONENTIAL AND LOGARITHMIC FUNCTION

- Exponential function

$$f(x) = a^x$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x \left(a^h - 1\right)}{h} \end{aligned}$$

$$\begin{aligned} f'(x) &= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \\ &= f'(0)\, a^x \end{aligned}$$

DERIVATIVE OF EXPONENTIAL AND LOGARITHMIC FUNCTION

- Exercise (10 minutes): Given $f(x) = \frac{1}{1+e^{-x}}$, Determine derivative
  - Answer $f(x)(1 - f(x))$

DERIVATIVE OF EXPONENTIAL AND LOGARITHMIC FUNCTION

- Three definitions of the natural number

1. $\mathbf{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
2. $\mathbf{e}$ is the unique positive number for which $\lim_{h\to 0}\frac{\mathbf{e}^h-1}{h} = 1$
3. $\mathbf{e} = \sum_{n=0}^{\infty}\frac{1}{n!}$

- For the natural exponential function $f(x) = e^x$

$$f'(0) = \lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

DERIVATIVE OF EXPONENTIAL AND LOGARITHMIC FUNCTION

- Hence

$$f(x) = \mathbf{e}^x \qquad \Rightarrow \qquad f'(x) = \mathbf{e}^x$$

$$f(x) = a^x \qquad \Rightarrow \qquad f'(x) = a^x \ln(a)$$

DERIVATIVE OF EXPONENTIAL AND LOGARITHMIC FUNCTION

- Logarithmic function

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\begin{aligned}\frac{d}{dx}(\log_a x) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \frac{1}{\ln a}\frac{d}{dx}(\ln x) \\ &= \frac{1}{x\ln a}\end{aligned}$$

DERIVATIVES OF INVERSE TRIGONOMETRY FUNCTION

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

DERIVATIVE OF HYPERBOLIC FUNCTION

$$\sinh x = \frac{\mathbf{e}^x - \mathbf{e}^{-x}}{2} \qquad \cosh x = \frac{\mathbf{e}^x + \mathbf{e}^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

CHAIN RULE

- Given two differentiable functions $f(x)$ and $g(x)$
  - A composite function $F(x)$ is defined as

$$F(x) = (fog)(x) = f(g(x))$$

  - Derivative of $F(x)$ is defined as

$$F'(x) = f'(g(x))g'(x)$$

  - If $y = f(u)$ and $u = g(x)$ then derivative of y is defined as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

CHAIN RULE

- Example: Use chain rule to differentiate $R(z) = \sqrt{5z - 8}$

$$f(z) = \sqrt{z} \qquad g(z) = 5z - 8$$

$$f'(z) = \frac{1}{2\sqrt{z}} \qquad g'(z) = 5$$

$$\begin{aligned} R'(z) &= f'(g(z))\; g'(z) \\ &= f'(5z-8)\; g'(z) \\ &= \frac{1}{2}(5z-8)^{-\frac{1}{2}}\;(5) \\ &= \frac{1}{2\sqrt{5z-8}}\;(5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

HIGHER ORDER OF DIFFERENTIATION

- Given a differentiable function

$$f(x) = 5x^3 - 3x^2 + 10x - 5$$

- 1^{st} order derivative

$$f'(x) = 15x^2 - 6x + 10$$

- 2^{nd} order derivative

$$f''(x) = \left(f'(x)\right)' = 30x - 6$$

LOGARITHMIC DIFFERENTIATION

- Given a function $y = \dfrac{x^5}{(1-10x)\sqrt{x^2+2}}$
  - Take the natural logarithmic function on both sides

$$\ln y = \ln\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right)$$

because

$$\frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx}$$
$$= \frac{y'}{y}$$

$$\ln y = \ln(x^5) - \ln\left((1-10x)\sqrt{x^2+2}\right)$$
$$\ln y = \ln(x^5) - \ln(1-10x) - \ln\left(\sqrt{x^2+2}\right)$$

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}(x^2+2)^{-\frac{1}{2}}(2x)}{(x^2+2)^{\frac{1}{2}}}$$
$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

Hence

$$y' = y\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$
$$= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$

RECAP

- Summary

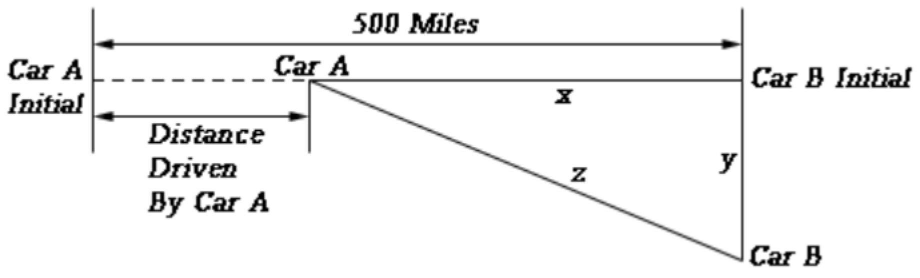
$\frac{d}{dx}\left(a^b\right) = 0$	This is a constant
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	Power Rule
$\frac{d}{dx}\left(a^x\right) = a^x \ln a$	Derivative of an exponential function
$\frac{d}{dx}\left(x^x\right) = x^x\left(1 + \ln x\right)$	Logarithmic Differentiation

APPLICATION OF DERIVATIVES

- Rates of Change
- Critical Points
- Minimum and Maximum Values
- Finding Absolute Extrema
- The Shape of Graph
- The Mean Value Theorem
- Optimization Problems
- L'Hospital's Rule and Indeterminate Forms
- Linear Approximation
- Differentials
- Newton's Method

APPLICATION OF DERIVATIVES

- **Rate of Changes: Example**
  - Two cars start out 500 miles apart
  - Car A is to the west of Car B and starts driving to the east (i.e. towards Car B) at 35 mph and at the same time Car B starts driving south at 50 mph
  - Question 1: After 3 hours of driving at what rate is the distance between the two cars changing?
  - Question 2: Is it increasing or decreasing?



APPLICATION OF DERIVATIVES

- **Rate of Changes: Solution**

  Let y be the distance driven by Car B

  Let x be the distance separating Car A from Car B's initial position

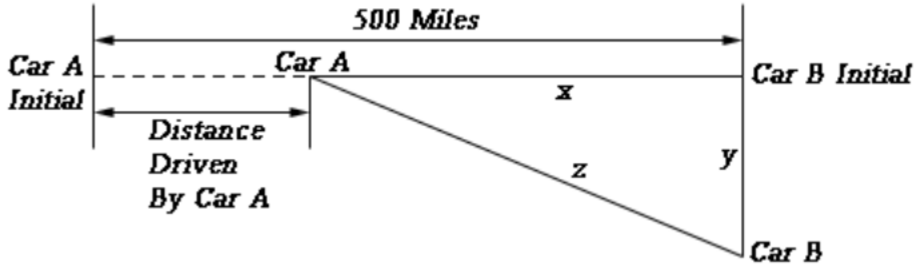
  Let z be the distance separating the two cars

  After 3 hours driving time with have the following values of x and y

$$x = 500 - 35\,(3) = 395 \qquad\qquad y = 50\,(3) = 150$$

  Apply Pythagorean theorem to find z at this time

$$z^2 = 395^2 + 150^2 = 178525 \qquad \Rightarrow \qquad z = \sqrt{178525} = 422.5222$$



APPLICATION OF DERIVATIVES

- **Rate of Changes: Solution (Cont.)**

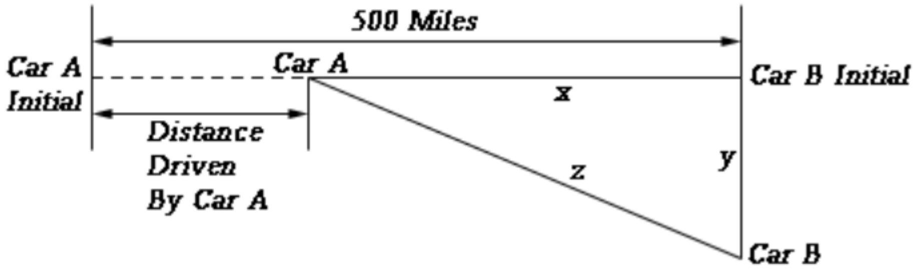
Let x', y' and z' be the rate of distance change over time

The rate of change for the distance between two car after 3 hours is z' where $x' = -35$ and $y' = 50$

$$z = f(x, y) = \sqrt{x^2 + y^2}$$

$$z^2 = x^2 + y^2$$

$$2zz' = 2xx' + 2yy'$$



Substitute the values of x, y and z

$$(422.522)z' = (395)(-35) + (150)(50)$$

$z' = -14.9696$ The distance between 2 cars after 3 hours is decreasing

APPLICATION OF DERIVATIVES

- Critical Points

  Definition: a critical point is a point in the domain of the function where the function is either not differentiable or the derivative is equal to zero

- Example
  - $f(x) = x^2 + 2x + 3$
    - $f(x)$ is differentiable everywhere
    - $f'(x) = 2x + 2$
    - $f(x)$ has a unique critical point -1, because it makes $f'(x) = 0$

APPLICATION OF DERIVATIVES

- Critical Points

  Definition: a critical point is a point in the domain of the function where the function is either not differentiable or the derivative at $x = c$ makes $f'(c) = 0$

- Example
  - $f(x) = x^{\frac{2}{3}}$
    - $f(x)$ is defined for all x and differentiable for $x \neq 0$
    - $f'(x) = \frac{2x^{\frac{-1}{3}}}{3}$
    - $f'(0) = 0$,The critical point is $x = 0$

APPLICATION OF DERIVATIVES

- Critical Points

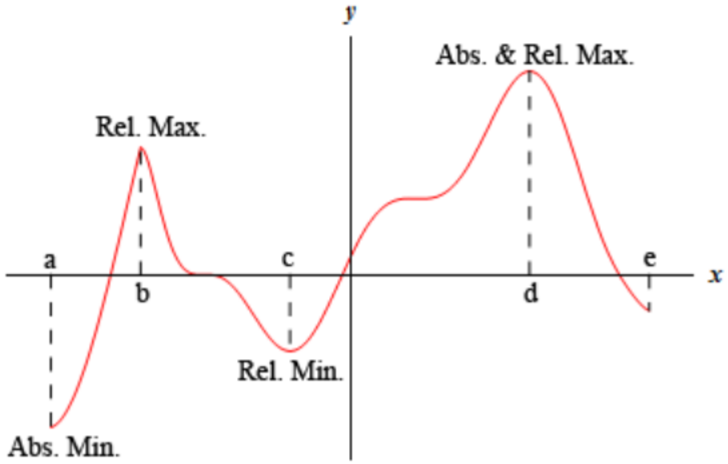
  Definition: a critical point is a point in the domain of the function where the function is either not differentiable or the derivative is equal to zero

- Example
  - $f(x) = \frac{1}{x}$
    - $f(x)$ is differentiable everywhere except $x = 0$
    - $f'(x) = -x^{-2}$
    - $f(x)$ does not have critical point because $x = 0$ is not included in the function domain

APPLICATION OF DERIVATIVES

- **Maximum and Minimum Values**

  Definitions:
  $f(x)$ has an absolute (or global) maximum at $x = c$ if and only if $f(x) \leq f(c)$ for all x
  $f(x)$ has a relative (or local) maximum at $x = c$ if and only if $f(x) \leq f(c)$ for all x in a given interval
  $f(x)$ has an absolute (or global) minimum at $x = c$ if and only if $f(x) \geq f(c)$ for all x
  $f(x)$ has a relative (or local) maximum at $x = c$ if and only if $f(x) \geq f(c)$ for all x in a given interval



APPLICATION OF DERIVATIVES

- Finding Absolute Extrema of $f(x)$ in $[a, b]$

- Example: Find the absolute extrema of $g(t) = 2t^3 + 3t^2 - 12t + 4$ on $[-4, 2]$

Determine derivative of $g(t)$

$$g'(t) = 6t^2 + 6t - 12 = 6(t+2)(t-1)$$

The critical points are $t = -2$ and $t = 1$ which are in the interval $[-4, 2]$

Evaluate the values of $g(t)$ at each critical point and interval

$$g(-2) = 24 \qquad g(1) = -3$$
$$g(-4) = -28 \qquad g(2) = 8$$

APPLICATION OF DERIVATIVES

- The shape of graph

Definitions: Given a function $f(x)$

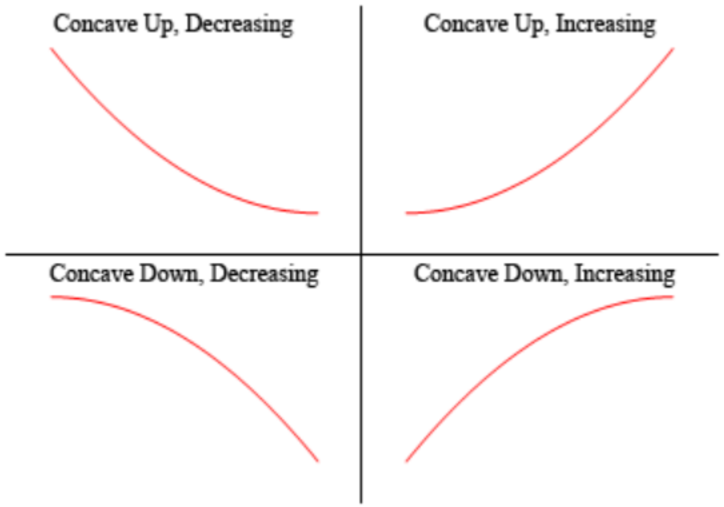
Given any x_1 and x_2 from an interval I where $x_1 < x_2$

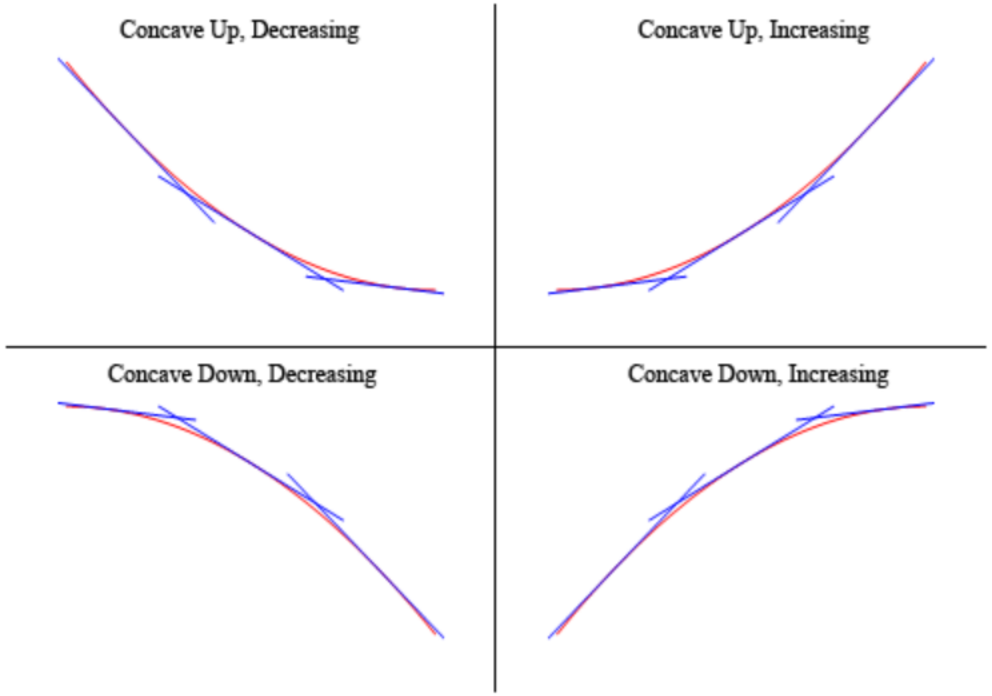
If $f(x_1) < f(x_2)$ then $f(x)$ is **increasing** on I

If $f(x_1) > f(x_2)$ then $f(x)$ is **decreasing** on I

APPLICATION OF DERIVATIVES

- The shape of graph (cont.)





APPLICATION OF DERIVATIVES

- The Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following.

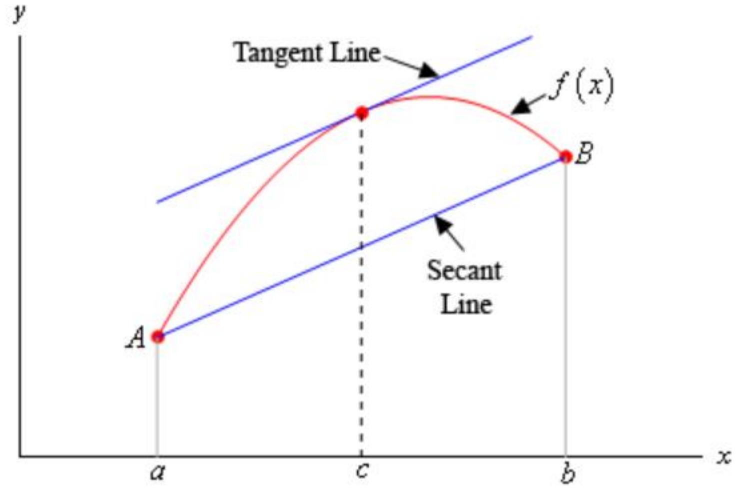
1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).

Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

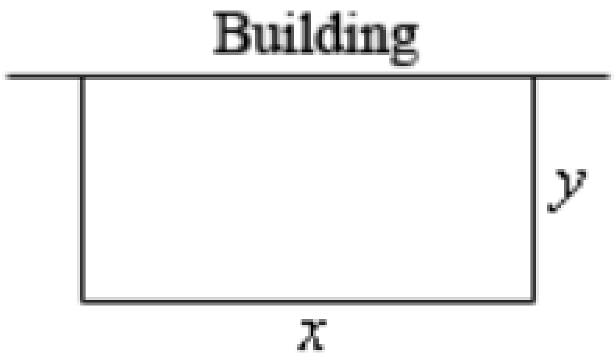
Or,

$$f(b) - f(a) = f'(c)(b - a)$$



APPLICATION OF DERIVATIVES

- Optimization
  - Goal: Maximize or minimize objective subject to constraints
  - Intuitive Example: We need to enclose a rectangular field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won’t need any fencing. Determine the dimensions of the field that will enclose the largest area.



Objective: Maximize area $A = xy$

Subject to the constraint (Fence material quantity): $x + 2y = 500$

$$x = 500 - 2y$$

Therefore

$$A(y) = (500 - 2y)y$$

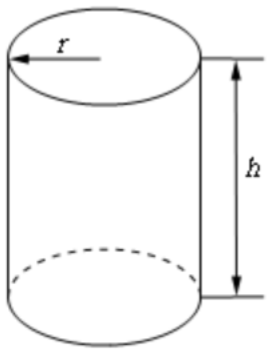
$$= 500y - 2y^2$$

$$A'(y) = 500 - 4y = 0$$

$$\therefore y = 125 \quad x = 250$$

APPLICATION OF DERIVATIVES

- Optimization (Cont.)
  - Exercise (10 minutes): A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.



APPLICATION OF DERIVATIVES

- Indeterminate Forms
  - If the limit of function has indeterminate forms such as

$$(0)(\pm\infty) \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0} \qquad \infty - \infty$$

  - Algebra is required to determine such limit

- L'Hospital's Rule

Suppose that we have one of the following cases,

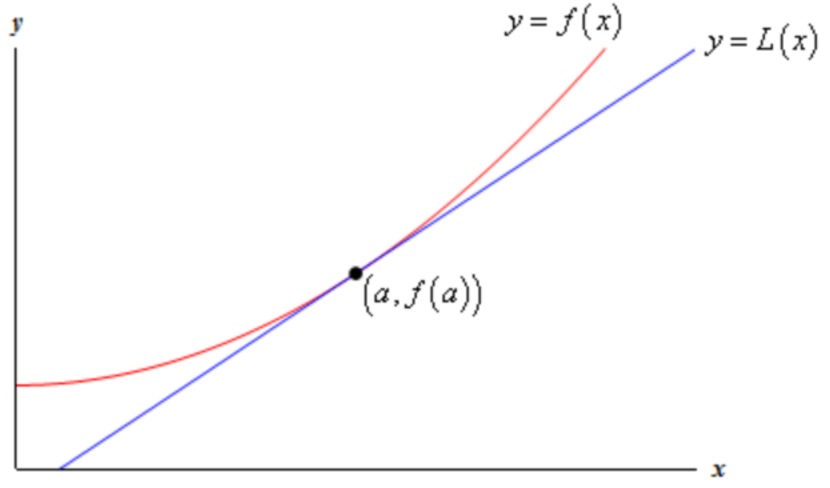
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{OR} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

APPLICATION OF DERIVATIVES

- Linear Approximation
  - Given a function, $f(x)$, we can find its tangent line $L(x)$ at $x = a$



$$L(x) = f(a) + f'(a)(x - a)$$

APPLICATION OF DERIVATIVES

- Differential
  - Given a function $y = f(x)$, we call dy and dx *differentials* and the relationship between them is given by,

$$dy = f'(x)\,dx \quad \text{or} \quad df = f'(x)\,dx$$

  - Example: Compute dy and Δy if $y = \cos(x^2 + 1) - x$ as x changes from $x = 2$ to $x = 2.03$

Compute the actual change (Δy) in y:

$$\Delta y = \cos\left((2.03)^2 + 1\right) - 2.03 - \left(\cos\left(2^2 + 1\right) - 2\right) = 0.083581127$$

Define dy

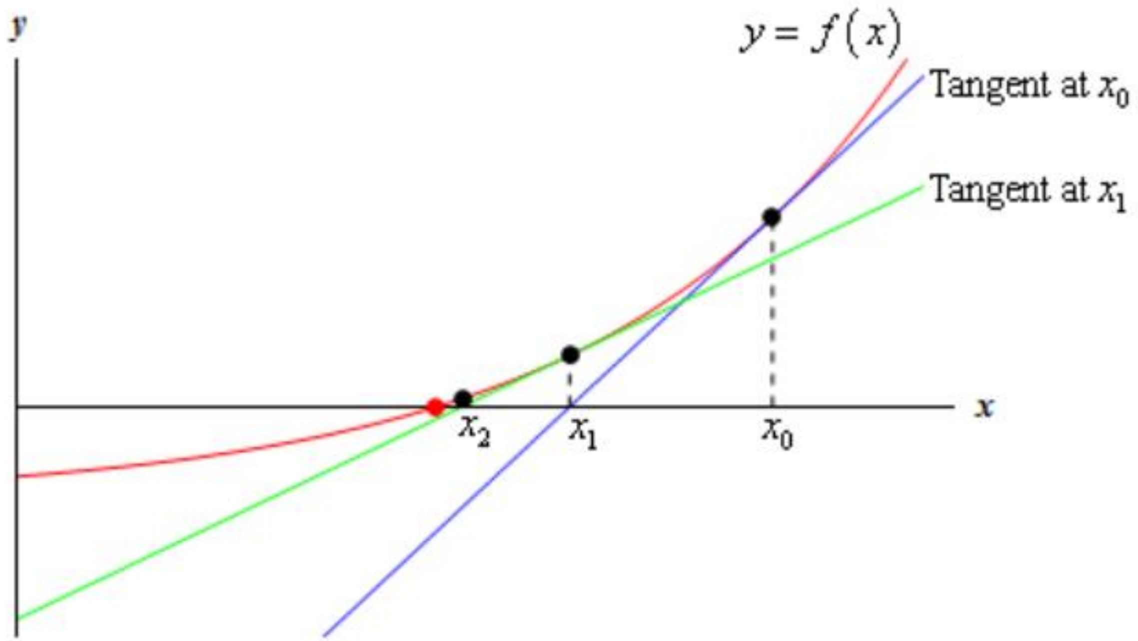
$$dy = \left(-2x\sin\left(x^2 + 1\right) - 1\right)dx$$

Substitute dx which is $\Delta x = 0.03.$ This gives approximation change of y

$$dy = \left(-2\,(2)\sin\left(2^2 + 1\right) - 1\right)(0.03) = 0.085070913$$

APPLICATION OF DERIVATIVES

- Newton's Method



The tangent line equation:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Approximate x_1 on the tangent line obtained from x_0

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Do it repeatedly:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

If x_n is an approximation a solution of $f(x) = 0$ and if $f'(x_n) \neq 0$, the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

INTEGRALS

INDEFINITE INTEGRALS

- Definition

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)\,dx = F(x) + c, \qquad c \text{ is any constant}$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and the "c" is called the **constant of integration**.

INDEFINITE INTEGRALS

- Properties

$\int kf(x)dx = k\int f(x)dx$ where k is a constant

$$\int \big(f(x) \pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx$$

COMPUTING INDEFINITE INTEGRALS

- Formulas

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\int k\,dx = kx + c, \qquad c \text{ and } k \text{ are constants}$$

$$\int \sin x\,dx = -\cos x + c$$
$$\int \sec^2 x\,dx = \tan x + c$$
$$\int \csc^2 x\,dx = -\cot x + c$$

$$\int \cos x\,dx = \sin x + c$$
$$\int \sec x \tan x\,dx = \sec x + c$$
$$\int \csc x \cot x\,dx = -\csc x + c$$

$$\int \mathbf{e}^x\,dx = \mathbf{e}^x + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + c$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + c$$
$$\int \sinh x\,dx = \cosh x + c$$
$$\int \operatorname{sech}^2 x\,dx = \tanh x + c$$
$$\int \operatorname{csch}^2 x\,dx = -\coth x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$$
$$\int \cosh x\,dx = \sinh x + c$$
$$\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$$
$$\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}x + c$$

SUBSTITUTION RULE

- Definition

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \quad \text{where, } u = g(x)$$

- Make integral easier
  - Example: $\int 18x^2 \sqrt[4]{6x^3+5}dx$

    Let $u = 6x^3 + 5$ then $du = 18x^2dx$

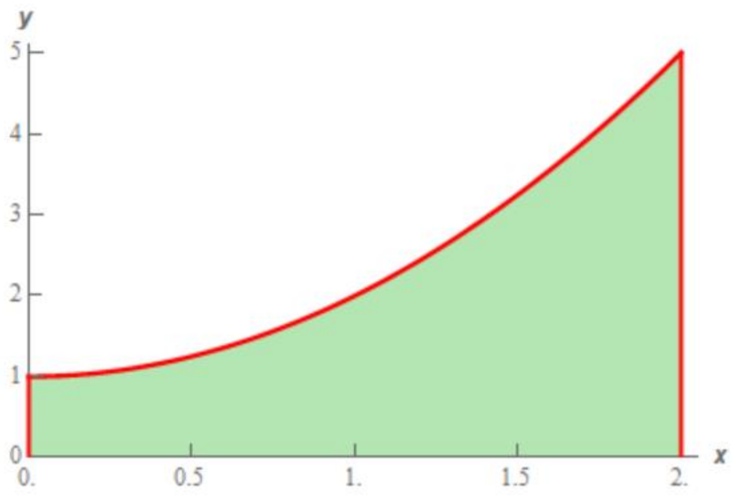
$$\begin{aligned}\int 18x^2\, \sqrt[4]{6x^3+5}\, dx &= \int \left(6x^3+5\right)^{\frac{1}{4}}\, \left(18x^2 dx\right)\\ &= \int u^{\frac{1}{4}}\, du\\ &= \frac{4}{5}u^{\frac{5}{4}} + c = \frac{4}{5}\left(6x^3+5\right)^{\frac{5}{4}} + c\end{aligned}$$

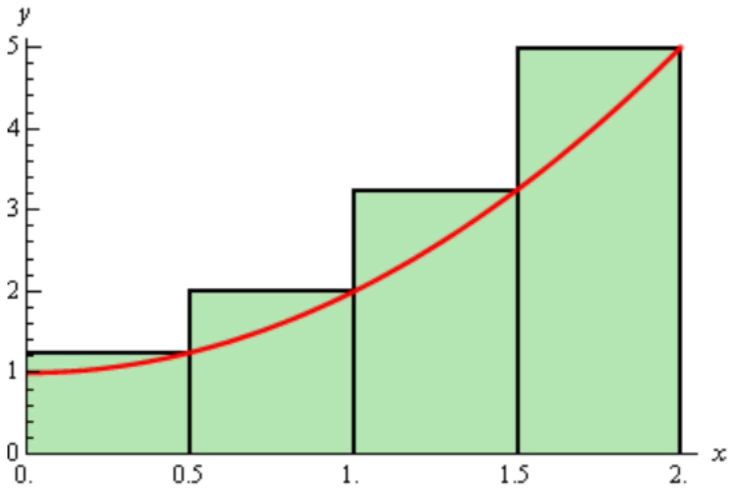
SUBSTITUTION RULE

- Exercise (10 minutes): $\int e^{2t} + \sec(2t)\tan(2t)\, dt$

AREA PROBLEM

- Given a function $f(x) = x^2 + 1$ that is positive in some interval $[a, b]$

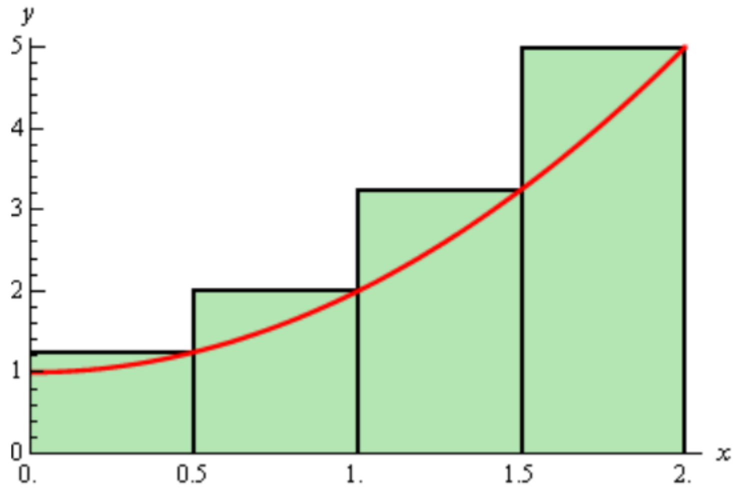




$$\Delta x = \frac{b-a}{n}$$

AREA PROBLEM

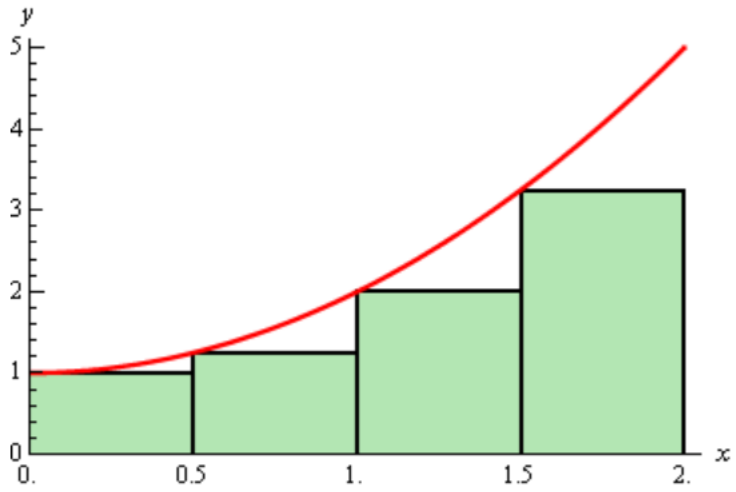
- Given a function $f(x) = x^2 + 1$ that is positive in some interval $[a, b]$



$$
\begin{aligned}
A_r &= \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f(2) \\
&= \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) + \frac{1}{2}(5) \\
&= 5.75
\end{aligned}
$$

AREA PROBLEM

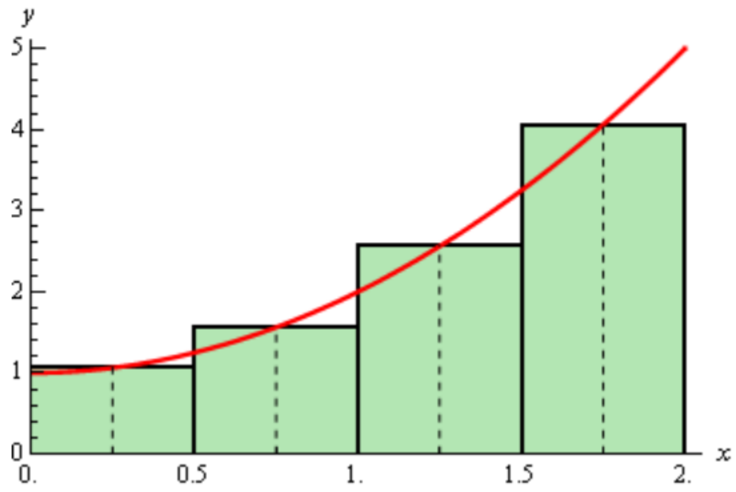
- Given a function $f(x) = x^2 + 1$ that is positive in some interval $[a, b]$



$$
\begin{aligned}
A_l &= \frac{1}{2} f(0) + \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) \\
&= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) \\
&= 3.75
\end{aligned}
$$

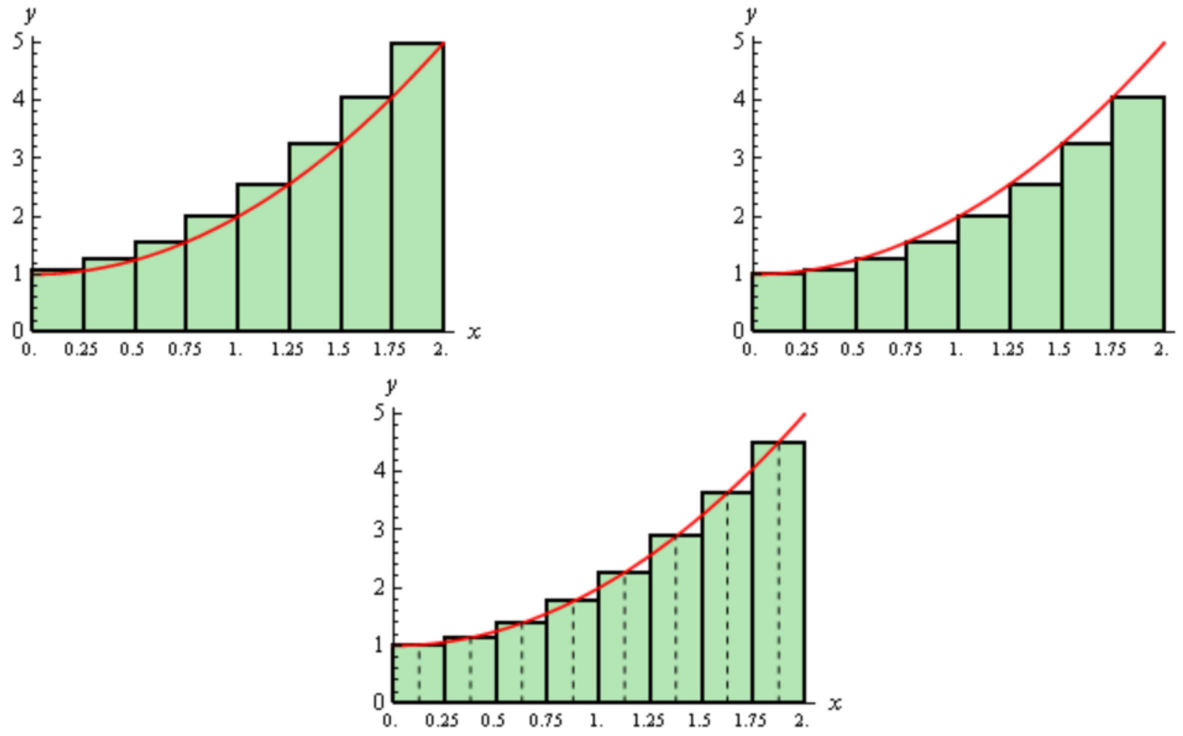
AREA PROBLEM

- Given a function $f(x) = x^2 + 1$ that is positive in some interval $[a, b]$



$$\begin{aligned} A_m &= \frac{1}{2} f\left(\frac{1}{4}\right) + \frac{1}{2} f\left(\frac{3}{4}\right) + \frac{1}{2} f\left(\frac{5}{4}\right) + \frac{1}{2} f\left(\frac{7}{4}\right) \\ &= \frac{1}{2}\left(\frac{17}{16}\right) + \frac{1}{2}\left(\frac{25}{16}\right) + \frac{1}{2}\left(\frac{41}{16}\right) + \frac{1}{2}\left(\frac{65}{16}\right) \\ &= 4.625 \end{aligned}$$

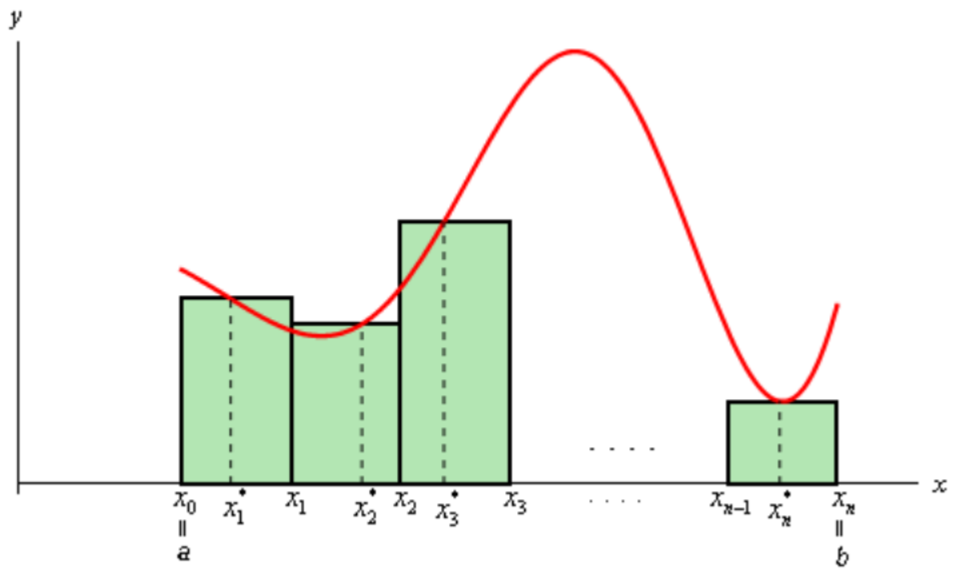
AREA PROBLEM



$A_r = 5.1875 \qquad A_l = 4.1875 \qquad A_m = 4.65625$

AREA PROBLEM

- In general, given a function $f(x) > 0$ on $[a, b]$



$$\Delta x = \frac{b-a}{n}$$

Note that the subintervals don't have to be equal length

$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_i &= a + i\Delta x \\ &\vdots \\ x_{n-1} &= a + (n-1)\Delta x \\ x_n &= a + n\Delta x \qquad = b \end{aligned}$$

The area under the curve on the given interval is then approximately,

$$A \approx f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_i^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x$$

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

DEFINITE INTEGRALS

- Definitions

Given a function $f\left(x\right)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of $f\left(x\right)$ from a to b** is

$$\int_a^b f\left(x\right)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

DEFINITE INTEGRALS

- Properties

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. We can break up definite integrals across a sum or difference.

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a, c]$ and $[c, b]$. Note however that c doesn't need to be between a and b.

6. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

DEFINITE INTEGRALS

- Properties

7. $\displaystyle\int_a^b c\,dx = c\,(b-a)$, c is any number.

8. If $f(x) \geq 0$ for $a \leq x \leq b$ then $\displaystyle\int_a^b f(x)\ dx \geq 0$.

9. If $f(x) \geq g(x)$ for$a \leq x \leq b$then $\displaystyle\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$.

10. If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m\,(b-a) \leq \int_a^b f(x)\ dx \leq M\,(b-a)$.

11. $\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\,dx$

DEFINITE INTEGRALS

- Computing

Suppose $f(x)$ is a continuous function on $[a, b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

DEFINITE INTEGRALS

- Exercise (5 minutes): $$\int_1^2 \frac{2w^5 - w + 3}{w^2}\, dw$$

SUBSTITUTION RULE FOR DEFINITE INTEGRALS

- Example $\displaystyle\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt$

Solution I

$$u = 1-4t^3 \qquad du = -12t^2dt \qquad \Rightarrow \qquad t^2dt = -\frac{1}{12}du$$

$$\begin{aligned}
\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{6}\int_{-2}^{0} u^{\frac{1}{2}}\,du \\
&= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{-2}^{0} \\
&= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) \\
&= \frac{1}{9}\left(33\sqrt{33}-1\right)
\end{aligned}$$

SUBSTITUTION RULE FOR DEFINITE INTEGRALS

- Example $\displaystyle\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt$

Solution 2

$$u = 1 - 4t^3 \qquad du = -12t^2dt \qquad \Rightarrow \qquad t^2dt = -\frac{1}{12}du$$

$$t = -2 \qquad \Rightarrow \qquad u = 1 - 4(-2)^3 = 33$$

$$t = 0 \qquad \Rightarrow \qquad u = 1 - 4(0)^3 = 1$$

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{6}\int_{33}^{1} u^{\frac{1}{2}}\,du \\ &= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1} \\ &= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) = \frac{1}{9}\left(33\sqrt{33} - 1\right)\end{aligned}$$

APPLICATION OF INTEGRALS

- Average Function Value
- Area Between Curve
- Volumes of Solids of Revolution

APPLICATION OF INTEGRALS

- Average Function Value

The average value of a function $f(x)$ over the interval $[a,b]$ is given by,

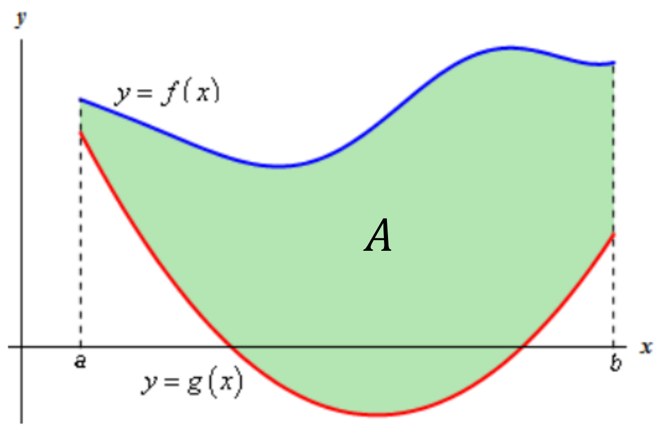
$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

- Example $f(t) = t^2 - 5t + 6\cos(\pi t)$ on $\left[-1, \frac{5}{2}\right]$

$$\begin{aligned} f_{avg} &= \frac{1}{\frac{5}{2}-(-1)}\int_{-1}^{\frac{5}{2}} t^2 - 5t + 6\cos(\pi t)\,dt \\ &= \frac{2}{7}\left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + \frac{6}{\pi}\sin(\pi t)\right)\Bigg|_{-1}^{\frac{5}{2}} \\ &= \frac{12}{7\pi} - \frac{13}{6} \\ &= -1.620993 \end{aligned}$$

APPLICATION OF INTEGRALS

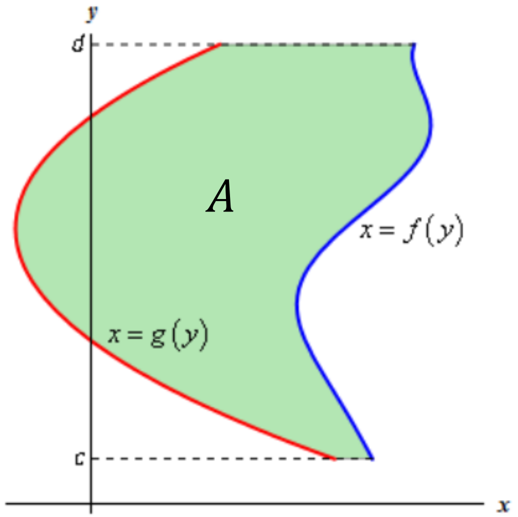
- Area Between Curve



$$A = \int_a^b f(x) - g(x)\ dx$$

APPLICATION OF INTEGRALS

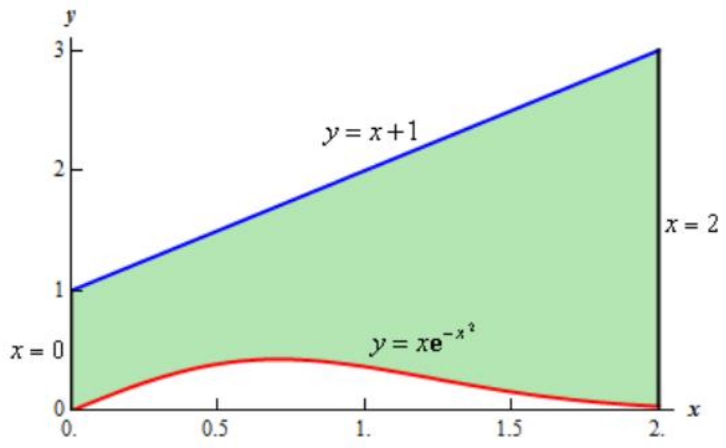
- Area Between Curve



$$A = \int_c^d f(y) - g(y)\, dy$$

APPLICATION OF INTEGRALS

- Example Determine the area of the region bounded by $y = x\mathbf{e}^{-x^2}$, $y = x + 1$, $x = 2$, and the y-axis.

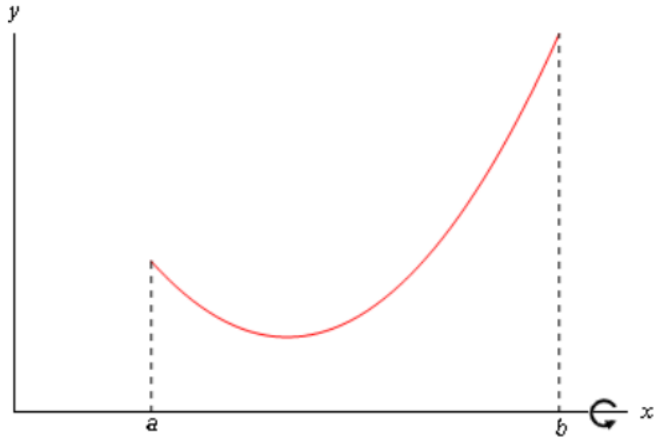


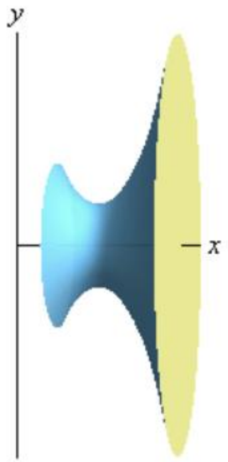
$$
\begin{aligned}
A &= \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx \\
&= \int_0^2 x + 1 - x\mathbf{e}^{-x^2}\, dx \\
&= \left(\frac{1}{2}x^2 + x + \frac{1}{2}\mathbf{e}^{-x^2} \right) \Bigg|_0^2 \\
&= \frac{7}{2} + \frac{\mathbf{e}^{-4}}{2} = 3.5092
\end{aligned}
$$

APPLICATION OF INTEGRALS

- Volumes of Solids of Revolution

Given a function $y = f(x)$ on $[a, b]$





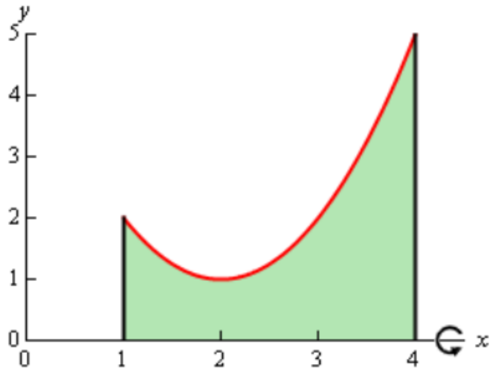
$$V = \int_a^b A(x)\, dx \qquad V = \int_c^d A(y)\, dy$$

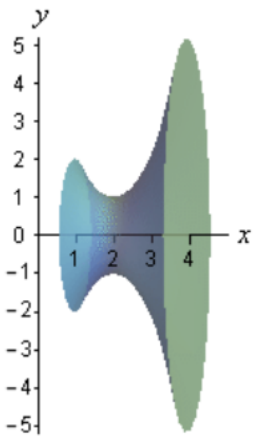
where
$A(x)$ is the cross-sectional area over x-axis
$A(y)$ is the cross-sectional area over y-axis

APPLICATION OF INTEGRALS

- Example: Determine the volume of the solid obtained by rotating the region bounded by

$$y = x^2 - 4x + 5$$
$$x = 1$$
$$x = 4$$



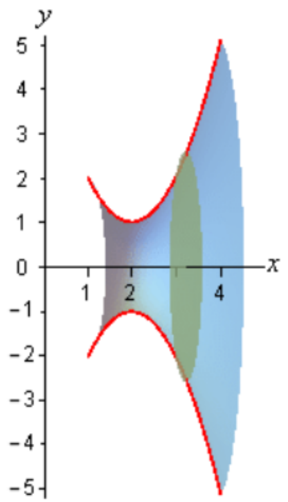


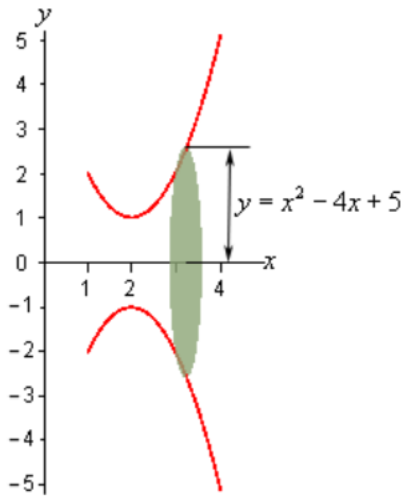
APPLICATION OF INTEGRALS

- Example: Determine the volume of the solid obtained by rotating the region bounded by

$$y = x^2 - 4x + 5$$
$$x = 1$$
$$x = 4$$

$$A(x) = \pi\left(x^2 - 4x + 5\right)^2 = \pi\left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)$$





$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \pi \int_1^4 x^4 - 8x^3 + 26x^2 - 40x + 25\,dx \\ &= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Big|_1^4 \\ &= \frac{78\pi}{5} \end{aligned}$$