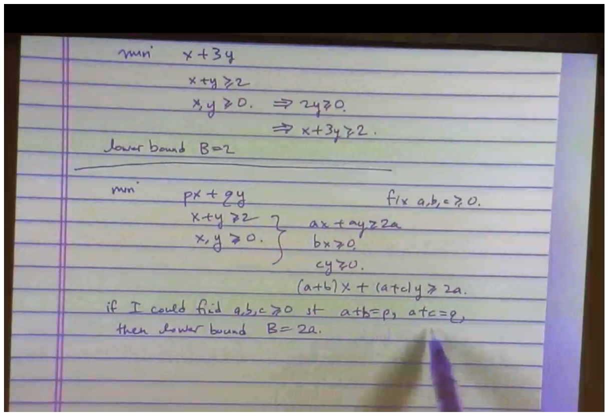
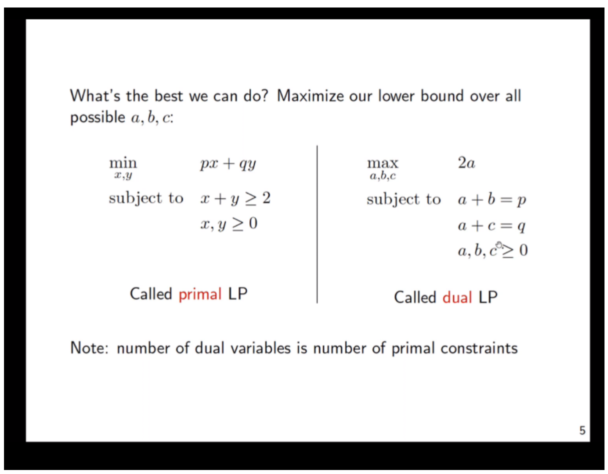
Duality

- editorial: beautiful? deep idea but very simple mathematical concept that lead to these deep conclusions

Introduction to duality via linear programs





Approach
1. find a lower bound for the criterion function
2. find the max of this lower bound by creating a new LP on the constraints

Algorithm to generate dual form?
1. define multipliers on the different inequalities
2. add the new statements together
3. map to the variables in the original formulation
4. maximise the lower bound

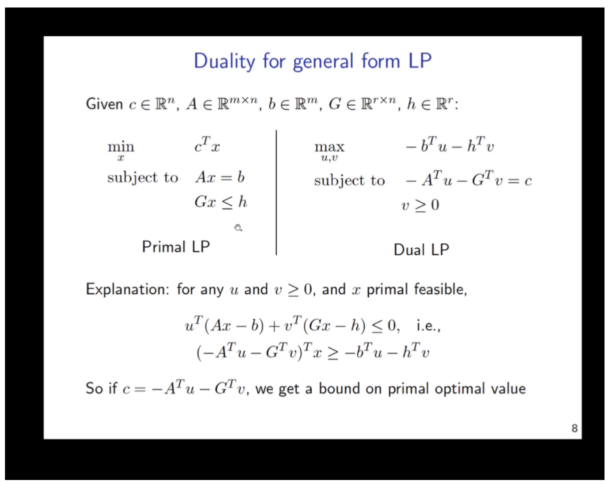
Questions
1. what if there is no a, b, c that satisifies these constraints?

- e.g. if p was negative? and a,b, c ⩾ 0?
- the bound would be -inf - it is a trivial lower bound

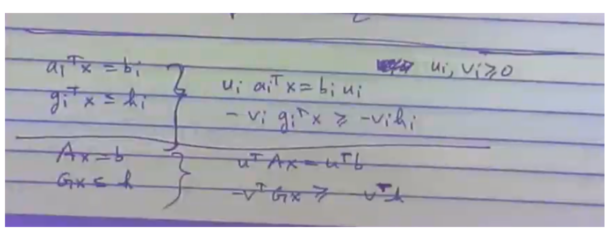
2. Will this give us something tight in general?
   - in general for LPs, the dual is always a tight bound - pretty much always

Outline
1. Duality in general LPs
2. Maxflow and mincut
3. Second take on duality - Lagrangian
4. Matrix games

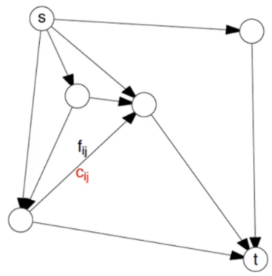
1. Duality in general form LPs



length of u = # of eqaulity constraints, length of v = # of inequality constraints



2. Example: max flow/min cut equality



Given graph $G = (V, E)$, define flow f_{ij}, $(i, j) \in E$ to satisfy:

- $f_{ij} \geq 0$, $(i, j) \in E$
- $f_{ij} \leq c_{ij}$, $(i, j) \in E$
- $\displaystyle\sum_{(i,k)\in E} f_{ik} = \sum_{(k,j)\in E} f_{kj}$, $k \in V\backslash\{s, t\}$

Max flow problem: find flow that maximizes total value of the flow from s to t. I.e., as an LP:

$$\begin{aligned} \max_{f\in\mathbb{R}^{|E|}} \quad & \sum_{(s,j)\in E} f_{sj} \\ \text{subject to} \quad & 0 \leq f_{ij} \leq c_{ij} \quad \text{for all } (i,j) \in E \\ & \sum_{(i,k)\in E} f_{ik} = \sum_{(k,j)\in E} f_{kj} \quad \text{for all } k \in V \setminus \{s,t\} \end{aligned}$$

Derive the dual, in steps:

- Note that

$$\sum_{(i,j)\in E} \Big(-a_{ij} f_{ij} + b_{ij}(f_{ij} - c_{ij}) \Big) + \sum_{k\in V\backslash\{s,t\}} x_k \Big(\sum_{(i,k)\in E} f_{ik} - \sum_{(k,j)\in E} f_{kj} \Big) \leq 0$$

  for any $a_{ij}, b_{ij} \geq 0$, $(i,j) \in E$, and x_k, $k \in V \setminus \{s,t\}$
- Rearrange as

$$\sum_{(i,j)\in E} M_{ij}(a, b, x) f_{ij} \leq \sum_{(i,j)\in E} b_{ij} c_{ij}$$

  where $M_{ij}(a, b, x)$ collects terms multiplying f_{ij}

- Want to make LHS in previous inequality equal to primal objective, i.e., $\begin{cases} M_{sj} = b_{sj} - a_{sj} + x_j & \text{want this} = 1 \\ M_{it} = b_{it} - a_{it} - x_i & \text{want this} = 0 \\ M_{ij} = b_{ij} - a_{ij} + x_j - x_i & \text{want this} = 0 \end{cases}$
- We've shown that

$$\text{primal optimal value} \le \sum_{(i,j)\in E} b_{ij}c_{ij},$$

subject to a, b, x satisfying constraints. Hence dual problem is (minimize over a, b, x to get best upper bound):

$$\begin{aligned} \min_{b\in\mathbb{R}^{|E|},\, x\in\mathbb{R}^{|V|}} \quad & \sum_{(i,j)\in E} b_{ij}c_{ij} \\ \text{subject to} \quad & b_{ij} + x_j - x_i \ge 0 \quad \text{for all } (i,j)\in E \\ & b \ge 0,\ x_s = 1,\ x_t = 0 \end{aligned}$$

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- eliminate a because a only appears in the equality constraints and it is constrained to be ⩾ 0, so you can eliminate it by just making these inequality constraints themselves - combine them
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Suppose that at the solution, it just so happened that

$$x_i \in \{0,1\} \quad \text{for all } i \in V$$

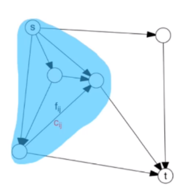
Let $A = \{i : x_i = 1\}$, $B = \{i : x_i = 0\}$; note $s \in A$, $t \in B$. Then

$$b_{ij} \ge x_i - x_j \quad \text{for } (i,j)\in E, \quad b \ge 0$$

imply that $b_{ij} = 1$ if $i \in A$ and $j \in B$, and 0 otherwise. Moreover, the objective $\sum_{(i,j)\in E} b_{ij}c_{ij}$ is the capacity of cut defined by A, B

I.e., we've argued that the dual is the LP relaxation of the min cut problem:

$$\begin{aligned} \min_{b\in\mathbb{R}^{|E|},\, x\in\mathbb{R}^{|V|}} \quad & \sum_{(i,j)\in E} b_{ij}c_{ij} \\ \text{subject to} \quad & b_{ij} \ge x_i - x_j \\ & b_{ij}, x_i, x_j \in \{0,1\} \\ & \text{for all } i,j \end{aligned}$$



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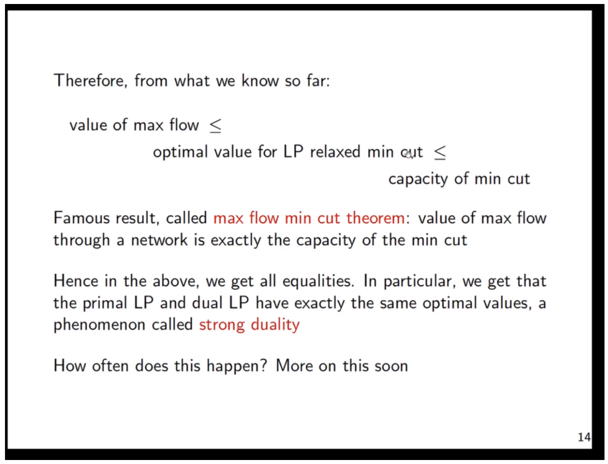
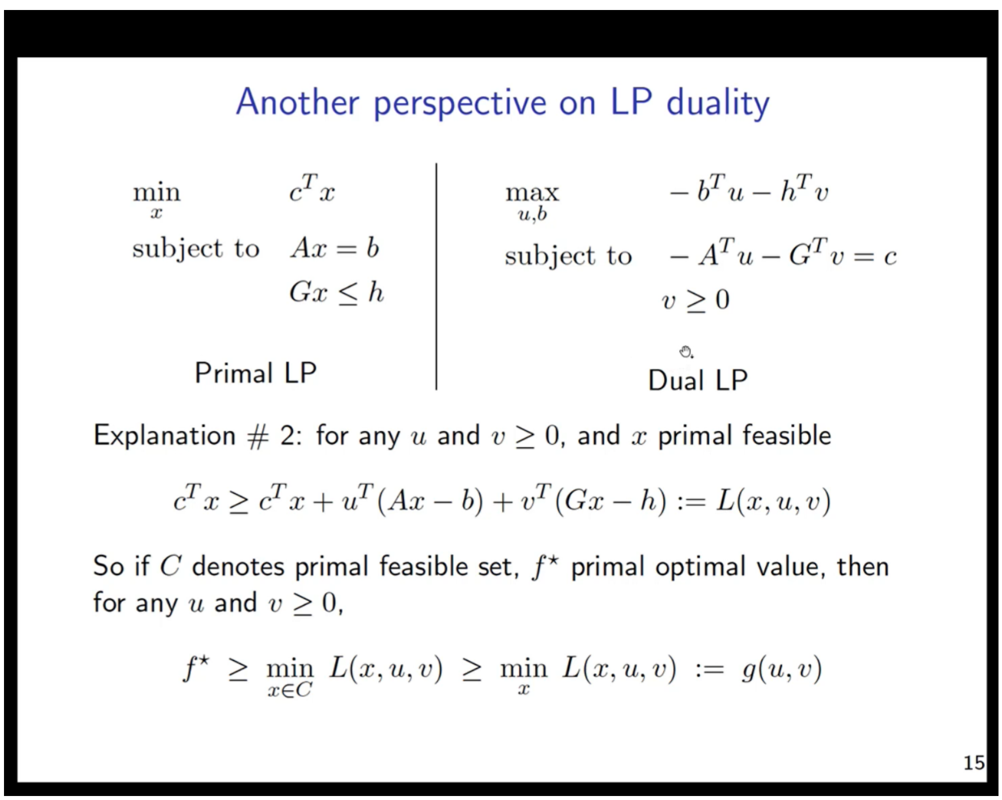


- even though mincut is combinatorial and non-convex, its convex relaxation is tight - it is equal to its convex relazation
- By max flow/min cut algorithm, the dual is tight - this is strong duality

3. Lagrangian duality

- arguments here apply to non-linear convex programs as well



The question is: how can you go from the primal to the dual?

- define L() = Langrangian, g() = min value of the Langrangian

- $g(u,\ v)\ =\ x^*\ for\ unconstrained\ x^*\ that\ minimizes\ L(u,\ v,\ x)\ =\ c^T x\ \ +\ u^T\left(Ax\ \ -b\right)\ +\ v^T\left(Gx\right.$
- if the constraint is obeyed i.e. $c\ =\ -A^T u\ -\ G^T v$ then
  - x is is in the feasible set

    $g(u,\ v)\ =\ min_x\ L(x,\ u,\ v)\ =\ \ min_x\left\{c^T x\ \ +\ u^T\left(Ax\ \ -b\right)\ +\ v^T\left(Gx\ \ -h\right)\right\}$
  - $=\ min_x\left\{c^T x\ +\ u^T Ax\ -\ u^T b\ +\ v^T Gx\ -\ v^T h\right\}$

    $=\ -u^T b\ -\ v^T h$
- If this constraint is NOT fulfilled, then L(x) is just a linear function in x with a minimum of -inf

In other words, $g(u, v)$ is a lower bound on $f^\star$ for any u and $v \geq 0$

Note that

$$g(u,v) = \begin{cases} -b^T u - h^T v & \text{if } c = -A^T u - G^T v \\ -\infty & \text{otherwise} \end{cases}$$

Now we can maximize $g(u, v)$ over u and $v \geq 0$ to get the tightest bound, and this gives exactly the dual LP as before

This last perspective is actually completely general and applies to arbitrary optimization problems (even nonconvex ones)

Example: mixed strategies for matrix games

Setup: two players,  vs.  , and a payout matrix P

		R			
		1	2	...	n
	1	P_{11}	P_{12}	...	P_{1n}
J	2	P_{21}	P_{22}	...	P_{2n}
	...				
	m	P_{m1}	P_{m2}	...	P_{mn}

Game: if J chooses i and R chooses j, then J must pay R amount P_{ij} (don't feel bad for J—this can be positive or negative)

They use mixed strategies, i.e., each will first specify a probability distribution, and then

$$x: \quad \mathbb{P}(\text{J chooses } i) = x_i,\ i = 1, \ldots m$$
$$y: \quad \mathbb{P}(\text{R chooses } j) = y_j,\ j = 1, \ldots, n$$

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