

## Duality

- editorial: beautiful? deep idea but very simple mathematical concept that lead to these deep conclusions

## Introduction to duality via linear programs

min  $x+3y$   
 $x+y \geq 2$   
 $x, y \geq 0 \Rightarrow x \geq 0, y \geq 0 \Rightarrow x+3y \geq 2$ .  
Lower bound  $B=2$

min  $px + qy$   
 $x+y \geq 2$   
 $x, y \geq 0 \Rightarrow x \geq 0, y \geq 0 \Rightarrow$   
 $\begin{cases} px + qy \geq 2 \\ a+b \geq p \\ a+c \geq q \\ a, b, c \geq 0 \end{cases}$   
 $(a+b)x + (a+c)y \geq 2a$ .  
if I could find  $a, b, c \geq 0$  st  $a+b=p$ ,  $a+c=q$ ,  
then lower bound  $B=2a$ .

What's the best we can do? Maximize our lower bound over all possible  $a, b, c$ :

$\begin{array}{ll} \min_{x,y} & px + qy \\ \text{subject to} & x+y \geq 2 \\ & x, y \geq 0 \end{array}$ <p>Called <b>primal LP</b></p>	$\begin{array}{ll} \max_{a,b,c} & 2a \\ \text{subject to} & a+b = p \\ & a+c = q \\ & a, b, c \geq 0 \end{array}$ <p>Called <b>dual LP</b></p>
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Note: number of dual variables is number of primal constraints

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## Approach

- find a lower bound for the criterion function
- find the max of this lower bound by creating a new LP on the constraints

## Algorithm to generate dual form?

- define multipliers on the different inequalities
- add the new statements together
- map to the variables in the original formulation
- maximise the lower bound

## Questions

- what if there is no  $a, b, c$  that satisfies these constraints?

- e.g. if  $p$  was negative? and  $a, b, c \geq 0$ ?
  - the bound would be  $-\infty$  - it is a trivial lower bound
2. Will this give us something tight in general?
- in general for LPs, the dual is always a tight bound - pretty much always

## Outline

1. Duality in general LPs
2. Maxflow and mincut
3. Second take on duality - Lagrangian
4. Matrix games

## 1. Duality in general form LPs

**Duality for general form LP**

Given  $c \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ ,  $G \in \mathbb{R}^{r \times n}$ ,  $h \in \mathbb{R}^r$ :

$\begin{array}{ll} \min_x & c^T x \\ \text{subject to} & Ax = b \\ & Gx \leq h \\ & v \end{array}$	$\begin{array}{ll} \max_{u,v} & -b^T u - h^T v \\ \text{subject to} & -A^T u - G^T v = c \\ & v \geq 0 \end{array}$
Primal LP	Dual LP

Explanation: for any  $u$  and  $v \geq 0$ , and  $x$  primal feasible,

$$u^T(Ax - b) + v^T(Gx - h) \leq 0, \quad \text{i.e.,}$$

$$(-A^T u - G^T v)^T x \geq -b^T u - h^T v$$

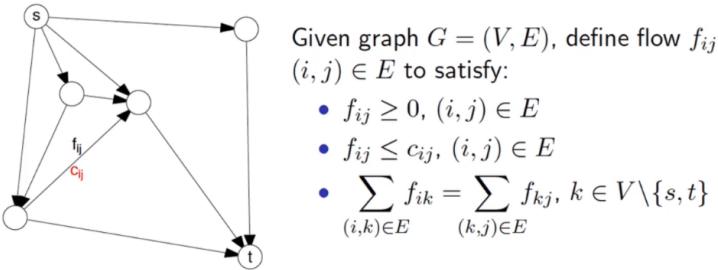
So if  $c = -A^T u - G^T v$ , we get a bound on primal optimal value

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length of  $u$  = # of equality constraints, length of  $v$  = # of inequality constraints

$$\begin{aligned}
 \left. \begin{array}{l} a_i^T x = b_i \\ g_i^T x \leq h_i \end{array} \right\} & \quad u_i; a_i^T x = b_i; u_i \quad \text{if } u_i, v_i \geq 0 \\
 & \quad -v_i; g_i^T x \geq -v_i h_i \\
 \left. \begin{array}{l} Ax = b \\ Gx \leq h \end{array} \right\} & \quad u^T A x = u^T b \\
 & \quad -v^T G x \geq -v^T h
 \end{aligned}$$

## 2. Example: max flow/min cut equality



**Max flow problem:** find flow that maximizes total value of the flow from  $s$  to  $t$ . I.e., as an LP:

$$\begin{aligned} \max_{f \in \mathbb{R}^{|E|}} \quad & \sum_{(s,j) \in E} f_{sj} \\ \text{subject to} \quad & 0 \leq f_{ij} \leq c_{ij} \quad \text{for all } (i, j) \in E \\ & \sum_{(i,k) \in E} f_{ik} = \sum_{(k,j) \in E} f_{kj} \quad \text{for all } k \in V \setminus \{s, t\} \end{aligned}$$

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Derive the dual, in steps:

- Note that

$$\begin{aligned} \sum_{(i,j) \in E} \left( -a_{ij}f_{ij} + b_{ij}(f_{ij} - c_{ij}) \right) \\ + \sum_{k \in V \setminus \{s,t\}} x_k \left( \sum_{(i,k) \in E} f_{ik} - \sum_{(k,j) \in E} f_{kj} \right) \leq 0 \end{aligned}$$

for any  $a_{ij}, b_{ij} \geq 0$ ,  $(i, j) \in E$ , and  $x_k$ ,  $k \in V \setminus \{s, t\}$

- Rearrange as

$$\sum_{(i,j) \in E} M_{ij}(a, b, x) f_{ij} \leq \sum_{(i,j) \in E} b_{ij} c_{ij}$$

where  $M_{ij}(a, b, x)$  collects terms multiplying  $f_{ij}$

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- Want to make LHS in previous inequality equal to primal

objective, i.e., 
$$\begin{cases} M_{sj} = b_{sj} - a_{sj} + x_j & \text{want this } = 1 \\ M_{it} = b_{it} - a_{it} - x_i & \text{want this } = 0 \\ M_{ij} = b_{ij} - a_{ij} + x_j - x_i & \text{want this } = 0 \end{cases}$$

- We've shown that

$$\text{primal optimal value} \leq \sum_{(i,j) \in E} b_{ij} c_{ij},$$

subject to  $a, b, x$  satisfying constraints. Hence dual problem is (minimize over  $a, b, x$  to get best upper bound):

$$\begin{aligned} \min_{b \in \mathbb{R}^{|E|}, x \in \mathbb{R}^{|V|}} \quad & \sum_{(i,j) \in E} b_{ij} c_{ij} \\ \text{subject to} \quad & b_{ij} + x_j - x_i \geq 0 \quad \text{for all } (i, j) \in E \\ & b \geq 0, x_s = 1, x_t = 0 \end{aligned}$$

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- eliminate  $a$  because  $a$  only appears in the equality constraints and it is constrained to be  $\geq 0$ , so you can eliminate it by just making these inequality constraints themselves  
- combine them
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Suppose that at the solution, it just so happened that

$$x_i \in \{0, 1\} \quad \text{for all } i \in V$$

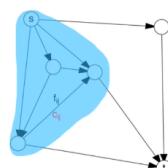
Let  $A = \{i : x_i = 1\}$ ,  $B = \{i : x_i = 0\}$ ; note  $s \in A$ ,  $t \in B$ . Then

$$b_{ij} \geq x_i^* - x_j^* \quad \text{for } (i, j) \in E, \quad b \geq 0$$

imply that  $b_{ij} = 1$  if  $i \in A$  and  $j \in B$ , and 0 otherwise. Moreover, the objective  $\sum_{(i,j) \in E} b_{ij} c_{ij}$  is the capacity of cut defined by  $A, B$

I.e., we've argued that the dual is the LP relaxation of the min cut problem:

$$\begin{aligned} \min_{b \in \mathbb{R}^{|E|}, x \in \mathbb{R}^{|V|}} \quad & \sum_{(i,j) \in E} b_{ij} c_{ij} \\ \text{subject to} \quad & b_{ij} \geq x_i - x_j \\ & b_{ij}, x_i, x_j \in \{0, 1\} \\ & \text{for all } i, j \end{aligned}$$



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Therefore, from what we know so far:

$$\begin{aligned} \text{value of max flow} &\leq \\ \text{optimal value for LP relaxed min cut} &\leq \\ \text{capacity of min cut} \end{aligned}$$

Famous result, called **max flow min cut theorem**: value of max flow through a network is exactly the capacity of the min cut

Hence in the above, we get all equalities. In particular, we get that the primal LP and dual LP have exactly the same optimal values, a phenomenon called **strong duality**

How often does this happen? More on this soon

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- even though mincut is combinatorial and non-convex, its convex relaxation is tight - it is equal to its convex relaxation
- By max flow/min cut algorithm, the dual is tight - this is strong duality

### 3. Lagrangian duality

- arguments here apply to non-linear convex programs as well

#### Another perspective on LP duality

$\begin{array}{ll} \min_x & c^T x \\ \text{subject to} & Ax = b \\ & Gx \leq h \end{array}$ <p style="text-align: center;">Primal LP</p>	$\begin{array}{ll} \max_{u,b} & -b^T u - h^T v \\ \text{subject to} & -A^T u - G^T v = c \\ & v \geq 0 \end{array}$ <p style="text-align: center;">Dual LP</p>
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Explanation # 2: for any  $u$  and  $v \geq 0$ , and  $x$  primal feasible

$$c^T x \geq c^T x + u^T (Ax - b) + v^T (Gx - h) := L(x, u, v)$$

So if  $C$  denotes primal feasible set,  $f^*$  primal optimal value, then for any  $u$  and  $v \geq 0$ ,

$$f^* \geq \min_{x \in C} L(x, u, v) \geq \min_x L(x, u, v) := g(u, v)$$

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The question is: how can you go from the primal to the dual?

- define  $L()$  = Langrangian,  $g()$  = min value of the Langrangian

- $g(u, v) = x^*$  for unconstrained  $x^*$  that minimizes  $L(u, v, x) = c^T x + u^T(Ax - b) + v^T(Gx - h)$
- if the constraint is obeyed i.e.  $c = -A^T u - G^T v$  then
  - $x$  is in the feasible set
$$\begin{aligned} g(u, v) &= \min_x L(x, u, v) = \min_x \left\{ c^T x + u^T(Ax - b) + v^T(Gx - h) \right\} \\ &= \min_x \left\{ \cancel{c^T x} + u^T Ax - u^T b + \cancel{v^T Gx} - v^T h \right\} \\ &= -u^T b - v^T h \end{aligned}$$
- If this constraint is NOT fulfilled, then  $L(x)$  is just a linear function in  $x$  with a minimum of  $-\infty$

In other words,  $g(u, v)$  is a lower bound on  $f^*$  for any  $u$  and  $v \geq 0$

Note that

$$g(u, v) = \begin{cases} -b^T u - h^T v & \text{if } c = -A^T u - G^T v \\ -\infty & \text{otherwise} \end{cases}$$

Now we can maximize  $g(u, v)$  over  $u$  and  $v \geq 0$  to get the tightest bound, and this gives exactly the dual LP as before

This last perspective is actually **completely general** and applies to arbitrary optimization problems (even nonconvex ones)

### Example: mixed strategies for matrix games

Setup: two players,  vs.  , and a payout matrix  $P$

		R			
		1	2	...	n
J	1	$P_{11}$	$P_{12}$	...	$P_{1n}$
	2	$P_{21}$	$P_{22}$	...	$P_{2n}$
...					
m		$P_{m1}$	$P_{m2}$	...	$P_{mn}$

Game: if J chooses  $i$  and R chooses  $j$ , then J must pay R amount  $P_{ij}$ . (don't feel bad for J—this can be positive or negative)

They use **mixed strategies**, i.e., each will first specify a probability distribution, and then

$$\begin{aligned} x: \quad & \mathbb{P}(J \text{ chooses } i) = x_i, \quad i = 1, \dots, m \\ y: \quad & \mathbb{P}(R \text{ chooses } j) = y_j, \quad j = 1, \dots, n \end{aligned}$$

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