I. SUBSYSTEM 2 - TROLLING MOTOR

A trolling motor is made up of two main components, a propeller and a motor. The power developed by a trolling motor is available at the propeller shaft in the form of torque. This torque is then converted into thrust, a reactional force caused by a pressure differential that pushes the boat, Figure 1 [1][2]. The objective for this subsystem was to maximise this thrust force.

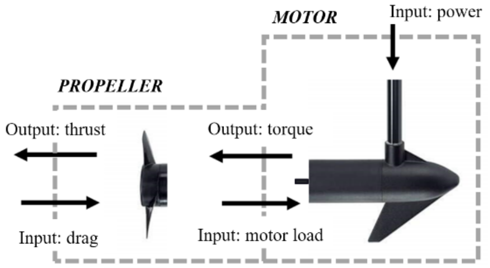

Fig. 1. System decomposition of a trolling motor [3].

The thrust created by a trolling motor, is affected by variables relating to the motor and propeller design. The following section will elaborate on the modelling of thrust using first principles and empirical correction factors, as well as the methods used to identify optimum quantities for variables such as propeller pitch, propeller diameter, blade number, chord length and voltage provided to motor; in order to maximise thrust while meeting functional criteria (constraints).

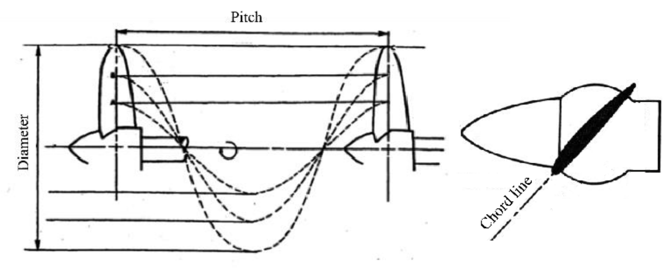

Fig. 2. Typical propeller operating through one revolution (left). Diagram with chord line annotated (right) [4].

A. Optimisation formulation

Below is are the constraints, parameters and objective function presented in its negative null form. On a subsystem level resistive force of the boat and drag force from the boat was not considered.

$$\min\ f(\mathbf{x},\mathbf{p}) = F_{\text{motor}} = -3.859\ x\ 10^{-12}\ \pi\rho(K_v v)^2 d^{3.5}\sqrt{p} \tag{3.1}$$

where $\mathbf{x} = (d, p, v, b, c) \quad \in \chi \in \mathbb{R}^n$ (3.2)

$\mathbf{p} = (\rho, Kv)$ (3.3)

s.t.

$$h_1(\rho) = \rho - 1000 = 0 \tag{3.4}$$
$$h_2(Kv) = K_v - 150 = 0 \tag{3.5}$$

$$g_1(p,d) = 0.85 - p\ /\ d \le 0 \tag{3.6}$$
$$g_2(p,d) = p\ /\ d - 1.5 \le 0 \tag{3.7}$$
$$g_3(d,b,c) = 0.35 - bc\ /\ \pi d \le 0 \tag{3.8}$$
$$g_4(v) = 12 - v \le 0 \tag{3.9}$$
$$g_5(v) = v - 24 \le 0 \tag{3.10}$$
$$g_6(b) = 1 - b \le 0 \tag{3.11}$$
$$g_7(b) = b - 3 \le 0 \tag{3.12}$$
$$g_8(c) = 1 - c \le 0 \tag{3.13}$$
$$g_9(c) = c - 4 \le 0 \tag{3.14}$$
$$g_{10}(p) = 1 - p \le 0 \tag{3.15}$$
$$g_{11}(p) = p - 10 \le 0 \tag{3.16}$$
$$g_{12}(d) = 1 - d \le 0 \tag{3.17}$$
$$g_{13}(d) = d - 12 \le 0 \tag{3.18}$$

Below are the chosen design variables and their relationships with thrust force and the other variables, Table I.

TABLE I. TABLE OF DESIGN VARIABLES

Variable	Symbol	Description of relationship with thrust and other variables	Source
Propeller Diameter (Inches)	d	The propeller diameter is proportional to the thrust produced, as a larger diameter distributes more power and thrust on a larger volume of fluid.	[5]
Pitch Length (Inches)	p	The pitch effectively converts torque to thrust, it determines how far you move as a single rotation (Figure 2). The pitch has effect on the revolutions per minute (RPM). The ratio between the pitch and diameter is also crucial for the functionality of the propeller.	[2][6]
Voltage (V)	v	The voltage supplied to the motor is proportional to both RPM of the motor as well as the power supplied to the motor; therefore, proportional to the thrust generated by the motor.	[7] [8]
Blade number	b	The blade number is related to the solidity of the propeller, rotor solidity ratio, a ratio which also incorporates the diameter of the propeller. Increasing the blade number increases the surface area to propel against fluid, therefore maximising the thrust produced.	[9]
Chord length	c	The chord length influences the rotor solidity as the efficiency of the propeller. Viscous losses also scale with chord length. Chord line is shown on Figure 2.	[5] [9] [10]

B. Modelling approach and assumptions

1) Objective equation formulation

Considering Newton's second law, we can define the thrust force (F_{motor}) to be the change in momentum of an object, momentum is an object's mass (m) times it's velocity (V). The change in velocity (ΔV) is shown by velocity of exit(V_e) subtracted from the freestream velocity(V_f) (3.19) [11][12].

$$F_{\text{motor}} = m\Delta V = m(V_e - V_f) \tag{3.19}$$

For a moving fluid it is important that we incorporate the mass flow rate (m_{dot}), which is the amount of mass moving through a given plane over a certain amount of time. This is equal to the cross-sectional area (A) through which the fluid is flowing times the velocity of the fluid (V_e), multiplied by density of freshwater (ρ). Where rotor disc area (A) is simplified to the area of a circle (3.20) [10][12].

$$m_{dot} = \rho A V_e = \rho \frac{\pi\, d^2}{4} V_e \tag{3.20}$$

By substitution we get the following equation (3.21) which summarises theoretical dynamic propeller thrust.

$$F_{\text{motor}} = \rho \frac{\pi d^2}{4} V_e (V_e - V_f) \tag{3.21}$$

Pitch speed (V_p) is a function of RPM of the propeller as well as pitch (p), which represents the theoretical distance forward that the propeller moves in one revolution, equation (3.22). The propeller travels at pitch speed when the propeller is most efficient. Using this assumption and conversion rates we can deduce equation (3.22)[12][13].

$$V_p = 0.000423\, p\, RPM \tag{3.22}$$

Equation (3.21) can be simplified to static thrust, which incorporates design variables that the designer can change; by substituting pitch speed for exit velocity and equating free stream velocity to 0, as well as incorporating conversion multipliers for diameter (3.23).

$$F_{\text{motor}} = \rho \frac{\pi (0.0254\ d)^2}{4} (0.000423\, p\, RPM)^2 \tag{3.23}$$

Calculating dynamic thrust is highly dependent on inflow velocity, pressure changes and often it is calculated using momentum theory which greatly simplifies the design parameters of a propeller to the diameter or area of propeller [1][14]. Due to this a static thrust equation was used. Through use of correction factors based on empirical data, a thrust equation can be obtained that is seen as accurate within +/- 26% of a most cases (at high speeds) [7][12][15].

Equation (3.24) is the objective equation in its negative null form, with the motor velocity constant (K_v) and voltage (v) substituted for RPM and empirical correction factors incorporated. A coefficient of 0.8 was also incorporated, assuming the efficiency of the propeller is 80% [15][16][17]. This equation was then simplified (3.1).

$$F_{\text{motor}} = -0.8\, \rho \frac{\pi (0.0254\, d)^2}{4} (0.000423\, p\, K_v\, v)^2 \left(\frac{d}{3.29546\, p}\right)^{1.5} \tag{3.24}$$

2) *Design constraints and parameters*

Constraints were used to bound variables, as well as incorporating further functional design criteria, for example maintaining a suitable value for the rotor solidity ratio (g_3).

TABLE II. FUNCTIONAL CONSTRAINT TABLE

Constraint	Constraint description, assumptions and tradeoffs	Source
$\boldsymbol{g_1(p, d)}$	Pitch to diameter ratio constraint. Created to reduce likelihood of stalling and ensure pitch ratio is within optimal range.	[5] [6]
$\boldsymbol{g_2(p, d)}$	Pitch to diameter ratio constraint. Created to ensure trolling motor is within suitable shaft speed and ensure pitch ratio is within optimal range.	[2]
$\boldsymbol{g_3(d, b, c)}$	Rotor solidity constraint. Created to improve rotor solidity and reduce risk of cavitation.	[18]
$\boldsymbol{g_4, g_5(v)}$	Voltage bounding. Informed by motor voltages on the market (benchmarking) that voltage values would work with available circuit breakers. Maximum 24V constraint ensures that kayak isn't travelling at unsafe speeds for our boat length and assumed weight.	[19] [20]
$\boldsymbol{g_6, g_7(b)}$	Blade number bounding. A propeller with more blades will perform better however and increase surface area in contact with the fluid, however more blades means reduced chord which can cause issues with cavitation and fatigue.	[5][9]
$\boldsymbol{g_8, g_9(c)}$	Chord length bounding. Large chord lengths can help reduce cavitation however reduce efficiency of a propeller due to viscous losses. Minimum chord length to allow for blade solidity and reduce cavitation.	[10]
$\boldsymbol{g_{10}, g_{11}(p)}$	Pitch bounding. Large pitch may have good efficiency, but blades tend to stall. A large pitch will also reduce RPM however a small pitch can also ruin the engine.	[2][5]
$\boldsymbol{g_{12}, g_{13}(d)}$	Diameter bounding. Similar bounding to pitch as a propeller with the same pitch and diameter length is most efficient. Maximum value diameter is larger than pitch as it is more important to absorb torque. The diameter range is also similar to propellers on the market (benchmarking).	[2] [20]

TABLE III. PARAMETER TABLE

Constraint	Constraint description and assumptions	Source
$\boldsymbol{h_1(\rho)}$	Density of freshwater is 1000 kg/m^3	[21]
$\boldsymbol{h_2(K_v)}$	Motor velocity constant was assumed to be 150, based on motor data and forums (benchmarking).	[17] [22]

Alongside previously mentioned assumptions the following subsystem assumptions were also considered, Table VI.

TABLE IV. ADDITIONAL SUBSYSTEM ASSUMPTIONS

No.	Description	Source
1	The propeller imparts a uniform force to the water passing through it.	[1]
2	The thrust generated by propeller is uniformly distributed over the entire disk.	[1]
3	The flow is frictionless.	[1]
4	There is an unlimited supply of water available to the propeller.	[1]
5	The propeller effeciency was assumed to be 80% rather than a function of thrust, diameter, speed etc.	[2] [5]
6	Drag force, viscous losses and resistive force of the boat were not considered when maximising thrust, beyond the incorporation of 80% efficiency.	
7	K_v was assumed to be constant.	[17] [22]
8	Pitch length did not affect the RPM (K_vv) of the propeller	[2]
9	The fundamental principles behind airplane and marine propellers are the same (except for the difference in fluid densities).	[23]

C. *Exploring the problem space*

Monotonicity of the objective and constraint functions were exploited in order to reduce the problem space and check the minimisation problem is well-bounded, Table V. The non-active constraints were eliminated, and voltage was solved to be 24V.

TABLE V. MONOTONICITY TABLE

Function	Design variables					Active
	d	***p***	***v***	***b***	***c***	
f	-	-	-			
$g_1(p,d)$	+	-				*Yes*
$g_2(p,d)$	-	+				*Yes*
$g_3(d,b,c)$	+			-	-	*Yes*
$g_4(v)$			-			*No*
$g_5(v)$			+			*Yes*
$g_6(b)$				-		*No*
$g_7(b)$				+		*Yes*
$g_8(c)$					-	*No*
$g_9(c)$					+	*Yes*
$g_{10}(p)$		-				*No*
$g_{11}(p)$		+				*Yes*
$g_{12}(d)$	-					*No*
$g_{13}(d)$	+					*Yes*

D. Optimisation

The minimisation problem was optimised using MATLAB's fmincon. Constrained non-linear gradient-based methods called Sequential Quadratic Programming algorithm (SQP) and Interior Point were used. Both methods use quasi-Newton methods and identify local minimums.

MATLAB'S GlobalSearch was used to identify if the minimums produced were global minimums.

TABLE VI. RESULTS TABLE

Algorithm	Optimal value				Performance
	diameter	***pitch***	***blade number***	***chord***	***Time (s)***
SQP	10.588	9	3	4	1.519
Interior Point	10.588	9	2.989	3.9844	1.945
Global Search	10.588	9	2.976	3.918	5.298

All three algorithms converged to the same optimal values when rounded to one decimal place (Table VI) and outputted the final optimal thrust value to be 1820.679 N, using the voltage value of 24V which was solved using monotonicity analysis.

Following the optimisation, sensitivity analysis was performed. This was done by examining the Lagrange multipliers (returned after SQP and Interior Point optimisation). It seems the parameter K_v had a relatively high Lagrange multiplier (24.276, 3rd largest upper bound multiplier). Previously, it was decided to assume the value in order to simplify the optimisation problem, despite knowing that the constant K_v may vary as voltage changes. The Lagrange multiplier value emphasises to the need to reconsider assuming a value for this parameter.

This statement is further emphasised by Figure 3, which indicates the parametric plots of pitch and potential K_v values, illustrating that given the potential K_v range and pitch range, K_v could have more impact on the output thrust.

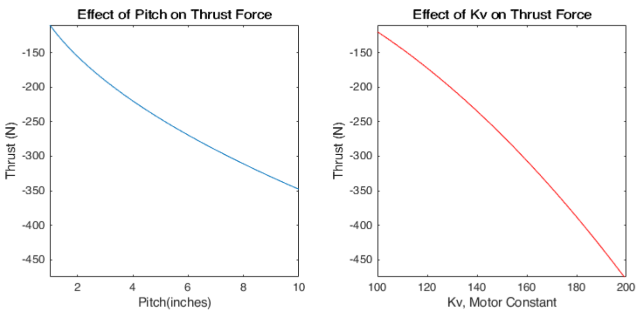

Fig. 3. Parametric plot comparison of pitch and K_v

E. Discussion and Conclusion

In terms of the algorithm performances, the Global Search solver was most computationally expensive, as suspected; as it identifies a local minimum and explores for other start points that are likely to improve the best local minimum. Whereas both SQP and Interior Points only identify local minimums. In addition to this, the SQP algorithm was the most computationally inexpensive as it takes larger steps in the optimisation process. Based on performance and results, SQP was the most effective algorithm.

A challenge was formulating a representative yet simple model to optimise. The objective function was derived from large assumptions. For example, assuming pitch speed is equated to exit velocity, assuming efficiency is not a function of the design variables and neglecting drag; these assumptions are not representative of real life [5][12]. In addition to this, the correction factors used are less accurate for slow speed propellers.

Further steps to improve the optimisation problem include:

- Calculating my own empirical correction factors using a propeller test rig, rather than market data and air propeller data.
- Incorporating the relationship between the design variables and drag.
- Incorporating a propeller efficiency equation that is a function of design variables into my objective equation.
- Identifying how K_v can change with different voltage values.
- Investigate further into blade element theory to identify relationships with other variables and thrust, for example pitch angle.

F. Conclusion

Overall the optimisation was successful, as the three algorithms identified consistent logical optimum values. However, the optimisation using the fmincon solvers tended to the upper bounds of the variables, these combinations of variables are less available and are generally more expensive combination of options. Despite being successful from the perspective of a user that has not taken cost into account, this is not representative of most customers of trolling motors

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NOMENCLATURE

Symbol	***Description***	***Units***
d	Propeller diameter	inches
p	Pitch	inches
b	Blade number	
c	Chord length	inches
v	voltage	volts
$\mathbf{K_v}$	Motor velocity constant	
m	Mass	kg
$\mathbf{\Delta V}$	Change in velocity	m/s
$\mathbf{V_e}$	Velocity of exit	m/s
$\mathbf{V_f}$	Freestream velocity	m/s
$\mathbf{m_{dot}}$	Mass flow rate	Kg/m
A	Cross sectional area	m^2
$\mathbf{V_p}$	Pitch speed	m/s
RPM	Revolutions per minute	rpm