Basic Facts about Numbers

Rahul Saha

Splash

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1 is not a prime!!

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It is called the **Fundamental Theorem of Arithmetic**!!

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This means a and b must themselves be perfect squares!

$$abc = 2020^3$$

and a, b, c don't have any common factors, what can we say about a, b and c?

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The correct condition for a, b, c to be cubes is for *every two* of them to have no common factors.

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and no two of a, b, c don't have any common factors, what can we say about a, b and c?

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Note that then we have

$$(-i)^2 = i^2 = -1$$

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Example. $1 + 2i$, $\sqrt{3} + \pi i$ etc.

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$$x^2 + x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$x = 1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$$

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So our solutions are $1, \omega, \omega^2$.

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Answer. As before the solutions are $1, \omega, , \omega^2, \omega^3, \omega^4$. Since we know the roots of the polynomial $x^5 - 1$, we have

$$x^5 - 1 = (x-1)(x-\omega)(x-\omega^2)(x-\omega^3)(x-\omega^4)$$

Let p be a prime. Suppose ω is a non-real solution of $x^p - 1$. Then all the solutions are $1, \omega, \omega^2, \ldots, \omega^{p-1}$.

And that's it!!

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4. Sum and difference of multiples of some number n are also multiples of n (Example: $3 + 6 = 9$)

5. If $x^p - 1 = 0$ (where p is a prime) and ω is a non real solution, then all the solutions are $1, \omega, \ldots, \omega^{p-1}$ and

$$x^p - 1 = (x - 1)(x - \omega) \ldots (x - \omega^{p-1})$$