Rahul Saha (rsaha@princeton.edu)

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McConnell Summer Reading Group
Representaion Theory

Multilinear Maps.

Bilinear Map — A bilinear map is a function $f : V \times V \to U$ that satisfies the following conditions:

1. $f(v_1 + v_2, w) = f(v_1, w) + f(v2, w)$
2. $f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$
3. $\lambda f(v, w) = f(\lambda v, w) = f(v, \lambda w)$

Multilinear Map — A multilinear map is a vunction $f : V \times \ldots \times V \to U$ that satisfies linearity in each of its components when the other components are kept fixed.

Alternating Multilinear Map — An alternating multilinear map is a multilinear map that evaluates to zero whenever two of the input vectors are equal. In particular, this means

$$f(\ldots, v, \ldots, w, \ldots) + f(\ldots, w, \ldots, v, \ldots) = 0$$

Symmetric Multilinear Map — A symmetric multilinear map is a multilinear map that is invariant under permutation of the inputs.

Universal Mapping Property

A universal mapping property is a property that is satisfied by a morphism.

$$\begin{tikzcd}
 & \frac{G}{[G,G]} \arrow[dd, dotted, "\tilde{f}"] \\
G \arrow[ur, "\pi"] \arrow[dr, "f"'] & \\
 & A
\end{tikzcd}$$

Consider the abelianization $\frac{G}{[G,G]}$ of a group G. Homomorphisms $G \to A$ with abelian A are the "same" as homomorphisms $\frac{G}{[G,G]} \to A$ because every homomorphism $f : G \to A$ is the commposite of a canonical homomorphism $\pi : G \to \frac{G}{[G,G]}$ with a unique homomorphism $\tilde{f} : \frac{G}{[G,G]} \to A$.

Tensor Products

Resources. $\boxed{\text{Keith Conrad's Notes}}$

Let R be a ring. Let M and N be R-modules.

Then the tensor product $M \otimes_R N$ is defined as an R-module equipped with a bilinear map $M \times N \to M \otimes_R N$ such that for each bilinear map $M \times N \to P$ factors through $M \otimes_R N$.

$$\begin{tikzcd}
 & M \otimes_R N \arrow[dd, "L"] \\
M \times N \arrow[ur, "\otimes"] \arrow[dr, "B"'] & \\
 & P
\end{tikzcd}$$

It is important to note that B is a bilinear map while L is a linear map.

Properties of Tensor products.

$$V \otimes W = W \otimes V$$
$$(V \otimes W) \otimes U = V \otimes (W \otimes U)$$
$$(V \oplus W) \otimes U = (V \otimes U) \oplus (W \otimes U)$$

Exterior Powers

The n-th exterior power of a vector space V is the wedge of n copies of V. It is denoted by $\bigwedge^n V$.

It is defined the same way as a tensor product, using a universal property. In this case, the universal property factors alternating maps $V^n \to U$.

$$\begin{tikzcd}
 & \bigwedge^n V \arrow[dd, dotted, "L"] \\
V^n \arrow[ur, "A"] \arrow[dr, "A'"'] & \\
 & U
\end{tikzcd}$$

Here, A and A' are alternating maps.

Properties of Exterior Powers.

$$\bigoplus_{a=0}^{n}\left(\bigwedge^{a} V \otimes \bigwedge^{n-a} W\right) = \bigwedge^{n}(V \oplus W)$$

Symmetric Powers

The n-th symmetric sum is denoted by $\mathrm{Sym}^n(V)$.

Likewise the space has a universal mapping property. It factors symmetric multilinear maps.

$$\begin{tikzcd} & \bigwedge^n V \arrow[dd, dotted, "L"] \\ V^n \arrow[ur, "S"] \arrow[dr, "S'"'] & \\ & U \end{tikzcd}$$

Here, S and S' are symmetric maps.

Properties of Symmetric Powers.

$$\bigoplus_{a=0}^{n}\big(\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^{n-a}(W)\big) = \mathrm{Sym}^n(V \oplus W)$$

Group Representations.

GL_n denotes the complex general linear group, i.e. the set of all $n \times n$ invertible matrices with complex entries.

A **matrix representation** of a group G is a homomorphism $R : G \to GL_n$. The number n is the dimension of the representation.

We use the notation R_g instead of $R(g)$ for the image of a group element g. Each R_g is an invertible matrix, and the statement that R is a homomorphism reads

$$R_{gh} = R_g R_h$$

It is essential to work as much as possible without fixing a basis, and to facilitate this, we use the idea of a representation of a group on a vector space V. We denote by

$$GL(V)$$

the group of invertible linear operators on V, the law of composition being composition of operators. We will assume that V is a finite dimensional complex vector space for now, and not the zero space.

Representation — A **representation** of a group G on a complex vector space V is a homomorphism

$$\rho : G \to GL(V)$$

So a representation assigns a linear operator to every group element.

A change in basis in V by a matrix P changes the matrix representation R associated to ρ to a conjugate representation $R' = P^{-1}RP$.

G-linear map — An isomorphism from one representation $\rho : LG \to GL(V)$ of a group G to another representation $\rho' : G \to GL(V')$ is an isomorphism of vector spaces $T : V \to V'$, an invertible linear transformation, that is compatible with the operators of G:

$$T(gv) = gT(v)$$

This is called a **G-linear map.**

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\psi"'] & V \arrow[d, "\psi"] \\ W \arrow[r, "\rho(g)"] & W \end{tikzcd}$$

Subrepresentation — A subrepresentation of a representation $\rho : G \to GL(V)$ is its restriction $\rho^w : G \to GL(V)$ is its restriction $\rho^w : G \to GL(W)$ to a subspace $W \subset V$ that is G-invariant. This means $\rho_g W = W$. If no such W exsits then ρ is irreducible.

Examples of Representations.

Quotient Representation — If $\rho : G \to V$ has ρ^W as a subrepresentation, then it also has $\rho_{V/W} \to GL(V/W)$ as a representation.

Direct Sum — If ρ and σ are representations to V and W, then $\rho \oplus \sigma : G \to V \oplus W$ is a representation.

Tensor Product — $\rho \otimes \sigma : G \to GL(V \otimes W)$ is defined by

$$\rho \otimes \sigma(g)(v \otimes w) = \rho(g)(v) \otimes \sigma(g)(v)$$

Tensor Power — The n-th tensor power $V^{\otimes n}$ is also a representation.

Irreducible Representations

> If W is a subrepresentation of a representation V of a finite group G, then there is a complementary invariant subspace $W^\perp$ of V such that $V = W \oplus W^\perp$

Unitary Representations.

Let V be a Hermitian space — acomplex vector space together with a positive definite Hermitian form $\langle , \rangle$. A unitary operator T on V is a linear operator with the property

$$\langle Tv, Tw \rangle = \langle v, w \rangle$$

for all v and w in V. If A is the matrix of a linear poerator T with respect to an orthonormal basis, then T is unitary if and only if A is a unitary matrix $A^* = A^{-1}$.

A representation ρ on a Hermitian space V is called unitary if ρ_g is a unitary operator for every g. We can write this condition as

$$\langle gv, gw \rangle = \langle v, w \rangle$$

> Let ρ be a unitary representation of G on a Hermitian space V, and let W be a G-invariant subspace. The orthogonal complement $W^\perp$ is also G-invariant, and ρ is the direct sum of its restrictions to the Hermitian spaces W and $W^\perp$. These restrictions are also unitary representations.

Proof — It is true that $V = W \oplus W^\perp$. Since ρ is unitary, it preserves orthogonality. If W is invariant and $u \perp W$, then $gu \perp gW = W$. This means that $u \in W^\perp$, then $gu \in W^\perp$. $\square$

Using this fact and induction, we get the stronger result.

> Every unitary representation $\rho : G \to GL(V)$ on a Hermitian vector space V is an orthogonal sum of irreducible representations.

Now we will turn the condition for a unitary representation around, and think of it as a condition on the Hermitian form instead of on the representation. If we are given the representation we are free to choose the form so that the space becomes Hermitian.

> Let $\rho : G \to GL(V)$ be a representation of a finite group on a vector space V. There exists a G-invariant, positive definite Hermitian form on V.

Proof — Consider an aribtrary positive definite Hermitian form on V that we denote by $\{,\}$. For example, we may choose a basis for V and use it to transfer the standard Hermitian form X^*Y on $\mathbb{C}^n$ over to V. Then we use the averaging provess to construct another form. The averaged form is defined as

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} \{gv, gw\}$$

We claim that this form is Hermitian, positive definite and G-invariant. We will only show the G-invariance.

Let h be an element of G. We want to show that $\langle hv, hw \rangle = \langle v, w \rangle$. Then we have

$$\langle hv, hw \rangle = \frac{1}{|G|} \sum_{g} \{ghv, ghw\} = \frac{1}{|G|} \sum_{g} \{g'v, g'w\} = \frac{1}{|G|} \sum_{g} \{gv, gw\} = \langle v, w \rangle$$

$\square$

Combining these, we get Maschke's theorem.

> **(Maschke's Theorem)** Every representation of a finite group G is a direct sum of irreducible representations.

> **(Maschke's Theorem for Modules)** Let V be a representation of G on a finite dimensional complex vector space i.e. a $\mathbb{C}[G]$-module. Each $\mathbb{C}[G]$ submodule W of V has as complementary $\mathbb{C}[G]$-submodule W' so that $V = W \oplus W'$.

This is called semisimplicity or complete reducibility.

Counterexample to Maschke's Theorem (G not finite)

Consider the group $(\mathbb{R}, +)$ and the two-dimensional representation given by

$$\rho : a \mapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in GL(\mathbb{R}^2)$$

If this representation were decomposable, it would be decomposable as the sum of two one dimensional representations i.e. for some linearly independent vectors $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}$,

$$\rho_a v = \lambda v$$

$$\rho_a w = \mu w$$

where λ and μ are constants. The first equation tells us

$$v_1 + av_2 = \lambda v_1$$

$$v_2 = \lambda v_2$$

so $\lambda = 1, v_2 = 0$. We reach a similar conclusion with $\mu = 1$ and $w_2 = 0$. But then v and w are not linearly independent, a contradiction.

It's important to note that this representation is *not* irreducible; the x-axis is preserved by the action of G. So it has a subrepresentation, but there is no complementary subrepresentation which makes ρ decomposable and herein lies the contradiction.

Schur's Lemma

If V and W are irreducible representations of G and $\phi : V \to W$ is a G-module homomorphism, then

1. Either φ is an isomorphism, or $\varphi = 0$
2. If $V = W$, then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$, I the identity matrix.

This now allows us to determine the extent to which a decomposition of a representation into irreducible representations is unique.

Decomposition of Representations

For any representation V of a fintie group G, there is a decomposition

$$V = V_1^{\oplus a_1} \oplus \ldots \oplus V_k^{\oplus a_k}$$

where the V_i are distinct irreducible representations. The deocompositon of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

A decomposition may also be written as

$$V = a_1 V_1 \oplus \ldots \oplus a_k V_k = a_1 V_1 + \ldots + a_k V_k$$

$V^{\oplus a_i}$ has no canonical decomposition

We see that the irreducible factors V_i are unique, and their multiplicities a_i are unique. However, note that the direct sum $V_i^{\oplus a_i}$ has no canonical decomposition when $a_i > 1$. For example, consider the cyclic group C_2 and suppose it acts on $\mathbb{R}^2$ by fixing the x-coordinate and negating the y -coordinate. This decomposes as the direct sum of the trivial representation and the sign/alternating composition, and this decomposition is canonical: the trivial and sign rep'ns be defined on the x-axis and y -axis respectively.

However, consider the action of C_2 on $\mathbb{R}^2$ defined by negating both coordinates. Then the representation decomposes into two copies of the sign representation, and these copies can act on any two independent one-dimensional subspaces of $\mathbb{R}^2$, akin to picking a basis, which we can do in many ways.

Regular Representation

The regular representation R corresponds to the action of G on iteself. Consider the vector space V with basis elements indexed by g, $\{e_g : g \in G\}$, and G acts on this vector space by

$$g \cdot \sum a_g e_x = \sum a_g e_x$$

Alternatively R is the space of complex-valued functions on G, where an element of G acts on a function α by $(g\alpha)(h) = \alpha(g^{-1}h)$.

1. Verify that these two definitions agree, by identifying the element e_x with the characteristic function which takes on the value 1 on x and 0 on other elements of G.

2. The space of functions on G can also be made into a G-module by the rule $(g\alpha)(h) = \alpha(hg)$. Show that this is an isomorphic representation.

Representations of Abelian Groups

Group Ring $\mathbb{C}[G]$ — The group ring $\mathbb{C}[G]$ is the set of formal sums of the form

$$a_0g_0 + \ldots + a_ng_n$$

where $a_i \in \mathbb{C}, g_i \in G$. This set forms a ring via addition which is commutative, and multiplicative through the distributive property, by allowing the a_i to commute with the g_i.

$\mathbb{C}[G]$ **module** — A $\mathbb{C}[G]$ module V is a module over the ring $\mathbb{C}[G]$.

Keeping these in mind, let G be abelian. Then for any representation ρ, the action of g commutes with the action of any other g, thus by the second part of Schur's lemma, each $\rho(g)$ is a homothety, a scalar multiple of the identity. It follows each irreducible representation must be one-dimensional.

Representations of S_3

The symmetric groups have two one dimensional representations. The first is the trivial representation. The second is the alternating representation, which sends even permutations to $+1$ and odd permuations to -1.

Consider the natural permutation representation given by

$$\rho((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rho((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $(1,1,1)^T$ is an eigenvector of the matrices, and therefore spans a subspace that remains invariant under the permutations. The representation spanned by this vector is isomorphic to the trivial representation.

The complementary subrepresentation must therefore be 2 dimensional. This representation is called the standard representation and is irreducible.

We will use this to describe arbitrary representations of S_3.

The useful idea is to exploit what we know about abelian groups. Consider the cyclic subgroup H (and hence abelian) of S_3 generated by $\tau = (123)$. Let (W, ρ) be an arbitrary representation of G (here, W is the vector space and ρ is the representation).

Consider W to be a representation of the action of H instead (probably reducible). Since H is abelian, we have that W can be decomposed into irreducible one-dimensional representations. This means that ρ_g is diagonalizable.

In particular, W is spanned by τ-eigenvectors, where the eigenvalues are third roots of unities. Since W is three dimensional, write this as

$$W = \operatorname{span}(v_1) \oplus \operatorname{span}(v_1) \oplus \operatorname{span}(v_1)$$

We further have $\tau v_i = \omega^i v_i$.

Now we investigate how $\sigma = (12)$ act on our basis $\{v_i\}$. We have the relation

$$\sigma\tau\sigma = \tau^2$$

which gives us

$$\begin{aligned} \tau(\sigma(v)) &= \sigma\left(\tau^2(v)\right) \\ &= \sigma\left(\omega^{2i}(v)\right) \\ &= \omega^{2i}(\sigma(v)) \end{aligned}$$

Therefore, if v is a τ-eigenvector with eigenvalue ω^i , σ takes it to an eigenvector of τ with eigenvalue $\omega^2 i$.

Explicit Standard Representation of S_3

Let $\alpha = (\omega, 1, \omega^2)^T$ and $\beta = (1, \omega, \omega^2)^T$ be in W where W is the permutation representation on the set of three points. This basis spans a space that is complementary to the subspace generated by $(1, 1, 1)^T$, and we observe the following relations.

$$\rho_\tau(\alpha) = \rho_{(123)}(\alpha) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ 1 \\ \omega^2 \end{bmatrix} = \begin{bmatrix} \omega^2 \\ \omega \\ 1 \end{bmatrix} = \omega\alpha$$

$$\rho_\tau(\beta) = \omega^2\beta$$

$$\rho_\sigma(\alpha) = \beta$$

$$\rho_\sigma(\beta) = \alpha$$

This proves that the subspace that these basis vectors span is invariant under the permutations. Therefore our basis must span a copy of the standard representation. Let us denote it by (V, ρ). Using this basis,

$$\rho_\tau = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \rho_\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Given an eigenvector v of τ, there are three possibilities.

1. Eigenvalue is ω, ω^2 and $v, \sigma v$ span a copy of V
2. Eigenvalue is 1 and $v, \sigma v$ are independent. Then $v + \sigma v$ span the trivial representation (we denote this by U). Meanwhile $v - \sigma v$ span alternating representatio (we denote this by U')
3. The eigenvalue is 1 and $v, \sigma v$ are dependent. Then v spans U if $v = \sigma v$, or v spans U' if $v = -\sigma v$.

This decomposes any represnetation entirely, thus we are done.

For any arbitrary representation W, we have $W \cong U^a \oplus U'^b \oplus V^c$ based on our three cases.

1. $c =$ number of pairs of τ-eigenvectors with eigenvalues ω, ω^2.
2. $a + c =$ number of σ-eigenvectors with eigenvalue 1.
3. $b + c =$ number of σ-eigenvectors with eigenvalue 1.

Decomposing $V \otimes V$

$V \otimes V$ has basis of τ-eigenvectors $\alpha \otimes \alpha, \alpha \otimes \beta, \beta \otimes \alpha, \beta \otimes \beta$. This means

$$V \otimes V \cong U \oplus U' \oplus V$$

Decomposing $\mathrm{Sym}^2(V)$

$$\mathrm{Sym}^2(V) \cong U \oplus U'$$

Decomposing $\mathrm{Sym}^3(V)$

$\mathrm{Sym}^3(V)$ has basis $\alpha \bullet \alpha \bullet \alpha, \alpha \bullet \alpha \bullet \beta, \alpha \bullet \beta \bullet \beta, \beta \bullet \beta \bullet \beta$. This means

$$\mathrm{Sym}^3(V) \cong U \oplus U' \oplus V$$

Characters

We have seen that spaces and vectors that remain invariant are very useful for classifying representations. We will formalize this intuition to work with characters.

Characters — The character χ of a representation ρ is the complex valued function whose domain is the group G, defined by

$$\chi(g) = \text{trace } \rho_g$$

If R is the matrix representation obtained from ρ by a choice of basis, then χ is also the character of R. The dimension of the vector space V is callede the dimension of the representation and the character. The character of an irreducible representation is called an irreducible character.

$$\begin{aligned}\chi_V\left(hgh^{-1}\right) &= \operatorname{Tr}\left(h|v \cdot g|v \cdot h^{-1} \mid v\right) \\ &= \operatorname{Tr}(g \mid v) = \chi_V(g)\end{aligned}$$

This means it suffices to define trace on just the conjugacy classes.

> Given two characters χ_V, χ_W of representations, we have
>
> 1. $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$
> 2. $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$

Proof — The first part is obvious. For the second, note that for any two basis vectors v, w of V, W, the value along the trace of g corresponds to $v \otimes w$ is the product of the values $gv|_V, rw|_W$ along the traces of g for v and w in V and W, respectively, as $g(v\otimes w) = gv \otimes gw$. Thus the image of the character on g is equal to

$$\left(gv_1|_{v_1} + \ldots + gv_a|_{v_a}\right)\left(gw_1|_{w_1} + \ldots + gw_b|_{w_b}\right)$$

> Given two characters χ_W, χ_V of representations, we have $\chi_{V^*} = \overline{\chi_V}$.

Proof — Recall that

$$\rho^*(g) = \rho(g^{-1})^T$$

This combined with the fact that the eigenvalues have to be roots of unity concludes our proof.

> Any representation is determined by its character

<u>Intuition</u> — Since we have the traces of $g, g^2, \ldots$, we have the sum of k-th powers of eigenvalues. Using Newton sums, we can extract the eigenvalues themselves, which gives us linear operators that are equal upto some jordan normal form.

For a representation V,

$$\chi_{\bigwedge^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$$

Character Table

By Maschke's theorem, we get the characters of every representation from the characters of irreducible representations. These irreducible representations are written in a table, called the character table.

As an example consider character table of the tetrahedral group T of 12 rotational symmetries of a tetrahedron.

	(1) 1	(4) x	(4) x^2	(3) z
χ_1	1	1	1	1
χ_2	1	a	b	c
χ_3	1	a'	b'	c'
χ_4	3	0	0	-1

Notice that the columns of the table are orthogonal. This is true in general.

The First Projection Formula

Suppose we wish to find the number of copies of the trivial representation in an arbitrary representation. Recall that the decomposition of a representation splits into G-invariant subspaces, so this value will be the dimension of

$$V^G = \{v \in V \mid gv = v \quad \forall g \in G\}$$

We build a function on V whose image is V^G using our averaging trick.

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g(v)$$

This preserves V^G and the image is contained in V^G as well, as showed before.

Then $\dim V^G$ will be the trace of φ, as the subspaces of irreducible representations will map to zero.

$$\begin{aligned} \dim V^G &= \mathrm{Tr}(\varphi) \\ &= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \end{aligned}$$

Thus, if an irreducible representation is non-trivial, the sum of the characters over all the elements is zero.

If V is the permutation representation associated to the action of a group G on a finite set X, show that $\chi_V(g)$ is the number if elements of X fixed by g.

Proof — Recall that the character is equal to the trace. In the matrix representation of g, the k-th row determines where k is mapped to, namely it maps to the index whose entry in that row is 1. Thus the 1 will be in the k-th column if and only if it is fixed.

$\mathbf{Hom}(V, W)$

$\mathrm{Hom}(V, W)$ is a rep with a G-action, $g\varphi = g \circ \varphi \circ g^{-1}$

$$\varphi \in \mathrm{Hom}(V, W)^G \iff g\varphi = \varphi \iff \varphi \text{ is G-Linear}$$

> $\mathrm{Hom}(V, W)^G$ is the vector space of all G-Linear maps.

Next we prove a lemma (this is essentially Schur's lemma written differently)

> Let V and W be irreducible representations. Then
>
> $$\dim \mathrm{Hom}(V, W)^G = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases}$$

Proof — If $V \ncong W$, then Schur's lemma implies that $\mathrm{Hom}(V, W)^G = 0$.

If $V \cong W$, let $T_1, T_2 \in \mathrm{Hom}(V, W)^G$ be nonzero elements. T_1, T_2 are isomorphisms from Schur's lemma. Thus $T_1 \circ T_2^{-1}$ is now a G-linear V-automorphism. So $T_1 \circ T_2^{-1} = \lambda \cdot I$ by Schur's lemma for some $\lambda \in \mathbb{C}$. This implies $T_1 = \lambda T_2$ i.e. $\mathrm{Hom}(V, W)^G$ is one-dimensional. $\square$

Now we use this to prove a wonderful result in the next section. We can use this to define the Hermitian product of two characters. Things simplify once we treat characters as vectors.

Orthogonality of Characters

Let V and W be irreps. Then

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\cdot\chi_W(g)=\begin{cases}1 & \text{if } V\cong W\\ 0 & \text{if } V\ncong W\end{cases}$$

Proof — Since $\mathrm{Hom}(V,W)\cong V^*\otimes W$, the character of $\mathrm{Hom}(V,W)$ is given by

$$\chi_{\mathrm{Hom}(V,W)}=\overline{\chi_V}\cdot\chi_W$$

Now for the clever part

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V}(g)\cdot\chi_W(g)=\frac{1}{|G|}\sum\chi_{\mathrm{Hom}(V,W)}(g)=\dim\mathrm{Hom}(V,W)^G$$

which concludes our proof by using the lemma in the last section.

This motivates our following construction.

Class Functions as Inner Product Space

Let $\mathbb{C}_{\text{class}}(G)$ be the vector space of complex-valued class functions on G. For any two functions $\alpha,\beta\in\mathbb{C}_{\text{class}}(G)$, we define the Hermitian inner product

$$\langle\alpha,\beta\rangle_G=\frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}\cdot\beta(g)$$

This new concept lets us write our previous theorem in a compact form.

Let V and W be irreps. Then $\langle\chi_V,\chi_W\rangle_G=\mathbb{1}_{V\cong W}$

Irreducible characters are orthonormal with respect to $\langle .,.\rangle_G$

Example

S_3		1 1	3 (12)	2 (123)
Triv	V_1	1	1	1
Alt	V_2	1	-1	1
Std	V_3	2	0	-1

The hermitian products give us —

$$\langle \chi_1, \chi_2 \rangle_G = 1 \cdot 1 + 3 \cdot (-1) + 2 \cdot 1 = 0$$

$$\langle \chi_2, \chi_3 \rangle_G = 1 \cdot 2 + 3 \cdot 0 + 2 \cdot (-1) = 0$$

$$\langle \chi_3, \chi_1 \rangle_G = 1 \cdot 2 + 3 \cdot 0 + 2 \cdot (-1) = 0$$

$$\langle \chi_1, \chi_1 \rangle_G = \langle \chi_2, \chi_2 \rangle_G = \langle \chi_3, \chi_3 \rangle_G = 1$$

An interesting observation is that the columns are also orthogonal, but with respect to the regular Euclidean dot product. This is easy to view once we think of the character table as a matrix.

Another thing of note is that the number of characters is equal to the number of different conjugacy classes. Since the characters are orthonormal, they span a subspace of the class functions whose dimension is equal to the number of irreducible representations. But the dimension of the space of class functions is the number of conjugacy classes, so this proves an upper bound. Equality will be handled later.

Character = Representation

> Any representation is determined by its character . In particular, the multiplicity a_i of V_i in V is $\langle \chi_V, \chi_{V_i} \rangle_G$.

Proof — Suppose $V = \oplus V_i^{\oplus a_i}$ giving us $\chi_V = \sum\limits_i a_i \chi_i$. Since the λ_i are linearly independent, the a_i are uniquely determined. In particular, orthonormality gives us

$$\langle \chi_V, \chi_{V_i} \rangle_G = \sum_j a_j \langle \chi_{V_j}, \chi_{V_i} \rangle_G = a_i$$

Irreducibility Criterion

A representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle_G = 1$.

Proof — Let $V = \oplus V_i^{\oplus a_i}$. Then $\chi_V = \sum a_i \lambda_i$ which gives us

$$\begin{aligned}\langle \chi_V, \chi_V \rangle_G &= \sum_i a_i^2 \langle \chi_{V_i}, \chi_{V_i} \rangle_G + \sum_{i \neq j} a_i a_j \left\langle \chi_{V_i}, \chi_{V_j} \right\rangle_G \\ &= \sum_i a_i^2\end{aligned}$$

This can equal 1 precisely when one of the squares is a 1 and the others are zero, meaning V is irreducible. □

We now revisit an old concept again for some refresher.

The Regular Representation

The regular representation is the representation of a group G acting on itself by left multiplication. The following is easy to check by counting the number of fixed points upon multiplication by g.

The character of R (the regular representation) is

$$\chi_R(g) = \begin{cases} 0 & \text{if } g \neq e \\ |G| & \text{if } g = e \end{cases}$$

An interesting consequence of this is the following.

any irreducible representation V appears in the regular representation $\dim V$ times.

The sketch of the proof is to use characters to find the multiplicity a_i. This tells us that $R = \oplus V_i^{\oplus \dim V_i}$ and so

$$|G| = \sum_i \dim V_i^2$$

Character Table of S_4

Conjugacy classes correspond to the cycle type of a permutation, which corresponds to partitions of 4.

- $1+1+1+1$, $\# = 1$
- $1+1+2$, $\# = \binom{4}{2} = 6$
- $2+2$, $\# = \frac{1}{2}\binom{4}{2} = 6$
- $1+3$, $\# = \binom{4}{3}2! = 8$
- 4, $\# = 3! = 5$

Note that the dimension 1 representations are the trivial (V_1) and alternating (V_2) representations. The characters are $(1,1,1,1,1)$ and $(1,-1,1,-1,1)$.

The standard representation given by

$$V_3 = \left\{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 \in \mathbb{C}^4 : \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0\right\}$$

is three dimensional. If χ_P is the permutation representation, then we have

$$\chi_3 = \chi_P + \chi_1$$

and counting the number of fixed points, we have

$$\chi_P = (4,2,0,1,0)$$

Therefore

$$\chi_3 = \chi_P - \chi_1 = (3,1,-1,0,1)$$

Finally, let S be the sum of squares of the remaining irreps. Then we have $S + 1^2 + 1^2 + 3^2 = |G| \implies S = 13$ and the only possibility is $2^2 + 3^2 = 13$. Let the irrep of dimension 2 be V_4 and dimension 3 be V_5.

		1	6	3	8	6
S_4		1	(12)	(12)(34)	(123)	(1234)
Triv	V_1	1	1	1	1	1
Alt	V_2	1	-1	1	1	-1
Std	V_3	3	1	-1	0	-1
	V_4	2	?	?	?	?
	V_5	3	?	?	?	?

We can try making educated guesses about what the other characters are. One of them is $U = V_3 \otimes V_2$ which has character $\chi_U = \chi_3 \cdot \chi_2 = (3, -1, -1, 0, 1)$. Since $\langle \chi_U, \chi_U \rangle_G = 1$, we have U is irreducible. So $V_5 = U$.

We can figure out the remaining row by using orthonormality of columns.

		1	6	3	8	6
S_4		1	(12)	(12)(34)	(123)	(1234)
Triv	V_1	1	1	1	1	1
Alt	V_2	1	-1	1	1	-1
Std	V_3	3	1	-1	0	-1
	V_4	2	0	2	-1	0
	V_5	3	-1	-1	0	1

and we are done!!

Manifolds

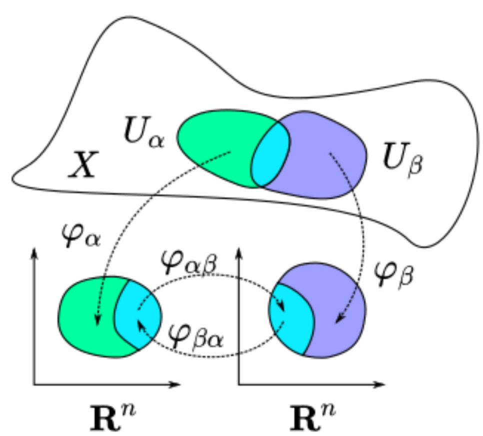
Separable Space

A topological space that contains a countable dense subset is called a separable space. Any separable Hausdorff space is uncountabe. The reals form a separable space with $\mathbb{Q}$ as the countable dense subset.

A differentiable manifold of dimension n is a separable Hausdorff space M together with a family $\{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ such that

- $\{U_\alpha\}_{\alpha \in A}$ is a covering of M by open sets.
- u_α is a homeomorphism of U_α onto an open subset of Euclidean n-space $\mathbb{R}^n$
- If $\alpha, \beta \in A$, then the transiiton map $u_\beta \cdot u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U_\alpha \cap U_\beta)$ is a C^∞ map of domains in $\mathbb{R}^n$.
- $\{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ is maximal for the first three properties. We call U_α a coordinate neighborhood on M, and we call (U_α, u_α) a local coordinate system or a chart. The collection of charts is an atlas.

A maximal atlas exists by Zorn's lemma. However, it is rarely ever required to work with a maximal atlas.



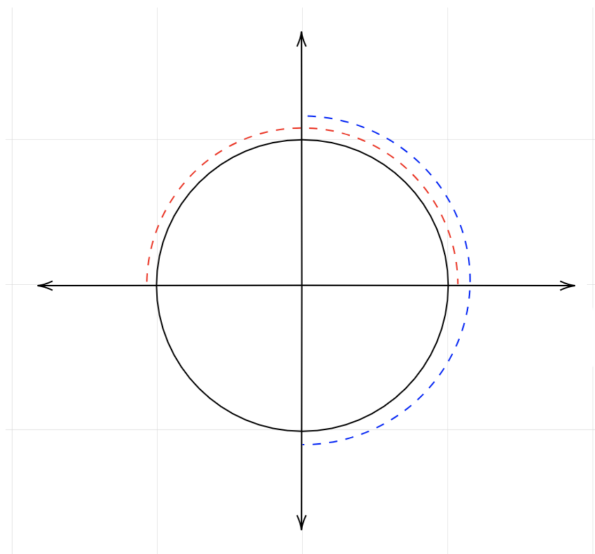
Example

Let $M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, the unit circle. We can cover M with four open sets U_α which are upper, right, lower, and left open circles respectively:

$$U_1 = \{(x,y) \in M : y > 0\}$$

$$U_2 = \{(x,y) \in M : x > 0\}$$
$$U_3 = \{(x,y) \in M : y < 0\}$$
$$U_4 = \{(x,y) \in M : x < 0\}$$

Define maps u_α as the projection from U_α onto the x- or y-coordinate, so u_1, u_3 are projections onto the x-coordinate, u_2, u_4 are projections onto the y-coordinate. All four u_α have image $(-1,1)$.



Then we have the transition map $u_2 u_1^{-1} : u_1(U_1 \cap U_2) \to u_2(U_1 \cap U_2)$ via $x \to \sqrt{1-x^2}$. We can verify this is C^∞ on $(0,1)$ but not differentiable at $x = 1$, and its inverse is also C^∞ on $(0,1)$ but not differentiable at $y = 1$. The U_α form an atlas of the unit circle.

Differentiable Maps

Let M and N be differentiable manifolds. A map $f : M \to N$ is called differentiable if it is differentiable in terms of any system of local coordinates, i.e. $u_\beta \circ f \circ u_\alpha^{-1}$ is differentiable for all $\alpha, \beta \in A$.

$$u_\alpha^{-1} : \mathbb{R}^n \to M$$

$$f : M \to N$$

$$u_\beta : N \to \mathbb{R}^n$$

Diffeomorphism

A function $f : M \to N$ is a diffeomorphism if both f and f^{-1} are differentiable homeomorphisms.

Product Manifold

If M and N are differentiable manifolds defined by local coordinate systems $\{U_\alpha, u_\alpha\}_{\alpha \in A}$ and $\{V_\beta, v_\beta\}_{\beta \in B}$, then the product manifold $M \times N$ is the set $M \times N$ with local coordinate system $\{U_\alpha \times V_\beta, u_\alpha \times v_\beta\}_{(\alpha,\beta) \in A \times B}$

Armed with these definitions, we are now ready to define Lie Groups.

Lie Group

Lie Group

A Lie group is a smooth i.e. C^∞ manifold where multiplication and inverse operations are smooth maps i.e.

$$\times : G \times G \to G$$

and

$$\iota : G \to G$$

are differential maps.

Map of Lie Groups

A map or morphism between two Lie Groups G and H is a map $\rho : G \to H$ that is both differentiable and a group homomorphism i.e. for $g_1, g_2 \in G$,

$$\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$$

Now we are going to define what Lie subgroups are. However, we need to make a distinction between an embedded/regular submanifold of a manifold M and an immersed submanifold. For that, we will need to define an immersion.

Immersion

Let M, N be differentiable manifolds. The differentiable function $f : M \to N$ is an immersion if

$$D_p f : T_p M \to T_p N$$

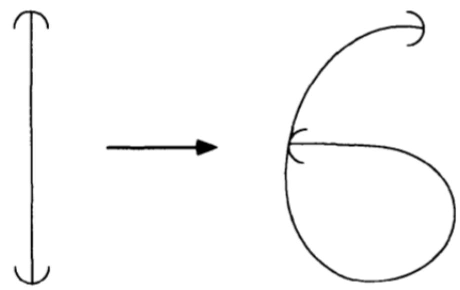
is an injective function at every point p of M, where $T_p X$ denotes the tangent space of a manifold X at a point p in X. The functin f itself is not necessarily injective. Equivalently, f is an immersion if

$$\text{rank } D_p f = \dim M$$

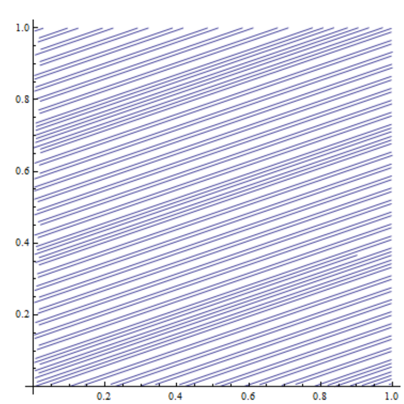
Immersed Submanifold

An immersed submanifold of a manifold M is the image S of an injective immersion map $f : N \to M$ with the topology on N. Note that general, the topology on N will not agree with the subspace topology, so generally S is not a submanifold of M in the subspace topology.

On the other hand, an embedded or regular submanifold (what Fulton and Harris call a closed submanifold) is a subset of M which has a manifold structure with respect to the subspace topology inherited from M.



Consider the immersion of some open interval in $\mathbb{R}$ into $\mathbb{R}^2$. This fails to be a submanifold under the subspace topology, since at the point p where the endpoint almost meets, consider the open neighborhood X under the subspace topology. If we remove the point p, then X will have three connected components, but such a space cannot be homeomorphic to $\mathbb{R}^n$ for any n.



Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be a torus. Let X be the image of a line with slope π. Then this image is dense in M. But under the subspace topology, this is not a manifold, since it consists of dense disconnected segments.

Lie Algebra

Property 1. of Wedge Product

Let $e_1, \ldots, e_n$ be a basis of $\mathbb{R}^n$, and A a linear operator. Then we have

$$Ae_1 \wedge Ae_2 \wedge \ldots \wedge Ae_n = \det(A)\ e_1 \wedge e_2 \wedge \ldots \wedge e_n$$

Proof Sketch

$$\begin{aligned}\mathbf{v} \wedge \mathbf{w} &= (a\mathbf{e}_1 + b\mathbf{e}_2) \wedge (c\mathbf{e}_1 + d\mathbf{e}_2) \\ &= ac\mathbf{e}_1 \wedge \mathbf{e}_1 + ad\mathbf{e}_1 \wedge \mathbf{e}_2 + bc\mathbf{e}_2 \wedge \mathbf{e}_1 + bd\mathbf{e}_2 \wedge \mathbf{e}_2 \\ &= (ad - bc)\mathbf{e}_1 \wedge \mathbf{e}_2\end{aligned}$$

Follow this with a recursion.

Property 2. of Wedge Product

Let X be a linear operator in $\mathbb{R}^2$ and e_1, e_2 a basis. Then

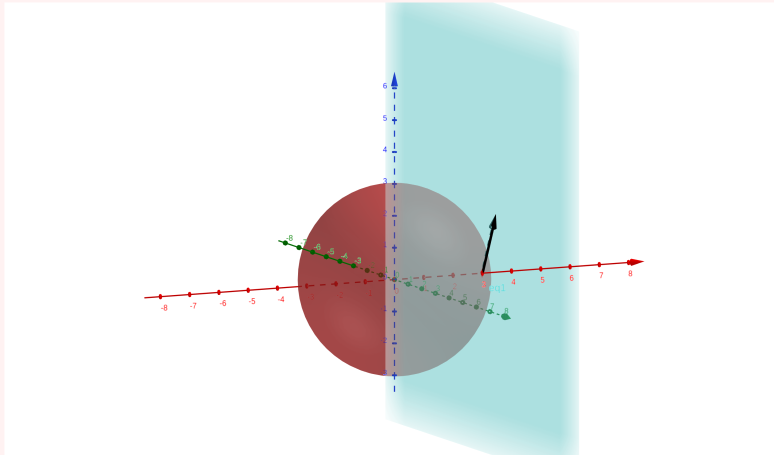
$$\mathrm{X}\, e_1 \wedge e_2 + e_1 \wedge \mathrm{X}\, e_2 = \mathrm{Trace}(\mathrm{X}) \cdot (e_1 \wedge e_2)$$

General version of Property 2.

Let X be a linear operator in $\mathbb{R}^n$ and $e_1, \ldots, e_n$ a basis. Then

$$\sum e_1 \wedge \cdots \wedge \mathrm{X}\,(e_i) \wedge \cdots \wedge e_n = \mathrm{Trace}(\mathrm{X}) \cdot (e_1 \wedge \cdots \wedge e_n)$$

Arcs



Arcs of a manifold M are one dimensional submanifolds of M. They can be characterized as a map A from $[0,1]$ to the manifold, sending $t \to A_t$.

Example 1.

Find the Lie algebra of the Special Linear Group $SL_2\mathbb{R}$.

Suppose $A : [-1,1] \to SL_n\mathbb{R}$ is an arc in $SL_n\mathbb{R}$ with $A_0 = I$ and $A'_0 = X$ at $t = 0$. Pick any basis e_1, e_2 of $\mathbb{R}^2$. This gives us

$$A_t\,(e_1) \wedge A_t\,(e_2) \equiv \det(A_t)\, e_1 \wedge e_2 \equiv e_1 \wedge e_2$$

Therefore the expression on the left remains invariant as t changes, therefore its derivative with respect to t is 0.
So differentiating and evaluating at $t = 0$,

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} (A_t\,(e_1) \wedge A_t\,(e_2)) \\ &= \mathrm{X}\, e_1 \wedge e_2 + e_1 \wedge \mathrm{X}\, e_2 \\ &= \mathrm{Trace(X)} \cdot (e_1 \wedge e_2) \end{aligned}$$

So X has trace 0!

Example 2.

Find the Lie algebra of the Special Linear Group $SL_n\mathbb{R}$.

Suppose $A : [-1, 1] \to SL_n\mathbb{R}$ is an arc in $SL_n\mathbb{R}$ with $A_0 = I$ and $A_0' = X$ at $t = 0$. Pick any basis $e_1, \ldots, e_n$ of $V = \mathbb{R}^n$. This gives us

$$A_t\,(e_1) \wedge \cdots \wedge A_t\,(e_n) \equiv \det(A_t)\, e_1 \wedge \cdots \wedge e_n \equiv e_1 \wedge \cdots \wedge e_n$$

Therefore the expression on the left remains invariant as t changes, therefore its derivative with respect to t is 0.
So differentiating and evaluating at $t = 0$,

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} (A_t\,(e_1) \wedge \cdots \wedge A_t\,(e_n)) \\ &= \sum e_1 \wedge \cdots \wedge \mathrm{X}\,(e_i) \wedge \cdots \wedge e_n \\ &= \mathrm{Trace(X)} \cdot (e_1 \wedge \cdots \wedge e_n) \end{aligned}$$

This implies that X must have trace 0, and so

$$\mathfrak{sl}_n = \text{vector space of trace 0 matrices}$$

Example 3.

Find the Lie algebra of Orthogonal Group $O_n\mathbb{R}$

Let A be an arc with $A_0 = I$ and $A'_0 = X$. Suppose Q is the symmetric quadratic form that O_n preserves. For every pair of vectors v, w, we have

$$Q(A_t(v), A_t(w)) = Q(v, w)$$

Differentiating this and evaluating at 0,

$$Q(X(v), w) + Q(v, X(w)) = 0$$

We can make this condition prettier by considering

$$Q(v, w) = v^T M w$$

for a symmetric $n \times n$ matrix M. Since A is orthogonal, it satisfies

$$A^T M A = M$$

Differentiating we get

$$X^T M + M X = 0$$

which characterizes all such matrices X.

Example 4.

Find the Lie algebra of Symplectic Group $SP_{2n}\mathbb{R}$

Let A be an arc with $A_0 = I$ and $A'_0 = X$. Suppose Q is the skew-symmetric quadratic form that SP_{2n} preserves. We get the condition

$$Q(X(v), w) + Q(v, X(w)) = 0$$

$$Q(X(v), w) = Q(X(w), v)$$

Motivation

Let $G = \mathbb{R}_{>0}$ under multiplication. This is a Lie group of dimension 1, connected, abelian. Let m_g be the map $x \mapsto gx$. The derivative

$$(m_g)_* = \frac{d}{dx}(gx) = g$$

Let X be a fixed tangent vector at $g = 1$. Consider a vector field

$$v_X(g) = (m_g)_*(X) = gX$$

The tangent vector based at a general point g is gX. This is a left-invariant vector field. For this G, every vector in a left-invariant vector field is g times longer at g than it was at 1.

Think of X as a constant scalar which is the length (with $\pm$ sign) of a vector based at 1. If we try integrating this vector field, we are looking for a function y, whose tangent vector at any point is the vector assigned by the vector field.

$$\frac{dy}{dx} = Xy$$

Lucky for us, we know how to solve this differential equation. The solution is

$$y = e^{Xx}$$

Note that this map is a homomorphism, in the sense that

$$y(a + b) = y(a)y(b)$$

We call y a one-parameter subgroup of G with tangent vector X.

Formalizing the Idea

Suppose X is an element of the lie algebra $\mathfrak{g}$. We can definte the vector field

$$v_X(g) = (m_g)_*(X)$$

This vector field remains invariant under the map m_g for all g. This is called a left-invariant vector field on G. The main takeaway is the existence of this vector field.

So now, we have a vector field v on a manifold M and a point $p \in M$. We can now integrate over this vector field. We construct an antiderivative φ.

This map $\varphi : I \to M$ is differentiable, where I is some open neighborhood containing 0. The tangent vector at any point is the vector assigned to that point by v

$$\varphi'(t) = v(\varphi(t))$$

for all $t \in I$.

Suppose the manifold is G, with $\varphi(0) = p$ where p is the identity, and v_X the vector field associated to X. We claim that $\varphi : I \to G$ is a homomorphism. It suffices to show that

$$\varphi(s + t) = \varphi(s)\varphi(t)$$

whenever $s, t, s + t \in I$. To prove this, we fix s and let t vary. So suppose α and β are arcs so that

$$\alpha(t) = \varphi(s)\varphi(t)$$
$$\beta(t) = \varphi(s + t)$$

$$\alpha(t) = \varphi(s)\varphi(t)$$
$$\beta(t) = \varphi(s + t)$$

We have $\alpha(0) = \beta(0)$ and by the invariance of v_X, we have

$$\alpha'(t) = v_X(\alpha(t))$$

$$\beta'(t) = v_X(\beta(t))$$

By the uniqueness of the integral curve, we have $\alpha(t) = \beta(t)$ for all t.

We can extend φ uniquely to all of $\mathbb{R}$, with

$$\varphi_X'(t) = v_X(\varphi)(t) = (m_{\varphi(t)})_*(X)$$

One Parameter Subgroup

The Lie group map $\varphi_X : \mathbb{R} \to G$ is called the one-parameter subgroup of G with tangent vector X at the identity.

Exponential Function

We define the exponential map
$$\exp : \mathfrak{g} \to G$$
by
$$\mathrm{exp(X) = \varphi_X(1)}$$

We have
$$\varphi_{\lambda x} = \varphi_X(\lambda t)$$
so the exponential map restricted to the lines through the origin in $\mathfrak{g}$ gives the one-parameter subgroup of G. We have the charcterization:

Proposition

The exponential map is the unique map taking 0 to e whose differential at the origin
$$(\exp_*)_0 : T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$$
is the identity, and whose restrictions to the lines through the origin in $\mathfrak{g}$ are one-parameter subgroups of G.

First Principle

If G is connected, then the map ψ is determined by its differential $d\psi_e$ at the identity.

Proof

Since the differential of the exponential map at the origin in $\mathfrak{g}$ is an isomorphism, the image of exp will contain a neighborhood of the identity in G. If G is connected, this will generate all of G, as required.

Campbell Hausdorff Formula

Power series of exp in $GL_n\mathbb{R}$

We have

$$\exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \ldots$$

Power series of the inverse function in $GL_n\mathbb{R}$

We can also define the inverse of the map.

$$\log(g) = (g - I) - \frac{(g-I)^2}{2} + \frac{(g-I)^3}{3} - \cdots \in \mathfrak{gl}_n\mathbb{R}$$

Example

$$\exp\left(\begin{bmatrix} 0 & 0 \\ at & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ at & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ at & 1 \end{bmatrix}$$

Example

$$\begin{aligned} \exp\left(\begin{bmatrix} bt & 0 \\ 0 & ct \end{bmatrix}\right) &= \begin{bmatrix} 1 + bt + \frac{(bt)^2}{2} + \cdots & 0 \\ 0 & 1 + ct + \frac{(ct)^2}{2} + \cdots \end{bmatrix} \\ &= \begin{bmatrix} e^{bt} & 0 \\ 0 & e^{ct} \end{bmatrix} \end{aligned}$$

Bilinear Operation $*$

Now we can define a new bilinear operation on $\mathfrak{gl}_n\mathbb{R}$. We set

$$X * Y = \log(\exp(X)\exp(Y))$$

Substituting the power series for exp and log, we expand to get

$$\begin{aligned} X * Y &= (X + Y) + \left(-\frac{(X+Y)^2}{2} + \left(\frac{X^2}{2} + X \cdot Y + \frac{Y^2}{2}\right)\right) + \cdots \\ &= X + Y + \frac{1}{2}[X, Y] + \cdots \end{aligned}$$

Campbell-Hausdorff Formula

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [Y, X]] + \cdots$$

Second Principle

Proposition

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then the subgroup of the group G generated by $\exp(\mathfrak{h})$ is an immersed subgroup H with tangent space $T_eH = \mathfrak{h}$.

Proof

Note that the subgroup generated by $\exp(\mathfrak{h})$ is the same as the sub-group generated by $\exp(U)$, where U is any neighborhood of the origin in $\mathfrak{h}$. It will suffice, then (see Exercise 8.42), to show that the image of $\mathfrak{h}$ under the exponential map is "locally" closed under multiplication, i.e., that for a sufficiently small disc $\Delta \subset \mathfrak{h}$, the product $\exp(\Delta) \cdot \exp(\Delta)$ (that is, the set of pairwise products $\exp(X) \cdot \exp(Y)$ for $X, Y \in \Delta$) is contained in the image of $\mathfrak{h}$ under the exponential map. Assume that G can be realized as a subgroup of a general linear group $GL_n\mathbb{R}$, so that we can use the power series expansion. So it suffices to do the case $G = GL_n\mathbb{R}$, but the Campbell Hausdorff formula is just a sum over Lie brackets, i.e. elements of $\mathfrak{h}$, so we are done.

Second Principle

Let G and H be Lie groups with G simply connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. A linear map $\alpha : \mathfrak{g} \to \mathfrak{h}$ is the differential of a map $A : G \to H$ of Lie groups if and only if α is a map of Lie algebras.

Proof

To see this, consider the product $G \times H$. Its Lie algebra is just $\mathfrak{g} \oplus \mathfrak{h}$. Let $\mathfrak{i} \subset \mathfrak{g} \oplus \mathfrak{h}$ be the graph of the map α. Then the hypothesis that α is a map of Lie algebras is equivalent to the statement that $\mathfrak{i}$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$; and given this, by the proposition there exists an immersed Lie subgroup $J \subset G \times H$ with tangent space $T_eJ = \mathfrak{i}$.

Look now at the map $\pi : J \to G$ given by projection on the first factor. By hypothesis, the differential of this map $d\pi_e : i \to g$ is an isomorphism, so that the map $J \to G$ is an isogeny; but since G is simply connected it follows that π is an isomorphism. The projection $\eta : G \cong J \to H$ on the second factor is then a Lie group map whose differential at the identity is α.

Rough Classifiction of Lie Algebras

Center of a Lie Algebra

The center $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined to be the subspace of $\mathfrak{g}$ of elements $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. $\mathfrak{g}$ is abelian if $[\mathfrak{g}, \mathfrak{g}] = 0$

Ideal

A lie subalgebra $\mathfrak{h} \in \mathfrak{g}$ is called an ideal if it satisfies the condition

$$[X, Y] \in \mathfrak{h} \text{ for all } X \in \mathfrak{h}, Y \in \mathfrak{g}$$

Theorem

Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra. Show that the subgroup of G generated by exponentiating the Lie subalgebra $Z(\mathfrak{g})$ is the connected component of the identity in the center $Z(G)$ of G.

Ideal $\iff$ Normal

Let G be a connected Lie Group, $H \subset G$ be a connected subgroup and $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras. Then H is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Some properties of normal subgroups have analogues for ideals. For instance, $\mathfrak{g}/\mathfrak{h}$ has a compatible bracket operation if and only if $\mathfrak{h}$ is an ideal.

Simple Lie Algebras

A Lie Algebra $\mathfrak{g}$ is simple if $\dim \mathfrak{g} > 1$ and it contains no nontrivial ideals.

Simple Lie algebras are building blocks of general lie algebras.

We will define two chains of subalgebras.

Lower Central Series — The subalgebras $\mathfrak{D}_k\mathfrak{g}$, defined by

$$\mathfrak{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

$$\mathfrak{D}_k\mathfrak{g} = [\mathfrak{g}, \mathfrak{D}_{k-1}\mathfrak{g}]$$

Derived Series — The subalgebras $\mathfrak{D}^k\mathfrak{g}$, defined by

$$\mathfrak{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

$$\mathfrak{D}^k\mathfrak{g} = [\mathfrak{D}^{k-1}\mathfrak{g}, \mathfrak{D}^{k-1}\mathfrak{g}]$$

These two definitions are motivated by the equivalent definitions for groups. The following definitions are also motivated by the corresponding groups.

Nilpotent — $\mathfrak{D}_k\mathfrak{g} = 0$ for some k.

Solvable — $\mathfrak{D}^k\mathfrak{g} = 0$ for some k.

Semisimple — $\mathfrak{g}$ has no non-zero solvable ideals.

Example — Nilpotent Lie Algebra

$$\mathfrak{n}_n\mathbb{R} = \{\text{strictly upper-triangular } n \times n \text{ matrices}\}$$

Then we have $\mathfrak{D}_k\mathfrak{g}$ will merely be all matrices with zeros above the main diagonal that are manhattan distance at most k. This implies $\mathfrak{D}_n\mathfrak{g} = 0$

$$\begin{bmatrix} 0 & \bullet & \bullet & \bullet \\ & 0 & \bullet & \bullet \\ & & 0 & \bullet \\ & & & 0 \end{bmatrix} \xrightarrow{\mathrm{ad}_X} \begin{bmatrix} 0 & 0 & \bullet & \bullet \\ & 0 & 0 & \bullet \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \xrightarrow{\mathrm{ad}_X} \begin{bmatrix} 0 & 0 & 0 & \bullet \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \xrightarrow{\mathrm{ad}_X} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

Example — Solvable Lie Algebra

$$\mathfrak{b}_n = \{\text{upper-triangular } n \times n \text{ matrices}\}$$

Note that we have the relation

$$\mathfrak{D}^k\mathfrak{b}_n\mathbb{R} = \mathfrak{n}_{2^{k-1}n}\mathbb{R}$$

This example is important for a reason - the converse is also in some sense true. Any solvable lie algebra has an upper triangular representation in terms of some suitable basis.

Exact Sequence

Consider a sequence of groups and morphisms

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} G_n$$

This sequence is called **exact** if the image of each morphism is equal to the kernel of the next.

$$\text{im }(f_k) = \ker(f_{k+1})$$

A short exact sequence is of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

A short exact sequence is called **split** if there exists a homomorphism $h : C \to B$ such that the composition $g \circ h$ is the identity map on C.

Splitting Lemma

If A, B, C are abelian groups in a short exact sequence that splits, then

$$B \cong A \oplus C$$

Consider the example

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Radical of $\mathfrak{g}$

Given two solvable ideals in a Lie algebra, their sum is once again solvable.

$$\frac{(\mathfrak{a} + \mathfrak{b})}{\mathfrak{b}} = \frac{\mathfrak{a}}{(\mathfrak{a} \cap \mathfrak{b})}$$

Therefore, summing over all solvable ideals, we get a **maximal solvable ideal**, otherwise called the **radical** of $\mathfrak{g}$. This is denoted by $\mathbf{Rad}(\mathfrak{g})$.

The quotient $\mathfrak{g}/\mathbf{Rad}(\mathfrak{g})$ is semisimple. Therefore, the following sequence is a short exact sequence.

$$0 \to \mathbf{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathbf{Rad}(\mathfrak{g}) \to 0$$

with the first algebra solvable and the last one semisimple. This sequence splits, this is called the Levi decomposition.

Engel and Lie

Engel's Theorem

Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be any Lie subalgebra such that every $X \in \mathfrak{g}$ is a nilpotent endomorphism of V. Then there exists a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.

Since $\mathfrak{g}$ kills v, it will act on the quotient $\overline{V}V/\text{span}(v)$, and by induction we can find a basis $\overline{v}_2, \ldots, \overline{v}_n$ for $\overline{V}$ so that the action is strictly upper triangular.

Lie's Theorem

Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a complex solvable Lie algebra. Then there exists a non-zero vector $v \in V$ that is an eigenvector of X for all $X \in \mathfrak{g}$.

Consequence of Lie's Theorem

Let $\mathfrak{g}$ be a complex Lie algebra $\mathfrak{g}_{ss} = \mathfrak{g}/\textbf{Rad}(\mathfrak{g})$. Every irreducible representation of $\mathfrak{g}$ is of the form $V = V_0 \otimes L$, where V_0 is an irreducible representation of $\mathfrak{g}_{ss}$, and L is a one dimensional representation.

Semisimplicity

Failure of Complete Reducibility

Let $\rho : \mathbb{R} \to GL(\mathbb{R}^2)$ be the representation

$$\rho : a \mapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

It has $\mathbb{R}e_1$ as the unique subrepresentation of dimension 1. Therefore it is indecomposable.

We can get past this bad news by restricting our focus to semisimple Lie algebras.

Complete Reducibility

Let V be a representation of a semisimple Lie algebra $\mathfrak{g}$ and $W \subset V$ be a subspace invariant under the action of $\mathfrak{g}$. Then there exists a subspace $W^\perp \subset V$ complementary to W and invariant under $\mathfrak{g}$.

Jordan Decomposition

Note that any endomorphism X of a complex vector space can be uniquely written in the form $X = X_s + X_n$ where X_s is diagonalizable and X_n is nilpotent and the two commute. To generalize this for semisimple algebras, we have the following.

Preservation of Jordan Decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra. For any element $X \in \mathfrak{g}$, there exist X_s and X_n in $\mathfrak{g}$ such that for any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ we have

$$\rho(X)_s = \rho(X_s)$$

$$\rho(X)_n = \rho(X_n)$$

Weyl's Unitary Trick

If $\mathfrak{g}$ is any complex semisimple Lie algebra, there exists a unique real Lie algebra $\mathfrak{g}_0$ with complexification $\mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}$ such that the simply connected form of the Lie algebra $\mathfrak{g}_0$ is a compact Lie group G.

semisimple $= \bigoplus$ simple

Killing Form

Let L be any Lie algebra. If $x, y \in L$, define

$$\kappa(x, y) = \mathrm{Trace}(\mathrm{ad}_x \circ \mathrm{ad}_y)$$

. Then κ is a symmetric bilinear form called the **Killing Form**.

We will state the following properties of the killing form without proof.

- A lie algebra is semisimple if and only if the killing form is non-degenerate
- Let L be a Lie algebra such that $\kappa(x, y) = 0$ for all $x \in [L, L]$ and $y \in L$. Then L is solvable.

Every semisimple lie algebra is the direct sum of simple lie algebras.

Proof — Let I be a minimal nonzero ideal. Then $I^\perp$ is also an ideal. From Cartan's criterion, $I \cap I^\perp = \emptyset$ since it is solvable. So $L = I \oplus I^\perp$ since κ is non-degenerate on I. Both I and $I^\perp$ are semisimple, so I is necessarily simple. By induction $I^\perp$ is the direct sum of simple ideals.

Classification of simple Lie algebras

Classification

With five exceptions, every simple complex Lie algebra is isomorphic to $\mathfrak{sl}_n\mathbb{C}, \mathfrak{so}_n\mathbb{C}, \mathfrak{sp}_{2n}\mathbb{C}$.