Crypto!!!

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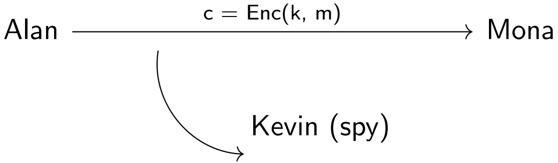

Problem

Suppose Kevin is spying on the channel between Alan and Mona. In order to encrypt the messages, Alan and Mona have to agree on a key for encrypting and decrypting their messages. But with Kevin spying on the channel, how do they send each other a key?

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Solution. Computational complexity!!

Fact.

Suppose you're given a prime p, and you pick some number a modulo a prime p. Kevin is given the equation

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This is called the **discrete log problem**.

Turns out that the discrete log problem is actually super hard, in fact it is NP hard. So unless Kevin can solve NP hard problems, I would wager a guess that he has no chance of solving the problem. But we still love him anyway.

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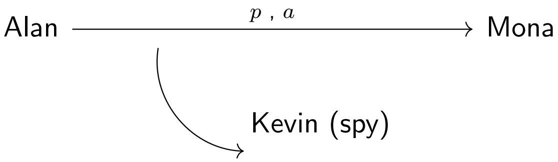

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$$
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& \text{Kevin (spy)} &
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So Alan and Mona can now use $a^{mn} \pmod p$ as their key, and Kev would be none the wiser.

Formalizing Encryption — Syntax

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$$\mathsf{Enc} : K \times M \to C$$

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Formalizing Encryption — Correctness

For all keys $k \in K$, messages $m \in M$, we have Dec(k, Enc(k, m)) $=$ m.

Formalizing Encryption — Semantic Security

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- Plaintext comes from an arbitrary distribution
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- Seeing the ciphertext should not reveal any more information
- Model unknown key by assuming it is chosen uniformly at random

Imitiation Game.

Sometimes it is the people no one imagines anything of who do the things that no one can imagine. - Alan Turing, re Kevin (?)

Substitution Cipher.

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VB V NVRRWO LC CVKR, U RJUFE RJVR EWYUF UB V XFXBXVQ EUFZ LC GWOBLF
Hint: Y = V

Transposition Cipher

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Transposition Cipher

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3. Ciphertext $C = \{a, \ldots, z\}^n$

Use the inverse permutation to decrypt!

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Use a different substitution for each character, never use the same key twice.

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$$\begin{aligned}\mathsf{Enc}(k, m_0) - \mathsf{Enc}(k, m_1) \\ = (k + m_0) - (k + m_1) \\ = m_0 - m_1\end{aligned}$$

This is sufficient to recover both messages.

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There exists a function (called the **simulator**) $S : \{0,1\}^* \to \{0,1\}^*$ such that

$$\Pr[A(\mathsf{Enc}(k,m), I(m)) = f(m)]$$

$$= \Pr[S(I(m)) = f(m)]$$

where probabilities are taken over $k \in K, m \in D$.

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1. We need Kevin in the definition.

2. The description is unnecessarily complicated (see Kevin)

More Notation

Let us say two random variables X, Y over a finite set S have identical distributions. Then we write

$$X \overset{\mathsf{d}}{=} Y$$

Perfect Shannon Secrecy (Shannon '49)

A scheme (End, Dec) has perfect secrecy if, for any two messages $m_0, m_1 \in M$ we have

$$\mathsf{Enc}(k, m_0) \stackrel{\mathsf{d}}{=} \mathsf{Enc}(k, m_1)$$

Both sides contain a random variable over uniform distribution of the key k.

Semantic Security = Perfect Security

A scheme (Enc, Dec) is semantically secure if and only if it has perfect shannon secrecy.

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Proof. Boring probability bash, see lecture slide.

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Solution.

"As a matter of fact, I think that Kevin is a unusual kind of person."

What's Next?

1. Randomized Encryption
2. Limitations of information theoretic security
3. Pseudorandom generators