Expanding variational autoencoders for learning and exploiting latent representations in search distributions

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ABSTRACT

In the past, evolutionary algorithms (EAs) that use probabilistic modeling of the best solutions have incorporated latent or hidden variables to the models as a more accurate way to represent the search distributions. Recently, a number of neural-network models that compute approximations of posterior (latent variable) distributions, have been introduced. In this paper we investigate the use of the variational autoencoder (VAE), a class of neural-network based generative model, for modeling and sampling search distributions as part of an estimation of distribution algorithm. We show that VAE can capture dependencies between decision variables and objectives. This feature is proven to improve the optimizing capacity of model based EAs. Furthermore, we extend the original VAE model by adding a new, fitness-approximating network component. We show that it is possible to adapt the architecture of these models and we present evidence of how to extend VAEs to better fulfill the requirements of probabilistic modeling in EAs.

1 INTRODUCTION

In addition to their primary goal of finding high-quality solutions in the decision space, one desideratum of efficient Evolutionary Algorithms (EAs) when applied to black-box domains is to exhibit some modeling capacity. In the single-objective scenario, this capacity is related to the question of capturing and representing the dependencies between the problem variables in the selected solutions. In multi-objective optimization problems (MOPs), modeling can also include the relationship between decision variables and objectives, particularly to accurately model this relationship in the constrained space of non-dominated solutions.

Information about interactions between decision variables and objectives can be used to design factorized distribution algorithms [15] or gray-box optimization methods that exploit the problem structure [4]. Models can also be useful if they serve to generate solutions similar to the optimal ones, i.e., if they can be applied for *generative modeling*. In EAs, modeling of variables and objective interactions has been mainly applied in estimation of distribution algorithms (EDAs) [19, 25].

A variational autoencoder (VAE) [17] is a neural network approach that focuses on learning a generative model of the data. This approach assumes that the information being treated has a latent representation. One of the main characteristics of the VAE is that it learns this latent representation in a process in which samples similar to the input data are incrementally generated. In this paper we investigate the suitability of VAEs as a model to capture the dependencies between variables and objectives in an EA. These dependencies are later used for generating high-quality solutions. We also introduce new variants of the VAE models that allow to explicitly represent the relationship between the decision variables and objectives.

The main contributions of this paper are the following: 1) We apply VAE to learn probability distributions in EAs and use it as a generative model to produce new samples. 2) We introduce new types of VAE-related models that, similarly to previous generators, include encoder and decoder components, but add a third *approximator* component. 3) We show that it is feasible to incorporate the VAE-extended models to EDAs. We have not found any reference to previous applications of VAE for modeling in EAs or the use of neural networks for explicitly capturing and modeling the dependencies between decision variables and objectives.

The paper is organized as follows: In the next section the question of modeling search distributions is presented and discussed. Section 3 describes the VAE components and the loss function optimized by this neural network. Our proposal for using VAEs within EAs is presented in Section 4. This section also introduces two new extensions to VAEs. Related work is analyzed in Section 5. The experiments that illustrate the differences in the capabilities of the VAE variants for generative modeling, and the behavior of VAEs when applied as EDA components are presented in Section 6. The conclusions of the paper and some topics of future research are presented in Section 7.

2 PROBABILISTIC MODELING OF SEARCH DISTRIBUTIONS

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an n-dimensional vector of continuous variables. We will use $\mathbf{x} = (x_1, \ldots, x_n)$ to denote an assignment to the variables. S will denote a set of indices in $\{1, \ldots, n\}$, and X_S (respectively x_S) a subset of the variables of $\mathbf{X}$ (respectively $\mathbf{x}$) determined by the indices in S.

The idea behind EDAs and other model-based EAs is to define search distributions that associate a probability $p(\mathbf{x})$ to the points of the search space. Points with higher fitness will have a higher probability. Two crucial steps for EDAs are to learn a compact representation of the search distribution (learning step) and use this representation to sample new solutions (sampling step).

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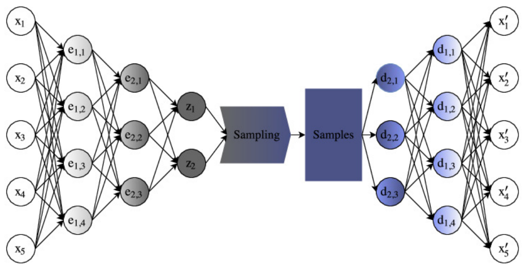


Figure 1: VAE architecture.

Let us consider a set $\mathscr{X} = \{\mathbf{x}^i\}_{i=1}^N$ consisting of N solutions. We assume that there exists some unknown distribution $p(\mathbf{x})$ (optimal search distribution) that *causes* -and therefore, perfectly fits- $\boldsymbol{X}$. However, this distribution is unknown, and can only be approximated with the data in $\mathscr{X}$, $p_d(\mathbf{x})$. We are interested in generating new samples that are similar to those in $\mathscr{X}$. This can be approached as a *generative modeling* problem in which we would first approximate the search distribution $p(\mathbf{x})$ and then generate new samples from our approximation.

In some modeling problems, it is convenient to assume the existence of an unobserved (latent) continuous random variable $\mathbf{Z}$ that influences the generation process. For instance, the generation process could consist of a first step where a value $\mathbf{z}$ is generated from some distribution $p_z(\mathbf{z})$ (the prior distribution), and a second step where some conditional distribution $p(\mathbf{x}|\mathbf{z})$ (decoding distribution) is used to generate $\mathbf{x}$. In search distributions, this generation process may not have been part of the search strategy, however, addressing the modeling problem using the latent variable formalism can lead to good approximations of $p(\mathbf{x})$ and effective methods for generating samples that resemble those in $\mathscr{X}$.

A number of works have proposed the use of latent representations as a more accurate way to model dependencies. In Boltzmann Machines [2] and Restricted Boltzmann Machines (RBMs) [33], latent variables are added to the model to capture a higher order representation of the patterns in the data. Recent deep neural networks models such as deep belief nets [31] and deep Boltzmann machines also incorporate several layers of hidden variables to improve representation.

Nevertheless, latent models exhibit a number of drawbacks. They can be very difficult to learn, and for certain models, exact inference is intractable. In EDAs, inference is mainly involved as a step in the generation of new solutions. For complex models with latent variables, Markov chain Monte Carlo (MCMC) algorithms would be required to generate new solutions. In this paper, we propose the use of the VAE, for which learning is simpler, and MCMC methods are not required.

3 VARIATIONAL AUTOENCODERS (VAES)

VAE [17] is a generative model that explicitly represents a latent variable $\mathbf{z}$. A variational autoencoder can be split into two components: an *Encoder* and a *Decoder*. The general goal of the model is to approximate the probability distribution:

$$p(\mathbf{x}) = \int_{\mathbf{z}} p(\mathbf{x}|\mathbf{z};\theta)p(\mathbf{z})dz \qquad (1)$$

where θ comprises all the decoder parameters.

The key idea behind the variational autoencoder is to attempt to sample values of $\mathbf{z}$ that are likely to have produced $\mathbf{x}$. For that purpose, it learns an encoding function $q(\mathbf{z}|\mathbf{x})$ that serves as an approximation of the prior distribution $p(\mathbf{z})$.

VAE is considered a *generative* model since it can be used to generate new samples that resemble those used for training it. Both the encoder (E) and the decoder (D) are implemented as neural networks, more specifically as multi-layer perceptrons (MPLs).

The encoder receives a vector $\mathbf{x}$ from the dataset as input and produces a vector $\mathbf{z}$, that is interpreted as a mean μ and a standard deviation σ coding. These μ and σ are then used to parametrize a normal distribution, before it is sampled, $\mathcal{N}(\mu,\sigma)$.

Latent variable $\mathbf{z}$ is assumed to follow a multivariate normal distribution with mean 0 and diagonal covariance matrix I, $\mathcal{N}(0,I)$.

The samples obtained from $\mathcal{N}(\mu,\sigma)$ are fed to the decoder, which is asked to reproduce, from each of the samples, the same image that was inputed to the encoder. This is, $\tilde{\mathbf{x}}$ according to $\tilde{\mathbf{x}} \sim p_d(\mathbf{x}|\mathbf{z};\theta)$.

Training a VAE pursues two goals: 1) Optimize E to obtain an accurate approximation of the latent variable distribution $\mathbf{z}$. 2) Optimize D to obtain an accurate reconstruction during the decoding phase. An example of the VAE architecture is shown in Figure 1.

3.1 Learning VAEs

As in other neural network models, VAE uses the back-propagation algorithm, assisted by a gradient descent optimizer, to optimize the parameters of the network (weights and biases). The optimizer attempts to find the minimum of the loss function. VAE minimizes the following upper-bound on the negative log-likelihood of $\mathbf{z}$:

$$\mathbb{E}_{\mathbf{x}\sim p_{data}(\mathbf{x})}[\mathbb{E}_{\mathbf{z}\sim p_z(\mathbf{z})}[-log(p(\mathbf{x}|\mathbf{z}))] + KL(q(\mathbf{z}|\mathbf{x})||p(\mathbf{z}))] \qquad (2)$$

where $\mathbb{E}_{\mathbf{x}\sim p_{data}(\mathbf{x})}$ and $\mathbb{E}_{\mathbf{z}\sim p_z(\mathbf{z})}$ respectively represent the expectations with respect to the original data distribution p_{data} and the latent variable distribution $p_z(\mathbf{z})$. The KL function stands for the Kullback-divergence metric [18] .

The first term of the equation represents the goal of maximizing the quality of the reconstruction of the original data given the latent variable. The second encodes the goal of accurately approximating the prior distribution $p(\mathbf{z})$.

The general question of learning the optimal VAE architecture for a given problem (i.e., the number of layers, activation functions, etc.) is open. Usually, the architecture is set by hand.

4 VAES FOR MODEL-BASED OPTIMIZATION

In this section, we introduce the models that are later tested in the experiment section. The models proposed can serve several purposes -such as image generation, or data augmentation-, however, our interest lays on optimization. Therefore, the development of this work is addressed from this perspective. When considered in the context of search distributions, we expect an “ideal" VAE model to simultaneously satisfy the following conditions:

(1) It can accurately learn a (latent) model from the best solutions. This is, the model should assign more sampling probability to those solutions that are used in the training phase.
(2) It can be efficiently used as a generator of solutions that resemble those in the original data.
(3) It learns a mapping between decision variables and the objective variable(s):
   (a) The values of the decision variables can be predicted using the values of the objective variables (Fitness-conditioned sampling).
   (b) The values of the objectives can be predicted from the values of the decision variables (Fitness-estimation).

Properties 1 and 2 are those usually required by any probabilistic model used in EDAs, and met by regular VAEs. These models are *implicitly* biased towards sampling solutions with high fitness values, since only this kind of solutions are used for training. Property 3a) would serve to *explicitly* bias the sampling of solutions to try to obtain variable configurations that produce a given fitness value. This property can be particularly useful for sampling specific areas of the Pareto front in MOPs. Property 3b) can be useful when the model is applied as a surrogate or to evaluate only those solutions that are predicted by the model to be of high fitness.

4.1 Extending the VAE model

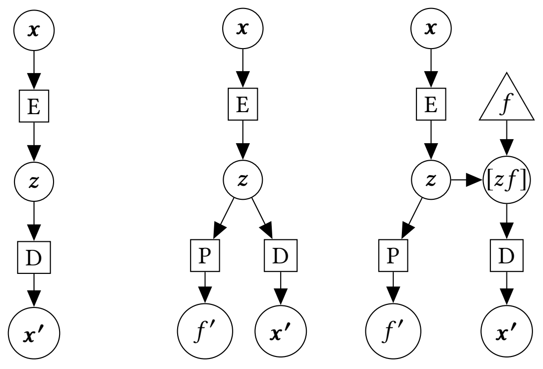
The original VAE model does not exhibit all the ideal above-mentioned properties. Therefore, we propose two extensions of the model. These extensions correspond in fact to new generative models that incorporate an additional neural network component and modified loss functions. However, the general principles of VAEs are kept.

4.1.1 Extended VAE. The E-VAE is a variation of the regular VAE which consists of the common encoder-decoder combination network, plus a third one, the *predictor*. This third component behaves as another decoder that tries to predict the fitness value $f(\mathbf{x})$ associated to a solution $\mathbf{x}$. The input to this second decoder is the same $\mathbf{z}$ that the encoder produces. We assume that if $\mathbf{z}$ has sufficient information to reconstruct the original solution, it can also be used to approximate $f(\mathbf{x})$. The two decoders are expected to learn different parameters since they solve different decoding tasks.

The E-VAE structure is trained simultaneously by minimizing an expansion of the loss function described by Equation (2), to which a new term that specifies the optimality goal for the third network is added. This term is the mean squared error (MSE) between the fitness values predicted by the network $\hat{f}(\mathbf{x})$, and $f(\mathbf{x})$.

We expect that the inclusion of a second decoder will also work as an indirect regularizer for the latent representation that $\mathbf{z}$ encodes since now it has to be informative enough to accomplish two different tasks. The third network additionally provides another asset to the model. In cases in which the evaluation function being optimized by the EDA is excessively costly to compute, it can be used by the algorithm as a fitness surrogate. This application would discard solutions predicted to have poor fitness values and avoid evaluating them, saving computational time.

A diagram of E-VAE is shown in Figure 2b. It can be compared to the regular VAE shown in Figure 2a. Note on the diagram the characteristic feature of E-VAE, the same $\mathbf{z}$ samples are used to produce a new solution, and an approximation of the fitness value corresponding to that solution.



(a) Regular VAE **(b) E-VAE** **(c) CE-VAE**

Figure 2: Diagrams of the three models tested in this work, regular VAE and its two variants. f corresponds to the value(s) the samples are biased towards.

4.1.2 Conditioned, Extended VAE. The CE-VAE pursues the same goal as E-VAE, to sample solutions using the information known about the fitness of the solutions. However, in this model, this information is exploited explicitly, again, to bias the sampling process towards the highest quality solutions.

The overall structure is the same as the one for the E-VAE. Nevertheless, the input of the original decoder is a concatenation of the $\mathbf{z}$ samples and $f(\mathbf{x})$. Figure 2c illustrates the differences between E-VAE and CE-VAE. Notice in the figure that these differences are located in the right branches of the two models.

The random component in the procedure of sampling from $\mathbf{z}$ leaves a margin to the decoder to produce, given a single $\mathbf{z}$, various solutions. This structure aims at exploiting this characteristic, as the introduction of explicit fitness values that belong in the higher section of that margin would likely improve the input it has been given. Ideally, the introduction of these fitness values would affect the solution generation procedure, producing an improved outcome. Additionally, note that the third *predictor* network from the E-VAE structure is transferred to this one. This way, the advantages mentioned in the previous model, such as the possibility of using a surrogate fitness evaluation, are kept.

We designed a set of Python classes that implement the different VAE models. The network architectures are defined using the `tensorflow` library [1]. The VAEs used in this work are extensions of the VAE implementation[1].

5 RELATED WORK

In this section we review a number of papers that are related to our work.

[1]https://jmetzen.github.io

5.1 Latent models and generative approaches in EDAs

There are a number of strategies applied in EDAs that implicitly assume or model hidden variables. The most extensively applied model of this type are the mixtures of distributions, which have been applied in the discrete [3, 26] and continuous domains [5, 8]. A limitation of the mixtures models is that, in most cases, the number of components have to be fixed in advance. Also, mixture-learning algorithms such as expectation-maximization (EM) can be computationally very costly.

RBMs have been also applied to model binary problems in single-objective [29] and multi-objective EDAs [32, 35]. In comparison to RBMs, VAEs are easier to train and the sampling process is much faster.

Generative adversarial nets (GANs) [10] and denoising autoencoders [6, 27] are exemplars of the few neural-based used as models of an EDA. GAN-EDA was introduced in [28] and applied to single objective problems. However, the algorithm did not produce results competitive to other simpler approaches, neither in the number of fitness evaluations nor in the computational time [28].

5.2 Explicitly modeling interactions between decision variables and objectives

In [15], dependencies between objectives and variables are modeled using multi-dimensional Gaussian networks. In the network, decision variables and objectives are represented with nodes. The authors propose to use fake values of the objectives for biasing the generation of the decision variables. More specifically, the information about the objective variables is given as evidence, and then the inference problem of generating configurations that produce strong performances for the decision variables is addressed.

Martins et al. [24] propose to encode in a Bayesian network not only decision variables and objectives but also the parameters of local optimization procedure. In this way, dependencies can be also exploited at the time of selecting the most effective parameters for the local optimizers.

None of these previous works have used latent or generative models to explicitly represent the relationships between objectives and variables. Furthermore, in these previous works, no modifications or extensions were proposed to the models to better represent the specific characteristics of search distributions.

5.3 Multi-network models

In the area of deep neural networks, some recent works propose modifications and extensions to multi-network generative models. Although we have not found previous examples of VAE variants with three or more models, there are such multi-network models developed from GANs. In [16], an example of a GAN with two discriminators (ordinary GAN has a single discriminator) and one generator is proposed. The so-called DiscoGAN model was designed to discover relations between two unpaired, unlabeled datasets.

Furthermore, recent work has developed the integration or combination of different classes of generative models in more flexible, adjustable models, comprising heterogenous neural networks [13, 21, 23]

6 EXPERIMENTS

To test the validity of the proposed models to drive an EDA, two key issues need to be addressed. The first one, addressed as a preliminary step, is to evaluate the capacity of the models to fulfill two key tasks necessary to succeed when placed in an evolutionary framework: sampling both *accurate* and *diverse* points similar to the input data. This evaluation is conducted isolated from the EDA optimization context and therefore is valuable to assess the quality of the introduced models in other applications where it is possible to classify the variables in two classes: a set of independent variables (our decision variables in the optimization scenario) and dependent variables (the objective variables in the optimization scenario). After completing the assessment of the models, we evaluate their use within the context of EDAs.

6.1 Preliminary experiments

Evaluating the quality of generative models is difficult since the generated data should be similar but not identical to the inputs. This evaluation can be difficult to implement automatically for certain problems. For example, it is not trivial to design scores that accurately evaluate the quality of generated images since there is an important perceptual and subjective component involved.

In this paper we test the generative models in the task of generating solutions from the Pareto set (PS) of a bi-objective function. We expect that good generators will produce solutions well spread in the PS, and we will measure the quality of these solutions using the images of those points, i.e., the Pareto front (PF) approximations of the model-generated points.

As bi-objective problems, we use a number of instances originally introduced in [22] to investigate the difficulty of MOPs with complex PSs. More specifically, we select functions $\{F1, \ldots, F9\} \setminus F6$ from those introduced in [22]. F6 is excluded because it is a tri-objective function.

The experiment consists of:

(1) Generate 1000 points in the Pareto set of a given function $f \in \{F1, \ldots F9\} \setminus F6$. The fitness values of the points are computed.
(2) Generate a triple of models (VAE, E-VAE, CE-VAE) whose encoder and decoder have the same architectures. To do this, the number of layers, neurons in layers, and activation functions in the encoder and decoder networks are chosen randomly, and replicated for all three models. Analogously, the architecture configuration of the *predictor* in E-VAE and CE-VAE match too.
(3) The generated models are trained with the 1000 solutions sampled from the PS of one of the functions. E-VAE and CE-VAE are provided the fitness values corresponding to these solutions too.
(4) Each of the trained models is used to sample another set of 1000 solutions.
(5) The similarity between the points in the true PS and in the approximated PSs corresponding to the samples generated by each model are computed using the inverted generational distance (IGD) [7].

The IGD metric measures the smallest distance between two sets of solutions. One of the sets is the one containing the points known

Table 1: Table showing feasible assignments for the VAE components when generated at random.

Latent-var	Act. func.	Loss func.
uniform	None	MSE
normal	relu	Gaussian. Div.
	elu	
Weights-init	softplus	
xavier [9]	softsign	
random	sigmoid	
uniform	tahn	

to be in the Pareto set (R), while the other are the points sampled from the model A. IGD measures the diversity of A, as well as it closeness to R. It is computed as:

$$IGD = \frac{1}{|R|}\left(\sum_{r\in R} min_{a\in A} d(r,a)^p\right)^{\frac{1}{p}}$$
$$d(r,a) = \left(\sum_{k=1}^{m}(r_k - a_k)\right)^{\frac{1}{2}} \tag{3}$$

This experiment was performed 500 times for each function-model combination. The architectures of the networks in the models could have between one and ten hidden layers, with a maximum of 50 neurons in each layer. The rest of parameters that could be chosen in the random generation of the networks can be found in Table 1.

6.2 EDA scenario

The goal of this experiment is to investigate the behavior of the VAE and CE-VAE models when used as an EDA component. In this case we address a simplified protein-folding single-objective problem. We use three instances of this problem and compare the quality of the results obtained by the EDAs that use VAE and CE-VAE as models, to the Continuous Univariate Marginal Distribution Algorithm (UMDA$_c$) [20] -an algorithm that does not consider interactions between variables- and a random search.

6.2.1 *Off-lattice protein model.*

Off-lattice models are simplified protein models that do not follow a given lattice topology. The 2D or 3D coordinate in the real axis define the positions of the protein residues. Among the off-lattice models with known lowest energy states is the *AB* model [34], where *A* stands for hydrophobic and *B* for polar residues. The distances between consecutive residues along the chain are held fixed to $b = 1$, while non-consecutive residues interact through a modified Lennard-Jones potential. There is an additional energy contribution from each angle θ_i between successive bonds. The energy function for a chain of n residues, that is to be minimized, is shown in equation (4).

$$E = \sum_{i=2}^{n-1} E_1(\theta_i) + \sum_{i=1}^{n-2}\sum_{j=i+2}^{n} E_2(r_{ij}, \zeta_i, \zeta_j), \tag{4}$$

where

$$E_1(\theta_i) = \frac{1}{4}(1 - cos\theta_i) \tag{5}$$

and

$$E_2(r_{ij}, \zeta_i, \zeta_j) = 4(r_{ij}^{-12} - C(\zeta_i, \zeta_j) r_{ij}^{-6}) \tag{6}$$

Here, r_{ij} is the distance between residues i and j (with $i < j$). Each ζ_i is either A or B, and $C(\zeta_i, \zeta_j)$ is $+1$, $+\frac{1}{2}$, and $-\frac{1}{2}$ respectively, for *AA*, *BB*, and *AB* pairs, giving strong attraction between *AA* pairs, weak attraction between *BB* pairs, and weak repulsion between *A* and *B* [12].

We consider Fibonacci sequences defined recursively by

$$S_0 = A, \quad S_1 = B, \quad S_{i+1} = S_{i-1} * S_i \tag{7}$$

where $*$ is the concatenation operator.

A 2D off-lattice solution of the *AB* model can be represented as a set of $n - 2$ angles. Angles are represented as continuous values in $[0, 2\pi]$. The first two residues can be fixed at positions $(0, 0)$ and $(1, 0)$. The other $n - 2$ residues are located from the angles with respect to the previous bond. We look for the set of angles that defines the optimal off-lattice configuration. As instances, we consider Fibbonacci sequences for numbers $(6, 7, 8)$. The respective sizes of these sequences, in the same order, are $n \in (8, 13, 21)$.

The EDAs driven by both VAE and CE-VAE are used to minimize the energy function, shown in Equation (4).

6.2.2 *Parameters of the algorithms.*

The pseudocode of the EDA is shown in Algorithm 1. It makes use of the following functions:

- create_model(pop[, fitnesses]): Given a set of solutions, this functions returns a VAE, CE-VAE, or a set of means and variances (for the UMDA$_c$) trained with those solutions. In the case of the CE-VAE, the fitness values associated to the solutions are also inputs of the function.
- sample_model(model[, fitnesses]): Given the VAE CE-VAE, or the set of means and variances, this function generates samples (new solutions) from it. Equivalently, arbitrary fitness values are provided when CE-VAE is the model chosen.
- best(pop, fitnesses): Given a population, and their respective fitness values, this function returns the best individuals using tournament selection, where the tournament size is $s_t = 10$.

```
pop = generate_population();
while halting condition is not met do
    fitness = evaluate_population(pop);
    selected_pop, selected_fit = select_solutions(pop, fitness);
    model = create_model(selected_pop, selected_fit);
    offs = sample_model(model);
    pop = offs + best(pop, fitnesses)
end
```

Algorithm 1: Pseudo-code for a generic EDA.

The structures chosen for this experimental phase are fixed to a reasonable number of hidden layers ($h_l = 3$), and neurons in each of these layers ($h_n = 50$). The population of the algorithm (number of solutions being evaluated in each iteration) is set to $N = 50$, 100, and 200 points. The stop condition is a maximum number of $n_{gen} = 300$ generations.

Algorithm 1 is run 30 times, and the energy value of the best individual found during the search is used to evaluate the performance of the algorithm.

6.3 Results for the comparison of the models

To evaluate the results of the first experiments, we will determine the rank of the three models in terms of the IGD values computed from the PS approximations they generate.

For each of the 8 functions, and each of the 500 executions of related models (with identical encoder and decoder structures), we proceed as follows: Each time the IGD generated by one model was better (lower) than the other, the winner is awarded one point. This results in a table similar to Table 2 which contains the summary of the comparison for the 500 trials for function $F9$. When read by rows, the table shows in how many pairings the model in the row has obtained a better IGD than the model in that column. For example, it can be seen in Table 2 that CE-VAE obtained a better score than VAE in 294 occasions, and performed worse in 206. The tables for all functions have been summarized in Table 3, by adding the individual tables by rows. Following the example, it can be observed in the summary table the way the cell value corresponding to CE-VAE $F9$ is computed as $294 + 304 = 598$.

Additionally, in order to evaluate the margins by which a model outperforms another, we have formed a similar table, in which, instead of adding points, a *performance* index is computed, as $-\log\left(\frac{IGD_i}{IGD_j}\right)$ (where i, j are the model index) is added in cell (i, j). An example for $F9$ can be seen in Table 4, and its summarized version in Table 5. Again, the larger the number, the better the performance.

Judging by the results shown in Table 3, we can see some regularities. Excluding $F8$, CE-VAE has better IGDs overall. In some cases, such as $F1$, $F4$, and $F7$, the most complex model has the upper hand in 160-260 more occasions that its peers. Then, for other functions, $F2$, $F3$, $F5$, and $F9$, this difference is lowered to 100-150. Finally, $F8$ seems to be an outlier, as it shows very close results between all models, awarding a narrow edge to the E-VAE. The comparisons between E-VAE and VAE do not display that much margin as the CE-VAE showed. In some cases, as in $F1$, $F3$, $F8$ and $F9$, the differences where not superior to 20, while in the rest, they did not exceed 65.

From this comparison, we can extract that the sole addition of the *approximator* does not improve the sampling capability of the model. However, the explicit addition of arbitrary (in this case, chosen from the Pareto front) fitness values seems to be key to obtain better samples, as in all but one functions CE-VAE obtained clearly better results.

Following to Table 5, it can be seen how the superiority of the CE-VAE is maintained regularly across all comparisons. The largely unbalanced values in this table denote that, not only CE-VAE was better in most cases, but when its IGD was worse, it was similar to the superior one. However, a contradiction between the two tables is found when comparing the results achieved for $F8$. When comparing the instances in which one model produced better results than the other, $F8$ showed abnormally balanced results. When comparing the quality of all the solutions, however, this balance is broken, and results more similar to those generated from the rest of the functions were found.

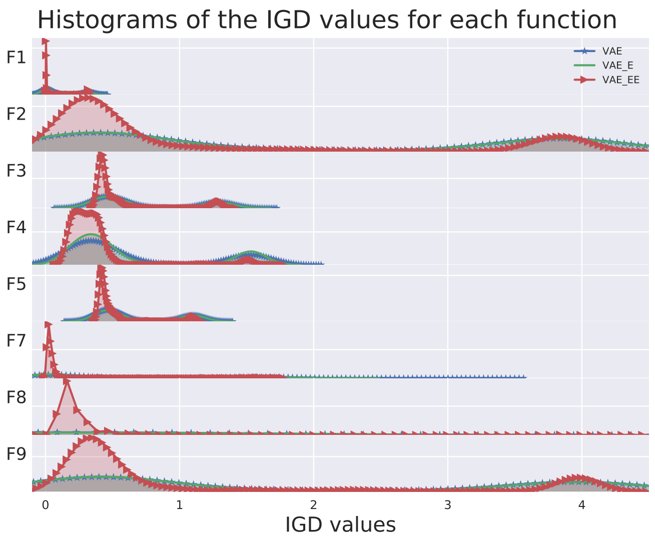


Figure 3: Density function of all the 500 IGDs generated for each function, by each variation of the VAE. For $F8$, a reduced number of configurations that produced an IGD value near 100 were generated. These have been omitted in the representation for a more detailed perception.

The combination of the results in both tables for $F8$ shows that, even though the scoring was paired, when VAE or E-VAE beat CE-VAE the results were close. However, when it was CE-VAE that produced better results, the difference was much larger. This means that the top solutions produced by the CE-VAE are much better than those generated by VAE.

To back up these conclusions, we have subjected these results to a statistical test. The selected test was the Wilcoxon signed rank test [36], and was applied pairwise. The results back up what is seen in the tables, as the test shows p-values near 10^{-7}, when comparing the CE-VAE with the other models, and only shows statistical difference for $F1$ between the VAE and E-VAE.

$F8$ seems to be the most complicated function to be optimized, as the IGD values range from fractional numbers up to numbers larger than 10. The rest of the functions rarely obtained IGD values above 4, and none above 5. This can be clearly seen in Figure 3. This figure shows the density function of the whole set of IGDs generated in the experiments. It is apparent that $F2$, $F7$ and $F9$ have similar difficulties, as well as $F1$, $F3$, $F4$ and $F5$. $F8$ however clearly poses other kind of difficulty to the algorithm. In any case, it can be seen how the line representing the CE-VAE has sampled more solutions in the section where the top points are found. Meanwhile, the lines representing the sampling by VAE and E-VAE, stay close to each other. It can be observed too, how CE-VAE has sampled more bad solutions in $F8$ compared to its competitors, which resulted in the abnormal behavior displayed in previous tables.

As a summary, we can state that CE-VAE produces better results than VAE regularly, and its top results are much better than the best

Structure	VAE	E-VAE	CE-VAE
VAE	0	247	206
E-VAE	253	0	196
CE-VAE	294	304	0

Table 2: Example of score computation in the comparison of the three models for function $F9$. Each time a IGD generated by a model x is lower than any of those generated by any of its counters y, 1 is added in cell (x, y). A large positive number denotes better performance.

Str.	F1	F2	F3	F4	F5	F7	F8	F9
VAE	403	502	461	462	480	449	488	453
E-VAE	413	437	465	414	427	411	508	449
CE-VAE	678	560	568	621	589	635	499	598

Table 3: Scores for all functions. These scores were computed by adding all the scores in each row of the score matrix for each function. See an example of these matrices in Table 2. These numbers were rounded to integers. A larger value denotes a better performance.

Structure	VAE	E-VAE	CE-VAE
VAE	0	45.2308	-89.7701
E-VAE	-45.2308	0	-135.001
CE-VAE	89.7701	135.001	0

Table 4: Performance index example for $F9$. The lower the number, the better the performance by the model

Str.	F1	F2	F3	F4	F5	F7	F8	F9
VAE	-274	43	25	14	54	-272	-129	-44
E-VAE	-648	-231	-169	-272	-224	-378	-324	-180
CE-VAE	923	187	143	257	169	651	453	224

Table 5: Performance index for all functions. These scores were computed by adding all the indices in each row of the index matrix for each function. See an example of these matrices in Table 4. These numbers were rounded to integers. A larger value denotes a better performance.

ones produced by the VAE. Taking into consideration these results, we would expect that the CE-VAE would have a good performance when being used as a model running an EDA.

6.4 Results in the optimization scenario

We present and discuss the results for the second experiment. Firstly, we have empirically set the number of iterations over the set of known points that the VAE models are awarded to 5. The following Table 6 Shows the mean results of some of the runs, discerned by problem dimension and search algorithm in the columns, and population sizes in the rows. The results of each run were filtered as a previous step to computing the mean, since some searches performed randomly and guided by the UMDA$_c$ got stuck very early in the optimization in local optima distant from the real optimal values. This filter consisted on deleting any value above 0. The filter resulted in deleting 94.4% of the random searches, 24.1% of the searches by UMDA$_c$, and below 4.1% of the three VAE model-guided searches.

From this table, we can firstly see the effects of the solution filter, since the random search was unable to create any solution that matched the requirement for dimensions 21 and 34. For dimensions 13, as much as 15 runs in total were used. This shows the inefficacy of a random search in such a difficult problem. Focusing in the comparison between the models, it can be seen how, for small population sizes, the VAE-based searches and the UMDA$_c$ exchange the top performer condition, depending on the number of individuals in a generation, and the dimension of the problem. For dimension 13, with 50 individuals per population, it seems like the VAE guided search holds a performance gap to the rest of the models and the UMDA$_c$, that have a similar performance. However, when doubling the number of individuals, the E-VAE and the CE-VAE have a clear advantage over the VAE, and, more clearly, over the UMDA$_c$. When doubling again, the UMDA$_c$ continues its progression, while, the VAE-based searches, suffer a major drop-off, probably due to *over-training* of the net, which causes a large narrowing in the diversity of the solutions sampled by the models. However, with an inferior size of population (100 over 200), the VAE-based searches outperformed the best UMDA$_c$ performances.

A similar patter can be extracted from the other two dimensions of the problem tested. The UMDA$_c$ improves its mean performance as the number of individuals per generation is increased. Meanwhile, the VAE suffers from *under-training* or *over-training* when the population size is not the adequate. Results somewhat balanced can be found when the populations consist of 50 individuals, even though they favor the UMDA$_c$. With 100 individuals, the VAE-based searches have a slight upper-hand, while they fail to scale when the population size in 200.

7 CONCLUSIONS

This paper is focused on testing the utility of VAEs as models to drive EDAs. More specifically, two variations of the common VAE are proposed in this work, and all three of them are evaluated in a two-step analysis to prove its effectiveness. The two proposed models, E-VAE (implicitly) and CE-VAE (explicitly), exploit extra information about the fitness function when sampling new solutions.

The first preliminary evaluation determined that the presented model that explicitly uses the extra information was able to produce more distributed samples in the set of top performing solutions, compared to the regular VAE. This is a key feature that is necessary in any model intended to be used in an evolutionary algorithm.

The results provided by the models when placed in an evolutionary framework showed the elevated sensibility to different parameter combinations, since a wrong setting in either the training method -the amount of training- can lead to fast convergence in values relatively far away from the real optimal values. In some cases, a deficiency in this matter could lead the model to be outclassed by simple algorithms that do not consider variable interactions in problems that have that characteristic.

Dimension	13					21					34				
Pop. size	Rnd	UMDA$_c$	VAE	E-VAE	CE-VAE	Rnd	UMDA$_c$	VAE	E-VAE	CE-VAE	Rnd	UMDA$_c$	VAE	E-VAE	CE-VAE
50	-0.108	-0.402	-0.457	-0.413	-0.416	-	-0.710	-0.679	-0.577	-0.636	-	-0.440	-0.368	-0.317	-0.259
100	-0.190	-0.545	-0.918	-1.031	-1.085	-	-0.982	-1.162	-1.211	-1.200	-	-0.673	-0.628	-0.774	-0.708
200	-0.173	-0.922	-0.488	-0.508	-0.553	-	-1.601	-0.359	-0.343	-0.257	-	-1.461	-0.280	-0.271	-0.296

Table 6: Overall mean results for the 5 search algorithms, for 3 problem dimensions, with 3 different population sizes.

7.1 Future work

While we have focused on problems with continuous representations, there are some proposals that extend the application of VAEs to discrete problems [30]. Similarly, variants of VAE, such the one that adds noise to the original data could be investigated [14].

More work investigating the training technique of the VAE models in evolutionary framework needs to be carried off. The results in the preliminary section suggest that these models have the capacity of performing well in evolutionary methods, and therefore finding and solving the issues that caused their poor performance are worth investigating.

Additionally, the VAEs generated at each generation were randomly initialized. Running a search algorithm to optimize the structures of the neural networks inside the VAEs would help the sampling quality. Simple local searches would probably be the best option, as over-complicating the optimization with a costly internal search at each generation would make the EDA infeasible.

Covariance matrix adaptation [11] is another evolutionary approach in which the validity of the VAEs in optimization problems can be tested.

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