Assignment 3

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Question 1

a)

A Runs Test tests a set of binary variables $X_1, ..., X_n$ to verify if the variables occur randomly.

H_0: variables occur randomly, i.e. knowing $X_1, ..., X_n$ does not help predict X_{n+1}.

H_A: variables are not random, i.e. knowing some part of the sequence can help predict subsequent variables.

As the variables are binary, they will take the value 0 or 1. The number of 0s is n_0 and the number of 1s is n_1, where:

$n_0 = n - \sum_{i=1}^{n} X_i$

$n_1 = \sum_{i=1}^{n} X_i$

To perform a Runs Test the observations are combined into one collection of $n = n_0 + n_1$ observations and arranged in increasing order of magnitude or observation. They are labeled according to which set they originally came from. A run is a group of two or more sequential values of 0 or 1.

Let R denote the number of runs in the combined ordered sample of $X \in \{0, 1\}$. Under H_0, R can be approximated as a normally distributed random variable, assuming both n_0 and n_1 are sufficiently large.

$R = 1 + \sum_{i=2}^{n} I_{(X_i, X_{i-1})}$, where $I_{(X_i, X_{i-1})} = 0$ if $X_i = X_{i-1}$ and $I_{(X_i, X_{i-1})} = 1$ if $X_i \neq X_{i-1}$

$\bar{R} = \dfrac{2n_0n_1}{n} + 1$

$Var(\bar{R}) = \dfrac{2n_0n_1(2n_0n_1 - n)}{n^2(n-1)}$

With test statistic $Z = \dfrac{R - \bar{R}}{\sqrt{Var(\bar{R})}}$ where $Z \sim N(0, 1)$

b)

A small number of runs (a small value for R) would indicate that X_i is more likely to be the same as X_{i-1}. A large number means that X is fluctuating regularly between values and X is less likely to be the same as X_{i-1}.

c)

```
## 0 healthy, 1 has disease
X <- "HHHDDDHDDHHHHHDHHHHHHHHHHDDDHHHHHDDDHHHHHDHHHDHH" %>%
  stringr::str_split("") %>% unlist() %>% `==`("D") %>% as.integer()

n <- length(X)
n_0 <- n - sum(X)
n_1 <- sum(X)
```

```r
R <- 1 + sum(X[2:n] != X[1:n-1])
R_est <- 1 + 2*n_0*n_1/n
R_var <- (2*n_0*n_1*(2*n_0*n_1 - n))/(n^2*(n - 1))
Z <- (R - R_est)/sqrt(R_var)
p <- pnorm(Z)
cbind(R, R_est, R_var, Z, p)
```

```
##       R    R_est    R_var         Z          p
## [1,] 15 20.65957 7.974767 -2.004125 0.02252834
```

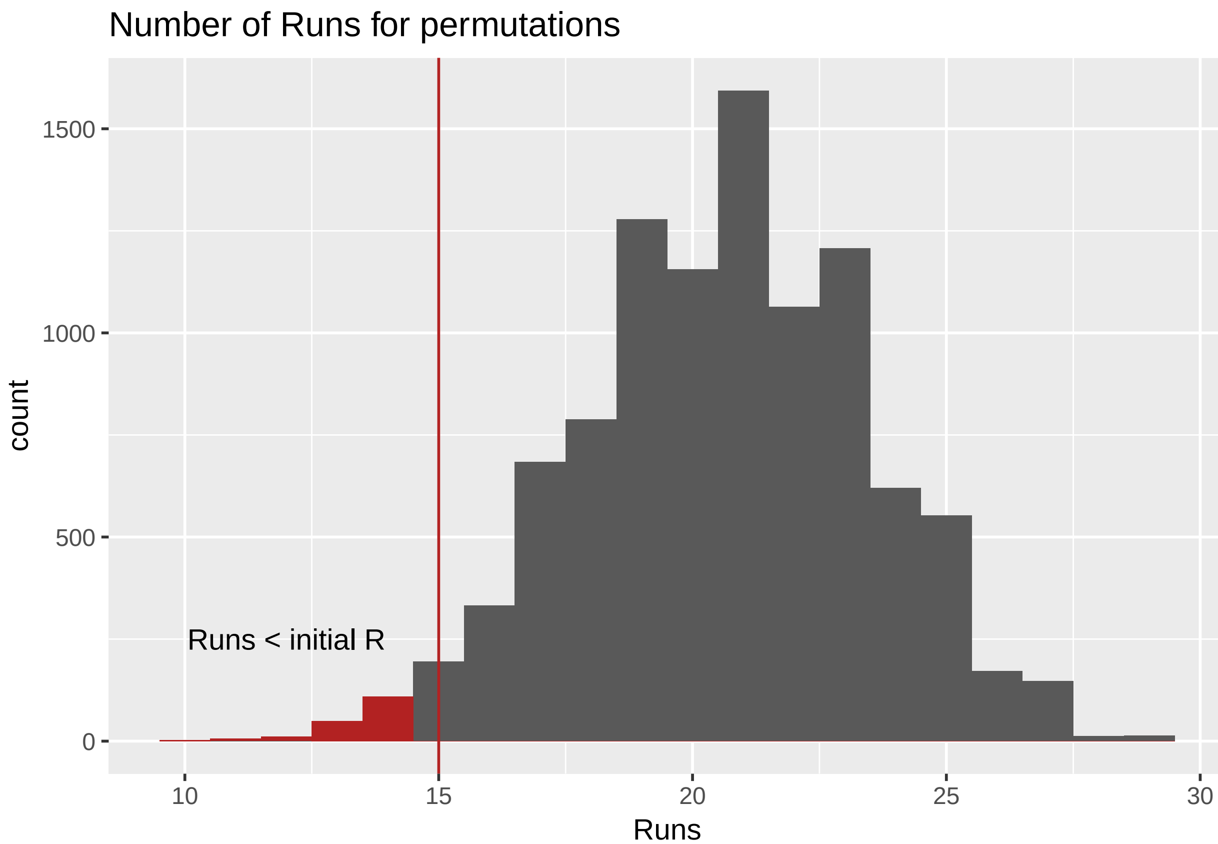
With p-value of 0.0225 we reject H_0 at the 5% level. We conclude that values are not randomly ordered.

d)

```r
set.seed(101)

calc_R <- function(input) {
  1 + sum(input[2:length(input)] != input[1:length(input)-1])
}

sample_R <- function(i, input) {
  input %>% sample() %>% calc_R()
}

initial_R <- calc_R(X)

N <- 1e4
perm_R <- 1:N %>% sapply(sample_R, input = X)

p <- length(perm_R[perm_R < initial_R])/length(perm_R)
p
```

```
## [1] 0.0179
```

```r
ggplot() +
  geom_histogram(data = data.frame(x = perm_R[perm_R >= initial_R]),
                 aes(x = x), binwidth = 1) +
  geom_histogram(data = data.frame(x = perm_R[perm_R < initial_R]),
                 aes(x = x), binwidth = 1, fill = "firebrick") +
  geom_vline(xintercept = initial_R, colour = "firebrick") +
  ggtitle("Number of Runs for permutations") +
  xlab("Runs") +
  geom_text(aes(x = 12, y = 250), label = "Runs < initial R")
```



Because the number of runs is not a continuous variable, there is some ambiguity around whether the p-value should be calculated comparing values < or <= or even using some half point. I have decided to use < as it gives the test the highest power.

Given this we find a p-value of 0.0179. We reject H_0 at the 5% level. We conclude that values are not randomly ordered.