

Exact Analytical Solution for a Particle in a Velocity-Dependent Force Field

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Abstract

Introducing an integrating factor to solve a first-order linear differential equation, in the case of a *vector* equation, also serves as a generator for rotations. In other words, the exponential $\exp(J\omega t)$, where J is a matrix for the vector product, serves a dual role: as an integrating factor to analytically solve the differential equation, and at the same time consolidates the rotations to simple sine and cosine functions, which are readily integrable. This technique¹ is explained and then applied to three velocity-dependent forces found in classical mechanics: the Lorentz force, the Coriolis force, and the centrifugal force.

¹ J. M. Yáñez, G. Gutiérrez, F. González-Cataldo, and D. Laroze, “An exact solution for a particle in a velocity-dependent force field,” [Am. J. Phys.](#) **89**(12), 1103–1112 (2021).

The coordinate-free formula for the counterclockwise rotation of a vector \mathbf{v} about the unit vector $\hat{\mathbf{n}}$ through the angle θ is (see the [Appendix](#) for a derivation of this formula)

$$R_{\hat{\mathbf{n}}}(\theta)\mathbf{v} = \mathbf{v} + \hat{\mathbf{n}} \times \mathbf{v} \sin \theta + \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{v})(1 - \cos \theta). \quad (1)$$

Let us transform this formula from a vector equation to a matrix equation.^{1,2,3} Define the matrix operator $J(\hat{\mathbf{n}})$ as follows.

$$J(\hat{\mathbf{n}})\mathbf{v} \equiv \hat{\mathbf{n}} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ n_1 & n_2 & n_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\mathbf{i}}(n_2 v_3 - n_3 v_2) + \hat{\mathbf{j}}(n_3 v_1 - n_1 v_3) + \hat{\mathbf{k}}(n_1 v_2 - n_2 v_1) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors along x , y , and z axes, respectively, so that

$$J(\hat{\mathbf{n}}) \equiv \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}.$$

We see that $[J(\hat{\mathbf{n}})]_{ij} = -\epsilon_{ijk}n_k$, $\det J = 0$, which implies J has no inverse, $\text{tr } J = 0$, and $J^\dagger = -J$. Also, most importantly for our purposes,

$$\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{v}) = J(\hat{\mathbf{n}})(\hat{\mathbf{n}} \times \mathbf{v}) = J(\hat{\mathbf{n}})J(\hat{\mathbf{n}})\mathbf{v} = J^2(\hat{\mathbf{n}})\mathbf{v},$$

so that

$$J^3(\hat{\mathbf{n}})\mathbf{v} = \hat{\mathbf{n}} \times [\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{v})] = \hat{\mathbf{n}} \times [\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{v}) - \mathbf{v}] = -\hat{\mathbf{n}} \times \mathbf{v} = -J(\hat{\mathbf{n}})\mathbf{v}.$$

Hence,

$$J^3 = -J, \quad J^4 = -J^2, \quad J^5 = -J^3 = J, \quad J^6 = J^2, \quad \dots \quad (2)$$

For completeness,

$$J^2(\hat{\mathbf{n}}) \equiv \begin{bmatrix} -(1 - n_1^2) & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & -(1 - n_2^2) & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & -(1 - n_3^2) \end{bmatrix},$$

although we will not need the explicit form of either J or J^2 . Now, since the exponential of a matrix is defined by its power series, and making use of eq. (2), we have

$$\begin{aligned} e^{J\theta} &= I + J\theta + \frac{(J\theta)^2}{2!} + \frac{(J\theta)^3}{3!} + \frac{(J\theta)^4}{4!} + \frac{(J\theta)^5}{5!} + \frac{(J\theta)^6}{6!} + \dots \\ &= I + J \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \mp \dots \right) + J^2 \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} \mp \dots \right) \\ &= I + J \sin \theta + J^2(1 - \cos \theta) \end{aligned}$$

so that J is the *generator of rotations* and we can express the rotation operator in eq. (1) as

$$R_{\hat{\mathbf{n}}}(\theta) = e^{J(\hat{\mathbf{n}})\theta} = I + J(\hat{\mathbf{n}}) \sin \theta + J^2(\hat{\mathbf{n}})(1 - \cos \theta). \quad (3)$$

Lorentz Force, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

The equation of motion for a charged particle of mass m and charge q in the presence of an electric field \mathbf{E} and magnetic field \mathbf{B} , is

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

which we can rewrite as

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{E} - \frac{qB}{m}\hat{\mathbf{b}} \times \mathbf{v} = \frac{\omega}{B}\mathbf{E} - \omega\hat{\mathbf{b}} \times \mathbf{v} = \frac{\omega}{B}\mathbf{E} - \omega J(\hat{\mathbf{b}})\mathbf{v}$$

where $\omega \equiv qB/m$, $\hat{\mathbf{b}} = \mathbf{B}/B$, and $J(\hat{\mathbf{b}})\mathbf{v} = \hat{\mathbf{b}} \times \mathbf{v}$, or

$$\frac{d\mathbf{v}}{dt} + \omega J(\hat{\mathbf{b}})\mathbf{v} = \frac{\omega}{B}\mathbf{E}.$$

An integrating factor for this first-order linear differential equation is $e^{J(\hat{\mathbf{b}})\omega t}$, which gives

$$\frac{d}{dt} \left(e^{J(\hat{\mathbf{b}})\omega t} \mathbf{v} \right) = \frac{\omega}{B} e^{J(\hat{\mathbf{b}})\omega t} \mathbf{E}.$$

Integrating,

$$e^{J(\hat{\mathbf{b}})\omega t} \mathbf{v}(t) = \mathbf{v}_0 + \frac{\omega}{B} \left(\int_0^t e^{J(\hat{\mathbf{b}})\omega \xi} d\xi \right) \mathbf{E},$$

and

$$\mathbf{v}(t) = e^{-J(\hat{\mathbf{b}})\omega t} \mathbf{v}_0 + \frac{\omega}{B} \left(\int_0^t e^{-J(\hat{\mathbf{b}})\omega t} e^{J(\hat{\mathbf{b}})\omega \xi} d\xi \right) \mathbf{E}.$$

Let $\tau = t - \xi$, $d\tau = -d\xi$, then the solution for the velocity is

$$\mathbf{v}(t) = e^{-J(\hat{\mathbf{b}})\omega t} \mathbf{v}_0 + \frac{\omega}{B} \left(\int_0^t e^{-J(\hat{\mathbf{b}})\omega \tau} d\tau \right) \mathbf{E}.$$

Integrating once more, the solution for the position is

$$\mathbf{r}(t) = \mathbf{r}_0 + \left(\int_0^t e^{-J(\hat{\mathbf{b}})\omega \tau} d\tau \right) \mathbf{v}_0 + \frac{\omega}{B} \left(\int_0^t \int_0^\tau e^{-J(\hat{\mathbf{b}})\omega \xi} d\xi d\tau \right) \mathbf{E}.$$

Making use of eq. (3), this can be written as

$$\mathbf{r}(t) = \mathbf{r}_0 + \left(\int_0^t R_{\hat{\mathbf{b}}}(-\omega \tau) d\tau \right) \mathbf{v}_0 + \frac{\omega}{B} \left(\int_0^t \int_0^\tau R_{\hat{\mathbf{b}}}(-\omega \xi) d\xi d\tau \right) \mathbf{E}.$$

Also from eq. (3),

$$\begin{aligned} \int_0^t R_{\hat{\mathbf{b}}}(-\omega \tau) d\tau &= \int_0^t [I - \sin(\omega \tau) J(\hat{\mathbf{b}}) + (1 - \cos(\omega \tau)) J^2(\hat{\mathbf{b}})] d\tau \\ &= \left[\tau + \left(\frac{\cos \omega \tau}{\omega} \right) J(\hat{\mathbf{b}}) + \left(\tau - \frac{\sin \omega \tau}{\omega} \right) J^2(\hat{\mathbf{b}}) \right]_0^t \\ &= t + \left(\frac{\cos \omega t - 1}{\omega} \right) J(\hat{\mathbf{b}}) + \left(t - \frac{\sin \omega t}{\omega} \right) J^2(\hat{\mathbf{b}}) \end{aligned}$$

and

$$\begin{aligned} \int_0^t \int_0^\tau R_{\hat{\mathbf{b}}}(-\omega \xi) d\xi d\tau &= \int_0^t \left[\tau + \left(\frac{\cos \omega \tau - 1}{\omega} \right) J(\hat{\mathbf{b}}) + \left(\tau - \frac{\sin \omega \tau}{\omega} \right) J^2(\hat{\mathbf{b}}) \right] d\tau \\ &= \left[\frac{\tau^2}{2} + \left(\frac{\sin \omega \tau}{\omega^2} - \frac{\tau}{\omega} \right) J(\hat{\mathbf{b}}) + \left(\frac{\tau^2}{2} + \frac{\cos \omega \tau}{\omega^2} \right) J^2(\hat{\mathbf{b}}) \right]_0^t \\ &= \frac{t^2}{2} + \left(\frac{\sin \omega t}{\omega^2} - \frac{t}{\omega} \right) J(\hat{\mathbf{b}}) + \left(\frac{t^2}{2} + \frac{\cos \omega t - 1}{\omega^2} \right) J^2(\hat{\mathbf{b}}) \end{aligned}$$

so that

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_0 + \left(\int_0^t R_{\hat{\mathbf{b}}}(-\omega\tau) d\tau \right) \mathbf{v}_0 + \frac{\omega}{B} \left(\int_0^t \int_0^\tau R_{\hat{\mathbf{b}}}(-\omega\xi) d\xi d\tau \right) \mathbf{E} \\
&= \mathbf{r}_0 + \left[t + \left(\frac{\cos \omega t - 1}{\omega} \right) J(\hat{\mathbf{b}}) + \left(t - \frac{\sin \omega t}{\omega} \right) J^2(\hat{\mathbf{b}}) \right] \mathbf{v}_0 \\
&\quad + \frac{\omega}{B} \left[\frac{t^2}{2} + \left(\frac{\sin \omega t}{\omega^2} - \frac{t}{\omega} \right) J(\hat{\mathbf{b}}) + \left(\frac{t^2}{2} + \frac{\cos \omega t - 1}{\omega^2} \right) J^2(\hat{\mathbf{b}}) \right] \mathbf{E} \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \left(\frac{\cos \omega t - 1}{\omega} \right) \hat{\mathbf{b}} \times \mathbf{v}_0 + \left(t - \frac{\sin \omega t}{\omega} \right) \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{v}_0) \\
&\quad + \frac{\omega}{B} \left[\frac{\mathbf{E} t^2}{2} + \left(\frac{\sin \omega t}{\omega^2} - \frac{t}{\omega} \right) \hat{\mathbf{b}} \times \mathbf{E} + \left(\frac{t^2}{2} + \frac{\cos \omega t - 1}{\omega^2} \right) \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{E}) \right] \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{\omega t^2}{2B} (\mathbf{E} + \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{E})) + \left(\frac{1 - \cos \omega t}{\omega} \right) \left(-\hat{\mathbf{b}} \times \mathbf{v}_0 - \frac{1}{B} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{E}) \right) \\
&\quad + \left(t - \frac{\sin \omega t}{\omega} \right) \left(\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{v}_0) - \frac{1}{B} \hat{\mathbf{b}} \times \mathbf{E} \right) \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{\omega t^2}{2B} \hat{\mathbf{b}} (\hat{\mathbf{b}} \cdot \mathbf{E}) + \left(\frac{1 - \cos \omega t}{\omega} \right) \left(-\hat{\mathbf{b}} \times \mathbf{v}_0 - \frac{1}{B} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{E}) \right) \\
&\quad + \left(t - \frac{\sin \omega t}{\omega} \right) \left(\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{v}_0) - \frac{1}{B} \hat{\mathbf{b}} \times \mathbf{E} \right)
\end{aligned}$$

Exact analytical solution for the case of the Lorentz force, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{\omega t^2}{2B} \hat{\mathbf{b}} (\hat{\mathbf{b}} \cdot \mathbf{E}) + \left(\frac{1 - \cos \omega t}{\omega} \right) \left(-\hat{\mathbf{b}} \times \mathbf{v}_0 - \frac{1}{B} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{E}) \right) \\
&\quad + \left(t - \frac{\sin \omega t}{\omega} \right) \left(\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{v}_0) - \frac{1}{B} \hat{\mathbf{b}} \times \mathbf{E} \right)
\end{aligned}$$

where $\omega \equiv qB/m$ and $\hat{\mathbf{b}} \equiv \mathbf{B}/B$.

Let us consider a particular case with the electric field along the z -axis and the magnetic field along the x -axis: $\mathbf{E} = E\hat{\mathbf{k}}$, $\mathbf{B} = B\hat{\mathbf{i}}$, and with the initial conditions that the charged particle is placed at the origin with zero initial velocity.⁴ The position of the particle as a function of time is

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{\omega t^2}{2B} \hat{\mathbf{b}} (\hat{\mathbf{b}} \cdot \mathbf{E}) + \left(\frac{1 - \cos \omega t}{\omega} \right) \left(-\hat{\mathbf{b}} \times \mathbf{v}_0 - \frac{1}{B} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{E}) \right) + \\
&\quad \left(t - \frac{\sin \omega t}{\omega} \right) \left(\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{v}_0) - \frac{1}{B} \hat{\mathbf{b}} \times \mathbf{E} \right) \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \left(\frac{1 - \cos \omega t}{\omega} \right) \left(-\hat{\mathbf{i}} \times \mathbf{v}_0 - \frac{E}{B} \hat{\mathbf{k}} \right) + \left(t - \frac{\sin \omega t}{\omega} \right) \left(\hat{\mathbf{i}} \times (\hat{\mathbf{i}} \times \mathbf{v}_0) + \frac{E}{B} \hat{\mathbf{j}} \right) \\
&= \left(\frac{1 - \cos \omega t}{\omega} \right) \frac{E}{B} \hat{\mathbf{k}} + \left(t - \frac{\sin \omega t}{\omega} \right) \frac{E}{B} \hat{\mathbf{j}}
\end{aligned}$$

Thus we have

$$y = (\omega t - \sin \omega t) \frac{E}{\omega B} \quad \text{and} \quad z = (1 - \cos \omega t) \frac{E}{\omega B}$$

Define $R \equiv \frac{E}{\omega B}$, then we have

$$y = R\omega t - R \sin \omega t \quad \text{and} \quad z = R - R \cos \omega t$$

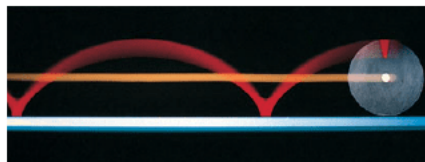
We can combine these two equations by solving the first for $\sin \omega t$ and the second for $\cos \omega t$ and making use of the trigonometric identity $\sin^2 \omega t + \cos^2 \omega t = 1$ to get

$$(y - R\omega t)^2 + (z - R)^2 = R^2.$$

This is the equation of a circle with center at $(0, R\omega t, R)$, moving along the y -axis at a speed of $v = \omega R$. As shown in the following figure, it follows the path of a cycloid.

Here is a figure, taken from *Halliday & Resnick Fundamentals of Physics (10th Edition)*, by Jearl Walker, that shows how a cycloid can be generated.

Figure 11-2 A time-exposure photograph of a rolling disk. Small lights have been attached to the disk, one at its center and one at its edge. The latter traces out a curve called a cycloid.



Richard Megna/Fundamental Photographs

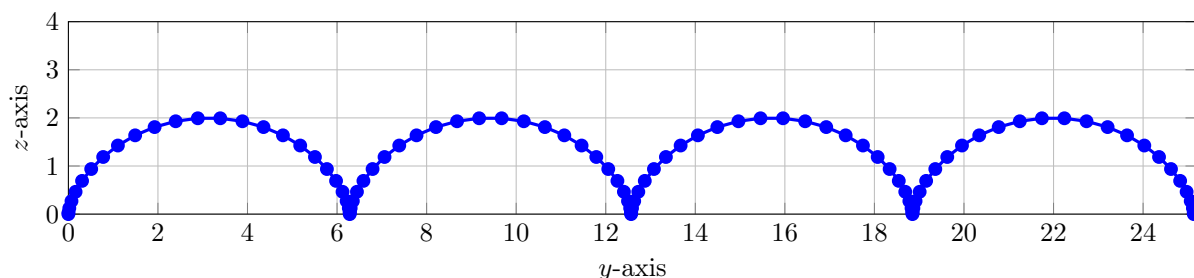


Figure 1. Trajectory of a charged particle with the electric field along the z -axis (up) and the magnetic field along the x -axis (out of the page). The charged particle begins at the origin with zero initial velocity and moves along a cycloid in the y - z plane.

The particle trajectory was generated by the following program, which makes use of the C++ Vector class Vector.h.

Listing 1. lorentz.cpp

```
1 // lorentz.cpp: Solution to equation of motion for Lorentz force
2
3 #include "Vector.h"
4 #include <iostream>
5 #include <cmath>
6 using namespace va; // vector algebra namespace
7 using namespace std;
8
9 int main( void ) {
10
11     const Vector ihat( 1., 0., 0. ); // unit vector along x-axis
12     const Vector jhat( 0., 1., 0. ); // unit vector along y-axis
13     const Vector khat( 0., 0., 1. ); // unit vector along z-axis
14
15     const double M = 1.;
16     const double Q = 1.;
17     const double EE = 1.;
18     const double BB = 1.;
19     const double OMEGA = Q * BB / M;
20     const double T = 8. * M.PI / ( OMEGA );
21     double t, theta;
22
23     const Vector r0( 0., 0., 0. );
24     const Vector v0( 0., 0., 0. );
25     const Vector E( 0., 0., EE ); // electric field along z-axis
26     const Vector B( BB, 0., 0. ); // magnetic field along x-axis
27     const Vector b = unit( B ); // unit vector along magnetic field
28
29     Vector r;
30     cout << "t" << "\t" << "y" << "\t" << "z" << endl;
31     for ( t = 0; t <= T; t += 0.01 * T ) {
32
33         theta = OMEGA * t;
34         r = r0 + v0 * t + 0.5 * ( OMEGA * t * t / BB ) * b * ( b * E ) +
35             ( ( 1. - cos( theta ) ) / OMEGA ) * ( - ( b ^ v0 ) - ( b ^ ( b ^ E ) ) / BB ) +
36             ( t - sin( theta ) / OMEGA ) * ( ( b ^ ( b ^ v0 ) ) - ( b ^ E ) / BB );
37         cout << t << "\t" << r * jhat << "\t" << r * khat << endl;
38     }
39     return EXIT_SUCCESS;
40 }
```

Coriolis Force, $\mathbf{F} = m\mathbf{g} - 2m\mathbf{\Omega} \times \mathbf{v}$

The equation of motion for a particle of mass m in the presence of gravity while in a non-inertial frame of reference with rotational velocity $\mathbf{\Omega}$ is

$$m \frac{d\mathbf{v}}{dt} = m\mathbf{g} - 2m\mathbf{\Omega} \times \mathbf{v},$$

which we can rewrite as

$$\frac{d\mathbf{v}}{dt} + 2\mathbf{\Omega} J(\hat{\omega})\mathbf{v} = \mathbf{g},$$

where $\hat{\omega} = \mathbf{\Omega}/\Omega$ and $J(\hat{\omega})\mathbf{v} = \hat{\omega} \times \mathbf{v}$. An integrating factor is $e^{2J(\hat{\omega})\Omega t}$, which gives

$$\frac{d}{dt} \left(e^{2J(\hat{\omega})\Omega t} \mathbf{v} \right) = e^{2J(\hat{\omega})\Omega t} \mathbf{g}$$

and integrates to

$$e^{2J(\hat{\omega})\Omega t} \mathbf{v}(t) = \mathbf{v}_0 + \left(\int_0^t e^{2J(\hat{\omega})\Omega \xi} d\xi \right) \mathbf{g}$$

or

$$\mathbf{v}(t) = e^{-2J(\hat{\omega})\Omega t} \mathbf{v}_0 + \left(\int_0^t e^{-2J(\hat{\omega})\Omega t} e^{2J(\hat{\omega})\Omega \xi} d\xi \right) \mathbf{g}.$$

Changing variables with $\tau = t - \xi$, $d\tau = -d\xi$, we have

$$\mathbf{v}(t) = e^{-2J(\hat{\omega})\Omega t} \mathbf{v}_0 + \left(\int_0^t e^{-2J(\hat{\omega})\Omega \tau} d\tau \right) \mathbf{g},$$

and the solution for the position vector is

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 + \left(\int_0^t e^{-2J(\hat{\omega})\Omega \tau} d\tau \right) \mathbf{v}_0 + \left(\int_0^t \int_0^\tau e^{-2J(\hat{\omega})\Omega \xi} d\xi d\tau \right) \mathbf{g} \\ &= \mathbf{r}_0 + \left(\int_0^t R_{\hat{\omega}}(-2\Omega \tau) d\tau \right) \mathbf{v}_0 + \left(\int_0^t \int_0^\tau R_{\hat{\omega}}(-2\Omega \xi) d\xi d\tau \right) \mathbf{g}. \end{aligned}$$

Evaluating the integrals, we have

$$\begin{aligned} \int_0^t R_{\hat{\omega}}(-2\Omega \tau) d\tau &= \int_0^t [I - \sin(2\Omega \tau) J(\hat{\omega}) + (1 - \cos(2\Omega \tau)) J^2(\hat{\omega})] d\tau \\ &= \left[\tau + \left(\frac{\cos 2\Omega \tau}{2\Omega} \right) J(\hat{\omega}) + \left(\tau - \frac{\sin 2\Omega \tau}{2\Omega} \right) J^2(\hat{\omega}) \right]_0^t \\ &= t + \left(\frac{\cos 2\Omega t - 1}{2\Omega} \right) J(\hat{\omega}) + \left(t - \frac{\sin 2\Omega t}{2\Omega} \right) J^2(\hat{\omega}) \\ \int_0^t \int_0^\tau R_{\hat{\omega}}(-2\Omega \xi) d\xi d\tau &= \int_0^t \left[\tau + \left(\frac{\cos 2\Omega \tau - 1}{2\Omega} \right) J(\hat{\omega}) + \left(\tau - \frac{\sin 2\Omega \tau}{2\Omega} \right) J^2(\hat{\omega}) \right] d\tau \\ &= \left[\frac{\tau^2}{2} + \left(\frac{\sin 2\Omega \tau}{4\Omega^2} - \frac{\tau}{2\Omega} \right) J(\hat{\omega}) + \left(\frac{\tau^2}{2} + \frac{\cos 2\Omega \tau}{4\Omega^2} \right) J^2(\hat{\omega}) \right]_0^t \\ &= \frac{t^2}{2} + \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega} \right) J(\hat{\omega}) + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2} \right) J^2(\hat{\omega}) \end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_0 + \left[t + \left(\frac{\cos 2\Omega t - 1}{2\Omega} \right) J(\hat{\boldsymbol{\omega}}) + \left(t - \frac{\sin 2\Omega t}{2\Omega} \right) J^2(\hat{\boldsymbol{\omega}}) \right] \mathbf{v}_0 \\
&\quad + \left[\frac{t^2}{2} + \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega} \right) J(\hat{\boldsymbol{\omega}}) + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2} \right) J^2(\hat{\boldsymbol{\omega}}) \right] \mathbf{g} \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 + \left(\frac{\cos 2\Omega t - 1}{2\Omega} \right) \hat{\boldsymbol{\omega}} \times \mathbf{v}_0 + \left(t - \frac{\sin 2\Omega t}{2\Omega} \right) \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \\
&\quad + \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega} \right) \hat{\boldsymbol{\omega}} \times \mathbf{g} + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2} \right) \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{g}) \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 + \frac{1}{2} [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{g})] t^2 - \left(\frac{1 - \cos 2\Omega t}{2\Omega} \right) \left(\hat{\boldsymbol{\omega}} \times \mathbf{v}_0 + \frac{1}{2\Omega} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{g}) \right) \\
&\quad - \left(t - \frac{\sin 2\Omega t}{2\Omega} \right) \left(\frac{1}{2\Omega} \hat{\boldsymbol{\omega}} \times \mathbf{g} - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right)
\end{aligned}$$

Exact analytical solution for the case of the Coriolis force, $\mathbf{F} = m\mathbf{g} - 2m\boldsymbol{\Omega} \times \mathbf{v}$

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 + \frac{1}{2} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{g}) t^2 \\
&\quad - \left(\frac{1 - \cos 2\Omega t}{2\Omega} \right) \left(\hat{\boldsymbol{\omega}} \times \mathbf{v}_0 + \frac{1}{2\Omega} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{g}) \right) \\
&\quad - \left(t - \frac{\sin 2\Omega t}{2\Omega} \right) \left(\frac{1}{2\Omega} \hat{\boldsymbol{\omega}} \times \mathbf{g} - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right)
\end{aligned}$$

This is an exact solution for all values of Ω , but in the case when $\Omega \ll 1$, we may expand this in powers of Ω to compare to previous approximations. We have

$$\begin{aligned}
\left(\frac{1 - \cos 2\Omega t}{2\Omega} \right) &= \frac{1}{2\Omega} \left(\frac{4\Omega^2}{2!} t^2 - \frac{16\Omega^4}{4!} t^4 \pm \dots \right) = \Omega t^2 - \frac{1}{3} \Omega^3 t^4 \pm \dots \\
\left(t - \frac{\sin 2\Omega t}{2\Omega} \right) &= t - \frac{1}{2\Omega} \left(2\Omega t - \frac{8\Omega^3}{3!} t^3 \pm \dots \right) = \frac{2}{3} \Omega^2 t^3 \mp \dots
\end{aligned}$$

and we find, to first order in Ω ,

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 - \Omega \left(\hat{\boldsymbol{\omega}} \times \mathbf{v}_0 t^2 + \frac{1}{3} \hat{\boldsymbol{\omega}} \times \mathbf{g} t^3 \right) + \mathcal{O}(\Omega^2) \quad (4)$$

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 - \boldsymbol{\Omega} \times \mathbf{v}_0 t^2 - \frac{1}{3} \boldsymbol{\Omega} \times \mathbf{g} t^3 + \mathcal{O}(\Omega^2), \quad (5)$$

which agrees with the approximation in *Mechanics* by Landau and Lifshitz.⁵ In particular, we verify that $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2$ when $\boldsymbol{\Omega} \rightarrow 0$.

The paths of a particle under the force of gravity and deviated by the Coriolis force are shown in Figure 2.

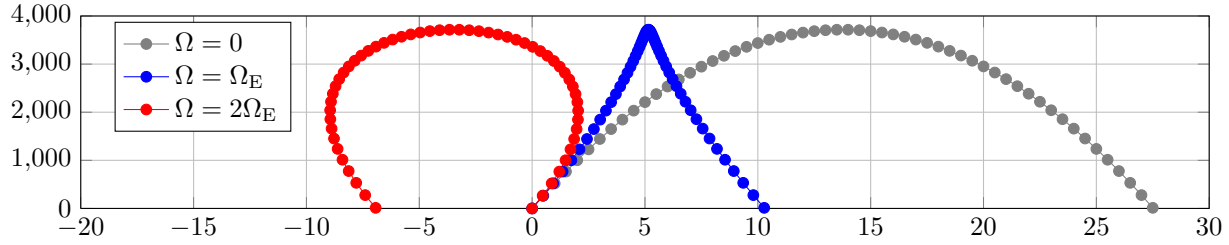


Figure 2. Trajectories of a particle under the force of gravity alone (solid gray) and under gravity and deviated by the Coriolis force (colored circles). The z axis (vertical) corresponds to the radial direction of the Earth. The positive y axis (horizontal) corresponds to East. The same initial conditions lead to a qualitatively different trajectory (red) when the angular speed of the Earth, Ω_E , is doubled. Initial velocity, $\mathbf{v}_0 = (0, 0.5, 270)$ m/s, and latitude = 30° in all three cases.

These particle trajectories were generated by the following program, which makes use of the C++ Vector class `Vector.h`.

Listing 2. coriolis.cpp

```

1 // coriolis.cpp
2
3 #include "Vector.h"
4 #include <iostream>
5 using namespace va;
6 using namespace std;
7
8 int main( void ) {
9
10     const Vector ihat( 1., 0., 0. ); // points south
11     const Vector jhat( 0., 1., 0. ); // points east
12     const Vector khat( 0., 0., 1. ); // points up (radially from the earth)
13
14     const double VX      = 0., VY = 0.5, VZ = 260., H = 0.;
15     const double ALPHA   = atan( VZ / VY );
16     const double V0      = sqrt( VX * VX + VY * VY + VZ * VZ );
17     const double OMEGA_EARTH = 7.29e-5, G = 9.81, LAT = 30. * D2R;
18     const double OMEGA    = OMEGA_EARTH;
19     const double TWO_OMEA = 2. * OMEGA_EARTH;
20     const double COS_LAT   = cos( LAT ), SIN_LAT = sin( LAT );
21     const double COS_ALPHA = cos( ALPHA ), SIN_ALPHA = sin( ALPHA );
22     const Vector omega      = Vector( -COS_LAT, 0., SIN_LAT ); // unit vector
23     const Vector g          = -G * khat;
24     const Vector r0         = Vector( 0., 0., H );
25     const Vector v0         = Vector( 0., V0 * COS_ALPHA, V0 * SIN_ALPHA );
26     double T;
27     if ( H == 0. )
28         T = 2. * V0 * SIN_ALPHA / ( G - ( 2. * OMEGA ) * COS_LAT * V0 * COS_ALPHA );
29     else
30         T = sqrt( 2. * H / G );
31
32     double f1 = 0.5 * ( 1. - cos( 2. * OMEGA * T ) ) / OMEGA;
33     double f2 = T - 0.5 * sin( 2. * OMEGA * T ) / OMEGA;
34
35     Vector w1 = omega ^ g;
36     Vector w2 = omega ^ w1;
37     Vector g1 = ( omega ^ g ) / ( 2. * OMEGA );
38     Vector g2 = omega ^ g1;
39     Vector v1 = omega ^ v0;
40     Vector v2 = omega ^ v1;
41
42     Vector r, r1, r2, r_prime, g1_prime, g2_prime;
43     double t2, f1_prime, f2_prime;
44     cout << "t" << "\t" << "y0" << "\t" << "y1" << "\t" << "y2" << "\t" << "z" << endl;
45     for ( double t = 0.; t <= T; t += 1. ) {
46
47         t2 = t * t;
48         f1 = 0.5 * ( 1. - cos( 2. * OMEGA * t ) ) / OMEGA;
49         f2 = t - 0.5 * sin( 2. * OMEGA * t ) / OMEGA;
50         f1_prime = 0.5 * ( 1. - cos( 2. * TWO_OMEA * t ) ) / TWO_OMEA;
51         f2_prime = t - 0.5 * sin( 2. * TWO_OMEA * t ) / TWO_OMEA;
52         g1_prime = ( omega ^ g ) / ( 2. * TWO_OMEA );
53         g2_prime = omega ^ g1_prime;
54         r1 = r0 + v0 * t + 0.5 * g * t2;
55         r2 = 0.5 * w2 * t2;
56         r = r1 + r2 - f1 * ( v1 + g2 ) - f2 * ( g1 - v2 );
57         r_prime = r1 + r2 - f1_prime * ( v1 + g2_prime ) - f2_prime * ( g1_prime - v2 );
58         cout << t << "\t" // elapsed time
59              << r1 * jhat << "\t" // distance east with OMEGA = 0
60              << r * jhat << "\t" // distance east
61              << r_prime * jhat << "\t" // distance east with OMEGA = 2. * OMEGA_EARTH
62              << r1 * khat << endl; // height
63     }
64     return EXIT_SUCCESS;
65 }

```


Coriolis + Centrifugal Force, $\mathbf{F} = m\mathbf{g} - 2m\mathbf{\Omega} \times \mathbf{v} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$

The rate of change of any vector in an inertial frame of reference (*fixed frame*) is equal to the rate of change of the vector in the *rotating frame* plus the vector product of the angular velocity, $\mathbf{\Omega}$, with the given vector.⁶ Thus we have the operator equation:

$$\left(\frac{d}{dt}\right)_{\text{fixed}} = \left(\frac{d}{dt}\right)_{\text{rotating}} + \mathbf{\Omega} \times$$

Applied to the position vector, we have

$$\begin{aligned}\dot{\mathbf{r}}_{\text{fixed}} &= \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \mathbf{\Omega} \times \right] \mathbf{r} = \dot{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{r} \\ \ddot{\mathbf{r}}_{\text{fixed}} &= \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \mathbf{\Omega} \times \right] (\dot{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{r}) \\ &= \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \mathbf{\Omega} \times \right] \dot{\mathbf{r}} + \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \mathbf{\Omega} \times \right] (\mathbf{\Omega} \times \mathbf{r}) \\ &= \ddot{\mathbf{r}} + \mathbf{\Omega} \times \dot{\mathbf{r}} + \dot{\mathbf{\Omega}} \times \mathbf{r} + \mathbf{\Omega} \times \dot{\mathbf{r}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \\ &= \ddot{\mathbf{r}} + \underbrace{2\mathbf{\Omega} \times \dot{\mathbf{r}} + \dot{\mathbf{\Omega}} \times \mathbf{r}}_{\text{Coriolis force}} + \underbrace{\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})}_{\text{Centrifugal force}}\end{aligned}$$

The total force is

$$\mathbf{F} = m\mathbf{a}_{\text{fixed}} = m\mathbf{a}_{\text{rotating}} + 2m\mathbf{\Omega} \times \mathbf{v}_{\text{rotating}} + m\dot{\mathbf{\Omega}} \times \mathbf{r} + m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$$

For an observer in the rotating frame, the effective force is

$$\mathbf{F}_{\text{eff}} \equiv m\mathbf{a}_{\text{rotating}} = m\mathbf{g} - 2m\mathbf{\Omega} \times \dot{\mathbf{r}} - m\dot{\mathbf{\Omega}} \times \mathbf{r} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$$

Thus, the equation of motion in the *rotating frame* is

$$\begin{aligned}\ddot{\mathbf{r}} + 2\mathbf{\Omega} \times \dot{\mathbf{r}} + \dot{\mathbf{\Omega}} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) &= \mathbf{g} \\ \ddot{\mathbf{r}} + 2\mathbf{\Omega}J(\hat{\omega})\dot{\mathbf{r}} + [\dot{\mathbf{\Omega}}J(\hat{\omega}) + \mathbf{\Omega}^2J^2(\hat{\omega})]\mathbf{r} &= \mathbf{g}\end{aligned}$$

We rewrite this as³

$$\ddot{\mathbf{r}} + 2A(t)\dot{\mathbf{r}} + B(t)\mathbf{r} = \mathbf{g} \tag{6}$$

where $A(t)$ and $B(t)$ are the matrices $A(t) \equiv \mathbf{\Omega}J(\hat{\omega})$ and $B(t) \equiv \dot{\mathbf{\Omega}}J(\hat{\omega}) + \mathbf{\Omega}^2J^2(\hat{\omega})$. Let $\mathbf{r} = V(t)\mathbf{y}$, then

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{V}\mathbf{y} + V\dot{\mathbf{y}} \\ \ddot{\mathbf{r}} &= \ddot{V}\mathbf{y} + 2\dot{V}\dot{\mathbf{y}} + V\ddot{\mathbf{y}},\end{aligned}$$

which transforms eq. (6) into

$$\begin{aligned}\ddot{V}\mathbf{y} + 2\dot{V}\dot{\mathbf{y}} + V\ddot{\mathbf{y}} + 2A(\dot{V}\mathbf{y} + V\dot{\mathbf{y}}) + BV\mathbf{y} &= \mathbf{g} \\ V\ddot{\mathbf{y}} + 2(\dot{V} + AV)\dot{\mathbf{y}} + (\ddot{V} + 2A\dot{V} + BV)\mathbf{y} &= \mathbf{g}\end{aligned} \tag{7}$$

And now we choose V to eliminate the $\dot{\mathbf{y}}$ term:

$$\dot{V} + AV = 0 \quad \implies \quad \dot{V} = -AV \quad \implies \quad V = e^{-At}.$$

Also

$$\dot{V} = -AV \quad \implies \quad \ddot{V} = -\dot{A}V - A\dot{V} = -\dot{A}V + A^2V,$$

so that eq. (7) becomes

$$\begin{aligned} V\ddot{\mathbf{y}} + (-\dot{A}V + A^2V - 2A^2V + BV)\mathbf{y} &= \mathbf{g} \\ V\ddot{\mathbf{y}} + (B - A^2 - \dot{A})V\mathbf{y} &= \mathbf{g} \end{aligned} \quad (8)$$

Now, using the expressions for A and B , we find that the coefficient of \mathbf{y} also vanishes:

$$B - A^2 - \dot{A} = \dot{\Omega}J(\hat{\omega}) + \Omega^2J^2(\hat{\omega}) - \Omega^2J^2(\hat{\omega}) - \dot{\Omega}J(\hat{\omega}) = 0$$

Thus, eq. (8) reduces to simply

$$V\ddot{\mathbf{y}} = \mathbf{g}$$

and therefore we have

$$\ddot{\mathbf{y}} = V^{-1}\mathbf{g} = e^{At}\mathbf{g} = e^{J(\hat{\omega})\Omega t}\mathbf{g} = [I + J(\hat{\omega})\sin\Omega t + J^2(\hat{\omega})(1 - \cos\Omega t)]\mathbf{g}$$

For convenience, let's write J for $J(\hat{\omega})$. Starting with

$$\ddot{\mathbf{y}} = [I + \sin\Omega tJ + (1 - \cos\Omega t)J^2]\mathbf{g}$$

and integrating over time twice, we get

$$\begin{aligned} \dot{\mathbf{y}}(t) &= \dot{\mathbf{y}}(0) + \left(\int_0^t [I + \sin\Omega\tau J + (1 - \cos\Omega\tau)J^2]d\tau \right) \mathbf{g} \\ &= \dot{\mathbf{y}}(0) + \mathbf{g}t + \frac{1 - \cos\Omega t}{\Omega}J\mathbf{g} + \left(t - \frac{\sin\Omega t}{\Omega} \right) J^2\mathbf{g} \\ \mathbf{y}(t) &= \mathbf{y}(0) + \dot{\mathbf{y}}(0)t + \frac{1}{2}\mathbf{g}t^2 + \frac{t - \Omega^{-1}\sin\Omega t}{\Omega}J\mathbf{g} + \frac{\Omega\frac{1}{2}t^2 + \Omega^{-1}(\cos\Omega t - 1)}{\Omega}J^2\mathbf{g} \\ &= \mathbf{y}(0) + \dot{\mathbf{y}}(0)t + \frac{1}{2}\mathbf{g}t^2 + \frac{\Omega t - \sin\Omega t}{\Omega^2}J\mathbf{g} + \frac{\Omega^2\frac{1}{2}t^2 + \cos\Omega t - 1}{\Omega^2}J^2\mathbf{g} \end{aligned}$$

Now, $\mathbf{r}_0 = \mathbf{y}(0)$ and $\mathbf{v}_0 \equiv \dot{\mathbf{r}}_0 = \dot{\mathbf{y}}(0) - \Omega J\mathbf{y}(0)$, so that we can make the replacements

$$\begin{aligned} \mathbf{y}(0) &\implies \mathbf{r}_0 \\ \dot{\mathbf{y}}(0) &\implies \mathbf{v}_0 + \Omega J\mathbf{r}_0 \end{aligned}$$

Also, $\mathbf{r}(t)$ and $\mathbf{y}(t)$ are related by the transformation $\mathbf{r}(t) = e^{-J\Omega t}\mathbf{y}(t)$. Thus, we have

$$\begin{aligned} \mathbf{r}(t) &= e^{-J\Omega t} \left[\mathbf{r}_0 + \mathbf{v}_0 t + \Omega J\mathbf{r}_0 t + \frac{1}{2}\mathbf{g}t^2 + \frac{\Omega t - \sin\Omega t}{\Omega^2}J\mathbf{g} + \frac{\frac{1}{2}\Omega^2 t^2 + \cos\Omega t - 1}{\Omega^2}J^2\mathbf{g} \right] \\ &= e^{-J\Omega t} \left\{ \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 - \frac{1}{\Omega^2} \underbrace{[I + \sin\Omega tJ + (1 - \cos\Omega t)J^2]}_{e^{J\Omega t}} \mathbf{g} + \Omega J\mathbf{r}_0 t + \frac{1}{\Omega^2} \left[I + \Omega tJ + \frac{1}{2}\Omega^2 t^2 J^2 \right] \mathbf{g} \right\} \\ &= -\frac{\mathbf{g}}{\Omega^2} + [I - \sin\Omega tJ + (1 - \cos\Omega t)J^2] \left\{ (\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2) + \Omega J\mathbf{r}_0 t + \frac{1}{\Omega^2}(I + \Omega tJ + \frac{1}{2}\Omega^2 t^2 J^2)\mathbf{g} \right\} \\ &= -\frac{\mathbf{g}}{\Omega^2} + (\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2) - \sin\Omega tJ(\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2) + (1 - \cos\Omega t)J^2(\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2) \\ &\quad + \Omega J\mathbf{r}_0 t - \Omega \sin\Omega tJ^2\mathbf{r}_0 t - (1 - \cos\Omega t)\Omega J\mathbf{r}_0 t + \frac{1}{\Omega^2}(I + \Omega tJ + \frac{1}{2}\Omega^2 t^2 J^2)\mathbf{g} \\ &\quad - \sin\Omega t\frac{1}{\Omega^2}(J + \Omega tJ^2 - \frac{1}{2}\Omega^2 t^2 J)\mathbf{g} + (1 - \cos\Omega t)\frac{1}{\Omega^2}(J^2 - \Omega tJ - \frac{1}{2}\Omega^2 t^2 J^2)\mathbf{g} \end{aligned}$$

where we used $J^3 = -J$ and $J^4 = -J^2$. Collecting terms, we have

$$\begin{aligned}
\mathbf{r}(t) &= -\frac{\mathbf{g}}{\Omega^2} + \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \frac{\mathbf{g}}{\Omega^2} \\
&\quad + J \left[-\sin \Omega t (\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2) + \Omega \mathbf{r}_0 t - \Omega \mathbf{r}_0 t + \cos \Omega t \Omega \mathbf{r}_0 t + \frac{\mathbf{g}t}{\Omega} - \frac{\sin \Omega t}{\Omega^2} \mathbf{g} + \frac{1}{2} \sin \Omega t \mathbf{g}t^2 - \frac{\mathbf{g}t}{\Omega} + \frac{\cos \Omega t}{\Omega} \mathbf{g}t \right] \\
&\quad + J^2 \left[(1 - \cos \Omega t)(\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2) - \Omega \sin \Omega t \mathbf{r}_0 t + \frac{1}{2}\mathbf{g}t^2 - \frac{\Omega t \sin \Omega t}{\Omega^2} \mathbf{g} + \frac{1 - \cos \Omega t}{\Omega^2} \mathbf{g} - (1 - \cos \omega t) \frac{1}{2}\mathbf{g}t^2 \right] \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + J \left[-\sin \Omega t (\mathbf{r}_0 + \mathbf{v}_0 t) + \Omega \cos \Omega t \mathbf{r}_0 t - \frac{\sin \Omega t}{\Omega^2} \mathbf{g} + \frac{\cos \Omega t}{\Omega} \mathbf{g}t \right] \\
&\quad + J^2 \left[(1 - \cos \Omega t)(\mathbf{r}_0 + \mathbf{v}_0 t) - \Omega \sin \Omega t \mathbf{r}_0 t + \frac{1}{2}\mathbf{g}t^2 + \frac{1 - \cos \Omega t - \Omega t \sin \Omega t}{\Omega^2} \mathbf{g} \right] \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + J \left[-\sin \Omega t (\mathbf{r}_0 + \mathbf{v}_0 t) + \Omega \cos \Omega t \mathbf{r}_0 t + \frac{\Omega t \cos \Omega t - \sin \Omega t}{\Omega^2} \mathbf{g} \right] \\
&\quad + J^2 \left[(1 - \cos \Omega t)(\mathbf{r}_0 + \mathbf{v}_0 t) - \Omega \sin \Omega t \mathbf{r}_0 t + \frac{1}{2}\mathbf{g}t^2 + \frac{1 - \cos \Omega t - \Omega t \sin \Omega t}{\Omega^2} \mathbf{g} \right] \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + J \left[-\sin \Omega t (\mathbf{r}_0 + \mathbf{v}_0 t) + \Omega t \cos \Omega t \mathbf{r}_0 + \frac{\Omega t \cos \Omega t - \sin \Omega t}{\Omega^2} \mathbf{g} \right] \\
&\quad + J^2 \left[\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 - \cos \Omega t (\mathbf{r}_0 + \mathbf{v}_0 t) - \Omega t \sin \Omega t \mathbf{r}_0 + \frac{\mathbf{g}}{\Omega^2} - \frac{\cos \Omega t + \Omega t \sin \Omega t}{\Omega^2} \mathbf{g} \right] \\
&= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \frac{\mathbf{g}}{\Omega^2})] - \sin \Omega t (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) - \cos \Omega t [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0)] \\
&\quad - \Omega t \sin \Omega t \frac{1}{\Omega} (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) - \Omega t \cos \Omega t \frac{1}{\Omega} [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0)] + \Omega t \cos \Omega t (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) - \Omega t \sin \Omega t [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0)] \\
&\quad - \sin \Omega t (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) - \cos \Omega t [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2})] + \Omega t \cos \Omega t (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) - \Omega t \sin \Omega t [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2})]
\end{aligned}$$

Introducing $\mathbf{r}_0^* \equiv \mathbf{r}_0 + \mathbf{g}/\Omega^2$, we can write this as

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0^* + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2)] - \sin \Omega t (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0^*) - \cos \Omega t [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0^*)] \\
&\quad - \Omega t \sin \Omega t \frac{1}{\Omega} (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) - \Omega t \cos \Omega t \frac{1}{\Omega} [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0)] + \Omega t \cos \Omega t (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0^*) - \Omega t \sin \Omega t [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0^*)]
\end{aligned}$$

Exact analytical solution for the case of both the Coriolis and the Centrifugal force
 $\mathbf{F} = m\mathbf{g} - 2m\boldsymbol{\Omega} \times \mathbf{v} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0^* + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2)] - \sin \Omega t (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0^*) - \cos \Omega t [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0^*)] \\
&\quad - \Omega t \sin \Omega t \left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0^*) + \frac{1}{\Omega} (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right] + \Omega t \cos \Omega t \left[(\hat{\boldsymbol{\omega}} \times \mathbf{r}_0^*) - \frac{1}{\Omega} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right]
\end{aligned}$$

Again, this result is exact. And again, let us expand in powers of Ω to compare to previous approximations.

$$\begin{aligned}
\mathbf{r}(t) = & \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0 + \frac{\mathbf{g}}{\Omega^2} + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2)] \\
& - \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \dots \right) [\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0 + \frac{\mathbf{g}}{\Omega^2})] \\
& - \left(1 - \frac{(\Omega t)^2}{2!} \pm \dots \right) [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0 + \frac{\mathbf{g}}{\Omega^2}))] \\
& - \Omega t \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \dots \right) \left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0 + \frac{\mathbf{g}}{\Omega^2})) + \frac{1}{\Omega} (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right] \\
& + \Omega t \left(1 - \frac{(\Omega t)^2}{2!} \pm \dots \right) \left[(\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0 + \frac{\mathbf{g}}{\Omega^2})) - \frac{1}{\Omega} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right]
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}(t) = & \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2)] + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) \\
& - \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \dots \right) [\hat{\boldsymbol{\omega}} \times \mathbf{r}_0 + \hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}] \\
& - \left(1 - \frac{(\Omega t)^2}{2!} + \frac{(\Omega t)^4}{4!} \mp \dots \right) [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2})] \\
& - \Omega t \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \dots \right) \left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) + \frac{1}{\Omega} (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right] \\
& + \Omega t \left(1 - \frac{(\Omega t)^2}{2!} \pm \dots \right) \left[(\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) + (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) - \frac{1}{\Omega} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right]
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}(t) = & \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2)] + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) - \Omega [\hat{\boldsymbol{\omega}} \times \mathbf{r}_0 + \hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}] t + \frac{(\Omega t)^3}{3!} \hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2} \\
& - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) + \Omega^2 [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2})] \frac{t^2}{2} - \Omega^4 [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2})] \frac{t^4}{24} \\
& - \Omega^2 \left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) + \frac{1}{\Omega} (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right] t^2 + \Omega^4 [\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2})] \frac{t^4}{6} \\
& + \Omega \left[(\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) + (\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) - \frac{1}{\Omega} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right] t - \Omega^3 \left[(\hat{\boldsymbol{\omega}} \times \frac{\mathbf{g}}{\Omega^2}) - \frac{1}{\Omega} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) \right] \frac{t^3}{2} + \mathcal{O}(\Omega^3)
\end{aligned}$$

Thus, the second-order approximation reduces to

$$\begin{aligned}
\mathbf{r}(t) = & \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 - \Omega \left(\hat{\boldsymbol{\omega}} \times \mathbf{v}_0 t^2 + \frac{1}{3} \hat{\boldsymbol{\omega}} \times \mathbf{g} t^3 \right) \\
& + \Omega^2 \left(-\frac{1}{2} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{r}_0) t^2 + \frac{1}{2} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{v}_0) t^3 + \frac{1}{8} \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{g}) t^4 \right) + \mathcal{O}(\Omega^3)
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mathbf{r}(t) = & \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2 - \boldsymbol{\Omega} \times \mathbf{v}_0 t^2 - \frac{1}{3} \boldsymbol{\Omega} \times \mathbf{g} t^3 \\
& - \frac{1}{2} \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_0) t^2 + \frac{1}{2} \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{v}_0) t^3 + \frac{1}{8} \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{g}) t^4 + \mathcal{O}(\Omega^3)
\end{aligned} \tag{10}$$

We see that if we neglect terms of $\mathcal{O}(\Omega^2)$ and higher that it reduces to that found for the Coriolis force, eqs. (4) and (5). We also verify that $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{g} t^2$ when $\Omega \rightarrow 0$.

Let's compute the deflection of a particle falling freely under the Earth's gravity from a height h and at a northern latitude λ . We take the z -axis to be vertically outward from the surface of the earth, the x -axis to point south, and the y -axis to point east, to form a right-handed coordinate system, as shown in Figure 3.

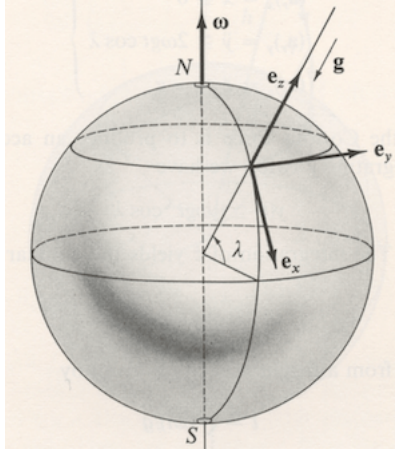


Figure 3. Earth-frame coordinate system reproduced from J. B. Marion, *Classical Dynamics of Particles and Systems*.⁵ Replace $\omega \rightarrow \hat{\omega}$, $\hat{e}_x \rightarrow \hat{i}$, $\hat{e}_y \rightarrow \hat{j}$, and $\hat{e}_z \rightarrow \hat{k}$ to be consistent with the notation used here.

We have $\hat{\omega} = -\cos \lambda \hat{i} + \sin \lambda \hat{k}$, $\mathbf{r}_0 = h\hat{k}$, $\mathbf{v}_0 = 0$, and $\mathbf{g} = -g\hat{k}$, so that

$$\begin{aligned} \mathbf{r}(t) &= h\hat{k} - \frac{1}{2}gt^2\hat{k} + \Omega \frac{1}{3}(\hat{\omega} \times \hat{k})gt^3 - \Omega^2 \frac{1}{2}[\hat{\omega} \times (\hat{\omega} \times \hat{k})]ht^2 - \Omega^2 \frac{1}{8}[\hat{\omega} \times (\hat{\omega} \times \hat{k})]gt^4 \\ &= (h - \frac{1}{2}gt^2)\hat{k} + \Omega \frac{1}{3}(\hat{\omega} \times \hat{k})gt^3 - \Omega^2[\hat{\omega} \times (\hat{\omega} \times \hat{k})](\frac{1}{2}ht^2 + \frac{1}{8}gt^4) \end{aligned}$$

The total time of fall is $t = \sqrt{2h/g}$, so that $h - \frac{1}{2}gt^2 = 0$, and $\frac{1}{2}ht^2 + \frac{1}{8}gt^4 = \frac{h^2}{g} + \frac{1}{2}\frac{h^2}{g} = \frac{3}{2}\frac{h^2}{g}$. Therefore,

$$\mathbf{r} = \Omega \frac{1}{3}(\hat{\omega} \times \hat{k})g \left(\frac{2h}{g}\right)^{3/2} - \Omega^2 \frac{3}{2}[\hat{\omega} \times (\hat{\omega} \times \hat{k})]\frac{h^2}{g} \quad (11)$$

Now

$$\hat{\omega} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos \lambda & 0 & \sin \lambda \\ 0 & 0 & 1 \end{vmatrix} = \cos \lambda \hat{j}$$

and

$$\hat{\omega} \times (\hat{\omega} \times \hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos \lambda & 0 & \sin \lambda \\ 0 & \cos \lambda & 0 \end{vmatrix} = -\sin \lambda \cos \lambda \hat{i} - \cos^2 \lambda \hat{k}$$

so that eq. (11) becomes

$$\begin{aligned} \mathbf{r} &= \frac{1}{3}\sqrt{\frac{8h^3}{g}}\Omega \cos \lambda \hat{j} + \frac{3}{2}\frac{h^2}{g}\Omega^2(\sin \lambda \cos \lambda \hat{i} + \cos^2 \lambda \hat{k}) \\ &= \frac{3}{2}\frac{h^2\Omega^2}{g}\sin \lambda \cos \lambda \hat{i} + \frac{1}{3}\sqrt{\frac{8h^3}{g}}\Omega \cos \lambda \hat{j} + \frac{3}{2}\frac{h^2\Omega^2}{g}\cos^2 \lambda \hat{k} \end{aligned}$$

For a height $h = 100$ m, latitude $\lambda = 45^\circ$, and $\Omega = 7.29 \times 10^{-5}$ rad/s for the earth's rotational velocity, we get a (first-order) easterly deflection (along \hat{j}) of 1.55 cm, and a (second-order) southerly deflection (along \hat{i}) of 4×10^{-4} cm.

The following program uses the exact formula to compute the position of the particle when it hits the ground and also the approximate values cited above.

Listing 3. centrifugal.cpp

```

1 // centrifugal.cpp
2
3 #include "Vector.h"
4 #include <iostream>
5 #include <stdlib.h>
6 #include <cmath>
7 using namespace va; // vector algebra namespace
8 using namespace std;
9
10 int main( void ) {
11
12     Vector ihat( 1., 0., 0. ), jhat( 0., 1., 0. ), khat( 0., 0., 1. ); // unit vectors for cartesian coordinate system
13
14     const double OMEGA = 7.29e-5; // Earth's rotational velocity
15     const double G = 9.81; // acceleration due to gravity (m/s^2)
16     const double H = 100.; // height (m)
17     const double LAT = 45. * D2R; // northern latitude (deg converted to rad)
18     const double OMEGA2 = OMEGA * OMEGA;
19     const double T = sqrt( 2. * H / G ); // time to fall
20     const double T2 = T * T;
21     const double T3 = T * T2;
22     const double T4 = T2 * T2;
23     const double THETA = OMEGA * T;
24     const double COS = cos( THETA );
25     const double SIN = sin( THETA );
26     const double COS_LAT = cos( LAT );
27     const double SIN_LAT = sin( LAT );
28
29     const Vector omega = Vector( -COS_LAT, 0., SIN_LAT ); // unit vector for Earth's rotation velocity
30     const Vector g = -G * khat;
31     const Vector r0 = Vector( 0., 0., H );
32     const Vector r0_star = r0 + g / OMEGA2;
33     const Vector v0 = Vector( 0., 0., 0. );
34
35     const Vector R1 = r0 + v0 * T + 0.5 * g * T2;
36     const Vector R2 = R1 + g / OMEGA2;
37     const Vector R3 = omega ^ R2;
38     const Vector R4 = omega ^ R3;
39     const Vector R5 = omega ^ r0_star;
40     const Vector R6 = omega ^ R5;
41
42     const Vector W1 = omega ^ g;
43     const Vector W2 = omega ^ W1;
44     const Vector R0 = omega ^ ( omega ^ r0 );
45     const Vector V1 = omega ^ v0;
46     const Vector V2 = omega ^ V1;
47
48     Vector r_exact = R1 + R4 - SIN * R5 - COS * R6 - THETA * SIN * ( R6 + V1 / OMEGA ) + THETA * COS * ( R5 - V2 / OMEGA );
49     Vector r_approx = R1 - OMEGA * ( V1 * T2 + W1 * T3 / 3. ) + OMEGA2 * ( -0.5 * R0 * T2 + 0.5 * V2 * T3 + 0.125 * W2 * T4 );
50
51     cout.precision(6);
52     cout << "r_exact (cm) = " << std::fixed << r_exact * 100. << endl; // output in cm
53     cout << "r_approx (cm) = " << std::fixed << r_approx * 100. << endl; // output in cm
54
55     return EXIT_SUCCESS;
56 }

```

The output from running ./centrifugal is

```

r_exact (cm) = 0.000400 1.551678 0.000400
r_approx (cm) = 0.000406 1.551679 0.000406

```

References

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Appendix: Rotation of a Vector

We want to derive a formula for the counterclockwise rotation, through the angle θ , of an arbitrary vector \mathbf{a} about the unit vector $\hat{\mathbf{u}}$. We start with the identity

$$\mathbf{a} = \underbrace{\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}}_{\mathbf{a}_{\perp}} + \underbrace{(\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}}_{\mathbf{a}_{\parallel}},$$

where the third term on the right is the component of \mathbf{a} that is parallel to $\hat{\mathbf{u}}$ and so will remain unchanged after a rotation about $\hat{\mathbf{u}}$:

$$\mathbf{a}'_{\parallel} = \mathbf{a}_{\parallel}.$$

The first two terms form the component of \mathbf{a} that is perpendicular to $\hat{\mathbf{u}}$ and will be rotated into

$$\mathbf{a}'_{\perp} = [\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}] \cos \theta + \hat{\mathbf{u}} \times [\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}] \sin \theta.$$

Hence,

$$\begin{aligned} \mathbf{a}' &\equiv R_{\hat{\mathbf{u}}}(\theta) \mathbf{a} \\ &= \mathbf{a}'_{\perp} + \mathbf{a}'_{\parallel} \\ &= [\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}] \cos \theta + \hat{\mathbf{u}} \times [\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}] \sin \theta + (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} \\ &= [\mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}] \cos \theta + \hat{\mathbf{u}} \times \mathbf{a} \sin \theta + (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}. \end{aligned}$$

Now,

$$\hat{\mathbf{u}} \times (\mathbf{a} \times \hat{\mathbf{u}}) = \mathbf{a}(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}) - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{a}) = \mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}},$$

and therefore

$$\begin{aligned} \mathbf{a}' &= \hat{\mathbf{u}} \times (\mathbf{a} \times \hat{\mathbf{u}}) \cos \theta + \hat{\mathbf{u}} \times \mathbf{a} \sin \theta + \mathbf{a} - \hat{\mathbf{u}} \times (\mathbf{a} \times \hat{\mathbf{u}}) \\ &= -\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \mathbf{a}) \cos \theta + \hat{\mathbf{u}} \times \mathbf{a} \sin \theta + \mathbf{a} + \hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \mathbf{a}) \\ &= \mathbf{a} + \hat{\mathbf{u}} \times \mathbf{a} \sin \theta + \hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \mathbf{a}) (1 - \cos \theta). \end{aligned}$$

Thus, we have

Rotation of a Vector

$$R_{\hat{\mathbf{u}}}(\theta) \mathbf{a} = \mathbf{a} + \hat{\mathbf{u}} \times \mathbf{a} \sin \theta + \hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \mathbf{a}) (1 - \cos \theta)$$