Calculating the Location of Lagrangian Point L2

Richard Saucier

January 2022

Abstract

With the launching of the James Webb Space Telescope, and positioning it at the Lagrangian point L2, you may be wondering how to calculate the location of this equilibrium point. This was one of the problems in a standard physics textbook[1] and we show how we may solve this problem.

[1] J. B. Marion, *Classical Dynamics of Particles and Systems* (Academic Press, New York, 1965).

We adapt the problem from the 1965 edition of the book by Marion, as the newer editions leave out this problem. Refer to Fig. 10-11.

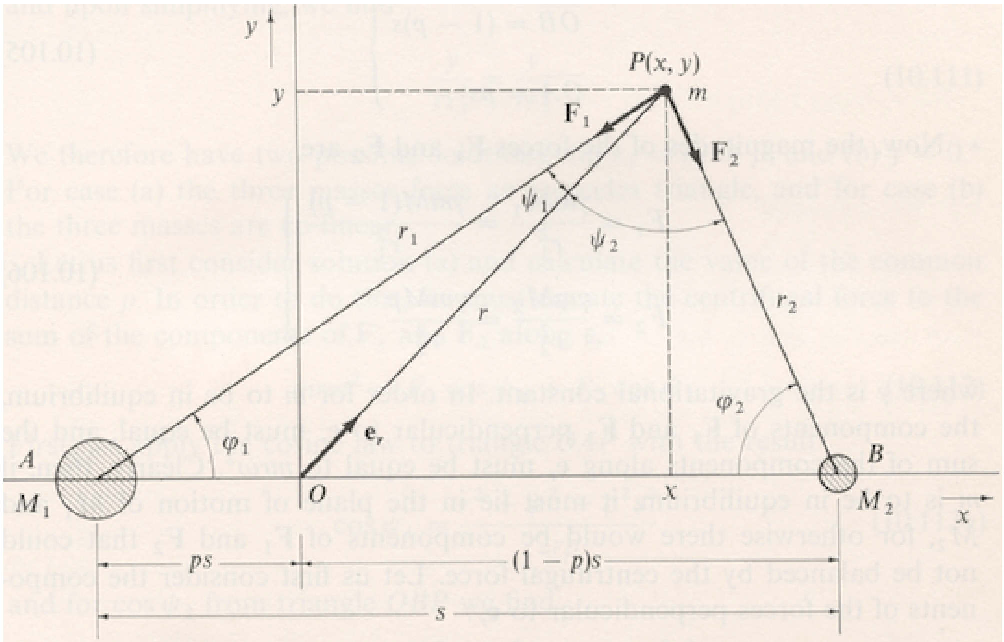

FIG. 10-11

We have two large masses, M_1 and M_2, located at A and B, respectively, and a much smaller mass m located at point P. We will be taking M_1 as the Sun, M_2 as the Earth, and m as the JWST. Since there is no known closed-form solution to the three-body problem, we will be finding approximate solutions, using the fact that $m \ll M_2 \ll M_1$.

The masses M_1 and M_2 are assumed to be moving in circular orbits about the center of mass, which is at the origin O in Fig. 10-11. By the definition of the center of mass,

$$M_1 ps = M_2(1-p)s.$$

Solving this for p gives

$$p = \frac{M_2}{M},$$

where $M \equiv M_1 + M_2$. Thus, $M_1 = (1-p)M$ and $M_2 = pM$. Let ω represent the angular velocity of the motion. Then if m is to be in a position of equilibrium, it must also be revolving about O with a constant angular velocity ω. The masses M_1 and M_2 provide the centripetal force, so we have

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 + mr\omega^2 \hat{\boldsymbol{e}}_r = 0,$$

where the magnitudes of the forces are

$$\begin{aligned} F_1 &= G\frac{mM_1}{r_1^2} = G\frac{mM(1-p)}{r_1^2} \\ F_2 &= G\frac{mM_2}{r_2^2} = G\frac{mMp}{r_2^2} \end{aligned} \tag{1}$$

In order for m to be in equilibrium, the components of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ perpendicular to $\hat{\boldsymbol{e}}_r$ must be equal, and the sum of the components along $\hat{\boldsymbol{e}}_r$ must be equal to $mr\omega^2$. The first of these requires

$$F_1 \sin\psi_1 = F_2 \sin\psi_2. \tag{2}$$

Applying the law of sines to the triangle OAP, we have

$$\frac{\sin\psi_1}{ps} = \frac{\sin\phi_1}{r}.$$

But we also have

$$\sin\phi_1 = \frac{y}{r_1}.$$

Combining the two gives

$$\sin\psi_1 = \frac{psy}{rr_1}. \tag{3}$$

Similarly, for triangle BOP,

$$\frac{\sin\psi_2}{(1-p)s} = \frac{\sin\phi_2}{r}$$

and

$$\sin\phi_2 = \frac{y}{r_2},$$

so that

$$\sin\psi_2 = \frac{(1-p)sy}{rr_2}. \tag{4}$$

Therefore, from eqs. (3) and (4), we have

$$\frac{\sin\psi_1}{\sin\psi_2} = \frac{psy}{rr_1}\frac{rr_2}{(1-p)sy} = \frac{pyr_2}{(1-p)yr_1},$$

and from eqs. (1) and (2), we have

$$\frac{\sin\psi_1}{\sin\psi_2} = \frac{F_2}{F_1} = \frac{GmMp}{r_2^2}\frac{r_1^2}{GmM(1-p)} = \frac{pr_1^2}{(1-p)r_2^2}.$$

Equating these two then gives

$$\frac{y}{r_1^3} = \frac{y}{r_2^3}.$$

We therefore have two possible solutions: (1) $r_1 = r_2$, and (2) $y = 0$. For the first case the three mases form an equilateral triangle, which would correspond to S_1 and S_2 in Fig. 10-12, and for the second case the three masses are co-linear, corresponding to U_1, U_2, and U_3 in Fig. 10-12. Here we are interested in the second solution, $y = 0$.

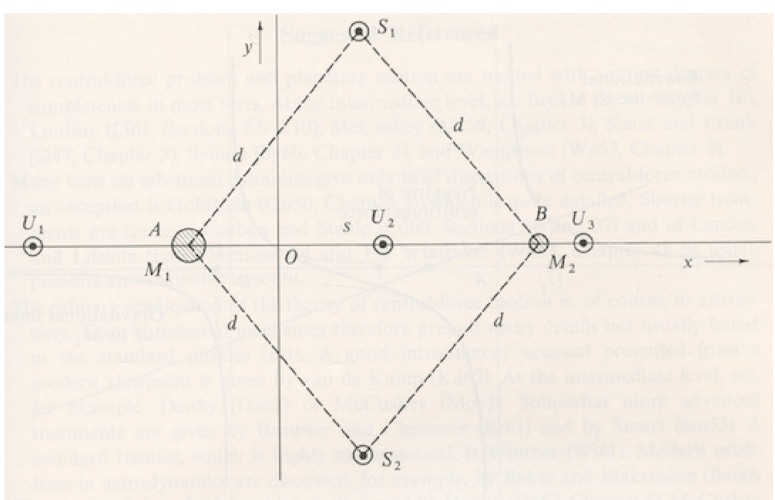

FIG. 10-12

Next we need to calculate the value of ω^2. We can do this by equating the gravitational and centrifugal forces on, say M_1 (refer to Fig. 10-11):

$$G\frac{M_1M_2}{s^2} = M_1 ps\omega^2,$$

which gives

$$\omega^2 = \frac{GM_2}{ps^3} = \frac{GM}{s^3}. \tag{5}$$

Now, we want to show that if the mass M_2 is much smaller than M_1 (which implies $p \ll 1$), then the equilibrium points U_2 and U_3 are approximately symmetrical about M_2 and lie at distances from M_2 approximately equal to $s\sqrt[3]{p/3}$.

The gravitational forces provide the centripetal force, so we have from eqs. (1) and (5),

$$\frac{GmM(1-p)}{r_1^2} \pm \frac{GmMp}{r_2^2} = m\omega^2 r = m\frac{GM}{s^3}[(1-p)s \pm \Delta],$$

where the positive sign corresponds to the position U_3, and the negative sign to position U_2, in Fig. 10-12, and Δ is the positive distance of m from M_2. This simplifies to

$$\frac{1-p}{r_1^2} \pm \frac{p}{r_2^2} = \frac{(1-p)s \pm d}{s^3}.$$

Now, $r_1 = s \pm \Delta$ and $r_2 = \Delta$, so this becomes

$$\frac{1-p}{(s\pm\Delta)^2} \pm \frac{p}{\Delta^2} = \frac{(1-p)s \pm \Delta}{s^3} \qquad \Longrightarrow \qquad \frac{1-p}{s^2(1\pm\Delta/s)^2} \pm \frac{1}{s^2}\frac{p}{(\Delta/s)^2} = \frac{1-p\pm\Delta/s}{s^2}$$

or

$$(1-p)(1\pm\Delta/s)^{-2} \pm \frac{p}{(\Delta/s)^2} = 1-p\pm\Delta/s.$$

Since $\Delta/s \ll 1$, we can use the binomial expansion,

$$(1\pm\Delta/s)^{-2} = 1 \mp (2)(\Delta/s) \pm \frac{1}{2!}(2)(3)(\Delta/s)^2 \mp \frac{1}{3!}(2)(3)(4)(\Delta/s)^3 \pm \cdots$$

to get

$$\begin{aligned}
\pm\frac{p}{(\Delta/s)^2} &= 1-p\pm\Delta/s - (1-p)(1\pm\Delta/s)^{-2} \\
&= 1-p\pm\Delta/s - (1-p)(1\mp 2\Delta/s \pm\cdots) \\
&= 1-p\pm\Delta/s - 1 \pm 2\Delta/s + p \mp 2p\Delta/s \pm \cdots \\
&= (\pm 3 \mp 2p)\Delta/s \pm \cdots
\end{aligned}$$

so, the approximate solutions are

$$(\Delta/s)^3 = \frac{\pm p}{\pm 3 \mp 2p} = \frac{p}{3-2p} \quad \text{and} \quad (\Delta/s)^3 = \frac{-p}{-3+2p} = \frac{p}{3-2p}$$

Thus we have the same solution in both cases. And since $p \ll 1$, we have

$$(\Delta/s)^3 \approx \frac{p}{3}\left(1+\frac{2}{3}p\right) \approx \frac{p}{3} = \frac{M_2}{3M}$$

so that

$$\boxed{\Delta \approx s\sqrt[3]{\frac{M_2}{3M}}}$$

Thus, the Lagrangian points L_1 and L_2 are approximately equidistant from the earth. Now we put in some numbers:

$$\begin{aligned} M_1 \equiv M_{\text{sun}} &\approx 2 \times 10^{30} \text{ kg} \\ M_2 \equiv M_{\text{earth}} &\approx 6 \times 10^{24} \text{ kg} \\ s &\approx 150 \times 10^{6} \text{ km} \end{aligned}$$

and find that the distance from the earth is

$$\Delta \approx 150 \times 10^{6} \text{ kg} \left(\frac{6 \times 10^{24} \text{ kg}}{3(2 \times 10^{30} \text{ kg})} \right)^{1/3} = 150 \times 10^{4} \text{ km} = 1.5 \text{ million km} = 931,677 \text{ miles},$$

where L_1 is in Region 1 and L_2 is in region 2. Travel time for a signal between the Earth and the JWST, located at L_2, is 5 seconds.