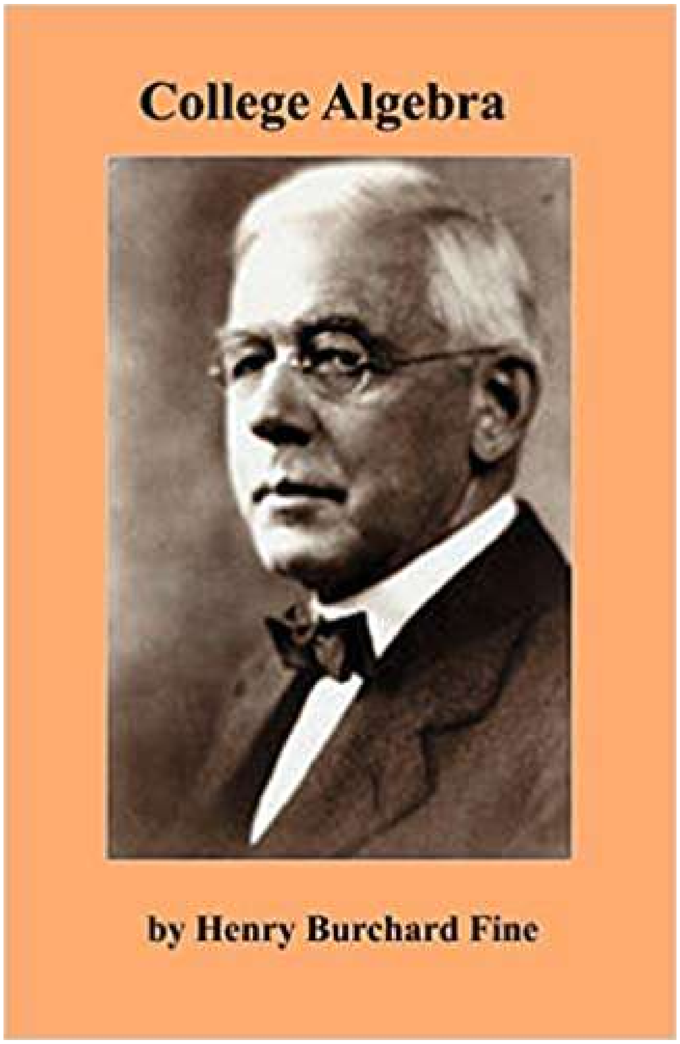
Problems and Solutions from



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Contents

Memorize the following formulas:

For *every* value of n,

$$(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})(a - b) = a^n - b^n$$

For example,

$$\begin{aligned}(a + b)(a - b) &= a^2 - b^2\\ (a^2 + ab + b^2)(a - b) &= a^3 - b^3\\ (a^3 + a^2b + ab^2 + b^3)(a - b) &= a^4 - b^4\end{aligned}$$

For every *odd* value of n,

$$(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1})(a + b) = a^n + b^n$$

For example,

$$\begin{aligned}(a^2 - ab + b^2)(a + b) &= a^3 + b^3\\ (a^4 - a^3b + a^2b^2 - ab^3 + b^4)(a + b) &= a^5 + b^5\end{aligned}$$

For every *even* value of n,

$$(a^{n-1} - a^{n-2}b + \cdots + ab^{n-2} - b^{n-1})(a + b) = a^n - b^n$$

For example,

$$\begin{aligned}(a - b)(a + b) &= a^2 - b^2\\ (a^3 - a^2b + ab^2 - b^3)(a + b) &= a^4 - b^4\end{aligned}$$

Memorize

$$\begin{aligned}(a + b)(a - b) &= a^2 - b^2\\ (a - b)^2 &= a^2 - 2ab + b^2\\ (a + b)^2 &= a^2 + 2ab + b^2\\ (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\ (a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\\ (a + b + c)^2 &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\\ (a + b + c)^3 &= a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2a + 3b^2c + 3c^2a + 3c^2b + 6abc\end{aligned}$$

Let's use these identities to simplify the expression

$$\frac{(x^3 - y^3)(x^3 + y^3)}{(x - y)(x^2 - xy + y^2)}$$

We have $x^3 - y^3 = (x^2 + xy + y^2)(x - y)$ and $x^3 + y^3 = (x^2 - xy + y^2)(x + y)$, so that

$$\frac{(x^3 - y^3)(x^3 + y^3)}{(x - y)(x^2 - xy + y^2)} = \frac{(x^2 + xy + y^2)(x - y)(x^2 - xy + y^2)(x + y)}{(x - y)(x^2 - xy + y^2)} = (x^2 + xy + y^2)(x + y)$$

Product of any two polynomials $(a_0x^n + a_1x^{n-1} + \cdots + a_n)(b_0x^m + b_1x^{m-1} + \cdots + b_m)$

Rule to find the coefficient of each term: Find the difference between the degree of the product and the degree of the term, and then form and add all the products a_hb_k in which $h + k$ equals the difference. One can also use the **Method of Detached Coefficients** to multiply two polynomials.

Square of any polynomial

The square of any polynomial is equal to the sum of the squares of all its terms together with twice the products of every two of its terms.

The Remainder Theorem

When a polynomial in x is divided by $x - b$, a remainder is obtained which is equal to the result of substituting b for x in the dividend, so that if $f(x)$ denotes the dividend, $f(b)$ will denote the remainder. Note that synthetic division by b will return the value of $f(b)$ as the remainder.

Polynomials with integral coefficients

If a polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ with integral coefficients has $x - b$ as a factor, then b must be a factor of the constant term a_n. Thus, first try the factors of a_n; only these can be factors of $f(x)$. The polynomial could have factors of the form $\alpha x - \beta$, but if so, then α must be a factor of a_0 and β a factor of a_n.

Chapter 4

Systems of Simultaneous Simple Equations, Exercise XI (p. 150)

Problem 1. Find three numbers whose sum is 20 and such that (1) the first plus twice the second plus three times the third equals 44 and (2) twice the sum of the first and second minus four times the third equals -14.

Solution. Let a, b, c represent the three numbers. Then we have

$$\begin{aligned} a + b + c &= 20 \\ a + 2b + 3c &= 44 \\ 2(a+b) - 4c &= -14 \end{aligned}$$

Let's rewrite this as a matrix equation.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 20 \\ 44 \\ -14 \end{bmatrix}$$

We solve this using Cramer's rule. The determinant of the coefficient matrix is $-8-6+6+4+2-4=-6$. Cramer's rule gives

$$\begin{aligned} a &= -\frac{1}{6}\det\begin{bmatrix} 20 & 1 & 1 \\ 44 & 2 & 3 \\ -14 & 2 & -4 \end{bmatrix} = \frac{-160-120-42+176+88+28}{-6} = \frac{30}{6} = 5 \\ b &= -\frac{1}{6}\det\begin{bmatrix} 1 & 20 & 1 \\ 1 & 44 & 3 \\ 2 & -14 & -4 \end{bmatrix} = \frac{-176+42+120+80-14-88}{-6} = \frac{36}{6} = 6 \\ c &= -\frac{1}{6}\det\begin{bmatrix} 1 & 1 & 20 \\ 1 & 2 & 44 \\ 2 & 2 & -14 \end{bmatrix} = \frac{-28-88+88+14+40-80}{-6} = \frac{54}{6} = 9 \end{aligned}$$

Thus, the three numbers are 5, 6, and 9.

Problem 2. The sum of three numbers is 51. If the first number be divided by the second, the quotient is 2 and the remainder 5; but if the second number be divided by the third, the quotient is 3 and the remainder 2. What are the numbers?

Solution. Let a, b, and c be the three numbers. We have $a+b+c=51$, $a=2b+5$, and $b=3c+2$, or

$$\begin{aligned} a + b + c &= 51 \\ a - 2b &= 5 \\ b - 3c &= 2, \end{aligned}$$

which we can write as the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 51 \\ 5 \\ 2 \end{bmatrix}$$

The determinant of the coefficient matrix is $6 + 3 + 1 = 10$. Cramer's rule gives

$$a = \frac{1}{10} \det \begin{bmatrix} 51 & 1 & 1 \\ 5 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} = \frac{306 + 15 + 5 + 4}{10} = \frac{330}{10} = 33$$

$$b = \frac{1}{10} \det \begin{bmatrix} 1 & 51 & 1 \\ 1 & 5 & 0 \\ 0 & 2 & -3 \end{bmatrix} = \frac{-15 + 153 + 2}{10} = \frac{140}{10} = 14$$

$$c = \frac{1}{10} \det \begin{bmatrix} 1 & 1 & 51 \\ 1 & -2 & 5 \\ 0 & 1 & 2 \end{bmatrix} = \frac{-4 - 5 - 2 + 51}{-6} = \frac{40}{10} = 4$$

Thus, the numbers are 33, 14, and 4.

Problem 3. Find a number of two digits from the following data: (1) twice the first digit plus three times the second equals 37; (2) if the order of the digits be reversed, the number is diminished by 9.

Solution. Let ab represent the two-digit number. Then we have $2a+3b = 37$ and $10b+a = 10a+b-9 \implies a - b = 1$. We express this as a matrix equation:

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 37 \\ 1 \end{bmatrix}.$$

Cramer's rule gives

$$a = -\frac{1}{5} \det \begin{bmatrix} 37 & 3 \\ 1 & -1 \end{bmatrix} = 8 \quad \text{and} \quad b = -\frac{1}{5} \det \begin{bmatrix} 2 & 37 \\ 1 & 1 \end{bmatrix} = 7,$$

so the number is 87.

Problem 4. A owes \$5000 and B owes \$3000. A could pay all his debts if besides his own money he had $\frac{2}{3}$ of B's; and B could pay all but 100 of his debts if besides his own money he had $\frac{1}{2}$ of A's. How much money has each?

Solution. Let a be the amount of money that A has and let b be the amount of money B has. Then we have $a + \frac{2}{3}b = 5000$ and $\frac{1}{2}a + b = 2900$, or, clearing of fractions, $3a + 2b = 15000$ and $a + 2b = 5800$. We express this with the matrix equation

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 15000 \\ 5800 \end{bmatrix}.$$

Cramer's rule gives

$$a = \frac{1}{4} \det \begin{bmatrix} 15000 & 2 \\ 5800 & 2 \end{bmatrix} = 4600 \quad \text{and} \quad b = \frac{1}{4} \det \begin{bmatrix} 3 & 15000 \\ 1 & 5800 \end{bmatrix} = 600,$$

so A has \$4600 and B has \$600.

Problem 5. Find the fortunes of three men, A, B, and C, from the following data: A and B together have p dollars, B and C, q dollars; C and A, r dollars. What conditions must p, q, and r satisfy in order that the solution found may be an admissible one?

Solution. Let a, b, and c be the fortunes of A, B, and C, respectively. Then we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

The determinant of the coefficient matrix is 2, so Cramer's rule gives

$$a = \frac{1}{2} \det \begin{bmatrix} p & 1 & 0 \\ q & 1 & 1 \\ r & 0 & 1 \end{bmatrix} = \frac{p+r-q}{2}, \; b = \frac{1}{2} \det \begin{bmatrix} 1 & p & 0 \\ 0 & q & 1 \\ 1 & r & 1 \end{bmatrix} = \frac{q-r+p}{2}, \; c = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & p \\ 0 & 1 & q \\ 1 & 0 & r \end{bmatrix} = \frac{r+q-p}{2}.$$

Thus, A has $(p+r-q)/2$ dollars, B has $(q-r+p)/2$ dollars, and C has $(r+q-p)/2$ dollars. Of course, each of these must be non-negative.

Problem 6. A sum of money at simple interest amounts to \$2556.05 in 2 years and to \$2767.10 in 4 years. What is the sum of money, and what is the rate of interest?

Solution. Let P be the sum of money, and let r be the simple rate of interest. Then, the amount after t years is given by the formula $A = P(1+rt)$, so that we have

$$P(1+2r) = 2556.05 \quad \text{and} \quad P(1+4r) = 2767.10$$

Dividing the first equation by the second gives $\dfrac{1+2r}{1+4r} = \dfrac{2556.05}{2767.10} \equiv a$, so that

$$\frac{1+2r}{1+4r} = \frac{2556.05}{2767.10} \equiv a \implies 1+2r = a+4ar \implies 1-a = 4ar-2r = (4a-2)r \implies r = \frac{1-a}{4a-2} = 0.045$$

Substituting this into the first equation gives $P = \dfrac{2556.05}{1+(2)(0.045)} = 2345$. Thus, the sum of money is \$2345 and the interest rate is 4.5%.

Problem 8. Find the area of a rectangle from the following data: if 6 inches be added to its length and 6 inches to its breadth, the one becomes $\frac{3}{2}$ of the other, and the area of the rectangle is increased by 84 square inches.

Solution. Let l represent the length and w the width. Then we have

$$\begin{aligned} l+6 = \frac{3}{2}(w+6) &\implies 2l-3w = 6 \\ (l+6)(w+6) = lw+84 &\implies l+w = 8 \end{aligned}$$

We write this as a matrix equation:

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Cramer's rule gives

$$l = \frac{1}{5} \det \begin{bmatrix} 6 & -3 \\ 8 & 1 \end{bmatrix} = 6 \quad \text{and} \quad w = \frac{1}{5} \det \begin{bmatrix} 2 & 6 \\ 1 & 8 \end{bmatrix} = 2 \tag{4.1}$$

Therefore, the area of the rectangle is $lw = 6 \cdot 2 = 12$ square inches.

Problem 9. A gave B as much money as B had; then B gave A as much money as A had left; finally A gave B as much money as B then had left. A then had \$16 and B \$24. How much had each originally?

Solution. Let a represent the amount of money that A had and b the amount that B had. Then we have

$$\begin{aligned} A\!:\; a-b+a-b-[b+b-(a-b)] &= 16 \implies 3a-5b = 16 \\ B\!:\; b+b-(a-b)+[b+b-(a-b)] &= 24 \implies -2a+6b = 24 \end{aligned}$$

We write this as a matrix equation:

$$\begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 16 \\ 24 \end{bmatrix}$$

Cramer's rule gives

$$a = \frac{1}{8} \det \begin{bmatrix} 16 & -5 \\ 24 & 6 \end{bmatrix} = 27 \quad \text{and} \quad b = \frac{1}{8} \det \begin{bmatrix} 3 & 16 \\ -2 & 24 \end{bmatrix} = 13 \tag{4.2}$$

Thus, A had \$27 and B had \$13.

Problem 10. A and B together can do a certain piece of work in $5\frac{1}{7}$ days; A and C, in $4\frac{4}{5}$ days. All three of them work at it for 2 days when A drops out and B and C finish it in $1\frac{9}{17}$ days. How long would it take each man separately to do the piece of work?

Solution. Let a represent the number of days it takes A, separately, to complete the work, similarly for b and c. Then the amount of work that a can accomplish in one day is $\frac{1}{a}$. Thus, we have

$$5\frac{1}{7}\left(\frac{1}{a}+\frac{1}{b}\right) = 1 \implies \frac{36}{7}\left(\frac{1}{a}+\frac{1}{b}\right) = 1 \implies 36\frac{1}{a} + 36\frac{1}{b} = 7$$
$$4\frac{4}{5}\left(\frac{1}{a}+\frac{1}{c}\right) = 1 \implies \frac{24}{5}\left(\frac{1}{a}+\frac{1}{c}\right) = 1 \implies 24\frac{1}{a} + 24\frac{1}{c} = 5$$
$$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \frac{26}{17}\left(\frac{1}{b}+\frac{1}{c}\right) = 1 \implies 34\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + 26\left(\frac{1}{b}+\frac{1}{c}\right) = 17 \implies 34\frac{1}{a} + 60\frac{1}{b} + 60\frac{1}{c} = 17$$

Writing this in matrix form, we have

$$\begin{bmatrix} 36 & 36 & 0 \\ 24 & 0 & 24 \\ 34 & 60 & 60 \end{bmatrix} \begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 17 \end{bmatrix}$$

Cramer's rule gives

$$\frac{1}{a} = \frac{1}{-74304} \det \begin{bmatrix} 7 & 36 & 0 \\ 5 & 0 & 24 \\ 17 & 60 & 60 \end{bmatrix} = \frac{-6192}{-74304} = \frac{1}{12}$$
$$\frac{1}{b} = \frac{1}{-74304} \det \begin{bmatrix} 36 & 7 & 0 \\ 24 & 5 & 24 \\ 36 & 17 & 60 \end{bmatrix} = \frac{-8256}{-74304} = \frac{1}{9}$$
$$\frac{1}{c} = \frac{1}{-74304} \det \begin{bmatrix} 36 & 36 & 7 \\ 24 & 0 & 5 \\ 36 & 60 & 17 \end{bmatrix} = \frac{-9288}{-74304} = \frac{1}{8}$$

Thus, it would take A 12 days, B 9 days, and C 8 days separately to complete the work.

Problem 11. Two points move at constant rates along the circumference of a circle whose length is 150 feet. When they move in opposite senses they meet every 5 seconds; when they move in the same sense they are together every 25 seconds. What are their rates?

Solution. Let the rates be given by v_1 and v_2. The time for a complete revolution of the circle for each is $\tau_1 = \dfrac{150}{v_1}$ and $\tau_2 = \dfrac{150}{v_2}$. The time between meetings when moving in opposite and in the same senses, respectively, are

$$t = \frac{150 - v_2 \cdot 5}{v_1} = 5 \quad \text{and} \quad t = \frac{150 + v_2 \cdot 25}{v_1} = 25$$

or

$$5v_1 + 5v_2 = 150 \quad \text{and} \quad 25v_1 - 25v_2 = 150$$

Solving simultaneously, we get $v_1 = 18$ and $v_2 = 12$.

Problem 12. It would take two freight trains whose lengths are 240 yards and 200 yards respectively 25 seconds to pass one another when moving in opposite directions; but were the trains moving in the same direction, it would take the faster one $3\frac{3}{4}$ minutes to pass the slower one. What are the rates of the trains in miles per hour?

Solution. Let v_1 and v_2 be the speed of the two trains. When moving in opposite directions, we have, converting yards to feet, $(v_1+v_2)25 = 3(240+200) \implies 25v_1+25v_2 = 1320$, and when moving in the same direction, $(v_1 - v_2)225 = 1320 \implies 225v_1 - 225v_2 = 1320$. Multiplying the first equation by 9 gives

$$\begin{aligned} 225v_1 + 225v_2 &= 11880 \\ 225v_1 - 225v_2 &= 1320 \end{aligned}$$

Adding and subtracting, we find $v_1 = \frac{13200}{450}$ f/s and $v_2 = \frac{10560}{450}$ f/s. Using 88 f/s = 60 mph to convert form f/s to mph, we get

$$v_1 = \frac{13200}{450} \cdot \frac{60}{88} = 20 \text{ mph} \quad \text{and} \quad v_2 = \frac{10560}{450} \cdot \frac{60}{88} = 16 \text{ mph}.$$

Problem 13. Two steamers, A and B, ply between cities C and D, which are 200 miles apart. The steamer A can start from C 1 hour later than B, overtake B in 2 hours, and having reached D and made a 4 hours' wait there, on its return trip meet B 10 miles from D. What are the rates of A and B?

Solution. Let a be the rate of A and b be the rate of B. At any time t after A leaves C, and after a 1 hour head start of B, we have at for the distance A travels and $b(1+t)$ for the distance B travels. Since A overtakes B after 2 hours, we have $2a = 3b$. When they meet up again, B will have traveled $b(t+1) = 200 - 10 = 190$. During this time, A will have rested at D for 4 hours so that $a(t-4) = 200 + 10 = 210$. Making use of the relation $2a = 3b$, we have

$$\begin{aligned} b(t+1) = 190 &\implies 3b(t+1) = 2a(t+1) = 570 &&\implies 2at = 570 - 2a \\ a(t-4) = 210 &\implies 2a(t-4) = 420 &&\implies 2at = 420 + 8a \end{aligned}$$

So we have, $570 - 2a = 420 + 8a \implies 10a = 150 \implies a = 15$. Thus, A's rate is 15 mph and B's rate is $\frac{2}{3}15 = 10$ mph.

Problem 14. In a half-mile race A can beat B by 20 yards and C by 30 yards. By how many yards can B beat C?

Solution. Let a be the speed of A, b be the speed of B, c be the speed of C. If t is the time it takes a to run the 880 yards (half-mile), then we have $at = 880$, $bt = 860$, and $ct = 850$. Therefore,

$$\frac{c}{b} = \frac{c/a}{b/a} = \frac{850/880}{860/880} = \frac{850}{860}$$

So that when b gets to the finish line, i.e., 880 yards, c will be at $\dfrac{850}{860} \cdot 880 = 869 + \frac{660}{860}$. So B beats C by $880 - (869 + \frac{33}{43}) = 11 - \frac{33}{43} = 10\frac{10}{43}$ yards.

Problem 15. A and B run two 440-yard races. In the first race A gives B a start of 20 yards and beats him by 2 seconds. In the second race A gives B a start of 4 seconds and beats him by 6 yards. What are the rates of A and B?

Solution. Let a be the rate of A, b be the rate of B, and t be the time for a to run 440 yards. Then $b = \dfrac{440-20}{t+2} = \dfrac{440-6}{t+4} \implies 420t + 4(420) = 434t + 2(434) \implies 14t = 812 \implies t = 58$. Thus, $a = \dfrac{440}{58} = 7 + \frac{34}{58} = 7\frac{17}{29}$ and $b = \dfrac{420}{60} = 7$.

Problem 16. Two passengers together have 500 pounds of baggage. One pays \$1.25 , the other \$1.75 for excess above the weight allowed. If the baggage had belonged to one person, he would have had to pay \$4. How much baggage is allowed free to a single passenger, and what is the charge for excess weight?

Solution. Let f be the free limit, x the amount above the free limit for one of the passengers, y be the amount above the free limit for the other passenger, and let α represent the charge per pound above the free limit. Then

$$f + x + f + y = 500 \qquad x\alpha = 1.25 \qquad y\alpha = 1.75 \qquad (f + x + y)\alpha = 4.$$

From the last three equations,

$$\frac{f + x + y}{x + y} = \frac{4}{3}, \tag{4.3}$$

which gives $3f = x + y$. Substituting this into the first equation then gives $f = 100$. Then, from the last three equations, $100\alpha + 3 = 4$, so that $\alpha = 1/100$. Thus, the free weight is 100 pounds and the charge for excess weight is a penny per pound.

Problem 17. Given three alloys of the following composition: A, 5 parts (by weight) gold, 2 silver, 1 lead; B, 2 parts gold, 5 silver, 1 lead; C, 3 parts gold, 1 silver, 4 lead. To obtain 9 ounces of an alloy containing equal quantities (by weight) of gold, silver, and lead, how many ounces of A, B, and C must be taken and melted together?

Solution. Let g be the amount of gold, s the amount of silver, and l the amount of lead (by weight). Let a be the amount of A, b be the amount of B, and c the amount of C that must be added together. Then we have

$$\begin{aligned}
&\text{A: } 5g + 2s + 1l, \quad \text{B: } 2g + 5s + 1l, \quad \text{C: } 3g + 1s + 4l \\
&a(5g + 2s + 1l) + b(2g + 5s + 1l) + c(3g + 1s + 4l) \\
&5a + 2b + 3c = 2a + 5y + c = a + b + 4c = 1
\end{aligned}$$

We can express this with the matrix equation

$$\begin{bmatrix} 5 & 2 & 3 \\ 2 & 5 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving this with Cramer's rule gives $a = 1/72$, $b = 11/72$, $c = 15/72$. The normalization is of no consequence, only the relative percentages. So we need 1 part of A, 11 parts of B and 15 parts of C, or to make 9 ounces, 1/3 A, 11/3 B, and 15/3 C.

Problem 18. A and B are alloys of silver and copper. An alloy which is 5 parts A and 3 parts B is 52% silver. One which is 5 parts A and 11 parts B is 42% silver. What are the percentages of silver in A and B respectively?

Solution. Let x represent the percentage of silver in A and y represent the percentage of silver in B. Then we have

$$\frac{5}{8}x + \frac{3}{8}y = 0.52 \quad \text{and} \quad \frac{5}{16}x + \frac{11}{16}y = 0.42$$

Multiplying the second equation by 2 and then subtracting the first equation from it gives $y = 0.84 - 0.52 = 0.32$, and then we find $x = 0.64$. Thus, the percentage of silver in A is 32% and in B is 64%.

Problem 19. A marksman who is firing at a target 500 yards distant hears the bullet strike $2\frac{2}{5}$ seconds after he fires. An observer distant 600 yards from the target and 210 yards from the marksman hears the bullet strike $2\frac{1}{10}$ seconds after he hears the report of the rifle. Find the velocity of sound and the velocity of the bullet, assuming that both of these velocities are constant.

Solution. Let v_s be the velocity of sound, v_b be the velocity of the bullet. We shall work in feet instead of yards. For the marksman,

$$v_b t_1 = 1500 \quad \text{and} \quad v_s t_2 = 1500 \quad \text{where} \quad t_1 + t_2 = \frac{12}{5} \tag{4.4}$$

For the observer,

$$v_s t_3 = 630, \quad v_b t_1 = 1500, \quad v_s t_4 = 1800 \quad \text{where} \quad -t_3 + t_1 + t_4 = \frac{21}{10} \tag{4.5}$$

t_3 the time it takes for the sound of the rifle firing to reach him is a delay and so is negative. We can rewrite the equations as

$$\begin{aligned} \frac{1500}{v_b} + \frac{1500}{v_s} &= \frac{12}{5} \\ -\frac{630}{v_s} + \frac{1500}{v_b} + \frac{1800}{v_s} &= \frac{21}{10} \end{aligned}$$

These equations are linear in $1/v_b$ and $1/v_s$ and we can express them as the matrix equation

$$\begin{bmatrix} 1500 & 1500 \\ 1500 & 1170 \end{bmatrix} \begin{bmatrix} 1/v_b \\ 1/v_s \end{bmatrix} = \begin{bmatrix} 24/10 \\ 21/10 \end{bmatrix}$$

Solving this with Cramer's rule leads to $v_s = 1100$ f/s and $v_b = 1447\frac{7}{19}$.

Problem 20. A tank is supplied by two pipes, A and B, and emptied by a third pipe, C. If the tank be full and all the pipes be opened, the tank will be emptied in 3 hours; if A and C alone be opened, in 1 hour; if B and C alone be opened, in 45 minutes. If A supplies 100 more gallons a minute than B does, what is the capacity of the tank, and how many gallons a minute pass through each of the pipes?

Solution. We work in units of minutes rather than hours. Let a be the number of gallons per minute that flow through pipe A, and similarly for b and c. Also, let V be the volume of the tank. Then we have

$$\begin{aligned} 180(c - a - b) &= V \\ 60(c - a) &= V \\ 45(c - b) &= V \end{aligned}$$

Making use of $b = a - 100$ to replace b throughout, we have

$$\begin{aligned} V + 360a - 180c &= 18000 \\ V + 60a - 60c &= 0 \\ V + 45a - 45c &= 4500, \end{aligned}$$

which we can express with the matrix equation

$$\begin{bmatrix} 1 & 360 & -180 \\ 1 & 60 & -60 \\ 1 & 45 & -45 \end{bmatrix} \begin{bmatrix} V \\ a \\ c \end{bmatrix} = \begin{bmatrix} 18000 \\ 0 \\ 4500 \end{bmatrix}$$

The determinant of the coefficient matrix is -2700, and Cramer's rule gives

$$\begin{aligned} V &= \frac{1}{-2700} \det \begin{bmatrix} 18000 & 360 & -180 \\ 0 & 60 & -60 \\ 4500 & 45 & -45 \end{bmatrix} = \frac{-48600000}{-2700} = 18000 \\ a &= \frac{1}{-2700} \det \begin{bmatrix} 1 & 18000 & -180 \\ 1 & 0 & -60 \\ 1 & 4500 & -45 \end{bmatrix} = \frac{-810000}{-2700} = 300 \\ c &= \frac{1}{-2700} \det \begin{bmatrix} 1 & 360 & 18000 \\ 1 & 60 & 0 \\ 1 & 45 & 4500 \end{bmatrix} = \frac{-1620000}{-2700} = 600 \end{aligned}$$

And since a delivers 100 gallon per minute more than b, $b = 200$. In summary, the tank volume is 18,000 gallons, 300 gallons per minute flows through A, 200 through B, and 600 through C.

4.1 Problems Illustrating the Method of Undetermined Coefficients, Exercise XII (p. 154)

Problem 1. Express $3x^3 - x^2 + 2x - 5$ as a polynomial in $x - 2$.

Solution.

$$\begin{aligned}
3x^3 - x^2 + 2x - 5 &= a(x-2)^3 + b(x-2)^2 + c(x-2) + d \\
&= a[x^3 + 3x^2(-2) + 3x(-2)^2 + (-2)^3] + b(x^2 - 4x + 4) + c(x-2) + d \\
&= a[x^3 - 6x^2 + 12x - 8] + b(x^2 - 4x + 4) + c(x-2) + d \\
&= ax^3 + (-6a + b)x^2 + (12a - 4b + c)x + (-8a + 4b - 2c + d),
\end{aligned}$$

which requires $a = 3$, $-18 + b = -1 \implies b = 17$, $36 - 68 + c = 2 \implies c = 34$, $-24 + 68 - 68 + d = -5 \implies d = 19$. Thus,

$$3x^3 - x^2 + 2x - 5 = 3(x-2)^3 + 17(x-2)^2 + 34(x-2) + 19$$

Problem 2. Express $4x^2 + 8x + 7$ as a polynomial in $2x + 3$.

Solution.

$$\begin{aligned}
4x^2 + 8x + 7 &= a(2x+3)^2 + b(2x+3) + c \\
&= a(4x^2 + 12x + 9) + b(2x+3) + c \\
&= 4ax^2 + (12a + 2b)x + 9a + 3b + c,
\end{aligned}$$

which requires $a = 1$, $12 + 2b = 8 \implies b = -2$, $9 - 6 + c = 7 \implies c = 4$. Thus,

$$4x^2 + 8x + 7 = (2x+3)^2 - 2(2x+3) + 4$$

Problem 3. Find $f(x) = ax^2 + bx + c$ such that $f(-1) = 11$, $f(1) = -5$, $f(5) = 6$.

Solution. We get $f(-1) = a - b + c = 11$, $f(1) = a + b + c = -5$, $f(5) = 25a + 5b + c = 6$, which we can express with the matrix equation

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ 6 \end{bmatrix}.$$

Solving this with Cramer's rule gives $a = \frac{43}{24}$, $b = -\frac{192}{24}$, $c = \frac{29}{24}$, so that the function is

$$f(x) = \frac{43}{24}x^2 - \frac{192}{24}x + \frac{29}{24} = \frac{43}{24}x^2 - 8x + \frac{29}{24}.$$

Problem 4. Find $f(x) = ax^3 + bx^2 + cx + d$ such that $f(0) = 5$, $f(-1) = 1$, $f(1) = 9$, $f(2) = 31$.

Solution. We get $f(0) = d = 5$, $f(-1) = -a + b - c = -4$, $f(1) = a + b + c = 4$, $f(2) = 8a + 4b + 2c = 26$, which we can express with the matrix equation

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ 26 \end{bmatrix}.$$

Solving this with Cramer's rule gives $a = \dfrac{36}{12} = 3$, $b = \dfrac{0}{12} = 0$, $c = \dfrac{12}{12} = 1$, so that the function is

$$f(x) = 3x^3 + x + 5.$$

Problem 5. Find $f(x, y) = ax + by + c$ such that

$$f(0,0) = 4, \qquad f(4,4) = 0, \qquad f(1,0) = 6.$$

Solution. We have $f(0,0) = c = 4$, $f(4,4) = 4a + 4b + 4 = 0$, $f(1,0) = a + 4 = 6 \implies a = 2 \implies 8 + 4b + 4 = 0 \implies b = -3$. Therefore, $f(x, y) = 2x - 3y + 4$.

Problem 6. Find a simple equation $ax + by + 1 = 0$ two of whose solutions are $x = 3$, $y = 1$ and $x = 4$, $y = -1$.

Solution. We have $3a + b + 1 = 0$ and $4a - b + 1 = 0$. Adding gives $7a + 2 = 0 \implies a = -2/7$. Substituting into the first equation gives $-6/7 + b + 7/7 = 0 \implies b = -1/7$. Thus, the equation is $-\frac{2}{7}x - \frac{1}{7}y + 1 = 0$ or $2x + y - 7 = 0$.

Problem 7. Can a simple equation $ax + by + c = 0$ be found which has the three solutions $x = 3$, $y = 1$; $x = 4$, $y = -1$; $x = 1$, $y = 1$?

Solution. Since two points determine a line, the third point won't satisfy the equation unless it lies along that line. Fitting an equation to the first two points, we have

$$\frac{y-1}{x-3} = \frac{-1-1}{4-3} \implies 2x + y - 7 = 0$$

and it is easy to see that $(1, 1)$ is not a solution. So the answer is no.

Problem 8. Find the simple equation whose graph is the straight line determined by the points $(2, 3)$, $(-4, 5)$.

Solution. Since a straight line has constant slope, we must have

$$\frac{y-3}{x-2} = \frac{5-3}{-4-2} = -\frac{1}{3} \implies 3y - 9 = -x + 2 \implies x + 3y - 11 = 0.$$

Problem 9. Determine c so that the graph of $3x + y + c = 0$ will pass through the point $(-2, 3)$.

Solution. We must have $(3)(-2) + 3 + c = 0 \implies c = 3$. Thus, the equation is $3x + y + 3 = 0$.

Problem 10. Find two simple equations, $ax + by + 1 = 0$ (1), $a'x + b'y + 1 = 0$ (2), such that both are satisfied by $x = 2$, $y = 3$ and also (1) by $x = 7$, $y = 5$ and (2) by $x = 3$, $y = 7$.

Solution. For equation (1), we have $2a + 3b = -1$, $7a + 5b = -1$, or

$$\begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Solving with Cramer's rule gives $a = 2/11$, $b = -5/11$, so that $\dfrac{2}{11}x - \dfrac{5}{11}y + 1 = 0 \implies 2x - 5y + 11 = 0$. For equation (2), we have $2a' + 3b' = -1$, $3a' + 7b' = -1$, or

$$\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Solving with Cramer's rule gives $a = -4/5$, $b = 1/5$, so that $-\frac{4}{5}x + \frac{1}{5}y + 1 = 0 \implies 4x - y - 5 = 0$.

Problem 11. Find the equation $x^3 + bx^2 + cx + d = 0$ whose roots are -2, 1, and 3.

Solution.

$$\begin{aligned}(x+2)(x-1)(x-3) = 0 &\implies x^3 + (2-1-3)x^2 + [(2)(-1) + (2)(-3) + (-1)(-3)]x + (2)(-1)(-3) = 0 \\ &\implies x^3 - 2x^2 - 5x + 6 = 0\end{aligned}$$

Problem 12. Find an equation of the form $x^2 + bxy + cx + dy = 0$ which has the solutions $x = 1$, $y = 0$; $x = 2$, $y = 1$; $x = -2$, $y = 1$.

Solution. We have $1 + c = 0 \implies c = -1$, $4 + 2b - 2 + d = 0$, $4 - 2b + 2 + d = 0$. Adding the last two equations gives $8 + 2d = 0 \implies d = -4 \implies b = 1$. Therefore, the equation is $x^2 + xy - x - 4y = 0$.

Problem 13. Express $3x + 2y - 3$ in the form

$$a(x + y - 1) + b(2x - y + 2) + c(x + 2y - 3),$$

where a, b, and c denote constants.

Solution. Comparing the two equations, we must have $a + 2b + c = 3$, $a - b + 2c = 2$, and $-a + 2b - 3c = -3$, which we can express with the matrix equation

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$$

Solving this with Cramer's rule gives $a = -1$, $b = 1$, $c = 2$. Thus,

$$3x + 2y - 3 = -(x + y - 1) + (2x - y + 2) + 2(x + 2y - 3).$$

Expression of One Polynomial in Terms of Another

Let A and B denote two polynomials in x, A of higher degree than B. Divide A by B and call the quotient Q, the remainder R; then

$$A \equiv QB + R.$$

If Q is not of lower degree than B, divide Q by B and call the quotient Q_1, the remainder R_1; then

$$Q \equiv Q_1 B + R_1.$$

Similarly, if Q_1 is not of lower degree than B, divide Q_1 by B and call the quotient Q_2, the remainder R_2; then

$$Q_1 \equiv Q_2 B + R_2.$$

Suppose that Q_2 is of lower degree than B. We then have

$$\begin{aligned} A &\equiv QB + R \\ &\equiv [Q_1 B + R_1]B + R \\ &\equiv [(Q_2 B + R_2)B + R_1]B + R \\ &\equiv Q_2 B^3 + R_2 B^2 + R_1 B + R, \end{aligned}$$

where all the coefficients Q_2, R_2, R_1, R are of lower degree than B.

Chapter 6

Factors of Rational Integral Expression

6.1 Expression of One Polynomial in Terms of Another, Exercise XV (p. 176)

Problem 1. Express $x^4 + x^3 - 1$ in terms of $x^2 + 1$.

Solution. Using synthetic division, we find $x^4 + x^3 - 1 = (x^2 + 1)(x^2 + x - 1)$ with a remainder $R = -x$. And $x^2 + x - 1 = 1(x^2 + 1) + x - 2$, so that $Q_1 = 1$ and $R_1 = x - 2$. Thus,

$$x^4 + x^3 - 1 = (x^2 + 1)^2 + (x - 2)(x^2 + 1) - x$$

Problem 2. Express $4x^4 + 2x^3 + 4x^2 + x + 6$ in terms of $2x^2 + 1$.

Solution. Using synthetic division, we find $4x^4 + 2x^3 + 4x^2 + x + 6 = (2x^2 + x + 1)(2x^2 + 1)$ with a remainder $R = 5$. And $2x^2 + x + 1 = 1(2x^2 + 1) + x$, so that $Q_1 = 1$ and $R_1 = x$. Thus,

$$4x^4 + 2x^3 + 4x^2 + x + 6 = (2x^2 + 1)^2 + x(2x^2 + 1) + 5$$

Problem 3. Express $2x^7 - 3x^6 + 2x^5 + 5x^4 - x^2 + 6$ in terms of $x^3 - x^2 + x + 3$.

Solution. Using synthetic division, we find $2x^7 - 3x^6 + 2x^5 + 5x^4 - x^2 + 6 = 2x^4 - x^3 - x^2 - x + 3$ with a remainder $R = 6x^2 - 3$. And $2x^4 - x^3 - x^2 - x + 3 = (2x + 1)(x^3 - x^2 + x + 3) - 2x^2 - 8x$, so that $Q_1 = 2x + 1$ and $R_1 = -2x^2 - 8x$. Thus,

$$2x^7 - 3x^6 + 2x^5 + 5x^4 - x^2 + 6 = (2x + 1)(x^3 - x^2 + x + 3)^2 - (2x^2 + 8x)(x^3 - x^2 + x + 3) + 6x^2 - 3$$

Problem 4. Express $x^5 + x^3y^2 + x^2y^3 + y^5$ in terms of $x^2 + xy + y^2$.

Solution. Using synthetic division, we find $x^5 + x^3y^2 + x^2y^3 + y^5 = x^3 - yx^2 + y^2x + y^3$ with a remainder $R = -2y^4x$. And $x^3 - yx^2 + y^2x + y^3 = (x - 2y)(x^2 + xy + y^2) + 2y^2x + 3y^3$, so that $Q_1 = x - 2y$ and $R_1 = 2y^2x + 3y^3$. Thus,

$$x^5 + x^3y^2 + x^2y^3 + y^5 = (x - 2y)(x^2 + xy + y^2)^2 + (2xy^2 + 3y^3)(x^2 + xy + y^2) - 2xy^4$$

Problem 5. Express $2x^3 - 8x^2 + x + 6$ in terms of $x - 3$.

Solution. Using synthetic division, $2x^3 - 8x^2 + x + 6 = (2x^2 - 2x - 5)(x - 3) - 9$, so that $Q = 2x^2 - 2x - 5$ and $R = -9$. And $2x^2 - 2x - 5 = (2x + 4)(x - 3) + 7$, so that $Q_1 = 2x + 4$ and $R_1 = 7$. Also $2x + 4 = 2(x - 3) + 10$, so that $Q_2 = 2$ and $R_2 = 10$. Thus,

$$2x^3 - 8x^2 + x + 6 = 2(x - 3)^3 + 10(x - 3)^2 + 7(x - 3) - 9$$

Problem 6. Express $x^5 + 3x^4 - 6x^3 + 2x^2 - 3x + 7$ in terms of $x + 2$.

Solution. Using synthetic division, we find $x^5+3x^4-6x^3+2x^2-3x+7 = (x^4+x^3-8x^2+18x-39)(x+2)+85$, so that $Q = x^4+x^3-8x^2+18x-39$ and $R = 85$. And $x^4+x^3-8x^2+18x-39 = (x^3-x^2-6x+30)(x+2)-99$, so that $Q_1 = x^3 - x^2 - 6x + 30$ and $R_1 = -99$. Also $x^3 - x^2 - 6x + 30 = (x^2 - 3x)(x + 2) + 30$, so that $Q_2 = x^2 - 3x$ and $R_2 = 30$. Also $x^2 - 3x = (x - 5)(x + 2) + 10$, so that $Q_3 = x - 5$ and $R_3 = 10$. Finally, $x - 5 = 1(x + 2) - 7$, so that $Q_4 = 1$ and $R_4 = -7$. Thus,

$$x^5 + 3x^4 - 6x^3 + 2x^2 - 3x + 7 = (x + 2)^5 - 7(x + 2)^4 + 10(x + 2)^3 + 30(x + 2)^2 - 99(x + 2) + 85$$

Problem 7. Express $x^3 + 9x^2 + 27x$ in terms of $x + 3$.

Solution. Using synthetic division, we find $x^3+9x^2+27x = (x^2+6x+9)(x+3)-27$, so that $Q = x^2+6x+9$ and $R = -27$. And $x^2 + 6x + 9 = (x + 3)(x + 3)$, so that $Q_1 = x + 3$ and $R_1 = 0$. Thus,

$$x^3 + 9x^2 + 27x = (x + 3)(x + 3)^2 - 27 = (x + 3)^3 - 27$$

Problem 8. Express $x^3 + 3x^2 + x - 1$ in terms of $x + 1$.

Solution. Using synthetic division, we find $x^3 + 3x^2 + x - 1 = (x^2 + 2x - 1)(x + 1)$, so that $Q_1 = x^2 + 2x - 1$ and $R_1 = 0$. And $x^2 + 2x - 1 = (x + 1)(x + 1) - 2$, so that $Q_2 = x + 1$ and $R_2 = -2$. Also $x + 1 = 1(x + 1)$, so that $Q_3 = 1$ and $R_3 = 0$. Thus,

$$x^3 + 3x^2 + x - 1 = (x + 1)^3 - 2(x + 1)$$

6.2 Factoring, Exercise XVI (p. 180)

Factor the following expressions.

Problem 1. $6x^4y^3z^2 - 12x^2y^4z + 8x^2y^3$

Solution. $6x^4y^3z^2 - 12x^2y^4z + 8x^2y^3 = 2x^2y^3(3x^2z^2 - 6yz + 4)$

Problem 2. $2n^2 + (n - 3)n$

Solution. $2n^2 + (n - 3)n = 2n^2 + n^2 - 3n = 3n(n - 3)$

Problem 3. $ab - a + b - 1$

Solution. $ab - a + b - 1 = a(b - 1) + b - 1 = (a + 1)(b - 1)$

Problem 4. $mx - nx - mn + n^2$

Solution. $mx - nx - mn + n^2 = m(x - n) + n(n - x) = (m - n)(x - n)$

Problem 5. $3xy - 2x - 12y + 8$

Solution. $3xy - 2x - 12y + 8 = 3y(x - 4) + 2(4 - x) = (3y - 2)(x - 4)$

Problem 6. $10xy + 5y^2 + 6x + 3y$

Solution. $10xy + 5y^2 + 6x + 3y = 2x(3 + 5y) + (5y + 3)y = (5y + 3)(y + 2x)$

Problem 7. $x^3y^2 - x^2y^3 + 2x^2y - 2xy^2$

Solution. $x^3y^2 - x^2y^3 + 2x^2y - 2xy^2 = x^2y^2(x - y) + 2xy(x - y) = (x - y)(x^2y^2 + 2xy) = xy(xy + 2)(x - y)$

Problem 8. $x^4 + x^3 + x^2 + x$

Solution. $x^4 + x^3 + x^2 + x = x(x^3 + x^2 + x + 1) = x[x^2(x + 1) + x + 1] = x(x + 1)(x^2 + 1)$

Problem 9. $ac + bd - (bc + ad)$

Solution. $ac + bd - (bc + ad) = a(c - d) + b(d - c) = (a - b)(c - d)$

Problem 10. $a^2c - abd - abc + a^2d$

Solution. $a^2c - abd - abc + a^2d = a^2(c + d) - ab(d + c) = a(a - b)(c + d)$

Problem 11. $ad + ce + bd + ae + cd + be$

Solution. $ad + ce + bd + ae + cd + be = e(c + a + b) + d(a + b + c) = (d + e)(a + b + c)$

Problem 12. $a^2 + cd - ab - bd + ac + ad$

Solution. $a^2 + cd - ab - bd + ac + ad = a(a - b + c) + d(c - b + a) = (a + d)(a - b + c)$

6.3 Factoring, Exercise XVIII (p. 185)

Factor the following expressions.

Problem 1. $x^4 - x^3 + x - 1$

Solution. $x^4 - x^3 + x - 1 = x^3(x - 1) + x - 1 = (x^3 + 1)(x - 1) = (x + 1)(x^2 - x + 1)(x - 1)$

Problem 2. $x^5 - x^3 - 8x^2 + 8$

Solution. $x^5 - x^3 - 8x^2 + 8 = x^3(x^2 - 1) - 8x(x^2 - 1) = (x^3 - 8)(x^2 - 1) = (x - 2)(x^2 + 2x + 4)(x - 1)(x + 1)$

Problem 3. $x^4 - 2x^3 + 2x - 1$

Solution. $x^4 - 2x^3 + 2x - 1 = x^4 - 1 - 2x(x^2 - 1) = (x^2 - 1)(x^2 + 1) - 2x(x^2 - 1) = (x^2 - 1)(x^2 + 1 - 2x) = (x^2 - 1)(x - 1)^2 = (x - 1)^3(x + 1)$

Problem 4. $x^3 - 7x^2 - 4x + 28$

Solution. $x^3 - 7x^2 - 4x + 28 = x(x^2 - 4) - 7(x^2 - 4) = (x - 7)(x - 2)(x + 2)$

Problem 5. $x^6 - x^4y^2 - x^2y^4 + y^6$

Solution. $x^6 - x^4y^2 - x^2y^4 + y^6 = x^2(x^4 - y^4) + y^2(y^4 - x^4) = (x^2 - y^2)(x^2 - y^2)(x^2 + y^2) = (x^2 - y^2)^2(x^2 + y^2) = (x - y)^2(x + y)^2(x^2 + y^2)$

Problem 6. $x^3 + 2x^2 + 3x + 2$

Solution. $x^3 + 2x^2 + 3x + 2 = (x^2 + 2x + 1) + x^3 + x^2 + x + 1 = (x + 1)^2 + x^2(x + 1) + x + 1 = (x + 1)[x + 1 + x^2 + 1] = (x + 1)(x^2 + x + 2)$

Problem 7. $x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 1$

Solution. $x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 1 = x^5 + 1 + 2x(x^3 + 1) + 3x^2(x + 1) =$
$(x+1)(x^4 - x^3 + x^2 - x + 1) + 2x(x+1)(x^2 - x + 1) + 3x^2(x+1) = (x+1)[x^4 - x^3 + x^2 - x + 1 + 2x^3 - 2x^2 + 2x + 3x^2] =$
$(x + 1)(x^4 + x^3 + 2x^2 + x + 1)$

Problem 8. $x^4 + 4x^3 + 10x^2 + 12x + 9$

Solution. $x^4+4x^3+10x^2+12x+9=(x^2+2x)^2+6x^2+12x+9=(x^2+2x)^2+6(x^2+2)+9=(x^2+2x+3)^2$

6.4 Factoring, Exercise XIX (p. 190)

Factor the following expressions.

Problem 1. $x^2-14x+48$

Solution $x^2-14x+48=(x-6)(x-8)$

Problem 2. $x^2-21x-120$

Soilution. $x^2-21x-120=\left(x-\frac{21}{2}\right)^2-\frac{441}{4}-\frac{480}{4}=\left(x-\frac{21}{2}\right)^2-\frac{921}{4}=\left(x-\frac{21}{2}-\frac{\sqrt{921}}{2}\right)\left(x-\frac{21}{2}+\frac{\sqrt{921}}{2}\right)$

Problem 3. $5x^2-53x-22$

Solution. $5x^2-53x-22=\frac{1}{5}\left[(5x)^2-53(5x)-110\right]=\frac{1}{5}\left[(5x-55)(5x+2)\right]=(x-11)(5x+2)$

Problem 4. $16x^2+64x+63$

Solution. $16x^2+64x+63=(4x)^2+16(4x)+63=(4x+8)^2-64+63=(4x+8-1)(4x+8+1)=(4x+7)(4x+9)$

Problem 5. $54x^2-21x+2$

Solution. $54x^2-21x+2=(6)(9)x^2-(3)(7)x+2=6(3x)^2-7(3x)+2=6\left[(3x)^2-\frac{7}{6}(3x)+\frac{2}{6}\right]=6\left[\left(3x-\frac{7}{12}\right)^2-\frac{49}{144}+\frac{48}{144}\right]=6\left[\left(3x-\frac{7}{12}-\frac{1}{12}\right)\left(3x-\frac{7}{12}+\frac{1}{12}\right)\right]=6\left[\left(3x-\frac{8}{12}\right)\left(3x-\frac{6}{12}\right)\right]=6\left[\left(3x-\frac{2}{3}\right)\left(3x-\frac{1}{2}\right)\right]=(9x-2)(6x-1)$

Problem 6. $12x^2+20xy-8y^2$

Solution. $12x^2+20xy-8y^2=3(2x)^2+10y(2x)-8y^2=3\left[(2x)^2+\frac{10}{3}y(2x)-\frac{8}{3}y^2\right]=3\left[\left(2x+\frac{5}{3}y\right)^2-\frac{25}{9}y^2-\frac{24}{9}y^2\right]=3\left[\left(2x+\frac{5}{3}y\right)^2-\left(\frac{7}{3}\right)^2y^2\right]=3\left[\left(2x+\frac{5}{3}y-\frac{7}{3}y\right)\left(2x+\frac{5}{3}y+\frac{7}{3}y\right)\right]=3\left[\left(2x-\frac{2}{3}y\right)(2x+4y)\right]=(6x-2y)(2x+4y)$

Problem 7. x^4-13x^2+36

Solution $x^4-13x^2+36=(x^2-9)(x^2-4)=(x-3)(x+3)(x-2)(x+2)$

Problem 8. $x^3y-3x^2y^2-18xy^3$

Solution. $x^3y-3x^2y^2-18xy^3=xy[x^2-3xy-18y^2]=xy\left[\left(x-\frac{3}{2}y\right)^2-\frac{9}{4}y^2-\frac{72}{4}y^2\right]=xy\left[\left(x-\frac{3}{2}y\right)^2-\left(\frac{9}{2}y^2\right)^2\right]=xy(x-6y)(x+3y)$

Problem 9. x^2-3x+3

Solution. $x^2-3x+3=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+\frac{12}{4}=\left(x-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=\left(x-\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)\left(x-\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)$

Problem 10. $3x^2+2x-3$

Solution. $3x^2+2x-3=\frac{1}{3}[(3x)^2+2(3x)-9]=\frac{1}{3}[(3x+1)^2-1-9]=\frac{1}{3}(3x+1-\sqrt{10})(3x+1+\sqrt{10})$

Problem 11. $x^2-4xy-2y^2$

Solution. $x^2-4xy-2y^2=(x-2y)^2-4y^2-2y^2=(x-2y-\sqrt{6}y)(x-2y+\sqrt{6}y)$

Problem 12. $x^2 - 6ax - 9b^2 - 18ab$

Solution. $x^2 - 6ax - 9b^2 - 18ab = (x-3a)^2 - 9a^2 - 9b^2 - 18ab = (x-3a)^2 - 9(a+b)^2 =$
$(x-3a-3a-3b)(x-3a+3a+3b) = (x-6a-3b)(x+3b)$

Problem 13. $abx^2 - (a^2+b^2)x - (a^2-b^2)$

Solution. $abx^2 - (a^2+b^2)x - (a^2-b^2) = \frac{1}{ab}[(abx)^2 - (a^2+b^2)(abx) - ab(a^2-b^2)] \equiv \frac{1}{ab}[(abx+l)(abx+m)]$, where

$$\begin{aligned} l+m &= -(a^2+b^2) \\ lm &= -ab(a^2-b^2) = -ab(a-b)(a+b) = (ab-b^2)(-a^2-ab), \end{aligned}$$

which gives $l = ab - b^2$ and $m = -a^2 - ab$. Thus,

$abx^2 - (a^2+b^2)x - (a^2-b^2) = \dfrac{1}{ab}[(abx+l)(abx+m)] = \dfrac{1}{ab}(abx+ab-b^2)(abx-a^2-ab)] =$
$(ax+a-b)(bx-a-b)$

Problem 14. $x^2 + bd + dx + bx + cx^2 + cdx$

Solution. $x^2 + bd + dx + bx + cx^2 + cdx = [x^2 + (b+d)x + bd] + cx(x+d) = (x+b)(x+d) + cx(x+d) =$
$(x+cx+b)(x+d) = [(1+c)x+b](x+d)$

Problem 15. $x^2 - 8xy + 15y^2 + 2x - 4y - 3$

Solution.

$$\begin{aligned} x^2 - 8xy + 15y^2 + 2x - 4y - 3 &= (x-4y)^2 - 16y^2 + 15y^2 + 2x - 4y - 3 \\ &= (x-4y)^2 - y^2 + 2x - 4y - 3 \\ &= (x-4y-y)(x-4y+y) + 2x - 4y - 3 \\ &= (x-5y)(x-3y) + 2x - 4y - 3 \\ &\equiv (x-5y+l)(x-3y+m) \\ &= (x-5y)(x-3y) + (l+m)x + (-3l-5m)y + lm, \end{aligned}$$

which requires

$$l+m=2, \quad -3l-5m=-4, \quad lm=-3 \quad \Longrightarrow \quad l=3, \quad m=-1$$

Thus,

$x^2 - 8xy + 15y^2 + 2x - 4y - 3 = (x-5y+3)(x-3y-1).$

Problem 16. $x^2 + 3xy + 2y^2 + 3zx + 5yz + 2z^2$

Solution

$$\begin{aligned} x^2 + 3xy + 2y^2 + 3zx + 5yz + 2z^2 &= (x+y)(x+2y) + 3zx + 5yz + 2z^2 \\ &\equiv (x+y+lz)(x+2y+mz) \\ &= (x+y)(z+2y) + (l+m)xz + (2l+m)yz + lmz^2, \end{aligned}$$

which requires

$$l+m=3, \quad 2l+m=5, \quad lm=2 \quad \Longrightarrow \quad l=2, \quad m=1$$

Thus,

$x^2 + 3xy + 2y^2 + 3zx + 5yz + 2z^2 = (x+y+2z)(x+2y+z).$

6.5 Factoring Polynomials and Solving Equations, Exercise XX (p. 194)

Factor the following expressions.

Problem 1. $x^3 - 7x + 6$

Solution. $x - 1$ is a factor since the sum of the coefficients is zero, and $x^3 - 7x + 6 = (x - 1)(x^2 + x - 6) = (x - 1)(x + 3)(x - 2)$.

Problem 2. $x^3 + 6x^2 + 11x + 6$

Solution. $x + 1$ is a factor since $f(-1) = 0$. Possible factors are ± 1, ± 2, ± 3, ± 6 and we find $x^3 + 6x^2 + 11x + 6 = (x + 1)(x^2 + 5x + 6) = (x + 1)(x + 2)(x + 3)$.

Problem 3. $x^4 - 10x^3 + 35x^2 - 50x + 24$

Solution. $x - 1$ is a factor since the sum of the coefficients is zero, and $x^4 - 10x^3 + 35x^2 - 50x + 24 = (x - 1)(x^3 - 9x^2 + 26x - 24) = (x - 1)(x - 2)(x^2 - 7x + 12) = (x - 1)(x - 2)(x - 3)(x - 4)$.

Problem 4. $x^4 - 2x^2 + 3x - 2$

Solution. $x - 1$ is a factor since the sum of the coefficients is zero. Possible factors are ± 1, ± 2, and we find $x^4 - 2x^2 + 3x - 2 = (x - 1)(x^3 + x^2 - x + 1) = (x - 1)(x + 2)(x^2 - x + 1)$.

Problem 5. $6x^3 - 13x^2 - 14x - 3$

Solution. Possible factors are of the form $\alpha x - \beta$, where α must be a factor of 6: ± 1, ± 2, ± 3, ± 6, and β must be a factor of 3: ± 1, ± 3, so that β/α: ± 1, $\pm 1/2$, $\pm 3/2$. We find $6x^3 - 13x^2 - 14x - 3 = (x - 3)(6x^2 + 5x + 1)$. Although we can apply the quadratic formula to the quadratic factor, we will factor it directly as follows.

$$
\begin{aligned}
6x^2 + 5x + 1 &= \frac{1}{6}[(6x)^2 + 5(6x) + 6] \\
&= \frac{1}{6}[(6x + \frac{5}{2})^2 - \frac{25}{4} + \frac{24}{4}] \\
&= \frac{1}{6}[(6x + \frac{5}{2})^2 - \frac{1}{4}] \\
&= \frac{1}{6}(6x + \frac{5}{2} - \frac{1}{2})(6x + \frac{5}{2} + \frac{1}{2}) \\
&= \frac{1}{6}(6x + 2)(6x + 3) \\
&= (3x + 1)(2x + 1)
\end{aligned}
$$

Therefore, $6x^3 - 13x^2 - 14x - 3 = (x - 3)(3x + 1)(2x + 1)$.

Problem 6. $2x^3 - 5x^2y - 2xy^2 + 2y^3$

Solution.

$$
\begin{aligned}
2x^3 - 5x^2y - 2xy^2 + 2y^3 &= (x - \frac{y}{2})(2x^2 - 4xy - 4y^2) \\
&= (2x - y)(x^2 - 2xy - 2y^2) \\
&= (2x - y)[(x - y)^2 - y^2 - 2y^2] \\
&= (2x - y)[(x - y)^2 - 3y^2] \\
&= (2x - y)(x - y - \sqrt{3}y)(x - y + \sqrt{3}y) \\
&= (2x - y)[x - (1 + \sqrt{3})y][x - (1 - \sqrt{3})y]
\end{aligned}
$$

Problem 7. $2x^4 - x^3 - 9x^2 + 13x - 5$

Solution. We know that $x - 1$ is a factor since the coefficients sum to zero. This gives $2x^4 - x^3 - 9x^2 + 13x - 5 = (x-1)(2x^3 + x^2 - 8x + 5)$. The cubic also has $x - 1$ as a factor, which gives $2x^4 - x^3 - 9x^2 + 13x - 5 = (x-1)^2(2x^2 + 3x - 5)$. Once again, $x - 1$ is a factor of the quadratic, and we are left with $2x^4 - x^3 - 9x^2 + 13x - 5 = (x-1)^3(2x+5)$.

Problem 8. $4x^6 - 41x^4 + 46x^2 - 9$

Solution. Let $y = x^2$, which gives $4y^3 - 41y^2 + 46y - 9$. We know that $y - 1$ is a factor since the coefficients sum to zero, and we have $4y^3 - 41y^2 + 46y - 9 = (y-1)(4y^2 - 37y + 9) = (y-1)(4y-1)(y-9)$. Therefore, $4x^6 - 41x^4 + 46x^2 - 9 = (x^2-1)(4x^2-1)(x^2-9) = (x-1)(x+1)(2x-1)(2x+1)(x-3)(x+3)$.

Problem 9. $6x^5 + 19x^4 + 22x^3 + 23x^2 + 16x + 4$

Solution. Synthetic division gives $6x^5+19x^4+22x^3+23x^2+16x+4 = (x+\frac{2}{3})(6x^4+15x^3+12x^2+15x+6) = (x+\frac{2}{3})(x+2)(6x^3+3x^2+6x+3) = (x+\frac{2}{3})(x+2)(x+\frac{1}{2})(6x^2+6) = (3x+2)(x+2)(2x+1)(x^2+1)$.

Problem 10. $5x^6 - 7x^5 - 8x^4 - x^3 + 7x^2 + 8x - 4$

Solution. The sum of the coefficients equals zero, so $x - 1$ is a factor.

$$\begin{aligned}
5x^6 - 7x^5 - 8x^4 - x^3 + 7x^2 + 8x - 4 &= (x-1)(5x^5 - 2x^4 - 10x^3 - 11x^2 - 4x + 4) \\
&= (x-1)(x-2/5)(5x^4 - 10x^2 - 15x - 10) \\
&= (x-1)(5x-2)(x^4 - 2x^2 - 3x - 2) \\
&= (x-1)(5x-2)(x+1)(x^3 - x^2 - x - 2) \\
&= (x-1)(5x-2)(x+1)(x-2)(x^2 + x + 1)
\end{aligned}$$

Solve the following equations.

Problem 11. $x^2 - 4x - 12 = 0$

Solution. $x^2 - 4x - 12 = 0 \implies (x-6)(x+2) = 0 \implies x = -2,\ 6.$

Problem 12. $6x^2 - 7x + 2 = 0$

Solution. $6x^2 - 7x + 2 = 0 \implies (3x-2)(2x-1) = 0 \implies x = 2/3,\ 1/2.$

Problem 13. $x^2 - 5x = 14$

Solution. $x^2 - 5x = 14 \implies x^2 - 5x - 14 = 0 \implies (x-7)(x+2) = 0 \implies x = 7,\ -2.$

Problem 14. $x^2 + 6x = 2$

Solution. $x^2 + 6x = 2 \implies x^2 + 6x - 2 = 0 \implies (x+3)^2 - 11 = 0 \implies x = -3 \pm \sqrt{11}.$

Problem 15. $x^3 - 9x^2 + 26x = 24$

Solution.

Problem 16. $x^4 + 2x^3 - 4x^2 - 2x + 3 = 0$

Solution. The sum of the coefficients equals zero, so $x-1$ is a factor.

$$\begin{aligned} x^4+2x^3-4x^2-2x+3=0 &\implies (x-1)(x^3+3x^2-x-3)=0 \\ &\implies (x-1)(x-1)(x^2+4x+3)=0 \\ &\implies (x-1)^2(x+3)(x+1)=0 \implies x=1,\,1,\,-3,\,-1 \end{aligned}$$

Problem 17. $x^3-1=0$

Solution.

$$\begin{aligned} x^3-1=0 &\implies (x-1)(x^2+x+1)=0 \\ &\implies (x-1)[(x+1/2)^2+3/4]=0 \\ &\implies (x-1)(x+1/2-\sqrt{3}i/2)(x+1/2+\sqrt{3}i/2)=0 \implies x=1,\,(1\pm\sqrt{3}i)/2 \end{aligned}$$

Problem 18. $10x^3-9x^2-3x+2=0$

Solution. The sum of the coefficients equals zero, so $x-1$ is a factor.

$$\begin{aligned} 10x^3-9x^2-3x+2=0 &\implies (x-1)(10x^2+x-2) \\ &\implies (x-1)(5x-2)(2x+1)=0 \implies x=1,\,2/5,\,-1/2 \end{aligned}$$

6.6 Factoring, Exercise XXI (p. 195)

The followiing expressions can be factored by methods explained in the present chapter. Carry the factorization as far as possible without introducing irrational or imaginary coefficients.

Problem 1. $6xy+15x-4y-10$

Solution. $6xy+15x-4y-10=(3x+l)(2y+m)=6xy+3mx+2ly+lm=(3x-2)(2y+5)$

Problem 2. $a^2bc-ac^2d-ab^2d+bcd^2$

Solution. $a^2bc-ac^2d-ab^2d+bcd^2=ab(ac-bd)-cd(ac-bd)=(ab-cd)(ac-bd)$

Problem 3. $a^3(a-b)+b^3(b-a)$

Solution. $a^3(a-b)+b^3(b-a)=(a-b)(a^3-b^3)=(a-b)(a-b)(a^2+ab+b^2)=(a-b)^2(a^2+ab+b^2)$

Problem 4. a^5-81ab^4

Solution. $a^5-81ab^4=a(a^4-9^2b^4)=a(a^2-9b^2)(a^2+9b^2)=a(a-3b)(a+3b)(a^2+9b^2)$

Problem 5. $a^4b-a^2b^3+a^3b^2-ab^4$

Solution $a^4b-a^2b^3+a^3b^2-ab^4=ab[a^3-ab^2+a^2b-b^3]=ab(a+b)(a^2-b^2)=ab(a+b)^2(a-b)$

Problem 6. $3abx^2-6axy+bxy-2y^2$

Solution.

$$\begin{aligned}
3abx^2 - 6axy + bxy - 2y^2 &= 3abx^2 + (b-6a)xy - 2y^2 \\
&= \frac{1}{3ab}\left[(3abx + \frac{b-6a}{2}y)^2 - \frac{(b-6a)^2}{4}y^2 - \frac{24ab}{4}y^2\right] \\
&= \frac{1}{3ab}\left[(3abx + \frac{b-6a}{2}y)^2 - \frac{(b^2 - 12ab + 36a^2 + 24ab)}{4}y^2\right] \\
&= \frac{1}{3ab}\left[(3abx + \frac{b-6a}{2}y)^2 - \frac{(b+6a)^2}{4}y^2\right] \\
&= \frac{1}{3ab}\left[\left(3abx + \frac{b-6a}{2}y - \frac{b+6a}{2}y\right)\left(3abx + \frac{b-6a}{2}y + \frac{b+6a}{2}y\right)\right] \\
&= \frac{1}{3ab}[(3abx - 6ay)(3abx + by)] = (bx - 2y)(3ax + y)
\end{aligned}$$

Problem 7. $3x^6 - 192y^6$

Solution. $3x^6 - 192y^6 = 3(x^6 - 64y^6) = 3[(x^3)^2 - (2^3y^3)^2] = 3(x^3 - 8y^3)(x^3 + 8y^3) =$ $3(x^3 - 8y^3)(x + 2y)(x^2 - 2xy + 4y^2) = 3(x + 2y)(x^2 - 2xy + 4y^2)(x - 2y)(x^2 + 2xy + 4y^2)$

Problem 8. $(x^2 + x)^3 - 8$

Solution. $(x^2 + x)^3 - 8 = (x^2 + x)^3 - 2^3 = (x^2 + x - 2)[(x^2 + x)^2 + 2(x^2 + x) + 4] =$ $(x - 1)(x + 2)(x^4 + 2x^3 + x^2 + 2x^2 + 2x + 4) = (x - 1)(x + 2)(x^4 + 2x^3 + 3x^2 + 2x + 4)$

Problem 9. $64x^6y^3 - y^{15}$

Solution. $64x^6y^3 - y^{15} = y^3(2^6x^6 - y^{12}) = y^3[(2^3x^3)^2 - (y^6)^2] = y^3[(2x)^3 - (y^2)^3][(2x)^3 + (y^2)^3] =$ $y^3(2x - y^2)(4x^2 + 2xy^2 + y^4)(2x + y^2)(4x^2 - 2xy^2 + y^4)$

Problem 10. $x^2 - (a - b)x - ab$

Solution. $x^2 - (a - b)x - ab = (x - a)(x + b)$

Problem 11. $x^{2n} - 3x^n - 18$

Solution. $x^{2n} - 3x^n - 18 = (x^n - 6)(x^n + 3)$

Problem 12. $x - x^2 + 42$

Solution. $x - x^2 + 42 = -(x^2 - x - 42) = -(x - 7)(x + 6)$

Problem 13. $3x^4 + 3x^3 - 24x - 24$

Solution. $3x^4 + 3x^3 - 24x - 24 = (x + 1)(3x^3 - 24) = 3(x + 1)(x^3 - 2^3) = 3(x + 1)(x^2 + 2x + 4)(x - 2)$

Problem 14. $x^5 - 9x^3 + 8x^2 - 72$

Solution. $x^5 - 9x^3 + 8x^2 - 72 = (x^3 + 8)(x^2 - 9) = (x + 2)(x^2 - 2x + 4)(x - 3)(x + 3)$

Problem 15. $2xc - a^2 + x^2 - 2ab + c^2 - b^2$

Solution. $2xc - a^2 + x^2 - 2ab + c^2 - b^2 = x^2 + 2cx + c^2 - a^2 - 2ab - b^2 = (x + c)^2 - (a + b)^2 =$ $(x + c - a - b)(x + c + a + b)$

Problem 16. $x^2(x^2 - 20) + 64$

Solution. $x^2(x^2-20)+64 = x^4-20x^2+64 = (x^2-10)^2-100+64 = (x^2-10)^2-36 = (x^2-10-6)(x^2-10+6) = (x^2-16)(x^2-4) = (x-4)(x+4)(x-2)(x+2)$

Problem 17. $a^2-2ab+b^2-5a+5b+6$

Solution. $a^2-2ab+b^2-5a+5b+6 = (a-b)^2-5(a-b)+6 = (a-b-3)(a-b-2)$

Problem 18. $x^4-10x^2y^2+9y^4$

Solution. $x^4-10x^2y^2+9y^4 = (x^2-5y^2)^2-25y^4+9y^4 = (x^2-5y^2)^2-16y^4 = (x^2-5y^2-4y^2)(x^2-5y^2+4y^2) = (x^2-9y^2)(x^2-y^2) = (x-3y)(x+3y)(x-y)(x+y)$

Problem 19. $6x^2-7xy-5y^2-4x-2y$

Solution.

$$\begin{aligned}
6x^2-7xy-5y^2-4x-2y &= \frac{1}{6}[(6x)^2-7y(6x)-30y^2-24x-12y] \\
&= \frac{1}{6}[(6x-\frac{7}{2}y)^2-\frac{49}{4}y^2-\frac{120}{4}y^2-24x-12y] \\
&= \frac{1}{6}[(6x-\frac{7}{2}y-\frac{13}{2}y)(6x-\frac{7}{2}+\frac{13}{2}y)-24x+12y] \\
&= \frac{1}{6}[(6x-10y)(6x+3y)-24x-12y] \\
&= \frac{1}{6}[(6x-10y)(6x+3y)-4(6x+3y] \\
&= \frac{1}{6}[(6x-10y-4)(6x+3y)] \\
&= (3x-5y-2)(2x+y)
\end{aligned}$$

Problem 20. $x^4-(a^2+b^2)x^2+a^2b^2$

Solution. $x^4-(a^2+b^2)x^2+a^2b^2 = (x^2-a^2)(x^2-b^2) = (x-a)(x+a)(x-b)(x+b)$

Problem 21. $4(xz+uy)^2-(x^2-y^2+z^2-u^2)^2$

Solution.

$$\begin{aligned}
4(xz+uy)^2-(x^2-y^2+z^2-u^2)^2 &= [2(xz+uy)-(x^2-y^2+z^2-u^2)][2(xz+uy)+(x^2-y^2+z^2-u^2)] \\
&= -[x^2-y^2+z^2-u^2-2xz-2uy][x^2-y^2+z^2-u^2+2xz+2uy] \\
&= -[(x-z)^2-(y+u)^2][(x+z)^2-(y-u)^2] \\
&= -(x-z-y-u)(x-z+y+u)(x+z-y+u)(x+z+y-u)
\end{aligned}$$

Problem 22. $14x^2+19x-3$

Solution. $14x^2+19x-3 = \frac{1}{14}[(14x)^2+19(4x)-42] = \frac{1}{14}[(14x+\frac{19}{2})^2-\frac{361}{4}-\frac{168}{4}] =$

$\frac{1}{14}(14x+\frac{19}{2}-\frac{23}{2})(14x+\frac{19}{2}+\frac{23}{2}) = \frac{1}{14}(14x-2)(14x+21) = (7x-1)(2x+3)$

Problem 23. $1+19y-66y^2$

Solution. $1+19y-66y^2 = -[66y^2-19y-1] = -\frac{1}{66}[(66y)^2-19(66y)-66] = -\frac{1}{66}[(66y-\frac{19}{2})^2-\frac{361}{4}-\frac{264}{4}] =$

$-\frac{1}{66}[(66y-\frac{19}{2}-\frac{25}{2})(66y-\frac{19}{2}+\frac{25}{2})] = -\frac{1}{66}[(66y-22)(66y+3)] = -(3y-1)(22y+1) = (1-3y)(1+22y)$

Problem 24. $xy^3+55x^2y^2+204x^3y$

Solution. $xy^3 + 55x^2y^2 + 204x^3y = xy[y^2 + 55xy + 204x^2] = xy[(y + \frac{55}{2}x)^2 - \frac{3025}{4}x^2 + \frac{816}{4}x^2] =$

$xy(y + \frac{55}{2} - \frac{47}{2}x)(y + \frac{55}{2}x + \frac{47}{2}x) = xy(y + 4x)(y + 51x)$

Problem 25. $a^4 - 18a^2b^2c^2 + 81b^4c^4$

Solution. $a^4 - 18a^2b^2c^2 + 81b^4c^4 = (a^2 - 9b^2c^2)^2 - 81b^4c^4 + 81b^4c^4 = (a - 3bc)^2(a + 3bc)^2$

Problem 26. $(x^2 - 7x)^2 + 6x^2 - 42x$

Solution. $(x^2 - 7x)^2 + 6x^2 - 42x = x^4 - 14x^3 + 49x^2 + 6x^2 - 42x = x(x^3 - 14x^2 + 55x - 42) =$
$x(x - 1)(x^2 - 13x + 42) = x(x - 1)(x - 7)(x - 6)$

Problem 27. $8(x + y)^3 - 27(x - y)^3$

Solution. $8(x + y)^3 - 27(x - y)^3 = [2(x + y) - 3(x - y)][4(x + y)^2 + 2(x + y)3(x - y) + 9(x - y)^2] =$
$[2x + 2y - 3x + 3y][4x^2 + 8xy + 4y^2 + 6x^2 - 6y^2 + 9x^2 - 18xy + 9y^2] = (-x + 5y)(19x^2 - 10xy + 7y^2)$

Problem 28. $(x - 2y)x^3 - (y - 2x)y^3$

Solution. $(x - 2y)x^3 - (y - 2x)y^3 = x^4 - y^4 - 2xy(x^2 - y^2) = (x^2 - y^2)(x^2 + y^2) - 2xy(x^2 - y^2) =$
$(x^2 - y^2)(x^2 - 2xy + y^2) = (x - y)(x + y)(x - y)^2 = (x + y)(x - y)^3$

Problem 29. $x^2 + a^2 - bx - ab + 2ax$

Solution. $x^2 + a^2 - bx - ab + 2ax = x^2 + (2a - b)x + a^2 - ab = (x + l)(x + m)$, which requires $l + m = 2a - b$, $lm = a(a - b) \implies l = a$, $m = a - b$, and therefore, $x^2 + a^2 - bx - ab + 2ax = (x + a)(x + a - b)$.

Problem 30. $x^5 - y^5 - (x - y)^5$

Solution. $x^5 - y^5 - (x - y)^5 = (x - y)[x^4 + x^3y + x^2y^2 + xy^3 + y^4 - (x - y)^4] =$
$(x - y)[x^4 + x^3y + x^2y^2 + xy^3 + y^4 - (x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4)] = (x - y)[5x^3y - 5x^2y^2 + 5xy^3] =$
$5xy(x - y)(x^2 - xy + y^2)$

Problem 31. $x^5 - x^4 - 2x^3 + 2x^2 + x - 1$

Solution. $x^5 - x^4 - 2x^3 + 2x^2 + x - 1 = (x^5 - 1) - x(x^3 - 1) - 2x^2(x - 1) =$
$(x-1)(x^4+x^3+x^2+x+1)-x(x-1)(x^2+x+1)-2x^2(x-1) = (x-1)[x^4+x^3+x^2+x+1-x^3-x^2-x-2x^2] =$
$(x - 1)[x^4 - 2x^2 + 1] = (x - 1)(x^2 - 1)^2 = (x - 1)(x - 1)^2(x + 1)^2 = (x - 1)^3(x + 1)^2$

Problem 32. $b^4 + b^2 + 1$

Solution. $b^4 + b^2 + 1 = b^4 + 2b^2 + 1 - b^2 = (b^2 + 1)^2 - b^2 = (b^2 + 1 - b)(b^2 + 1 + b) = (b^2 - b + 1)(b^2 + b + 1)$

Problem 33. $2x^2 + 7xy + 3y^2 + 9x + 2y - 5$

Solution. $2x^2+7xy+3y^2+9x+2y-5 = (2x+y+l)(x+3y+m) = 2x^2+7xy+3y^2+(l+2m)x+(3l+m)y+lm$ requires $l+2m = 9$, $3l+m = 2$, $lm = -5 \implies l = -1$, $m = 5$, and therefore $2x^2+7xy+3y^2+9x+2y-5 =$
$(2x + y - 1)(x + 3y + 5)$.

Problem 34. $a^4 + 4$

Solution. $a^4 + 4 = a^4 + 4a^2 + 4 - 4a^2 = (a^2 + 2)^2 - 4a^2 = (a^2 + 2 - 2a)(a^2 + 2 + 2a) =$
$(a^2 - 2a + 2)(a^2 + 2a + 2)$

Problem 35. $x^2 - xy - 2y^2 + 4xz - 5yz + 3z^2$

Solution. $x^2 - xy - 2y^2 + 4xz - 5yz + 3z^2 = (x - 2y)(x + y) + 4xz - 5yz + 3z^2 = (x - 2y + lz)(x + y + mz) =$

$(x-2y)(x+y)+(l+m)xz+(l-2m)yz+lmz^2$, which requires $l+m=4$, $l-2m=-5$, $lm=3 \implies m=3$, $l=1$, and therefore, $x^2-xy-2y^2+4xz-5yz+3z^2=(x-2y+z)(x+y+3z)$.

Problem 36. $4a^4+3a^2b^2+9b^4$

Solution. $4a^4+3a^2b^2+9b^4=(2a^2+3b^2)^2-9a^2b^2=(2a^2+3b^2-3ab)(2a^2+3b^2+3ab)$

Problem 37. $x^2-8ax-40ab-25b^2$

Solution. $x^2-8ax-40ab-25b^2=(x-4a)^2-16a^2-40ab-25b^2=(x-4a)^2-(4a+5b)^2=$ $(x-4a-4a-5b)(x-4a+4a+5b)=(x-8a-5b)(x+5b)$

Problem 38. x^8+x^4+1

Solution. $x^8+x^4+1=(x^4+1)^2-x^4=(x^4+1-x^2)(x^4+1+x^2)=(x^4-x^2+1)[(x^2+1)^2-x^2]=$ $(x^4-x^2+1)(x^2+1-x)(x^2+1+x)=(x^4-x^2+1)(x^2-x+1)(x^2+x+1)$.

Problem 39. $(x^2+2x-1)^2-(x^2-2x+1)^2$

Solution. $(x^2+2x-1)^2-(x^2-2x+1)^2=(x^2+2x-1-x^2+2x-1)(x^2+2x-1+x^2-2x+1)=$ $(4x-2)(2x^2)=4x^2(2x-1)$

Problem 40. $(ax+by)^2-(bx+ay)^2$

Solution. $(ax+by)^2-(bx+ay)^2=(ax+by-bx-ay)(ax+by+bx+ay)=[(a-b)x-(a-b)y][(a+b)x+(a+b)y]=$ $(a-b)(x-y)(a+b)(x+y)$

Problem 41. $x^3-ax^2-b^2x+ab^2$

Solution. $x^3-ax^2-b^2x+ab^2=(x-a)(x^2-b^2)=(x-a)(x-b)(x+b)$

Problem 42. $x^4+bx^3-a^3x-a^3b$

Solution. $x^4+bx^3-a^3x-a^3b=(x-a)[x^3+(a+b)x^2+a(a+b)x+a^2b]=(x-a)(x+b)[x^2+ax+a^2]$

Problem 43. $a^2-9b^2+12bc-4c^2$

Solution. $a^2-9b^2+12bc-4c^2=(a-3b)(a+3b)+12bc-4c^2=(a-3b+2c)(a+3b-2c)$

Problem 44. $8a^3+12a^2+6a+1$

Solution. $8a^3+12a^2+6a+1=(2a)^3+3(2a)^2+3(2a)+1=(2a+1)^3$

Problem 45. $x^4-2x^3+3x^2-2x+1$

Solution. $x^4-2x^3+3x^2-2x+1=(x^2+lx+1)(x^2+mx+1)=x^4+(l+m)x^3+(lm+2)x^2+(l+m)x+1$, which requires $l+m=-2$, $lm+2=3 \implies l=m=-1$, and therefore, $x^4-2x^3+3x^2-2x+1=(x^2-x+1)^2$.

Problem 46. $(ax+by)^2+(bx-ay)^2$

Solution. $(ax+by)^2+(bx-ay)^2=a^2x^2+2abxy+b^2y^2+b^2x^2-2abxy+a^2y^2=(a^2+b^2)(x^2+y^2)$

Problem 47. $4x^5+4x^4-37x^3-37x^2+9x+9$

Solution. $4x^5+4x^4-37x^3-37x^2+9x+9=(x+1)(4x^4-37x^2+9)=(x+1)\frac{1}{4}[(2x)^4-37(2x)^2+36]=$ $(x+1)\frac{1}{4}[(2x)^2-1][(2x)^2-36]=(x+1)[(2x)^2-1](x^2-9)=(x+1)(2x-1)(2x+1)(x-3)(x+3)$

Problem 48. x^4-4x+3

Solution. $x^4 - 4x + 3 = (x-1)(x^3 + x^2 + x - 3) = (x-1)(x-1)(x^2 + 2x + 3) = (x-1)^2(x^2 + 2x + 3)$

Problem 49. $x^2 + 5ax + 6a^2 - ab - b^2$

Solution. $x^2 + 5ax + 6a^2 - ab - b^2 = x^2 + 5ax + (2a-b)(3a+b) = (x + 2a - b)(x + 3a + b)$

Problem 50. $15x^3 + 29x^2 - 8x - 12$

Solution. $15x^3 + 29x^2 - 8x - 12 = (x+2)(15x^2 - x - 6) = (x+2)(3x-2)(5x+3)$

Problem 51. $abcx^2 + (a^2b^2 + c^2)x + abc$

Solution. $abcx^2 + (a^2b^2 + c^2)x + abc \equiv (abx + l)(cx + m) = abcx^2 + (cl + abm)x + lm$, which requires $cl + abm = a^2b^2 + c^2$, $lm = abc \implies l = c$, $m = ab$, and therefore, $abcx^2 + (a^2b^2 + c^2)x + abc = (abx + c)(cx + ab)$.

Problem 52. $2x^3 - ax^2 - 5a^2x - 2a^3$

Solution. $2x^3 - ax^2 - 5a^2x - 2a^3 = (x+a)(2x^2 - 3ax - 2a^2) = (x+a)\frac{1}{2}[(2x)^2 - 3a(2x) - 4a^2] = (x+a)\frac{1}{2}[(2x - \frac{3}{2}a)^2 - \frac{9}{4}a^2 - \frac{16}{4}a^2] = (x+a)\frac{1}{2}[(2x - \frac{3}{2}a - \frac{5}{2}a)(2x - \frac{3}{2}a + \frac{5}{2}a)] = (x+a)\frac{1}{2}[(2x - 4a)(2x + a)] = (x+a)(x-2a)(2x+a)$

Problem 53. $(a-b)x^2 + 2ax + (a+b)$

Solution. $(a-b)x^2 + 2ax + (a+b) = \frac{1}{a-b}\{[(a-b)x]^2 + 2a[(a-b)x] + a^2 - b^2\} = \frac{1}{a-b}\{[(a-b)x + a]^2 - a^2 + a^2 + b^2\} = \frac{1}{a-b}[(a-b)x + a - b][(a-b)x + a + b] = (x+1)[(a-b)x + a + b]$

Problem 54. $x^{15} - y^{15}$

Solution. First approach. Let $a \equiv x^5$ and $b \equiv y^5$. Then we have

$$\begin{aligned}
x^{15} - y^{15} &= a^3 - b^3 \\
&= (a-b)(a^2 + ab + b^2) \\
&= (x^5 - y^5)(x^{10} + x^5y^5 + y^{10}) \\
&= (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x^{10} + x^5y^5 + y^{10})
\end{aligned}$$

Second (more challenging) approach.

$$\begin{aligned}
x^{15} - y^{15} &= (x^3)^5 - (y^3)^5 \\
&= (x^3 - y^3)[(x^3)^4 + (x^3)^3(y^3) + (x^3)^2(y^3)^2 + (x^3)(y^3)^3 + (y^3)^4] \\
&= (x-y)(x^2 + xy + y^2)[(x^3)^4 + (x^3)^3(y^3) + (x^3)^2(y^3)^2 + (x^3)(y^3)^3 + (y^3)^4] \\
&= (x-y)(x^2 + xy + y^2)(x^{12} + x^9y^3 + x^6y^6 + x^3y^9 + y^{12})
\end{aligned}$$

Now, since we know that $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ is a factor of $x^{15} - y^{15}$, we check to see if it is a factor of $x^{12} + x^9y^3 + x^6y^6 + x^3y^9 + y^{12}$. And we find that indeed it is:

$$\frac{x^{12} + x^9y^3 + x^6y^6 + x^3y^9 + y^{12}}{x^4 + x^3y + x^2y^2 + xy^3 + y^4} = x^8 - x^7y + x^5y^3 - x^4y^4 + x^3y^5 - xy^7 + y^8$$

Therefore, our final result is

$$\begin{aligned}
x^{15} - y^{15} &= (x-y)(x^2 + xy + y^2)(x^{12} + x^9y^3 + x^6y^6 + x^3y^9 + y^{12}) \\
&= (x-y)(x^2 + xy + y^2)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x^8 - x^7y + x^5y^3 - x^4y^4 + x^3y^5 - xy^7 + y^8)
\end{aligned}$$

Problem 55. $x^4 - 6x^3 + 7x^2 + 6x - 8$

Solution. $x^4 - 6x^3 + 7x^2 + 6x - 8 = (x-2)(x^3 - 4x^2 - x + 4) = (x-2)(x-4)(x^2 - 1) = (x-2)(x-4)(x-1)(x+1)$

Problem 56. $4x^3 - 3x - 1$

Solution. $4x^3 - 3x - 1 = (x-1)(4x^2 + 4x + 1) = (x-1)(2x+1)^2$

Problem 57. $3x^5 - 10x^4 - 8x^3 - 3x^2 + 10x + 8$

Solution. $3x^5 - 10x^4 - 8x^3 - 3x^2 + 10x + 8 = (x-1)(3x^4 - 7x^3 - 15x^2 - 18x - 8) =$
$(x-1)(x-4)(3x^3 + 5x^2 + 5x + 2) = (x-1)(x-4)(3x+2)(x^2 + x + 1)$

Problem 58. $5x^4 + 24x^3 - 15x^2 - 118x + 24$

Solution. $5x^4 + 24x^3 - 15x^2 - 118x + 24 = (x-2)(5x^3 + 34x^2 + 53x - 12) = (x-2)(x+3)(5x^2 + 19x - 4) =$
$(x-2)(x+3)(5x-1)(x+4)$

Problem 59. $a^2bc + ac^2 + acd - abd - cd - d^2$

Solution. $a^2bc + ac^2 + acd - abd - cd - d^2 = ac(ab + c + d) - d(ab + c + d) = (ac - d)(ab + c + d)$

Problem 60. $x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2$

Solution. We reason as follows. The expression is completely symmetric in x, y, and z, so whatever we do with one of these we must do to all three. At the same time we need to introduce a negative sign, so we try the factors $(-x+y+z)(x-y+z)(x+y-z)$ and since this is a quartic, we need another factor, and we try $(-x-y-z)$. Hence, we want to check if the product of these four will give us the given expression. We have

$$\begin{aligned}
(-x-y-z)(-x+y+z)(x-y+z)(x+y-z) &= (x+y+z)(x-y-z)(x-y+z)(x+y-z) \\
&= (x+y+z)(x+y-z)(x-y-z)(x-y+z) \\
&= [(x+y)^2 - z^2][(x-y)^2 - z^2] \\
&= (x+y)^2(x-y)^2 - [(x+y)^2 + (x-y)^2]z^2 + z^4 \\
&= [(x+y)(x-y)]^2 - [x^2 + 2xy + y^2 + x^2 - 2xy + y^2]z^2 + z^4 \\
&= (x^2 - y^2)^2 - 2x^2z^2 - 2y^2z^2 + z^4 \\
&= x^4 - 2x^2y^2 + y^4 - 2x^2z^2 - 2y^2z^2 + z^4 \\
&= x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2
\end{aligned}$$

Therefore, $x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = (-x-y-z)(-x+y+z)(x-y+z)(x+y-z)$.

Chapter 7

Highest Common Factor and Lowest Common Multiple

7.1 Highest Common Factor, Exercise XXII (p. 204)

Find the HCF (highest common factor) of the following.

Problem 1. $10x^3y^2z^5$, $4x^5yz^3$, $6x^4y^3z^5$, and $8x^4y^4z^4$

Solution. x^3yz^3

Problem 2. $(a+b)^2(a-b)$, $(a+b)(a-b)^2$, and a^3b-ab^3

Solution. $a^3b-ab^3 = ab(a^2-b^2) = ab(a-b)(a+b)$, so the HCF is $(a+b)(a-b)$

Problem 3. y^4+y^2+1, and y^2-y+1

Solution. $y^4+y^2+1 = (y^2-y+1)(y^2+y+1)$, so HCF is y^2-y+1

Problem 4. a^2-1, a^2+2a+1, and a^3+1

Solution. $a^2-1 = (a-1)(a+1)$, $a^2+2a+1 = (a+1)^2$, $a^3+1 = (a+1)(a^2-a+1)$, so the HCF is $a+1$

Problem 5. x^3-1 and x^3+ax^2-ax-1

Solution. $x^3-1 = (x-1)(x^2+x+1)$ and $x^3+ax^2-ax-1 = (x-1)[x^2+(a+1)x+1]$, so the HCF is $x-1$

Problem 6. x^4-y^4, x^6+y^6, and $x^3+x^2y+xy^2+y^3$

Solution.

$$x^4-y^4 = (x-y)(x^3+x^2y+xy^2+y^3) = (x-y)(x+y)(x^2+y^2)$$
$$x^6+y^6 = [(x^2)^3+(y^2)^3] = (x^2+y^2)[(x^2)^2-(x^2)(y^2)+(y^2)^2] = (x^2+y^2)(x^4-x^2y^2+y^4)$$
$$x^3+x^2y+xy^2+y^3 = (x+y)(x^2+y^2)$$

Therefore, the HCF is x^2+y^2

Problem 7. x^2+5x+6, x^2+x-2, and $x^2-14x-32$

Solution. $x^2+5x+6 = (x+2)(x+3)$, $x^2+x-2 = (x-1)(x+2)$, and $x^2-14x-32 = (x+2)(x-16)$, so the HCF is $x+2$

Problem 8. $(x-1)(x-2)$ and $5x^4-15x^3+8x^2+6x-4$

Solution. $5x^4-15x^3+8x^2+6x-4=(x-1)(x-2)(5x-2)$, so the HCF is $(x-1)(x-2)$

Problem 9. x^3-1 and x^3-4x^2-4x-5

Solution. $x^3-1=(x-1)(x^2+x+1)$ and $x^3-4x^2-4x-5=(x-5)(x^2+x+1)$, so the HCF is x^2+x+1

Problem 10. $(x^2-1)^2(x+1)^2$ and $(x^3+5x^2+7x+3)(x^2-6x-7)$

Solution. $(x^2-1)^2(x+1)^2=(x-1)^2(x+1)^2(x+1)^2$ and $(x^3+5x^2+7x+3)(x^2-6x-7)=(x+1)(x^2+4x+3)(x-7)(x+1)$, so the HCF is $(x+1)^2$

Problem 11. $(x-1)^2(x-2)^2$ and $(x^2-3x+2)(2x^3-5x^2+5x-6)$

Solution. $(x^2-3x+2)(2x^3-5x^2+5x-6)=(x-2)(x-1)(x-2)(2x^2-x+3)$, so the HCF is $(x-1)(x-2)^2$

Problem 12. $2x^3-3x^2-11x+6$ and $4x^3+3x^2-9x+2$

Solution. $2x^3-3x^2-11x+6=(x-3)(2x^2+3x-2)=(x-3)(x+2)(2x-1)$ and $4x^3+3x^2-9x+2=(x+2)(x-1)(4x-1)$, so the HCF is $x+2$

Problem 13. x^3-2x^2-2x-3 and $2x^3+x^2+x-1$

Solution. $x^3-2x^2-2x-3=(x-3)(x^2+x+1)$ and $2x^3+x^2+x-1=(x^2+x+1)(2x-1)$, so the HCF is x^2+x+1

Problem 14. $3x^3+2x^2-19x+6$ and $2x^3+x^2-13x+6$

Solution. $3x^3+2x^2-19x+6=(x-2)(x+3)(3x-1)$ and $2x^3+x^2-13x+6=(x-2)(x+3)(2x-1)$, so the HCF is $(x-2)(x+3)=x^2+x-6$

Problem 15. $x^4-x^3-3x^2+x+2$ and $2x^4+3x^3-x^2-3x-1$

Solution. $x^4-x^3-3x^2+x+2=(x-1)(x-2)(x+1)^2$ and $2x^4+3x^3-x^2-3x-1=(x-1)(x+1)^2(2x+1)$, so the HCF is $(x-1)(x+1)^2=x^3+x^2-x-1$

Problem 16. $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$

Solution.

$$\frac{6x^3-26x^2+46x-42}{6x^3+x^2-44x+21} \implies q=1 \text{ and } r=-27x^2+90x-63$$
$$\frac{6x^3+x^2-44x+21}{3x^2-10x+7} \implies q=2x+7 \text{ and } r=12x-28$$
$$\frac{12x^2-40x+28}{12x-28}=x-1$$

Therefore, the HCF is $3x-7$

Problem 17. $3x^3+8x^2-4x-15$ and $6x^4+10x^3-3x^2-2x+5$

Solution. $3x^3+8x^2-4x-15=(3x+5)(x^2+x-3)$ and $6x^4+10x^3-3x^2-2x+5=(3x+5)(2x^3-x+1)$, so the HCF is $3x+5$

Problem 18. $6x^5+7x^4-9x^3-7x^2+3x$ and $6x^5+7x^4+3x^3+7x^2-3x$

Solution. $6x^5+7x^4-9x^3-7x^2+3x=x(x-1)(x+1)(3x-1)(2x+3)$ and $6x^5+7x^4+3x^3+7x^2-3x=$

$x(3x-1)(2x+3)(x^2+1)$, so the HCF is $x(3x-1)(2x+3) = 6x^3+7x^2-3x$

Problem 19. $6x^4-3x^3+7x^2+x-3$ and $2x^4+3x^3+7x^2+3x+9$

Solution. $6x^4-3x^3+7x^2+x-3 = (2x^2-x+3)(3x^2-1)$ and $2x^4+3x^3+7x^2+3x+9 = (2x^2-x+3)(x^2+2x+3)$, so the HCF is $2x^2-x+3$

Problem 20. $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$

Solution. $6x^5-4x^4-11x^3-3x^2-3x-1 = (x+1)(3x+1)(2x^3-4x^2+x-1)$ and $4x^4+2x^3-18x^2+3x-5 = (2x+5)(2x^3-4x^2+x-1)$, so the HCF is $2x^3-4x^2+x-1$

Problem 21. $x^5-x^3-4x^2-3x-2$ and $5x^4-3x^2-8x-3$

Solution. $x^5-x^3-4x^2-3x-2 = (x-2)(x^2+x+1)^2$ and $5x^4-3x^2-8x-3 = (x^2+x+1)(5x^2-5x-3)$, so the HCF is x^2+x+1

Problem 22. $3x^3-x^2-12x+4$, x^3-2x^2-5x+6 and $7x^3+19x^2+8x-4$

Solution. $3x^3-x^2-12x+4 = (x+2)(3x^2-7x+2)$, $x^3-2x^2-5x+6 = (x-1)(x-3)(x+2)$ and $7x^3+19x^2+8x-4 = (x+2)(7x^2+5x-2)$, so the HCF is $x+2$

Problem 23. $x^3+ax^2-3x-3a$, x^3-x^2-3x+3 and x^3+x^2-3x-3

Solution. $x^3+ax^2-3x-3a = (x^2-3)(x+a)$, $x^3-x^2-3x+3 = (x-1)(x^2-3)$ and $x^3+x^2-3x-3 = (x^2-3)(x+1)$, so the HCF is x^2-3

Problem 24. $7x^4y-6x^3y^2-18x^2y^3+4xy^4$ and $14x^3y-19x^2y^2-32xy^3+28y^4$

Solution.

$7x^4y-6x^3y^2-18x^2y^3+4xy^4 = xy(7x^3-6x^2y-18xy^2+4y^3)$ and
$14x^3y-19x^2y^2-32xy^3+28y^4 = y(14x^3-19x^2y-32xy^2+28y^3)$.
Define $f(x,y) \equiv 7x^3-6x^2y-18xy^2+4y^3$ and $g(x,y) \equiv 14x^3-19x^2y-32xy^2+28y^3$.

We find that $g(2y,y) = 14(2y)^3-19(2y)^2y-32(2y)y^2+28y^3 = [14(8)-19(4)-32(2)+28]y^3 = 0$, which means $x-2y$ is a factor of $g(x,y)$. We also find that it is a factor of $f(x,y)$:
$f(2y,y) = 7(2y)^3-6(2y)^2y-18(2y)y^2+4y^3 = [7(8)-6(4)-18(2)+4]y^3 = 0$, and by division, we get

$$\begin{aligned} 7x^3-6x^2y-18xy^2+4y^3 &= (x-2y)(7x^2+8xy-2y^2) \\ 14x^3-19x^2y-32xy^2+28y^3 &= (x-2y)(14x^2+9xy-14y^2). \end{aligned}$$

Thus, the HCF is $y(x-2y) = xy-2y^2$.

Problem 25. $x(x-1)(x^3+4x^2+4x+3)$ and $(x-1)(x+3)(12x^3+x^2+x-1)$

Solution. $x(x-1)(x^3+4x^2+4x+3) = x(x-1)(x+3)(x^2+x+1)$ and $12x^3+x^2+x-1$ doesn't have any real roots, so the HCF is $(x-1)(x+3)$

Problem 26. $4x^3-8x^2-3x+9$ and $(2x^2-x-3)(2x^2-7x+6)$

Solution. $4x^3-8x^2-3x+9 = (2x-3)(2x^2-x-3) = (2x-3)^2(x+1)$ and $(2x^2-x-3)(2x^2-7x+6) = (x+1)(2x-3)(2x-3)(x-2) = (x+1)(2x-3)^2(x-2)$, so the HCF is $(2x-3)^2(x+1)$

Chapter 8

Rational Fractions

8.1 Rational Fractions, Exercise XXIV (p. 215)

Reduce the following fractions to their lowest terms.

Problem 1. $\dfrac{x^5y^3 - 4x^3y^5}{x^3y^2 - 2x^2y^3}$

Solution. $\dfrac{x^5y^3 - 4x^3y^5}{x^3y^2 - 2x^2y^3} = \dfrac{x^3y^3(x^2 - 4y^2)}{x^2y^2(x - 2y)} = \dfrac{xy(x - 2y)(x + 2y)}{x - 2y} = xy(x + 2y)$

Problem 2. $\dfrac{(x^6 - y^6)(x + y)}{(x^3 + y^3)(x^4 - y^4)}$

Solution. $\dfrac{(x^6 - y^6)(x + y)}{(x^3 + y^3)(x^4 - y^4)} = \dfrac{(x^3 - y^3)(x^3 + y^3)(x + y)}{(x^3 + y^3)(x^4 - y^4)} = \dfrac{(x^3 - y^3)(x + y)}{(x + y)(x^3 - x^2y + xy^2 - y^3)} =$
$\dfrac{x^3 - y^3}{x^3 - x^2y + xy^2 - y^3} = \dfrac{(x - y)(x^2 + xy + y^2)}{(x - y)(x^2 + y^2)} = \dfrac{x^2 + xy + y^2}{x^2 + y^2}$

Problem 3. $\dfrac{x^2 - 4x - 21}{x^2 + 2x - 63}$

Solution. $\dfrac{x^2 - 4x - 21}{x^2 + 2x - 63} = \dfrac{(x + 3)(x - 7)}{(x - 7)(x + 9)} = \dfrac{x + 3}{x + 9}$

Problem 4. $\dfrac{3x^2 - 8x - 3}{3x^2 + 7x + 2}$

Solution. $\dfrac{3x^2 - 8x - 3}{3x^2 + 7x + 2} = \dfrac{(3x + 1)(x - 3)}{(3x + 1)(x + 2)} = \dfrac{x - 3}{x + 2}$

Problem 5. $\dfrac{3x^2 - 18bx + 27b^2}{2x^2 - 18b^2}$

Solution. $\dfrac{3x^2 - 18bx + 27b^2}{2x^2 - 18b^2} = \dfrac{3(x^2 - 6bx + 9b^2)}{2(x^2 - 9b^2)} = \dfrac{3(x - 3b)(x - 3b)}{2(x - 3b)(x + 3b)} = \dfrac{3(x - 3b)}{2(x + 3b)}$

Problem 6. $\dfrac{5x^2 + 6ax + a^2}{5x^2 + 2ax - 3a^2}$

Solution. $\dfrac{5x^2+6ax+a^2}{5x^2+2ax-3a^2} = \dfrac{(5x+a)(x+a)}{(5x-3a)(x+a)} = \dfrac{5x+a}{5x-3a}$

Problem 7. $\dfrac{(x^2-25)(x^2-8x+15)}{(x^2-9)(x^2-7x+10)}$

Solution. $\dfrac{(x^2-25)(x^2-8x+15)}{(x^2-9)(x^2-7x+10)} = \dfrac{(x-5)(x+5)(x-3)(x-5)}{(x-3)(x+3)(x-5)(x-2)} = \dfrac{(x+5)(x-5)}{(x+3)(x-2)}$

Problem 8. $\dfrac{15x^2-46x+35}{10x^2-29x+21}$

Solution. $\dfrac{15x^2-46x+35}{10x^2-29x+21} = \dfrac{(5x-7)(3x-5)}{(5x-7)(2x-3)} = \dfrac{3x-5}{2x-3}$

Problem 9. $\dfrac{x^4+x^2y^2+y^4}{(x^3+y^3)(x^3-y^3)}$

Solution. $\dfrac{x^4+x^2y^2+y^4}{(x^3+y^3)(x^3-y^3)} = \dfrac{(x^2-xy+y^2)(x^2+xy+y^2)}{(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)} = \dfrac{1}{(x+y)(x-y)}$

Problem 10. $\dfrac{x^2-y^2+z^2+2xz}{x^2+y^2-z^2+2xy}$

Solution. $\dfrac{x^2-y^2+z^2+2xz}{x^2+y^2-z^2+2xy} = \dfrac{(x+z)^2-y^2}{(x+y)^2-z^2} = \dfrac{(x+z-y)(x+z+y)}{(x+y-z)(x+y+z)} = \dfrac{x-y+z}{x+y-z}$

Problem 11. $\dfrac{(1+xy)^2-(x+y)^2}{1-x^2}$

Solution. $\dfrac{(1+xy)^2-(x+y)^2}{1-x^2} = \dfrac{(1+xy-x-y)(1+xy+x+y)}{(1-x)(1+x)} = \dfrac{(1-x)(1-y)(1+x)(1+y)}{(1-x)(1+x)} =$
$(1-y)(1+y)$

Problem 12. $\dfrac{2mx-my-12nx+6ny}{6mx-3my-2nx+ny}$

Solution. $\dfrac{2mx-my-12nx+6ny}{6mx-3my-2nx+ny} = \dfrac{2x(m-6n)-y(m-6n)}{2x(3m-n)-y(3m-n)} = \dfrac{(m-6n)(2x-y)}{(3m-n)(2x-y)} = \dfrac{m-6n}{3m-n}$

Problem 13. $\dfrac{2x^3+7x^2-7x-12}{2x^3+3x^2-14x-15}$

Solution. $\dfrac{2x^3+7x^2-7x-12}{2x^3+3x^2-14x-15} = \dfrac{(x+1)(2x^2+5x-12)}{(x+1)(2x^2+x-15)} = \dfrac{(2x-3)(x+4)}{(2x-5)(x+3)}$

Problem 14. $\dfrac{x^3-8x^2+19x-12}{2x^3-13x^2+17x+12}$

Solution. $\dfrac{x^3-8x^2+19x-12}{2x^3-13x^2+17x+12} = \dfrac{(x-1)(x-3)(x-4)}{(x-3)(2x+1)(x-4)} = \dfrac{x-1}{2x+1}$

Problem 15. $\dfrac{x^4+x^3+5x^2+4x+4}{2x^4+2x^3+14x^2+12x+12}$

Solution. $\dfrac{x^4+x^3+5x^2+4x+4}{2x^4+2x^3+14x^2+12x+12} = \dfrac{x^4+x^3+5x^2+4x+4}{2(x^4+x^3+7x^2+6x+6)} = \dfrac{(x^2+x+1)(x^2+4)}{2(x^2+x+1)(x^2+6)} = \dfrac{x^2+4}{2(x^2+6)}$

Problem 16. $\dfrac{x^3-2x^2-x-6}{x^4+3x^3+8x^2+8x+8}$

Solution. $\dfrac{x^3-2x^2-x-6}{x^4+3x^3+8x^2+8x+8} = \dfrac{(x-3)(x^2+x+2)}{(x^2+x+2)(x^2+2x+4)} = \dfrac{x-3}{x^2+2x+4}$

Problem 17. $\dfrac{(x^2+c^2)^2-4b^2x^2}{x^4+4bx^3+4b^2x^2-c^4}$

Solution. $\dfrac{(x^2+c^2)^2-4b^2x^2}{x^4+4bx^3+4b^2x^2-c^4} = \dfrac{(x^2+c^2-2bx)(x^2+c^2+2bx)}{(x^2+2bx+c^2)(x^2+2bx-c^2)} = \dfrac{x^2-2bx+c^2}{x^2+2bx-c^2}$

Problem 18. $\dfrac{(a-b)^3+(b-c)^3+(c-a)^3}{(a-b)(b-c)(c-a)}$

Solution.

$$\begin{aligned}
\frac{(a-b)^3+(b-c)^3+(c-a)^3}{(a-b)(b-c)(c-a)} &= \frac{a^3-3a^2b+3ab^2-b^3+b^3-3b^2c+3bc^2-c^3+c^3-3c^2a+3ca^2-a^3}{(ab-ac-b^2+bc)(c-a)} \\
&= \frac{-3a^2b+3ab^2-3b^2c+3bc^2-3c^2a+3ca^2}{abc-a^2b-ac^2+a^2c-b^2c+ab^2+bc^2-abc} \\
&= 3\frac{ab(b-a)+bc(c-b)+ca(a-c)}{ab(b-a)+bc(c-b)+ca(a-c)} = 3
\end{aligned}$$

8.2 Rational Fractions, Exercise XXV (p. 222)

Simplify the following expressions.

Problem 1. $\dfrac{1}{2a-3b}+\dfrac{1}{2a+3b}-\dfrac{6b}{4a^2-9b^2}$

Solution. $\dfrac{1}{2a-3b}+\dfrac{1}{2a+3b}-\dfrac{6b}{4a^2-9b^2} = \dfrac{2a+3b+2a-3b-6b}{4a^2-9b^2} = \dfrac{4a-6b}{4a^2-9b^2} = \dfrac{2(2a-3b)}{(2a-3b)(2a+3b)} = \dfrac{2}{2a+3b}$

Problem 2. $\dfrac{1}{x+1}+\dfrac{1}{x^2-1}+\dfrac{1}{x^3+1}$

Solution. $\dfrac{1}{x+1}+\dfrac{1}{x^2-1}+\dfrac{1}{x^3+1} = \dfrac{1}{x+1}+\dfrac{1}{(x-1)(x+1)}+\dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{x-1+1}{(x+1)(x-1)}+\dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{x}{(x+1)(x-1)}+\dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{x(x^2-x+1)+x-1}{(x+1)(x-1)(x^2-x+1)} = \dfrac{x^3-x^2+2x-1}{(x+1)(x-1)(x^2-x+1)}$

Problem 3. $\dfrac{1}{x^2-3x+2}+\dfrac{1}{x^2-5x+6}-\dfrac{1}{x^2-4x+3}$

Solution. $\dfrac{1}{x^2-3x+2}+\dfrac{1}{x^2-5x+6}-\dfrac{1}{x^2-4x+3} = \dfrac{1}{(x-2)(x-1)}+\dfrac{1}{(x-3)(x-2)}+\dfrac{1}{(x-3)(x-1)} =$

$$\frac{x-3+x-1+x-2}{(x-1)(x-2)(x-3)} = \frac{3x-6}{(x-1)(x-2)(x-3)} = \frac{3}{(x-1)(x-3)}$$

Problem 4. $\dfrac{x+1}{(x-1)(x-2)} - \dfrac{x+2}{(2-x)(x-3)} + \dfrac{x+3}{(3-x)(1-x)}$

Solution. $\dfrac{x+1}{(x-1)(x-2)} - \dfrac{x+2}{(2-x)(x-3)} + \dfrac{x+3}{(3-x)(1-x)} = \dfrac{(x+1)(x-3)+(x+2)(x-1)+(x+3)(x-2)}{(x-1)(x-2)(x-3)} =$
$\dfrac{x^2-2x-3+x^2+x-2+x^2+x-6}{(x-1)(x-2)(x-3)} = \dfrac{3x^2-11}{(x-1)(x-2)(x-3)}$

Problem 5. $\dfrac{1}{x+b} - \dfrac{1}{x+c} + \dfrac{1}{x-b} - \dfrac{1}{x-c}$

Solution. $\dfrac{1}{x+b} - \dfrac{1}{x+c} + \dfrac{1}{x-b} - \dfrac{1}{x-c} = \dfrac{x-b+x+b}{(x+b)(x-b)} - \dfrac{x-c+x+c}{(x+c)(x-c)} = \dfrac{2x}{x^2-b^2} - \dfrac{2x}{x^2-c^2} =$
$2x\left(\dfrac{1}{x^2-b^2} - \dfrac{1}{x^2-c^2}\right) = 2x\left(\dfrac{x^2-c^2-x^2+b^2}{(x^2-b^2)(x^2-c^2)}\right) = \dfrac{2x(b^2-c^2)}{(x^2-b^2)(x^2-c^2)}$

Problem 6. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(b-c)(b-a)} - \dfrac{c}{(c-a)(c-b)}$

Solution. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(b-c)(b-a)} - \dfrac{c}{(c-a)(c-b)} = \dfrac{-a(b-c)-b(c-a)-c(a-b)}{(a-b)(b-c)(c-a)} =$
$\dfrac{-ab+ac-bc+ab-ac+bc}{(a-b)(b-c)(c-a)} = 0$

Problem 7. $\dfrac{yz(x+a)}{(x-y)(x-z)} + \dfrac{zx(y+a)}{(y-z)(y-x)} + \dfrac{xy(z+a)}{(z-x)(z-y)}$

Solution.

$\dfrac{yz(x+a)}{(x-y)(x-z)} + \dfrac{zx(y+a)}{(y-z)(y-x)} + \dfrac{xy(z+a)}{(z-x)(z-y)} =$
$\dfrac{-yz(x+a)(y-z)-zx(y+a)(z-x)-xy(z+a)(x-y)}{(x-y)(y-z)(z-x)} =$
$\dfrac{-xyz[y-z+z-x+x-y]+a[yz(z-y)+xz(x-z)+xy(y-x)]}{(x-y)(y-z)(z-x)} =$
$a\dfrac{yz(z-y)+xz(x-z)+xy(y-x)}{(x-y)(y-z)(z-x)} =$
$a\dfrac{yz(z-y)+xz(x-z)+xy(y-x)}{yz(z-y)+xz(x-z)+xy(y-x)} = a,$

since

$$\begin{aligned}(x-y)(y-z)(z-x) &= xyz - x^2y - xz^2 + x^2z - y^2z + y^2x + yz^2 - xyz \\ &= y^2x - x^2y + x^2z - z^2x + yz^2 - y^2z \\ &= xy(y-x) + xz(x-z) + yz(z-y).\end{aligned}$$

Problem 8. $x + \dfrac{1}{3-2x} - \dfrac{8x^4-33x}{8x^3-27} - \dfrac{2x+6}{4x^2+6x+9}$

Solution. $x + \dfrac{1}{3-2x} - \dfrac{8x^4-33x}{8x^3-27} - \dfrac{2x+6}{4x^2+6x+9} = \dfrac{3x-2x^2+1}{3-2x} - \dfrac{2x+6}{4x^2+6x+9} - \dfrac{8x^4-33x}{8x^3-27} =$

$$\frac{(-2x^2+3x+1)(4x^2+6x+9)+(2x+6)(2x-3)}{(3-2x)(4x^2+6x+9)} - \frac{8x^4-33x}{8x^3-27} = \frac{-8x^4+8x^2+39x-9}{-8x^3+27} - \frac{8x^4-33x}{8x^3-27} =$$
$$\frac{8x^4-8x^2-39x+9-8x^4+33x}{8x^3-27} = \frac{-8x^2-6x+9}{8x^3-27}$$

Problem 9. $\left(x+\frac{1}{x}\right)^2 + \left(y+\frac{1}{y}\right)^2 + \left(xy+\frac{1}{xy}\right)^2 - \left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(xy+\frac{1}{xy}\right)$

Solution. Expanding the triple product, we have

$$\begin{aligned}\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(xy+\frac{1}{xy}\right) &= x^2y^2+1+\frac{x}{y}xy+\frac{x}{y}\frac{1}{xy}+\frac{y}{x}xy+\frac{y}{x}\frac{1}{xy}+1+\frac{1}{x^2y^2} \\ &= x^2y^2+\frac{1}{x^2y^2}+x^2+\frac{1}{x^2}+y^2+\frac{1}{y^2}+2\end{aligned}$$

so that

$$\left(x+\frac{1}{x}\right)^2 + \left(y+\frac{1}{y}\right)^2 + \left(xy+\frac{1}{xy}\right)^2 - \left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(xy+\frac{1}{xy}\right) =$$
$$x^2+2+\frac{1}{x^2}+y^2+2+\frac{1}{y^2}+x^2y^2+2+\frac{1}{x^2y^2}-x^2y^2-\frac{1}{x^2y^2}-x^2-\frac{1}{x^2}-y^2-\frac{1}{y^2}-2=4$$

Problem 10. $\frac{(a+b)^3-c^3}{a+b-c}+\frac{(b+c)^3-a^3}{b+c-a}+\frac{(c+a)^3-b^3}{c+a-b}$

Solution. Using the identity $x^3-y^3=(x-y)(x^2+xy+y^2)$, we have

$$\frac{(a+b)^3-c^3}{a+b-c}+\frac{(b+c)^3-a^3}{b+c-a}+\frac{(c+a)^3-b^3}{c+a-b} =$$
$$(a+b)^2+(a+b)c+c^2+(b+c)^2+(b+c)a+a^2+(c+a)^2+(c+a)b+b^2 =$$
$$a^2+2ab+b^2+ac+bc+c^2+b^2+2bc+c^2+ab+ac+a^2+c^2+2ac+a^2+bc+ab+b^2 =$$
$$3a^2+4ab+3b^2+4ac+4bc+3c^2 = 3(a^2+b^2+c^2)+4(ab+bc+ca)$$

Problem 11. $\frac{x^2-4}{x^3-3x^2-x+6}-\frac{3x^2-14x-5}{3x^3-2x^2-10x-3}$

Solution. $\frac{x^2-4}{x^3-3x^2-x+6}-\frac{3x^2-14x-5}{3x^3-2x^2-10x-3} = \frac{(x-2)(x+2)}{(x-2)(x^2-x-3)}-\frac{(3x+1)(x-5)}{(3x+1)(x^2-x-3)} =$
$$\frac{x+2-x+5}{x^2-x-3} = \frac{7}{x^2-x-3}$$

Problem 12. $\frac{1}{x^4-4x^2-x+2}+\frac{1}{2x^4-3x^3-5x^2+7x-2}+\frac{1}{2x^4+3x^3-2x^2-2x+1}$

Solution. $\frac{1}{x^4-4x^2-x+2}+\frac{1}{2x^4-3x^3-5x^2+7x-2}+\frac{1}{2x^4+3x^3-2x^2-2x+1} =$
$$\frac{1}{x^4-4x^2-x+2}+\frac{1}{(x-2)(2x^3+x^2-3x+1)}+\frac{1}{(x+1)(2x^3+x^2-3x+1)} =$$
$$\frac{1}{x^4-4x^2-x+2}+\frac{1}{2x^3+x^2-3x+1}\left[\frac{1}{x-2}+\frac{1}{x+1}\right] = \frac{1}{x^4-4x^2-x+2}+\frac{2x-1}{(2x^3+x^2-3x+1)(x-2)(x+1)} =$$
$$\frac{1}{x^4-4x^2-x+2}+\frac{1}{(x^2+x-1)(x-2)(x+1)} = \frac{1}{(x^2+x-1)(x^2-x-2)}+\frac{1}{(x^2+x-1)(x-2)(x+1)} =$$

$\dfrac{2}{(x^2+x-1)(x-2)(x+1)} =$

Problem 13. $\left(a^4 - \dfrac{1}{a^4}\right) \div \left(a - \dfrac{1}{a}\right)$

Solution. $\left(a^4 - \dfrac{1}{a^4}\right) \div \left(a - \dfrac{1}{a}\right) = \dfrac{a^4 - \dfrac{1}{a^4}}{a - \dfrac{1}{a}} = a^3 + a^2\dfrac{1}{a} + a\dfrac{1}{a^2} + \dfrac{1}{a^3} = a^3 + a + \dfrac{1}{a} + \dfrac{1}{a^3} = \dfrac{a^6 + a^4 + a^2 + 1}{a^3}$

Problem 14. $\left(\dfrac{1}{a^3} - \dfrac{1}{a^2} + \dfrac{1}{a}\right)\left(a^4 + a^3\right)$

Solution. $\left(\dfrac{1}{a^3} - \dfrac{1}{a^2} + \dfrac{1}{a}\right)\left(a^4 + a^3\right) = a + 1 - a^2 - a + a^3 + a^2 = a^3 + 1$

Problem 15. $\dfrac{x^2-5x+6}{x^2+3x-4} \cdot \dfrac{x^2+7x+12}{x^2-8x+15} \div \dfrac{x^2+x-6}{x^2-4x-5}$

Solution. $\dfrac{x^2-5x+6}{x^2+3x-4} \cdot \dfrac{x^2+7x+12}{x^2-8x+15} \div \dfrac{x^2+x-6}{x^2-4x-5} = \dfrac{x^2-5x+6}{(x+4)(x-1)} \cdot \dfrac{(x+4)(x+3)}{(x-5)(x-3)} \cdot \dfrac{x^2-4x-5}{x^2+x-6} =$
$\dfrac{x^2-5x+6}{(x-1)} \cdot \dfrac{(x+3)}{(x-5)(x-3)} \cdot \dfrac{(x-5)(x+1)}{(x+3)(x-2)} = \dfrac{(x-3)(x-2)(x-5)(x+1)}{(x-1)(x-5)(x-3)(x-2)} = \dfrac{x+1}{x-1}$

Problem 16. $\dfrac{1}{x} - \left\{1 - \left[\dfrac{x-1}{x} + \dfrac{1}{2}\left(\dfrac{x-1}{x+1} - \dfrac{(x-2)(x-3)}{x(x+1)}\right)\right]\right\}$

Solution. $\dfrac{1}{x} - \left\{1 - \left[\dfrac{x-1}{x} + \dfrac{1}{2}\left(\dfrac{x-1}{x+1} - \dfrac{(x-2)(x-3)}{x(x+1)}\right)\right]\right\} = \dfrac{1}{x} - \left\{1 - \left[\dfrac{x-1}{x} + \dfrac{1}{2}\dfrac{x^2-x-x^2+5x-6}{x(x+1)}\right]\right\} =$
$\dfrac{1}{x} - \left\{1 - \left[\dfrac{x-1}{x} + \dfrac{4x-6}{2x(x+1)}\right]\right\} = \dfrac{1}{x} - \left\{1 - \left[\dfrac{x^2-1+2x-3}{x(x+1)}\right]\right\} = \dfrac{1}{x} - \left\{1 - \dfrac{x^2+2x-4}{x(x+1)}\right\} =$
$\dfrac{1}{x} - \left\{\dfrac{x^2+x-x^2-2x+4}{x(x+1)}\right\} = \dfrac{1}{x} - \dfrac{-x+4}{x(x+1)} = \dfrac{x+1+x-4}{x(x+1)} = \dfrac{2x-3}{x(x+1)}$

Problem 17. $\dfrac{ax+x^2}{2b-cx} \cdot \dfrac{2bx^2-cx^3}{(a+x)^2}$

Solution. $\dfrac{ax+x^2}{2b-cx} \cdot \dfrac{2bx^2-cx^3}{(a+x)^2} = \dfrac{x}{a+x}\dfrac{x^2(cx-2b)}{cx-2b} = \dfrac{x^3}{x+a}$

Problem 18. $(x^2-y^2-z^2+2yz) \div \dfrac{x-y+z}{x-y-z}$

Solution. $(x^2-y^2-z^2+2yz) \div \dfrac{x-y+z}{x-y-z} = (x^2-y^2-z^2+2yz)\dfrac{x-y-z}{x-y+z} = [x-(y-z)][x+(y-z)]\dfrac{x-y-z}{x-y+z} =$
$(x+y-z)(x-y-z) = (x-z)^2 - y^2 = x^2 - y^2 + z^2 - 2xz$

Problem 19. $\left(\dfrac{a+b}{a-b} - \dfrac{a^3+b^3}{a^3-b^3}\right)\left(\dfrac{a+b}{a-b} + \dfrac{a^2+b^2}{a^2-b^2}\right)$

Solution. $\left(\dfrac{a+b}{a-b} - \dfrac{a^3+b^3}{a^3-b^3}\right)\left(\dfrac{a+b}{a-b} + \dfrac{a^2+b^2}{a^2-b^2}\right) = \left(\dfrac{a+b}{a-b} - \dfrac{(a+b)(a^2-ab+b^2)}{(a-b)(a^2+ab+b^2)}\right)\left(\dfrac{a+b}{a-b} + \dfrac{a^2+b^2}{a^2-b^2}\right) =$

$\left(\dfrac{a+b}{a-b}\right)\left[1-\dfrac{a^2-ab+b^2}{a^2+ab+b^2}\right]\left(\dfrac{(a+b)^2}{a^2-b^2}+\dfrac{a^2+b^2}{a^2-b^2}\right)=\left(\dfrac{a+b}{a-b}\right)\dfrac{2ab}{a^2+ab+b^2}\left(\dfrac{a^2+2ab+b^2+a^2+b^2}{a^2-b^2}\right)=$
$\dfrac{a+b}{a-b}2ab\dfrac{2}{a^2-b^2}=\dfrac{4ab}{(a-b)^2}$

Problem 20. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y+z}}{\dfrac{1}{x}+\dfrac{1}{y+z}}\div\dfrac{\dfrac{1}{y}-\dfrac{1}{x+z}}{\dfrac{1}{y}+\dfrac{1}{x+z}}$

Solution. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y+z}}{\dfrac{1}{x}+\dfrac{1}{y+z}}\div\dfrac{\dfrac{1}{y}-\dfrac{1}{x+z}}{\dfrac{1}{y}+\dfrac{1}{x+z}}=\dfrac{\dfrac{1}{x}-\dfrac{1}{y+z}}{\dfrac{1}{x}+\dfrac{1}{y+z}}\cdot\dfrac{\dfrac{1}{y}+\dfrac{1}{x+z}}{\dfrac{1}{y}-\dfrac{1}{x+z}}=\dfrac{\dfrac{y+z-x}{x(y+z)}}{\dfrac{y+z+x}{x(y+z)}}\cdot\dfrac{\dfrac{x+z+y}{y(x+z)}}{\dfrac{x+z-y}{y(x+z)}}=\dfrac{(y+z-x)(y+z+x)}{(x+z+y)(x+z-y)}=$
$\dfrac{-x+y+z}{x-y+z}$

Problem 21. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}\div\dfrac{\dfrac{1}{a^4}-\dfrac{1}{b^4}}{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^2}$

Solution. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}\div\dfrac{\dfrac{1}{a^4}-\dfrac{1}{b^4}}{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^2}=\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}\cdot\dfrac{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^2}{\dfrac{1}{a^4}-\dfrac{1}{b^4}}=\dfrac{a^2+b^2}{a^2-b^2}\cdot\dfrac{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^2}{\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)}=$
$\dfrac{a^2+b^2}{a^2-b^2}\cdot\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\left(\dfrac{1}{a}-\dfrac{1}{b}\right)\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)}=\dfrac{a^2+b^2}{a^2-b^2}\cdot\dfrac{a+b}{b-a}\cdot\dfrac{a^2b^2}{a^2+b^2}=\dfrac{1}{a-b}\cdot\dfrac{1}{b-a}a^2b^2=-\left(\dfrac{ab}{a-b}\right)^2$

Problem 22. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}$

Solution. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}=\dfrac{x-2}{x-2-\dfrac{x(x-2)}{x(x-2)-(x-1)}}=\dfrac{x-2}{x-2-\dfrac{x^2-2x}{x^2-3x+1}}=$
$\dfrac{(x-2)(x^2-3x+1)}{(x-2)(x^2-3x+1)-x^2+2x}=\dfrac{(x-2)(x^2-3x+1)}{x^3-5x^2+7x-2-x^2+2x}=\dfrac{(x-2)(x^2-3x+1)}{x^3-6x^2+9x-2}=$
$\dfrac{(x-2)(x^2-3x+1)}{(x-2)(x^2-4x+1)}=\dfrac{x^2-3x+1}{x^2-4x+1}$

Problem 23. $x+\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}$

Solution. $x+\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}=x+\dfrac{1}{x+\dfrac{x}{x^2+1}}=x+\dfrac{x^2+1}{x^3+x+x}=\dfrac{x^4+2x^2+x^2+1}{x^3+2x}=\dfrac{x^4+3x^2+1}{x^3+2x}$

8.3 Limits of Rational Fractions, Exercise XXVI (p. 230)

Assign the appropriate values values to the following expressions.

Problem 1. $\dfrac{x^2-5x+6}{x^2-6x+8}$ when $x \to 2$

Solution. $\dfrac{x^2-5x+6}{x^2-6x+8} = \dfrac{(x-2)(x-3)}{(x-2)(x-4)} = \dfrac{x-3}{x-4} \to \dfrac{1}{2}$ as $x \to 2$

Problem 2. $\dfrac{x^3-3x^2+2}{x^3-2x+1}$ when $x \to 1$

Solution. $\dfrac{x^3-3x^2+2}{x^3-2x+1} = \dfrac{(x-1)(x^2-2x-2)}{(x-1)(x^2+x-1)} = \dfrac{x^2-2x-2}{x^2+x-1} \to -3$ as $x \to 1$

Problem 3. $\dfrac{x^2-1}{x^2-2x+1}$ when $x \to 1$

Solution. $\dfrac{x^2-1}{x^2-2x+1} = \dfrac{(x-1)(x+1)}{(x-1)^2} = \dfrac{x+1}{x-1} \to \infty$ as $x \to 1$

Problem 4. $\dfrac{x^2-2ax+a^2}{x^2-(a+b)x+ab}$ when $x \to a$

Solution. $\dfrac{x^2-2ax+a^2}{x^2-(a+b)x+ab} = \dfrac{(x-a)^2}{(x-a)(x-b)} = \dfrac{x-a}{x-b} \to 0$ as $x \to a$

Problem 5. $\dfrac{(3x+1)(x+2)^2}{(x^2-4)(x^2+3x+2)}$ when $x \to -2$

Solution. $\dfrac{(3x+1)(x+2)^2}{(x^2-4)(x^2+3x+2)} = \dfrac{(3x+1)(x+2)^2}{(x-2)(x+2)(x+2)(x+1)} = \dfrac{3x+1}{(x-2)(x+1)} \to \dfrac{-5}{(-4)(-1)} = -\dfrac{5}{4}$ as $x \to -2$

Problem 6. $\dfrac{x^3-x^2-x+1}{x^3-3x^2+3x-1}$ when $x \to 1$

Solution. $\dfrac{x^3-x^2-x+1}{x^3-3x^2+3x-1} = \dfrac{(x-1)(x^2-1)}{(x-1)(x^2-2x+1)} = \dfrac{(x-1)(x+1)}{(x-1)^2} = \dfrac{x+1}{x-1} \to \infty$ as $x \to 1$

Problem 7. $\dfrac{3x^2-x+5}{2x^2+6x-7}, \dfrac{x^2+1}{x}, \dfrac{3x}{x^2+1}, \dfrac{(2x^2+1)(x^3-5)}{(x^4+1)(x-6)}$ when $x \to \infty$

Solution.

$\dfrac{3x^2-x+5}{2x^2+6x-7} = \dfrac{3-1/x+5/x^2}{2+6/x-7/x^2} \to \dfrac{3}{2}$ as $x \to \infty$

$\dfrac{x^2+1}{x} = x + \dfrac{1}{x} \to \infty$ as $x \to 1$ as $x \to \infty$

$\dfrac{3x}{x^2+1} = \dfrac{3/x}{1+1/x^2} \to 0$ as $x \to \infty$

$\dfrac{(2x^2+1)(x^3-5)}{(x^4+1)(x-6)} = \dfrac{(2+1/x^2)(1-5/x^3)}{(1+1/x^4)(1-6/x)} \to 2$ as $x \to \infty$

Problem 8. $\dfrac{x-1}{x^2-9} - \dfrac{x-2}{x(x-3)}$ when $x \to 3$

Solution. $\dfrac{x-1}{x^2-9} - \dfrac{x-2}{x(x-3)} = \dfrac{x-1}{(x-3)(x+3)} - \dfrac{x-2}{x(x-3)} = \dfrac{1}{x-3}\left[\dfrac{x-1}{x+3} - \dfrac{x-2}{x}\right] = \dfrac{1}{x-3} \cdot \dfrac{x^2-x-(x^2+x-6)}{x(x+3)} =$
$\dfrac{1}{x-3} \cdot \dfrac{-2x+6}{x(x+3)} = \dfrac{-2}{x(x+3)} \to -\dfrac{1}{9}$ as $x \to 3$

Problem 9. $\dfrac{1}{x-1} + \dfrac{2}{x(x-1)}$ when $x \to 1$

Solution. $\dfrac{1}{x-1} + \dfrac{2}{x(x-1)} = \dfrac{1}{x-1}\left[1 + \dfrac{2}{x}\right] \to \infty$ as $x \to 1$

Problem 10. $\dfrac{x^2 + \dfrac{x+1}{x-2}}{x^2 + \dfrac{x-1}{x-2}}$ when $x \to 2$

Solution. $\dfrac{x^2 + \dfrac{x+1}{x-2}}{x^2 + \dfrac{x-1}{x-2}} = \dfrac{x^2(x-2)+x+1}{x^2(x-2)+x-1} = \dfrac{x^3-2x^2+x+1}{x^3-2x^2+x-1} \to \dfrac{8-8+2+1}{8-8+2-1} \to 3$ when $x \to 2$

Problem 11. $\dfrac{\dfrac{x}{x-1} - \dfrac{x}{x+1}}{\dfrac{3x+1}{x^2+1}}$ when $x \to \infty$

Solution. $\dfrac{\dfrac{x}{x-1} - \dfrac{x}{x+1}}{\dfrac{3x+1}{x^2+1}} = \dfrac{x(x+1-x+1)}{x^2-1} \cdot \dfrac{x^2+1}{3x+1} = \dfrac{2x(x^2+1)}{(x^2-1)(3x+1)} = \dfrac{2(1+1/x^2)}{(1-1/x^2)(3+1/x)} \to$
$\dfrac{2}{3}$ as $x \to \infty$

8.4 Fractional Equations, Exercise XXVII (p. 235)

Solve the following fractional equations for x.

Problem 1.
$\dfrac{6x-1}{3x+2} - \dfrac{4x-7}{2x-5} = 0$

Solution.

$(6x-1)(2x-5) - (4x-7)(3x+2) = 12x^2 - 32x + 5 - 12x^2 + 13x + 14 = 19x - 19 = 0 \implies \boldsymbol{x = 1}$.

Problem 2.
$\dfrac{6x}{5x-1} + \dfrac{8}{3-15x} = \dfrac{1}{6}$

Solution.
$\dfrac{18x - 90x^2 + 40x - 8}{15x - 75x^2 - 3 + 15x} = \dfrac{1}{6} \implies 6(-90x^2 + 58x - 8) = -75x^2 + 30x - 3 \implies -465x^2 + 318x - 45 = 0$

$$\implies 155x^2 - 106x + 15 = 0 \implies \boldsymbol{x = \frac{106 \pm 44}{310} = \frac{15}{31}, \quad \frac{62}{310}}.$$

Problem 3.

$$\frac{4}{x-2} - \frac{1}{x-4} = \frac{4}{x^2-6x+8}$$

Solution.

$$\frac{4}{x-2} - \frac{1}{x-4} = \frac{4}{x^2-6x+8} \implies \frac{4}{x-2} - \frac{1}{x-4} = \frac{4}{(x-2)(x-4)} \implies 4(x-4)-(x-2)=4 \implies \boldsymbol{x=6}.$$

Problem 4.

$$\frac{3}{2x+3} + \frac{1}{x-5} - \frac{8}{2x^2-7x-15} = 0$$

Solution.

$$\frac{3}{2x+3} + \frac{1}{x-5} - \frac{8}{(2x+3)(x-5)} = 0 \implies \frac{3x-15+2x+3-8}{(2x+3)(x-5)} = 0 \implies \frac{5x-20}{(2x+3)(x-5)} = 0 \implies \boldsymbol{x=4}.$$

Problem 5.

$$\frac{1}{(x+1)(x-3)} + \frac{2}{(x-3)(x+2)} + \frac{3}{(x+2)(x+1)} = 0$$

Solution.

$$\begin{aligned}\frac{1}{(x+1)(x-3)} + \frac{2x+2+3x-9}{(x-3)(x+1)(x+2)} = 0 &\implies \frac{1}{(x+1)(x-3)} + \frac{5x-7}{(x-3)(x+1)(x+2)} = 0\\ &\implies \frac{1}{(x+1)(x-3)}\left[\frac{x+2+5x-7}{x+2}\right] = 0\\ &\implies \frac{6x-5}{(x+1)(x=3)(x+2)} = 0 \implies \boldsymbol{x=\frac{5}{6}}.\end{aligned}$$ 7

Problem 6.

$$\frac{2}{x^2-1} - \frac{2}{x^2+4x-5} + \frac{3}{x^2+6x+5} = 0$$

Solution.

$$\begin{aligned}\frac{2}{x^2-1} - \frac{2}{x^2+4x-5} + \frac{3}{x^2+6x+5} = 0 &\implies \frac{2}{(x-1)(x+1)} - \frac{2}{(x+5)(x-1)} + \frac{3}{(x+5)(x+1)} = 0\\ &\implies \frac{2}{(x-1)(x+1)} + \frac{1}{x+5}\left[\frac{-2x-2+3x-3}{(x-1)(x+1)}\right] = 0\\ &\implies \frac{1}{x^2-1}\left[2+\frac{x-5}{x+5}\right] \implies \frac{2x+10+x-5}{(x^2-1)(x+5)} = 0\\ &\implies \frac{3x+5}{(x^2-1)(x+5)} = 0 \implies \boldsymbol{x=\frac{5}{3}}.\end{aligned}$$

Problem 7.

$$\frac{x+1}{3x+1} + \frac{2x}{5-6x} = \frac{5}{5+9x-18x^2}$$

Solution.

$$\frac{x+1}{3x+1} + \frac{2x}{5-6x} = \frac{5}{5+9x-18x^2} \implies \frac{\overbrace{5x-6x^2+5-6x+6x^2+2x}^{x+5}}{(3x+1)(5-6x)} = \frac{5}{(3x+1)(5-6x)} \implies \boldsymbol{x=0}$$

Problem 8.

$$\frac{x+a}{b(x+b)}+\frac{x+b}{a(x+a)}=\frac{a+b}{ab}$$

Solution.

$$\begin{aligned}
\frac{x+a}{b(x+b)}+\frac{x+b}{a(x+a)}=\frac{a+b}{ab} &\implies \frac{a(x+a)^2+b(x+b)^2}{ab(x+a)(x+b)}=\frac{a+b}{ab}\cdot\frac{x^2+(a+b)x+ab}{(x+a)(x+b)}\\
&\implies \frac{(a+b)x^2+2(a^2+b^2)x+a^3+b^3}{ab(x+a)(x+b)}=\frac{(a+b)x^2+(a+b)^2x+ab(a+b)}{ab(x+a)(x+b)}\\
&\implies (a^2-2ab+b^2)x+a^3-a^2b-ab^2+b^3=0\\
&\implies (a-b)^2x+(a^2-b^2)(a-b)=0\\
&\implies (a-b)^2x+(a-b)^2(a+b)=0 \implies \boldsymbol{x=-(a+b)}
\end{aligned}$$

Problem 9.

$$\frac{x^3+1}{x+1}-\frac{x^3-1}{x-1}=20$$

Solution.

$$\frac{x^3+1}{x+1}-\frac{x^3-1}{x-1}=20 \implies (x^2-x+1)-(x^2+x+1)=20 \implies -2x=20 \implies \boldsymbol{x=-10}$$

Problem 10.

$$\frac{x^2+2x+1}{x^2+5x+4}+\frac{x-1}{x^2+3x-4}=0$$

Solution.

$$\frac{x^2+2x+1}{x^2+5x+4}+\frac{x-1}{x^2+3x-4}=0 \implies \frac{(x+1)^2}{(x+1)(x+4)}+\frac{x-1}{(x+4)(x-1)}=0 \implies \frac{x+1+1}{x+4}=0 \implies \boldsymbol{x=-2}$$

Problem 11.

$$\frac{x-8}{x-3}-\frac{x-9}{x-4}=\frac{x+7}{x+8}-\frac{x+2}{x+3}$$

Solution.

$$\frac{x^2-12x+32-(x^2-12x+27)}{(x-3)(x-4)}=\frac{x^2+10x+21-(x^2+10x+16)}{(x+3)(x+8)} \implies \frac{5}{(x-3)(x-4)}=\frac{5}{(x+3)(x+8)} \implies$$

$$x^2+11x+24=x^2-7x+12 \implies 18x=-12 \implies \boldsymbol{x=-\frac{2}{3}}.$$

Problem 12.

$$\frac{x+7}{x+6}+\frac{x+9}{x+8}=\frac{x+10}{x+9}+\frac{x+6}{x+5}$$

Solution.

$$\frac{x+7}{x+6}+\frac{x+9}{x+8}=\frac{x+10}{x+9}+\frac{x+6}{x+5} \implies \frac{x+7}{x+6}-\frac{x+6}{x+5}=\frac{x+10}{x+9}-\frac{x+9}{x+8} \implies$$

$$\frac{x^2+12x+35-(x^2+12x+36)}{(x+5)(x+6)}=\frac{x^2+18x+80-(x^2+18x+81)}{(x+9)(x+8)} \implies$$

$$\frac{-1}{(x+5)(x+6)}=\frac{-1}{(x+9)(x+8)} \implies x^2+17x+72=x^2+11x+30 \implies 6x=-42 \implies \boldsymbol{x=-7}.$$

Problem 13.

$$\frac{x^3+2}{x-2}-\frac{x^3-2}{x+2}-\frac{15}{x^2-4}=4x$$

Solution.

$$\frac{x^3+2}{x-2}-\frac{x^3-2}{x+2}-\frac{15}{x^2-4}=4x \implies \frac{x^4+2x^3+2x+4-(x^4-2x^3-2x+4)}{x^2-4}-\frac{15}{x^2-4}=\frac{4x^3-16}{x^2-4} \implies$$
$$4x^3+4x-15=4x^3-16x \implies 20x=15 \implies \boldsymbol{x=\frac{3}{4}}.$$

Problem 14.

$$\frac{1}{x-1}-\frac{x-2}{x^2-1}+\frac{3x^2+x}{1-x^4}=0$$

Solution.

$$\frac{1}{x-1}-\frac{x-2}{x^2-1}+\frac{3x^2+x}{1-x^4}=0 \implies \frac{x+1}{x^2-1}-\frac{x-2}{x^2-1}-\frac{3x^2+x}{x^4-1}=0 \implies \frac{3(x^2+1)-(3x^2+x)}{x^4-1}=0 \implies$$
$$\frac{3-x}{x^4-1}=0 \implies \boldsymbol{x=3}.$$

Problem 15.

$$\frac{3}{x^3-8}+\frac{2x+5}{2x^2+4x+8}-\frac{1}{x-2}=0$$

Solution.

$$\frac{3}{x^3-8}+\frac{2x+5}{2x^2+4x+8}-\frac{1}{x-2}=0 \implies \frac{3}{(x-2)(x^2+2x+4)}-\frac{1}{x-2}+\frac{2x+5}{2(x^2+2x+4)}=0 \implies$$
$$\frac{3-(x^2+2x+4)}{(x-2)(x^2+2x+4)}+\frac{2x+5}{2(x^2+2x+4)}=0 \implies \frac{-x^2-2x-1}{(x-2)(x^2+2x+4)}+\frac{2x+5}{2(x^2+2x+4)}=0 \implies$$
$$\frac{-2x^2-4x-2+2x^2+x-10}{2(x-2)(x^2+2x+4)}=0 \implies \frac{-3x-12}{2(x-2)(x^2+2x+4)}=0 \implies \boldsymbol{x=-4}.$$

Problem 16.

$$\frac{ax+c}{x-p}+\frac{bx+d}{x-q}=a+b$$

Solution.

$$\begin{aligned}
\frac{ax+c}{x-p}+\frac{bx+d}{x-q}=a+b &\implies (ax+c)(x-q)+(bx+d)(x-p)=(a+b)[x^2-(p+q)x+pq]\\
&\implies ax^2-aqx+cx-cq+bx^2-bpx+dx-dp=(a+b)[x^2-(p+q)x+pq]\\
&\implies (a+b)x^2+(c+d-aq-bp)x-cq-dp=(a+b)[x^2-(p+q)x+pq]\\
&\implies (c+d-aq-bp+ap+aq+bp+bq)x=cq+dp+apq+bpq\\
&\implies (c+d+ap+bq)x=(c+ap)q+(d+bq)p\\
&\implies \boldsymbol{x=\frac{(c+ap)q+(d+bq)p}{c+ap+d+bq}}
\end{aligned}$$

Problem 17.

$$\frac{x^2+7x-8}{x-1}+\frac{x^2+x+3}{x+2}+\frac{2x^2-x+7}{x+3}=4x$$

Solution.

$$\frac{x^2+7x-8}{x-1}+\frac{x^2+x+3}{x+2}+\frac{2x^2-x+7}{x+3}=4x \implies$$
$$\frac{x^2+x+3}{x+2}+\frac{2x^2-x+7}{x+3}=4x-\frac{(x-1)(x+8)}{x-1} \implies$$
$$(x^2+x+3)(x+3)+(2x^2-x+7)(x+2)=[4x-(x+8)](x^2+5x+6) \implies$$
$$x^3+4x^2+6x+9+2x^3+3x^2+5x+14=3x^3+7x^2-22x-48 \implies 33x=-71 \implies \boldsymbol{x=-\frac{71}{33}}.$$

Problem 18.

$$\frac{x^2-ax+2bx-2ab}{x-a}+\frac{b^2-x^2}{x-2b}+\frac{3c^2}{x-2c}=0$$

Solution.

$$\frac{x^2-ax+2bx-2ab}{x-a}+\frac{b^2-x^2}{x-2b}+\frac{3c^2}{x-2c}=0 \implies \frac{(x-a)(x+2b)}{x-a}+\frac{b^2-x^2}{x-2b}+\frac{3c^2}{x-2c}=0 \implies$$
$$\frac{x^2-4b^2+b^2-x^2}{x-2b}+\frac{3c^2}{x-2c}=0 \implies \frac{-3b^2}{x-2b}+\frac{3c^2}{x-2c}=0 \implies \frac{3c^2(x-2b)-3b^2(x-2c)}{(x-2b)(x-2c)}=0 \implies$$
$$\frac{(3c^2-3b^2)x-6bc^2+6b^2c}{(x-2b)(x-2c)}=0 \implies \boldsymbol{x=\frac{6bc(c-b)}{3(c-b)(c+b)}=\frac{2bc}{b+c}}$$

Problem 19.

$$\frac{(x-a)^2}{(x-b)(x-c)}+\frac{(x-b)^2}{(x-c)(x-a)}+\frac{(x-c)^2}{(x-a)(x-b)}=3$$

Solution. Multiply through by $(x-a)(x-b)(x-c)$ to get

$$(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c) \implies (x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c)=0$$

Using

$$(x-a)^3=x^3-3ax^2+3a^2x-a^3$$

and similarly for $(x-b)^3$ and $(x-c)^3$, along with

$$(x-a)^3+(x-b)^3+(x-c)^3=3x^3-3(a+b+c)x^2+3(a^2+b^2+c^2)x-(a^3+b^3+c^3),$$

we get

$$x^3-3ax^2+3a^2x-a^3+x^3-3bx^2+3b^2x-b^3+x^3-3cx^2+3c^2x-c^3-$$
$$3x^3+3(a+b+c)x^2-3(a^2+b^2+c^2)x+(a^3+b^3+c^3)=0 \implies$$
$$3(a^2+b^2+c^2-ab-ac-bc)x-(a^3+b^3+c^3)+3abc=0 \implies$$
$$3[(a+b+c)^2-3(ab+ac+bc)]x=(a+b+c)^3-3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)-9abc$$

and since

$$(a+b+c)[(a+b+c)^2-3(ab+ac+bc)]=(a+b+c)^3-3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)-9abc,$$

we have

$$3[(a+b+c)^2-3(ab+ac+bc)]x=(a+b+c)[(a+b+c)^2-3(ab+ac+bc)] \implies 3x=(a+b+c)$$

Thus,

$$\boldsymbol{x = \frac{a+b+c}{3}}.$$

Problem 20.

$\frac{3x+2}{x^2+x} - \frac{x-5}{x^2-1} - \frac{x-2}{x^2-x} = 0$

Solution.

$\frac{3x+2}{x^2+x} - \frac{x-5}{x^2-1} - \frac{x-2}{x^2-x} = 0 \implies \frac{3x+2}{x(x+1)} - \frac{x-5}{(x-1)(x+1)} - \frac{x-2}{x(x-1)} = 0 \implies$
$\frac{(3x+2)(x-1)-(x-2)(x+1)}{x(x^2-1)} - \frac{x-5}{(x-1)(x+1)} = 0 \implies \frac{3x^2-x-2-x^2+x+2}{x(x^2-1)} - \frac{x-5}{x^2-1} \implies$
$\frac{2x}{x^2-1} - \frac{x-5}{x^2-1} = 0 \implies \boldsymbol{x = -5}$

Problem 21.

$\frac{a}{x+2} + \frac{a}{x-2} - \frac{x+6}{x^2-4} = 0$

Solution.

$\frac{a}{x+2} + \frac{2}{x-2} - \frac{x+6}{x^2-4} = 0 \implies \frac{ax-2a+2x+4-x-6}{x^2-4} = 0 \implies \frac{(1+a)x-2a-2}{(x-2)(x+2)} = 0 \implies$
$\frac{(1+a)x-2(1+a)}{(x-2)(x+2)} = 0 \implies \frac{(1+a)(x-2)}{(x-2)(x+2)} = 0 \implies \frac{1+a}{x+2} = 0 \implies$ **no root**

Problem 22.

$\frac{3x+y-1}{x-y+2} = \frac{6}{7}$ and $\frac{x+9}{y+4} = \frac{x+3}{y+3}$

Solution.

$\frac{3x+y-1}{x-y+2} = \frac{6}{7}$ and $\frac{x+9}{y+4} = \frac{x+3}{y+3} \implies 21x+7y-7 = 6x-6y+12$ and $xy+3x+9y+27 = xy+4x+3y+12$
$\implies 15x+13y = 19$ and $x-6y = 15 \implies \begin{bmatrix} 15 & 13 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix} \implies \boldsymbol{x = 3}$ **and** $\boldsymbol{y = -2}$

Problem 23.

$\frac{y-2}{x-3} + \frac{x-y}{x^2-9} = \frac{y-4}{x+3}$ and $\frac{2}{x^2-2x} + \frac{3}{xy-2y} + \frac{9}{xy} = 0$

Solution.

$\frac{y-2}{x-3} + \frac{x-y}{x^2-9} = \frac{y-4}{x+3} \implies \frac{x-y}{x^2-9} = \frac{y-4}{x+3} - \frac{y-2}{x-3} \implies x-y = (x-3)(y-4)-(x+3)(y-2) \implies$
$x-y = xy-4x-3y+12-xy+2x-3y+6 \implies 3x+5y = 18$ and $\frac{2}{x^2-2x} + \frac{3}{xy-2y} + \frac{9}{xy} = 0 \implies$
$\frac{2}{x(x-2)} + \frac{3}{y(x-2)} + \frac{9}{xy} = 0 \implies \frac{1}{x-2}\left[\frac{2}{x} + \frac{3}{y}\right] + \frac{9}{xy} = 0 \implies \frac{1}{x-2} \cdot \frac{3x+2y}{xy} = -\frac{9}{xy} \implies$
$3x+2y = -9(x-2) \implies 12x+2y = 18 \implies \begin{bmatrix} 3 & 5 \\ 12 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \end{bmatrix} \implies \boldsymbol{x = 1}$ **and** $\boldsymbol{y = 3}$

Problem 24.

$$\frac{xy}{x+y}=a,\quad \frac{yz}{y+z}=b,\quad \frac{zx}{z+x}=c$$

Solution.

$$\frac{xy}{x+y}=a,\quad \frac{yz}{y+z}=b,\quad \frac{zx}{z+x}=c \implies \frac{x+y}{xy}=\frac{1}{a},\quad \frac{y+z}{yz}=\frac{1}{b},\quad \frac{z+x}{zx}=\frac{1}{c} \implies$$
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{a},\quad \frac{1}{y}+\frac{1}{z}=\frac{1}{b},\quad \frac{1}{x}+\frac{1}{z}=\frac{1}{c} \implies \begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}1/x\\1/y\\1/z\end{bmatrix}=\begin{bmatrix}1/a\\1/b\\1/c\end{bmatrix} \implies$$
$$\frac{1}{x}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{c}-\frac{1}{b}\right),\quad \frac{1}{y}=\frac{1}{2}\left(\frac{1}{b}-\frac{1}{c}+\frac{1}{a}\right),\quad \frac{1}{z}=\frac{1}{2}\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{a}\right) \implies$$
$$\frac{1}{x}=\frac{bc+ab-ac}{2abc},\quad \frac{1}{y}=\frac{ac-ab+bc}{2abc},\quad \frac{1}{z}=\frac{ab+ac-bc}{2abc} \implies$$
$$\boldsymbol{x=\frac{2abc}{bc+ab-ac},\quad y=\frac{2abc}{ac-ab+bc},\quad z=\frac{2abc}{ab+ac-bc}}$$

Problem 25.

$$\frac{2}{x+2y}+2y+2z=3,\quad \frac{y+z}{2}-\frac{5}{z-3x}=\frac{7}{2},\quad \frac{4}{z-3x}-\frac{2}{x+2y}=-1$$

Solution.

$$\frac{2}{x+2y}+2y+2z=3,\quad \frac{y+z}{2}-\frac{5}{z-3x}=\frac{7}{2},\quad \frac{4}{z-3x}-\frac{2}{x+2y}=-1 \implies$$
$$\frac{2}{x+2y}=3-2y-2z,\quad \frac{10}{z-3x}=y+z-7,\quad \frac{2}{x+2y}-\frac{4}{z-3x}=1 \implies$$
$$3-2y-2z-\frac{4}{10}(y+z-7)=1 \implies \boldsymbol{y+z=2} \implies 1-\frac{5}{z-3x}=\frac{7}{2} \implies \boldsymbol{3x-z=2} \implies$$
$$\frac{2}{x+2y}+2(2)=3 \implies \boldsymbol{x+2y=-2} \implies \begin{bmatrix}0&1&1\\3&0&-1\\1&2&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\2\\-2\end{bmatrix} \implies \boldsymbol{x=2,\ y=-2,\ z=4}$$

Partial Fractions

Every partial fraction whose numerator and denominator have real coefficients is equal to a definite sum of partial fractions related as follows to the factors $x - a$ and $x^2 + px + q$ of its denominator.

1. For every factor $x - a$ occurring once there is a single fraction of the form $A/(x - a)$, where A is a real constant.
2. For every factor $x - a$ occurring r times there is a group of r fractions of the form
$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r},$$
where A_1, A_2, …, A_r are real constants.
3. For every factor x^2+px+q occurring once there is a single fraction of the form $(Dx+E)/(x^2+px+q)$, where D and E are real constants.
4. For every factor $x^2 + px + q$ occurring r times there is a group of r factors of the form
$$\frac{D_1x+E_1}{(x^2+px+q)} + \frac{D_2x+E_2}{(x^2+px+q)^2} + \cdots + \frac{D_rx+E_r}{(x^2+px+q)^r},$$
where D_1, E_1, D_2, E_2, …, D_r, E_r denote real constants.

8.5 Partial Fractions, Exercise XXVIII (p. 244)

Resolve the following into the simplest partial fractions whose denominators have real coefficients.

Problem 1.

$$\frac{2x+11}{(x-2)(x+3)}$$

Solution.

$$\frac{2x+11}{(x-2)(x+3)} \equiv \frac{A}{x-2} + \frac{B}{x+3} \implies 2x+11 = A(x+3) + B(x-2)$$

Setting $x = 2$ gives $15 = A(5)$ or $A = 3$ and setting $x = -3$ gives $5 = B(-5)$ or $B = -1$. Thus,

$$\frac{2x+11}{(x-2)(x+3)} \equiv \frac{3}{x-2} - \frac{1}{x+3}$$

Problem 2.

$$\frac{6x-1}{(2x+1)(3x-1)}$$

Solution.

$$\frac{6x-1}{(2x+1)(3x-1)} \equiv \frac{A}{2x+1} + \frac{B}{3x-1} \implies 6x-1 = A(3x-1) + B(2x+1)$$

Setting $x = -1/2$ gives $-4 = A(-5/2)$ or $A = 8/5$ and setting $x = 1/3$ gives $1 = B(5/3)$ or $B = 3/5$. Thus,

$$\frac{6x-1}{(2x+1)(3x-1)} \equiv \frac{8}{5(2x+1)} + \frac{3}{5(3x-1)}$$

Problem 3.

$$\frac{4x}{(x+1)(x+2)(x+3)}$$

Solution.

$$\frac{4x}{(x+1)(x+2)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \implies 4x = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Setting $x = 1$ gives $-4 = A(1)(2)$ or $A = -2$, setting $x = -2$ gives $-8 = B(-1)(1)$ or $B = 8$, and setting $x = -3$ gives $-12 = C(-2)(-1)$ or $C = -6$. Thus,

$$\frac{4x}{(x+1)(x+2)(x+3)} \equiv -\frac{2}{x+1} + \frac{8}{x+2} - \frac{6}{x+3}$$

Problem 4.

$$\frac{x^2+2x+3}{(x-1)(x-2)(x-3)(x-4)}$$

Solution.

$$\frac{x^2+2x+3}{(x-1)(x-2)(x-3)(x-4)} \equiv \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} + \frac{D}{x-4} \implies$$
$$x^2+2x+3 = A(x-2)(x-3)(x-4) + B(x-1)(x-3)(x-4) + C(x-1)(x-2)(x-4) + D(x-1)(x-2)(x-3)$$

Setting $x = 1$ gives $A = -1$, setting $x = 2$ gives $B = 11/2$, setting $x = 3$ gives $C = -9$, and setting $x = 4$ gives $D = 9/2$. Thus,

$$\frac{x^2+2x+3}{(x-1)(x-2)(x-3)(x-4)} \equiv -\frac{1}{x-1} + \frac{11}{2(x-2)} - \frac{9}{x-3} + \frac{9}{2(x-4)}$$

Problem 5.

$$\frac{x^2+2}{1+x^3}$$

Solution.

$$\frac{x^2+2}{1+x^3} = \frac{x^2+2}{(x+1)(x^2-x+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \implies x^2+2 = A(x^2-x+1) + (Bx+C)(x+1)$$

Setting $x = -1$ gives $A = 1$. Transposing this term to the left-hand side gives $x+1 = (Bx+C)(x+1) \implies B = 0$, $C = 1$. Thus,

$$\frac{x^2+2}{1+x^3} \equiv \frac{1}{x+1} + \frac{1}{x^2-x+1}$$

Problem 6.

$$\frac{8x+2}{x-x^3}$$

Solution.

$$\frac{8x+2}{x-x^3} = \frac{8x+2}{x(1-x)(1+x)} \equiv \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x} \implies 8x+2 = A(1-x^2) + Bx(1+x) + Cx(1-x)$$

Setting $x = 0$ gives $A = 2$, setting $x = 1$ gives $B = 5$, and setting $x = -1$ gives $C = 3$. Thus,

$$\frac{8x+2}{x-x^3} = \frac{2}{x} - \frac{5}{x-1} + \frac{3}{x+1}$$

Problem 7.

$\dfrac{x^3 - x^2 - 5x + 4}{x^2 - 3x + 2}$

Solution.

$\dfrac{x^3 - x^2 - 5x + 4}{x^2 - 3x + 2} = x + 2 - \dfrac{x}{(x-1)(x-2)}$ and $\dfrac{x}{(x-1)(x-2)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x-2} \implies x = A(x-2) + B(x-1)$

Setting $x = 1$ gives $A = -1$ and setting $x = 2$ gives $B = 2$. Thus,

$$\frac{x^3 - x^2 - 5x + 4}{x^2 - 3x + 2} \equiv x + 2 + \frac{1}{x-1} - \frac{2}{x-2}$$

Problem 8.

$\dfrac{2x^3 - x^2 + 1}{(x-2)^4}$

Solution. We express the numerator in terms of $x - 2$. Using synthetic division, we find

$2x^3 - x^2 + 1 = 2(x-2)^3 + 11(x-2)^2 + 20(x-2) + 13$

Hence,

$$\frac{2x^3 - x^2 + 1}{(x-2)^4} \equiv \frac{2}{x-2} + \frac{11}{(x-2)^2} + \frac{20}{(x-2)^3} + \frac{13}{(x-2)^4}$$

Problem 9.

$\dfrac{x-1}{2x^3 - 5x^2 - 12x}$

Solution.

$\dfrac{x-1}{2x^3 - 5x^2 - 12x} = \dfrac{x-1}{x(x-4)(2x+3)} \equiv \dfrac{A}{x} + \dfrac{B}{x-4} + \dfrac{C}{2x+3} \implies$
$x - 1 = A(x-4)(2x+3) + Bx(2x+3) + Cx(x-4)$

Setting $x = 0$ gives $A = 1/12$, setting $x = 4$ gives $B = 3/44$, and setting $x = -3/2$ gives $C = -10/33$. Thus,

$$\frac{x-1}{2x^3 - 5x^2 - 12x} \equiv \frac{1}{12x} + \frac{3}{44(x-4)} - \frac{10}{33(2x+3)}$$

Problem 10.

$\dfrac{6}{2x^4 - x^2 - 1}$

Solution.

$\dfrac{6}{2x^4 - x^2 - 1} = \dfrac{6}{(x-1)(x+1)(2x^2+1)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{2x^2+1} \implies$
$6 = A(x+1)(2x^2+1) + B(x-1)(2x^2+1) + (Cx+D)(x^2-1)$

Setting $x = 1$ gives $A = 1$, setting $x = -1$ gives $B = -1$, setting $x = 0$ gives $D = -4$, and setting $x = 2$ gives $C = 0$. Thus,

$$\frac{6}{2x^4 - x^2 - 1} \equiv \frac{1}{x-1} - \frac{1}{x+1} - \frac{4}{2x^2+1}$$

Problem 11.

$$\frac{2x^3 - 3x^2 + 4x - 5}{(x+3)^5}$$

Solution. Using synthetic division to expand the numerator in terms of $x + 3$, we get

$$\begin{aligned}\frac{2x^3 - 3x^2 + 4x - 5}{(x+3)^5} &\equiv \frac{2(x+3)^3 - 21(x+3)^2 + 76(x+3) - 98}{(x+3)^5} \\ &\equiv \frac{2}{(x+3)^2} - \frac{21}{(x+3)^3} + \frac{76}{(x+3)^4} - \frac{98}{(x+3)^5}\end{aligned}$$

Problem 12.

$$\frac{2x^2 + x + 1}{(x^2+1)(x^2+2)}$$

Solution.

$$\begin{aligned}\frac{2x^2 + x + 1}{(x^2+1)(x^2+2)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} \implies x^2 + x + 1 &= (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \\ &= (A+C)x^3 + (B+D)x^2 + (2A+C)x + 2B + D\end{aligned}$$

Comparing coefficients on both sides of the equation, we get

$$\begin{aligned}A + C &= 0 \\ B + D &= 1 \\ 2A + C &= 1 \\ 2B + D &= 1,\end{aligned}$$

which gives $A = 1$, $B = 0$, $C = -1$, $D = 1$. Thus,

$$\frac{2x^2 + x + 1}{(x^2+1)(x^2+2)} \equiv \frac{x}{x^2+1} - \frac{x-1}{x^2+2}$$

Problem 13.

$$\frac{x^2 + 6x - 1}{(x-3)^2(x-1)}$$

Solution.

$$\frac{x^2 + 6x - 1}{(x-3)^2(x-1)} \equiv \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} \implies x^2 + 6x - 1 = A(x-3)^2 + B(x-1)(x-3) + C(x-1)$$

Setting $x = 1$ gives $A = 3/2$, and setting $x = 3$ gives $C = 13$. This gives

$$\begin{aligned}2(x^2 + 6x - 1) - 3(x^2 - 6x + 9) - 26(x-1) &= 2B(x^2 - 4x + 3) \text{ or} \\ -x^2 + 4x - 3 &= B(2x^2 - 8x + 6) \implies B = -1/2\end{aligned}$$

Thus,

$$\frac{x^2+6x-1}{(x-3)^2(x-1)} \equiv \frac{3}{2(x-1)} - \frac{1}{2(x-3)} + \frac{13}{(x-3)^2}$$

Problem 14.

$\dfrac{3x-1}{(x-2)(x^2+1)}$

Solution.

$\dfrac{3x-1}{(x-2)(x^2+1)} \equiv \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+1} \implies 3x-1 = A(x^2+1)+(Bx+C)(x-2)$

Setting $x=2$ gives $A=1$, and then we have

$3x-1-x^2-1 = Bx^2-2Bx+Cx-2C \implies -x^2+3x-2 = Bx^2+(C-2B)x-2C,$

which gives $B=-1$, and $C=1$. Thus,

$$\frac{3x-1}{(x-2)(x^2+1)} \equiv \frac{1}{x-2} - \frac{x-1}{x^2+1}$$

Problem 15.

$\dfrac{2x^5-x+1}{(x^2+x+1)^3}$

Solution.

$\dfrac{2x^5-x+1}{(x^2+x+1)^3} \equiv \dfrac{Ax+B}{x^2+x+1} + \dfrac{Cx+D}{(x^2+x+1)^2} + \dfrac{Ex+F}{(x^2+x+1)^3} \implies$

$$\begin{aligned}2x^5-x+1 &= (Ax+B)(x^2+x+1)^2+(Cx+D)(x^2+x+1)+Ex+F\\ &= (Ax+B)(x^4+x^2+1+2x^3+2x^2+2x)+(Cx+D)(x^2+x+1)+Ex+F\\ &= Ax^5+(2A+B)x^4+(3A+2B+C)x^3+(2A+3B+C+D)x^2+(A+2B+C+D+E)x+B+D+F\end{aligned}$$

Comparing coefficients, we find $A=2$, $B=-4$, $C=2$, $D=6$, $E=-3$, and $F=-1$. Thus,

$$\frac{2x^5-x+1}{(x^2+x+1)^3} \equiv \frac{2x-4}{x^2+x+1} + \frac{2x+6}{(x^2+x+1)^2} - \frac{3x+1}{(x^2+x+1)^3}$$

Problem 16.

$\dfrac{2x^2-x+1}{(x^2-x)^2}$

Solution.

$\dfrac{2x^2-x+1}{(x^2-x)^2} \equiv \dfrac{2x^2-x+1}{x^2(x-1)^2} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2} \implies$

$2x^2-x+1 = Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2$

Setting $x=0$ gives $B=1$, and setting $x=1$ gives $D=2$. We then have

$2x^2-x+1-x^2+2x-1-2x^2 = Ax^3-2Ax^2+Ax+Cx^3-Cx^2 \implies -x^2+x = (A+C)x^3-(2A+C)x^2+Ax,$

which gives $A = 1$, and $C = -1$. Thus,

$$\frac{2x^2 - x + 1}{(x^2 - x)^2} \equiv \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

Problem 17.

$$\frac{3x^2 - x + 2}{(x^2+2)(x^2-x-2)}$$

Solution.

$$\frac{3x^2 - x + 2}{(x^2+2)(x^2-x-2)} \equiv \frac{3x^2 - x + 2}{(x-2)(x+1)(x^2+2)} \equiv \frac{A}{x-2} + \frac{B}{x+1} + \frac{Cx+D}{x^2+2} \implies$$
$$3x^2 - x + 2 = A(x+1)(x^2+2) + B(x-2)(x^2+2) + (Cx+D)(x-2)(x+1)$$

Setting $x = 2$ gives $A = 2/3$ and setting $x = -1$ gives $B = -2/3$. Multiplying through by 3 and transposing the A and C terms, we get

$$9x^2 - 3x + 6 - 2(x^3 + x^2 + 2x + 2) + 2(x^3 - 2x^2 + 2x - 4) = 3(Cx+D)(x^2 - x - 2) \implies$$
$$3x^2 - 3x - 6 = 3Cx^3 + (3D - 3C)x^2 + (-6C - 3D)x - 6D,$$

which gives $C = 0$ and $D = 1$. Thus,

$$\frac{3x^2 - x + 2}{(x^2+2)(x^2-x-2)} \equiv \frac{2}{3(x-2)} - \frac{2}{3(x+1)} + \frac{1}{x^2+2}$$

Problem 18.

$$\frac{x^2 + px + q}{(x-a)(x-b)(x-c)}$$

Solution.

$$\frac{x^2 + px + q}{(x-a)(x-b)(x-c)} \equiv \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \implies$$
$$x^2 + px + q = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$$

Setting $x = a$ gives $a^2 + pa + q = A(a-b)(a-c)$, setting $x = b$ gives $b^2 + pb + q = B(b-a)(b-c)$, and setting $x = c$ gives $c^2 + pc + q = C(c-a)(c-b)$. Thus,

$$\frac{x^2 + px + q}{(x-a)(x-b)(x-c)} \equiv \frac{a^2 + pa + q}{(a-b)(a-c)} \cdot \frac{1}{x-a} + \frac{b^2 + pb + q}{(b-a)(b-c)} \cdot \frac{1}{x-b} + \frac{c^2 + pc + q}{(c-a)(c-b)} \cdot \frac{1}{x-c}$$

Problem 19.

$$\frac{2x^2 - 3x - 2}{x(x-1)^2(x+3)^3}$$

Solution.

$$\frac{2x^2 - 3x - 2}{x(x-1)^2(x+3)^3} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{x+3} + \frac{E}{(x+3)^2} + \frac{F}{(x+3)^3} \implies 2x^2 - 3x - 2 =$$
$$A(x-1)^2(x+3)^3 + Bx(x-1)(x+3)^3 + Cx(x+3)^3 + Dx(x-1)^2(x+3)^2 + Ex(x-1)^2(x+3) + Fx(x-1)^2$$

Setting $x = 0$ gives $A = -2/27$, setting $x = 1$ gives $C = -3/64$, and setting $x = -3$ gives $F = -25/48$. Next we expand the right-hand side to get

$$\begin{aligned}
2x^2 - 3x - 2 = {} & (x^3 + 9x^2 + 27x + 27)[(Ax^2 - 2Ax + A + Bx^2 - Bx + Cx] + Dx(x^2 - 2x + 1)(x^2 + 6x + 9)+ \\
& (Ex^2 + 3Ex)(x^2 - 2x + 1) + Fx(x^2 - 2x + 1) \\
= {} & (x^3 + 9x^2 + 27x + 27)[(A + B)x^2 + (-2A - B + C)x + A]+ \\
& Dx^5 + 6Dx^4 + 9Dx^3 - 2Dx^4 - 12Dx^3 - 18Dx^2 + Dx^3 + 6Dx^2 + 9Dx+ \\
& Ex^4 - 2Ex^3 + Ex^2 + 3Ex^3 - 6Ex^2 + 3Ex + Fx^3 - 2Fx^2 + Fx \\
= {} & (A + B)x^5 + (-2A - B + C)x^4 + Ax^3 + (9A + 9B)x^4 + (-18A - 9B + 9C)x^3 + 9Ax^2+ \\
& (27A + 27B)x^3 + (-54A - 27B + 27C)x^2 + 27Ax + (27A + 27B)x^2 + (-54A - 27B + 27C)x + 27A+ \\
& Dx^5 + (4D + E)x^4 + (-2D + E + F)x^3 + (-12D - 5E - 2F)x^2 + (9D + 3E + F)x \\
= {} & (A + B + D)x^5 + (-2A - B + C + 9A + 9B + 4D + E)x^4+ \\
& (A - 18A - 9B + 9C + 27A + 27B - 2D + E + F)x^3+ \\
& (9A - 54A - 27B + 27C + 27A + 27B - 12D - 5E - 2F)x^2+ \\
& (27A - 54A - 27B + 27C + 9D + 3E + F)x + 27A \\
= {} & (A + B + D)x^5 + (7A + 8B + C + 4D + E)x^4 + (10A + 18B + 9C - 2D + E + F)x^3+ \\
& (-18A + 27C - 12D - 5E - 2F)x^2 + (-27A - 27B + 27C + 9D + 3E + F)x + 27A
\end{aligned}$$

Comparing powers of x, we have

$$\begin{aligned}
A + B + D &= 0 \\
7A + 8B + C + 4D + E &= 0 \\
10A + 18B + 9C - 2D + E + F &= 0 \\
-18A + 27C - 12D - 5E - 2F &= 2 \\
-27A - 27B + 27C + 9D + 3E + F &= -3 \\
27A &= -2
\end{aligned}$$

And using the values found for A, C and F, these become

$$\begin{aligned}
B + D &= 2/27 \\
8B + 4D + E &= 977/1728 \\
18B - 2D + E &= 2909/1728 \\
-12D - 5E &= 1539/1728 \\
-27B + 9D + 3E &= -369/1728
\end{aligned}$$

Since there are three unknowns, we are free to use any three of the five equations. Let us choose the first, second and fourth. This gives the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 \\ 8 & 4 & 1 \\ 0 & 12 & 5 \end{bmatrix} \begin{bmatrix} B \\ D \\ E \end{bmatrix} = \begin{bmatrix} 128/1728 \\ 977/1728 \\ -1539/1728 \end{bmatrix}$$

Using Cramer's rule, we get $B = 25/256$, $D = -163/6912$, and $E = -35/288$. Thus,

$$\frac{2x^2 - 3x - 2}{x(x-1)^2(x+3)^3} = -\frac{2}{27x} + \frac{25}{256(x-1)} - \frac{3}{64(x-1)^2} - \frac{163}{6912(x+3)} - \frac{35}{288(x+3)^2} - \frac{25}{48(x+3)^3}$$

Problem 20.

$$\frac{x^3 + x + 3}{x^4 + x^2 + 1}$$

Solution.

$$\frac{x^3+x+3}{x^4+x^2+1} = \frac{x^3+x+3}{(x^2+x+1)(x^2-x+1)} \equiv \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1} \implies$$

$$\begin{aligned} x^3+x+3 &= (Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1) \\ &= (A+C)x^3+(-A+B+C+D)x^2+(A-B+C+D)x+B+D \end{aligned}$$

Equating corresponding powers of x, we have

$$\begin{aligned} A+C &= 1 \\ -A+B+C+D &= 0 \\ A-B+C+D &= 1 \\ B+D &= 3 \end{aligned}$$

which gives $A=2$, $B=3/2$, $C=-1$, and $D=3/2$. Thus,

$$\frac{x^3+x+3}{x^4+x^2+1} \equiv \frac{4x+3}{2(x^2+x+1)} - \frac{2x-3}{2(x^2-x+1)}$$

Chapter 9

Symmetric Functions

Symmetric Functions

The necessary and sufficient condition that an integral function of certain letters, as x, y, z, be symmetric with respect to these letters is that all its terms of the same type shall have the same coefficients. This implies that if the symmetric function contains one terms of a certain type, it must contain *all* terms of that type. Thus, the most general symmetric and homogeneous functions of first, second and third degree are

1. $a(x + y + z)$
2. $a(x^2 + y^2 + z^2) + b(xy + xz + yz)$
3. $a(x^3 + y^3 + z^3) + b(x^2y + y^2x + x^2z + z^2x + y^2z + z^2y) + cxyz$

9.1 Symmetric Functions, Exercise XXX (p. 251)

Factor the following expressions.

Problem 1.

$x^2(y - z) + y^2(z - x) + z^2(x - y)$

Solution. Setting $x = y$ gives $y^2(y - z) + y^2(z - y) = 0$, so the expression is divisible by $x - y$. And since the expression is cyclic in x, y and z, it is also divisible by $y - z$ and $z - x$, and hence by their product. This gives

$x^2(y - z) + y^2(z - x) + z^2(x - y) = k(x - y)(y - z)(z - x),$

where k is a constant to be determined. Let $x = 1$, $y = 2$, $z = 0$, then $(1)(2) + (4)(-1) = k(-1)(2)(-1)$ gives $k = -1$. Thus,

$$x^2(y - z) + y^2(z - x) + z^2(x - y) = -(x - y)(y - z)(z - x)$$

Problem 2.

$yz(y - z) + zx(z - x) + xy(x - y)$

Solution. Setting $x = y$ gives $yz(y - z) + zy(z - y) = 0$, so the expression is divisible by $x - y$. And since the expression is cyclic in x, y and z, it is also divisible by $y - z$ and $z - x$, and hence by their product. This gives

$yz(y - z) + zx(z - x) + xy(x - y) = k(x - y)(y - z)(z - x),$

where k is a constant to be determined. Let $x = 1$, $y = 2$, $z = 0$, then $(2)((-1) = k(-1)(2)(-1)$ gives $k = -1$. Thus,

$$yz(y-z) + zx(z-x) + xy(x-y) = -(x-y)(y-z)(z-x)$$

Problem 3.

$(y-z)^3 + (z-x)^3 + (x-y)^3$

Solution. Setting $x = y$ gives $(y-z)^3 + (z-y)^3 = 0$, so the expression is divisible by $x - y$. And since the expression is cyclic in x, y and z, it is also divisible by $y - z$ and $z - x$, and hence by their product. This gives

$(y-z)^3 + (z-x)^3 + (x-y)^3 = k(x-y)(y-z)(z-x)$,

where k is a constant to be determined. Let $x = 1$, $y = 2$, $z = 0$, then $(8) + (-1) + (-1) = k(-1)(2)(-1)$ gives $k = 3$. Thus,

$$(y-z)^3 + (z-x)^3 + (x-y)^3 = 3(x-y)(y-z)(z-x)$$

Problem 4.

$x(y-z)^3 + y(z-x)^3 + z(x-y)^3$

Solution. Setting $x = y$ gives $y(y-z)^3 + y(z-y)^3 = 0$, so the expression is divisible by $x - y$. And since the expression is cyclic in x, y and z, it is also divisible by $y - z$ and $z - x$, and hence by their product. This gives

$x(y-z)^3 + y(z-x)^3 + z(x-y)^3 = (x-y)(y-z)(z-x)[k(x+y+z)]$,

where k is a constant to be determined. Let $x = 1$, $y = 2$, $z = 0$, then $(1)(8)+(2)(-1) = (-1)(2)(-1)k(1+2)$ gives $k = 1$. Thus,

$$x(y-z)^3 + y(z-x)^3 + z(x-y)^3 = (x-y)(y-z)(z-x)(x+y+z)$$

Problem 5.

$x^2(y-z)^3 + y^2(z-x)^3 + z^2(x-y)^3$

Solution. Setting $x = y$ gives $y^2(y-z)^3 + y^2(z-y)^3 = 0$. And since the expression is cyclic in x, y, z, it also vanishes for $y = z$ and $z = x$. Therefore, $x - y$, $y - z$, $z - x$ each divide the expression, and hence their product does as well. But $(x-y)(y-z)(z-x)$ is of third degree while the expression is of fifth degree. Thus,

$x^2(y-z)^3 + y^2(z-x)^3 + z^2(x-y)^3 = (x-y)(y-z)(z-x)[k(x^2+y^2+z^2) + l(xy+yz+zx)]$

where k and l are constants. To determine their value, let us set $x = 1$, $y = 2$, $z = 0$ and $x = 1$, $y = 2$, $z = 3$. This gives

$$\begin{aligned} 1(8) + 4(-1) &= (-1)(2)(-1)[k(1+4) + l(2)] & \implies 2 = 5k + 2l \\ 1(-1) + 4(8) + 9(-1) &= (-1)(-1)(2)[k(1+4+9) + l(2+6+3)] & \implies 11 = 14k + 11l \end{aligned}$$

We can write this as the matrix equation

$$\begin{bmatrix} 5 & 2 \\ 14 & 11 \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$$

Using Cramer's rule, we have

$$\det \begin{bmatrix} 5 & 2 \\ 14 & 11 \end{bmatrix} = 27, \quad k = \frac{1}{27} \det \begin{bmatrix} 2 & 2 \\ 11 & 11 \end{bmatrix} = 0, \quad l = \frac{1}{27} \det \begin{bmatrix} 5 & 2 \\ 14 & 11 \end{bmatrix} = 1$$

Therefore,

$$x^2(y-z)^3 + y^2(z-x)^3 + z^2(x-y)^3 = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Problem 6.

$x^4(y^2-z^2) + y^4(z^2-x^2) + z^4(x^2-y^2)$

Solution.

$x^4(y^2-z^2) + y^4(z^2-x^2) + z^4(x^2-y^2) = x^4(y-z)(y+z) + y^4(z-x)(z+x) + z^4(x-y)(x+y)$

Setting $x = y$ gives $y^4(y^2-z^2) + y^4(z^2-y^2) = 0$, and the same with setting $x = -y$. Since the expression is symmetric in x, y, and z, we see that it is divisible by the product $(x^2-y^2)(y^2-z^2)(z^2-x^2)$. Thus,

$x^4(y^2-z^2) + y^4(z^2-x^2) + z^4(x^2-y^2) = k(x-y)(x+y)(y-z)(y+z)(z-x)(z+x)$

To find k, set $x = 1$, $y = 2$, $z = 0$ to get $1(4) + 16(-1) = k(1-2)(1+2)(2)(2)(-1)(1)$ to get $k = -1$. Thus,

$$x^4(y^2-z^2) + y^4(z^2-x^2) + z^4(x^2-y^2) = -(x-y)(x+y)(y-z)(y+z)(z-x)(z+x)$$

Problem 7.

$(x+y+z)^3 - x^3 - y^3 - z^3$

Solution. The function vanishes when $x = -y$, when $y = -z$, and when $z = -x$. Thus the function is divisible by $x+y$, by $y+z$, by $z+x$, and hence by their product as well. This product is homogeneous of third degree, the same as the original function. Thus, we have

$(x+y+z)^3 - x^3 - y^3 - z^3 = k(x+y)(y+z)(z+x)$,

where k is a constant. Setting $x = 1$, $y = 2$, $z = 0$, we find $k = 3$. Therefore,

$$(x+y+z)^3 - x^3 - y^3 - z^3 = 3(x+y)(y+z)(z+x)$$

Problem 8.

$(y-z)^5 + (z-x)^5 + (x-y)^5$

Solution. The function vanishes when $x = y$, when $y = z$, and when $z = x$. Thus it must be divisible by $(x-y)$, by $(y-z)$, and by $(z-x)$, and hence by the product $(x-y)(y-z)(z-x)$. This would leave a quotient of second degree. Thus,

$(y-z)^5 + (z-x)^5 + (x-y)^5 = (x-y)(y-z)(z-x)[k(x^2+y^2+z^2) + l(xy+yz+zx)]$

Now let $x = 1$, $y = 2$, $z = 0$ and $x = 1$, $y = 2$, $z = 3$. This gives, respectively, $15 = 5k+2l$ and $15 = 14k+11l$, or

$$\begin{bmatrix} 5 & 2 \\ 14 & 11 \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$$

Using Cramer's rule, we have

$$\det \begin{bmatrix} 5 & 2 \\ 14 & 11 \end{bmatrix} = 27, \quad k = \frac{1}{27} \det \begin{bmatrix} 15 & 2 \\ 15 & 11 \end{bmatrix} = 5, \quad l = \frac{1}{27} \det \begin{bmatrix} 5 & 15 \\ 14 & 15 \end{bmatrix} = -5$$

Therefore,

$$(y-z)^5 + (z-x)^5 + (x-y)^5 = 5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$$

Problem 9.

$(x+y+z)^5-(y+z-x)^5-(z+x-y)^5-(x+y-z)^5$

Solution. Setting $x=0$ gives $(y+z)^5-(y+z)^5-(z-y)^5-(y-z)^5=0$. Similarly, we find that the expression vanishes for both $y=0$ and $z=0$. Thus, the expression is divisible by the product xyz. Since this is of third degree and the given expression is of fifth degree, the factored form must be

$(x+y+z)^5-(y+z-x)^5-(z+x-y)^5-(x+y-z)^5=xyz[k(x^2+y^2+z^2)+l(xy+yz+zx)]$

Setting $x=1$, $y=2$, $z=0$ on both sides of this equation gives $1120=14k+11l$. And setting $x=1$, $y=2$, $z=3$ gives $1120=14k+l$. The solution to this system is $l=0$, $k=80$. Thus,

$$(x+y+z)^5-(y+z-x)^5-(z+x-y)^5-(x+y-z)^5=80xyz(x^2+y^2+z^2)$$

Problem 10.

$(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3$

Solution. Setting $x=y$ gives $(y-z)(y+z)^3+(z-y)(z+y)^3=0$. And since the expression is cyclic in x, y, and z, we see that it also vanishes when $y=z$ and $z=x$. Hence it is divisible by the product $(x-y)(y-z)(z-x)$. Since the given expression is of the fourth degree and this product is of the third degree, the factored form must be

$(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3=(x-y)(y-z)(z-x)[k(x+y+z)]$

Setting $x=1$, $y=2$, $z=0$ gives $(2)(2)^3+(-1)(1)^3+(-1)(3)^3=(-1)(2)(-1)k(1+2) \implies k=-2$. Therefore,

$$(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3=-2(x-y)(y-z)(z-x)(x+y+z)$$

Problem 11.

$x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz$

Solution. Setting $x=-y$, we get

$$-y(y+z)^2+y(z-y)^2+4y^2z=-y(y^2+2yz+z^2)+y(z^2-2zy+y^2)+4y^2z=0$$

And since the given expression is cyclic in x, y and z, it also vanishes for $y=-z$ and $z=-x$. Thus the expression is divisible by the product $(x+y)(y+z)(z+x)$ and is of the form

$$x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz=k(x+y)(y+z)(z+x)$$

Setting $x=1$, $y=2$, $z=0$ gives $1(2)^2+2(1)^2=k(1+2)(2)(1) \implies k=1$. Thus,

$$x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz=(x+y)(y+z)(z+x)$$

Problem 12.

$x^5(y-z)+y^5(z-x)+z^5(x-y)$

Solution. Setting $x=y$ gives $y^5(y-z)+y^5(z-y)=0$, so the expression is divisible by $x-y$. Since it is cyclic in x, y, and z, it is also divisible by $y-z$ and $z-x$, as well as the product of these three factors. But the product is of third degree while the given expression if of sixth degree. Thus it must have the form

$x^5(y-z)+y^5(z-x)+z^5(x-y)=$
$(x-y)(y-z)(z-x)[k(x^3+y^3+z^3)+l(x^2y+y^2x+x^2z+z^2x+y^2z+z^2y)+m(xyz)]$

Setting $x = 1$, $y = 2$, $z = 0$ gives $-5 = 3k + 2l$; setting $x = 1$, $y = 2$, $z = 3$ gives $-15 = 6k + 8l + 1m$; and setting $x = -1$, $y = 2$, $z = 1$ gives $-5 = 4k + 2l - 1m$. We can express this with the matrix equation

$$\begin{bmatrix} 3 & 2 & 0 \\ 6 & 8 & 1 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} k \\ l \\ m \end{bmatrix} = \begin{bmatrix} -5 \\ -15 \\ -5 \end{bmatrix}$$

We can solve this using Cramer's rule to get $k = -1$, $l = -1$, $m = -1$. Thus,

$$x^5(y-z)+y^5(z-x)+z^5(x-y) = -(x-y)(y-z)(z-x)(x^3+y^3+z^3+x^2y+y^2x+x^2z+z^2x+y^2z+z^2y+xyz)$$

Simplify the following fractional expressions.

Problem 13.

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}$$

Solution.

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)} = \frac{-a^4(b-c) - b^4(c-a) - c^4(a-b)}{(a-b)(b-c)(c-a)} \equiv \frac{-f(a,b,c)}{(a-b)(b-c)(c-a)}$$

Now consider the function $f(a,b,c) \equiv a^4(b-c) + b^4(c-a) + c^4(a-b)$, and let $a = b$. We get $f(b,b,c) = b^4(b-c) + b^4(c-b) = 0$, so that $a - b$ divides $f(a,b,c)$. In the same way, it is easily verified that $b - c$ and $c - a$ also divide f. Thus, f is divisible by the product $(a-b)(b-c)(c-a)$. But the function is homogeneous and of degree five, whereas this product is homogeneous and of degree three. Thus,

$$f(a,b,c) \equiv a^4(b-c) + b^4(c-a) + c^4(a-b) = (a-b)(b-c)(c-a)[k(a^2+b^2+c^2) + l(ab+bc+ca)]$$

To determine the constants k and l, let $a = 1$, $b = 2$, $c = 0$ to get $-7 = 5k + 2l$. And let $a = 1$, $b = 2$, $c = 3$ to get $-25 = 14k + 11l$. This gives us the matrix equation

$$\begin{bmatrix} 5 & 2 \\ 14 & 11 \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} -7 \\ -25 \end{bmatrix}$$

Using Cramer's rule, we have

$$\det \begin{bmatrix} 5 & 2 \\ 14 & 11 \end{bmatrix} = 27, \quad k = \frac{1}{27} \det \begin{bmatrix} -7 & 2 \\ -25 & 11 \end{bmatrix} = -1, \quad l = \frac{1}{27} \det \begin{bmatrix} 5 & -7 \\ 14 & -25 \end{bmatrix} = -1$$

Therefore

$$f(a,b,c) \equiv a^4(b-c) + b^4(c-a) + c^4(a-b) = (a-b)(b-c)(c-a)[-(a^2+b^2+c^2) - (ab+bc+ca)]$$

and

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)} = a^2 + b^2 + c^2 + ab + bc + ca$$

Problem 14.

$$\frac{x+a}{(a-b)(a-c)} + \frac{x+b}{(b-c)(b-a)} + \frac{x+c}{(c-a)(c-b)}$$

Solution.

$$\frac{x+a}{(a-b)(a-c)} + \frac{x+b}{(b-c)(b-a)} + \frac{x+c}{(c-a)(c-b)} = -\frac{(x+a)(b-c) + (x+b)(c-a) + (x+c)(a-b)}{(a-b)(b-c)(c-a)}$$

$$\begin{aligned}(x+a)(b-c)+(x+b)(c-a)+(x+c)(a-b) &= (b-c)x+ab-ac+(c-a)x+bc-ab+(a-b)x+ac-bc \\ &= 0\end{aligned}$$

Thus,

$$\frac{x+a}{(a-b)(a-c)}+\frac{x+b}{(b-c)(b-a)}+\frac{x+c}{(c-a)(c-b)}=0$$

Problem 15.

$$\frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2-ca}{(b-c)(b-a)}+\frac{c^2-ab}{(c-a)(c-b)}$$

Solution.

$$\frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2-ca}{(b-c)(b-a)}+\frac{c^2-ab}{(c-a)(c-b)}=-\frac{(a^2-bc)(b-c)+(b^2-ca)(c-a)+(c^2-ab)(a-b)}{(a-b)(b-c)(c-a)}$$

$$\begin{aligned}&(a^2-bc)(b-c)+(b^2-ca)(c-a)+(c^2-ab)(a-b)= \\ &a^2b-a^2c-b^2c+bc^2+b^2c-ab^2-ac^2+a^2c+ac^2-bc^2-a^2b+ab^2=0\end{aligned}$$

Thus,

$$\frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2-ca}{(b-c)(b-a)}+\frac{c^2-ab}{(c-a)(c-b)}=0$$

Problem 16.

$$\frac{(b+c)^2}{(a-b)(a-c)}+\frac{(c+a)^2}{(b-c)(b-a)}+\frac{(a+b)^2}{(c-a)(c-b)}$$

Solution.

$$\frac{(b+c)^2}{(a-b)(a-c)}+\frac{(c+a)^2}{(b-c)(b-a)}+\frac{(a+b)^2}{(c-a)(c-b)}=-\frac{(b+c)^2(b-c)+(c+a)^2(c-a)+(a+b)^2(a-b)}{(a-b)(b-c)(c-a)}$$

$$\begin{aligned}f(a,b,c) &\equiv (b+c)^2(b-c)+(c+a)^2(c-a)+(a+b)^2(a-b) \\ &= (b+c)(b^2-c^2)+(c+a)(c^2-a^2)+(a+b)(a^2-b^2) \\ &= b^3-bc^2+b^2c-c^3+c^3-a^2c+ac^2-a^3+a^3-ab^2+a^2b-b^3 \\ &= bc(b-c)+ac(c-a)+ab(a-b)\end{aligned}$$

Setting $a=b$, $f(b,b,c)=bc(b-c)+bc(c-b)=0$. Setting $b=c$, $f(a,c,c)=ac(c-a)+ac(a-c)=0$. Setting $c=a$, $f(a,b,a)=ba(b-a)+ab(a-b)=0$. Thus, $f(a,b,c)$ is divisible by $(a-b)$, $(b-c)$, $(c-a)$, and hence by the product of all three. Therefore,

$$f(a,b,c)=k(a-b)(b-c)(c-a),$$

where the constant k is to be determined. Setting $a=1$, $b=2$, $c=0$ gives $(1)(2)(1-2)=k(1-2)(2)(-1) \implies k=-1$. Thus,

$$\frac{(b+c)^2}{(a-b)(a-c)}+\frac{(c+a)^2}{(b-c)(b-a)}+\frac{(a+b)^2}{(c-a)(c-b)}=-\frac{f(a,b,c)}{(a-b)(b-c)(c-a)}=-\frac{-(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)}=1$$

Problem 17.

$$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-c)(b-a)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}$$

Solution. This problem is complicated and it's easy to get lost, so we will solve it in stages.

$$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-c)(b-a)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}=$$
$$\frac{-1}{(a-b)(b-c)(c-a)}\left[\frac{a^2(b-c)}{x-a}+\frac{b^2(c-a)}{x-b}+\frac{c^2(a-b)}{x-c}\right]\equiv\frac{-1}{(a-b)(b-c)(c-a)}\cdot f(x)$$

$$\begin{aligned}f(x)&\equiv\frac{a^2(b-c)}{x-a}+\frac{b^2(c-a)}{x-b}+\frac{c^2(a-b)}{x-c}\\&=\frac{a^2(b-c)(x-b)(x-c)+b^2(c-a)(x-a)(x-c)+c^2(a-b)(x-a)(x-b)}{(x-a)(x-b)(x-c)}\\&\equiv\frac{g(x)}{(x-a)(x-b)(x-c)}\end{aligned}$$

Setting $x=0$ in the function $g(x)$, we get

$$g(0)=a^2(b-c)bc+b^2(c-a)ac+c^2(a-b)ab=abc[ab-ac+bc-ab+ac-bc]=0$$

Thus, $g(x)$ is divisible by x, and since $g(x)$ is of the second degree in x, we must have $g(x)=kx^2$, where k is a constant to be determined. Comparing the coefficient of the x^2 term on both sides then gives $k=a^2(b-c)+b^2(c-a)+c^2(a-b)$. Thus, we have

$$g(x)=[a^2(b-c)+b^2(c-a)+c^2(a-b)]x^2$$

Now consider the coefficient,

Setting $a=b \implies a^2(b-c)+b^2(c-a)+c^2(a-b)=b^2(b-c)+b^2(c-b)=0$

so that the coefficient is divisible by $a-b$. In the same way, we find it is also divisible by $b-c$ and $c-a$, and therefore by their product. Thus, we must have

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=l(a-b)(b-c)(c-a),$$

where the constant l is to be determined. Setting $a=1$, $b=2$, $c=0$, gives

$$(1)(2)+4(-1)=-2=l(1-2)(2)(-1)=2l \implies l=-1$$

Thus,

$$g(x)=-(a-b)(b-c)(c-a)x^2$$

Putting everything together, we have

$$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-c)(b-a)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}=$$
$$\frac{-1}{(a-b)(b-c)(c-a)}\cdot f(x)=$$
$$\frac{-1}{(a-b)(b-c)(c-a)}\cdot\frac{g(x)}{(x-a)(x-b)(x-c)}=$$
$$\frac{-1}{(a-b)(b-c)(c-a)}\cdot\frac{-(a-b)(b-c)(c-a)}{(x-a)(x-b)(x-c)}x^2$$
$$\frac{x^2}{(x-a)(x-b)(x-c)}$$

Therefore,

$$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-c)(b-a)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}=\frac{x^2}{(x-a)(x-b)(x-c)}$$

Chapter 10

The Binomial Theorem

The Binomial Theorem for the Expansion of $(a+b)^n$

1. The number of terms on the right is $n+1$.
2. The exponents of a decrease by one and those of b increase by one from term to term, the sum in each term being n.
3. The first coefficient is 1, the second is n, and the rest of them may be found by the following rule: **Multiply the coefficient of any term by the exponent of a in the term and divide by the exponent of b increased by 1; the result will be the coefficient of the next term**.

The General Term

The $(r+1)$th term in the expansion of $(a+b)^n$ is

$$\boxed{\frac{n(n-1)(n-2)\cdots \text{to } r \text{ factors}}{1\cdot 2\cdot 3\cdots r}a^{n-r}b^r}$$

This, with a *minus* sign before it when r is odd, is also the $(r+1)$th term in the expansion of $(a-b)^n$.

For example, with the aid of the binomial theorem:

$$(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{1\cdot 2}b^2$$
$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{1\cdot 2}ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}b^3$$
$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4$$
$$(a+b)^5 = a^5 + 5a^4b + \frac{5\cdot 4}{1\cdot 2}a^3b^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}a^2b^3 + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}ab^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}b^5$$
$$(a+b)^6 = a^6 + 6a^5b + \frac{6\cdot 5}{1\cdot 2}a^4b^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}a^3b^3 + \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}a^2b^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}ab^5 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}b^6$$
$$\cdots$$
$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r}a^{n-r}b^r + \cdots + b^n$$

or, simply

$$
\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \\
(a+b)^7 &= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 \\
(a+b)^8 &= a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8 \\
(a+b)^9 &= a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9
\end{aligned}
$$

Since the number of terms is $n+1$, there will be one middle term when n is even, two when n is odd. The coefficients of the terms which follow the middle term, or terms, are the same as those which precede them, but in reverse order.

10.1 Binomial Theorem, Exercise XXXI (p. 259)

Expand the following by aid of the binomial theorem.

Problem 1. $(3x+2y)^3$

Solution. Using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we have

$$(3x+2y)^3 = (3x)^3 + 3(3x)^2(2y) + 3(3x)(2y)^2 + (2y)^3 = \mathbf{27x^3 + 54x^2y + 36xy^2 + 8y^3}$$

Problem 2. $(a-b)^8$

Solution.

$$(a-b)^8 = \mathbf{a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8}$$

Problem 3. $(1+2x^2)^7$

Solution. Using $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$, we have

$$
\begin{aligned}
(1+2x^2)^7 &= 1 + 7(2x^2) + 21(2x^2)^2 + 35(2x^2)^3 + 35(2x^2)^4 + 21(2x^2)^5 + 7(2x^2)^6 + (2x^2)^7 \\
&= \mathbf{1 + 14x^2 + 84x^4 + 280x^6 + 560x^8 + 672x^{10} + 448x^{12} + 128x^{14}}
\end{aligned}
$$

Problem 4. $(2+1/x)^4$

Solution. Using $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, we have

$$\left(2 + \frac{1}{x}\right)^4 = \mathbf{16 + \frac{32}{x} + \frac{24}{x^2} + \frac{8}{x^3} + \frac{1}{x^4}}$$

Problem 5. $(x-3/x)^6$

Solution. Using $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, we have

$$
\begin{aligned}
\left(x - \frac{3}{x}\right)^6 &= x^6 - 6x^5\frac{3}{x} + 15x^4\frac{9}{x^2} - 20x^3\frac{27}{x^3} + 15x^2\frac{81}{x^4} - 6x\frac{243}{x^5} + \frac{729}{x^6} \\
&= \mathbf{x^6 - 18x^4 + 135x^2 - 540 + \frac{1215}{x^2} - \frac{1458}{x^4} + \frac{729}{x^6}}
\end{aligned}
$$

Problem 6. $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)^5$

Solution. Using $(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, we have

$$\begin{aligned}\left(\frac{x}{y}-\frac{y}{x}\right)^5 &= \left(\frac{x}{y}\right)^5 - 5\left(\frac{x}{y}\right)^4\left(\frac{y}{x}\right) + 10\left(\frac{x}{y}\right)^3\left(\frac{y}{x}\right)^2 - 10\left(\frac{x}{y}\right)^2\left(\frac{y}{x}\right)^3 + 5\left(\frac{x}{y}\right)\left(\frac{y}{x}\right)^4 - \left(\frac{y}{x}\right)^5\\ &= \boldsymbol{\left(\frac{x}{y}\right)^5 - 5\left(\frac{x}{y}\right)^3 + 10\frac{x}{y} - 10\frac{y}{x} + 5\left(\frac{y}{x}\right)^3 - \left(\frac{y}{x}\right)^5}\end{aligned}$$

Problem 7. $(1-x+2x^2)^4$

Solution. $(1-x+2x^2)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$ with $a=1$ and $b=2x^2-x$ gives

$$\begin{aligned}(1-x+2x^2)^4 &= 1+4(2x^2-x)+6(2x^2-x)^2+4(2x^2-x)^3+(2x^2-x)^4\\ &= 1+4x(2x-1)+6x^2(2x-1)^2+4x^3(2x-1)^3+x^4(2x-1)^4\end{aligned}$$

Now,

$$\begin{aligned}&(2x-1)^2 = (2x)^2-2(2x)+1 = 4x^2-4x+1\\ &(2x-1)^3 = (2x)^3-3(2x)^2+3(2x)-1 = 8x^3-12x^2+6x-1\\ &(2x-1)^4 = (2x)^4-4(2x)^3+6(2x)^2-4(2x)+1 = 16x^4-32x^3+24x^2-8x+1\end{aligned}$$

so that

$$\begin{aligned}(1-x+2x^2)^4 &= 1+8x^2-4x+6x^2(4x^2-4x+1)+4x^3(8x^3-12x^2+6x-1)+x^4(16x^4-32x^3+24x^2-8x+1)\\ &= \boldsymbol{16x^8-32x^7+56x^6-56x^5+49x^4-28x^3+14x^2-4x+1}\end{aligned}$$

Problem 8. $(a^2+ax-x^2)^3$

Solution. $(a^2+ax-x^2)^3 = x^3+3x^2y+3xy^2+y^3$ with $x=a^2$ and $y=ax-x^2$ gives

$$(a^2+ax-x^2)^3 = (a^2)^3+3(a^2)^2(ax-x^2)+3(a^2)(ax-x^2)^2+(ax-x^2)^3$$

Now,

$$\begin{aligned}&(ax-x^2)^2 = x^2(a-x)^2 = x^2(a^2-2ax+x^2) = a^2x^2-2ax^3+x^4\\ &(ax-x^2)^3 = x^3(a-x)^3 = x^3(a^3-3a^2x+3ax^2-x^3) = a^3x^3-3a^2x^4+3ax^5-x^6\end{aligned}$$

so that

$$\begin{aligned}(a^2+ax-x^2)^3 &= (a^2)^3+3(a^2)^2(ax-x^2)+3(a^2)(ax-x^2)^2+(ax-x^2)^3\\ &= a^6+3a^4(ax-x^2)+3a^2(a^2x^2-2ax^3+x^4)+a^3x^3-3a^2x^4+3ax^5-x^6\\ &= \boldsymbol{a^6+3a^5x-5a^3x^3+3ax^5-x^6}\end{aligned}$$

Problem 9. Find the sixth term in $(1+x/2)^{11}$

Solution. Here $n=11$ and $r+1=6$ or $r=5$. Hence, the required term is

$$\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(1)^{11-6}\left(\frac{x}{2}\right)^5 = 462\frac{x^5}{32} = \boldsymbol{\frac{231}{16}x^5}$$

Problem 10. Find the eighth term in $(3a-4b)^{12}$

Solution. Here $n=12$ and $r+1=8$ or $r=7$. Hence, the required term is

$$\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a)^{12-7}(-4b)^7 = 792(243a^5)(-16384b^7) = \boldsymbol{-3153199104a^5b^7}$$

Problem 11. Find the middle term in $(a^2 - 2bc)^{10}$

Solution. Since $n = 10$ is even, there will be one middle term. It is the sixth term, so that $n = 10$ and $r + 1 = 6$ or $r = 5$. The middle term is

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(a^2)^{10-5}(-2bc)^5 = 252(-32)a^{10}b^5c^5 = \boldsymbol{-8064a^{10}b^5c^5}$$

Problem 12. Find the two middle terms in $(1 - x)^9$

Solution. There are 10 terms in the expansion. The two middle terms are the 5th and 6th. For the 5th term, $r + 1 = 5$ so $r = 4$, and the term is

$$\frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}(1)^{10-4}(-x)^4 = \boldsymbol{126x^4}$$

For the 5th term, $r + 1 = 6$ so $r = 5$, and the term is

$$\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(1)^{10-5}(-x)^5 = \boldsymbol{-126x^5}$$

Problem 13. Find the coefficient of x^5 in $(1 + x)^8$

Solution. Omitting the coefficient, the term is $(1)^{8-5}x^5$, which means $r = 5$, and the term is

$$\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(1)^{8-5}(x)^5 = \boldsymbol{56x^5}$$

Problem 14. Find the coefficient of x^4 in $(3 - 2x)^7$

Solution. Omitting the coefficient, the term is $(3)^{7-4}(-2x)^4$, which means $r = 4$, and the term is

$$\frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4}(3)^{7-4}(-2x)^4 = 35(3)^3(-2)^4x^4 = \boldsymbol{15120x^4}$$

Problem 15. Find the coefficient of x^8 in $(1 - x^2)^6$

Solution. Omitting the coefficient, the term is $(1)^{6-4}(-x^2)^4$, which means $r = 4$, and the term is

$$\frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4}(1)^{6-4}(x^2)^4 = \boldsymbol{15x^8}$$

Problem 16. Find the coefficient of x^3 in $(1 + 2x)^9 + (1 - 2x)^{11}$

Solution. We have $r = 3$ in both terms, which gives $(1)^{9-3}(2x)^3$ in the first term and $(1)^{11-3}(-2x)^3$ in the second term. Hence, the coefficient is

$$\frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3}(1)^{9-3}(2x)^3 + \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3}(1)^{11-3}(-2x)^3 = 84(8)x^3 + 165(-8)x^3 = \boldsymbol{-648\,x^3}$$

Problem 17. Find the constant term in $(x + 1/x)^{12}$.

Solution. Omitting the coefficient, the general term is of the form $x^{12-r}(1/x)^r$, and in order for this to be a constant, we must have $12 - r = r$, which gives $r = 6$. Or we could reason that there are $n + 1 = 13$ terms, with the constant terms in the middle so that $r + 1 = 7$ and $r = 6$. Thus, the term is

$$\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}x^{12-6}(1/x)^6 = \mathbf{924}$$

Problem 18. Find the coefficient of x^7 in $(2x - 1/x)^{15}$.

Solution. The general term in the expansion s of the form $(2x)^{15-r}(1/x)^r$, and we want this to be x^7, which requires $15 - r - r = 7 \implies r = 4$. Thus, the term is

$$\frac{15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4}(2x)^{15-4}(1/x)^4 = 1365 \cdot 2^{11}x^{11}x^{-4} = \boldsymbol{2795520\, x^7}$$

Problem 19. Find $(x+2y)(x-3y)(x-5y)$ by inspection.

Solution.

$$x^3 + (2y - 3y - 5y)x^2 + [(2y)(-3y) + (2y)(-5y) + (-3y)(-5y)]x + (2y)(-3y)(-5y) =$$
$$x^3 - 6yx^2 - y^2x + 30y^3 = \boldsymbol{x^3 - 6x^2y - xy^2 + 30y^3}$$

Problem 20. Find $(x+2)(x+3)(x-4)(x-5)$ by inspection.

Solution. $x^4 + (2+3-4-5)x^3 + [(2)(3) + (2)(-4) + (2)(-5) + (3)(-4) + (3)(-5) + (-4)(-5)]x^2$
$+ [(2)(3)(-4) + (2)(3)(-5) + (3)(-4)(-5)]x + (2)(3)(-4)(-5)] \implies \boldsymbol{x^4 - 4x^3 - 19x^2 + 46x + 120}$

Problem 21. What is the number of terms in the product $(a+b+c+d)(f+g+h)(k+l)(m+n+p+q)$?

Solution. $4 \cdot 3 \cdot 2 \cdot 4 = \mathbf{96}$.

Problem 22. Find the sum of the coefficients in the following products.

1. $(1 + x^2 + x^3 + x^4)^3$ 2. $(1 + 2x + x^2)^2(1 + x + 3x^3)^2$

Solution. Setting $x = 1$ gives the sum of the coefficients, so that $(1 + x^2 + x^3 + x^4)^3 \implies (1+1+1+1)^3 = 4^3 = 64$, and $(1 + 2x + x^2)^2(1 + x + 3x^3)^2 \implies (1+2+1)^2(1+1+3)^2 = 16 \cdot 25 = 400$.

Problem 23. What is the sum of the coefficients in the following symmetric functions of *four* letters a, b, c, d when expanded?

1. $\Sigma a^2 \cdot \Sigma a$ 2. $\Sigma a^3 \cdot \Sigma abc$ 3. $\Sigma ab \cdot \Sigma abc$

Solution. The sum of the coefficients can be obtained by setting $a = b = c = d = 1$. Accordingly,

$$\begin{aligned}
\Sigma a^2 \cdot \Sigma a &= (a^2 + b^2 + c^2 + d^2) \cdot (a + b + c + d) \implies (1+1+1+1) \cdot (1+1+1+1) = (4)(4) = \mathbf{16} \\
\Sigma a^3 \cdot \Sigma abc &= (a^3 + b^3 + c^3 + d^3)(abc + abd + acd + bcd) \implies (4)(4) = \mathbf{16} \\
\Sigma ab \cdot \Sigma abc &= (ab + ac + ad + bc + bd + cd)(abc + abd + acd + bcd) \implies (6)(4) = \mathbf{24}
\end{aligned}$$

Problem 24. Show that the sum of the coefficients in the expansion of $(a+b)^n$ is 2^n.

Solution. The sum of the coefficients in the expansion $(a+b)^n$ can be obtained by setting $a = b = 1$, which gives $(1+1)^n = 2^n$.

Problem 25. Show that in the expansion of $(a-b)^n$ the sum of the positive coefficients is numerically equal to the sum of the negative coefficients.

Solution. The sum of all the coefficients, positive and negative, can be obtained by setting $a = b = 1$, which gives $(1-1)^n = 0$. Hence, the sum of the positive coefficients is numerically equal to the sum of the negative coefficients.

Chapter 11

Evolution

11.1 Roots of Polynomials, Exercise XXXII (p. 269)

Making use of the following formulas

$$
\begin{aligned}
(a+b+c)^2 &= a^2+b^2+c^2+2ab+2ac+2bc \\
(a+b+c)^3 &= a^3+b^3+c^3+3a^2b+3a^2c+3b^2a+3b^2c+3c^2a+3c^2b+6abc \\
(a+b+c)^4 &= a^4+b^4+c^4+4a^3b+4a^3c+4b^3a+4b^3c+4c^3a+4c^3b \\
&\quad +6a^2b^2+6a^2c^2+6b^2c^2+12a^2bc+12ab^2c+12abc^2 \\
(a+b+c)^5 &= a^5+b^5+c^5+5a^4b+5a^4c+5b^4a+5b^4c+5c^4a+5c^4b \\
&\quad +10a^3b^2+10a^3c^2+10b^3a^2+10b^3c^2+10c^3a^2+10c^3b^2 \\
&\quad +20a^3bc+20b^3ac+20c^3ab+30ab^2c^2+30ba^2c^2+30ca^2b^2
\end{aligned}
$$

we derive the following formulas, which will prove useful in what follows:

$$(x^2+px+q)^2 = x^4+2px^3+(p^2+2q)x^2+2pqx+q^2 \tag{11.1}$$

$$(x^2+px+q)^3 = x^6+3px^5+(3q+3p^2)x^4+(p^3+6pq)x^3+(3p^2q+3q^2)x^2+3pq^2x+q^3 \tag{11.2}$$

$$
\begin{aligned}
(x^2+px+q)^4 &= x^8+4px^7+(4q+6p^2)x^6+(4p^3+12pq)x^5+(p^4+6q^2+12p^2q)x^4+(4p^3q+12pq^2)x^3 \\
&\quad +(4q^3+6p^2q^2)x^2+4pq^3x+q^4
\end{aligned} \tag{11.3}
$$

$$
\begin{aligned}
(x^2+px+q)^5 &= x^{10}+5px^9+(5q+10p^2)x^8+(10p^3+20pq)x^7+(5p^4+10q^2+30p^2q)x^6 \\
&\quad +(p^5+20p^3q+30pq^2)x^5+(5p^4q+10q^3+30p^2q^2)x^4+(10p^3q^2+20pq^3)x^3 \\
&\quad +(5q^4+10p^2q^3)x^2+5pq^4x+q^5
\end{aligned} \tag{11.4}
$$

Simplify the following expressions.

Problem 1. $\sqrt[3]{-\dfrac{27x^6y^{15}}{125a^9z^{12}}}$

Solution. $\sqrt[3]{-\dfrac{3^3x^6y^{15}}{5^3a^9z^{12}}} = \sqrt[3]{-\dfrac{3^3(x^2)^3(y^5)^3}{5^3(a^3)^3(z^4)^3}} = \sqrt[3]{-\dfrac{(3x^2y^5)^3}{(5a^3z^4)^3}} = \sqrt[3]{-\left(\dfrac{3x^2y^5}{5a^3z^4}\right)^3} = -\dfrac{3x^2y^5}{5a^3z^4}$

Problem 2. $\sqrt{\dfrac{529a^4b^6}{625c^2d^8}}$

Solution. $\sqrt{\dfrac{529a^4b^6}{625c^2d^8}} = \sqrt{\dfrac{23^2a^4b^6}{25^2c^2d^8}} = \sqrt{\dfrac{(23a^2b^3)^2}{(25cd^4)^2}} = \sqrt{\left(\dfrac{23a^2b^3}{25cd^4}\right)^2} = \dfrac{23a^2b^3}{25cd^4}$

Problem 3. $\sqrt[6]{(x^4y^2 - 2x^3y^3 + x^2y^4)^3}$

Solution. $\sqrt[6]{(x^4y^2 - 2x^3y^3 + x^2y^4)^3} = \sqrt[6]{[x^2y^2(x^2 - 2xy + y^2)]^3} = \sqrt[6]{[x^2y^2(x-y)^2]^3} = \sqrt[6]{x^6y^6(x-y)^6} = xy(x-y)$

Problem 4. Find the square root of $x^4 - 2x^3 + 3x^2 - 2x + 1$.

Solution. Making use of eq. (1), we have

$$x^4 - 2x^3 + 3x^2 - 2x + 1 = (x^2 + px + q)^2 = x^4 + 2px^3 + (p^2 + 2q)x^2 + 2pqx + q^2$$

This requires $2p = -2 \implies p = -1$ and $1 + 2q = 3 \implies q = 1$ and it is easy to check that the other coefficients are satisfied as well. Thus, the square root of $x^4 - 2x^3 + 3x^2 - 2x + 1$ is $\boldsymbol{x^2 - x + 1}$.

Problem 5. Find the square root of $x^2 - 2x^4 + 6x^3 - 6x + x^6 + 9$.

Solution. Let

$$\begin{aligned} x^6 - 2x^4 + 6x^3 + x^2 - 6x + 9 &= (x^3 + px^2 + qx + r)^2 \\ &= x^6 + p^2x^4 + q^2x^2 + r^2 + 2px^5 + 2qx^4 + 2rx^3 + 2pqx^3 + 2prx^2 + 2qrx \\ &= x^6 + 2px^5 + (p^2 + 2q)x^4 + (2r + 2pq)x^3 + (2pr + q^2)x^2 + 2qrx + r^2, \end{aligned}$$

which requires $r = 3$, $p = 0$, $2q = -2 \implies q = -1$, and these values are consistent with all the coefficients. Thus, the square root of $x^2 - 2x^4 + 6x^3 - 6x + x^6 + 9$ is $\boldsymbol{x^3 - x + 3}$.

Problem 6. Find the square root of $4x^6 + 12x^5y + 9x^4y^2 - 4x^3y^3 - 6x^2y^4 + y^6$.

Solution. Let

$$\begin{aligned} 4x^6 + 12x^5y + 9x^4y^2 - 4x^3y^3 - 6x^2y^4 + y^6 &= (2x^3 + px^2 + qx + r)^2 \\ &= 4x^6 + p^2x^4 + q^2x^2 + r^2 + 4px^5 + 4qx^4 + 4rx^3 + 2pqx^3 + 2prx^2 + 2qrx \\ &= 4x^6 + 4px^5 + (p^2 + 4q)x^4 + (4r + 2pq)x^3 + (q^2 + 2pr)x^2 + 2qrx + r^2, \end{aligned}$$

which requires $r = \pm y^3$, $4p = 12y \implies p = 3y$, $9y^2 + 4q = 9y^2 \implies q = 0$, $2pr = -6y^4$, and these values are consistent with all the coefficients. Thus, the square root of $4x^6 + 12x^5y + 9x^4y^2 - 4x^3y^3 - 6x^2y^4 + y^6$ is $\boldsymbol{2x^3 + 3x^2y - y^3}$.

Problem 7. Find the square root of $4x^2 - 20x + 13 + 30/x + 9/x^2$.

Solution.

$$\begin{aligned} 4x^2 - 20x + 13 + 30/x + 9/x^2 &= \frac{1}{x^2}(4x^4 - 20x^3 + 13x^2 + 30x + 9) \\ &= \frac{1}{x^2}(2x^2 + px + q)^2 \\ &= \frac{1}{x^2}(4x^4 + p^2x^2 + q^2 + 4px^3 + 4qx^2 + 2pqx) \\ &= \frac{1}{x^2}[4x^4 + 4px^3 + (p^2 + 4q)x^2 + 2pqx + q^2] \end{aligned}$$

Comparing coefficients of the same power of x, we get $p = -5$, $q = -3$. Thus, the square root of $4x^2 - 20x + 13 + 30/x + 9/x^2$ is $\boldsymbol{2x - 5 - 3/x}$.

Problem 8. Find the square root of $49 - 84x - 34x^2 + 60x^3 + 25x^4$.

Solution.

$$\begin{aligned}49-84x-34x^2+60x^3+25x^4 &= (5x^2+px+q)^2\\ &= 25x^4+p^2x^2+q^2+10px^3+10qx^2+2pqx\\ &= 25x^4+10px^3+(p^2+10q)x^2+2pqx+q^2\end{aligned}$$

Comparing corresponding coefficients, we find $p=6$ and $q=-7$. Thus, the square root of $49-84x-34x^2+60x^3+25x^4$ is $\boldsymbol{5x^2+6x-7}$.

Problem 9. Find the square root of $x^8+2x^7-x^6-x^4-6x^3+5x^2-4x+4$.

Solution.

$$x^8+2x^7-x^6-x^4-6x^3+5x^2-4x+4=(x^4+px^3+qx^2+rx+s)^2=$$
$$x^8+p^2x^6+q^2x^4+r^2x^2+s^2+2px^7+2qx^6+2rx^5+2sx^4+2pqx^5+2prx^4+2psx^3+2qrx^3+2qsx^2+2rsx=$$
$$x^8+2px^7+(p^2+2q)x^6+(2r+2pq)x^5+(q^2+2s+2pr)x^4+(2ps+2qr)x^3+(r^2+2qs)x^2+2rsx+s^2,$$

which gives $2p=2 \implies p=1$, $1+2q=-1 \implies q=-1$, $2r-2=0 \implies r=1$, $1+2s+2=-1 \implies s=-2$. Thus, the square root of $x^8+2x^7-x^6-x^4-6x^3+5x^2-4x+4$ is $\boldsymbol{x^4+x^3-x^2+x-2}$.

Problem 10. Find the square root of $(x^2+1)^2-4x(x^2-1)$.

Solution.

$$(x^2+1)^2-4x(x^2-1)=x^4-4x^3+2x^2+4x+1=(x^2+px+q)^2=x^4+2px^3+(p^2+2q)x^2+2pqx+q^2,$$

which requires $2p=-4 \implies p=-2$, $4+2q=2 \implies q=-1$. Thus, the square root of $(x^2+1)^2-4x(x^2-1)$ is $\boldsymbol{x^2-2x-1}$.

Problem 11. Find the square root of $4x^4+9x^2y^2-12x^3y+16x^2-24xy+16$.

Solution.

$$\begin{aligned}4x^4+9x^2y^2-12x^3y+16x^2-24xy+16 &= 4x^4-12yx^3+(9y^2+16)x^2-24yx+16\\ &= (2x^2+px+q)^2\\ &= 4x^4+4px^3+(p^2+4q)x^2+2pqx+q^2,\end{aligned}$$

which requires $4p=-12y \implies p=-3y$, $9y^2+4q=9y^2+16 \implies q=4$. Thus, the square root of $4x^4+9x^2y^2-12x^3y+16x^2-24xy+16$ is $\boldsymbol{2x^2-3xy+4}$.

Problem 12. Find the square root of $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}+2+2x^2+2y^2+x^2y^2$.

Solution.

$$\begin{aligned}\frac{x^2}{y^2}+\frac{y^2}{x^2}+2+2x^2+2y^2+x^2y^2 &= \frac{1}{x^2y^2}(x^4+y^4+2x^2y^2+2x^4y^2+2x^2y^4+x^4y^4)\\ &= \frac{1}{x^2y^2}[(1+2y^2+y^4)x^4+2y^2(1+y^2)x^2+y^4]\\ &= \frac{1}{x^2y^2}[(1+y^2)^2x^4+2y^2(1+y^2)x^2+y^4]\\ &= \frac{1}{x^2y^2}[(1+y^2)x^2+px+q]^2\\ &= \frac{1}{x^2y^2}[(1+y^2)^2x^4+p^2x^2+q^2+2p(1+y^2)x^3+2q(1+y^2)x^2+2pqx]\\ &= \frac{1}{x^2y^2}\{(1+y^2)^2x^4+2p(1+y^2)x^3+[p^2+2q(1+y^2)]x^2+2pqx+q^2\},\end{aligned}$$

which requires $2p(1+y^2) = 0 \implies p = 0$, $2q(1+y^2) = 2y^2(1+y^2) \implies q = y^2$. Thus, the square root of $\frac{x^2}{y^2} + \frac{y^2}{x^2} + 2 + 2x^2 + 2y^2 + x^2y^2$ is $\frac{1}{xy}[(1+y^2)x^2 + y^2] = (1+y^2)\frac{x}{y} + \frac{y}{x} = \boldsymbol{\frac{x}{y} + xy + \frac{y}{x}}$.

Skip problems 13 and 14.

Problem 15. Find the cube root of $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$.

Solution. Making use of eq. (2), we have

$$\begin{aligned} x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1 &= (x^2 + px + q)^3 \\ &= x^6 + 3px^5 + (3q + 3p^2)x^4 + (p^3 + 6pq)x^3 + (3p^2q + 3q^2)x^2 + 3pq^2x + q^3 \end{aligned}$$

This requires $3p = 3 \implies p = 1$, $3q + 3 = 6 \implies q = 1$ and it is easy to check the other coefficients are satisfied as well. Thus, the cube root of the given polynomial is $\boldsymbol{x^2 + x + 1}$.

Problem 16. Find the cube root of $27x^{12} + 27x^{10} - 18x^8 - 17x^6 + 6x^4 + 3x^2 - 1$.

Solution. $27x^{12} + 27x^{10} - 18x^8 - 17x^6 + 6x^4 + 3x^2 - 1 = (3x^4 + px^3 + qx^2 + rx + s)^3$, where p, q, r, and s are to be determined. Using the expansion of $(a+b+c+d+e)^3$ with $a = 3x^4$, $b = px^3$, $c = qx^2$, $d = rx$, $e = s$, and collecting terms, we get

$$\begin{aligned} (3x^4 + px^3 + qx^2 + rx + s)^3 = {} & 27x^{12} + 27px^{11} + (27q + 9p^2)x^{10} + (p^3 + 27r + 8pq)x^9 + (27s + 3p^2q + 9q^2 + 18pr)x^8 \\ & + (3p^2r + 3q^2p + 18ps + 18qr)x^7 + (q^3 + 3p^2s + 9r^2 + 18qs + 6pqr)x^6 \\ & + (3q^2r + 3r^2p + 18rs)x^5 + (3q^2s + 3r^2q + 9s^2 + 6prs)x^4 + (r^3 + 3s^2p + 6qrs)x^3 \\ & + (3r^2s + 3s^2q)x^2 + 3s^2rx + s^3 \end{aligned}$$

Comparing coefficients of like powers of x, we find $p = 0$, $q = 1$, $r = 0$, $s = -1$, and these values are consistent with all the coefficients. Thus, the cube root of $27x^{12} + 27x^{10} - 18x^8 - 17x^6 + 6x^4 + 3x^2 - 1$ is $\boldsymbol{3x^4 + x^2 - 1}$.

Problem 17. Find the cube root of $8x^6 - 36ax^5 + 90a^2x^4 - 135a^3x^3 + 135a^4x^2 - 81a^5x + 27a^6$.

Solution.

$$8x^6 - 36ax^5 + 90a^2x^4 - 135a^3x^3 + 135a^4x^2 - 81a^5x + 27a^6 = (2x^2 + px + q)^3 =$$
$$8x^6 + p^3x^3 + q^3 + 3(2x^2)^2px + 3(2x^2)^2q + 3p^2x^2(2x^2) + 3p^2x^2q + 3q^2(2x^2) + 3q^2px + 6(2x^2)pxq =$$
$$8x^6 + 12px^5 + (12q + 6p^2)x^4 + (p^3 + 12pq)x^3 + (3p^2q + 6q^2)x^2 + 3pq^2x + q^3$$

Comparing coefficients of like powers of x gives $12p = -36a \implies p = -3a$, $12q + 54a^2 = 90a^2 \implies q = 3a^2$, and we find that these values are consistent for all the coefficients. Thus, the cube root of $8x^6 - 36ax^5 + 90a^2x^4 - 135a^3x^3 + 135a^4x^2 - 81a^5x + 27a^6$ is $\boldsymbol{2x^2 - 3ax + 3a^2}$.

Problem 18. Find the cube root of $\frac{x^3}{y^3} + \frac{y^3}{x^3} + 3\frac{x^2}{y^2} + 3\frac{y^2}{x^2} + 6\frac{x}{y} + 6\frac{y}{x} + 7$.

Solution.

$$\begin{aligned} \frac{x^3}{y^3} + \frac{y^3}{x^3} + 3\frac{x^2}{y^2} + 3\frac{y^2}{x^2} + 6\frac{x}{y} + 6\frac{y}{x} + 7 &= \frac{1}{x^3y^3}(x^6 + y^6 + 3x^5y + 3y^5x + 6x^4y^2 + 6y^4x^2 + 7x^3y^3) \\ &= \frac{1}{(xy)^3}(x^6 + 3yx^5 + 6y^2x^4 + 7y^3x^3 + 6y^4x^2 + 3y^5x + y^6) \\ &= \frac{1}{(xy)^3}(x^2 + px + q)^3 \\ &= \frac{1}{(xy)^3}(x^6 + 3px^5 + (3q + 3p^2)x^4 + (p^3 + 6pq)x^3 + (3p^2q + 3q^2)x^2 + 3pq^2x + q^3) \end{aligned}$$

Comparing coefficients of like powers of x, requires $3p = 3y \implies p = y$, $3q + 3y^2 = 6y^2 \implies q = y^2$. Also, these values satisfies all the other coefficients. Thus, the cube root of $\frac{x^3}{y^3} + \frac{y^3}{x^3} + 3\frac{x^2}{y^2} + 3\frac{y^2}{x^2} + 6\frac{x}{y} + 6\frac{y}{x} + 7$ is $\frac{x^2 + yx + y^2}{xy} = \boldsymbol{\frac{x}{y} + 1 + \frac{y}{x}}$.

Skip problem 19.

Problem 20. Find the fourth root of $x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1$.

Solution. Making use of eq. (3), we have

$x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1 = (x^2 + px + q)^4 =$
$x^8 + 4px^7 + (4q + 6p^2)x^6 + (4p^3 + 12pq)x^5 + (p^4 + 6q^2 + 12p^2q)x^4 + (4p^3q + 12pq^2)x^3 + (4q^3 + 6p^2q^2)x^2 + 4pq^3x + q^4$

Comparing coefficients of like powers of x, this requires $4p = -4 \implies p = -1$, $4q + 6 = 10 \implies q = 1$. Furthermore, we find that these values are consistent with all the coefficients. Thus, the fourth root of $x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1$ is $\boldsymbol{x^2 - x + 1}$.

Problem 21. Find the fifth root of $x^{10} + 5x^9 + 15x^8 + 30x^7 + 45x^6 + 51x^5 + 45x^4 + 30x^3 + 15x^2 + 5x + 1$.

Solution. Let $x^{10} + 5x^9 + 15x^8 + 30x^7 + 45x^6 + 51x^5 + 45x^4 + 30x^3 + 15x^2 + 5x + 1 = (x^2 + px + q)^5$, which requires $5p = 5$, $5q + 10p^2 = 15$, $10p^3 + 20pq = 30$, $5p^4 + 10q^2 + 30p^2q = 45$, $p^5 + 20p^3q + 30pq^2 = 51$, $5p^4q + 10q^3 + 30p^2q^2 = 45$, $10p^3q^2 + 20pq^3 = 30$, $5q^4 + 10p^2q^3 = 15$, $5pq^4 = 5$, and $q^5 = 1$. From the first and last of these, we have $p = 1$, $q = 1$, and it is easily check that these values satisfy the other equations. Thus,

$$x^{10} + 5x^9 + 15x^8 + 30x^7 + 45x^6 + 51x^5 + 45x^4 + 30x^3 + 15x^2 + 5x + 1 = (\boldsymbol{x^2 + x + 1})^5$$

Problem 22. To make $x^4 + 6x^3 + 11x^2 + ax + b$ a perfect square, what values must be assigned to a and b?

Solution. Let $x^4 + 6x^3 + 11x^2 + ax + b = (x^2 + px + q)^2 = x^4 + 2px^3 + (p^2 + 2q)x^2 + 2pqx + q^2$. This requires $2p = 6 \implies \boldsymbol{p = 3}$, $p^2 + 2q = 11 \implies \boldsymbol{q = 1}$, $2pq = a$, and $q^2 = b$. Therefore, in order for the expression to be a perfect square it is necessary that $\boldsymbol{a = 6}$ and $\boldsymbol{b = 1}$.

Chapter 12

Irrational Functions. Radicals and Fractional Exponents

12.1 Reducing Radicals, Exercise XXXIII (p. 274)

Reduce each of the following radicals to its simplest form.

Problem 1. $\sqrt{18}$

Solution. $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$

Problem 2. $\sqrt{588}$

Solution. $\sqrt{588} = \sqrt{4 \cdot 49 \cdot 3} = 14\sqrt{3}$

Problem 3. $\sqrt[3]{-27^2}$

Solution. $\sqrt[3]{-27^2} = \sqrt[3]{-(3 \cdot 9)^2} = \sqrt[3]{-(3^3)^2} = \sqrt[3]{-(3^2)^3} = -(3)^2 = -9$

Problem 4. $\sqrt[9]{-1000}$

Solution. $\sqrt[9]{-1000} = \sqrt[9]{-(10)^3} = -\sqrt[3]{10}$

Problem 5. $\sqrt{\dfrac{3}{2}}$

Solution. $\sqrt{\dfrac{3}{2}} = \dfrac{\sqrt{6}}{2}$

Problem 6. $\sqrt[3]{\dfrac{3}{2}}$

Solution. $\sqrt[3]{\dfrac{3}{2}} = \sqrt[3]{\dfrac{3 \cdot 4}{8}} = \dfrac{1}{2}\sqrt[3]{12}$

Problem 7. $\sqrt[3]{\dfrac{3}{4}}$

Solution. $\sqrt[3]{\dfrac{3}{4}} = \dfrac{1}{2}\sqrt[3]{6}$

Problem 8. $\sqrt[5]{\dfrac{3}{16}}$

Solution. $\sqrt[5]{\dfrac{3}{16}} = \dfrac{1}{2}\sqrt[5]{6}$

Problem 9. $\sqrt[5]{25a^5b^{10}c^{15}d^6}$

Solution. $\sqrt[5]{25a^5b^{10}c^{15}d^6} = \sqrt[5]{25(ab^2c^3d)^5d} = ab^2c^3d\sqrt[5]{25d}$

Problem 10. $\sqrt[6]{128a^2b^4c^8}$

Solution. $\sqrt[6]{128a^2b^4c^8} = \sqrt[6]{2^7a^2b^4c^8} = 2c\sqrt[6]{2a^2b^4c^2}$

Problem 11. $\sqrt[12]{8x^6y^9z^{15}}$

Solution. $\sqrt[12]{8x^6y^9z^{15}} = \sqrt[12]{(2x^2y^3z^5)^3} = \sqrt[4]{2x^2y^3z^5} = z\sqrt[4]{2x^2y^3z}$

Problem 12. $\sqrt[2n]{25a^2b^4c^6}$

Solution. $\sqrt[2n]{25a^2b^4c^6} = \sqrt[2n]{(5ab^2c^3)^2} = \sqrt[n]{5ab^2c^3}$

Problem 13. $\sqrt[3n]{a^nb^{2n}c^{3n}}$

Solution. $\sqrt[3n]{a^nb^{2n}c^{3n}} = c\sqrt[3n]{(ab^2)^n} = c\sqrt[3]{ab^2}$

Problem 14. $\sqrt[n]{a^{2n+1}b^{3n+2}c^{4n}}$

Solution. $\sqrt[n]{a^{2n+1}b^{3n+2}c^{4n}} = \sqrt[n]{ab^2(a^2b^3c^4)^n} = a^2b^3c^4\sqrt[n]{ab^2}$

Problem 15. $\sqrt{x^2y^2 - x^2z^2}$

Solution. $\sqrt{x^2y^2 - x^2z^2} = \sqrt{x^2(y^2 - z^2)} = x\sqrt{y^2 - z^2}$

Problem 16. $\sqrt{(x^2 - y^2)(x + y)}$

Solution. $\sqrt{(x^2 - y^2)(x + y)} = \sqrt{(x - y)(x + y)^2} = (x + y)\sqrt{x - y}$

Problem 17. $\sqrt[3]{x^6 - x^3y^3}$

Solution. $\sqrt[3]{x^6 - x^3y^3} = \sqrt[3]{x^3(x^3 - y^3)} = x\sqrt[3]{x^3 - y^3}$

Problem 18. $\sqrt[4]{a^4b^4 - 2a^3b^5 + a^2b^6}$

Solution. $\sqrt[4]{a^4b^4 - 2a^3b^5 + a^2b^6} = \sqrt[4]{a^2b^4(a^2 - 2ab + b^2)} = \sqrt[4]{a^2b^4(a - b)^2} = \sqrt[4]{[ab^2(a - b)]^2} = \sqrt{ab^2(a - b)} = b\sqrt{a(a - b)}$

Problem 19. $\sqrt[3]{\dfrac{a^3 + b^3}{32ab^2}}$

Solution. $\sqrt[3]{\dfrac{a^3 + b^3}{32ab^2}} = \sqrt[3]{\dfrac{a^3 + b^3}{2^5ab^2}} = \sqrt[3]{\dfrac{2a^2b(a^3 + b^3)}{2^6a^3b^3}} = \dfrac{\sqrt[3]{2a^2b(a^3 + b^3)}}{4ab}$

Problem 20. $\sqrt{\dfrac{a + b}{a - b}}$

Solution. $\sqrt{\dfrac{a+b}{a-b}} = \dfrac{\sqrt{a^2-b^2}}{a-b}$

Problem 21. $\sqrt[3]{\dfrac{x^2-x+1}{9(x+1)^2}}$

Solution. $\sqrt[3]{\dfrac{x^2-x+1}{9(x+1)^2}} = \dfrac{\sqrt[3]{3(x+1)(x^2-x+1)}}{3(x+1)} = \dfrac{\sqrt[3]{3(x^3+1)}}{3(x+1)}$

Problem 22. $\sqrt[3]{1-\dfrac{a^3}{b^3}}$

Solution. $\sqrt[3]{1-\dfrac{a^3}{b^3}} = \dfrac{1}{b}\sqrt[3]{b^3-a^3}$

Problem 23. $\sqrt[3]{\dfrac{c^{n+3}}{a^{3n}b^{3n+2}}}$

Solution. $\sqrt[3]{\dfrac{c^{n+3}}{a^{3n}b^{3n+2}}} = \sqrt[3]{\dfrac{c^3c^nb}{(a^nb^n)^3b^3}} = \dfrac{c}{b}\dfrac{\sqrt[3]{c^nb}}{(ab)^n} = \dfrac{c}{a^nb^{n+1}}\sqrt[3]{bc^n}$

Problem 24. $\sqrt{\dfrac{a^2x^2}{b^3} - \dfrac{2ax}{b^2} + \dfrac{1}{b}}$

Solution. $\sqrt{\dfrac{a^2x^2}{b^3} - \dfrac{2ax}{b^2} + \dfrac{1}{b}} = \sqrt{\dfrac{1}{b}\left(\dfrac{a^2x^2}{b^2} - \dfrac{2ax}{b} + 1\right)} = \sqrt{\dfrac{1}{b}\left(\dfrac{ax}{b}-1\right)^2} = \left(\dfrac{ax}{b}-1\right)\dfrac{1}{b}\sqrt{b} = \dfrac{ax-b}{b^2}\sqrt{b}$

Bring the coefficients of the following under the radical sign.

Problem 25. $3a\sqrt{3a}$

Solution. $3a\sqrt{3a} = \sqrt{9a^2 3a} = \sqrt{27a^3}$

Problem 26. $\dfrac{a+b}{a-b}\sqrt{\dfrac{a-b}{a+b}}$

Solution. $\dfrac{a+b}{a-b}\sqrt{\dfrac{a-b}{a+b}} = \sqrt{\dfrac{(a+b)^2}{(a-b)^2}\cdot\dfrac{a-b}{a+b}} = \sqrt{\dfrac{a+b}{a-b}}$

Problem 27. $3ax\sqrt[4]{\dfrac{1}{27a^3x^3}}$

Solution. $3ax\sqrt[4]{\dfrac{1}{27a^3x^3}} = \sqrt[4]{\dfrac{3^4a^4x^4}{3^3a^3x^3}} = \sqrt[4]{3ax}$

Show that the following sets of radicals are similar.

Problem 28. $\sqrt{18}$, $\sqrt{50}$, $\sqrt{\dfrac{1}{8}}$

Solution. $\sqrt{18} = \sqrt{9\cdot 2} = 3\sqrt{2}$, $\sqrt{50} = \sqrt{25\cdot 2} = 5\sqrt{2}$, $\sqrt{\dfrac{1}{8}} = \sqrt{\dfrac{1}{2^3}} = \dfrac{1}{2}\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

Problem 29. $\sqrt[3]{24}$, $\sqrt[3]{192}$, $\sqrt[3]{\dfrac{8}{9}}$

Solution. $\sqrt[3]{24} = \sqrt[3]{3 \cdot 8} = 2\sqrt[3]{3}$, $\sqrt[3]{192} = \sqrt[3]{64 \cdot 3} = 4\sqrt[3]{3}$, $\sqrt[3]{\dfrac{8}{9}} = \sqrt[3]{\dfrac{2^3 \cdot 3}{3^3}} = \dfrac{2}{3}\sqrt[3]{3}$

Problem 30. $\sqrt{(x^3 - y^3)(x - y)}$, $\sqrt{x^4y^2 + x^3y^3 + x^2y^4}$

Solution.
$\sqrt{(x^3 - y^3)(x - y)} = \sqrt{(x - y)(x^2 + xy + y^2)(x - y)} = (x - y)\sqrt{x^2 + xy + y^2}$
$\sqrt{x^4y^2 + x^3y^3 + x^2y^4} = \sqrt{(x^2y^2)(x^2 + xy + y^2)} = xy\sqrt{x^2 + xy + y^2}$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$$

The $(r+1)$th term in the series expansion is

$$\frac{\overbrace{n(n-1)(n-2)\cdots}^{r \text{ factors}}}{r!}\, a^{n-r}b^r$$

12.2 The Binomial Theorem for Negative and Fractional Exponents, Exercise XXXVI (p 284)

Expand each of the following to *four* terms.

Problem 1. $(1 + x)^{1/3}$

Solution.

$$(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}x^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2 \cdot 3}x^3 = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$$

Problem 2. $(a^{2/3} + x^{-2/3})^{-1/2}$

Solution.

$$\begin{aligned}
(a^{2/3} + x^{-2/3})^{-1/2} &= a^{-1/3}[1 + (ax)^{-2/3}]^{-1/2} \\
&= a^{-1/3}[1 - \frac{1}{2}(ax)^{-2/3} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(ax)^{-4/3} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{2 \cdot 3}(ax)^{-2}] \\
&= a^{-1/3}[1 - \frac{1}{2}(ax)^{-2/3} + \frac{3}{8}(ax)^{-4/3} - \frac{5}{16}(ax)^{-2}] \\
&= a^{-1/3} - \frac{1}{2}a^{-1}x^{-2/3} + \frac{3}{8}a^{-5/3}x^{-4/3} - \frac{5}{16}a^{-7/3}x^{-2}
\end{aligned}$$

Problem 3. $\sqrt[3]{(27 - 2x)^2}$

Solution.

$$\begin{aligned}\sqrt[3]{(27-2x)^2} = \sqrt[3]{(3^3-2x)^2} = \sqrt[3]{3^6\left(1-\frac{2x}{27}\right)^2} &= 9\left(1-\frac{2x}{27}\right)^{2/3} \\ &= 9\left[1-\frac{2}{3}\left(\frac{2x}{27}\right)+\frac{(\frac{2}{3})(-\frac{1}{3})}{2}\left(\frac{2x}{27}\right)^2+\frac{(\frac{2}{3})(-\frac{1}{3})(-\frac{4}{3})}{2\cdot 3}\left(\frac{2x}{27}\right)^3\right] \\ &= 9\left[1-\frac{4x}{3^4}-\frac{4x^2}{3^8}+\frac{32x^3}{3^{13}}\right] = 9-\frac{4x}{3^2}-\frac{4x^2}{3^6}+\frac{32x^3}{3^{11}}\end{aligned}$$

Problem 4. $(a^m + x)^{1/m}$

Solution.

$$\begin{aligned}(a^m+x)^{1/m} &= a(1+a^{-m}x)^{1/m} \\ &= a\left[1+\frac{1}{m}a^{-m}x+\frac{\frac{1}{m}(\frac{1}{m}-1)}{2}a^{-2m}x^2+\frac{\frac{1}{m}(\frac{1}{m}-1)(\frac{1}{m}-2)}{2\cdot 3}a^{-3m}x^3\right] \\ &= a\left[1+\frac{x}{ma^m}+\frac{(1-m)x^2}{2m^2a^{2m}}+\frac{(1-m)(1-2m)}{6m^3a^{3m}}x^3\right] \\ &= a+\frac{1}{m}a^{1-m}x+\frac{(1-m)}{2!\,m^2}a^{1-2m}x^2+\frac{(1m)(1-2m)}{3!\,m^3}a^{1-3m}x^3\end{aligned}$$

Problem 5. $(a^{-1} - b^{-1/2})^{-4}$

Solution.

$$\begin{aligned}(a^{-1}-b^{-1/2})^{-4} &= a^4(1-ab^{-1/2})^{-4} \\ &= a^4\left[1+(-4)(-ab^{-1/2})+\frac{(-4)(-5)}{2}(-ab^{-1/2})^2+\frac{(-4)(-5)(-6)}{2\cdot 3}(-ab^{-1/2})^3\right] \\ &= a^4\left[1+\frac{4a}{b^{1/2}}+10\frac{a^2}{b}+20\frac{a^3}{b^{3/2}}\right] = a^4+4a^5b^{-1/2}+10a^6b^{-1}+20a^7b^{-3/2}\end{aligned}$$

Problem 6. $(\sqrt{x} + \sqrt[3]{y})^{-6}$

Solution.

$$\begin{aligned}(\sqrt{x}+\sqrt[3]{y})^{-6} &= x^{-3}(1+x^{-1/2}y^{1/3})^{-6} \\ &= x^{-3}\left[1-6x^{-1/2}y^{1/3}+\frac{(-6)(-7)}{2}(x^{-1/2}y^{1/3})^2+\frac{(-6)(-7)(-8)}{2\cdot 3}(x^{-1/2}y^{1/3})^3\right] \\ &= x^{-3}-6x^{-7/2}y^{1/3}+21x^{-4}y^{2/3}-56x^{-9/2}y\end{aligned}$$

Problem 7. $\dfrac{1}{2+3x}$

Solution.

$$\begin{aligned}\frac{1}{2+3x} &= (2+3x)^{-1} = \frac{1}{2}\left(1+\frac{3x}{2}\right)^{-1} \\ &= \frac{1}{2}\left[1-\frac{3x}{2}+\frac{(-1)(-2)}{2}\left(\frac{3x}{2}\right)^2+\frac{(-1)(-2)(-3)}{2\cdot 3}\left(\frac{3x}{2}\right)^3\right] = \frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3\end{aligned}$$

Problem 8. $\dfrac{1}{\sqrt[5]{(1+x)^2}}$

Solution.

$$\frac{1}{\sqrt[5]{(1+x)^2}} = (1+x)^{-2/5} = 1 - \frac{2}{5}x + \frac{(-\frac{2}{5})(-\frac{2}{5}-1)}{2}x^2 + \frac{(-\frac{2}{5})(-\frac{2}{5}-1)(-\frac{2}{5}-2)}{2\cdot 3}x^3 = 1 - \frac{2}{5}x + \frac{7}{5^2}x^2 - \frac{28}{5^3}x^3$$

Problem 9. $\left(\dfrac{1}{\sqrt{1+3\sqrt{x}}}\right)^3$

Solution.

$$\left(\frac{1}{\sqrt{1+3\sqrt{x}}}\right)^3 = (1+3x^{1/2})^{-3/2} = 1 - \frac{3}{2}(3x^{1/2}) + \frac{(-\frac{3}{2})(-\frac{3}{2}-1)}{2}(3x^{1/2})^2 + \frac{(-\frac{3}{2})(-\frac{3}{2}-1)(-\frac{3}{2}-2)}{2\cdot 3}(3x^{1/2})^3$$
$$= 1 - \frac{9}{2}x^{1/2} + \frac{15}{8}9x - \frac{105}{16}9x^{3/2} = 1 - \frac{9}{2}x^{1/2} + \frac{135}{8}x - \frac{945}{16}x^{3/2}$$

Problem 10. Find the tenth term in $(1+x)^{-3}$.

Solution. The tenth term has $r+1=10$, so that $r=9$, and the term is

$$\frac{(-3)(-4)(-5)(-6)(-7)(-8)(-9)(-10)(-11)}{9!}x^9 = -\frac{9!(10)(11)}{2\cdot 9!}x^9 = -55x^9$$

Problem 11. Find the seventh term in $(x^{-2}-2y^{1/3})^{3/4}$.

Solution. The seventh term has $r+1=7$, so that $r=6$, and the term is

$$x^{-3/2}\frac{(\frac{3}{4})(\frac{3}{4}-1)(\frac{3}{4}-2)(\frac{3}{4}-3)(\frac{3}{4}-4)(\frac{3}{4}-5)}{6!}(-2x^2y^{1/3})^6 = x^{-3/2}\frac{(3)(-1)(-5)(-9)(-13)(-17)}{4^6\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}2^6x^{12}y^2 =$$
$$-\frac{(3)(13)(17)}{2^{10}}x^{21/2}y^2 = -\frac{663}{1024}x^{21/2}y^2$$

Problem 12. Find the term involving $x^{9/2}$ in $(1-x^{1/2})^{1/4}$.

Solution. This requires $r=9$ and the term is

$$\frac{(\frac{1}{4})(\frac{1}{4}-1)(\frac{1}{4}-2)(\frac{1}{4}-3)(\frac{1}{4}-4)(\frac{1}{4}-5)(\frac{1}{4}-6)(\frac{1}{4}-7)(\frac{1}{4}-8)}{9!}(-x^{1/2})^9 =$$
$$\frac{(\frac{1}{4})(-\frac{3}{4})(-\frac{7}{4})(-\frac{11}{4})(-\frac{15}{4})(-\frac{19}{4})(-\frac{23}{4})(-\frac{27}{4})(-\frac{31}{4})}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}(-x^{9/2}) = \frac{3\cdot 7\cdot 11\cdot 15\cdot 19\cdot 23\cdot 27\cdot 31}{4^9\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}(-x^{9/2}) =$$
$$-\frac{33\cdot 19\cdot 23\cdot 31}{2^{25}}x^{9/2} = -\frac{447051}{33554432}x^{9/2}$$

Problem 13. Find the term involving x^{-2} in $x^{-3/2}(2+x^{-1/6})^{-3}$.

Solution. $x^{-3/2}(2+x^{-1/6})^{-3} = 2^{-3}x^{-3/2}\left(1+\dfrac{1}{2x^{1/6}}\right)^{-3}$ and we want $x^{-3/2}x^{-r/6} = x^{-2} \implies r = 3$. Therefore, the term is

$$2^{-3}x^{-3/2}\frac{(-3)(-4)(-5)}{2\cdot 3}\frac{1}{(2x^{1/6})^3} = -\frac{10}{2^6}x^{-2} = -\frac{5}{32}x^{-2}$$

Problem 14. By the method illustrated in §601, Ex. 4, find approximate values of the following.

(1) $\sqrt{99}$. **(2)** $\sqrt[3]{62}$. **(3)** $\sqrt[5]{31}$.

Solution.

(1) $$\sqrt{99} = \sqrt{100-1} = 10\left(1 - \frac{1}{100}\right)^{1/2} = 10\left[1 - \frac{1}{2}\left(\frac{1}{100}\right) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}\left(\frac{1}{100}\right)^2 + \cdots\right]$$
$$= 10\left[1 - \frac{1}{200} - \frac{1}{80000} - \cdots\right] = 9.9498$$

(2) $$\sqrt[3]{62} = \sqrt[3]{64-2} = 4\left(1 - \frac{2}{64}\right)^{1/3} = 4(1-2^{-5})^{1/3} = 4\left[1 + \frac{1}{3}(-2^{-5}) + \frac{(\frac{1}{3})(\frac{1}{3}-1)}{2}(-2^{-5})^2 + \cdots\right]$$
$$= 4\left[1 - \frac{1}{3}\frac{1}{32} - \frac{1}{9}\frac{1}{(32)^2} + \cdots\right] = 3.9578$$

(3) $$\sqrt[5]{31} = \sqrt[5]{32-1} = 2\left(1 - \frac{1}{32}\right)^{1/5} = 2\left[1 - \frac{1}{5}\frac{1}{32} + \frac{(\frac{1}{5})(-\frac{4}{5})}{2}\left(\frac{1}{32}\right)^2 + \cdots\right]$$
$$= 2\left[1 - \frac{1}{5}\frac{1}{32} - \frac{2}{25}\frac{1}{(32)^2} + \cdots\right] = 1.9873$$

12.3 Irrational Equations, Exercise XXXVIII (p. 290)

Solve the following equations for x.

Problem 1. $x^{1/4} = 4$.

Solution. $x^{1/4} = 4 \implies x = 4^4 = 256$.

Problem 2. $x^{-1/2} = 3$.

Solution. $x^{-1/2} = 3 \implies x^{1/2} = \frac{1}{3} \implies x = \frac{1}{9}$.

Problem 3. $x^{2/3} = 8$.

Solution. $x^{2/3} = 8 \implies x^{2/3} = 2^3 \implies x = (2^3)^{3/2} = \pm\sqrt{2^9} = \pm\sqrt{2^8 \cdot 2} = \pm 16\sqrt{2}$.

Problem 4. $(\sqrt{2x-1})^{1/3} = \sqrt{3}$.

Solution. $(\sqrt{2x-1})^{1/3} = \sqrt{3} \implies \sqrt{2x-1} = (\sqrt{3})^3 = 3^{3/2} \implies 2x - 1 = 3^3 = 27 \implies x = 14$.

Problem 5. $\sqrt{2 + \sqrt{3 + \sqrt{x}}} = 2$.

Solution. $\sqrt{2 + \sqrt{3 + \sqrt{x}}} = 2 \implies 2 + \sqrt{3 + \sqrt{x}} = 4 \implies 3 + \sqrt{x} = 4 \implies \sqrt{x} = 1 \implies x = 1$.

Problem 6. $\sqrt{ax} + \sqrt{bx} + \sqrt{cx} = d$.

Solution. $\sqrt{ax} + \sqrt{bx} + \sqrt{cx} = d \implies ax + bx + cx + 2(\sqrt{ax}\sqrt{bx} + \sqrt{ax}\sqrt{cx} + \sqrt{bx}\sqrt{cx}) = d^2 \implies [a + b + c + 2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})]x = d^2 \implies x = \left(\dfrac{d}{\sqrt{a} + \sqrt{b} + \sqrt{c}}\right)^2$.

Problem 7. $\sqrt{4x^2 + x + 10} = 2x + 1$.

Solution. $\sqrt{4x^2+x+10} = 2x+1 \implies 4x^2+x+10 = 4x^2+4x+1 \implies 3x = 9 \implies x = 3.$

Problem 8. $\sqrt{x+4}+\sqrt{x+11} = 7.$

Solution. $\sqrt{x+4}+\sqrt{x+11} = 7 \implies x+4+2\sqrt{(x+4)(x+11)}+x+11 = 49 \implies \sqrt{(x+4)(x+11)} = 17-x \implies (x+4)(x+11) = 289-34x+x^2 = x^2+15x+44 \implies 49x = 245 \implies x = 5.$

Problem 9. $\sqrt{4x+5}+\sqrt{x+1}-\sqrt{9x+10} = 0.$

Solution. $\sqrt{4x+5}+\sqrt{x+1}-\sqrt{9x+10} = 0.$

Problem 10. $\sqrt{x+1}+\dfrac{x-6}{\sqrt{x+2}} = 0.$

Solution. $\sqrt{x+1}+\dfrac{x-6}{\sqrt{x+2}} = 0 \implies \sqrt{(x+1)(x+2)} = 6-x \implies x^2+3x+2 = 36-12x+x^2$
$\implies x = \dfrac{34}{15}.$

Problem 11. $\sqrt{x^2+3x-1}-\sqrt{x^2-x-1} = 2.$

Solution.

$$
\begin{aligned}
\sqrt{x^2+3x-1}-\sqrt{x^2-x-1} = 2 &\implies x^2+3x-1+x^2-x-1-2\sqrt{(x^2+3x-1)(x^2-x-1)} = 4\\
&\implies x^2+x-3 = \sqrt{(x^2+3x-1)(x^2-x-1)}\\
&\implies x^4+x^2+9+2x^3-6x^2-6x = x^4-x^3-x^2+3x^3-3x^2-3x-x^2+x+1\\
&\implies x^4+2x^3-5x^2-6x+9 = x^4+2x^3-5x^2-2x+1 \implies 4x = 8\\
&\implies x = 2
\end{aligned}
$$

Problem 12. $\sqrt{x+7}+\sqrt{x-2} = \sqrt{x+2}+\sqrt{x-1}.$

Solution.

$$
\begin{aligned}
\sqrt{x+7}+\sqrt{x-2} = \sqrt{x+2}+\sqrt{x-1} &\implies x+7+x-2+2\sqrt{(x+7)(x-2)} = x+2+x-1+2\sqrt{(x+2)(x-1)}\\
&\implies 5+2\sqrt{(x+7)(x-2)} = 1+2\sqrt{(x+2)(x-1)}\\
&\implies 2 = \sqrt{(x+2)(x-1)}-\sqrt{(x+7)(x-2)}\\
&\implies 4 = (x+2)(x-1)+(x+7)(x-2)-2\sqrt{(x+2)(x-1)(x+7)(x-2)}\\
&\implies 4 = x^2+x-2+x^2+5x-14-2\sqrt{(x+2)(x-1)(x+7)(x-2)}\\
&\implies x^2+3x-10 = \sqrt{(x+2)(x-1)(x+7)(x-2)}\\
&\implies x^4+9x^2+100+6x^3-20x^2-60x = x^4+6x^3-11x^2-24x+28\\
&\implies 36x = 72 \implies x = 2
\end{aligned}
$$

Problem 13. $\dfrac{\sqrt{x+3}+\sqrt{x-5}}{\sqrt{x+3}-\sqrt{x-5}} = 2.$

Solution. $\dfrac{\sqrt{x+3}+\sqrt{x-5}}{\sqrt{x+3}-\sqrt{x-5}} = 2 \implies \sqrt{x+3}+\sqrt{x-5} = 2\sqrt{x+3}-2\sqrt{x-5} \implies \sqrt{x+3} = 3\sqrt{x-5} \implies x+3 = 9(x-5) \implies 8x = 48 \implies x = 6.$

Problem 14. $\dfrac{1}{\sqrt{x+1}}-\dfrac{1}{\sqrt{x-1}}+\dfrac{1}{\sqrt{x^2-1}} = 0.$

Solution.

$$
\begin{aligned}
\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x-1}} + \frac{1}{\sqrt{x^2-1}} = 0 &\implies \frac{1}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x+1}} \\
&\implies \frac{1}{x^2-1} = \frac{1}{x-1} + \frac{1}{x+1} - \frac{2}{\sqrt{(x-1)(x+1)}} \\
&\implies \frac{1}{x^2-1} = \frac{2x}{x^2-1} - \frac{2}{\sqrt{x^2-1}} \\
&\implies \frac{2}{\sqrt{x^2-1}} = \frac{2x-1}{x^2-1} \\
&\implies 2\sqrt{x^2-1} = 2x-1 \\
&\implies 4(x^2-1) = 4x^2-4x+1 \implies 4x = 5 \implies x = \frac{5}{4}.
\end{aligned}
$$

Solve the following for x and y.

Problem 15. $\begin{cases} \sqrt{x+17} + \sqrt{y-2} = \sqrt{x+5} + \sqrt{y+6} \\ \sqrt{y-x} = \sqrt{3-x} + \sqrt{y-3} \end{cases}$

Solution. By inspection we see that the second equation is satisfied by both $x = 3$ and by $y = 3$. We substitute each of these into the first equation to solve.

$$
\begin{aligned}
\boldsymbol{x = 3}:\ \sqrt{20} + \sqrt{y-2} = \sqrt{8} + \sqrt{y+6} &\implies \sqrt{20} - \sqrt{8} = \sqrt{y+6} - \sqrt{y-2} \\
&\implies 20 + 8 - 2\sqrt{160} = y + 6 + y - 2 - 2\sqrt{(y+6)(y-2)} \\
&\implies y + \sqrt{160} - 12 = \sqrt{(y+6)(y-2)} \\
&= y^2 + (\sqrt{160} - 12)^2 + 2(\sqrt{160} - 12)y = y^2 + 4y - 12 \\
&\implies (14 - \sqrt{160})y = 158 - 12\sqrt{160} \\
&\implies y = \frac{158 - 12\sqrt{160}}{14 - \sqrt{160}} \cdot \frac{14 + \sqrt{160}}{14 + \sqrt{160}} \\
&\implies y = \frac{73 - 10\sqrt{10}}{9}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{y = 3}:\ \sqrt{x+17} + 1 = \sqrt{x+5} + 3 &\implies 2 = \sqrt{x+17} - \sqrt{x+5} \\
&\implies 4 = x + 17 + x + 5 - 2\sqrt{(x+17)(x+5)} \\
&\implies \sqrt{(x+17)(x+5)} = x + 9 \\
&\implies x^2 + 22x + 85 = x^2 + 18x + 81 \\
&\implies 4x = -4 \implies x = -1
\end{aligned}
$$

Thus, the solutions are $x = 3$, $y = (73 - 10\sqrt{10})/9$ and $x = -1$, $y = 3$.

Problem 16. $\begin{cases} 3\sqrt{x-2y} - \sqrt{x+y-4} = 3 \\ \sqrt{x-2y} + 2\sqrt{x+y-4} = 8 \end{cases}$

Solution. Let us write this as a matrix equation:

$$
\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{x-2y} \\ \sqrt{x+y-4} \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}
$$

We can solve this with Cramer's rule to get $\sqrt{x-2y}=2$ and $\sqrt{x+y-4}=3$, or, $x-2y=4$ and $x+y=13$. Writing this as the matrix equation

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix},$$

Cramer's rule gives $x=10$ and $y=3$.

Problem 17. Show that $\sqrt{x+a}+\sqrt{x+b}+\sqrt{x+c}+\sqrt{x+d}=0$ will reduce to a rational equation of the first degree.

Solution. $\sqrt{x+a}+\sqrt{x+b}+\sqrt{x+c}+\sqrt{x+d}=0 \implies \sqrt{x+a}+\sqrt{x+b}=-(\sqrt{x+c}+\sqrt{x+d})$. Squaring both sides gives

$$\begin{aligned}
&x+a+x+b+2\sqrt{(x+a)(x+b)}=x+c+x+d+2\sqrt{(x+c)(x+d)} \implies \\
&a+b+2\sqrt{(x+a)(x+b)}=c+d+2\sqrt{(x+c)(x+d)} \implies \\
&2\sqrt{(x+a)(x+b)}=2\sqrt{(x+c)(x+d)}+c+d-a-b \implies \\
&4(x+a)(x+b)=4(x+c)(x+d)+(c+d-a-b)^2+4(c+d-a-b)\sqrt{(x+c)(x+d)} \implies \\
&4(c+d-a-b)\sqrt{(x+c)(x+d)}=4(x+a)(x+b)-4(x+c)(x+d)-(c+d-a-b)^2 \implies \\
&16(c+d-a-b)^2(x+c)(x+d)=16(x+a)^2(x+b)^2+16(x+c)^2(x+d)^2+(c+d-a-b)^4 \\
&-32(x+a)(x+b)(x+c)(x+d)+8(c+d-a-b)^2[-(x+a)(x+b)+(x+c)(x+d)] \implies \\
&32(x+a)(x+b)(x+c)(x+d)=16(x+a)^2(x+b)^2+16(x+c)^2(x+d)^2+(c+d-a-b)^4 \\
&-8(c+d-a-b)^2[(x+a)(x+b)+(x+c)(x+d)] \implies \\
&16[(x+a)(x+b)-(x+c)(x+d)]^2+(c+d-a-b)^2 \\
&-8(c+d-a-b)^2[(x+a)(x+b)+(x+c)(x+d)]=0 \implies \\
&16\{[\cancel{x^2}+(a+b)x+ab]-[\cancel{x^2}+(c+d)x+cd]\}^2+(c+d-a-b)^4 \\
&-8(c+d-a-b)^2[x^2+(a+b)x+ab+x^2+(c+d)x+cd]=0 \implies \\
&16[(a+b-c-d)x+ab-cd]^2+(c+d-a-b)^4-8(c+d-a-b)^2[2x^2+(a+b+c+d)x+ab+cd]=0 \implies \\
&16[(a+b-c-d)^2\cancel{x^2}+(ab-cd)^2+2(ab-cd)(a+b-c-d)x]+(c+d-a-b)^4 \\
&-8(c+d-a-b)^2[2\cancel{x^2}+(a+b+c+d)x+ab-cd]=0 \implies \\
&16(ab-cd)^2+32(ab-cd)(a+b-c-d)x+(c+d-a-b)^4-8(c+d-a-b)^2(a+b+c+d)xz \\
&-8(ab+cd)(c+d-a-b)^2=0 \implies \\
&[32(ab-cd)(a+b-c-d)-8(c+d-a-b)^2(a+b+c+d)]x=8(ab+cd)(c+d-a-b)^2-16(ab-cd)^2 \\
&-(c+d-a-b)^4
\end{aligned}$$

Thus, we get the rational expression for x:

$$\begin{aligned}
x &= \frac{8(ab+cd)(c+d-a-b)^2-16(ab-cd)^2-(c+d-a-b)^4}{32(ab-cd)(a+b-c-d)-8(c+d-a-b)^2(a+b+c+d)} \\
&= \frac{16(ab-cd)^2+(c+d-a-b)^2[(c+d-a-b)^2-8(ab+cd)]}{(c+d-a-b)[32(ab-cd)+8(c+d-a-b)(a+b+c+d)]}
\end{aligned}$$

Problem 18. Show that $\sqrt{ax+b}+\sqrt{cx+d}-\sqrt{ex+f}=0$ will reduce to a rational equation of the first degree if $\sqrt{a}+\sqrt{c}-\sqrt{e}=0$.

Solution.

$$\begin{aligned}\sqrt{ax+b}+\sqrt{cx+d}-\sqrt{ex+f}=0 &\implies ax+b+cx+d+2\sqrt{(ax+b)(cx+d)}=ex+f\\ &\implies 2\sqrt{(ax+b)(cx+d)}=(e-a-c)x+(f-b-d)\\ &\implies 4(ax+b)(cx+d)=(e-a-c)^2x^2+(f-b-d)^2+2(f-b-d)(e-a-c)x\end{aligned}$$

The coefficient of the x^2 term on the left side is $4ac$ and on the right side is $(e-a-c)^2$. If these two terms are equal, then we will have a linear equation for x. Thus, the condition for a rational equation of the first degree is

$$\begin{aligned}(e-a-c)^2=4ac &\implies e-a-c=2\sqrt{a}\sqrt{c}\\ &\implies e=(\sqrt{a}+\sqrt{c})^2\\ &\implies \sqrt{e}=\sqrt{a}+\sqrt{c}\\ &\implies \sqrt{a}+\sqrt{c}-\sqrt{e}=0.\end{aligned}$$

$$\begin{aligned}(a+b+c+d)^2&=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd\\ (a+b+c+d)^3&=a^3+b^3+c^3+d^3+3a^2b+3a^2c+3a^2d+3b^2a+3b^2c+3b^2d+3c^2a+3c^2b+3c^2d+3d^2a\\ &\quad+3d^2b+3d^2c+6abc+6abd+6acd+6bcd\end{aligned}$$

$$\begin{aligned}(a+b+c+d+e)^2&=a^2+b^2+c^2+d^2+e^2+2ab+2ac+2ad+2ae+2bc+2bd+2be+2cd+2ce+2de\\ (a+b+c+d+e)^3&=a^3+b^3+c^3+d^3+e^3+3a^2b+3a^2c+3a^2d+3a^2e+3b^2a+3b^2c+3b^2d+3b^2e\\ &\quad+3c^2a+3c^2b+3c^2d+3c^2e+3d^2a+3d^2b+3d^2c+3d^2e+3e^2a+3e^2b+3e^2c+3e^2d\\ &\quad+6abc+6abd+6abe+6acd+6ace+6ade+6bcd+6bce+6bde+6cde\end{aligned}$$

Square Roots of Binomial Surds

We have

$$(\sqrt{x}\pm\sqrt{y})^2=x+y\pm 2\sqrt{xy}.$$

Hence if $a+2\sqrt{b}$ denotes a given binomial surd, and we can find two *positve rational* numbers x and y such that

$$x+y=a \text{ and } xy=b,$$

then $\sqrt{x}+\sqrt{y}$ will be a square root of $a+2\sqrt{b}$ and $\sqrt{x}-\sqrt{y}$ will be a square root of $a-2\sqrt{b}$.

Formulas for x and y:

$$\sqrt{x}+\sqrt{y}=\sqrt{a+2\sqrt{b}} \quad\text{and}\quad \sqrt{x}-\sqrt{y}=\sqrt{a-2\sqrt{b}} \quad\implies\quad x-y=\sqrt{a^2-4b}.$$

$$\text{And with}\quad x+y=a \quad\implies\quad x=\frac{a+\sqrt{a^2-4b}}{2} \quad\text{and}\quad y=\frac{a-\sqrt{a^2-4b}}{2}.$$

12.4 Square Roots of Quadratic Surds, Exercise XXXIX (p. 293)

Find square roots of the following.

Problem 1. $9+\sqrt{56}$

Solution $9+\sqrt{56} = 9+2\sqrt{14} = 2+7+2\sqrt{(2)(7)} \implies \sqrt{9+\sqrt{56}} = \sqrt{2}+\sqrt{7}$.

Problem 2. $20+2\sqrt{96}$

Solution $20+2\sqrt{96} = 8+12+2\sqrt{(8)(12)} \implies \sqrt{20+2\sqrt{96}} = \sqrt{8}+\sqrt{12} = 2(\sqrt{2}+\sqrt{3})$.

Problem 3. $32-2\sqrt{175}$

Solution. $32-2\sqrt{175} = 25+7-2\sqrt{(25)(7)} \implies \sqrt{32-2\sqrt{175}} = \sqrt{25}-\sqrt{7} = 5-\sqrt{7}$

Problem 4. $1+\dfrac{2\sqrt{6}}{5}$

Solution. $1+\dfrac{2\sqrt{6}}{5} = \dfrac{5+2\sqrt{6}}{5} = \dfrac{2+3+2\sqrt{(2)(3)}}{5} \implies \sqrt{1+\dfrac{2\sqrt{6}}{5}} = \dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{5}} = \dfrac{\sqrt{10}+\sqrt{15}}{5}$.

Problem 5. $7-3\sqrt{5}$

Solution. $7-3\sqrt{5} = 7-\sqrt{45} = \dfrac{14-2\sqrt{45}}{2} = \dfrac{9+5-2\sqrt{(9)(5)}}{2} \implies \sqrt{7-3\sqrt{5}} = \dfrac{\sqrt{9}-\sqrt{5}}{\sqrt{2}} = \dfrac{\sqrt{18}-\sqrt{10}}{2} = \dfrac{3\sqrt{2}-\sqrt{10}}{2}$.

Problem 6. $8\sqrt{2}+2\sqrt{30}$

Solution. $8\sqrt{2}+2\sqrt{30} = 8\sqrt{2}+\sqrt{2}\sqrt{60} = \sqrt{2}(8+\sqrt{60}) = \sqrt{2}(8+2\sqrt{15}) = \sqrt{2}(5+3+2\sqrt{(5)(3)}) \implies \sqrt{8\sqrt{2}+2\sqrt{30}} = \sqrt[4]{2}(\sqrt{5}+\sqrt{3})$.

Problem 7. $2(a+\sqrt{a^2-b^2})$

Solution. $2(a+\sqrt{a^2-b^2}) = 2a+2\sqrt{(a-b)(a+b)} = (a-b)+(a+b)+2\sqrt{(a-b)(a+b)} \implies \sqrt{2(a+\sqrt{a^2-b^2})} = \sqrt{a-b}+\sqrt{a+b}$.

Problem 8. $b-2\sqrt{ab-a^2}$

Solution. $b-2\sqrt{ab-a^2} = a+(b-a)-2\sqrt{a(b-a)} \implies \sqrt{b-2\sqrt{ab-a^2}} = \sqrt{a}-\sqrt{b-a}$.

Simplify the following.

Problem 9. $\sqrt[4]{17+12\sqrt{2}}$

Solution. First, $17+12\sqrt{2} = 17+2\sqrt{72} = 9+8+2\sqrt{(9)(8)} \implies \sqrt{17+12\sqrt{2}} = \sqrt{9}+\sqrt{8} = 3+2\sqrt{2}$. But $3+2\sqrt{2} = 1+2+2\sqrt{(1)(2)} \implies \sqrt[4]{17+12\sqrt{2}} = \sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$.

Problem 10. $\sqrt{9+4\sqrt{4+2\sqrt{3}}}$

Solution. First, $4+2\sqrt{3} = 1+3+2\sqrt{(1)(3)} \implies \sqrt{4+2\sqrt{3}} = 1+\sqrt{3}$. Hence, $\sqrt{9+4\sqrt{4+2\sqrt{3}}} = \sqrt{9+4(1+\sqrt{3})} = \sqrt{9+4+4\sqrt{3}} = \sqrt{13+2\sqrt{12}} = \sqrt{1+12+2\sqrt{(1)(12)}} = 1+\sqrt{12} = 1+2\sqrt{3}$.

The powers of i

From the equation $i^2 = -1$ it follows that the even powers of i are either -1 or 1, and the odd powers either i or $-i$.

To find the value of i^n for any given value of n, divide n by 4. Then, according as the remainder is 0, 1, 2, 3, the value of i^n is 1, i, -1, $-i$.

Square Roots of Complex Numbers

We have

$$(\sqrt{x} \pm i\sqrt{y})^2 = x - y \pm 2i\sqrt{xy}.$$

Hence, if $a + bi$ denotes a given complex number in which b is *positive*, and we can find two positive numbers x and y such that

$$\boxed{x - y = a \quad \text{and} \quad 2\sqrt{xy} = b}$$

then $\sqrt{x} + i\sqrt{y}$ will be a square root of $a + bi$, and $\sqrt{x} - i\sqrt{y}$ will be a square root of $a - bi$. We may find such numbers x and y as follows. By hypothesis

$$\sqrt{x} + i\sqrt{y} = \sqrt{a + bi} \quad \text{and} \quad \sqrt{x} - i\sqrt{y} = \sqrt{a - bi}.$$

Multiplying together gives

$$x + y = \sqrt{a^2 + b^2}.$$

And since $x - y = a$, we get

$$x = \frac{a + \sqrt{a^2 + b^2}}{2} \quad \text{and} \quad y = \frac{-a + \sqrt{a^2 + b^2}}{2}.$$

And both these values are positive since $\sqrt{a^2 + b^2} > a$.

12.5 Square Roots of Complex Numbers, Exercise XL (p. 297)

Simplify the following.

Problem 1. $\sqrt{-49}$

Solution. $\sqrt{-49} = 7i$

Problem 2. $\sqrt{-18}$

Solution. $\sqrt{-18} = 3\sqrt{2}i$

Problem 3. $\sqrt{-8}\,\sqrt{-12}$

Solution. $\sqrt{-8}\,\sqrt{-12} = 2\sqrt{2}i\,2\sqrt{3}i = -4\sqrt{6}$

Problem 4. $\sqrt{-2^2}$

Solution. $\sqrt{-2^2} = i\sqrt{4} = 2i$

Problem 5. $(\sqrt{-2})^2$

Solution. $(\sqrt{-2})^2 = (i\sqrt{2})^2 = -2$

Problem 6. i^{12}

Solution. Dividing the exponent by 4 gives a quotient of 3 and a remainder of 0. Accordingly, $i^{12} = 1$

Problem 7. i^{-7}

Solution. $i^{-7} = \dfrac{1}{i^7} = \dfrac{1}{-i}$(since $7 \div 4$ gives a remainder of 3) $= \dfrac{i}{-i^2} = i$

Problem 8. i^{15}

Solution. $15 \div 4$ gives a remainder of 3. Hence, $i^{15} = -i$

Problem 9. $\sqrt{x-y} \cdot \sqrt{y-x}$

Solution. $\sqrt{x-y} \cdot \sqrt{y-x} = \sqrt{x-y} \cdot \sqrt{-(x-y)} = (x-y)i$

Problem 10. $(2+\sqrt{-3})(1+\sqrt{-2})$

Solution. $(2+\sqrt{-3})(1+\sqrt{-2}) = (2+\sqrt{3}i)(1+\sqrt{2}i) = 2-\sqrt{6}+(2\sqrt{2}+\sqrt{3})i$

Problem 11. $(\sqrt{-2})^7(\sqrt{-3})^9$

Solution. $(\sqrt{-2})^7(\sqrt{-3})^9 = (\sqrt{2}i)^7(\sqrt{3}i)^9 = [(\sqrt{2})^2]^3\sqrt{2}\,[(\sqrt{3})^2]^4\sqrt{3}\,i^{16} = 8\sqrt{2}{\cdot}81\sqrt{3} = 648\sqrt{6}$ since $i^{16} = 1$

Problem 12. $(1+2i)^3 + (1-2i)^3$

Solution. $(1+2i)^3 + (1-2i)^3 = 1+3(2i)+3(2i)^2+(2i)^3+1-3(2i)+3(2i)^2-(2i)^3 = 2+6(-4) = -22$

Problem 13. $\dfrac{a}{\sqrt{-a^2}} - \dfrac{b}{i\sqrt{b^2}}$

Solution. $\dfrac{a}{\sqrt{-a^2}} - \dfrac{b}{i\sqrt{b^2}} = \dfrac{a}{ia} - \dfrac{b}{ib} = \dfrac{1}{i} - \dfrac{1}{i} = 0$

Problem 14. $\dfrac{4+6i}{1+i} + \dfrac{4-6i}{1-i}$

Solution. $\dfrac{4+6i}{1+i} + \dfrac{4-6i}{1-i} = \dfrac{4+6i}{1+i} \cdot \dfrac{1-i}{1-i} + \dfrac{4-6i}{1-i} \cdot \dfrac{1+i}{1+i} = \dfrac{4+6+2i}{2} + \dfrac{4+6-2i}{2} = 10$

Problem 15. $(\sqrt{3+4i} + \sqrt{3-4i})^2$

Solution. $(\sqrt{3+4i} + \sqrt{3-4i})^2 = 3+4i+3-4i+2\sqrt{(3+4i)(3-4i)} = 6+2\sqrt{9+16} = 16$

Problem 16. $\dfrac{1+i^3}{1+i}$

Solution. $\dfrac{1+i^3}{1+i} = \dfrac{1-i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{1-1-2i}{2} = -i$

Problem 17. $\dfrac{a+bi}{a-bi}$

Solution. $\dfrac{a+bi}{a-bi} = \dfrac{a+bi}{a-bi} \cdot \dfrac{a+bi}{a+bi} = \dfrac{a^2-b^2+2abi}{a^2+b^2} = \dfrac{a^2-b^2}{a^2+b^2} + \dfrac{2ab}{a^2+b^2}i$

Problem 18. $\dfrac{9+3\sqrt{2}i}{(3+\sqrt{2}i)(1+\sqrt{2}i)}$

Solution. $\dfrac{9+3\sqrt{2}i}{(3+\sqrt{2}i)(1+\sqrt{2}i)} = \dfrac{3(3+\sqrt{2}i)}{(3+\sqrt{2}i)(1+\sqrt{2}i)} = \dfrac{3}{1+\sqrt{2}i}\cdot\dfrac{1-\sqrt{2}i}{1-\sqrt{2}i} = \dfrac{3-3\sqrt{2}i}{3} = 1-\sqrt{2}i$

Problem 19. $\dfrac{4}{1+\sqrt{-3}}$

Solution. $\dfrac{4}{1+\sqrt{-3}} = \dfrac{4}{1+\sqrt{3}i}\cdot\dfrac{1-\sqrt{3}i}{1-\sqrt{3}i} = \dfrac{4-4\sqrt{3}i}{4} = 1-\sqrt{3}i$

Problem 20. Find a fourth root of -16.

Solution. $\sqrt[4]{-16} = \sqrt{\sqrt{-16}} = \sqrt{i\sqrt{16}} = \sqrt{4i}$. Applying the formulas to $4i$, with $a = 0$, $b = 4$, we get $x = 2$ and $y = 2$, so that $\sqrt{4i} = \sqrt{2}+\sqrt{2}i = \sqrt{2}(1+i)$. A quicker way of solving this is as follows. $4i = 2-2+2\sqrt{(2)(2)}i \implies \sqrt{4i} = \sqrt{2}+\sqrt{2}i = \sqrt{2}(1+i)$.

Problem 21. Show that $(-1+\sqrt{3}i)/2$ is a cube root of 1.

Solution.

$(-1+\sqrt{3}i)^3 = (-1)^3+3(-1)^2(\sqrt{3}i)+3(-1)(\sqrt{3}i)^2+(\sqrt{3}i)^3 = -1+3\sqrt{3}i+9-3\sqrt{3}i = 8 \implies$
$[(-1+\sqrt{3}i)/2]^3 = 8/8 = 1$

Problem 22. Show that $(1+i)/\sqrt{2}$ is a fourth root of -1.

Solution. $(1+i)^4 = 1+4i+6i^2+4i^3+i^4 = 1+4i-6-4i+1 = -4 \implies \left(\dfrac{1+i}{\sqrt{2}}\right)^4 = \dfrac{-4}{4} = -1.$

Problem 23. Find real values of x and y satisfying the equation

$$3+2i+x(i-1)+2yi = (3i+4)(x+y)$$

Solution. Equating real and imaginary parts on both sides gives $3+2i+x(i-1)+2yi = (3i+4)(x+y) \implies 3+2i+xi-x+2yi = 3xi+3yi+4x+4y \implies 3-x+(2+x+2y)i = 4x+4y+(3x+3y)i \implies 3-x = 4x+4y$ and $2+x+2y = 3x+3y \implies 5x+4y = 3$ and $2x+y = 2 \implies x = 5/3$ and $y = -4/3$.

Find square roots of the following.

Problem 24. $5+12i$

Solution. $5+12i = 5+2\sqrt{36}i = 9-4+2\sqrt{(9)(4)}i \implies \sqrt{5+12i} = \sqrt{9}+\sqrt{4}i = 3+2i$

Problem 25. $2i$

Solution. Applying the formulas, with $a = 0$ and $b = 2$, we have $x = \dfrac{a+\sqrt{a^2+b^2}}{2} = 1$, $y = \dfrac{-a+\sqrt{a^2+b^2}}{2} = 1$, and therefore, $\sqrt{2i} = 1+i$. We could also derive this as follows. $2i = 1-1+2\sqrt{(1)(1)}i \implies \sqrt{2i} = \sqrt{1}+\sqrt{1}i = 1+i$.

Problem 26. $4ab+2(a^2-b^2)i$

Solution. $4ab+2(a^2-b^2)i = 4ab+2\sqrt{(a+b)^2(a-b)^2}i = (a+b)^2-(a-b)^2+2\sqrt{(a+b)^2(a-b)^2}i \implies \sqrt{4ab+2(a^2-b^2)i} = \sqrt{(a+b)^2}+\sqrt{(a-b)^2}i = (a+b)+(a-b)i$

Extra Problem. Find $\sqrt[4]{16i}$.

Solution. First,

$$\sqrt[4]{16i} = 2\sqrt[4]{i}$$

Next, we calculate the *square root* of i. Since $i = 1/2 - 1/2 + 2\sqrt{(1/2)(1/2)}i$, we have

$$\sqrt{i} = \sqrt{1/2} + \sqrt{1/2}i = (1+i)/\sqrt{2}.$$

Next we calculate the square root of $1+i$. With $a = 1$ and $b = 1$, we have $x = \dfrac{a+\sqrt{a^2+b^2}}{2} = \dfrac{1+\sqrt{2}}{2}$ and $y = \dfrac{-a+\sqrt{a^2+b^2}}{2} = \dfrac{-1+\sqrt{2}}{2}$, so that

$$\sqrt{1+i} = \frac{\sqrt{1+\sqrt{2}}}{\sqrt{2}} + \frac{\sqrt{-1+\sqrt{2}}}{\sqrt{2}}i.$$

Putting everything together, we have

$$\sqrt[4]{16i} = 2\sqrt[4]{i} = 2\sqrt{\frac{1+i}{\sqrt{2}}} = 2^{3/4}\sqrt{1+i} = 2^{3/4}\left(\frac{\sqrt{1+\sqrt{2}}}{\sqrt{2}} + \frac{\sqrt{-1+\sqrt{2}}}{\sqrt{2}}\,i\right)$$

$$= 2^{1/4}\left(\sqrt{1+\sqrt{2}} + \sqrt{-1+\sqrt{2}}\,i\right) = \sqrt{\sqrt{2}+2} + \sqrt{-\sqrt{2}+2}\,i = \sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}\,i$$

Chapter 13

Quadratic Equations

13.1 Quadratic Equations, Exercise XLI (p. 301)

Problem 1. $x^2 + 2x = 35$.

Solution. $x^2 + 2x = 35 \implies x^2 + 2x - 35 = 0 \implies (x+7)(x-5) = 0 \implies x = 5,\ x = -7$.

Problem 2. $4x^2 - 4x = 3$

Solution. $4x^2 - 4x = 3 \implies 4x^2 - 4x - 3 = 0 \implies (2x-3)(2x+1) = 0 \implies x = \frac{3}{2},\ x = -\frac{1}{2}$.

Problem 3. $x^2 = 10x - 18$

Solution. $x^2 = 10x - 18 \implies x^2 - 10x + 18 = 0 \implies x = \dfrac{10 \pm \sqrt{100 - 4(18)}}{2} = 5 \pm \sqrt{7}$.

Problem 4. $9x^2 + 6x + 5 = 0$.

Solution. $9x^2 + 6x + 5 = 0 \implies x = \dfrac{-6 \pm \sqrt{36 - 4(9)(5)}}{18} = \dfrac{-6 \pm 12i}{18} = \dfrac{-1 \pm 2i}{3}$.

Problem 5. $2x^2 + 3x - 4 = 0$.

Solution. $2x^2 + 3x - 4 = 0 \implies x = \dfrac{-3 \pm \sqrt{9 + 4(2)(4)}}{4} = \dfrac{-3 \pm \sqrt{41}}{4}$.

Problem 6. $(2x-3)^2 = 8x$.

Solution. $(2x-3)^2 = 8x \implies 4x^2 - 12x + 9 = 8x \implies 4x^2 - 20x + 9 = 0 \implies x = \dfrac{20 \pm \sqrt{400 - 4(4)(9)}}{8} = \dfrac{20 \pm 16}{8} \implies x = \frac{1}{2},\ \frac{9}{2}$.

Problem 7. $x^2 + 9x - 252 = 0$.

Solution. $x^2 + 9x - 252 = 0 \implies x = \dfrac{-9 \pm \sqrt{81 + 4(252)}}{2} = \dfrac{-9 \pm 33}{2} = 12,\ -21$.

Problem 8. $12x^2 + 56x - 255 = 0$.

Solution. $12x^2 + 56x - 255 = 0 \implies x = \dfrac{-56 \pm \sqrt{3136 + 4(12)(255)}}{24} = \dfrac{-56 \pm 124}{24} = \dfrac{17}{6},\ -\dfrac{15}{2}$.

Problem 9. $8x^2 - 82x + 207 = 0$.

Solution. $8x^2 - 82x + 207 = 0 \implies x = \dfrac{82 \pm \sqrt{6724 - 4(8)(207)}}{16} = \dfrac{82 \pm 10}{16} = \dfrac{23}{4}, \dfrac{9}{2}$.

Problem 10. $15x^2 - 86x - 64 = 0$.

Solution. $15x^2 - 86x - 64 = 0 \implies x = \dfrac{86 \pm \sqrt{7396 + 4(15)(64)}}{30} = \dfrac{86 \pm 106}{30} = -\dfrac{2}{3}, \dfrac{32}{5}$.

Problem 11. $x^2 - 3x - 1 + \sqrt{3} = 0$.

Solution. $x^2 - 3x - 1 + \sqrt{3} = 0 \implies x = \dfrac{3 \pm \sqrt{9 - 4(-1 + \sqrt{3})}}{2} = \dfrac{3 \pm \sqrt{13 - 4\sqrt{3}}}{2}$. We can simplify this as follows:

$$13 - 4\sqrt{3} = 12 - 2\sqrt{12} + 1 = (\sqrt{12} - \sqrt{1})^2 \implies \sqrt{13 - 4\sqrt{3}} = \sqrt{12} - 1 = 2\sqrt{3} - 1.$$

Therefore, $x = \dfrac{3 \pm (2\sqrt{3} - 1)}{2} = 1 + \sqrt{3}, 2 - \sqrt{3}$.

Problem 12. $x^2 - (6 + i)x + 8 + 2i = 0$.

Solution. $x^2 - (6 + i)x + 8 + 2i = 0 \implies x = \dfrac{6 + i \pm \sqrt{(6 + i)^2 - 4(8 + 2i)}}{2} = \dfrac{6 + i \pm \sqrt{3 + 4i}}{2}$. Now

$$3 + 4i = 4 - 1 + 2i\sqrt{4} = (\sqrt{4} + i\sqrt{1})^2 \implies \sqrt{3 + 4i} = 2 + i$$

so that $x = \dfrac{6 + i \pm (2 + i)}{2} = 4 + i, 2$.

Problem 13. $(x - 2)^2(x - 7) = (x + 2)(x - 3)(x - 6)$.

Solution. $(x-2)^2(x-7) = (x+2)(x-3)(x-6) \implies (x^2-4x+4)(x-7) = (x^2-x-6)(x-6) \implies x^3-11x^2+32x - 28 = x^3 - 7x^2 + 36 \implies 4x^2 - 32x + 64 = 0 \implies x^2 - 8x + 16 = 0 \implies (x - 4)^2 = 0 \implies x = 4, 4$.

Problem 14. $\dfrac{2x}{x + 2} + \dfrac{x + 2}{2x} = 2$.

Solution. Let $y \equiv \dfrac{2x}{x + 2}$, then $\dfrac{2x}{x + 2} + \dfrac{x + 2}{2x} = 2 \implies y + \dfrac{1}{y} = 2 \implies y^2 - 2y + 1 = (y - 1)^2 = 0 \implies y = \dfrac{2x}{x + 2} = 1 \implies 2x = x + 2 \implies x = 2, 2$.

Problem 15. $\dfrac{x + 1}{x} + 1 = \dfrac{x}{x - 1}$.

Solution. $\dfrac{x + 1}{x} + 1 = \dfrac{x}{x - 1} \implies (x-1)(x+1)+x(x-1) = x^2 \implies x^2-1+x^2-x = x^2 \implies x^2-x-1 = 0 \implies x = \dfrac{1 \pm \sqrt{1 + 4(1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$.

Problem 16. $\dfrac{3}{2(x^2 - 1)} + \dfrac{x}{4x + 4} = \dfrac{3}{8}$.

Solution. $\dfrac{3}{2(x^2 - 1)} + \dfrac{x}{4x + 4} = \dfrac{3}{8} \implies \dfrac{12}{x^2 - 1} + \dfrac{2x}{x + 1} = 3 \implies 12 + 2x(x - 1) = 3(x^2 - 1) \implies x^2 + 2x - 15 = 0 \implies (x + 5)(x - 3) = 0 \implies x = -5, 3$.

Problem 17. $\dfrac{3}{2x+1} - \dfrac{1}{4x-2} - \dfrac{2x}{1-4x^2} = \dfrac{7}{8}$.

Solution. $\dfrac{3}{2x+1} - \dfrac{1}{4x-2} - \dfrac{2x}{1-4x^2} = \dfrac{7}{8} \implies 24(1-2x) + 4(1+2x) - 16x = 7(1-4x^2) \implies 28x^2 - 56x + 21 = 0 \implies 4x^2 - 8x + 3 = 0 \implies x = \dfrac{8 \pm \sqrt{64 - 4(4)(3)}}{8} = \dfrac{8 \pm 4}{8} = \dfrac{3}{2}, \dfrac{1}{2}$. However, $\dfrac{1}{2}$ is not a solution since $\dfrac{1}{4x-2} \to \infty$ as $x \to \dfrac{1}{2}$. Thus, only $\dfrac{3}{2}$ is a solution.

Problem 18. $\dfrac{2x-1}{x-2} + \dfrac{3x+1}{x-3} = \dfrac{5x-14}{x-4}$.

Solution. Clearing of fractions will generate a cubic on both sides of the equation, but it is not hard to see that the cubic terms will cancel out and we will be left with a quadratic:

$$\begin{aligned}
\frac{2x-1}{x-2} + \frac{3x+1}{x-3} = \frac{5x-14}{x-4} &\implies \frac{(2x-1)(x-3) + (3x+1)(x-2)}{x^2 - 5x + 6} = \frac{5x-14}{x-4} \\
&\implies \frac{5x^2 - 12x + 1}{x^2 - 5x + 6} = \frac{5x-14}{x-4} \\
&\implies (5x^2 - 12x + 1)(x-4) = (5x-14)(x^2 - 5x + 6) \\
&\implies 5x^3 - 32x^2 + 49x - 4 = 5x^3 - 39x^2 + 100x - 84 \\
&\implies 7x^2 - 51x + 80 = 0 \\
&\implies x = \frac{51 \pm \sqrt{2601 - 4(7)(80)}}{14} = \frac{51 \pm 19}{14} = 5,\ \frac{16}{7}.
\end{aligned}$$

Problem 19. $\dfrac{x+1}{x(x-2)} - \dfrac{1}{2x-2} + \dfrac{1}{2x} = 0$.

Solution. $\dfrac{x+1}{x(x-2)} - \dfrac{1}{2x-2} + \dfrac{1}{2x} = 0 \implies \dfrac{2x+2}{x(x-2)} + \dfrac{1}{x} - \dfrac{1}{x-1} = 0 \implies \dfrac{2x+2}{x(x-2)} + \dfrac{x-1-x}{x(x-1)} = 0 \implies \dfrac{1}{x}\left[\dfrac{2x+2}{x-2} - \dfrac{1}{x-1}\right] = 0 \implies (2x+2)(x-1) - (x-2) = 0 \implies 2x^2 - 2x + 2x - 2 - x + 2 = 0 \implies 2x^2 - x = 0 \implies 2x - 1 = 0 \implies x = \dfrac{1}{2}$.

Problem 20. $\dfrac{4}{x-1} - \dfrac{1}{4-x} = \dfrac{3}{x-2} - \dfrac{2}{3-x}$.

Solution. $\dfrac{4}{x-1} - \dfrac{1}{4-x} = \dfrac{3}{x-2} - \dfrac{2}{3-x} \implies \dfrac{4}{x-1} + \dfrac{1}{x-4} = \dfrac{3}{x-2} + \dfrac{2}{x-3} \implies \dfrac{4x-16+x-1}{x^2-5x+4} = \dfrac{3x-9+2x-4}{x^2-5x+6} \implies (5x-17)(x^2-5x+6) = (5x-13)(x^2-5x+4) \implies 5x^3-42x^2+115x-102 = 5x^3-38x^2+85x-52 \implies 4x^2-30x+50 = 0 \implies 2x^2-15x+25 = 0 \implies x = \dfrac{15 \pm \sqrt{225-4(2)(25)}}{4} = \dfrac{15 \pm 5}{4} = 5, \dfrac{5}{2}$.

Problem 21. $\dfrac{x+3}{4(x+2)(3x-1)} + \dfrac{2x+1}{3(3x-1)(x+4)} - \dfrac{17x+7}{6(x+4)(x+2)} = 0$.

Solution. Multiply through by $(4)(3)(6)(x+2)(3x-1)(x+4)$ to get

$$\begin{aligned}
18(x+3)(x+4) + 24(2x+1)(x+2) - 12(17x+7)(3x-1) &= 0 \\
3(x^2 + 7x + 12) + 4(2x^2 + 5x + 2) - 2((51x^2 + 4x - 7) &= 0 \\
-91x^2 + 33x + 58 &= 0
\end{aligned}$$

and now the quadratic formula gives $x = \dfrac{-33 \pm \sqrt{1089 + 4(91)(58)}}{-182} = \dfrac{-33 \pm 149}{-182} = 1, -\dfrac{58}{91}$.

Problem 22. $\dfrac{x+7}{2x^2-7x+3} + \dfrac{x}{x^2-2x-3} + \dfrac{x+3}{2x^2+x-1} = 0.$

Solution. Factoring the denominators, we have

$$\begin{aligned}
\frac{x+7}{2x^2-7x+3} + \frac{x}{x^2-2x-3} + \frac{x+3}{2x^2+x-1} &= 0 \\
\frac{x+7}{(2x-1)(x-3)} + \frac{x}{(x-3)(x+1)} + \frac{x+3}{(2x-1)(x+1)} &= 0 \\
(x+7)(x+1) + x(2x-1) + (x+3)(x-3) &= 0 \\
x^2 + 8x + 7 + 2x^2 - x + x^2 - 9 &= 0 \\
4x^2 + 7x - 2 &= 0
\end{aligned}$$

and applying the quadratic formula, we get $x = \dfrac{-7 \pm \sqrt{49 + 4(4)(2)}}{8} = \dfrac{-7 \pm 9}{8} = -2, \dfrac{1}{4}$.

Problem 23. $3x^2 + (9a-1)x - 3a = 0.$

Solution. Applying the quadratic formula, we get

$$\begin{aligned}
x &= \frac{1 - 9a \pm \sqrt{(9a-1)^2 + 4(3)(3a)}}{6} \\
&= \frac{1 - 9a \pm \sqrt{81a^2 - 18a + 1 + 36a}}{6} \\
&= \frac{1 - 9a \pm \sqrt{81a^2 + 18a + 1}}{9} \\
&= \frac{1 - 9a \pm (9a+1)}{6} \\
&= -3a, \frac{1}{3}.
\end{aligned}$$

Problem 24. $x^2 - 2ax + a^2 - b^2 = 0.$

Solution. $x^2 - 2ax + a^2 - b^2 = 0 \implies (x-a)^2 - a^2 + a^2 - b^2 = (x-a)^2 - b^2 = 0 \implies (x-a-b)(x-a+b) = 0 \implies x = a+b, a-b.$

Problem 25. $c^2x^2 + c(a-b)x - ab = 0.$

Solution. Let $y \equiv cx$. Then we have $y^2 + (a-b)y - ab = (y+a)(y-b) = 0$. Therefore, $(cx+a)(cx-b) = 0 \implies x = -\dfrac{a}{c}, \dfrac{b}{c}$.

Problem 26. $x^2 - 4ax + 4a^2 - b^2 = 0.$

Solution. $x^2 - 4ax + 4a^2 - b^2 = 0 \implies (x-2a)^2 - 4a^2 + 4a^2 - b^2 = (x-2a)^2 - b^2 = 0 \implies (x-2a-b)(x-2a+b) = 0 \implies x = 2a+b, 2a-b.$

Problem 27. $x^2 - 6acx + a^2(9c^2 - 4b^2) = 0.$

Solution. $x^2 - 6acx + a^2(9c^2 - 4b^2) = 0 \implies (x-3ac)^2 - 9a^2c^2 + 9a^2c^2 - 4a^2b^2 = 0 \implies (x-3ac)^2 - 4a^2b^2 = (x-3ac-2ab)(x-3ac+2ab) = 0 \implies x = a(3c \pm 2b).$

Problem 28. $(a^2-b^2)x^2 - 2(a^2+b^2)x + a^2 - b^2 = 0.$

Solution. Applying the quadratic formula, we get

$$\begin{aligned}
x &= \frac{2(a^2+b^2) \pm \sqrt{4(a^2+b^2)^2 - 4(a^2-b^2)^2}}{2(a^2-b^2)} \\
&= \frac{2(a^2+b^2) \pm 2\sqrt{(a^2+b^2-a^2+b^2)(a^2+b^2+a^2-b^2)}}{2(a^2-b^2)} \\
&= \frac{2(a^2+b^2) \pm 2\sqrt{2b^2(2a^2)}}{2(a^2-b^2)} \\
&= \frac{(a^2+b^2) \pm 2ab}{(a-b)(a+b)} \\
&= \frac{a-b}{a+b}, \frac{a+b}{a-b}.
\end{aligned}$$

Problem 29. $\dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c} = 0.$

Solution.

$$\begin{aligned}
\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} &= 0 \\
(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) &= 0 \\
x^2 - (b+c)x + bc + x^2 - (a+c)x + ac + x^2 - (a+b)x + ab &= 0 \\
3x^2 - 2(a+b+c)x + bc + ac + ab &= 0
\end{aligned}$$

Applying the quadratic formula, we get

$$\begin{aligned}
x &= \frac{2(a+b+c) \pm \sqrt{4(a+b+c)^2 - 4(3)(bc+ac+ab)}}{6} \\
&= \frac{a+b+c \pm \sqrt{a^2+b^2+c^2+2ab+2ac+2bc-3bc-3ac-3ab}}{3} \\
&= \frac{a+b+c \pm \sqrt{a^2+b^2+c^2-bc-ac-ab}}{3}
\end{aligned}$$

Problem 30. $\dfrac{(x-a)^2-(x-b)^2}{(x-a)(x-b)} + \dfrac{4ab}{a^2-b^2} = 0.$

Solution.
$\dfrac{(x-a)^2-(x-b)^2}{(x-a)(x-b)}+\dfrac{4ab}{a^2-b^2}=0 \implies \dfrac{(b-a)(2x-a-b)}{x^2-(a+b)x+ab}+\dfrac{4ab}{a^2-b^2}=0 \implies$
$(a^2-b^2)(b-a)(2x-a-b)+4ab[x^2-(a+b)x+ab]=0 \implies$
$4abx^2+2[(a^2-b^2)(b-a)-2ab(a+b)]x+4a^2b^2-(a^2-b^2)(b-a)(b+a)=0 \implies$
$4abx^2+2[(a-b)(a+b)(b-a)-2ab(a+b)]x+4a^2b^2+(a^2-b^2)(a-b)(a+b)=0 \implies$
$4abx^2-2(a+b)[(a-b)^2+2ab]x+4a^2b^2+(a-b)^2(a+b)^2=0 \implies$
$4abx^2-2(a+b)(a^2+b^2)x+4a^2b^2+(a^2-b^2)^2=0 \implies$
$4abx^2-2(a+b)(a^2+b^2)x+(a^2+b^2)^2=0 \implies$
$x=\dfrac{2(a+b)(a^2+b^2)\pm\sqrt{4(a+b)^2(a^2+b^2)^2-16ab(a^2+b^2)^2}}{8ab} \implies$
$x=\dfrac{2(a+b)(a^2+b^2)\pm 2(a^2+b^2)\sqrt{(a+b)^2-4ab}}{8ab} \implies$
$x=\dfrac{2(a+b)(a^2+b^2)\pm 2(a^2+b^2)\sqrt{(a-b)^2}}{8ab} \implies$
$x=\dfrac{(a+b)(a^2+b^2)\pm(a^2+b^2)(a-b)}{4ab} \implies$
$x=\dfrac{a^2+b^2}{4ab}[a+b\pm(a-b)]=\dfrac{a^2+b^2}{2a},\ \dfrac{a^2+b^2}{2b}.$

Quadratic Equations (Word Problems), Exercise XLII (p. 302)

Problem 1. Find two consecutive integers whose product is 506.

Solution. $n(n+1)=560 \implies n^2+n-506=0 \implies n=\dfrac{-1\pm\sqrt{1+4(506)}}{2}=\dfrac{-1\pm 45}{2}=22$. Thus, the numbers are 22 and 23.

Problem 2. Find two consecutive integers the sum of whose squares is 481.

Solution. $n^2+(n+1)^2=481 \implies 2n^2+2n+1=481 \implies n^2+n-240=0 \implies n=\dfrac{-1\pm\sqrt{1+4(240)}}{2}=\dfrac{-1\pm 31}{2}=15$. Thus, the numbers are 15 and 16.

Problem 3. Find two consecutive integers the difference of whose cubes is 91.

Solution. $(n+1)^3-n^3=91 \implies n^3+3n^2+3n+1-n^3=91 \implies 3n^2+3n=90 \implies n^2+n-30=0 \implies n=\dfrac{-1\pm\sqrt{1+4(30}}{2}=\dfrac{-1\pm 11}{2}=5$. Thus, the numbers are 5 and 6.

Problem 4. Find three consecutive integers the sum of whose products by pairs is 587.

Solution. Let the integers be $n-1$, n, and $n+1$. Then $(n-1)n+(n-1)(n+1)+n(n+1)=3n^2-1=587 \implies 3n^2=588 \implies n^2=196 \implies n=14$. Thus, the numbers are 13, 14, and 15.

Problem 5. Find a number of two digits from the following data: the product of the digits is 48, and if the digits be interchanged the number is diminished by 18.

Solution. Let ab be the two-digit number. Then we have $a\cdot b=48$, and $10b+a=10a+b-18 \implies 9a=9b+18 \implies a=b+2$. Therefore, $b^2+2b-48=0 \implies b=\dfrac{-2\pm\sqrt{4+4(48)}}{2}=\dfrac{-2\pm 14}{2}$. The only acceptable solution is $b=6$, and that implies $a=8$. Thus the number is 86.

Problem 6. The numerator of a certain fraction exceeds its denominator by 2, and the fraction itself exceeds its reciprocal by 24/35. Find the fraction.

Solution. We have $\dfrac{n+2}{n} - \dfrac{n}{n+2} = \dfrac{24}{35} \implies 35[(n+2)^2 - n^2] - 24n(n+2) = 0 \implies 6n^2 - 23n - 35 = 0 \implies n = \dfrac{23 \pm \sqrt{(23)^2 + 4(6)(35)}}{12} = \dfrac{23 \pm 37}{12}$. The only acceptable solution is 5. Thus, the fraction is $\dfrac{7}{5}$.

Problem 7. A cattle dealer bought a certain number of steers for \$1200. Having lost 4 of them, he sold the rest for \$10 a head more than they cost him, and made \$260 by the entire transaction. How many steers did he buy?

Solution. Let n be the number of heads of cattle the farmer bought and x be the cost per head. Then $nx = 1260$ and $(n-4)(x+10) - 1260 = 260 \implies nx + 10n - 4x - 40 - nx = 260 \implies 10n - 4x = 300 \implies (10)(1260)/x - 4x = 300 \implies 12600 - 4x^2 = 300x \implies x^2 + 75x - 3150 = 0$, and the only acceptable solution of this quadratic is $x = [-75 + \sqrt{(75)^2 + 4(3150)}]/2 = 30$ for the original cost per head, and the number of heads of cattle is $n = 1260/30 = 42$.

Problem 8. A man sold some goods for \$48, and his gain per cent was equal to one half the cost of the goods in dollars. What was the cost of the goods?

Solution. Let x be the cost of the goods. Then we have

$$\frac{48-x}{x}100 = \frac{x}{2} \implies 200(48-x) = x^2 \implies x^2 + 200x - 9600 = 0 \implies x = \frac{-200 \pm \sqrt{(200)^2 + 4(9600)}}{2}$$
$$\implies x = \frac{-200 \pm 280}{2}$$

The only acceptable solution is $x = 40$. Thus, the cost of the goods was \$40.

Problem 9. If \$4000 amounts to \$4410 when put at compound interest for two years, interest being compounded annually, what is the rate of interest?

Solution. The amount at the end of n years is $A = P(1+r)^n$, where P is the principal and r is the rate. Solving this for the rate, we get $r = \left(\dfrac{A}{P}\right)^{1/n} - 1 = \left(\dfrac{4410}{4000}\right)^{1/2} - 1 = 0.05$. Thus, the rate of interest is 5%.

Problem 10. A man inherits \$25,000, but after a certain percentage has been deducted for the inheritance tax and then a percentage for fees at a rate one greater than that of the inheritance tax, he receives only \$22,800. What is the rate of the inheritance tax?

Solution. Let r be the rate of the inheritance tax. Then we have

$$\begin{aligned}
25000 - r25000 - (r + 1/100)(1-r)25000 = 22800 &\implies 25000[1 - r - (r+1/100)(1-r)] = 22800 \\
&\implies (1-r)(1-r-1/100) = \frac{22800}{25000} = 0.912 \\
&\implies (1-r)(0.99-r) = 0.912 \\
&\implies r^2 - 1.99r + 0.078 = 0 \\
&\implies r = \frac{1.99 \pm \sqrt{3.9601 - 4(0.078)}}{2} \\
&\implies r = \frac{1.99 \pm 1.91}{2} \\
&\implies r = 0.04,\ 1.95
\end{aligned}$$

The only acceptable solution is 0.04. Thus, the inheritance tax rate is 4%.

Problem 11. A man bought a certain number of \$50 shares for \$4500 when they were at a certain discount. Later he sold all but 10 of them for \$5850 when the premium was three times the discount at which he bought them. How many shares did he buy?

Solution. Let n be the number of shares and x be the discounted cost per share. Then $nx = 4500$. When he sold them, we have

$$\begin{aligned}
(n-10)[50+3(50-x)] = 5850 &\implies (n-10)(200-3x) - 5850 = 0 \\
&\implies 200n - 3(4500) - 2000 + 30x - 5850 = 0 \\
&\implies 30x + 200n - 213250 = 0 \\
&\implies 30x^2 + 200(4500) - 21350x = 0 \\
&\implies 30x^2 - 21350x + 900000 = 0 \\
&\implies x = \frac{21350 \pm \sqrt{(21350)^2 - 4(30)(900000)}}{60} \\
&\implies x = \frac{21350 \pm 18650}{60} = 45
\end{aligned}$$

Thus, the cost per share was \$45, the number of shares, $n = 100$, and he sold them for \$50+3(\$50−\$45) = \$65 per share.

Problem 12. The circumference of a hind wheel of a wagon exceeds that of a fore wheel by 8 inches, and in traveling 1 mile this wheel makes 88 less revolutions that a fore wheel. Find the circumference of each wheel.

Solution. Let x be the circumference of the hind wheel and y that of the fore wheel, so that $x - y = 8$ inches. Let n be the number of revolutions that the fore wheel makes in 1 mile. Then we have $(n-88)x = ny = 5280 \cdot 12 = 63360$ inches, and

$$(n-88)x = ny = n(x-8) \implies n = 11x$$

This gives

$$(11x-88)x = 63360 \implies 11x^2 - 88x - 63360 = 0 \implies x = \frac{88 \pm \sqrt{(88)^2 + 4(11)(63360)}}{22} = \frac{88 \pm 1672}{22} = 80$$

Thus, the circumference of the hind wheel is 80 inches and that of the fore wheel is 72 inches. The hind wheel makes $63360/80 = 792$ revolutions and the fore wheel makes $63360/72 = 880$ revolutions in 1 mile.

Problem 13. A square is surrounded by a border whose width lacks 1 inch of being one fourth of the length of a side of the square, and whose area in square inches exceeds the length of the perimeter of the square in inches by 64. Find the area of the square and that of the border.

Solution. Let x be a side length of the square. Then we have

area of border = area of larger square - area of smaller square $\implies \left[x + 2\left(\frac{x}{4} - 1\right)\right]^2 - x^2 = 4x + 64$

$$\begin{aligned}
&\implies x^2 + 4x\left(\frac{x}{4} - 1\right) + 4\left(\frac{x}{4} - 1\right)^2 - x^2 - 4x - 64 = 0 \\
&\implies x^2 - 4x + 4\left(\frac{x^2}{16} - \frac{x}{2} + 1\right) - 4x - 64 = 0 \\
&\implies x^2 - 4x + \frac{x^2}{4} - 2x + 4 - 4x - 64 = 0 \\
&\implies \frac{5x^2}{4} - 10x - 60 = 0 \\
&\implies 5x^2 - 40x - 240 = 0 \\
&\implies x = \frac{40 \pm \sqrt{(40)^2 + 4(5)(240)}}{10} \\
&\implies x = \frac{40 \pm 80}{10} = 12
\end{aligned}$$

Thus the side length of the smaller square is $x = 12$, giving that square an area of 144 square inches, and the border area is $4x + 64 = 4 \cdot 12 + 64 = 112$ square inches.

Problem 14. The corners of a square the length of whose side is 2 are cut off in such a way that a regular octagon remains. What is the length of a side of this octagon?

Solution. The octagon has all 8 of its sides the same length, which we label as x. Since the length of the square is 2, the length of each side that is cut off is $2 - x$, or $\frac{2-x}{2}$ on each side. Consider a corner that is cut off. It forms a right triangle of length $\frac{2-x}{2}$ on each side with the diagonal of length x. Thus, by the Pythagorean theorem, we have

$$2\left(\frac{2-x}{2}\right)^2 = x^2 \implies \frac{1}{2}(4 - 4x + x^2) = x^2 \implies 4 - 4x + x^2 = 2x^2 \implies x^2 + 4x - 4 = 0 \implies$$
$$x = \frac{-4 \pm \sqrt{16 + 4(4)}}{2} \implies x = -2 \pm 2\sqrt{2}$$

Thus, the length of a side of the octagon is $2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$.

Problem 15. A vintner draws a certain quantity of wine from a full cask containing 63 gallons. Having filled up the cask with water, he draws the same quantity as before and then finds that only 28 gallons of pure wine remain in the cask. How many gallons did he draw each time?

Solution. Let x be the amount he draws each time. After the first draw, he has $63 - x$ gallons of wine remaining and x gallons of water, so the fraction of wine now in the cask is $\frac{63-x}{63}$. When he draws the second time, he removes $x \cdot \frac{63-x}{63}$ pure wine, and thus the amount of wine remaining is

$$63 - x - x \cdot \frac{63 - x}{63} = 28 \implies 3969 - 63x - 63x + x^2 = 1764 \implies x^2 - 126x + 2205 = 0 \implies$$
$$x = \frac{126 \pm \sqrt{15876 - 4(2205)}}{2} \implies x = \frac{126 \pm 84}{2} \implies x = 105,\ 21$$

The only acceptable answer is $x = 21$ gallons.

Problem 16. A man travels 50 miles by train A, and then after a wait of 5 minutes returns by the train B, which runs 5 miles an hour faster than the train A. The entire journey occupies $2\frac{4}{9}$ hours. What are the rates of the two trains?

Solution. Let t_1 and t_2 be the times for train A and B respectively, and let a be the speed of train A. Then we have $at_1 = 50$, $(a+5)t_2 = 50$, and $t_1 + t_2 + \frac{5}{60} = 2\frac{4}{9}$, where the time is measured in hours. Therefore,

$$\frac{50}{a} + \frac{50}{a+5} + \frac{1}{12} - \frac{22}{9} = 0 \implies 50(a+5+a) - \frac{85}{36}a(a+5) = 0 \implies 20a + 50 - \frac{17}{36}(a^2+5a) = 0 \implies$$

$$720a + 1800 - 17a^2 - 85a = 0 \implies 17a^2 - 635a - 1800 = 0 \implies a = \frac{635 \pm \sqrt{(635)^2 + 4(17)(1800)}}{34}$$

$$\implies a = \frac{635 \pm 725}{34} \implies a = 40$$

Thus, the speed of train A is 40 miles per hour and that of train B is 45 miles per hour.

Problem 17. A pedestrian walked 6 miles in a certain interval of time. Had the time been 1/2 hour less, the rate would have been 2 miles per hour greater. Required the time and rate.

Solution. Let t be the time to walk 6 miles and let v be the rate. Then we have

$$v = \frac{6}{t}, \quad \text{and} \quad v + 2 = \frac{6}{t - \frac{1}{2}}$$

Therefore,

$$\frac{6}{t} + 2 = \frac{6}{t - 1/2} \implies 6(t - 1/2) + 2t(t - 1/2) = 6t \implies -3 + 2t^2 - t = 0 \implies t = \frac{1 \pm \sqrt{1 + 4(2)(3)}}{4}$$

$$\implies t = \frac{1 \pm 5}{4} \implies t = \frac{3}{2}$$

Thus, the time is $1\frac{1}{2}$ hour and the rate is $v = 4$ miles per hour.

Problem 18. A pedestrian walked 12 miles at a certain rate and then 6 miles farther at a rate 1/2 mile per hour greater. Had he walked the entire distance at the greater rate, his time would have been 20 minutes less. How long did it take him to walk the 18 miles?

Solution. Let v be the rate for the first 12 miles, t_1 be the time that takes, and t_2 be the time to walk the last 6 miles. Then we have

$$vt_1 = 12, \quad \left(v + \frac{1}{2}\right) t_2 = 6, \quad \left(v + \frac{1}{2}\right)\left(t_1 + t_2 - \frac{1}{3}\right) = 18$$

Substituting $t_1 = 12/v$ and $t_2 = 6/(v + 1/2)$ into the third equation gives

$$\left(v + \frac{1}{2}\right)\left(\frac{12}{v} + \frac{6}{v + \frac{1}{2}} - \frac{1}{3}\right) = 18 \implies \frac{12(v + \frac{1}{2})}{v} + 6 - \frac{1}{3}\left(v + \frac{1}{2}\right) - 18 = 0 \implies$$

$$12\left(v + \frac{1}{2}\right) - 12v - \frac{v^2}{3} - \frac{v}{6} = 0 \implies 72v + 36 - 72v - 2v^2 - v = 0 \implies 2v^2 + v - 36 = 0$$

$$v = \frac{-1 \pm \sqrt{1 + 4(2)(36)}}{4} \implies v = \frac{-1 \pm 17}{4} \implies v = 4$$

This gives $t_1 = \dfrac{12}{4} = 3$ and $t_2 = \dfrac{6}{4 + \frac{1}{2}} = \frac{4}{3}$, so that the total time is $4\frac{1}{3}$ hours or 4 hours and 20 minutes.

Problem 19. From the point of intersection of two straight roads which cross at right angles, two men, A and B, set out simultaneously, A on the one road at the rate of 3 miles per hour, B one the other at the rate of 4 miles per hour. After how many hours will they be 30 miles apart?

Solution. Since the paths form two sides of a right triangle, Pythagorean's theorem gives

$$(3t)^2 + (4t)^2 = 30^2 \implies (9 + 16)t^2 = 30^2 \implies t = \frac{30}{5} \implies t = 6,$$

so that after 6 hours they will be 30 miles apart.

Problem 20. If A and B walk on the roads just described, but at the rates of 2 and 3 miles per hour respectively, and A starts 2 hours before B, how long after B starts will they be 10 miles apart?

Solution. Again applying Pythagorean's theorem, we have

$[2(2+t)]^2 + (3t)^2 = 100 \implies 4(t^2+4t+4) + 9t^2 = 100 \implies 13t^2 + 16t - 84 = 0 \implies$
$t = \dfrac{-16 \pm \sqrt{(16)^2 + 4(13)(84)}}{26} \implies \dfrac{-16 \pm 68}{26} \implies t = 2,$

so that two hours after B starts they will be 10 miles apart.

Problem 21. If from a height of a feet a body be thrown vertically upward with an initial velocity of b feet per second, its height at the end of t seconds is given by the formula $h = a + bt - 16t^2$. The corresponding formula when the body is thrown vertically downward is $h = a - bt - 16t^2$.
(1) If a body be thrown vertically upward from the ground with an initial velocity of 32 feet per second, when will it be at a height of 7 feet? of 16 feet? Will it ever reach a height of 17 feet?
(2) A body is thrown from a height of 64 feet vertically downward with an initial velocity of 48 feet per second. When will it reach the height of 36 feet?
(3) If a body be dropped from a height of 36 feet, when will it reach the ground?

Solution.
(1) The height at any time t is $h = 32t - 16t^2$ or $16t^2 - 32t + h = 0$. Solving this for t gives

$$t = \frac{32 \pm \sqrt{(32)^2 - 4(16)h}}{32}$$

When $h = 7$, we get $t = \dfrac{32 \pm 24}{32} = 0.25$ seconds on the way up, and 1.75 seconds on the way down. When $h = 16$, we get $t = 1$ second. The maximum height is achieved when the discriminant vanishes, which gives $h_{\max} = \frac{(32)^2}{4(16)} = 16$, so the body never reaches 17 feet.
(2) The formula now is $36 = 64 - 48t - 16t^2$ or $16t^2 + 48t - 28 = 0$. Solving this for t gives

$$t = \frac{-48 \pm \sqrt{(48)^2 + 4(16)(28)}}{32} = \frac{-48 \pm 64}{32} = \frac{1}{2}$$

So it will be at a height of 36 feet 1/2 second after released.
(3) The formula now if $0 = 36 - 16t^2$, which gives $t = \sqrt{365/16} = 3/2$ second.

Chapter 15

Equations of Higher Degree which Can Be Solved by Means of Quadratics

15.1 Equations of Higher Degree Solvable by Quadratics, Example 7 (p. 311)

Solve the following equations.

Problem 1. $3x^4 - 29x^2 + 18 = 0$

Solution. $3x^4 - 29x^2 + 18 = 0 \implies (3x^2 - 2)(x^2 - 9) = 0 \implies \boldsymbol{x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}, \pm 3}$.

Problem 2. $x^4 - 6x^3 + 8x^2 + 3x = 2$

Solution. $x^4 - 6x^3 + 8x^2 + 3x = 2 \implies x^4 - 6x^3 + 8x^2 + 3x - 2 = 0 \implies (x^2 - 3x)^2 - 9x^2 + 8x^2 + 3x - 2 = 0 \implies (x^2 - 3x)^2 - x^2 + 3x - 2 = 0 \implies (x^2 - 3x)^2 - (x^2 - 3x) - 2 = 0$. Let $y = x^2 - 3x$, which gives $y^2 - y - 2 = 0 \implies (y + 1)(y - 2) = 0$. This gives two equations as follows: $x^2 - 3x + 1 = 0 \implies x = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$ and $x^2 - 3x - 2 = 0 \implies x = \frac{3 \pm \sqrt{9 + 4(2)}}{2} = \frac{3 \pm \sqrt{17}}{2}$. Thus, the solutions are $\boldsymbol{x = \frac{3 \pm \sqrt{5}}{2}, \frac{3 \pm \sqrt{17}}{2}}$.

Problem 3. $(x - a)(x + 2a)(x - 3a)(x + 4a) = 24a^4$

Solution. $(x - a)(x + 2a)(x - 3a)(x + 4a) = 24a^4 \implies [x^2 + ax - 2a^2][x^2 + ax - 12a^2] = 24a^4 \implies x^4 + 2ax^3 - 13a^2x^2 - 14a^3 + 24a^4 = 24a^4 \implies x(x^3 + 2ax^2 - 13a^2x - 14a^3) = 0 \implies x(x+a)(x^2 + ax - 14a^2) = 0 \implies \boldsymbol{x = 0, -a, \frac{-a \pm \sqrt{57}a}{2}}$

Problem 4. $(4x^2 + 2x)/(x^2 + 6) + (x^2 + 6)/(2x^2 + x) - 3 = 0$

Solution. Let $y = \frac{2x^2 + x}{x^2 + 6}$, which gives $2y + \frac{1}{y} - 3 = 0 \implies 2y^2 - 3y + 1 = 0 \implies (2y - 1)(y - 1) = 0$. This gives two equations: $\frac{4x^2 + 2x}{x^2 + 6} = 1 \implies 4x^2 + 2x = x^2 + 6 \implies 3x^2 + 2x - 6 = 0 \implies x = \frac{-2 \pm \sqrt{4 + 4(3)(6)}}{6} = \frac{-2 \pm 2\sqrt{19}}{6} = \frac{-1 \pm \sqrt{19}}{3}$ and $\frac{2x^2 + x}{x^2 + 6} = 1 \implies 2x^2 + x = x^2 + 6 \implies x^2 + x - 6 = 0 \implies (x + 3)(x - 2) = 0 \implies x = -3$, $x = 2$. Thus, the solutions are $\boldsymbol{x = \frac{-1 \pm \sqrt{19}}{3}, -3, 2}$.

15.2 Reciprocal Equations, Example 4 (p. 313)

Solve the following equations.

Problem 1. $x^3 - 2x^2 + 2x - 1 = 0$

Solution. $x^3 - 2x^2 + 2x - 1 = 0 \implies x^3 - 1 - 2x(x-1) = 0 \implies (x-1)(x^2+x+1) - 2x(x-1) = 0 \implies (x-1)(x^2-x+1) = 0$. Thus, the solutions are $\boldsymbol{x = 1, \dfrac{1 \pm \sqrt{3}i}{2}}$.

Problem 2. $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$

Solution. $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0 \implies x^2\left[x^2 - 4x + 5 - \dfrac{4}{x} + \dfrac{1}{x^2}\right] = 0 \implies x^2\left[\left(x^2 + \dfrac{1}{x^2}\right) - 4\left(x + \dfrac{1}{x}\right) + 5\right] = 0 \implies x^2\left[\left(x + \dfrac{1}{x}\right)^2 - 2 - 4\left(x + \dfrac{1}{x}\right) + 5\right] = 0 \implies x^2\left[\left(x + \dfrac{1}{x} - 2\right)^2 - 4 + 3\right] = 0 \implies x^2\left[\left(x + \dfrac{1}{x} - 2\right)^2 - 1\right] = 0$. This gives two equations: $x + 1/x - 2 - 1 = 0$ and $x + 1/x - 2 + 1 = 0$ or $x^2 - 3x + 1 = 0$ and $x^2 - x + 1 = 0$. Thus, $\boldsymbol{x = \dfrac{3 \pm \sqrt{5}}{2}, \dfrac{1 \pm \sqrt{3}i}{2}}$.

Problem 3. $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

Solution. $x^5 + x^4 + x^3 + x^2 + x + 1 = 0 \implies x^5 + 1 + x(x^3+1) + x^2(x+1) = 0 \implies (x+1)(x^4 - x^3 + x^2 - x + 1) + x(x+1)(x^2-x+1) + x^2(x+1) = 0 \implies (x+1)(x^4 + x^2 + 1) = 0 \implies (x+1)x^2(x^2 + 1 + 1/x^2) = 0 \implies x^2(x+1)[(x+1/x)^2 - 2 + 1] = 0 \implies x^2(x+1)[(x+1/x)^2 - 1] = 0 \implies x^2(x+1)(x+1/x-1)(x+1/x+1) = 0$. Thus, the solutions are $\boldsymbol{x = -1, \dfrac{1 \pm \sqrt{3}i}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}}$.

15.3 Binomial Equations, Example 3 (p. 313)

Solve the following equations.

Problem 1. $x^3 + 8 = 0$

Solution. $x^3 + 8 = 0 \implies (x+2)(x^2 - 2x + 4) = 0 \implies (x+2)[(x-1)^2 + 3] = 0$. Thus, the solutions are $\boldsymbol{x = -2, x = 1 \pm \sqrt{3}i}$.

Problem 2. $x^4 + 1 = 0$

Solution. Since zero is not a solution, we can divide by x^2 to get $x^2 + \dfrac{1}{x^2} = 0 \implies \left(x + \dfrac{1}{x}\right)^2 - 2 = 0$. This gives two equations: $x + 1/x - \sqrt{2} = 0$ and $x + 1/x + \sqrt{2} = 0$ or $x^2 - \sqrt{2}x + 1 = 0$ and $x^2 + \sqrt{2}x + 1 = 0$. Thus, the solutions are $\boldsymbol{x = \dfrac{\sqrt{2} \pm \sqrt{2}i}{2}, \dfrac{-\sqrt{2} \pm \sqrt{2}i}{2}}$.

Problem 3. $x^6 + 1 = 0$

Solution. $x^6 + 1 = 0 \implies \left(x^3 + \dfrac{1}{x^3}\right) = 0 \implies \left(x + \dfrac{1}{x}\right)^3 - 3x^2\dfrac{1}{x} - 3x\dfrac{1}{x^2} = 0 \implies \left(x + \dfrac{1}{x}\right)^3 - 3\left(x + \dfrac{1}{x}\right) = 0 \implies \left(x + \dfrac{1}{x}\right)\left[\left(x + \dfrac{1}{x}\right)^2 - 3\right] = 0 \implies (x^2 + 1)\left[\left(x + \dfrac{1}{x}\right)^2 - 3\right] = 0$. The second factor gives two equations: $x^2 - \sqrt{3}x + 1 = 0$ and $x^2 + \sqrt{3}x + 1 = 0$. Thus, the solutions are $\boldsymbol{x = \pm i, \dfrac{\sqrt{3} \pm i}{2}, \dfrac{-\sqrt{3} \pm i}{2}}$.

Alternate method of solution. Let $y = x^2$, then $x^6 + 1 = 0 \implies y^3 + 1 = 0 \implies (y+1)(y^2 - y + 1) = 0 \implies y = x^2 = -1, \dfrac{1 \pm \sqrt{3}i}{2}$. Since $\dfrac{1 \pm \sqrt{3}i}{2} = \dfrac{2 \pm 2\sqrt{3}i}{4} = \dfrac{3 - 1 \pm 2\sqrt{(3)(1)}i}{4} \implies \sqrt{\dfrac{1 \pm \sqrt{3}i}{2}} = \pm\dfrac{\sqrt{3} \pm i}{2}$.
Thus, again, the solutions are $\boldsymbol{x = \pm i, \pm\dfrac{\sqrt{3} \pm i}{2}}$.

15.4 Equations of Higher Degree Solvable by Quadratics, Exercise XLIV (p. 316)

Solve the following equations.

Problem 1. $4x^4 - 17x^2 + 18 = 0$

Solution. This is a quadratic in x^2, so that

$$x^2 = \frac{17 \pm \sqrt{17^2 - 4(4)(18)}}{8} = \frac{17 \pm 1}{8} = \frac{9}{4}, 2 \implies \boldsymbol{x = \pm\frac{3}{2}, \pm\sqrt{2}}$$

Problem 2. $3x^{3/2} - 4x^{3/4} = 7$

Solution. Let $y = x^{3/4}$, which gives $3y^2 - 4y - 7 = 0$ and $y = \dfrac{4 \pm \sqrt{16 + 4(3)(7)}}{6} = \dfrac{4 \pm 10}{6} = \dfrac{7}{3}, -1$. Thus, the only real solution is $x = \left(\dfrac{7}{3}\right)^{4/3} = \dfrac{7}{3}\sqrt[3]{\dfrac{7}{3}} = \dfrac{7}{3}\dfrac{\sqrt[3]{63}}{3} = \boldsymbol{\dfrac{7}{9}\sqrt[3]{63}}$.

Problem 3. $(x^2 - 4)(x^2 - 9) = 7x^2$

Solution. $(x^2 - 4)(x^2 - 9) = 7x^2 \implies x^4 - 13x^2 + 36 = 7x^2 \implies x^4 - 20x^2 + 36 = 0$. This is a quadratic in x^2 and its solution is $x^2 = \dfrac{20 \pm \sqrt{400 - 4(36)}}{2} = \dfrac{20 \pm 16}{2} = 18, 2$. Thus, $x = \sqrt{18} = \boldsymbol{\pm 3\sqrt{2}, \pm\sqrt{2}}$.

Problem 4. $(2x^2 - x - 3)(3x^2 + x - 2)^2 = 0$

Solution. $(2x^2 - x - 3)(3x^2 + x - 2)^2 = 0 \implies (2x - 3)(x + 1)(3x - 2)^2(x + 1)^2 = 0$. Thus, the solutions are $\boldsymbol{x = 3/2, -1, 2/3, 2/3, -1, -1}$.

Problem 5. $x^4 + x^3 + x^2 + 3x - 6 = 0$

Solution. By synthetic division, we find $x^4 + x^3 + x^2 + 3x - 6 = 0 \implies (x - 1)(x + 2)(x^2 + 3)$. Thus, the solutions are $\boldsymbol{x = 1, -2, \pm\sqrt{3}i}$

Problem 6. $x^4 - 2x^3 + x^2 + 2x - 2 = 0$

Solution. By synthetic division, we find $x^4 - 2x^3 + x^2 + 2x - 2 = 0 \implies (x - 1)(x + 1)(x^2 - 2x + 2) = 0$. The solutions are $\boldsymbol{x = 1, -1, 1 \pm i}$.

Problem 7. $(3x^2 - 2x + 1)(3x^2 - 2x - 7) + 12 = 0$

Solution.

$$\begin{aligned}(3x^2-2x+1)(3x^2-2x-7)+12=0 &\implies (3x^2-2x)^2-6(3x^2-2x)-7+12=0\\ &\implies (3x^2-2x-3)^2-9+5=0\\ &\implies (3x^2-2x-3-2)(3x^2-2x-3+2)=0\\ &\implies (3x^2-2x-5)(3x^2-2x-1)=0\\ &\implies [(3x)^2-2(3x)-15]\,[(3x)^2-2(3x)-3]=0\\ &\implies [(3x-1)^2-16]\,[(3x-1)^2-4]=0\\ &\implies (3x-1-4)(3x-1+4)(3x-1-2)(3x-1+2)=0\\ &\implies (3x-5)(3x+3)(3x-3)(3x+1)=0\end{aligned}$$

Thus, the solutions are $\boldsymbol{x = 5/3,\ -1,\ 1,\ -1/3}$.

Problem 8. $x^4-12x^3+33x^2+18x-28=0$

Solution. By synthetic division we find $x^4-12x^3+33x^2+18x-28=0 \implies (x+1)(x-7)(x^2-6x+4) = 0 \implies (x+1)(x-7)[(x-3)^2-5]=0$. Thus, the solutions are $\boldsymbol{x = -1,\ 7,\ 3\pm\sqrt{5}}$.

Problem 9. $4x^4+4x^3-x^2-x-2=0$

Solution. The possible rational factors are $\pm\dfrac{1,2}{1,2,4}$, but none of these work. Next we seek the square root of the expression. We have

$$(2x^2+px+q)^2 = 4x^4+p^2x^2+q^2+4px^3+4qx^2+2pqx = 4x^4+4px^3+(p^2+4q)x^2+2pqx+q^2,$$

which requires $4p=4 \implies p=1, 1+4q=-1 \implies q=-1/2$. This gives $2pq=2(1)(-1/2)=-1$, which is correct for the coefficient of x. However, the constant term comes out wrong: $q^2=1/4\neq -2$. Not all is lost, though. What we have is

$$\begin{aligned}4x^4+4x^3-x^2-x-2 &= 4x^4+4x^3-x^2-x+1/4-9/4\\ &= (2x^2+x-1/2)^2-9/4\\ &= (2x^2+x-1/2-3/2)(2x^2+x-1/2+3/2)\\ &= (2x^2+x-2)(2x^2+x+1)\end{aligned}$$

So we have factored the quartic into the product of two quadratics: $4x^4+4x^3-x^2-x-2=0 \implies (2x^2+x-2)(2x^2+x+1)=0$.

Alternative method of getting this result:

$$\begin{aligned}4x^4+4x^3-x^2-x-2=0 &\implies 4x^4+4x^3+x^2-2x^2-x-2=0\\ &\implies (2x^2+x)^2-(2x^2+x)-2=0\\ &\implies (2x^2+x-1/2)^2-1/4-8/4=0\\ &\implies (2x^2+x-1/2-3/2)(2x^2+x-1/2+3/2)=0\\ &\implies (2x^2+x-2)(2x^2+x+1)=0\end{aligned}$$

The solutions to the two quadratics are straightforward, and we get $\boldsymbol{x = \dfrac{-1\pm\sqrt{17}}{4},\ \dfrac{-1\pm\sqrt{7}i}{4}}$.

Problem 10. $x^4-2x^3+2x^2-2x+1=0$

Solution. Since the sum of the coefficients equals zero, we know that $x-1$ is a factor, so that $x^4-2x^3+2x^2-2x+1=0 \implies (x-1)(x^3-x^2+x-1)=0$. Once again, the sum of the coefficients equals zero so that $x-1$ is a factor of the cubic and we have $(x-1)^2(x^2+1)=0$. Thus, the solutions are $\boldsymbol{x = 1,\ 1,\ \pm i}$.

Problem 11. $x^4 + x^3 + 2x^2 + x + 1 = 0$

Solution.

$$\begin{aligned} x^4 + x^3 + 2x^2 + x + 1 = 0 &\implies x^4 + 2x^2 + 1 + x^3 + x = 0 \\ &\implies (x^2+1)^2 + x(x^2+1) = 0 \\ &\implies (x^2+1)(x^2+x+1) = 0 \end{aligned}$$

Thus, the solutions are $\boldsymbol{x = \pm i, \ \dfrac{-1 \pm \sqrt{3}i}{2}}$.

Problem 12. $x^5 - 11x^4 + 36x^3 - 36x^2 + 11x - 1 = 0$

Solution. Since the coefficients sum to zero, we know that $x - 1$ is a factor, and synthetic division gives $x^5 - 11x^4 + 36x^3 - 36x^2 + 11x - 1 = 0 \implies (x-1)(x^4 - 10x^3 + 26x^2 - 10x + 1) = 0$. Now,

$$\begin{aligned} x^4 - 10x^3 + 26x^2 - 10x + 1 &= (x^4+1) - 10(x^3+x) + 26x^2 \\ &= x^2(x^2 + 1/x^2) - 10x^2(x + 1/x) + 26x^2 \\ &= x^2[(x^2 + 1/x^2) - 10(x + 1/x) + 26] \\ &= x^2[(x + 1/x)^2 - 2 - 10(x + 1/x) + 26] \\ &= x^2[(x + 1/x)^2 - 10(x + 1/x) + 24] \\ &= x^2[(x + 1/x) - 6][(x + 1/x) - 4] \\ &= (x^2 + 1 - 6x)(x^2 + 1 - 4x) \\ &= (x^2 - 6x + 1)(x^2 - 4x + 1) \end{aligned}$$

so we have

$$x^5 - 11x^4 + 36x^3 - 36x^2 + 11x - 1 = 0 \implies (x-1)(x^2 - 6x + 1)(x^2 - 4x + 1) = 0.$$

Thus, the solutions are $\boldsymbol{x = 1, \ x = 3 \pm 2\sqrt{2}, \ 2 \pm \sqrt{3}}$.

Problem 13. $x^5 - 243 = 0$

Solution. Let $x = \sqrt[5]{243}y = 3y \implies x^5 - 243 = 0 \implies 3^5y^5 - 3^5 = 0 \implies y^5 - 1 = 0$. Now, $y^5 - 1 = (y-1)(y^4 + y^3 + y^2 + y + 1)$, so we need to find the solution to the reciprocal equation

$$\begin{aligned} y^4 + y^3 + y^2 + y + 1 = 0 &\implies (y^4 + 1) + (y^3 + y) + y^2 = 0 \\ &\implies y^2[(y^2 + 1/y^2) + (y + 1/y) + 1] = 0 \\ &\implies y^2[(y + 1/y)^2 - 2 + (y + 1/y) + 1] = 0 \\ &\implies y^2[(y + 1/y)^2 + (y + 1/y) - 1] = 0 \\ &\implies y^2[(y + 1/y + 1/2)^2 - 1/4 - 1] = 0 \\ &\implies y^2[(y + 1/y + 1/2)^2 - (\sqrt{5/4})^2] = 0 \\ &\implies y^2(y + 1/y + 1/2 - \sqrt{5}/2)(y + 1/y + 1/2 + \sqrt{5}/2) = 0 \\ &\implies \left(y^2 + 1 + \frac{1-\sqrt{5}}{2}y\right)\left(y^2 + 1 + \frac{1+\sqrt{5}}{2}y\right) = 0 \\ &\implies \left(y^2 + \frac{1-\sqrt{5}}{2}y + 1\right)\left(y^2 + \frac{1+\sqrt{5}}{2}y + 1\right) = 0 \end{aligned}$$

Since $x = 3y$, the solutions are $\boldsymbol{x = 3, \ 3\left(\dfrac{-1 + \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}}{4}\right), \ 3\left(\dfrac{-1 - \sqrt{5} \pm i\sqrt{10 - 2\sqrt{5}}}{4}\right)}$.

Problem 14. $(2x-1)^3 = 1$

Solution. Let $y = 2x-1$, then we have $y^3 - 1 = 0 \implies (y-1)(y^2+y+1) = 0$, which has solutions $y = 1,\ \dfrac{-1\pm\sqrt{3}i}{2}$. Solving for x, we get $\boldsymbol{x = 1,\ \dfrac{1\pm\sqrt{3}i}{4}}$.

Problem 15. $(1+x)^3 = (1-x)^3$

Solution. $(1+x)^3 = (1-x)^3 \implies 1+3x+3x^2+x^3 = 1-3x+3x^2-x^3 \implies 2x^3+6x = 0 \implies x(x^2+3) = 0$. thus, the solutions are $\boldsymbol{x = 0,\ \pm\sqrt{3}i}$.

Problem 16. $(x-2)^4 - 81 = 0$

Solution. $(x-2)^4 - 81 = 0 \implies [(x-2)^2-9][(x-2)^2+9] = 0 \implies x-2 = \pm 3,\ x-2 = \pm 3i$. Thus, the solutions are $\boldsymbol{x = -1,\ 5,\ 2\pm 3i}$.

Problem 17. $(a+x)^3 + (b+x)^3 = (a+b+2x)^3$

Solution.

$$\begin{aligned}(a+x)^3 + (b+x)^3 &= (a+b+2x)^3\\ &= [(a+x)+(b+x)]^3\\ &= (a+x)^3 + 3(a+x)^2(b+x) + 3(a+x)(b+x)^2 + (b+x)^3\end{aligned}$$

$\implies 3(a+x)(b+x)[(a+x)+(b+x)] = 0 \implies (x+a)(x+b)(2x+a+b) = 0$. Thus, the solutions are $\boldsymbol{x = -a,\ -b,\ -(a+b)/2}$.

Problem 18. $(a-x)^4 - (b-x)^4 = (a-b)(a+b-2x)$

Solution.

$$\begin{aligned}&(a-x)^4 - (b-x)^4 = (a-b)(a+b-2x) \implies\\ &[(a-x)^2 - (b-x)^2]\,[(a-x)^2 + (b-x)^2] = (a-b)(a+b-2x) \implies\\ &[a^2 - 2ax + x^2 - (b^2 - 2bx + x^2)]\,[a^2 - 2ax + x^2 + b^2 - 2bx + x^2] = (a-b)(a+b-2x) \implies\\ &[a^2 - b^2 - 2(a-b)x]\,[2x^2 - 2(a+b)x + a^2 + b^2] = (a-b)(a+b-2x) \implies\\ &(a-b)[a+b-2x]\,[2x^2 - 2(a+b)x + a^2 + b^2] = (a-b)(a+b-2x) \implies\\ &(a-b)[a+b-2x]\,[2x^2 - 2(a+b)x + a^2 + b^2 - 1] = 0\end{aligned}$$

The solutions to the quadratic factor are $x = \dfrac{2(a+b)\pm\sqrt{4(a+b)^2 - 4(2)(a^2+b^2-1)}}{4} = \dfrac{a+b\pm\sqrt{2-(a-b)^2}}{2}$.
Thus, the solutions are $\boldsymbol{x = \dfrac{a+b}{2},\ \dfrac{a+b\pm\sqrt{2-(a-b)^2}}{2}}$.

Problem 19. $\dfrac{x^2+3x+1}{4x^2+6x-1} - 3\dfrac{4x^2+6x-1}{x^2+3x+1} - 2 = 0$

Solution. We recognize this as a reciprocal equation.

$$\frac{x^2+3x+1}{4x^2+6x-1} - 3\frac{4x^2+6x-1}{x^2+3x+1} - 2 = 0 \implies \frac{4x^2+6x-1}{x^2+3x+1}\left[\left(\frac{x^2+3x+1}{4x^2+6x-1}\right)^2 - 3 - 2\frac{x^2+3x+1}{4x^2+6x-1}\right] = 0 \implies$$

$$\frac{4x^2+6x-1}{x^2+3x+1}\left[\left(\frac{x^2+3x+1}{4x^2+6x-1} - 1\right)^2 - 1 - 3\right] = 0 \implies \frac{4x^2+6x-1}{x^2+3x+1}\left(\frac{x^2+3x+1}{4x^2+6x-1} - 3\right)\left(\frac{x^2+3x+1}{4x^2+6x-1} + 1\right) = 0$$

$$\implies \frac{4x^2+6x-1}{x^2+3x+1}\cdot\frac{-11x^2-15x+4}{4x^2+6x-1}\cdot\frac{5x^2+9x}{4x^2+6x-1} = 0 \implies \frac{(11x^2+15x-4)(5x^2+9x)}{(x^2+3x+1)(4x^2+6x-1)} = 0$$

Thus, the solutions are $\boldsymbol{x = 0,\ -9/5,\ \dfrac{-15 \pm \sqrt{401}}{22}}$.

Problem 20. $x^2 + \dfrac{1}{x^2} = a^2 + \dfrac{1}{a^2}$

Solution.

$x^2 + \dfrac{1}{x^2} = a^2 + \dfrac{1}{a^2} \implies \left(x + \dfrac{1}{x}\right)^2 - 2 = \left(a + \dfrac{1}{a}\right)^2 - 2 \implies x + \dfrac{1}{x} = \pm\left(a + \dfrac{1}{a}\right)$

positive root: $x + \dfrac{1}{x} = +\left(a + \dfrac{1}{a}\right) \implies ax^2 - (1 + a^2)x + a = 0 \implies x = \dfrac{1 + a^2 \pm (1 - a^2)}{2a}$

negaitve root: $x + \dfrac{1}{x} = -\left(a + \dfrac{1}{a}\right) \implies ax^2 + (1 + a^2)x + a = 0 \implies x = \dfrac{-(1 + a^2) \pm (1 - a^2)}{2a}$

Thus, the solutions are $\boldsymbol{x = 1/a,\ a,\ -a,\ -1/a}$.

Problem 21. $3x^2 - 2x - 5\sqrt{3x^2 - 2x + 3} + 9 = 0$

Solution. Let $y = \sqrt{3x^2 - 2x + 3}$, which gives $y^2 - 5y + 6 = 0 \implies (y - 3)(y - 2) = 0$.

$y = 3$: $3x^2 - 2x + 3 = 9 \implies 3x^2 - 2x - 6 = 0 \implies x = \dfrac{2 \pm \sqrt{4 + 4(3)(6)}}{6} = \dfrac{1 \pm \sqrt{19}}{3}$

$y = 2$: $3x^2 - 2x + 3 = 4 \implies 3x^2 - 2x - 1 = 0 \implies x = \dfrac{2 \pm \sqrt{4 + 4(3)(1)}}{6} = \dfrac{2 \pm 4}{6} = 1, -\dfrac{1}{3}$

Thus, the solutions are $\boldsymbol{x = \dfrac{1 \pm \sqrt{19}}{3},\ 1,\ -\dfrac{1}{3}}$.

Problem 22. $4x^2 - 2x - 1 = \sqrt{2x^2 - x}$

Solution. Let $y = \sqrt{2x^2 - x}$, which gives $2y^2 - y - 1 = (y - 1)(2y + 1) = 0$.

$y = 1$: $2x^2 - x = 1 \implies 2x^2 - x - 1 = 0 \implies (x - 1)(2x + 1) = 0$

$y = -1/2$: $2x^2 - x = \dfrac{1}{4} \implies 8x^2 - 4x - 1 = 0 \implies x = \dfrac{4 \pm \sqrt{16 + 4(8)(1)}}{16} = \dfrac{1 \pm \sqrt{3}}{4}$

Thus, the solutions are $\boldsymbol{x = 1,\ -1/2,\ \dfrac{1 \pm \sqrt{3}}{4}}$.

Problem 23. $\sqrt{3 - x} + \sqrt{2 - x} = \sqrt{5 - 2x}$

Solution.

$\sqrt{3 - x} + \sqrt{2 - x} = \sqrt{5 - 2x} \implies \sqrt{3 - x} + \sqrt{2 - x} = \sqrt{(3 - x) + (2 - x)} \implies$

$3 - x + 2\sqrt{(3 - x)(2 - x)} + 2 - x = 3 - x + 2 - x \implies (3 - x)(2 - x) = 0$

Thus, the solutions are $\boldsymbol{x = 2,\ 3}$.

Problem 24. $\sqrt{2x + 3} + \sqrt{3x - 5} - \sqrt{x + 1} - \sqrt{4x - 3} = 0$

Solution.

$\sqrt{2x + 3} + \sqrt{3x - 5} - \sqrt{x + 1} - \sqrt{4x - 3} = 0 \implies \sqrt{2x + 3} + \sqrt{3x - 5} = \sqrt{x + 1} + \sqrt{4x - 3} \implies$

$2x + 3 + 2\sqrt{(2x + 3)(3x - 5)} + 3x - 5 = x + 1 + 2\sqrt{(x + 1)(4x - 3)} + 4x - 3 \implies$

$(2x + 3)(3x - 5) = (x + 1)(4x - 3) \implies 6x^2 - x - 15 = 4x^2 + x - 3 \implies 2x^2 - 2x - 12 = 0 \implies$

$(x + 2)(x - 3) = 0$

Thus, the solutions are $\boldsymbol{x = -2,\ 3}$.

Problem 25. $\dfrac{x^2-x+1}{x-1} - x = \sqrt{\dfrac{6}{x}}$

Solution.

$$\frac{x^2-x+1}{x-1} - x = \sqrt{\frac{6}{x}} \implies x + \frac{1}{x-1} - x = \sqrt{\frac{6}{x}} \implies (x-1)^2 = \frac{x}{6} \implies 6x^2 - 13x + 6 = 0 \implies$$
$$x = \frac{13 \pm \sqrt{169 - 4(6)(6)}}{12} = \frac{13 \pm 5}{12} = 2/3,\ 3/2.$$

The only acceptable solution is $\boldsymbol{x = \dfrac{3}{2}}$.

Problem 26. $\sqrt{x} + \sqrt{x - \sqrt{1-x}} = 1$

Solution.

$$\begin{aligned}
\sqrt{x} + \sqrt{x - \sqrt{1-x}} = 1 &\implies \sqrt{x - \sqrt{1-x}} = 1 - \sqrt{x} \\
&\implies x - \sqrt{1-x} = 1 - 2\sqrt{x} + x \\
&\implies -\sqrt{1-x} = 1 - 2\sqrt{x} \\
&\implies \sqrt{x} - \sqrt{1-x} = 1 - \sqrt{x} \\
&\implies x - 2\sqrt{x(1-x)} + 1 - x = 1 - 2\sqrt{x} + x \\
&\implies -2\sqrt{x(1-x)} = x - 2\sqrt{x} \\
&\implies -2\sqrt{1-x} = \sqrt{x} - 2 \\
&\implies 2(1 - \sqrt{1-x}) = \sqrt{x} \\
&\implies 4(1 - 2\sqrt{1-x} + 1 - x) = x \\
&\implies 4 - 8\sqrt{1-x} + 4 - 4x = x \\
&\implies 8\sqrt{1-x} = -5x + 8 \\
&\implies 64(1-x) = 25x^2 - 80x + 64 \\
&\implies 25x^2 - 80x + 64x = 0 \\
&\implies 25x = 16
\end{aligned}$$

Thus, the solution is $\boldsymbol{x = \dfrac{16}{25}}$.

Problem 27. $\sqrt{x+3} - \sqrt{x^2+3x} = 0$

Solution. $\sqrt{x+3} - \sqrt{x^2+3x} = 0 \implies \sqrt{x+3} = \sqrt{x}\sqrt{x+3} \implies x+3 = x(x+3) \implies (x-1)(x+3) = 0.$ Thus, the solutions are $\boldsymbol{x = 1,\ -3}$.

Problem 28. $\sqrt[4]{x^3} - 5\sqrt{x} + 6\sqrt[4]{x} = 0$

Solution. First note that $x = 0$ is a solution. Now we find any other solutions.

$$\begin{aligned}\sqrt[4]{x^3} - 5\sqrt{x} + 6\sqrt[4]{x} = 0 &\implies x^{3/4} + 6x^{1/4} = 5x^{1/2} \\ &\implies x^{3/2} + 12x + 36x^{1/2} = 25x \\ &\implies x^{3/2} + 36x^{1/2} = 13x \\ &\implies x^{1/2}(x+36) = 13x \\ &\implies (x+36) = 13x^{1/2} \\ &\implies (x+36)^2 = 169x \\ &\implies x^2 + 72x + 1296 = 169x \\ &\implies x^2 - 97x + 1296 = 0\end{aligned}$$

The solutions of the quadratic are $x = \dfrac{97 \pm \sqrt{(97)^2 - 4(1296)}}{2} = \dfrac{97 \pm 65}{2}$. Thus, the solutions are $\boldsymbol{x = 0,\ 16,\ 81}$.

Problem 29. $\sqrt{\dfrac{2x-5}{x-2}} - 3\sqrt{\dfrac{x-2}{2x-5}} + 2 = 0$

Solution. Notice that this is a reciprocal equation. Accordingly, let $y = \sqrt{\dfrac{2x-5}{x-2}}$, then we have

$$y - \frac{3}{y} + 2 = 0 \implies y^2 + 2y - 3 = (y+3)(y-1) = 0 \implies \sqrt{\frac{2x-5}{x-2}} = 1 \implies 2x - 5 = x - 2 \implies \boldsymbol{x = 3}.$$

Problem 30. $\dfrac{\sqrt{x-1} - \sqrt{x+1}}{\sqrt{x-1} + \sqrt{x+1}} = x - 3$

Solution. First we rationalize the denominator.

$\dfrac{\sqrt{x-1} - \sqrt{x+1}}{\sqrt{x-1} + \sqrt{x+1}} \cdot \dfrac{\sqrt{x-1} - \sqrt{x+1}}{\sqrt{x-1} - \sqrt{x+1}} = x - 3 \implies \dfrac{x - 1 - 2\sqrt{x^2-1} + x + 1}{x - 1 - (x+1)} = x - 3 \implies$
$\sqrt{x^2-1} - x = x - 3 \implies x^2 - 1 = 4x^2 - 12x + 9 \implies 3x^2 - 12x + 10 = 0 \implies$
$x = \dfrac{12 \pm \sqrt{(12)^2 - (4)(3)(10)}}{6} = \boldsymbol{\dfrac{6 \pm \sqrt{6}}{3}}$. However, only the positive square root satisfies the given equation.

Problem 31. $\sqrt{5x^2 - 6x + 1} - \sqrt{5x^2 + 9x - 2} = 5x - 1$

Solution.

$\sqrt{5x^2 - 6x + 1} - \sqrt{5x^2 + 9x - 2} = 5x - 1 \implies \sqrt{(5x-1)(x-1)} - \sqrt{(5x-1)(x+2)} = 5x - 1 \implies$
$\boldsymbol{x = 1/5}$ and $\sqrt{x-1} - \sqrt{x+2} = \sqrt{5x-1} \implies x - 1 - 2\sqrt{(x-1)(x+2)} + x + 2 = 5x - 1 \implies$
$-2\sqrt{(x-1)(x+2)} = 3x - 2 \implies 4(x^2 + x - 2) = 9x^2 - 12x + 4 \implies 5x^2 - 16x + 12 = 0 \implies$
$x = \dfrac{16 \pm \sqrt{(16)^2 - (4)(5)(12)}}{10} = 2,\ \dfrac{6}{5}$. However, only $\boldsymbol{x = \dfrac{1}{5}}$ satisfies the given equation.

Problem 32. $\dfrac{\sqrt{2x-1} + \sqrt{3x}}{\sqrt{2x-1} - \sqrt{3x}} + 3 = 0$

Solution. First we rationalize the denominator.

$\dfrac{\sqrt{2x-1}+\sqrt{3x}}{\sqrt{2x-1}-\sqrt{3x}}\cdot\dfrac{\sqrt{2x-1}+\sqrt{3x}}{\sqrt{2x-1}+\sqrt{3x}}+3=0 \implies \dfrac{2x-1+3x+2\sqrt{3x(2x-1)}}{2x-1-3x}+3=0 \implies$
$\dfrac{5x-1+2\sqrt{3x(2x-1)}}{x+1}=3 \implies 2\sqrt{3x(2x-1)}=3x+3-5x+1=-2x+4 \implies$
$\sqrt{3x(2x-1)}=2-x \implies 6x^2-3x=4-4x+x^2 \implies 5x^2+x-4=0 \implies$
$x=\dfrac{-1\pm\sqrt{1+(4)(5)(4)}}{10}=\mathbf{-1,\ \dfrac{4}{5}}$. However, we reject $x=-1$ since it gives a denominator of zero.

Problem 33. $\sqrt[3]{x}+\sqrt[3]{2-x}=2$

Solution.

$\sqrt[3]{x}+\sqrt[3]{2-x}=2 \implies (2-x)^{1/3}=2-x^{1/3} \implies 2-x=2^3-3\cdot 2^2x^{1/3}+3\cdot 2x^{2/3}-x \implies$
$2-x=8-12x^{1/3}+6x^{2/3}-x \implies 6x^{2/3}-12x^{1/3}+6=0 \implies x^{2/3}-2x^{1/3}+1=0$

Let $y=x^{1/3}$, then we have $y^2-2y+1=(y-1)^2=0 \implies y=1 \implies \boldsymbol{x=1}$.

Problem 34. $(x+a)^{1/3}+(x+b)^{1/3}+(x+c)^{1/3}=0$

Solution. Starting with $(x+a)^{1/3}+(x+b)^{1/3}=-(x+c)^{1/3}$ and cubing both sides gives

$(x+a)+3(x+a)^{2/3}(x+b)^{1/3}+3(x+a)^{1/3}(x+b)^{2/3}+(x+b)=-(x+c) \implies$
$3x+a+b+c+3[(x+a)^{2/3}(x+b)^{1/3}+(x+a)^{1/3}(x+b)^{2/3}]=0 \implies$
$x+\dfrac{a+b+c}{3}+(x+a)^{1/3}(x+b)^{1/3}[(x+a)^{1/3}+(x+b)^{1/3}]=0 \implies$
$x+\dfrac{a+b+c}{3}=(x+a)^{1/3}(x+b)^{1/3}(x+c)^{1/3} \implies$
$\left(x+\dfrac{a+b+c}{3}\right)^3=(x+a)(x+b)(x+c) \implies$
$x^3+(a+b+c)x^2+\dfrac{(a+b+c)^2}{3}x+\dfrac{(a+b+c)^3}{27}=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc \implies$
$9(a+b+c)^2x+(a+b+c)^3=27(ab+ac+bc)x+27abc \implies$
$[9(a+b+c)^2-27(ab+ac+bc)]x=27abc-(a+b+c)^3 \implies$

$$\boldsymbol{x=\frac{27abc-(a+b+c)^3}{9(a+b+c)^2-27(ab+ac+bc)}}$$

Problem 35. $x(x-1)(x-2)(x-3)=6\cdot 5\cdot 4\cdot 3$

Solution.

$[x(x-3)][(x-1)(x-2)]-360=0 \implies (x^2-3x)(x^2-3x+2)-360=0 \implies$
$(x^2-3x)^2+2(x^2-3x)-360=0 \implies (x^2-3x+1)^2-1-360=0 \implies x^2-3x+1=\pm 19$
$x^2-3x-18=0 \implies x=\dfrac{3\pm\sqrt{9+(4)(18)}}{2}=\mathbf{6,\ -3}$
$x^2-3x+20=0 \implies x=\dfrac{3\pm\sqrt{9-(4)(20)}}{2}=\boldsymbol{\dfrac{3\pm\sqrt{71}i}{2}}$

Problm 36. $(x+a)^2+4(x+a)\sqrt{x}=a^2-4a\sqrt{x}$

Solution.

$(x+a)^2 + 4(x+a)\sqrt{x} = a^2 - 4a\sqrt{x} \implies (x+a)^2 - a^2 + 4x\sqrt{x} + 8a\sqrt{x} = 0 \implies$
$x(x+2a) + 4(x+2a)\sqrt{x} = 0 \implies (x+2a)(x+4\sqrt{x}) = 0 \implies (x+2a)\sqrt{x}(\sqrt{x}+4) = 0 \implies \boldsymbol{x = 0,\ -2a}$

Problem 37. $\sqrt[3]{1 + \left(\dfrac{2x}{x^2-1}\right)^2} + \sqrt[3]{1 + \dfrac{2}{x^2-1}} = 6$

Solution.

$$\left[\frac{(x^2-1)^2 + 4x^2}{(x^2-1)^2}\right]^{1/3} + \left[\frac{x^2-1+2}{x^2-1}\right]^{1/3} = 6 \implies \left[\frac{(x^2+1)^2}{(x^2-1)^2}\right]^{1/3} + \left[\frac{x^2+1}{x^2-1}\right]^{1/3} = 6$$

Let $y = \left(\dfrac{x^2+1}{x^2-1}\right)^{1/3}$, then we have $y^2 + y - 6 = 0$, so that $y = \dfrac{-1 \pm \sqrt{1+(4)(6)}}{2} = \dfrac{-1 \pm 5}{2} = -3,\ 2.$

Taking each of these solution in turn, we get

$$\frac{x^2+1}{x^2-1} = -27 \implies 28x^2 = 26 \implies x = \pm\sqrt{\frac{13}{14}} = \pm\frac{\mathbf{\sqrt{182}}}{\mathbf{14}}$$
$$\frac{x^2+1}{x^2-1} = 8 \implies 7x^2 = 9 \implies x = \pm\frac{3}{\sqrt{7}} = \pm\frac{\mathbf{3\sqrt{7}}}{\mathbf{7}}$$

Chapter 16

Simultaneous Equations which Can be Solved by Means of Quadratics

Infinite Roots

The formulas for the roots of the quadratic $ax^2 + bx + c = 0$ are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

As it stands, the limit as $a \to 0$ is problematic. However,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}} = -\frac{2c}{b + \sqrt{b^2 - 4ac}} \to -\frac{c}{b} \text{ as } a \to 0$$
$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b + \sqrt{b^2 - 4ac}}{-b + \sqrt{b^2 - 4ac}} = -\frac{2c}{b - \sqrt{b^2 - 4ac}} \to \infty \text{ as } a \to 0$$

and if both $a \to 0$ and $b \to 0$, then both $a \to \infty$ and $\beta \to \infty$. Therefore,

one root of $ax^2 + bx + c = 0$ becomes infinite when a vanishes, and both roots become infinite when a and b (but not c) vanish simultaneously.

16.1 Simultaneous Quadratic Equations, Exercise XLV (p. 320)

Solve the following pairs of equations.

Problem 1. $\begin{cases} 7x^2 - 6xy = 8, \\ 2x - 3y = 5. \end{cases}$

Solution. $2x - 3y = 5 \implies 3y = 2x - 5 \implies 7x^2 - 6xy = 8 \implies 7x^2 - 2x(2x-5) = 8 \implies 3x^2 + 10x - 8 = 0 \implies x = \dfrac{-10 \pm \sqrt{100 + 4(3)(8)}}{6} = \dfrac{-10 \pm 14}{6} = -4,\ \dfrac{2}{3} \implies (x, y) = (-4, -13/3),\ (2/3, -11/9)$.

Problem 2. $\begin{cases} xy = 1, \\ 3x - 5y = 2. \end{cases}$

Solution. $xy = 1 \implies y = 1/x \implies 3x - 5y = 2 \implies 3x - 5/x = 2 \implies 3x^2 - 2x - 5 = (3x-5)(x+1) = 0 \implies x = 5/3,\ -1 \implies (x, y) = (5/3, 3/5),\ (-1, -1)$.

Problem 3. $\begin{cases} x^2 + x = 4y^2, \\ 3x + 6y = 1. \end{cases}$

Solution. $3x + 6y = 1 \implies y = (1 - 3x)/6 \implies x^2 + x = 4y^2 \implies x^2 + x = 4(1 - 3x)^2/36 \implies 36x^2 + 36x = 4 - 24x + 36x^2 \implies 60x = 4 \implies x = 1/15 \implies (x, y) = (1/15, 2/15)$. Also, since the coefficient of $x^2 \to 0$ in the quadratic, there is also a solution that $\to \infty$.

Problem 4. $\begin{cases} 3x^2 - 3xy - y^2 - 4x - 8y + 3 = 0, \\ 3x - y - 8 = 0. \end{cases}$

Solution. $3x - y - 8 = 0 \implies y = 3x - 8 \implies 3x^2 - 3xy - y^2 - 4x - 8y + 3 = 0 \implies 3x^2 - 3x(3x - 8) - (3x - 8)^2 - 4x - 8(3x - 8) + 3 = 0 \implies 3x^2 - 9x^2 + 24x - 9x^2 + 48x - 64 - 4x - 24x + 64 + 3 = 0 \implies -15x^2 + 44x + 3 = 0 \implies 15x^2 - 44x - 3 = 0 \implies x = \dfrac{44 \pm \sqrt{1936 + 4(15)(3)}}{30} = \dfrac{44 \pm 46}{30} = 3,\ -1/15 \implies (x, y) = (3, 1),\ (-1/15, -41/5)$.

Problem 5. $\begin{cases} x^2 + 5y^2 - 8x - 7y = 0, \\ x + 3y = 0. \end{cases}$

Solution. $x + 3y = 0 \implies x = -3y \implies x^2 + 5y^2 - 8x - 7y = 0 \implies 9y^2 + 5y^2 + 24y - 7y = 0 \implies 14y^2 + 17y = y(14y + 17) = 0 \implies y = 0,\ -17/14 \implies (x, y) = (0, 0),\ (51/14, -17/14)$.

Problem 6. $\begin{cases} 2x^2 - xy - 3y = 0, \\ 7x - 6y - 4 = 0. \end{cases}$

Solution. $7x - 6y - 4 = 0 \implies y = (7x - 4)/6 \implies 2x^2 - xy - 3y = 0 \implies 2x^2 - x(7x - 4)/6 - 3(7x - 4)/6 = 0 \implies 12x^2 - 7x^2 + 4x - 21x + 12 = 0 \implies 5x^2 - 17x + 12 = 0 \implies x = \dfrac{17 \pm \sqrt{289 - 4(5)(12)}}{10} = \dfrac{17 \pm 7}{10} = 1,\ 12/5 \implies (x, y) = (1, 1/2),\ (12/5, 32/15)$.

Problem 7. $\begin{cases} x^2 + 3xy + 2y^2 - 1 = 0, \\ x + y = 0. \end{cases}$

Solution. $x + y = 0 \implies y = -x \implies x^2 + 3xy + 2y^2 - 1 = 0 \implies x^2 - 3x^2 + 2x^2 - 1 = 0 \implies -1 = 0 \implies$ no finite solution.

Problem 8. $\begin{cases} 2x + 3y = 37, \\ 1/x + 1/y = 14/45. \end{cases}$

Solution. $2x + 3y = 37 \implies y = (37 - 2x)/3 \implies 1/x + 1/y = 14/45 \implies 1/x + 3/(37 - 2x) = 14/45 \implies 45/x + 135/(37 - 2x) = 14 \implies 45(37 - 2x) + 135x = 14x(37 - 2x) \implies 1665 - 90x + 135x = 518x - 28x^2 \implies 28x^2 - 473x + 1665 = 0 \implies x = \dfrac{473 \pm \sqrt{223729 - 4(28)(1665)}}{56} = \dfrac{473 \pm 193}{56} = \dfrac{666}{56}, \dfrac{280}{56} = \dfrac{333}{28}, 5 \implies (x, y) = (333/28, 185/42),\ (5, 9)$.

Problem 9. $\begin{cases} 1/y - 3/x = 1, \\ 7/xy - 1/y^2 = 12. \end{cases}$

Solution. Let $p \equiv 1/x$, $q \equiv 1/y$. Then we have $q - 3p = 1, 7pq - q^2 = 12$. Now, $q - 3p = 1 \implies p = (q - 1)/3 \implies 7q(q - 1)/3 - q^2 = 12 \implies 7q^2 - 7q - 3q^2 = 36 \implies 4q^2 - 7q - 36 = 0 \implies q = \dfrac{7 \pm \sqrt{49 + 4(4)(36)}}{8} = \dfrac{7 \pm 25}{8} = 4,\ -9/4 \implies (p, q) = (1, 4),\ (-13/12, -9/4) \implies (x, y) = (1, 1/4),\ (-12/13, -4/9)$.

Problem 10. $\begin{cases} x^2+xy+2=0, \\ (3x+y)(2x+y-1)=0. \end{cases}$

Solution. $3x+y=0 \implies y=-3x \implies x^2+xy+2=0 \implies x^2=1 \implies x=\pm 1 \implies$ $\boldsymbol{(x,y)=(1,-3),\ (-1,3)}$. And $2x+y-1=0 \implies y=1-2x \implies x^2+xy+2=0 \implies x^2+x(1-2x)+2=0 \implies x^2-x-2=0 \implies x=\dfrac{1\pm\sqrt{1+4(2)}}{2}=\dfrac{1\pm 3}{2}=-1,\,2 \implies \boldsymbol{(x,y)=(-1,3),\ (2,-3)}$.

Problem 11. $\begin{cases} x^2+y^2-8=0, \\ (x+1)^2=(y-1)^2. \end{cases}$

Solution. $(x+1)^2=(y-1)^2 \implies x+1=\pm(y-1) \implies y=x+2$ and $y=-x$. Now, $y=x+2 \implies x^2+y^2-8=0 \implies x^2+y^2-8=0 \implies x^2+(x+2)^2-8=0 \implies x^2+2x-2=0 \implies x=\dfrac{-2\pm\sqrt{4+4(2)}}{2}=-1\pm\sqrt{3} \implies (x,y)=(-1-\sqrt{3},1-\sqrt{3}),\,(-1+\sqrt{3},1+\sqrt{3})$. And $y=-x \implies x^2+x^2-8=0 \implies x=\pm 2 \implies (x,y)=(2,-2),\,(-2,2)$.

Problem 12. $\begin{cases} x^2-xy-2y^2+y=0, \\ ((x-2y)(x+y-3)=0. \end{cases}$

Solution. $x-2y=0 \implies x=2y \implies x^2-xy-2y^2+y=0 \implies 4y^2-2y^2-2y^2+y=0 \implies y=0 \implies (x,y)=(0,0)$. And $x+y-3=0 \implies x=3-y \implies x^2-xy-2y^2+y=0 \implies (3-y)^2-(3-y)y-2y^2+y=0 \implies 9-6y+y^2-3y+y^2-2y^2+y=0 \implies -8y+9=0 \implies y=9/8 \implies (x,y)=(15/8,9/8)$, and since the coefficient of x^2 is zero, there are two infinite solutions as well.

Problem 13. Determine m so that the two solutions of the pair $y^2+4x+4=0$, $y=mx$ shall be equal.

Solution. $y=mx \implies y^2+4x+4=0 \implies m^2x^2+4x+4=0 \implies x=\dfrac{-4\pm\sqrt{16-4m^2(4)}}{2m^2}=\dfrac{-2\pm 2\sqrt{1-m^2}}{m^2}$, so the two solutions will be equal when $m=\pm 1$. The two solutions will be either $(x,y)=(-2,2)$ or $(x,y)=(-2,-2)$.

Problem 14. Determine m and c so that both solutions of the pair

$$x^2+xy-2y^2+x=0, \quad y=mx+c$$

shall be infinite.

Solution. $y=mx+c \implies x^2+xy-2y^2+x=0 \implies x^2+x(mx+c)-2(mx+c)^2+x=0 \implies x^2+mx^2+cx-2m^2x^2-4mcx-2c^2+x=0 \implies (1+m-2m^2)x^2+(c-4mc+1)x-2c^2=0$. Both solutions will be infinite when both the coefficient of x^2 and the coefficient of x go to zero. (Only one solution will be infinite if the coefficient of x^2 is zero and the coefficient of x is non-zero.) This will be the case when $2m^2-m-1=0$ and $c=1/(4m-1)$. Now, $2m^2-m-1=0 \implies x=\dfrac{1\pm\sqrt{1+4(2)}}{4}=\dfrac{1\pm 3}{4}=1,\,-\dfrac{1}{2} \implies (m,c)=(1,1/3),\,(-1/2,-1/3)$.

Problem 15. By the method of §650, Ex. 2, show that $2x-y+4$ is a factor of $2x^2+xy-y^2+10x+y+12$.

Solution. $2x-y+4 \implies y=2x+4 \implies 2x^2+xy-y^2+10x+y+12=0 \implies 2x^2+x(2x+4)-(2x+4)^2+10x+2x+4+12=0 \implies 0=0$, so this is an identity, which implies that $2x-y+4$ is a factor of $2x^2+xy-y^2+10x+y+12$. Indeed, by straightforward division, we find $2x^2+xy-y^2+10x+y+12=(2x-y+4)(x+y+3)$.

Problem 16. Show that the pair $xy=1$, $xy+x+y=0$ has not more than two finite solutions, and that the pair $x^2y+xy=1$, $x^2y+y^2=2$ has not more than four finite solutions.

Solution. $xy = 1 \implies y = 1/x \implies xy + x + y = 0 \implies 1 + x + 1/x = 0 \implies x^2 + x + 1 = 0 \implies x = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}i}{2} \implies (x, y) = ((-1 \pm \sqrt{3}i)/2, 2/(-1 \pm \sqrt{3}i))$. Since the equation $x^2 + x + 1 = 0$ is not an identity, it does not have infinitely many solutions. The fact that it's a quadratic means it has only two finite solutions.

Now, $x^2y + xy = 1 \implies y = 1/(x^2+x) \implies x^2y + y^2 = 2 \implies \dfrac{x^2}{x^2+x} + \dfrac{1}{(x^2+x)^2} = 2 \implies x^2(x^2+x) + 1 = 2(x^2+x)^2 \implies x^4 + x^3 + 1 = 2x^4 + 4x^3 + 2x^2 \implies x^4 + 3x^3 + 2x^2 - 1 = 0$. Since this is not an identity, there are not infinitely many solutions. The fact that it's a fourth-degree equation indicates that it has no more than four finite solutions.

16.2 Pairs of Equations which Can Be Solved by Factorization, Addition, or Subtraction. (p. 323)

Example 2. Solve $\begin{cases} x^2 - 3xy + 2y^2 + 3x - 3y = 0, \\ 2x^2 + xy - y^2 + x - 2y + 3 = 0. \end{cases}$

Solution. $x^2 - 3xy + 2y^2 + 3x - 3y = 0 \implies (x-y)(x-2y+3) = 0$. Setting $x = y$ in the second equation gives $2x^2 + x^2 - x^2 + x - 2x + 3 = 2x^2 - x + 3 = 0 \implies x = \dfrac{1 \pm \sqrt{1 - 4(2)(3)}}{4} = \dfrac{1 \pm \sqrt{23}i}{4} \implies (x, y) = (1 \pm \sqrt{23}i)/4, (1 \pm \sqrt{23}i)/4)$. And setting $x = 2y - 3$ in the second equation gives $2(2y-3)^2 + y(2y-3) - y^2 + 2y - 3 + 3 = 0 \implies$

16.3 Simultaneous Quadratic Equations, Exercise XLVI (p. 324)

Problem 1. $\begin{cases} x^2 + 3y^2 = 31, \\ 7x^2 - 2y^2 = 10. \end{cases}$

Solution. $\begin{bmatrix} 1 & 3 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \begin{bmatrix} 31 \\ 10 \end{bmatrix}$. The determinant of the coefficient matrix is -23. Cramer's rule then gives $x^2 = \dfrac{1}{-23} \det \begin{bmatrix} 31 & 3 \\ 10 & -2 \end{bmatrix} = \dfrac{-92}{-23} = 4$ and $y^2 = \dfrac{1}{-23} \det \begin{bmatrix} 1 & 31 \\ 7 & 10 \end{bmatrix} = \dfrac{-207}{-23} = 9$. Therefore, $(x, y) = (\pm 2, \pm 3)$.

Problem 2. $\begin{cases} \dfrac{36}{x^2} + \dfrac{1}{y^2} = 18, \\ \dfrac{1}{y^2} - \dfrac{4}{x^2} = 8. \end{cases}$

Solution. $\begin{bmatrix} 36 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1/x^2 \\ 1/y^2 \end{bmatrix} = \begin{bmatrix} 18 \\ 8 \end{bmatrix}$. The determinant of the coefficient matrix is 40. Cramer's rule gives $\dfrac{1}{x^2} = \dfrac{1}{40} \det \begin{bmatrix} 18 & 1 \\ 8 & 1 \end{bmatrix} = \dfrac{10}{40} = \dfrac{1}{4}$ and $\dfrac{1}{y^2} = \dfrac{1}{40} \det \begin{bmatrix} 36 & 18 \\ -4 & 8 \end{bmatrix} = \dfrac{360}{40} = 9$. Therefore, $(x, y) = (\pm 2, \pm 1/3)$.

Problem 3. $\begin{cases} y^2 + xy + 6 = 0, \\ y^2 - y - 2 = 0. \end{cases}$

Solution. The second equation is easily factored: $y^2 - y - 2 = (y-2)(y+1) = 0$. First, set $y = 2$ in the first equation to give $4 + 2x + 6 = 0 \implies x = -5$. Next, set $y = -1$ in the first equation to give $1 - x + 6 = 0 \implies x = 7$. Thus, the solutions are $(x, y) = (-5, 2)$, $(7, -1)$.

Problem 4. $\begin{cases} x^2 + y^2 - 3x + 2y - 39 = 0, \\ 3x^2 - 17xy + 10y^2 = 0. \end{cases}$

Solution. We factor the second equation: $3x^2 - 17xy + 10y^2 = (3x - 2y)(x - 5y) = 0$. Set $x = \dfrac{2}{3}y$ and the first equation becomes $\dfrac{4}{9}y^2 + y^2 - 2y + 2y - 39 \implies 13y^2 = 351 \implies y^2 = 27 \implies y = \pm 3\sqrt{3}$. Next, set $x = 5y$ and the first equation becomes $25y^2 + y^2 - 15y + 2y - 39 = 0 \implies 26y^2 - 13y - 39 = 0 \implies 2y^2 - y - 3 = 0 \implies y = \dfrac{1 \pm \sqrt{1 + 4(2)(3)}}{4} = \dfrac{1 \pm 5}{4} = -1, \dfrac{3}{2}$. Thus, the solutions are $(x, y) = (2\sqrt{3}, 3\sqrt{3}), (-2\sqrt{3}, -3\sqrt{3}), (-5, -1), (15/2, 3/2)$.

Problem 5. $\begin{cases} y^2 - x^2 - 5 = 0, \\ 4x^2 + 4xy + y^2 + 4x + 2y = 3. \end{cases}$

Solution. $4x^2 + 4xy + y^2 + 4x + 2y = 3 \implies (2x + y)^2 + 2(2x + y) = 3 \implies (2x + y + 1)^2 - 1 = 3 \implies (2x + y + 1)^2 = 4 \implies 2x + y + 1 = \pm 2$. Now, $2x + y + 1 = 2 \implies y = 1 - 2x \implies (1 - 2x)^2 - x^2 - 5 = 3x^2 - 4x - 4 = 0 \implies (x - 2)(3x + 2) = 0 \implies x = 2, -2/3$. And, $2x + y + 1 = -2 \implies y = -3 - 2x \implies (3 + 2x)^2 - x^2 - 5 = 3x^2 + 12x + 4 = 0 \implies x = \dfrac{-12 \pm \sqrt{144 - 4(3)(4)}}{6} = \dfrac{-12 \pm 4\sqrt{6}}{6} = \dfrac{-6 \pm 2\sqrt{6}}{3}$. Thus, the solutions are $(x, y) = (2, -3), (-2/3, 7/3), ((-6 \pm 2\sqrt{6})/3, (3 \mp 4\sqrt{6})/3)$.

Problem 6. $\begin{cases} x^2 + 5xy - 2x + 3y + 1 = 0, \\ 3x^2 + 15xy - 7x + 8y + 4 = 0. \end{cases}$

Solution. Multiplying the first equation by -3 and adding it to the second equation gives $-x - y + 1 = 0 \implies y = 1 - x \implies x^2 + 5x(1 - x) - 2x + 3(1 - x) + 1 = 0 \implies -4x^2 + 4 = 0 \implies x = \pm 1$ and $(x, y) = (1, 0), (-1, 2)$. And the fact that $-x - y + 1 = 0$ has no x^2 nor y^2 term indicates that there are two infinite roots as well.

Problem 7. $\begin{cases} x^2 - 15xy - 3y^2 + 2x + 9y = 98, \\ 5xy + y^2 - 3y = -21 \end{cases}$

Solution. Multiplying the second equation by 3 and adding it to the first equation gives $x^2 + 2x = 35 \implies x^2 + 2x - 35 = (x + 7)(x - 5) = 0 \implies x = 5, -7$. Substituting $x = 5$ into the first equation gives $25 - 75y - 3y^2 + 10 + 9y = 98 \implies 3y^2 + 66y + 63 = 0 \implies y^2 + 22y + 21 = (y + 21)(y + 1) = 0 \implies y = -21, -1 \implies (x, y) = (5, -21), (5, -1)$. And substituting $x = -7$ into the first equation gives $49 + 105y - 3y^2 - 14 + 9y = 98 \implies 3y^2 - 114y + 63 = 0 \implies y^2 - 38y + 21 = 0 \implies y = \dfrac{38 \pm \sqrt{1444 - 4(21)}}{2} = \dfrac{38 \pm \sqrt{1360}}{2} = 19 \pm 2\sqrt{85} \implies (x, y) = (-7, 19 \pm 2\sqrt{85})$.

Problem 8. $\begin{cases} 2x^2 + 3xy - 4y^2 = 25, \\ 15x^2 + 24xy - 31y^2 = 200 \end{cases}$

Solution. Multiply the first equation by 8 and then subtract the second equation from it to get $x^2 - y^2 = 0 \implies x = \pm y$. Substituting $x = y$ into the first equation gives $y^2 = 25 \implies y = \pm 5 \implies (x, y) = (5, 5), (-5, -5)$. And substituting $x = -y$ into the first equation gives $y^2 = -5 \implies y = \pm\sqrt{5}i \implies (x, y) = (\sqrt{5}i, -\sqrt{5}i), (-\sqrt{5}i, \sqrt{5}i)$.

Problem 9. $\begin{cases} x(x + 3y) = 18, \\ x^2 - 5y^2 = 4 \end{cases}$

Solution. We can eliminate the constant term in both equation by multiplying the first equation by 2 and the second equation by 9 to get $2x^2 + 6xy = 9x^2 - 45y^2 \implies 7x^2 - 6xy - 45y^2 = 0 \implies x = \dfrac{6y \pm \sqrt{36y^2 + 4(7)(45y^2)}}{14} = \dfrac{6y \pm 36y}{14} = 3y, -15y/7$. Substituting $x = 3y$ into the second equation gives $4y^2 = 4 \implies y = \pm 1 \implies (x, y) = (3, 1), (-3, -1)$. And substituting $x = -15y/7$ into the second equation gives $\frac{225}{49}y^2 - 5y^2 = 4 \implies -20y^2 = 196 \implies y = \frac{7i}{\sqrt{5}} = \frac{7\sqrt{5}i}{5} \implies (x, y) = (-3\sqrt{5}i, 7\sqrt{5}i/5)$.

Problem 10. $\begin{cases} x^2 - 3xy + 3y^2 = x^2y^2, \\ 7x^2 - 10xy + 4y^2 = 12x^2y^2 \end{cases}$

Solution. Multiply the first equation by 12 and equate it to the second equation, which gives $7x^2 - 10xy + 4y^2 = 12x^2 - 36xy + 36y^2$ or $5x^2 - 26xy + 32y^2 = 0$. The quadratic formula gives $x = \frac{26y \pm \sqrt{676y^2 - 4(5)(32y^2)}}{10} = \frac{26y \pm 6y}{10} = \frac{16y}{5}, 2y$. Substituting $x = 16y/5$ into the first equation gives $91y^2 = 256y^4 \implies y = 0, 0, \pm\sqrt{91}/16 \implies (x, y) = (0, 0), (0, 0), (\pm\sqrt{91}/5, \pm\sqrt{91}/16)$. And substituting $x = 2y$ into the first equation gives $y^2 = 4y^4 \implies y = 0, 0, \pm 1/2 \implies (x, y) = (0, 0), (0, 0), (\pm 1, \pm 1/2)$.

Problem 11. $\begin{cases} x^2 + xy + y^2 = 38, \\ x^2 - xy + y^2 = 14 \end{cases}$

Solution. Adding the two equations and dividing by 2 gives $x^2 + y^2 = 26 \implies y = \pm\sqrt{26 - x^2}$. Substituting this expression into the first equation gives $x^4 - 26x^2 + 144 = 0 \implies x^2 = \frac{26 \pm \sqrt{676 - 4(144)}}{2} = \frac{26 \pm 10}{2} = 18, 8 \implies x = \pm 3\sqrt{2}, \pm 2\sqrt{2} \implies (x, y) = (\pm 3\sqrt{2}, \pm 2\sqrt{2}), (\pm 2\sqrt{2}, \pm 3\sqrt{2})$.

Problem 12. $\begin{cases} x^2 - xy + y^2 = 21(x - y), \\ xy = 20 \end{cases}$

Solution. From the second equation, $y = 20/x$. Substituting this into the first equation gives $x^2 - 20 + \frac{400}{x^2} = 21\left(x - \frac{20}{x}\right)$, or $x^4 - 21x^3 - 20x^2 + 420x + 400 = 0$. Using synthetic division, we find $(x-5)(x+4)(x^2 - 20x - 20) = 0$. Applying the quadratic formula to the last factor gives $x = \frac{20 \pm \sqrt{400 + 4(2)}}{2} = 10 \pm 2\sqrt{30} \implies y = \frac{20}{10 \pm 2\sqrt{30}} \cdot \frac{10 \mp 2\sqrt{30}}{10 \mp 2\sqrt{30}} = -10 \pm 2\sqrt{30}$. Thus, we have $(x, y) = (5, 4), (-4, -5), (10 \pm 2\sqrt{30}, -10 \pm 2\sqrt{30})$.

Problem 13. $\begin{cases} x^2 + y - 8 = 0, \\ y^2 + 15x - 46 = 0. \end{cases}$

Solution. From the first equation, $y = 8 - x^2$. Substituting this into the second equation gives $(8 - x^2)^2 + 15x - 46 = 0 \implies x^4 - 16x^2 + 15x + 18 = 0$. By synthetic division, we find $x^4 - 16x^2 + 15x + 18 = (x-3)(x-2)(x^2 + 5x + 3) = 0$, so that the solutions are $x = 3, 2, \frac{-5 \pm \sqrt{25 - 4(3)}}{2} = \frac{-5 \pm \sqrt{13}}{2}$. Therefore, $(x, y) = (3, -1), (2, 4), (-5 \pm \sqrt{13})/2, (-3 \mp 5\sqrt{13})/2)$.

Cube Roots of Unity

$$x^3 = +1 \implies (x^3 - 1) = 0 \implies (x - 1)(x^2 + x + 1) = 0 \implies x = \omega, \omega^2, \omega^3 = 1,$$
$$x^3 = -1 \implies (x^3 + 1) = 0 \implies (x - 1)(x^2 - x + 1) = 0 \implies x = -\omega, -\omega^2, -\omega^3 = -1,$$

where $\omega \equiv \frac{-1 \pm \sqrt{3}\, i}{2}$.

16.4 Symmetric Pairs of Equations, Exercise XLVIII (p. 328)

Problem 1. $\begin{cases} x+y=5, \\ xy+36=0. \end{cases}$

Solution. $(x-y)^2 = (x+y)^2 - 4xy = 25 - 4(-36) = 169 \implies x-y = \pm 13$. Thus, the solutions are $(x,y) = (9,-4), (-4,9)$.

Problem 2. $\begin{cases} x^2+y^2=200, \\ x+y=12. \end{cases}$

Solution. $2xy = (x+y)^2 - (x^2+y^2) = 144 - 200 = -56$, so that $(x-y)^2 = x^2+y^2-2xy = 200+56 = 256 \implies x-y = \pm 16$. Thus, the solutions are $(x,y) = (14,-2), (-2,14)$.

Problem 3. $\begin{cases} x^2+y^2=293, \\ xy=34. \end{cases}$

Solution. $(x+y)^2 = x^2+y^2+2xy = 293+2(34) = 361 \implies x+y = \pm 19$. And $(x-y)^2 = x^2+y^2-2xy = 293-2(34) = 225 \implies x-y = \pm 15$. Thus, the solutions are $(x,y) = (17,2), (2,17), (-2,-17), (-17,-2)$.

Problem 4. $\begin{cases} x^2+y^2=85, \\ x-y=7. \end{cases}$

Solution. $2xy = x^2+y^2-(x-y)^2 = 85-49 = 36$, so that $(x+y)^2 = x^2+y^2+2xy = 85+36 = 121 \implies x+y = \pm 11$. Thus, the solutions are $(x,y) = (9,2), (-2,-9)$.

Problem 5. $\begin{cases} x^3+y^3=513, \\ x+y=9. \end{cases}$

Solution. Let $x = u+v$, $y = u-v$. Then

$$(u+v)^3 + (u-v)^3 = u^3+3u^2v+3uv^2+v^3+u^3-3u^2v+3uv^2-v^3 = 2u^3+6uv^2 = 513 \text{ and } 2u = 9$$
$$\implies 2u(u^2+3v^2) = 513 \implies 9\left(\frac{81}{4}+3v^2\right) = 513 \implies 3v^2 + \frac{81}{4} = 57 \implies v^2 = \frac{147}{12} = \frac{49}{4} \implies v = \pm\frac{7}{2}$$

Thus, $(u,v) = (\frac{9}{2}, \pm\frac{7}{2})$ and the solutions are $(x,y) = (8,1), (1,8)$.

Problem 6. $\begin{cases} x^3+y^3=468, \\ x^2y+xy^2=420. \end{cases}$

Solution. Let $x = u+v$, $y = u-v$. Then

$$\begin{aligned} x^3+y^3 &= (u+v)^3 + (u-v)^3 \\ &= u^3+3u^2v+3uv^2+v3+u^3-3u^2v+3uv^2-v3 \\ &= 2u^3+6uv^2 \\ &= 2u(u^2+3v^2) = 468 \end{aligned} \tag{16.1}$$

$$\begin{aligned}
x^2y + xy^2 &= (u+v)^2(u-v) + (u+v)(u-v)^2 \\
&= (u^2+2uv+v^2)(u-v) + (u+v)(u^2-2uv+v^2) \\
&= u(2u^2+2v^2) - v(4uv) \\
&= 2u(u^2+v^2) - 4uv^2 \\
&= 2u(u^2+v^2-2v^2) \\
&= 2u(u^2-v^2) = 420 \qquad (16.2)
\end{aligned}$$

We have

$$\frac{2u(u^2+3v^2)}{2u(u^2-v^2)} = \frac{468}{420} = \frac{39}{35} \implies 35u^2 + 105v^2 = 39u^2 - 39v^2 \implies u^2 = 36v^2 \implies u = \pm 6v.$$

Substituting this into either eq. (1) or eq. (2) gives $v^3 = \pm 1$. Thus, we have

$$v^3 - 1 = 0 \implies (v-1)(v^2+v+1) = 0 \implies v = \frac{-1 \pm \sqrt{3}\,i}{2},\ 1 = \omega,\ \omega^2,\ \omega^3 = 1$$

$$v^3 + 1 = 0 \implies (v+1)(v^2-v+1) = 0 \implies v = \frac{1 \pm \sqrt{3}\,i}{2},\ -1 = -\omega,\ -\omega^2,\ -\omega^3 = -1,$$

where $\boxed{\omega \equiv \frac{-1+\sqrt{3}\,i}{2}}$. Thus,

$(u,v) = (6,1),\ (-6,1),\ (6\omega,\omega),\ (-6\omega,\omega), (-6,-1),\ (6,-1),\ (-6\omega^2,\omega^2),\ (6\omega^2,\omega^2),$

and the solutions are $(x,y) = (7,5),\ (5,7),\ (7\omega,5\omega),\ (5\omega,7\omega),\ (7\omega^2,5\omega^2),\ (5\omega^2,7\omega^2)$.

Problem 7. $\begin{cases} 27x^3 + 64y^3 = 65, \\ 3x + 4y = 5. \end{cases}$

Solution. Let $p \equiv 3x$, $q = 4y$. Then we have $p^3 + q^3 = 65$ and $p + q = 5$.

$$\begin{aligned}
p^2 + q^3 = (p+q)(p^2 - pq + q^2) &\implies p^2 - pq + q^2 = \frac{65}{5} = 13 \\
(p+q)^2 - (p^2-pq+q^2) = 3pq = 25 - 13 = 12 &\implies pq = 4 \\
(p-q)^2 = (p^2-pq+q^2) - pq = 13 - 4 = 9 &\implies p - q = \pm 3
\end{aligned}$$

Thus, $(p,q) = (3x,4y) = (4,1),\ (1,4)$ and so $(x,y) = (\frac{4}{3},\frac{1}{4}),\ (\frac{1}{3},1)$.

Problem 8. $\begin{cases} x^4 + y^4 = 82, \\ x - y = 2. \end{cases}$

Solution. Let $x = u + v$, $y = u - v$. Then we have

$$\begin{aligned}
x^4 + y^4 &= (u+v)^4 + (u-v)^4 \\
&= u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4 + u^4 - 4u^3v + 6u^2v^2 - 4uv^3 + v^4 \\
&= 2(u^4 + 6u^2v^2 + v^4) = 82 \implies u^4 + 6u^2v^2 + v^4 = 41 \\
x - y &= 2v = 2 \implies v = 1
\end{aligned}$$

Substituting $v = 1$ in the quartic for u gives

$$u^4 + 6u^2 - 40 = 0 \implies (u^2+10)(u^2-4) = 0 \implies u = \pm\sqrt{10}\,i,\ \pm 2$$

Thus, we have $(u, v) = (\sqrt{10}\,i, 1)$, $(-\sqrt{10}\,i, 1)$, $(2, 1)$, $(-2, 1)$, and the solutions are

$(x, y) = (1 + \sqrt{10}\,i, -1 + \sqrt{10}\,i)$, $(1 - \sqrt{10}\,i, -1 - \sqrt{10}\,i)$, $(3, 1)$, $(-1, -3)$.

Problem 9. $\begin{cases} x^5 + y^5 = 32, \\ x + y = 2. \end{cases}$

Solution. Let $x = u + v$, $y = u - v$. Then we have

$$\begin{aligned} x^5 + y^5 &= (u+v)^5 + (u-v)^5 \\ &= u^5 + 5u^4v + 10u^3v^2 + 10u^2v^3 + 5uv^4 + v^5 + u^5 - 5u^4v + 10u^3v^2 - 10u^2v^3 + 5uv^4 - v^5 \\ &= 2(u^5 + 10u^3v^2 + 5uv^4) = 32 \implies u^5 + 10u^3v^2 + 5uv^4 = 16 \\ x + y &= 2u = 2 \implies u = 1 \end{aligned}$$

Substituting $u = 1$ into the quintic reduces it to a quartic:

$$5v^4 + 10v^2 - 15 = 0 \implies v^4 + 2v^2 - 3 = 0 \implies (v^2+3)(v^2-1) = 0 \implies v = \pm\sqrt{3}\,i, \pm 1$$

Thus, we have $(u, v) = (1, \sqrt{3}\,i)$, $(1, -\sqrt{3}\,i)$, $(1, 1)$, $(1, -1)$, and the solutions are

$(x, y) = (1 + \sqrt{3}\,i, 1 - \sqrt{3}\,i)$, $(1 - \sqrt{3}\,i, 1 + \sqrt{3}\,i)$, $(2, 0)$, $(0, 2)$.

Problem 10. $\begin{cases} x + y = \dfrac{1}{2}, \\ 56(\dfrac{x}{y} + \dfrac{y}{x}) + 113 = 0. \end{cases}$

Solution. The second equation can be rewritten as

$$56(x^2 + y^2) + 113xy = 0 \implies 56(x+y)^2 + xy = 0 \implies xy = -56\left(\frac{1}{4}\right) = -14$$

Therefore,

$$(x-y)^2 = (x+y)^2 - 4xy = \frac{1}{4} - 4(-14) = \frac{1}{4} + 56 = \frac{225}{4} \implies x - y = \pm\frac{15}{2}$$

Thus, the solutions are $(x, y) = (4, -\frac{7}{2})$, $(-\frac{7}{2}, 4)$.

Problem 11. $\begin{cases} xy + x + y + 19 = 0, \\ x^2y + xy^2 + 20 = 0. \end{cases}$

Solution. The second equation may be written as $xy(x + y) + 20 = 0$. Multiplying the first equation by $x + y$ and making use of this, we get

$$xy(x+y) + (x+y)^2 + 19(x+y) = 0 \implies (x+y)^2 + 19(x+y) - 20 = (x+y+20)(x+y-1) = 0$$

So we have

$$\begin{aligned} &x + y = -20 \implies xy = 1 \text{ (from first eq.)} &&\implies (x-y)^2 = (x+y)^2 - 4xy = 400 - 4 = 396 \implies x - y = \pm 6\sqrt{11} \\ &x + y = 1 \implies xy = -20 \text{ (from first eq.)} &&\implies (x-y)^2 = (x+y)^2 - 4xy = 1 + 80 = 81 \implies x - y = \pm 9 \end{aligned}$$

Thus, we have

$$\begin{aligned} &x + y = -20 \text{ and } x - y = \pm 6\sqrt{11} \implies (x, y) = (-10 \pm 3\sqrt{11}, -10 \mp 3\sqrt{11}) \\ &x + y = 1 \text{ and } x - y = \pm 9 \implies (x, y) = (5, -4), (-4, 5) \end{aligned}$$

Problem 12. $\begin{cases} x^4 + y^4 - (x^2 + y^2) = 72, \\ x^2 + x^2y^2 + y^2 = 19. \end{cases}$

Solution. From the second equation, $x^2y^2 = 19 - (x^2 + y^2)$, so that

$$\begin{aligned} x^4 + y^4 &= (x^2 + y^2)^2 - 2x^2y^2 \\ &= (x^2 + y^2)^2 - 38 + 2(x^2 + y^2) \end{aligned}$$

and the first equation becomes

$$(x^2 + y^2)^2 - 38 + 2(x^2 + y^2) - (x^2 + y^2) = 72 \implies (x^2 + y^2)^2 + (x^2 + y^2) - 110 = 0 \implies$$
$$(x^2 + y^2 + 11)(x^2 + y^2 - 10) = 0 \implies x^2 + y^2 = -11,\ 10$$

Therefore, $x^2y^2 = 19 + 11,\ 19 - 10 = 30,\ 9 \implies xy = \pm\sqrt{30},\ \pm 3$. Thus, we have

$$(x + y)^2 = x^2 + y^2 + 2xy = -11 \pm 2\sqrt{30},\ -11 \pm 6,\ 10 \pm 2\sqrt{30},\ 10 \pm 6$$
$$(x - y)^2 = x^2 + y^2 - 2xy = -11 \mp 2\sqrt{30},\ -11 \mp 6,\ 10 \mp 2\sqrt{30},\ 10 \mp 6$$

Using $-11 \pm 2\sqrt{30} = -5 - 6 \pm 2\sqrt{(6)(5)} = (\sqrt{5}\,i)^2 + (\sqrt{6}\,i)^2 \mp 2(\sqrt{5}\,i)(\sqrt{6}\,i) = (\sqrt{5}\,i \mp \sqrt{6}\,i)^2$, the solutions are $(x, y) = (\pm\sqrt{5}\,i, \pm\sqrt{6}\,i)$, $(\pm\sqrt{6}\,i, \pm\sqrt{5}\,i)$, $(3, 1)$, $(1, 3)$, $(-3, -1)$, $(-1, -3)$.

Problem 13. $\begin{cases} x^2y + xy^2 = 30, \\ 1/x + 1/y = 3/10. \end{cases}$

Solution. From the second equation, we have $1/x + 1/y = 3/10 \implies 10(x + y) = 3xy$, so that

$$x^2y + xy^2 = 30 \implies xy(x + y) = 30 \implies 3xy(x + y) = 90 \implies 10(x + y)^2 = 90 \implies x + y = \pm 3$$
$$xy(x + y) = 30 \implies 10xy(x + y) = 300 \implies 3(xy)^2 = 300 \implies xy = \pm 10$$

And $(x - y)^2 = (x + y)^2 - 4xy = 9 \pm 40 = 49,\ -31 \implies x - y = \pm 7,\ \pm\sqrt{31}\,i$. Thus, the solutions are $(x, y) = (2, -5)$, $(-5, 2)$, $((3 \pm \sqrt{31}\,i)/2, (3 \mp \sqrt{31}\,i)/2)$.

Problem 14. $\begin{cases} x^2 + 3xy + y^2 + 2x + 2y = 8, \\ 2x^2 + 2y^2 + 3x + 3y = 14. \end{cases}$

Solution. We have

$$\begin{aligned} x^2 + 3xy + y^2 + 2x + 2y = 8 &\implies (x + y)^2 + 2(x + y) + xy = 8 \\ 2x^2 + 2y^2 + 3x + 3y = 14 &\implies 2(x + y)^2 + 3(x + y) - 4xy = 14 \end{aligned}$$

Multiplying the first equation by 4 and adding to the second equation eliminates the xy term to give $6(x + y)^2 + 11(x + y) - 46 = 0$, and this has the solution

$$x + y = \frac{-11 \pm \sqrt{121 + 4(6)(46)}}{12} = \frac{-11 \pm 35}{12} = 2,\ -\frac{23}{6} \tag{16.1}$$

Also,

$$x^2 + 3xy + y^2 + 2x + 2y = 8 \implies 2(x^2 + y^2) + 4(x + y) + 6xy = 16 \tag{16.2}$$
$$2x^2 + 2y^2 + 3x + 3y = 14 \implies 2(x^2 + y^2) + 3(x + y) = 14 \tag{16.3}$$

Subtracting eq. (3) from eq. (2) gives $x + y + 6xy = 2$. Making use of eq. (1), we have

$$xy = \frac{2 - (x + y)}{6} = \frac{2 - 2}{6},\ \frac{2 + \frac{23}{6}}{6} = 0,\ \frac{35}{36} \tag{16.4}$$

From eqs. (1) and (4), we have

$$(x-y)^2 = (x+y)^2 - 4xy = 4,\ \frac{529}{36} - \frac{140}{36} = \frac{389}{36} \implies x - y = \pm 2,\ \pm\frac{\sqrt{389}}{6}$$

Thus, the solutions are $(x, y) = (0, 2)$, $(2, 0)$, $\left(\dfrac{-23 \pm \sqrt{389}}{12}, \dfrac{-23 \mp \sqrt{389}}{12}\right)$.

Problem 15. $\begin{cases} x^3 = 5y, \\ y^3 = 5x. \end{cases}$

Solution. Subtracting and adding the two equations gives

$$x^3 - y^3 = 5(y-x) \implies x^3 - y^3 + 5(x-y) = 0 \implies (x-y)(x^2+xy+y^2+5) = 0 \tag{16.1}$$
$$x^3 + y^3 = 5(y+x) \implies x^3 + y^3 - 5(x+y) = 0 \implies (x+y)(x^2-xy+y^2-5) = 0 \tag{16.2}$$

Setting $x = y$ in eq. (2) gives $y^2 - 5 = 0 \implies (x, y) = (\pm\sqrt{5}, \pm\sqrt{5})$.
Setting $x = -y$ in eq. (1) gives $y^2 + 5 = 0 \implies (x, y) = (\pm\sqrt{5}\, i, \mp\sqrt{5}\, i)$.

To find additional roots, and to handle the other factor in eqs. (1) and (2), let $x = u + v$, $y = u - v$. Then we get

$$2v(3u^2 + v^2 + 5) = 0 \implies v = 0,\ v^2 = -3u^2 - 5 \tag{16.3}$$
$$2u(3v^2 + u^2 - 5) = 0 \implies u = 0,\ u^2 = -3v^2 + 5 \tag{16.4}$$

Substitute $u^2 = -3v^2 + 5$ into eq. (3) and substitute $v^2 = -3u^2 - 5$ into eq. (4) to get

$$3(-3v^2+5) + v^2 + 5 = 0 \implies -8v^2 + 20 = 0 \implies v^2 = \frac{10}{4} \implies v = \pm\frac{\sqrt{10}}{2}$$
$$3(-3u^2-5) + u^2 - 5 = 0 \implies -8u^2 - 20 = 0 \implies u^2 = -\frac{10}{4} \implies u = \pm\frac{\sqrt{10}\, i}{2}$$

Putting everything together, the solutions are $(x, y) = (0, 0)$, $(\pm\sqrt{5}, \pm\sqrt{5})$, $(\pm\sqrt{5}\, i, \mp\sqrt{5}\, i)$,

$$\left(\frac{\pm\sqrt{10}(1+i)}{2}, \frac{\mp\sqrt{10}(1-i)}{2}\right),\ \left(\frac{\pm\sqrt{10}(1-i)}{2}, \frac{\mp\sqrt{10}(1+i)}{2}\right).$$

16.5 Systems Involving More Than Two Unknown Letters, Exercise XLIX (p. 329)

Problem 1. $\begin{cases} x + y = 3, \\ y + z = 2, \\ x^2 - yz = 19. \end{cases}$

Solution. Solving the first two equations for x and z in terms of y, we have $x = 3 - y$, $z = 2 - y$, and the third equation becomes

$$(3-y)^2 - y(2-y) = 19 \implies 9 - 6y + y^2 - 2y + y^2 = 19 \implies 2y^2 - 8y - 10 = 0 \implies y^2 - 4y - 5 = (y-5)(y+1) = 0$$

Thus, the solutions are $(x, y, z) = (-2, 5, -3)$, $(4, -1, 3)$.

Problem 2. $\begin{cases} x(y+z) = 12, \\ y(z+x) = 6, \\ z(x+y) = 10. \end{cases}$

Solution. Define $a \equiv xy$, $b \equiv yz$, $c \equiv xz$, which allows us to write $a + c = 12$, $a + b = 6$, $b + c = 10$. We express this system with the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 10 \end{bmatrix}.$$

Solving with Cramer's rule, we get $a = 4$, $b = 2$, $c = 8$, or $xy = 4$, $yz = 2$, $xz = 8$. Therefore,

$$\begin{aligned} x^2 &= \frac{(xy)(xz)}{yz} = \frac{(4)(8)}{2} = 16 \implies x = \pm 4 \\ y^2 &= \frac{(xy)(yz)}{xz} = \frac{(4)(2)}{8} = \ \ 1 \implies y = \pm 1 \\ z^2 &= \frac{(yz)(xz)}{xy} = \frac{(2)(8)}{4} = \ \ 4 \implies x = \pm 2 \end{aligned}$$

Thus, the solutions are $(x, y, z) = (4, 1, 2)$, $(-4, -1, -2)$.

Problem 3. $\begin{cases} (y + b)(z + c) = a^2, \\ (z + c)(x + a) = b^2, \\ (x + a)(y + b) = c^2. \end{cases}$

Solution.

$$\begin{aligned} (x + a)^2 &= \frac{(x + a)(y + b)(z + c)(x + a)}{(y + b)(z + c)} = \frac{c^2 b^2}{a^2} \implies x + a = \pm \frac{bc}{a} \\ (y + b)^2 &= \frac{(y + b)(z + c)(x + a)(y + b)}{(z + c)(x + a)} = \frac{a^2 c^2}{b^2} \implies y + b = \pm \frac{ac}{b} \\ (z + c)^2 &= \frac{(z + c)(x + a)(y + b)(z + c)}{(x + a)(y + b)} = \frac{b^2 a^2}{c^2} \implies z + c = \pm \frac{ab}{c} \end{aligned}$$

Thus, $x = -\left(\dfrac{a^2 \pm bc}{a}\right)$, $y = -\left(\dfrac{b^2 \pm ac}{b}\right)$, $z = -\left(\dfrac{c^2 \pm ab}{c}\right)$.

16.6 Simultaneous Quadratic Equations, Exercise L (p. 330)

Solve the following systems of equations by any of the methods of the present chapter.

Problem 1. $\begin{cases} 7x^2 - 6xy = 8, \\ 2x - 3y = 5. \end{cases}$

Solution. From the second equation, $y = \dfrac{2x - 5}{3}$, and the first equation becomes

$$7x^2 + 2x(5 - x) = 8 \implies 3x^2 + 10x - 8 = 0 \implies x = \frac{-10 \pm \sqrt{100 + 4(3)(8)}}{6} = \frac{-10 \pm 14}{6} = \frac{2}{3}, -4$$

Thus, the solutions are $(x, y) = (2/3, -11/9)$, $(-4, -13/3)$.

Problem 2. $\begin{cases} x^2 + y^2 = 25, \\ x - y = 1. \end{cases}$

Solution. We have $2xy = x^2 + y^2 - (x - y)^2 = 25 - 1 = 24$, so that

$$(x + y)^2 = x^2 + y^2 + 2xy = 25 + 24 = 49 \implies x + y = \pm 7$$

Thus, the solutions are $(x, y) = (4, 3),\ (-3, -4)$.

Problem 3. $\begin{cases} x - y = a, \\ xy = (b^2 - a^2)/4. \end{cases}$

Solution. We have

$$(x+y)^2 = (x-y)^2 + 4xy = a^2 + (b^2 - a^2) = b^2 \implies x + y = \pm b$$

Thus, the solutions are $(x, y) = \left(\frac{a+b}{2}, \frac{b-a}{2}\right),\ \left(\frac{a-b}{2}, -\frac{b+a}{2}\right)$.

Problem 4. $\begin{cases} \dfrac{a}{x^2} + \dfrac{b}{y^2} = a^2 + b^2, \\ x^2 + y^2 = 0. \end{cases}$

Solution. We can express this as the matrix equation

$$\begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/x^2 \\ 1/y^2 \end{bmatrix} = \begin{bmatrix} a^2 + b^2 \\ 0 \end{bmatrix}$$

This can be easily solved with Cramer's rule to give $\frac{1}{x^2} = \frac{a^2+b^2}{a-b}$ and $\frac{1}{y^2} = \frac{a^2+b^2}{b-a}$. Thus, the solutions are $(x, y) = \left(\pm\sqrt{\frac{a-b}{a^2+b^2}}, \pm\sqrt{\frac{b-a}{a^2+b^2}}\right)$.

Problem 5. $\begin{cases} \dfrac{1}{y} - \dfrac{3}{x} = 1, \\ \dfrac{7}{xy} - \dfrac{1}{y^2} = 12. \end{cases}$

Solution. Define $p \equiv \frac{1}{x}$ and $q \equiv \frac{1}{y}$, which gives $-3p + q = 1$ and $7pq - q^2 = 12$. Substituting $p = (q-1)/3$ from the first equation into the second equation gives

$$7q\left(\frac{q-1}{3}\right) - q^2 = 12 \implies 7q^2 - 7q - 3q^2 = 36 \implies 4q^2 - 7q - 36 = (4q+9)(q-4) = 0 \implies q = -\frac{9}{4}, 4$$

Thus, $(p, q) = (-13/12, -4/9), (1, 4)$, and the solutions are $(x, y) = (-12/13, -9/4), (1, 1/4)$.

Problem 6. $\begin{cases} x + y = a + b, \\ \dfrac{a}{x+b} + \dfrac{b}{y+a} = 1. \end{cases}$

Solution. Let $p \equiv x + b$, $q \equiv y + a$. Then we have $p + q = 2(a+b)$, and $bp + aq = pq$. Using $q = 2a + 2b - p$ from the first equation in the second equation gives

$$\begin{aligned} bp + a(2a + 2b - p) = p(2a + 2b - p) &\implies p^2 - (3a+b)p + 2a(a+b) = 0 \\ &\implies p = \frac{3a + b \pm \sqrt{9a^2 + 6ab + b^2 - 8a^2 - 8ab}}{2} \\ &\implies p = \frac{3a + b \pm \sqrt{a^2 - 2ab + b^2}}{2} = \frac{3a + b \pm (a-b)}{2} \\ &\implies p = 2a,\ a + b \implies q = 2b,\ a + b. \end{aligned}$$

Thus, the solutions are $(x, y) = (2a - b, 2b - a),\ (a, b)$.

Problem 7. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = \dfrac{1001}{125}, \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{11}{5}. \end{cases}$

Solution. Define $p \equiv 1/x$ and $q \equiv 1/y$. Then we have

$$(p+q)(p^2-pq+q^2) = p^3+q^3 = \frac{1001}{125} \text{ and } p+q = \frac{11}{5} \implies p^2-pq+q^2 = \frac{5}{11}\cdot\frac{1001}{125} = \frac{91}{25}$$
$$3pq = (p+q)^2-(p^2-pq+q^2) = \frac{121}{25}-\frac{91}{25} = \frac{30}{25} \implies pq = \frac{10}{25} = \frac{2}{5}$$
$$(p-q)^2 = (p+q)^2-4pq = \frac{121}{25}-\frac{8}{5} = \frac{121-40}{25} = \frac{81}{25} \implies p-q = \pm\frac{9}{5}$$

Thus, we have $(p,q) = (2, 1/5),\ (1/5, 2)$, and the solutions are $(x,y) = (1/2, 5),\ (5, 1/2)$.

Problem 8. $\begin{cases} \dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 5, \\ \dfrac{ab}{xy} = 2. \end{cases}$

Solution. Define $p \equiv a/x$ and $q \equiv b/y$, which gives $p^2+q^2 = 5$ and $pq = 2$. Therefore,

$$(p+q)^2 = p^2+q^2+2pq = 5+4 = 9 \implies p+q = \pm 3$$
$$(p-q)^2 = p^2+q^2-2pq = 5-4 = 1 \implies p-q = \pm 1$$

Thus, $(p,q) = (2,1),\ (1,2),\ (-1,-2),\ (-2-1) \implies (x,y) = (a/2, b),\ (a, b/2),\ (-a, -b/2),\ (-a/2, -b)$.

Problem 9. $\begin{cases} x^2+y^2 = \dfrac{17}{4}xy, \\ x-y = \dfrac{3}{4}xy. \end{cases}$

Solution.

$$2xy = x^2+y^2-(x-y)^2 = \frac{17}{4}xy - \frac{9}{16}x^2y^2 \implies \frac{9}{4}xy = \frac{9}{16}(xy)^2 \implies xy = 0,\ 4$$
$$x-y = \frac{3}{4}xy \implies x-y = 3$$
$$(x+y)^2 = x^2+y^2+2xy = \frac{17}{4}4+2(4) = 25 \implies x+y = \pm 5$$

Thus, the solutions are $(x,y) = (4,1),\ (-1,-4),\ (0,0),\ (0,0)$.

Problem 10. $a(x+y) = b(x-y) = xy$.

Solution. $(a-b)x+(a+b)y = 0 \implies y = \dfrac{b-a}{b+a}x$. Now,

$$a(x+y) = xy \implies ax + a\frac{b-a}{b+a}x = \frac{b-a}{b+a}x^2 \implies [a(b+a)+a(b-a)]x = (b-a)x^2 \implies 2abx = (b-a)x^2$$
$$\implies x = 0,\ \frac{2ab}{b-a} \implies y = 0,\ \frac{2ab}{b+a}.$$

Thus, the solutions are $(x,y) = (0,0),\ \left(\dfrac{2ab}{b-a}, \dfrac{2ab}{b+a}\right)$.

Problem 11. $40xy = 21(x^2-y^2) = 210(x+y)$.

Solution.

$$21(x-y)(x+y) = 210(x+y) \implies x+y=0,\ x-y=10 \implies y = x-10 \implies 210x + 210(x-10) = 40x(x-10)$$
$$\implies 40x^2 - 820x + 2100 = 0 \implies 2x^2 - 41x + 105 = 0 \implies x = \frac{41 \pm \sqrt{1681 - 4(2)(105)}}{4} = \frac{41 \pm 29}{4} = \frac{35}{2},\ 3.$$

Thus, the solutions are $(x, y) = (35/2, 15/2)$, $(3, -7)$, $(0, 0)$, $(0, 0)$.

Problem 12. $\begin{cases} 4x^2 - 25y^2 = 0, \\ 2x^2 - 10y^2 - 3y = 4. \end{cases}$

Solution. Eliminate the x^2 term by multiplying the second equation by 2 and subtracting from the first equation to get $-5y^2 + 6y = -8$ or $5y^2 - 6y - 8 = 0$. The quadratic formula gives

$$y = \frac{6 \pm \sqrt{36 + 4(5)(8)}}{10} = \frac{6 \pm 14}{10} = 2,\ -\frac{4}{5}$$

Substituting each of these values into the first equation gives

$$\begin{aligned} y = 2{:}\quad & 4x^2 = 25(4) \implies x = \pm 5 \\ y = -\frac{4}{5}{:}\quad & 4x^2 = 25\left(\frac{16}{25}\right) \implies x = \pm 2 \end{aligned}$$

Thus, the solutions are $(x, y) = (\pm 5, 2)$, $(\pm 2, -4/5)$.

Problem 13. $\begin{cases} x^2 + 3xy - 9y^2 = 9, \\ x^2 - 13xy + 21y^2 = -9. \end{cases}$

Solution. Adding the two equations gives $2x^2 - 10xy + 12y^2 = x^2 - 5xy + 6y^2 = (x-2y)(x-3y) = 0 \implies x = 2y, 3y$. Substituting each of these in turn into the first equation gives

$$\begin{aligned} x = 2y{:}\quad & 4y^2 + 6y^2 - 9y^2 = 9 \implies y^2 = 9 \implies y = \pm 3 \\ x = 3y{:}\quad & 9y^2 + 9y^2 - 9y^2 = 9 \implies y^2 = 1 \implies y = \pm 1 \end{aligned}$$

Thus, the solutions are $(x, y) = (6, 3)$, $(-6, -3)$, $(3, 1)$, $(-3, -1)$.

Problem 14. $\begin{cases} x^2 - 7y^2 - 29 = 0, \\ x^2 - 6xy + 9y^2 - 2x + 6y = 3. \end{cases}$

Solution. The second equation can be written as

$$(x-3y)^2 - 2(x-3y) - 3 = 0 \implies x - 3y = \frac{4 \pm \sqrt{4 + 4(3)}}{2} = \frac{2 \pm 4}{2} = -1,\ 3$$

Substituting each of these into the first equation gives

$$x = 3y - 1{:}\quad (3y-1)^2 - 7y^2 - 29 = 0 \implies y^2 - 3y - 14 = 0 \implies y = \frac{3 \pm \sqrt{9 + 4(14)}}{2} = \frac{3 \pm \sqrt{65}}{2}$$

so that $\boldsymbol{(x, y) = \left(\dfrac{7 \pm 3\sqrt{65}}{2}, \dfrac{3 \pm \sqrt{65}}{2}\right)}$, and

$$x = 3y + 3{:}\quad (3y+3)^2 - 7y^2 - 29 = 0 \implies y^2 + 9y - 10 = (y+10)(y-1) = 0 \implies y = -10,\ 1$$

so that $\boldsymbol{(x, y) = (-27, -10),\ (6, 1)}$.

Problem 15. $\begin{cases} x/y + y/x = 65/28, \\ 2(x^2+y^2) + (x-y) = 34. \end{cases}$

Solution. Let $p \equiv x/y$. Then the first equation can be written as $p + 1/p = 65/28 \implies 28p^2 - 65p + 28 = 0 \implies p = \dfrac{65 \pm \sqrt{4225 - 4(28)(28)}}{56} = \dfrac{65 \pm 33}{56} = \dfrac{7}{4}, \dfrac{4}{7} \implies x = \dfrac{7}{4}y$, $x = \dfrac{4}{7}y$. Substituting each of these into the second equation gives

$$x = \frac{7}{4}y\text{:}\quad 2\left(\frac{49}{16}y^2 + y^2\right) + \frac{3}{4}y = 34 \implies 65y^2 + 6y - 272 = 0 \implies y = \frac{-6 \pm \sqrt{36 + 4(65)(272)}}{130} = 2, -\frac{136}{65}$$

so that $\boldsymbol{(x,y) = (7/2, 2), (-238/65, -136/65)}$, and

$$x = \frac{4}{7}y\text{:}\quad 2\left(\frac{16}{49}y^2 + y^2\right) - \frac{3}{7}y = 34 \implies 2\frac{65}{49}y^2 - \frac{3}{7}y - 34 = 0 \implies 130y^2 - 21y - 1666 = 0 \implies$$

$$y = \frac{21 \pm \sqrt{441 + 4(130)(1666)}}{260} = \frac{21 \pm 931}{260} = \frac{952}{260}, -\frac{910}{260} = \frac{238}{65}, -\frac{7}{2}$$

so that $\boldsymbol{(x,y) = (136/65, 238/65), (-2, -7/2)}$.

Problem 16. $\begin{cases} x^2y = a, \\ xy^2 = b. \end{cases}$

Solution. $\dfrac{x}{y} = \dfrac{x^2y}{xy^2} = \dfrac{a}{b} \implies x = \dfrac{a}{b}y$. Substituting this into the second equation gives

$$\frac{a}{b}y^3 = b \implies y^3 = \frac{b^2}{a} = \frac{a^2b^2}{a^3} \implies y = \frac{\sqrt[3]{a^2b^2}}{a} \implies x = \frac{\sqrt[3]{a^2b^2}}{b},$$

and since this is a cubic, the solutions are $(x,y) = (\alpha, \beta)$, $(\alpha\omega, \beta\omega)$, $(\alpha\omega^2, \beta\omega^2)$, where

$$\alpha \equiv \frac{\sqrt[3]{a^2b^2}}{b},\ \beta \equiv \frac{\sqrt[3]{a^2b^2}}{a}, \text{ and } \omega = \frac{-1 \pm \sqrt{3}\,i}{2}.$$

Problem 17. $\begin{cases} x^2y + xy^2 = a, \\ x^2y - xy^2 = b. \end{cases}$

Solution.

$$\frac{x^2y + xy^2}{x^2y - xy^2} = \frac{xy(x+y)}{xy(x-y)} = \frac{a}{b} \implies a(x-y) = b(x+y) \implies (a-b)x = (a+b)y \implies x = \frac{a+b}{a-b}y$$

Substituting into the first equation, we get

$$\left(\frac{a+b}{a-b}\right)^2 y^3 + \left(\frac{a+b}{a-b}\right) y^3 = a \implies \frac{a+b}{a-b}\left(\frac{a+b+a-b}{a-b}\right) y^3 = a \implies \frac{2(a+b)}{(a-b)^2}y^3 = 1 \implies$$

$$y^3 = \frac{(a-b)^2}{2(a+b)} = \frac{4(a+b)^2(a-b)^2}{8(a+b)^3} \implies y = \frac{\sqrt[3]{4(a^2-b^2)^2}}{2(a+b)} \implies x = \frac{\sqrt[3]{4(a^2-b^2)^2}}{2(a-b)}$$

And since this is a cubic, the full set of solutions are $(x,y) = (\alpha, \beta)$, $(\alpha\omega, \beta\omega)$, $(\alpha\omega^2, \beta\omega^2)$, where

$$\alpha = \frac{\sqrt[3]{4(a^2-b^2)^2}}{2(a-b)},\ \beta = \frac{\sqrt[3]{4(a^2-b^2)^2}}{2(a+b)}, \text{ and } \omega = \frac{-1 \pm \sqrt{3}\,i}{2}.$$

Problem 18. $\begin{cases} x = a(x^2+y^2), \\ y = b(x^2+y^2). \end{cases}$

Solution. $\dfrac{x}{y} = \dfrac{a}{b} \implies x = \dfrac{a}{b}y$, and the second equation becomes

$$y = b\left(\frac{a^2}{b^2}y^2 + y^2\right) = \frac{a^2+b^2}{b}y^2 \implies y = 0,\ \frac{b}{a^2+b^2}$$

Thus, the solutions are $(x, y) = (0, 0)$, $(a/(a^2 + b^2), b/(a^2 + b^2))$.

Problem 19. $\begin{cases} (x+y)/(x-y) = 5/3, \\ (2x+3y)(3x-2y) = 110a^2. \end{cases}$

Solution. The first equation gives $3x + 3y = 5x - 5y \implies x = 4y$. Substituting into the second equation, we have $8y + 3y)(12y - 2y) = (11y)(10y) = 110y^2 = 110a^2 \implies y = \pm a$. Thus the solutions are $(x, y) = (\pm 4a, \pm a)$.

Problem 20. $\begin{cases} 3(x^3 - y^3) = 13xy, \\ x - y = 1. \end{cases}$

Solution. Making use of the second equation, the first equation can be written as

$$3(x^3 - y^3) = 3(x-y)(x^2+xy+y^2) = 3(x^2+xy+y^2) = 13xy \implies 3[(y+1)^2 + (y+1)y + y^2] = 13(y+1)y$$
$$\implies 3(y^2 + 2y + 1 + y^2 + y + y^2) = 13y^2 + 13y \implies 9y^2 + 9y + 3 = 13y^2 + 13y \implies 4y^2 + 4y - 3 = 0$$
$$\implies y = \frac{-4 \pm \sqrt{16 + 4(4)(3)}}{8} = \frac{-4 \pm 8}{8} = -\frac{3}{2},\ \frac{1}{2}$$

Thus, the solutions are $(x, y) = (-1/2, -3/2)$, $(3/2, 1/2)$.

Problem 21. $\begin{cases} x^4 + y^4 = a^4, \\ x + y = a. \end{cases}$

Solution. Using $y = a - x$, we have

$$x^4 + a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4 = a^4 \implies 2x^4 - 4ax^3 + 6a^2x^2 - 4a^3x = 0 \implies$$
$$x(x^3 - 2ax^2 + 3a^2x - 2a^3) = 0 \implies x(x-a)(x^2 - ax + 2a^2) = 0 \implies$$
$$x = 0,\ a,\ \frac{a \pm \sqrt{a^2 - 4(2a^2)}}{2} = 0,\ a,\ a\left(\frac{1 \pm \sqrt{7}\,i}{2}\right)$$

Thus, the solutions are $(x, y) = (0, a)$, $(a, 0)$, $(a(1 \pm \sqrt{7}\,i)/2, a(1 \mp \sqrt{7}\,i)/2)$.

Problem 22. $\begin{cases} 21(x+y) = 10xy, \\ x + y + x^2 + y^2 = 68. \end{cases}$

Solution. We have

$$\begin{aligned} x + y + x^2 + y^2 = 68 &\implies x + y + (x+y)^2 - 2xy = 68 \\ &\implies 5(x+y)^2 + 5(x+y) - 21(x+y) - 340 = 0 \\ &\implies 5(x+y)^2 - 16(x+y) - 340 = 0 \\ &\implies x + y = \frac{16 \pm \sqrt{256 + 4(5)(340)}}{10} = \frac{16 \pm 84}{10} = 10,\ -\frac{34}{5} \qquad (16.1) \end{aligned}$$

Substituting these values into the first equation gives

$$10xy = 21(10) \implies xy = 21$$
$$10xy = -\frac{34}{5}(21) \implies xy = -\frac{357}{25}$$

Using each of these in turn gives

$$xy = 21: \quad (x-y)^2 = (x+y)^2 - 4xy = 100 - 4(21) = 16 \implies x - y = \pm 4 \tag{16.2}$$

$$xy = -\frac{34}{5}: \quad (x-y)^2 = (x+y)^2 - 4xy = \frac{1156}{25} + (4)\frac{357}{25} = \frac{2584}{25} \implies x - y = \pm\frac{2\sqrt{646}}{5} \tag{16.3}$$

Thus, from eqs. (1), (2), and (3), the solutions are $(x,y) = (7,3)$, $(3,7)$, $\left(\dfrac{-17 \pm \sqrt{646}}{5}, \dfrac{-17 \mp \sqrt{646}}{5}\right)$.

Problem 23. $x^2 + y^2 = xy = x + y$.

Solution.

$$xy = x + y \implies y = \frac{x}{x-1} \implies x^2 + \frac{x^2}{(x-1)^2} = x + \frac{x}{x-1} \implies x^2(x-1)^2 + x^2 = x(x-1)^2 + x(x-1)$$
$$\implies x^2(x^2 - 2x + 1) + x^2 = x(x^2 - 2x + 1) + x^2 - x \implies x^4 - 2x^3 + x^2 + x^2 = x^3 - 2x^2 + x + x^2 - x$$
$$\implies x^4 - 3x^3 + 3x^2 = 0 \implies x^2(x^2 - 3x + 3) = 0 \implies x = 0, \frac{3 \pm \sqrt{9 - 4(3)}}{2} = \frac{3 \pm \sqrt{3}\,i}{2}$$

The corresponding value of y is

$$y = \frac{\dfrac{3 \pm \sqrt{3}\,i}{2}}{\dfrac{3 \pm \sqrt{3}\,i}{2} - 1} = \frac{3 \pm \sqrt{3}\,i}{1 \pm \sqrt{3}\,i} \cdot \frac{1 \mp \sqrt{3}\,i}{1 \mp \sqrt{3}} = \frac{6 \mp 2\sqrt{3}\,i}{4} = \frac{3 \mp \sqrt{3}\,i}{2}.$$

Thus, the solutions are $(x,y) = (0,0)$, $\left(\dfrac{3 \pm \sqrt{3}\,i}{2}, \dfrac{3 \mp \sqrt{3}\,i}{2}\right)$.

Problem 24. $x^2 - xy + y^2 = 3a^2 = x^2 - y^2$.

Solution.

$$x^2 - xy + y^2 = x^2 - y^2 \implies 2y^2 = xy \implies y = 0, \frac{x}{2}$$

$$y = 0 \implies x^2 = 3a^2 \implies x = \pm\sqrt{3}a \implies (x,y) = (\pm\sqrt{a}, 0)$$
$$y = \frac{x}{2} \implies x^2 - \frac{x^2}{2} + \frac{x^2}{4} = \frac{3x^2}{4} = 3a^2 \implies x^2 = 4a^2 \implies x = \pm 2a \implies (x,y) = (\pm 2a, \pm a)$$

Thus, the solutions are $(x,y) = (\pm\sqrt{a}, 0)$, $(\pm 2a, \pm a)$.

Problem 25. $\begin{cases} x^2 + xy + y^2 = 21, \\ x + \sqrt{xy} + y = 7. \end{cases}$

Solution. Solving the second equation for xy, we have

$$\sqrt{xy} = (7 - x - y) \implies xy = 49 + x^2 + y^2 - 14x - 14y + 2xy \implies xy + x^2 + y^2 - 14x - 14y + 49 = 0$$

Subtracting this from the first equation gives $14x + 14y - 49 - 21 = 0 \implies 14(x+y) - 70 = 0 \implies x + y = 5$ and hence $(x+y)^2 = x^2 + y^2 + 2xy = 25$. Subtracting the first equation from this gives $xy = 4$ and we also have $(x-y)^2 = (x+y)^2 - 4xy = 25 - 16 = 9 \implies x - y = \pm 3$. Thus, the solutions are $(x,y) = (4,1)$, $(1,4)$.

Problem 26. $\begin{cases} 4x^2 - 3y^2 = 12(x-y), \\ xy = 0. \end{cases}$

Solution. The second equation tells us either $x = 0$ or $y = 0$.

$$x = 0 \implies -3y^2 = -12y \implies y = 0,\, 4 \implies (x, y) = (0, 0),\, (0, 4)$$
$$y = 0 \implies 4x^2 = 12x \implies x = 0,\, 3 \implies (x, y) = (0, 0),\, (3, 0)$$

Thus, the solutions are $(x, y) = (0, 0), (0, 0), (0, 4), (3, 0)$.

Problem 27. $\begin{cases} x^2 + y^2 = x + y + 20, \\ xy + 10 = 2(x + y). \end{cases}$

Solution. Combining the two via the constant term, we have

$$x^2 + y^2 = x + y + 4(x + y) - 2xy \implies (x + y)^2 = 5(x + y) \implies x + y = 0,\, 5.$$

Now, consider each of these values in turn.

$$x + y = 0 \implies x^2 + y^2 = 20 \text{ and } xy = -10 \implies (x - y)^2 = x^2 + y^2 - 2xy = 20 + 20 = 40 \implies x - y = \pm 2\sqrt{10}$$
$$x + y = 5 \implies x^2 + y^2 = 25 \text{ and } xy = 0 \implies (x - y)^2 = x^2 + y^2 - 2xy = 25 \implies x - y = \pm 5$$

Thus, the solutions are $(x, y) = (\pm\sqrt{10}, \mp\sqrt{10}),\, (5, 0),\, (0, 5)$.

Problem 28. $\begin{cases} x^2 + 4x - 3y = 0, \\ y^2 + 10x - 9y = 0. \end{cases}$

Solution. Combining the two equations, we have

$$x^2 + 4x - 3y = y^2 + 10x - 9y \implies x^2 - y^2 - 6x + 6y = 0 \implies (x - y)(x + y) - 6(x - y) = 0 \implies$$
$$(x - y)(x + y - 6) = 0 \implies x - y = 0,\, 6$$

Consider each of these values in turn:

$$\begin{aligned} x = y\text{:} \quad & y^2 + 10y - 9y = 0 \implies y(y + 1) = 0 \implies y = 0,\, -1 \implies (x, y) = (0, 0),\, (-1, -1) \\ y = 6 - x\text{:} \quad & x^2 + 4x - 3(6 - x) = 0 \implies x^2 + 4x - 18 + 3x = 0 \implies x^2 + 7x - 18 = (x + 9)(x - 2) = 0 \\ & \implies x = 2,\, -9 \implies (x, y) = (2, 4),\, (-9, 15) \end{aligned}$$

Thus, the solutions are $(x, y) = (0, 0),\, (-1, -1),\, (2, 4),\, (-9, 15)$.

Problem 29. $\begin{cases} 28(x^5 + y^5) = 61(x^3 + y^3), \\ x + y = 2. \end{cases}$

Solution. We make use of the fact that

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$$
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Substituting $y = 2 - x$, we have

$$\begin{aligned} x^4 - x^3y + x^2y^2 - xy^3 + y^4 &= x^4 - x^3(2 - x) + x^2(2 - x)^2 - x(2 - x)^3 + (2 - x)^4 \\ &= x^4 - 2x^3 + x^4 + x^2(4 - 4x + x^2) - x(8 - 12x + 6x^2 - x^3) + 16 - 32x + 24x^2 - 8x^3 + x^4 \\ &= 5x^4 - 20x^3 + 40x^2 - 40x + 16 \\ x^2 - xy + y^2 &= x^2 - x(2 - x) + (2 - x)^2 \\ &= x^2 - 2x + x^2 + 4 - 4x + x^2 \\ &= 3x^2 - 6x + 4 \end{aligned}$$

and so the first equation becomes

$$28(5x^4 - 20x^3 + 40x^2 - 40x + 16) = 61(3x^2 - 6x + 4) \implies 140x^4 - 560x^3 + 937x^2 - 754x + 204 = 0$$

By synthetic division we find $140x^4 - 560x^3 + 937x^2 - 754x + 204 = (x - 3/2)(x - 1/2)(140x^2 - 280x + 272)$. The quadratic may be written as $35x^2 - 70x + 68 = 0$ and has the solution

$$x = \frac{70 \pm \sqrt{4900 - 4(35)(68)}}{70} = \frac{70 \pm \sqrt{-4620}}{70} = 1 \pm \frac{\sqrt{1155}}{35}i.$$

Thus, the solutions are $(x, y) = (3/2, 1/2)$, $(1/2, 3/2)$, $(1 \pm \sqrt{1155}\, i/35, 1 \mp \sqrt{1155}\, i/35)$.

Problem 30. $\begin{cases} xy - x/y = a, \\ xy - y/x = 1/a. \end{cases}$

Solution. Let $p \equiv x/y$, then $a + p = 1/a + 1/p \implies p^2 + ap = p/a + 1 \implies ap^2 + (a^2 - 1)p - a = 0 \implies$

$$p = \frac{1 - a^2 \pm \sqrt{(1 - a^2)^2 + 4a^2}}{2a} = \frac{1 - a^2 \pm \sqrt{1 - 2a^2 + a^4 + 4a^2}}{2a} = \frac{1 - a^2 \pm (1 + a^2)}{2a} = -a,\ \frac{1}{a},\text{ so that}$$

$x = -ay$ and $x = \dfrac{y}{a}$.

$x = -ay$: $\quad -ay^2 + \dfrac{1}{a} = \dfrac{1}{a} \implies ay^2 = 0 \implies y = 0$ but this is not a solution since it leads to a divide by 0.

$$x = \frac{y}{a}: \quad \frac{y^2}{a} - \frac{1}{a} = a \implies \frac{y^2}{a} = a + \frac{1}{a} = \frac{1 + a^2}{a} \implies y = \pm\sqrt{1 + a^2}$$

Thus, the solutions are $(x, y) = \left(\dfrac{\pm\sqrt{1 + a^2}}{a}, \pm\sqrt{1 + a^2} \right)$.

Problem 31. $\begin{cases} (x + 1)^3 + (y - 2)^3 = 19, \\ x + y = 2. \end{cases}$

Solution. Since $y - 2 = -x$ from the second equation, the first equation becomes

$$\begin{aligned}(x + 1)^3 + (-x)^3 = (x + 1)^3 - x^3 &= (x + 1 - x)[(x + 1)^2 + (x + 1)x + x^2] \\ &= (1)(x^2 + 2x + 1 + x^2 + x + x^2) \\ &= 3x^2 + 3x + 1\end{aligned}$$

and we have $3x^2 + 3x + 1 = 19 \implies 3x^2 + 3x - 18 = 0 \implies x^2 + x - 6 = (x + 3)(x - 2) = 0 \implies x = -3,\ 2$. Thus, the solutions are $(x, y) = (-3, 5)$, $(2, 0)$.

Problem 32. $\begin{cases} x^2 + y = 8/3, \\ x + y^2 = 34/9. \end{cases}$

Solution. Substituting $y = 8/3 - x^2$ from the first equation into the second equation gives

$$\begin{aligned} x + \left(\frac{8}{3} - x^2\right)^2 = \frac{34}{9} &\implies x + \frac{64}{9} - \frac{16}{3}x^2 + x^4 = \frac{34}{9} \implies 9x + 64 - 48x^2 + 9x^4 = 34 \\ &\implies 9x^4 - 48x^2 + 9x + 30 = 0 \implies 3x^4 - 16x^2 + 3x + 10 = 0 \\ &\implies (x - 1)(x - 2)(3x^2 + 9x + 5) = 0 \text{ from synthetic division} \end{aligned}$$

The solution to the quadratic is $x = \dfrac{-9 \pm \sqrt{81 - 4(3)(5)}}{6} = \dfrac{-9 \pm \sqrt{21}}{6}$. The corresponding value of y is

$$y = \frac{8}{3} - \left(\frac{-9 \pm \sqrt{21}}{6}\right)^2 = \frac{96}{36} - \frac{81 \mp 18\sqrt{21} + 21}{36} = \frac{-6 \pm 18\sqrt{21}}{36} = \frac{-1 \pm 3\sqrt{21}}{6}.$$

Thus, the solutions are $(x, y) = (1, 5/3)$, $(2, -4/3)$, $\left(\dfrac{-9 \pm \sqrt{21}}{6}, \dfrac{-1 \pm 3\sqrt{21}}{6}\right)$.

Problem 33. $\begin{cases} y^2 - xy - yz = 3, \\ x + 4y + z = 14, \\ x - y + 2z = 0. \end{cases}$

Solution. First eliminate z from the first and last equation by using $\boxed{z = 14 - x - 4y}$ from the second equation:

$$y^2 - xy - y(14 - x - 4y) = 3 \implies 5y^2 - 14y - 3 = 0 \implies y = \frac{14 \pm \sqrt{196 + 4(5)(3)}}{10} = \frac{14 \pm 16}{10} = 3, -\frac{1}{5}$$

and $x - y + 2(14 - x - 4y) = 0 \implies \boxed{x = 28 - 9y}$.

Thus, the solutions are $(x, y, z) = (1, 3, 1)$, $(149/5, -1/5, -15)$.

Problem 34. $\begin{cases} x + y + z + u = 0, \\ 3x + z + u = 0, \\ 3y + 2z = 0, \\ x^2 + y^2 + zu = 5. \end{cases}$

Solution. First eliminate u from the second and fourth equations by using $\boxed{u = -3x - z}$ from the second equation:

$$\begin{cases} x + y + z + (-3x - z) = 0 \implies -2x + y = 0 \implies \boxed{y = 2x} \\ 3y + 2z = 0 \implies \boxed{2z = -3y} \\ x^2 + y^2 + z(-3x - z) = 5 \implies x^2 + y^2 - 3xz - z^2 = 5 \end{cases}$$

and now the last equation may be transformed as follows:

$$\begin{aligned} x^2 + y^2 - 3xz - z^2 = 5 &\implies 4x^2 + 4y^2 - 6x(-3y) - (-3y)^2 = 5 \implies 4x^2 + 4y^2 + 18xy - 9y^2 = 5 \\ &\implies 4x^2 - 5y^2 + 18xy = 5 \implies 4x^2 - 5(4x^2) + 18x(2x) = 5 \\ &\implies 4x^2 - 20x^2 + 36x^2 = 5 \implies 20x^2 = 5 \implies x^2 = \frac{1}{4} \implies x = \pm\frac{1}{2} \end{aligned}$$

Thus, the solutions are $(x, y, z, u) = (\pm 2, \pm 4, \mp 6, 0)$, $(\pm 1/2, \pm 1, \mp 3/2, 0)$.

Problem 35. $\begin{cases} (y + z)(x + y + z) = 10, \\ (z + x)(x + y + z) = 20, \\ (x + y)(x + y + z) = 20. \end{cases}$

Solution. We have

$$\frac{x+z}{y+z}=2 \text{ and } \frac{x+y}{y+z}=2 \implies x+z=2y+2z \text{ and } x+y=2y+2z \implies x+z=x+y \implies \boxed{z=y}$$
$$\implies x+y=4y \implies \boxed{x=3y}$$

Thus, we have

$$(y+z)(x+y+z)=10 \implies (y+y)(3y+y+y)=10 \implies 2y(5y)=10 \implies \boxed{y=\pm 1}$$

and hence he solutions are $(x,y)=(3,1,1)$, $(-3,-1-1)$.

Problem 36. $\begin{cases} x^2+y^2+z^2=6, \\ xy+yz+zx=-1, \\ 2x+y-2z=-3. \end{cases}$

Solution. Solving the third equation for y, we have $\boxed{y=2z-2x-3}$. Substituting this in the first equation gives

$$\begin{aligned} (2z-2x-3)^2=6-x^2-z^2 &\implies 4z^2+4x^2+9-8xz-12z+12x=6-x^2-z^2 \\ &\implies 5z^2+5x^2-8xz+12x-12z+3=0 \end{aligned} \tag{16.1}$$

From the first and second equation, we have

$$\begin{aligned} (x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)=6+2(-1)=4 &\implies x+y+z=\pm 2 \\ &\implies x+(2z-2x-3)+z=\pm 2 \\ &\implies -x+3z=3\pm 2 \\ &\implies \boxed{x=3z-(3\pm 2)} \end{aligned} \tag{16.2}$$

We consider each of these two expressions for x in turn as we substitute them in eq. (1).

$$x=3z-5: \quad 5z^2+5(9z^2-30z+25)-8(3z^2-5z)+12(3z-5)-12z+3=0 \implies 13z^2-43z+34=0$$
$$z=\frac{43\pm\sqrt{1849-4(13)(34)}}{26}=\frac{43\pm 9}{26}=2,\ \frac{17}{13}$$

From the boxed equations, we get $(x,y,z)=(1,-1,2)$, $(-14/13, 23/13, 17/13)$.

$$x=3z-1: \quad 5z^2+5(9z^2-6z+1)-8(3z^2-z)+12(3z-1)-12z+3=0 \implies 13z^2+z-2=0$$
$$z=\frac{-1\pm\sqrt{1+4(13)(2)}}{26}=\frac{-1\pm\sqrt{105}}{26}$$

And from the boxed equations, we get $(x,y)=((-29\pm 3\sqrt{105})/26, (-22\mp 4\sqrt{105})/26, (-1\pm\sqrt{105})/26)$.

16.7 Simultaneous Quadratic Equations (Word Problems), Exercise LI (p. 331)

Problem 1. The difference of the cubes of two numbers is 218 and the cube of their difference is 8. Find the numbers.

Solution. Let x and y be the two numbers. Then we have $x^3 - y^3 = 218$ and $(x-y)^3 = 8 \implies x - y = 2$ or $x = y + 2$, so that

$$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2 = \frac{218}{2} = 109 \implies y^2 + 4y + 4 + y(y+2) + y^2 = 109 \implies 3y^2 + 6y - 105 = 0$$

and the quadratic formula gives $x = \dfrac{-6 \pm \sqrt{36 + 4(3)(105)}}{6} = \dfrac{-6 \pm 36}{6} \implies x = 5$ and $y = 7$, so the numbers are 5 and 7.

Problem 2. The square of the sum of two numbers less their product is 63, and the difference of their cubes is 189. What are they?

Solution. Let x and y be the two numbers. Then $(x+y)^2 - xy = x^2 + xy + y^2 = 63$ and $x^3 - y^3 = 189$. Therefore,

$$x - y = \frac{x^3 - y^3}{x^2 + xy + y^2} = \frac{189}{63} = 3 \implies x = y + 3 \implies y^2 + 6y + 9 + y^2 + 3y + y^2 = 63 \implies y^2 + 3y - 18 = 0$$

and the quadratic formula gives $y = \dfrac{-3 \pm \sqrt{9 + 4(18)}}{2} = \dfrac{-3 \pm 9}{2} \implies y = 3$ and $x = 6$, so the two numbers are 3 and 6.

Problem 3. The sum of the terms of a certain fraction is 11, and the product of this fraction by one whose numerator and denominator exceed its numerator and denominator by 3 and 4 respectively is 2/3. Find the fraction.

Solution. Let $\dfrac{x}{y}$ be the fraction. Then we have $x + y = 11$ and $\dfrac{x}{y}\left(\dfrac{x+3}{y+4}\right) = \dfrac{2}{3}$. Clearing of fractions gives $3x^2 + 9x = 2y^2 + 8y$. Substituting $y = 11 - x$, we have $3x^2 + 9x = 242 - 44x + 2x^2 + 88 - 8x \implies x^2 + 61x - 330 = 0 \implies x = \dfrac{-61 \pm \sqrt{3721 + 4(330)}}{2} = \dfrac{-61 + 71}{2} = 5 \implies y = 6$. Thus, the fraction is $\dfrac{5}{6}$.

Problem 4. Separate 37 into three parts whose product is 1440 and such that the product of two of them exceeds three times the third by 12.

Solution. Set $37 = x + y + z$, then $xyz = 1440$ and $xy = 3z + 12$. We have $1440 = xyz = (3z+12)z \implies 3z^2 + 12z - 1440 = 0 \implies z = \dfrac{-12 \pm \sqrt{144 + 4(3)(1440)}}{6} = \dfrac{-12 \pm 132}{6} = 20$. This gives $x + y = 17$ and $xy = 72$, so that $x^2 + 2xy + y^2 = 289$ and $4xy = 288$. Subtracting, we get $(x-y)^2 = (x+y)^2 - 4xy = 289 - 288$ so that $x - y = \pm 1$. Therefore, $x = 9$, $y = 8$, and $z = 20$.

Problem 5. The diagonal of a rectangle is 13 feet long. If each side were 2 feet longer than it is, the area would be 38 square feet greater than it is. What are the sides?

Solution. Let x and y be the sides of the triangle, then the diagonal is $\sqrt{x^2 + y^2} = 13$, and we have $(x+2)(y+2) = xy + 38$, which simplifies to $x + y = 17$. Now, we have $x^2 + y^2 = 13^2 = 169$ and $(x+y)^2 = x^2 + y^2 + 2xy = 17^2 = 289$, so that $2xy = 289 - 169 = 120$. Therefore, $(x-y)^2 = x^2 + y^2 - 2xy = 169 - 120 = 49$, which means $x - y = 7$. From $x + y = 17$ and $x - y = 7$, we get $x = 12$ and $y = 5$.

Problem 6. The perimeter of a right-angled triangle is 36 inches long and the area of the triangle is 54 square inches. Find the lengths of the sides.

Solution. Let x be the base and y be the height, then $x + y + \sqrt{x^2 + y^2} = 36$ and $\dfrac{1}{2}xy = 54$. This gives $2xy = 216$, and

$$\begin{aligned} \cancel{x^2 + y^2} &= (36 - x - y)^2 = 1296 + \cancel{x^2 + y^2} - 72x - 72y + 2xy \\ 0 &= 1296 - 72(x+y) + 216 \implies x + y = 21 \end{aligned}$$

Therefore, $x^2 + y^2 = (36 - x - y)^2 = (36 - 21)^2 = 225 \implies (x - y)^2 = x^2 + y^2 - 2xy = 225 - 216 = 9$, which means $x - y = 3$. Finally, from $x + y = 21$ and $x - y = 3$, we get $x = 12$, $y = 9$, and the diagonal is $\sqrt{x^2 + y^2} = 15$.

Problem 7. The hypotenuse of a right-angled triangle is longer than the two perpendicular sides by 3 and 24 inches respectively. Find the sides of the triangle.

Solution. Let x be the base and y be the height of the triangle. Then we have $x + 3 = \sqrt{x^2 + y^2} = y + 24$, which gives $x = y + 21$ and $x^2 + y^2 = (x + 3)^2 = x^2 + 6x + 9$ or

$$y^2 = 6x + 9 = 6(y + 21) + 9 \implies y^2 - 6y - 135 = 0 \implies y = \frac{6 \pm \sqrt{36 + 4(135)}}{2} = \frac{6 \pm 24}{2} = 15,$$

and from this we get $x = 15 + 21 = 36$. Thus, the sides of the triangle are 15 and 36, and the diagonal is 39.

Problem 8. Find the dimensions of a room from the following data: its floor is a rectangle whose area is 224 square feet, and the area of two of its side walls are 126 and 144 square feet respectively.

Solution. Let x and y represent the width and depth of the room, and z represent the height. Then we have

$$xy = 224, \qquad xz = 126, \qquad yz = 144.$$

Therefore,

$$\begin{aligned} x^2 &= \frac{(xy)(xz)}{yz} = \frac{(224)(126)}{144} = 196 \implies x = 14 \\ y^2 &= \frac{(xy)(yz)}{xz} = \frac{(224)(144)}{126} = 256 \implies y = 16 \\ z^2 &= \frac{(xz)(yz)}{xy} = \frac{(126)(144)}{224} = 81 \implies z = 9 \end{aligned}$$

Problem 9. A rectangle is surrounded by a border whose width is 5 inches. The area of the rectangle is 168 square inches, that of the border 360 square inches. Find the length and breadth of the rectangle.

Solution. Let x and y be the dimensions of the inner rectangle. Then we have

$$xy = 168, \qquad (x + 10)(y + 10) - xy = 360.$$

The second equation simplifies to $x + y = 26$. Therefore,

$$(x - y)^2 = (x + y)^2 - 4xy = (26)^2 - 4(168) = 4 \implies x - y = 2.$$

From $x + y = 26$ and $x - y = 2$, we get $x = 14$ and $y = 12$.

Problem 10. In buying coal A gets 3 tons more for \$135 than B does and pays \$7 less for 4 tons than B pays for 5. Required the price each pays per ton.

Solution. Let a and b be the price per ton that A and B pays, respectively, and let n be the number of tons. Then we have

$$\begin{aligned} (n + 3)a &= 135 \\ nb &= 135 \\ 5b - 4a &= 7 \end{aligned}$$

From the first two equations, $na + 3a = nb$. Multiplying through by 5 and making use of $5b = 7 + 4a$ from the third equation, we get $5na+15a = 5nb = n(7+4a) = 7n+4na \implies na+15a = 7n \implies n(7-a) = 15a$. Substituting $n = 15a/(7-a)$ into the first equation then gives

$$3a + \frac{15a^2}{7-a} = 135 \implies 21a - 3a^2 + 15a^2 = 945 - 135a \implies 12a^2 + 156a - 945 = 0 \implies 4a^2 + 52a - 315 = 0$$

Applying the quadratic formula gives

$$a = \frac{-52 \pm \sqrt{2704 + 4(4)(315)}}{8} = \frac{-52 + 88}{8} = \frac{36}{8} = \frac{9}{2} \implies b = \frac{7 + 4n}{5} = \frac{7 + 18}{5} = 5 \implies n = \frac{135}{5} = 27$$

Thus, A pays \$4.50 per ton, B pays \$5.00 per ton, and the number of tons that cost each of them \$135 is 30 for A and 27 for B.

Problem 12. A man leaves \$60,000 to his children and grandchildren, seven in all. The children receive $\frac{1}{3}$ of it, which is \$2000 more apiece than the grandchildren get. How many children are there and how many grandchildren, and what does each receive?

Solution. Let n be the number of children, $7 - n$ the number of grandchildren, and let c be the amount each of the children receive, g be the amount each of the grandchildren receive. Then we have

$$20,000 = nc \qquad 40,000 = (7-n)g \qquad c = g + 2,000$$

$$20,000 = nc = ng + 2,000n = n\left(\frac{40,000}{7-n}\right) + 2,000n \implies 20 = \frac{40n}{7-n} + 2n \implies 140 - 20n = 40n + 14n - 2n^2$$

$$\implies 2n^2 - 74n + 140 = 0 \implies n^2 - 37n + 70 = 0 \implies n = \frac{37 \pm \sqrt{1369 - 4(70)}}{2} = \frac{37 \pm 33}{2} \implies n = 2,$$

and $g = 40,000/5 = 8,000$, $c = 8,000+2,000 = 10,000$. Thus, there are two children and five grandchildren. Each child receives \$10,000 and each grandchild receives \$8,000.

Problem 13. At his usual rate a man can row 15 miles downstream in 5 hours less time than it takes him to return. Could he double his rate, his time downstream would be only 1 hour less than his time upstream. What is his usual rate in dead water and what is the rate of the current?

Solution. Let v be his usual rate and u be the rate of the current. Then we have

$$(v+u)t_1 = 15 = (v-u)(t_1+5) \qquad \text{and} \qquad (2v+u)t_2 = 15 = (2v-u)(t_2+1) \tag{16.1}$$

Expanding and solving for the times, we have

$$\begin{aligned} vt_1 + ut_1 = vt_1 - ut_1 + 5(v-u) &\implies t_1 = \frac{5(v-u)}{2u} \\ 2vt_2 + ut_2 = 2vt_2 - ut_2 + 2v - u &\implies t_2 = \frac{2v-u}{2u} \end{aligned}$$

Substituting these times into eqs. (1) gives

$$\begin{aligned} (v+u)\frac{5(v-u)}{2u} = 15 &\implies 5(v^2 - u^2) = 30u \\ (2v+u)\frac{2v-u}{2u} = 15 &\implies 4v^2 - u^2 = 30u \end{aligned}$$

Therefore, $5v^2 - 5u^2 = 4v^2 - u^2 \implies v^2 = 4u^2 \implies v = 2u$, and substituting this into the above expressions, we easily find $v = 4$ and $u = 2$, so that his rate is 4 mph and the current rate is 2 mph.

Problem 14. Three men A, B, C together can do a piece of work in 1 hour 20 minutes. To do the work alone it would take C twice as long as A and 2 hours longer than B. How long would it take each man to do the work alone?

Solution. Let a, b, c be the work rates of A, B, and C respectively and let w be the total work done. We have $(a+b+c)\frac{4}{3} = w$ and $a = 2c$. Also, if t is the time it takes b to do the work alone, then $bt = w$ and so $c(t+2) = c(\frac{w}{b}+2) = w$. Solving this for b, we have $b = cw/(w-2c)$. Put this together, we have

$$\begin{aligned}
(a+b+c)\frac{4}{3} = w &\implies \left(2c + \frac{cw}{w-2c} + c\right)\frac{4}{3} = w \\
&\implies \left(3c + \frac{cw}{w-2c}\right)4 = 3w \\
&\implies 12c(w-2c) + 4cw = 3w(w-2c) \\
&\implies 12cw - 24c^2 + 4cw = 3w^2 - 6cw \\
&\implies 3w^2 - 22cw + 24c^2 = 0 \\
&\implies (3w-4c)(w-6c) = 0
\end{aligned}$$

The only acceptable solution is $6c = w$ since $\frac{4}{3}c = w$ would only be the case where C did all the work! This indicates that it takes C 6 hours to perform the work by himself. Since A works at twice the rate of C, it would take A 3 hours to perform the work, and since B can do the work two hours less than C, it would take him 4 hours alone.

Problem 15. Two bodies A and B are moving at constant rates and in the same direction around the circumference of a circle whose length is 20 feet. A makes one circuit in 2 seconds less time than B, and A and B are together every minute. What are their rates?

Solution. Let a be the rate of A, b be the rate of B, and let τ be the time it takes A to complete one cycle. Then we have

$$a\tau = 20, \qquad b(\tau+2) = 20, \qquad (a-b)60 = 20.$$

We first eliminate τ from the first two equations and rewrite the third equation:

$$b\left(\frac{20}{a} + 2\right) = 20 \implies 20b + 2ab = 20a \implies ab = 10(a-b) \tag{16.1}$$

$$(a-b)60 = 20 \implies a - b = \frac{1}{3} \tag{16.2}$$

Now, from eqs. (1) and (2), we get

$$\begin{aligned}
(a+b)^2 &= 4ab + (a-b)^2 \\
&= 40(a-b) + \frac{1}{9} \\
&= \frac{40}{3} + \frac{1}{9} = \frac{120}{9} + \frac{1}{9} = \frac{121}{9} \implies a + b = \frac{11}{3}
\end{aligned} \tag{16.3}$$

From eqs. (2) and (3), we get $a = 2$ feet/sec $= 24$ inches/sec and $b = \frac{5}{3}$ feet/sec $= 20$ inches/sec.

Problem 16. On two straight lines which meet at right angles at O the points A and B are moving toward O at constant rates. A is now 28 inches from O and B 9 inches; 2 seconds hence A and B will be 13 inches apart, and 3 seconds hence they will be 5 inches apart. At what rates are A and B moving?

Solution. Let the point A move along the x-axis with speed v and let the point B move along the y-axis with speed u. Then we have

$$x = vt - 28, \qquad y = ut - 9,$$

where t is the time. Since the x and y axes are perpendicular, the distance between the two points is given by $\sqrt{x^2+y^2}$. We have

$$\begin{aligned}
\text{at 2 seconds: } 169 &= (2v-28)^2+(2u-9)^2 \implies 4v^2+4u^2-112v-36u+696=0\\
\text{at 3 seconds: } 25 &= (3v-28)^2+(3u-9)^2 \implies 9v^2+9u^2-168v-54u+840=0
\end{aligned}$$

We eliminate the v^2+u^2 terms by multiplying the first equation by 9 and the second by 4 and subtracting to get $-336v-108u+2904=0$, which simplifies to $9u=242-28v$. We substitute this into the first equation, but first we divide through by 4:

$$\begin{aligned}
v^2+u^2-28v-9u+174=0 &\implies v^2+\left(\frac{242-28v}{9}\right)^2-28v-(242-28v)+174=0\\
&\implies 81v^2+58564-13552v+784v^2-2268v-19602+2268v+14094=0\\
&\implies 865v^2-13552v+53056=0\\
&\implies v=\frac{13552\pm\sqrt{(13552)^2-4(865)(53056)}}{170}=\frac{13552\pm 288}{1730}\\
&\implies v=8,
\end{aligned}$$

and $u=(242-28(8))/9=2$. Thus, A moves at the rate of 8 inches/sec and B moves at the rate of 2 inches/sec.

Problem 17. Three men A, B, and C set out at the same time to walk a certain distance. A walks $4\frac{1}{2}$ miles an hour and finishes the journey 2 hours before B. B walks 1 mile an hour faster than C and finishes the journey in 3 hours less time. What is the distance?

Solution. Let x be the distance to be found, let t be the time it takes C, and let c be the speed of C. Then we have

$$\frac{9}{2}[(t-3)-2]=x, \qquad (c+1)(t-3)=x, \qquad ct=x.$$

Making use of the third equation, the first two equations can be rewritten as

$$9t=2ct+45 \qquad \text{and} \qquad t=3c+3,$$

and now using the second equation in the first gives

$$9(3c+3)=2c(3c+3)+45 \implies 6c^2-21c+18=0 \implies 2c^2-7c+6=(2c-3)(c-2)=0$$

Thus, $c=2,\ \frac{3}{2}$, and the corresponding times are $t=9,\ \frac{15}{2}$. Therefore, the distance is either 18 miles or $11\frac{1}{4}$ miles.

Problem 18. Two couriers A and B start simultaneously from P and Q respectively and travel toward each other. When they meet A has traveled 12 miles farther than B. After their meeting A continues toward Q at the same rate as before, arriving in $4\frac{2}{3}$ hours. Similarly B arrives at P in $7\frac{5}{7}$ hours after the meeting. What is the distance from P to Q?

Solution. Let a be the speed of A and x the distance traveled at their meeting. Similarly let b be the speed of B and y the distance traveled at their meeting. If t is the time when they meet after starting, then we have $at=x=y+12$ and $bt=y$. Therefore, their speeds are related by $a\dfrac{y}{b}=y+12$, or

$$\left(\frac{a}{b}-1\right)y=12 \tag{16.1}$$

After their meeting, we have

$$a\frac{14}{3}=y, \qquad b\frac{54}{7}=y+12. \tag{16.2}$$

Substituting eqs. (2) into eq. (1) gives

$$\left[\frac{3y/14}{(7y+84)/54} - 1\right] y = 12$$

Expanding, we get $\left(\frac{3y}{14}\right) y - \left(\frac{7y+84}{54}\right) y = \frac{12}{54}(7y+84) \implies 162y^2 - 98y^2 - 1176y = 1176y + 14112$ or

$64y^2 - 2352y - 14112 = 0 \implies y = \dfrac{2352 \pm \sqrt{5531904 + 4(64)(14112)}}{128} = \dfrac{2352 \pm 3024}{128} \implies y = \dfrac{5376}{128} = 42,$

and therefore $x = 42 + 12 = 54$. The total distance is $x + y = 96$ miles. For the sake of completeness, $a = 3y/14 = 9$ mph and $b = 7x/54 = 7$ mph.

Chapter 24

Logarithms

24.1 Exponential and Logarithmic Equations, Exercise LXIV (p. 392)

Problem 1. Find $\log_5 555$, $\log_7 .0463$, $\log_{100} 47$.

Solution. Using the formula $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$ to convert logarithms from one base to another, we get $\log_5 555 = \dfrac{\log 555}{\log 5} = \dfrac{2.74429}{0.69897} = 3.926$.

Problem 2. Solve the following exponential equations.

(1) $3^x = 729$ **(2)** $a^{x^2+2} = a^{3x}$ **(3)** $213^x = 516^{-x+4}$

Solution.
(1) $3^x = 729 = 9^3 = (3^2)^3 = 3^6 \implies x = 6$.
(2) $a^{x^2+2} = a^{3x} \implies x^2 + 2 = 3x \implies x^2 - 3x + 2 = (x-2)(x-1) = 0 \implies x = 2$ or $x = 1$.
(3) $213^x = 516^{-x+4} \implies x \log 213 = (4-x)\log 516 \implies x(\log 213 + \log 516) = 4\log 516 \implies x = \dfrac{4\log 516}{\log 213 + \log 516} = \dfrac{4\log 516}{\log(213 \cdot 516)} = \dfrac{10.851}{5.041} = 2.152$.

Problem 3. Solve the following logarithmic equations.

(1) $\log x + \log(x+3) = 1$ **(2)** $\log x^2 + \log x = 2$
(3) $\log(1-2x)^3 - \log(3-x)^3 = 6$ **(4)** $x^{\log x} = 2$

Solution.
(1) $\log x + \log(x+3) = 1 \implies \log x(x+3) = 1 \implies x(x+3) = 10 \implies x^2 + 3x - 10 = 0 \implies x = \dfrac{-3 \pm \sqrt{9+40}}{2} = \dfrac{-3 \pm 7}{2} \implies x = -5,\ 2$.
(2) $\log x^2 + \log x = 2 \implies \log x^3 = 2 \implies x^3 = 10^2 \implies x = \sqrt[3]{100} = 4.642$.
(3) $\log(1-2x)^3 - \log(3-x)^3 = 6 \implies \log \dfrac{(1-2x)^3}{(3-x)^3} = 6 \implies \left(\dfrac{1-2x}{3-x}\right)^3 = 10^6 \implies \dfrac{1-2x}{3-x} = 10^2 \implies 1 - 2x = 300 - 100x \implies 98x = 299 \implies x = \dfrac{299}{98} = 3.051$.
(4) $x^{\log x} = 2 \implies (\log x)(\log x) = \log 2 \implies \log x = \sqrt{\log 2} \implies x = 10^{\sqrt{\log 2}} = 3.5372$.

Problem 4. Find the amount of \$7500 in thirty-five years at 5% compound interest, the interest being compounded annually.

Solution. The amount is given by the formula

$$A = P\,(1+r)^n = \$7500\,(1+0.05)^{35} = \$41,370.12$$

Problem 5. Find the amount of \$5500 in twenty years at 3% compound interest, the interest being compounded semiannually.

Solution. The amount is given by the formula

$$A = P\left(1+\frac{r}{2}\right)^{2n} = \$5500\left(1+\frac{0.03}{2}\right)^{40} = \$9,977.10$$

Problem 6. Show that a sum of money will more than double intself in fifteen years and that it will increase more than a hundredfold in ninety-five years at 5% compound interest.

Solution. The formula for compound interest, compounded annually, is $A = P(1+r)^n$, so that the multiplicative factor is $(1+r)^n$ for a given principal P. Thus, we get

$$\begin{aligned}(1+0.05)^{15} &= 2.0789 \text{ for 15 years at 5\%}\\ (1+0.05)^{95} &= 103.03 \text{ for 95 years at 5\%}\end{aligned}$$

Problem 7. What sum will amount to \$1250 if put at compound interest at 4% for fifteen years?

Solution. $A = P(1+r)^n \implies P = A/(1+r)^n \implies \$1250/(1+0.04)^{15} = \$694.08$.

Problem 8. A man invests \$200 a year in a savings bank which pays $3\frac{1}{2}$% per annum on all deposits. What will be the total amount due him at the end of twenty-five years?

Solution. In general, the amount after n years is $A = P[1+r+(1+r)^2+\cdots+(1+r)^n]$. We can sum this geometric series as follows:

$$\begin{aligned}S = 1+r+(1+r)^2+\cdots+(1+r)^n &\implies S(1+r) = (1+r)^2+(1+r)^3+\cdots+(1+r)^{n+1}\\ &\implies S(1+r)+(1+r)-(1+r)^{n+1} = S\\ &\implies S[(1+r)-1] = (1+r)^{n+1}-(1+r)\\ &\implies Sr = (1+r)[(1+r)^n-1]\\ &\implies S = \frac{1+r}{r}\,[(1+r)^n-1]\end{aligned}$$

Therefore, in our case,

$$A = \$200\,\frac{1.035}{0.035}\left[(1.035)^{25}-1\right] = \$8,062.62$$

Problem 9. What sum should be paid for an annuity of \$1200 a year to be paid for thirty years, money being supposed to be worth 4% per annum? What sum should be paid were this annuity to be perpetual?

Solution. In general, the amount is given by the formula $A\left[\dfrac{1}{1+r}+\dfrac{1}{(1+r)^2}+\cdots+\dfrac{1}{(1+r)^n}\right]$. We can sum this geometric series as follows.

$$
\begin{aligned}
S = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^n} &\implies S\frac{1}{1+r} = \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots + \frac{1}{(1+r)^{n+1}} \\
&\implies S\frac{1}{1+r} + \frac{1}{1+r} - \frac{1}{(1+r)^{n+1}} = S \\
&\implies \frac{1}{1+r} - \frac{1}{(1+r)^{n+1}} = S\left[1 - \frac{1}{1+r}\right] \\
&\implies S\left[\frac{r}{1+r}\right] = \frac{1}{1+r}\left[1 - \frac{1}{(1+r)^n}\right] \\
&\implies S = \frac{1}{r}\left[1 - \frac{1}{(1+r)^n}\right]
\end{aligned}
$$

And in our case, we get

$$\$1200\,\frac{1}{0.04}\left[1 - \frac{1}{(1.04)^{30}}\right] = \$20,750.44$$

If this annuity is to be perpetual, then we take the limit as $n \to \infty$ to get

$$\$1200\,\frac{1}{0.04} = \$30,000.$$

Chapter 25

Permutations and Combinations

Permutations and Combinations

The number of r-permutations of n items is given by

$$\begin{aligned} P_r^n &= n(n-1)(n-2)\cdots \text{ to } r \text{ factors} \\ &= n(n-1)(n-2)\cdots(n-r+1) \end{aligned}$$

The number of distinguishable n-permutations of n items of which p are alike, q others are alike, and so on, is given by

$$N = \frac{n!}{p!\,q!\,\cdots}$$

The number of combinations of n items taken r at a time is given by

$$C_r^n \equiv \binom{n}{r} = \frac{n}{r!(n-r)!}$$

Example 3. (p. 396) Show that $P_6^8 = 4 \cdot P_7^7$, and that $P_3^{16} = 2P_4^8$.

Solution.

$P_6^8 = \underbrace{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}_{\text{6 factors}} = \dfrac{8!}{2!} = 4 \cdot 7!$

$P_3^{16} = (16)(15)(14) = 4(8)(15)(7) = 2(8)(7)(30) = 2(8)(7)(6)(5) = 2P_4^8$

Example 4. (p. 396) If $P_4^{2n} = 127P_3^{2n}$, find n.

Solution. $P_4^{2n} = 127P_3^{2n} \implies (2n)(2n-1)(2n-2)(2n-3) = 127(2n)(2n-1)(2n-2) \implies 2n-3 = 127 \implies n = 65$.

Example 5. (p. 396) How many passenger tickets will a railway company need for use on a division on which there are twenty stations?

Solution. There are twenty stations for the departure, and that leaves 19 stations for the arrival, hence the answer is $20 \cdot 19 = 380$.

Example 6. (p. 396) In how many of the permutations of the letters *a, e, i, o, u, y*, taken all at a time, do the letters *a, e, i* stand together?

Solution. There are 4 starting positions for the 3 letters among the 6 letters. Then there are 3! permutations of the 3 letters and 3! permutation of the other 3 letters, hence the answer is $4\,3!\,3! = 144$.

Example 7. (p. 396) With the letters of the word *numerical* how many arrangements of five letters each can be formed in which the odd places are occupied by consonants?

Solution. The 9-letter word has 5 consonants and 4 vowels. For 5-letter words we need 3 consonants in positions 1, 3 and 5 and that leaves 6 letters for the remaining 2 positions, hence the answer is $P_3^5 P_2^6 = (5 \cdot 4 \cdot 3)(6 \cdot 5) = 1800$.

Example 8. (p. 396) Show that with the digits 0, 1, 2, ..., 9 it is possible to form $P_4^{10} - P_3^9$ numbers, each of which has four different figures.

Solution. We have P_4^{10} possible arrangements of the 10 digits to form numbers of length 4. However, we must exclude all those that begin with 0, which is P_3^9, so the answer is $P_4^{10} - P_3^9$.

Example 9. (p. 396) How many numbers all told can be formed with the digits 3, 4, 5, 7, 8, all the figures in each number being different?

Solution. We have 5 different number to form the figures, which can have lengths of 5, 4, 3, 2 or 1, hence the answer is

$$P_5^5 + P_4^5 + P_3^5 = P_2^5 + P_1^5 = 5! + 5 \cdot 4 \cdot 3 \cdot 2 + 5 \cdot 4 \cdot 3 + 5 \cdot 4 + 5 = 120 + 120 + 60 + 20 + 5 = 325$$

Example 10. (p. 396) In how many ways can seven boys be arranged in a row if one particular boy is not permitted to stand at either end of the row?

Solution. Among the $P_7^7 = 7!$ arrangements of the 7 boys, taken all at a time, we must exclude the cases where the particular boy is at the front or back of the line. For each of those cases, there are P_6^6 arrangements of the other 6 boys, hence the answer is $P_7^7 - 2P_6^6 = 7! - 2\,6! = 3600$.

Example 4. (p. 398) In how many ways can five pennies, six nickels, and four dimes be distributed among 15 children so that each may receive a coin?

Solution.

$$N = \frac{15!}{5!\,6!\,4!} = 630,630$$

Example 5. (p. 398) In a certain district of a town there are ten streets running north and south, and five running east and west. In how many ways can a person walk from the southwest corner of the district to the northeast corner, always taking the shortest course?

Solution. We can visualize this as a grid made up of x and y segments, where we need to move along 9 x-segments and 4 y-segments to arrive at our destination. This is tantamount to arranging a string of xs and ys that is 13 letters long and where all the xs are alike and all the ys are alike. Hence, the answer is

$$N = \frac{13!}{9!\,4!} = 715.$$

The general formula is

$$N = \frac{[(n-1)+(m-1)]!}{(n-1)!(m-1)!},$$

where n is the number of north-south streets and m is the number of east-west streets.

Example 6. (p. 401) Find the values of C_{15}^{17}, C_5^{10}, and C_{19}^{23}.

Solution.

$$C_{15}^{17} = C_{17-15}^{17} = C_2^{17} = \frac{17 \cdot 16}{1 \cdot 2} = 136$$
$$C_5^{10} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252$$
$$C_{19}^{23} = C_{23-19}^{23} = C_4^{23} = \frac{23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4} = 8855$$

Example 7. (p. 401) If $C_8^n = C_7^n$, find n.

Solution. $C_8^n = C_7^n \implies \dfrac{n!}{8!(n-8)!} = \dfrac{n!}{7!(n-7)!} \implies 7!(n-7)(n-8)! = 8 \cdot 7!(n-8)! \implies n-7 = 8 \implies n = 15$.

Example 8. (p. 401) If $2C_4^n = 5C_2^n$, find n.

Solution. $2C_4^n = 5C_2^n \implies 2\dfrac{n!}{4!(n-4)!} = 5\dfrac{n!}{2!(n-2)!} \implies 2 \cdot 2!(n-2)(n-3)(n-4)! = 5 \cdot 4!(n-4)! \implies 2(n-2)(n-3) = 5 \cdot 4 \cdot 3 = 60 \implies n^2 - 5n - 24 = 0 \implies n = \dfrac{5 \pm \sqrt{25 + 4(24)}}{2} = \dfrac{5 \pm 11}{2} = 8$.

Example 9. (p. 401) How many planes are determined by twelve points, no four of which lie in the same plane?

Solution. Three non-colinear points determine a plane, so this is tantamount to selecting 12 points, 3 at a time: $C_9^{12} = \dfrac{12!}{3!9!} = \dfrac{12 \cdot 11 \cdot 10}{2 \cdot 3} = 220$.

Example 10. (p. 401) How many parties of five men each can be chosen from a company of twelve men? In how many of these parties will a particular man A be included? From how many will A be excluded?

Solution. This is a combination of 12 items taken 5 at a time, so $C_5^{12} = \dfrac{12!}{5!7!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5!} = 792$ parties. If we include A, then we need to select 4 more from 11 items, hence $C_4^{11} = \dfrac{11!}{4!7!} = 330$. The number of parties where A is excluded is the difference: $792 - 330 = 462$.

Example 11. (p. 401) Of the parties described in the preceding example how many will include two particular men A and B? How many will include one but not both of them? How many will include neither of them?

Solution. Once we have selected A and B, we have 3 more to choose from a pool of 10, hence $C_3^{10} = \dfrac{10!}{3!7!} = 120$. There are 330 groups that have A included and 120 groups with both A and B include, so there are $330 - 120 = 210$ groups that have A but not B included. Similarly, there are 210 groups that have B but not A. Thus, there are $210 + 210 = 420$ groups with one of them but not both.

Example 12. (p. 401) From twenty Republicans and eighteen Democrats how many committees can be chosen, each consisting of four Republicans and three Democrats?

Solution. $C_4^{20} C_3^{18} = \dfrac{20!}{4!16!} \dfrac{18!}{3!15!} = 3,953,520$.

Example 13. (p. 401) With five vowels and fourteen consonants how many arrangements of letters can be formed, each consisting of three vowels and four consonants?

Solution. $C_3^5 C_4^{14} 7! = \dfrac{5!}{3!2!} \dfrac{14!}{4!10!} 7! = 10010 \cdot 7! = 50,450,400$.

Example 14. (p. 401) In how many ways can a pack of fifty-two cards be divided equally among four players, A, B, C, D? In how many ways can the cards be distributed into four piles containing thirteen each?

Solution. For player A, we select 13 cards from 52, for B we select 13 cards from the remaining 39, for C we select 13 from the remaining 26, and then D gets the last 13 cards. This gives $C_{13}^{52}C_{13}^{39}C_{13}^{26}C_{13}^{13} = \dfrac{52!}{13!39!}\dfrac{39}{13!26!}\dfrac{26!}{13!13!} = \dfrac{52!}{(13!)^4}$. The cards can be distributed into four piles containing 13 cards each in $\dfrac{52!}{(13!)^4}4!$ different ways.

Example 15. (p. 401) How many numbers, each of five figures, can be formed with the characters 2, 3, 4, 2, 5, 2, 3, 6, 7?

Solution. The characters are 2, 2, 2; 3, 3; 4; 5; 6; 7, so we have nine characters but only six are different. No single formula will apply. Instead we first classify and enumerate the arrangements.

1. The three 2's and the two 3's can be combined in $C_2^5 = 5!/(3!2!) = 10$ arrangements.
2. Combining the three 2's with two of the other 5 characters gives $C_2^5 \cdot 5!/3! = 10 \cdot 20 = 200$ arrangements.
3. Combining two pairs of 2's and 3's with one of the other four characters gives $4 \cdot 5!/(2!2!) = 4 \cdot 30 = 120$ arrangements.
4. Those having two characters alike and the other three different gives $2 \cdot C_3^5 \cdot 5!/2! = 2 \cdot 5!/(3!2!) \cdot 5!/2! = 2 \cdot 10 \cdot 60 = 1200$ arrangements.
5. Finally, those having five different characters of the six possible gives $5! \cdot C_5^6 = 120 \cdot 6 = 720$ arrangements.

Thus, the total number of arrangements is $10 + 200 + 120 + 1200 + 720 = 2250$.

25.1 Permutations and Combinations, Exercise LXV (p. 405)

Problem 1. If there are three roads leading from P to Q, two from Q to R, and four from R to S, by how many routes can a person travel for P to S?

Solution. $3 \times 2 \times 4 = 24$.

Problem 2. In how many ways can a company of five persons be arranged in six numbered seats?

Solution. We can select the unoccupied seat in 6 possible ways, and for each selection there are $P_5^5 = 5!$ permutations of the remaining seats, hence the answer is $6 \cdot 5! = 6! = 720$.

Problem 3. If eight runners enter a half-mile race, in how many ways can the first, second, and third places be won?

Solution. This is simply the number of 3-permutations of 8 items. Hence, $P_3^8 = 8 \cdot 7 \cdot 6 = 336$.

Problem 4. In how many ways can a four-oar crew be chosen from ten oarsmen and in how many ways can all these crews be arranged in the boat?

Solution. This is the number of combinations of 10 items taken 4 at a time: $C_4^{10} = 10 \cdot 9 \cdot 8 \cdot 7/4! = 210$. There are $4! \cdot C_4^{10} = 5040$ possible arrangements.

Problem 5. From a company of 100 soldiers how many pickets of three men can be chosen?

Solution. This is simply the number of combinations of 100 taken three at a time: $C_3^{100} = 100 \cdot 99 \cdot 98/3! = 161,700$.

Problem 6. Five baseball nines wish to arrange a schedule of games in which each nine shall meet every other nine three times. How many games must be scheduled?

Solution. This is simply three times the number of combinations of 5 teams taken 2 at a time: $3 \cdot C_2^5 = 3 \cdot 5 \cdot 4/2 = 30$.

Problem 7. In how many ways can the digits 1, 2, 1, 3, 2, 1, 5 be arranged, all the digits occurring in each arrangement?

Solution. We have 7 digits, which includes three 1's and two 2's, hence $7!/(3!\,2!) = 4320$.

Problem 8. Of the permutations of the letters in the word *factoring*, taken all at a time, (1) how many begin with a vowel and end with a consonant? (2) how many do not begin with f? (3) how many have vowels in the first three places?

Solution. We note that there are three vowels and six consonants in the nine letters of *factoring*. For (1), that gives $3 \cdot 6 \cdot P_7^7 = 18 \cdot 7! = 90,720$. For (2), we have 9! total permutations of the nine letters and 8! permutations that *do* begin with f, hence there are $9! - 8! = 322,560$ that *do not* begin with *f*. (3) The number that have vowels in the first three places, and hence six consonants in the last six places, is $3!6! = 6 \cdot 720 = 4320$.

Problem 9. In how many of the permutations just described do the vowels retain the order, a, o, i? In how many do the consonants retain the order f, c, t, r, n, g? In how many do both the vowels and the consonants retain these orders?

Solution. We want one particular order of the 3! possible for the vowels, so that gives $9!/3! = 60,480$. In this case, we want one particular order of the 6! possible for the consonants, so that gives $9!/6! = 504$. Finally, if both the vowels and consonants retain these orders, then that gives $9!/(3!\,6!) = 84$.

Problem 10. With the letters of the word *resident* how many permutations of five letters each can be formed in which the first, third, and fifth letters are vowels?

Solution. The word *resident* contains three vowels and 5 consonants, and since we need all three vowels, that leaves two consonants of the five remaining to make up the five letters. Hence, $3 \cdot P_2^5 = 3(5 \cdot 4) = 60$.

Problem 11. In how many ways can a baseball nine be selected from fifteen candidates of whom six are qualified to play in the outfield only and nine in the infield only?

Solution. A team of nine consists of three outfielders and six infielders. We select three outfielders from the six candidates and select six infielders from the nine candidates to give a total of

$$C_3^6 \cdot C_6^9 = \frac{6!}{3!3!} \cdot \frac{9!}{6!3!} = \frac{9!}{(3!)^3} = 1680.$$

Problem 12. In how many ways can two numbers whose sum is even be chosen from the numbers 1, 2, 3, 8, 9, 10?

Solution. We must select either two odd numbers or two even numbers, but not one even and one odd. Now there are three odd numbers and three even numbers, so that gives $C_2^3 + C_2^3 = 3 + 3 = 6$ possibilities.

Problem 13. How many numbers of one, two, or three figures can be formed with the digits 1, 2, 3, 4, 5, 6, 7 (1) when the digits may be repeated? (2) when they may not be repeated?

Solution. (1) $7 + 7^2 + 7^3 = 399$. (2) $P_1^7 + P_2^7 + P_3^7 = 7 + 7 \cdot 6 + 7 \cdot 6 \cdot 5 = 259$.

Problem 14. How many odd numbers, each having five different figures, can be formed with the digits 1, 2, 3, 4, 5, 6?

Solution. The number must end in either 1, 3, or 5. That leaves four possibilities from the remaining five. Hence the answer is $3 \cdot P_4^5 = 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 360$.

Problem 15. How many odd numbers without repeated digits are there between 3000 and 8000? How many of these are divisible by 5?

Solution. Of the 5 odd digits (1, 3, 5, 7, 9), for numbers between 3000 and 4000 we can use only four (1, 5, 7, 9) , similarly for numbers between 5000 and 6000 (1, 3, 7, 9) and between 7000 and 8000 (1, 3, 5, 9), whereas for numbers between 4000 and 5000 and between 6000 and 7000, we can use all five. That leaves eight digits to use for the second digit and then seven for third digit. Thus we have

$$\begin{aligned}
3000-4000: &\quad 8 \cdot 7 \cdot 4 = 224\\
4000-5000: &\quad 8 \cdot 7 \cdot 5 = 280\\
5000-6000: &\quad 8 \cdot 7 \cdot 4 = 224\\
6000-7000: &\quad 8 \cdot 7 \cdot 5 = 280\\
7000-8000: &\quad 8 \cdot 7 \cdot 4 = 224
\end{aligned}$$

and the total number is $3(224)+2(280) = 1232$. None of the 224 numbers in he range 5000 - 6000 end in 5 (by construction), one-fourth of those in the range 3000-4000 and 7000-8000 end in 5, and one-fifth of those in the range 4000 - 5000 and 6000-7000 end in 5. Hence, the total number divisible by 5 is $\frac{1}{4}2(224)+\frac{1}{5}2(280) = 224$.

Problem 16. In how many ways can a person invite one or more of five friends to dinner?

Solution. $C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = 2^5 - 1 = 31$.

Problem 17. In how many ways can fifteen apples be distributed among three boys so that one boy shall receive six, another five, and another four?

Solution. Starting with fifteen apples, we choose 6, that leaves 9 of which we choose 5, and that leaves four. Then there are 3! ways to distribute the three groups among the three boys. Thus,

$$3! \cdot C_6^{15} \cdot C_5^9 \cdot C_4^4 = 3! \cdot \frac{15!}{6!9!} \cdot \frac{9!}{5!4!} \cdot \frac{4!}{4!} = 3! \cdot \frac{15!}{6!5!4!} = 3! \cdot 630,630 = 3,783,780.$$

Problem 18. In how many ways can six positive and five negative signs be written in a row?

Solution. $\dfrac{11!}{6!\,5!} = 462$.

Problem 19. How many numbers of four figures each can be formed with the characters 1, 2, 3, 2, 3, 4, 2, 4, 5, 3, 6, 7?

Solution. The characters are 1; 2, 2, 2; 3, 3, 3; 4, 4; 5; 6; 7, so there are 12 characters, but only 7 are different.

Problem 20. From fifteen French and twelve German books eight French and seven German books are to be selected and arranged on a shelf. In how many ways can this be done?

Solution. The number of combinations of the French books is $C_8^{15} = \dfrac{15!}{8!7!} = 6435$, and the number of combinations of the German books is $C_7^{12} = \dfrac{12!}{7!5!} = 792$. The number of permutations of the 15 books selected is 15!, so the total number is $(6435)(792)15! = 5,096,520 \cdot 15!$.

Problem 21. From a complete suit of thirteen cards five are to be selected which shall include the king or queen, or both. In how many ways can this be done?

Solution. $C_3^{11} = \dfrac{11!}{3!8!} = 165$, $C_4^{11} = \dfrac{11!}{4!7!} = 330$, so that the total number is $165 + 2(330) = 825$.

Problem 22. In how many ways can four men be chosen from five Americans and six Englishman so as to include (1) only one Englishman? (2) at least one Englishman?

Solution.

(1) $C_1^6 C_3^5 = \dfrac{6!}{5!} \cdot \dfrac{5!}{3!2!} = 6 \cdot 10 = 60$

(2) $C_1^6 C_3^5 + C_2^6 C_2^5 + C_3^6 C_1^5 + C_4^6 = 60 + \dfrac{6!}{2!4!} \cdot \dfrac{5!}{3!2!} + \dfrac{6!}{3!3!} \cdot \dfrac{5!}{4!} + \dfrac{6!}{4!2!} = 60 + 15 \cdot 10 + 20 \cdot 5 + 15 = 325$

Problem 23. How many parallelograms are formed when a set of ten parallel lines is met by another set of twelve parallel lines?

Solution. $C_2^{10} \cdot C_2^{12} = \dfrac{10!}{8!2!} \cdot \dfrac{12!}{2!10!} = \dfrac{12!}{8!(2!)^2} = 2970.$

Problem 24. Given n points in a plane no three of which lie in the same straight line, except m which all lie in the same straight line. Show that the number of lines obtained by joining these points is $C_2^n - C_2^m + 1$.

Solution. If there weren't the three points that lie on the same line, the number of lines obtained by joining the points would be the number of combinations of n points taken 2 at a time, or C_2^n. But since there are m such points, we must subtract C_2^m, and add the one comprised of the m points, so that the total number of lines is $C_2^n - C_2^m + 1$.

Problem 25. Find the number of bracelets that can be formed by stringing together five like pearls, six like rubies, and five like diamonds.

Solution. $\dfrac{(5+6+5-1)!}{2} \dfrac{1}{5!} \dfrac{1}{6!} \dfrac{1}{5!} = 63,063.$

Problem 26. In how many ways can ten persons be arranged at two round tables, five at each table?

Solution. $C_5^{10} 4!4! = \dfrac{10!}{5!5!} 4!4! = \dfrac{10!}{5 \cdot 5} = 145,152.$

Problem 27. In how many ways can six ladies and five gentlemen arrange a game of lawn tennis, each side to consist of one lady and one gentleman?

Solution. $C_2^6 \cdot C_2^5 \cdot 2 = \dfrac{6!}{4!2!} \cdot \dfrac{5!}{2!3!} \cdot 2 = 15 \cdot 10 \cdot 2 = 300.$

Problem 28. In how many ways can fifteen persons vote to fill a certain office for which there are five candidates? In how many of these ways will the vote be equally divided among the five candidates?

Solution.

Problem 29. A boat crew consists of eight men two of whom are qualified to row on the stroke side only and one on the bow side only. In how many ways can the crew be arranged?

Solution. $C_2^5 \cdot C_3^3 \cdot 4! \cdot 4! = \dfrac{5!}{2!3!} \cdot \dfrac{3!}{3!} (4!)^2 = \dfrac{5 \cdot 4}{2} (24)^2 = 10(576) = 5760.$

Problem 30. How many baseball nines can be chosen from eighteen players of whom ten are qualified to play in the infield only, five in the outfield only, and three in any position?

Solution. Answer: $37,340$.

Problem 31. Show that the number of permutations of six different letters taken all at a time, when two of the letters are excluded each from a particular position, is $6! - 2 \cdot 5! + 4!$.

Solution.

Problem 32. How many combinations four at a time can be formed with the letters p, q, r, s, t, v when repetitions are allowed?

Solution.

Problem 33. How many different throws can be made with five dice?

Solution.

Problem 34. How many terms has each of the symmetric functions $\sum x^4y^3z^2u$, $\sum x^2y^2z^2u$, $\sum x^3y^3z^2u^2v$, the number of variables being ten?

Solution.

Problem 35. Show that the number of terms in a complete homogeneous function of the nth degree in four variables is $(n+1)(n+2)(n+3)/3!$.

Solution.

Chapter 28

Mathematical Induction

28.1 Mathematical Induction, Exercise LXIX (p. 425)

Prove the truth of the following formulas by the method of mathematical induction.

Problem 1. $a + ar + ar^2 + \cdots + ar^{n-1} = a(1 - r^n)/(1 - r)$.

Solution. For $n = 1$, the formula gives $a = a(1 - r)/(1 - r)$, which is true. Let's assume that the formula holds for an integer k. Adding ar^k to both sides then gives

$$a + ar + ar^2 + \cdots + ar^{k-1} + ar^k = a\frac{1 - r^k}{1 - r} + ar^k = a\frac{1 - r^k + (1 - r)r^k}{1 - r} = a\frac{1 - r^{k+1}}{1 - r},$$

which is what we get if we replace k with $k + 1$ in the given formula. Thus, by the principle of mathematical induction, it is true for all positive integers.

Problem 2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.

Solution. Checking the formula for $n = 1$, we have $1 = 1(2)(3)/6$, which is true. Let us assume that it holds for an integer k. Adding $(k + 1)^2$ to both sides then gives

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k + 1)^2 &= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 \\
&= \frac{k + 1}{6}(2k^2 + k + 6k + 6) \\
&= \frac{k + 1}{6}(2k^2 + 7k + 6) \\
&= \frac{k + 1}{6}(k + 2)(2k + 3) \\
&= \frac{(k + 1)[(k + 1) + 1][2(k + 1) + 1]}{6},
\end{aligned}$$

which is what we get if we replace k with $k + 1$ in the given formula. Thus, by the principle of mathematical induction, it is true for all positive integers.

Problem 3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = n^2(n + 1)^2/4$.

Solution. Checking the formula for $n = 1$, we have $1 = 1^2(1 + 1)^2/4$, which is true. Let us assume that the formula holds for an integer k. Adding $(k+1)^3$ to both sides then gives

$$\begin{aligned}
1^3+2^3+3^3+\cdots+k^3+(k+1)^3 &= \frac{k^2(k+1)^2}{4}+(k+1)^3\\
&= \frac{(k+1)^2}{4}(k^2+4k+4)\\
&= \frac{(k+1)^2(k+2)^2}{4}\\
&= \frac{(k+1)^2[(k+1)+1]^2}{4}
\end{aligned}$$

which is what we get if we replace k with $k+1$ in the given formula. Thus, by the principle of mathematical induction, it is true for all positive integers.

Problem 4. $1+3+6+\cdots+n(n+1)/2! = n(n+1)(n+2)/3!$.

Solution. Checking the formula for $n=1$ gives $1 = 1(1+1)(1+2)/3! = (2)(3)/3!$, which is true. Let us assume that the formula holds for an integer k. Adding $(k+1)^3$ to both sides then gives

$$\begin{aligned}
1+3+6+\cdots+\frac{k(k+1)}{2!}+\frac{(k+1)(k+2)}{2!} &= \frac{k(k+1)(k+2)}{3!}+\frac{(k+1)(k+2)}{2!}\\
&= \frac{(k+1)(k+2)}{3!}(k+3)\\
&= \frac{(k+1)[(k+1)+1][(k+1)+2]}{3!},
\end{aligned}$$

which is what we get if we replace k with $k+1$ in the given formula. Thus, by the principle of mathematical induction, it is true for all positive integers.

Chapter 29

Theory of Equations

29.1 Theory of Equations, Exercise LXXI (p. 435)

Problem 1. Two of the roots of $2x^3 - 7x^2 + 10x - 6 = 0$ are $1 \pm i$; find the third root.

Solution. When put in standard form $x^3 - \frac{7}{2}x^2 + 5x - 3 = 0$, we see that $p = -\frac{7}{2}$, and since $-p$ is the sum of the roots, we have

$$\frac{7}{2} = 1 + i + 1 - i + \alpha \implies \alpha = \frac{3}{2}$$

Thus, the third root is 3/2.

Problem 2. The roots of each of the following equations are in geometrical progression; find them.

(1) $8x^3 - 14x^2 - 21x + 27 = 0$ (2) $x^3 + x^2 + 3x + 27 = 0$

Solution. Let the roots be α/β, α, $\alpha\beta$.
(1) We have

$$\frac{14}{8} = \frac{\alpha}{\beta} + \alpha + \alpha\beta = \alpha\left(\frac{1}{\beta} + 1 + \beta\right), \quad -\frac{21}{8} = \frac{\alpha^2}{\beta} + \alpha^2 + \alpha^2\beta = \alpha^2\left(\frac{1}{\beta} + 1 + \beta\right), \quad -\frac{27}{8} = \frac{\alpha}{\beta}\alpha\alpha\beta = \alpha^3.$$

From the last equation, $\alpha = -\frac{3}{2} \implies \frac{1}{\beta} + 1 + \beta = -\frac{2}{3} \cdot \frac{14}{8} = -\frac{7}{6} \implies 6\beta^2 + 13\beta + 6 = 0 \implies (3\beta + 2)(2\beta + 3) = 0 \implies \beta = -\frac{2}{3}, -\frac{3}{2} \implies \frac{1}{\beta} + 1 + \beta = -\frac{7}{6}$ (since each value is the reciprocal of the other). Therefore, both of the values of β give rise to the same three roots, namely, $-\frac{3}{2}, 1, \frac{9}{4}$.

(2) We have

$$-1 = \frac{\alpha}{\beta} + \alpha + \alpha\beta = \alpha\left(\frac{1}{\beta} + 1 + \beta\right), \quad 3 = \frac{\alpha^2}{\beta} + \alpha^2 + \alpha^2\beta = \alpha^2\left(\frac{1}{\beta} + 1 + \beta\right), \quad -27 = \frac{\alpha}{\beta}\alpha\alpha\beta = \alpha^3.$$

Thus, $\alpha = -3$, and $\frac{1}{\beta} + 1 + \beta = -\frac{1}{3}(-1) = \frac{1}{3} \implies 3\beta^2 + 2\beta + 3 = 0 \implies \beta = \frac{-2 \pm \sqrt{4 - 4(3)(3)}}{6} = \frac{-2 \pm 4\sqrt{2}\,i}{6} = \frac{-1 \pm 2\sqrt{2}\,i}{3}$. The reciprocal is $\frac{3}{-1 \pm 2\sqrt{2}\,i} \cdot \frac{-1 \mp 2\sqrt{2}\,i}{-1 \mp 2\sqrt{2}\,i} = \frac{3(-1 \mp 2\sqrt{2}\,i)}{9} = \frac{-1 \mp 2\sqrt{2}\,i}{3}$, so the roots are $1 - 2\sqrt{2}\,i$, -3, $1 + 2\sqrt{2}\,i$.

Problem 3. The roots of each of the following equations are in arithmetical progression; find them.

(1) $x^3 + 6x^2 + 7x - 2 = 0$ (2) $x^3 - 9x^2 + 23x - 15 = 0$

Solution. Let the roots be $a-d$, a, $a+d$.
(1) We have

$$\begin{aligned} -6 &= a-d+a+a+d = 3a \implies a=-2 \\ 7 &= a^2-ad+a^2-d^2+a^2+ad = 3a^2-d^2 = 3(4)-d^2 \implies d^2=5 \implies d=\pm\sqrt{5} \\ 2 &= a(a^2-d^2) = -2(4-5) = 2 \end{aligned}$$

Thus, the roots are $-2-\sqrt{5}$, -2, $-2+\sqrt{5}$.

(2) We have

$$\begin{aligned} 9 &= 3a \implies a=3 \\ 23 &= 3a^2-d^2 = 27-d^2 \implies d^2=4 \implies d=\pm 2 \\ 15 &= a(a^2-d^2) = 3(9-4) = 15 \end{aligned}$$

Thus, the roots are 1, 3, 5.

Problem 4. Show that if one root of $x^3+px^2+qx+r=0$ be the negative of another root, $pq=r$.

Solution. Let the roots be$-\alpha$, α, β. We have

$$\begin{aligned} -p &= -\alpha+\alpha+\beta = \beta \\ q &= -\alpha^2-\alpha\beta+\alpha\beta = -\alpha^2 \\ -r &= (-\alpha)(\alpha)(\beta) = -\alpha^2\beta \end{aligned}$$

Thus, we see that $-r=-pq \implies pq=r$.

Problem 5. Find the condition that one root of $x^3+px^2+qx+r=0$ shall be the reciprocal of another root.

Solution. Let the roots be α, $1/\alpha$, β. Then we have

$$\begin{aligned} -p &= \alpha+\frac{1}{\alpha}+\beta \\ q &= 1+\alpha\beta+\frac{\beta}{\alpha} \\ -r &= \beta \end{aligned}$$

which gives $-p=\alpha+\dfrac{1}{\alpha}-r \implies \alpha+\dfrac{1}{\alpha}=r-p$, and $q=1-\alpha r-\dfrac{r}{\alpha} \implies 1-q=r\left(\alpha+\dfrac{1}{\alpha}\right)=r(r-p) \implies q+r(r-p)-1=0 \implies r^2-rp+q-1=0$.

Problem 6. Solve $x^4+4x^3+10x^2+12x+9=0$, having given that it has two double roots

Solution. Let the roots be α, α, β, β. Then we have

$$\begin{aligned} -4 &= 2(\alpha+\beta) \implies \alpha+\beta=-2 \\ 10 &= \alpha^2+\alpha\beta+\alpha\beta+\alpha\beta+\alpha\beta+\beta^2 = \alpha^2+4\alpha\beta+\beta^2 \\ -12 &= \alpha^2\beta+\alpha^2\beta+\alpha\beta^2+\alpha\beta^2 = 2\alpha\beta(\alpha+\beta) \\ 9 &= \alpha^2\beta^2 \implies \alpha\beta=3 \text{ (We must take the principal square root to be consistent with } -12=2\alpha\beta(\alpha+\beta).) \end{aligned}$$

Therefore, $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=4-(4)(3)=-8 \implies \alpha-\beta=\pm 2\sqrt{2}\,i$. Thus, $(\alpha,\beta)=(-1+\sqrt{2}\,, i, -1-\sqrt{2}\,i)$.

Problem 7. Solve the equation $14x^3 - 13x^2 - 18x + 9 = 0$, having given that its roots are in harmonic progression.

Solution. Let the roots be $\dfrac{1}{a-d}$, $\dfrac{1}{a}$, $\dfrac{1}{a+d}$. Then we have

$$\begin{aligned}
\frac{13}{14} &= \frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = \frac{a^2+ad+a^2-d^2+a^2-ad}{a(a^2-d^2)} = \frac{3a^2-d^2}{a(a^2-d^2)} \\
-\frac{18}{14} &= \frac{1}{a^2-ad} + \frac{1}{a^2-d^2} + \frac{1}{a^2+ad} = \frac{a^2+ad+a^2-ad}{a^4-a^2d^2} + \frac{1}{a^2-d^2} = \frac{2a^2}{a^2(a^2-d^2)} + \frac{1}{a^2-d^2} = \frac{3}{a^2-d^2} \\
-\frac{9}{14} &= \frac{1}{a(a^2-d^2)}
\end{aligned}$$

Dividing the second equation by the third, we get $3a = -2$ so $a = -2/3$. And dividing the first equation by the third, we get $3a^2 - d^2 = -13/9 \implies d^2 = 25/9 \implies d = 5/3$. Thus, the roots are $-3/7$, $-3/2$, 1.

Problem 8. Solve the equation $x^4 - x^3 - 56x^2 + 36x + 720 = 0$, having given that two of its roots are in the ratio $2 : 3$ and that the difference between the other two roots is 1.

Solution. Let the roots be 2α, 3α, β, $\beta+1$. Then we have

$$\begin{aligned}
1 &= 2\alpha + 3\alpha + \beta + \beta + 1 = 5\alpha + 2\beta + 1 \\
-56 &= 6\alpha^2 + 2\alpha\beta + 2\alpha\beta + 2\alpha + 3\alpha\beta + 3\alpha\beta + 3\alpha + \beta^2 + \beta = 6\alpha^2 + 10\alpha\beta + \beta^2 + 5\alpha + \beta \\
-36 &= 6\alpha^2\beta + 6\alpha^2\beta + 6\alpha^2 + 2\alpha\beta + 2\alpha + 3\alpha\beta^2 + 3\alpha = 12\alpha^2\beta + 6\alpha^2 + 3\alpha\beta^2 + 2\alpha\beta + 5\alpha \\
720 &= 6\alpha^2(\beta^2 + \beta)
\end{aligned}$$

The first equation gives $5\alpha + 2\beta = 0 \implies 2\beta = -5\alpha$, and then the fourth equation gives $120 = \alpha^2(\beta^2+\beta) \implies 480 = \alpha^2[(2\beta)^2 + 2(2\beta)] = \alpha^2[25\alpha^2 - 10\alpha] \implies 25\alpha^4 - 10\alpha^3 - 480 = 0$. With synthetic division, we find $25\alpha^4 - 10\alpha^3 - 480 = (\alpha+2)(25\alpha^3 - 60\alpha^2 + 120\alpha - 240) = 0$, so $\alpha = -2$ is a solution, which gives $\beta = 5$ and the roots are -4, -6, 5, 6.

Problem 9. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, find the equation whose roots are

(1) $-\alpha, -\beta, -\gamma$	(2) $k\alpha, k\beta, k\gamma$
(3) $1/\alpha, 1/\beta, 1/\gamma$	(4) $\alpha+k, \beta+k, \gamma+k$
(5) $\alpha^2, \beta^2, \gamma^2$	(6) $-1/\alpha^2, -1/\beta^2, -1/\gamma^2$

Solution. We seek the coefficients of the equation $x^3 + p'x^2 + q'x + r' = 0$.
(1)

$$\begin{aligned}
-p' &= -\alpha - \beta - \gamma = p \\
q' &= \alpha\beta + \alpha\gamma + \beta\gamma = q \\
-r' &= -\alpha\beta\gamma = r
\end{aligned}$$

Therefore, the equation is $x^3 - px^2 + qx - r = 0$.

(2)

$$\begin{aligned}
-p' &= k(\alpha + \beta + \gamma) = k(-p) \\
q' &= k^2(\alpha\beta + \alpha\gamma + \beta\gamma) = k^2 q \\
-r' &= k^3\alpha\beta\gamma = k^3(-r)
\end{aligned}$$

Therefore, the equation is $x^3 + kpx^2 + k^2qx + k^3r = 0$.

(3)

$$\begin{aligned}
-p' &= \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} \implies -\alpha\beta\gamma p' = \beta\gamma+\alpha\gamma+\alpha\beta = q \implies -p' = \frac{q}{-r} \implies p' = \frac{q}{r}\\
q' &= \frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma} \implies \alpha^2\beta^2\gamma^2 q' = \alpha\beta\gamma^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma = \alpha\beta\gamma(\gamma+\beta+\alpha) = pr \implies q' = \frac{p}{r}\\
-r' &= \frac{1}{\alpha\beta\gamma} = \frac{1}{-r} \implies r' = \frac{1}{r}
\end{aligned}$$

Therefore, the equation is $x^3+\frac{q}{r}x^2+\frac{p}{r}x+\frac{1}{r}=0$ or $rx^3+qx^2+px+1=0$.

(4)

$$\begin{aligned}
-p' &= \alpha+\beta+\gamma+3k = -p+3k\\
q' &= (\alpha+k)(\beta+k)+(\alpha+k)(\gamma+k)+(\beta+k)(\gamma+k)\\
&= \alpha\beta+(\alpha+\beta)k+k^2+\alpha\gamma+(\alpha+\gamma)k+k^2+\beta\gamma+(\beta+\gamma)k+k^2\\
&= q+2(-p)k+3k^2\\
-r' &= (\alpha+k)[\beta\gamma+(\beta+\gamma)k+k^2]\\
&= \alpha\beta\gamma+(\alpha\beta+\alpha\gamma)k+\alpha k^2+\beta\gamma k+(\beta+\gamma)k^2+k^3\\
&= -r+qk+(-p)k^2+k^3
\end{aligned}$$

Therefore, the equation is $x^3+(p-3k)x^2+(q-2pk+3k^2)x+r-qk+pk^2-k^3=0$.

(5)

$$\begin{aligned}
-p' &= \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma) = p^2-2q\\
q' &= \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = (\alpha\beta+\alpha\gamma+\beta\gamma)^2-2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2) = q^2-2(-r)(-p)\\
-r' &= \alpha^2\beta^2\gamma^2 = r^2
\end{aligned}$$

Therefore, the equation is $x^3+(2q-p^2)x^2+(q^2-2pr)x-r^2=0$.

(6)

$$\begin{aligned}
p' &= \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2}{\alpha^2\beta^2\gamma^2} = \frac{(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)}{r^2} = \frac{q^2-2(-r)(-p)}{r^2}\\
q' &= \frac{1}{\alpha^2\beta^2}+\frac{1}{\alpha^2\gamma^2}+\frac{1}{\beta^2\gamma^2} = \frac{\gamma^2+\beta^2+\alpha^2}{\alpha^2\beta^2\gamma^2} = \frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)}{r^2} = \frac{p^2-2q}{r^2}\\
r' &= \frac{1}{\alpha^2\beta^2\gamma^2} = \frac{1}{r^2}
\end{aligned}$$

Therefore, the equation is $x^3+\frac{q^2-2pr}{r^2}x^2+\frac{p^2-2q}{r^2}x+\frac{1}{r^2}=0$ or $r^2x^3+(q^2-2pr)x^2+(p^2-2q)x+1=0$.

Problem 10. If α, β, γ are the roots of $2x^3+x^2-4x+1=0$, find the values of

(1) $\alpha^2+\beta^2+\gamma^2$	(2) $\alpha^3+\beta^3+\gamma^3$
(3) $1/\beta\gamma+1/\gamma\alpha+1/\alpha\beta$	(4) $\alpha\beta^2+\beta\alpha^2+\beta\gamma^2+\gamma\beta^2+\gamma^2\alpha+\alpha^2\gamma$

Solution.
(1) We derive a general formula for $\alpha^2+\beta^2+\gamma^2$, starting with p^2:

$$\begin{aligned}
p^2 &= (\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)\\
&= \alpha^2+\beta^2+\gamma^2+2q
\end{aligned}$$

Hence $\boxed{\alpha^2+\beta^2+\gamma^2 = p^2-2q}$, and with $p=1/2$, $q=-2$, we get $\alpha^2+\beta^2+\gamma^2 = \frac{1}{4} = 4 = \frac{17}{4}$.

(2) First we derive a general formula for $\alpha^3+\beta^3+\gamma^3$.

$$\begin{aligned}
-p^3 = (\alpha+\beta+\gamma)^3 &= \alpha^3+\beta^3+\gamma^3+3\alpha^2\beta+3\alpha^2\gamma+3\beta^2\alpha+3\beta^3\gamma+3\gamma^2\alpha+3\gamma^2\beta+6\alpha\beta\gamma \\
&= \alpha^3+\beta^3+\gamma^3+3\alpha^2(\alpha+\beta+\gamma)-3\alpha^3+3\beta^2(\alpha+\beta+\gamma)-3\beta^3+3\gamma^2(\alpha+\beta+\gamma)-3\gamma^3-6r \\
&= \alpha^3+\beta^3+\gamma^3+3(\alpha^2+\beta^2+\gamma^2)(-p)-3(\alpha^3+\beta^3+\gamma^3)-6r \\
&= -2(\alpha^3+\beta^3+\gamma^3)-3p[(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)]-6r \\
&= -2(\alpha^3+\beta^3+\gamma^3)-3p[p^2-2q]-6r \\
&= -2(\alpha^3+\beta^3+\gamma^3)-3p^3+6pq-6r
\end{aligned}$$

Therefore, $2(\alpha^3+\beta^3+\gamma^3) = -2p^3+6(pq-r)$ or $\boxed{\alpha^3+\beta^3+\gamma^3 = -p^3+3(pq-r)}$. For this problem we have $p=1/2$, $q=-2$, $r=1/2$, so that $\alpha^3+\beta^3+\gamma^3 = -\frac{1}{8}+3\left(-1-\frac{1}{2}\right) = -\frac{37}{4}$.

(3)

$$\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r} = \frac{1/2}{1/2} = 1$$

(4)

$$\begin{aligned}
\alpha\beta^2+\beta\alpha^2+\beta\gamma^2+\gamma\beta^2+\gamma^2\alpha+\alpha^2\gamma &= \alpha\beta(\alpha+\beta)+\beta\gamma(\beta+\gamma)+\gamma\alpha(\gamma+\alpha) \\
&= \alpha\beta(\alpha+\beta+\gamma)-\alpha\beta\gamma+\beta\gamma(\alpha+\beta+\gamma)-\alpha\beta\gamma+\gamma\alpha(\alpha+\beta+\gamma)-\alpha\beta\gamma \\
&= (\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha+\beta+\gamma)-3\alpha\beta\gamma \\
&= -pq+3r \\
&= -\frac{1}{2}(-2)+3\frac{1}{2} = \frac{5}{2}.
\end{aligned}$$

Problem 11. If α, β, γ are the roots of $x^3-2x^2+x-3=0$, find the values of

(1) $\alpha/\beta\gamma+\beta/\gamma\alpha+\gamma/\alpha\beta$ (2) $\alpha\beta/\gamma+\beta\gamma/\alpha+\gamma\alpha/\beta$

(3) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$ (4) $(\beta^2+\gamma^2)(\gamma^2+\alpha^2)(\alpha^2+\beta^2)$

(5) $\alpha\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)+\beta\left(\frac{1}{\gamma}+\frac{1}{\alpha}\right)+\gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$

Solution. We have $p=-2$, $q=1$, $r=-3$, where

$$\begin{aligned}
-p &= \alpha+\beta+\gamma \\
q &= \alpha\beta+\alpha\gamma+\beta\gamma \\
-r &= \alpha\beta\gamma
\end{aligned}$$

(1)

$$\frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}+\frac{\gamma}{\alpha\beta} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma} = \frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)}{\alpha\beta\gamma} = \frac{p^2-2q}{-r} = \frac{4-2}{3} = \frac{2}{3}$$

(2)

$$\begin{aligned}
\frac{\alpha\beta}{\gamma}+\frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta} &= \frac{(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2}{\alpha\beta\gamma} = \frac{(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)}{\alpha\beta\gamma} = \frac{q^2-2(-r)(-p)}{-r} \\
&= \frac{q^2-2pr}{-r} = \frac{1-2(6)}{3} = -\frac{11}{3}
\end{aligned}$$

(3)

$$\begin{aligned}(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) &= (-p-\alpha)(-p-\beta)(-p-\gamma) = -(p+\alpha)(p+\beta)(p+\gamma)\\ &= -[p^3+(\alpha+\beta+\gamma)p^2+(\alpha\beta+\alpha\gamma+\beta\gamma)p+\alpha\beta\gamma]\\ &= -[p^3-p^3+pq-r] = r-pq = -3-(-2))(1) = -3+2 = -1\end{aligned}$$

(4) we make use of the relation

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma) = p^2-2q = 4-2 = 2$$

to write

$$\begin{aligned}(\beta^2+\gamma^2)(\gamma^2+\alpha^2)(\alpha^2+\beta^2) &= (2-\alpha^2)(2-\beta^2)(2-\gamma^2)\\ &= 8-4(\alpha^2+\beta^2+\gamma^2)+2(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)-\alpha^2\beta^2\gamma^2\\ &= 8-4(2)+2[(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)]-r^2\\ &= 2[q^2-2(-r)(-p)]-r^2\\ &= 2[q^2-2pr]-r^2 = 2[1-2(-2)(-3)]-9 = -31\end{aligned}$$

(5) We make use of the relation

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma} = \frac{q}{-r} = \frac{1}{3}$$

to write

$$\begin{aligned}\alpha\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)+\beta\left(\frac{1}{\gamma}+\frac{1}{\alpha}\right)+\gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) &= \alpha\left(\frac{1}{3}-\frac{1}{\alpha}\right)+\beta\left(\frac{1}{3}-\frac{1}{\beta}\right)+\gamma\left(\frac{1}{3}-\frac{1}{\gamma}\right)\\ &= \frac{\alpha}{3}-1+\frac{\beta}{3}-1+\frac{\gamma}{3}-1 = \frac{-p}{3}-3 = \frac{2}{3}-3 = -\frac{7}{3}.\end{aligned}$$

29.2 Theory of Equations, Exercise LXXII (p. 443)

Problem 1. Change the signs of the roots of $x^7+3x^4-2x^2+6x+7=0$.

Solution. This is effected by changing the sign of the coefficients of the odd power terms. Hence,

$$-x^7+3x^4-2x^2-6x+7=0 \implies x^7-3x^4+2x^2+6x-7=0$$

Problem 2. Multiply the roots of $2x^4+x^3-4x^2-6x+8=0$ by -2. Also divide them by 3.

Solution. This is effected by multiplying the first coefficient after the first by k, the next by k^2, and so on, taking into account missing terms as well. Thus, we have

$$2x^4+kx^3-4k^2x^2-6k^3x+8k^4=0 \implies 2x^4-2x^3-16x^2+48x+128=0 \text{ for } k=-2$$

$$\begin{aligned}2x^4+kx^3-4k^2x^2-6k^3x+8k^4=0 &\implies 2x^4+\frac{1}{3}x^3-\frac{4}{9}x^2-\frac{6}{27}x+\frac{8}{81}=0 \text{ for } k=\frac{1}{3}\\ &\implies 162x^4+27x^3-36x^2-18x+8=0\end{aligned}$$

Problem 3. In $5x^6-x^4+3x^3+9x+10=0$ replace each root by its reciprocal.

Solution. This is achieved by merely reversing the order of the coefficients. Hence, the equation is $10x^6+9x^5+3x^3-x^2+5=0$. To check on this, replace x with $1/x$ to get

$$5\frac{1}{x^6}-\frac{1}{x^4}+3\frac{1}{x^3}+9\frac{1}{x}+10=0 \implies 5-x^2+3x^3+9x^5+10x^6=0$$

Problem 4. Diminish the roots of $2x^5 + x^4 - 3x^2 + 6 = 0$ by 2. Also increase them by 1.

Solution. Synthetic division by 2 and by -1, respectively, gives

Diminish roots by 2: $2x^5 + 21x^4 + 88x^3 + 181x^2 + 180x + 74 = 0$

Increase roots by 1: $2x^5 - 9x^4 + 16x^3 - 17x^2 + 12x + 2 = 0$

Problem 5. Transform the equation $x^4 - x^3/3 + x^2/4 + x/25 - 1/48 = 0$ into another whose coefficients are integers, the leading one being 1.

Solution. This can be achieved by multiplying the roots by k, which transforms the equation into

$$x^4 - \frac{k}{3}x^3 + \frac{k^2}{4}x^2 + \frac{k^3}{25}x - \frac{k^4}{48} = 0,$$

and setting $k = 30$ gives

$$x^4 - 10x^3 + 225x^2 + 1080x - 16875 = 0.$$

Problem 6. Transform the equation $3x^4 - 36x^3 + x - 7 = 0$ into another which lacks the x^3 term.

Solution. Replacing x by $x + k$ gives $3(x + k)^4 - 36(x + k)^3 + x + k - 7 = 0$. Now let's work out the x^3 term: $3(4x^3k) - 36(x^3) = (12k - 36)x^3$. And this vanishes when $k = 3$. So we need to diminish the roots by 3, and this can be achieved by synthetic division to give

$$3x^4 - 162x^2 - 647x - 733 = 0$$

Problem 7. Transform the following into equations which lack the x term.

(1) $x^3 + 6x^2 + 9x + 10 = 0$ (2) $x^3 - x^2 - x - 3 = 0$

Solution.
(1) Replace x with $x + k$ to give

$$(x + k)^3 + 6(x + k)^2 + 9(x + k) + 10 = 0 \implies x^3 + 3kx^2 + 3k^2x + k^3 + 6(x^2 + 2kx + k^2) + 9(x + k) + 10 = 0$$

The x term is $(3k^2 + 12k + 9)x$ and vanishes when $k = \dfrac{-12 \pm \sqrt{144 - 4(3)(9)}}{6} = \dfrac{-12 \pm 6}{6} = -3, -1$. By synthetic division we find

$k = -3$: $x^3 - 3x^2 + 10 = 0$

$k = -1$: $x^3 + 3x^2 + 6 = 0$

(2) Again, replace x with $x + k$ to give

$$(x + k)^3 - (x + k)^2 - (x + k) - 3 = 0 \implies x^3 + 3kx^2 + 3k^2x + k^3 - (x^2 + 2kx + k^2) - (x + k) - 3 = 0$$

The x term is $(3k^2 - 2k - 1)x$ and vanishes when $k = \dfrac{2 \pm \sqrt{4 + 4(3)}}{6} = \dfrac{2 \pm 4}{6} = 1, -\dfrac{1}{3}$. By synthetic division we find

$k = 1$: $x^3 + 2x^2 - 4 = 0$

To avoid having to deal with fractions, we multiply the equation by 27 and set $y = 3x$, then instead of dividing by $x - 1/3$, we divide by $3x - 1$ or $y - 1$.

$$\begin{aligned} x^3 - x^2 - x - 3 = 0 &\implies 27x^3 - 27x^2 - 27x - 81 = 0 \\ &\implies (3x)^3 - 3(3x)^2 - 9(3x) - 81 = 0 \\ &\implies y^3 - 3y^2 - 9y - 81 = 0 \end{aligned}$$

and now synthetic division gives $y^3 - 6y^2 - 76 = 0 \implies 27x^3 - 54x^2 - 76 = 0$.

Problem 8. If the roots of $x^4 + x^3 - x + 2 = 0$ are α, β, γ, δ, find the equation whose roots are α^2, β^2, γ^2, δ^2.

Solution. Let the general equation be $x^4 + px^3 + qx^2 + rx + s = 0$, where

$$\begin{aligned}
-p &= \alpha + \beta + \gamma + \delta \\
q &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta \\
-r &= \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta \\
s &= \alpha\beta\gamma\delta
\end{aligned}$$

And for this particular case, $p = 1$, $q = 0$, $r = -1$, $s = 2$. And let $x^4 + p'x^3 + q'x^2 + r'x + s' = 0$ be the equation we are seeking, where

$$\begin{aligned}
-p' &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \\
&= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) \\
&= p^2 - 2q \qquad (29.1) \\
q' &= \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2 \\
&= (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^2 - 2(\alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha\beta^2\gamma + \alpha\beta^2\delta + \alpha\beta\gamma\delta + \alpha^2\gamma\delta + \alpha\beta\gamma^2 + \alpha\beta\gamma\delta + \\
&\qquad \alpha\gamma^2\delta + \alpha\beta\gamma\delta + \alpha\beta\delta^2 + \alpha\gamma\delta^2 + \beta^2\gamma\delta + \beta\gamma^2\delta + \beta\gamma\delta^2) \\
&= (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^2 - 2[(\alpha + \beta + \gamma + \delta)(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) - \alpha\beta\gamma\delta] \\
&= q^2 - 2[(-p)(-r) - s] \\
&= q^2 - 2pr + 2s \qquad (29.2) \\
-r' &= \alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 \\
&= (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)^2 - 2(\alpha^2\beta^2\gamma\delta + \alpha^2\beta\gamma^2\delta + \alpha\beta^2\gamma^2\delta + \alpha^2\beta\gamma\delta^2 + \alpha\beta^2\gamma\delta^2 + \alpha\beta\gamma^2\delta^2) \\
&= (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)^2 - 2\alpha\beta\gamma\delta(\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta) \\
&= r^2 - 2sq \qquad (29.3) \\
s' &= \alpha^2\beta^2\gamma^2\delta^2 \\
&= s^2 \qquad (29.4)
\end{aligned}$$

Thus, from eqs. (1)-(4), we have

$$p' = 2q - p^2, \qquad q' = q^2 - 2pr + 2s, \qquad r' = 2qs - r^2, \qquad s' = s^2$$

In general, if $x^4 + px^3 + qx^2 + rx + s = 0$ has four roots, α, β, γ, δ, then the equation that has the square of these roots, α^2, β^2, γ^2, δ^2, is

$$\boxed{x^4 + (2q - p^2)x^3 + (q^2 - 2pr + 2s)x^2 + (2qs - r^2)x + s^2 = 0}$$

For this particular problem, we get $x^4 - x^3 + 6x^2 - x + 4 = 0$.

Problem 9. If the roots of $x^4 + 3x^3 + 2x^2 - 1 = 0$ are α, β, γ, δ, find the equation whose roots are $\beta + \gamma + \delta$, $\alpha + \gamma + \delta$, $\alpha + \beta + \delta$, $\alpha + \beta + \gamma$.

Solution. Writing the given equation as $x^4 + px^3 + qx^2 + rx + s$, where $p = 3$, $q = 2$, $r = 0$, $s = -1$, and $x^4 + p'x^3 + q'x^2 + r'x + s' = 0$ as the equation to be found, we have

$$\begin{aligned}
-p &= \alpha + \beta + \gamma + \delta \\
q &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta \\
-r &= \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta \\
s &= \alpha\beta\gamma\delta
\end{aligned}$$

$$\begin{aligned}
-p' &= (-p-\alpha)+(-p-\beta)+(-p-\gamma)+(-p-\delta) = -4p-(\alpha+\beta+\gamma+\delta) = -4p+p = -3p\\
q' &= (p+\alpha)(p+\beta)+(p+\alpha)(p+\gamma)+(p+\alpha)(p+\delta)+(p+\beta)(p+\gamma)+(p+\beta)(p+\delta)+(p+\gamma)(p+\delta)\\
&= p^2+(\alpha+\beta)p+\alpha\beta+p^2+(\alpha+\gamma)p+\alpha\gamma+p^2+(\alpha+\delta)p+\alpha\delta+\\
&\quad p^2+(\beta+\gamma)p+\beta\gamma+p^2+(\beta+\delta)p+\beta\delta+p^2+(\gamma+\delta)p+\gamma\delta\\
&= 6p^2+(3\alpha+3\beta+3\gamma+3\delta)p+\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta\\
&= 6p^2+3(-p)p+q\\
&= 3p^2+q\\
-r &= -(p+\alpha)(p+\beta)(p+\gamma)-(p+\alpha)(p+\beta)(p+\delta)-(p+\alpha)(p+\gamma)(p+\delta)-(p+\beta)(p+\gamma)(p+\delta)\\
r &= p^3+(\alpha+\beta+\gamma)p^2+(\alpha\beta+\alpha\gamma+\beta\gamma)p+\alpha\beta\gamma+p^3+(\alpha+\beta+\delta)p^2+(\alpha\beta+\alpha\delta+\beta\delta)p+\alpha\beta\delta+\\
&\quad p^3+(\alpha+\gamma+\delta)p^2+(\alpha\gamma+\alpha\delta+\gamma\delta)p+\alpha\gamma\delta+p^3+(\beta+\gamma+\delta)p^2+(\beta\gamma+\beta\delta+\gamma\delta)p+\beta\gamma\delta\\
&= 4p^3+(3\alpha+3\beta+3\gamma+3\delta)p^2+(2\alpha\beta+2\alpha\gamma+2\alpha\delta+2\beta\gamma+2\beta\delta+2\gamma\delta)p+\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta\\
&= 4p^3+3(-p)p^2+2qp+(-r)\\
&= p^3+2qp-r\\
s' &= (p+\alpha)(p+\beta)(p+\gamma)(p+\delta)\\
&= p^4+(\alpha+\beta+\gamma+\delta)p^3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)p^2+(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)p+\alpha\beta\gamma\delta\\
&= p^4+(-p)p^3+qp^2-rp+s\\
&= qp^2-rp+s
\end{aligned}$$

In summary, $p' = 3p$, $q' = 3p^2+q$, $r = p^3+2qp-r$, $s' = qp^2-rp+s$, and for this particular equation, we have $p' = 9$, $q' = 29$, $r' = 39$, $s' = 17$, so the equation is $x^4+9x^3+29x^2+39x+17 = 0$.

Problem 10. If the roots of $x^3+px^2+qx+r=0$ are α, β, γ, find the equation whose roots are

$$(1)\ \frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta} \qquad\qquad (2)\ \frac{\alpha}{\beta+\gamma}, \frac{\beta}{\gamma+\alpha}, \frac{\gamma}{\alpha+\beta}.$$

Solution.
(1) We want the new roots to be $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}$, so we have

$$\begin{aligned}
-p' &= \frac{\alpha\beta}{\gamma}+\frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta} = \frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha\beta\gamma} = \frac{(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2(\alpha\beta^2\gamma+\alpha^2\beta\gamma+\beta\gamma^2\alpha)}{\alpha\beta\gamma}\\
&= \frac{q^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)}{\alpha\beta\gamma} = \frac{q^2-2(-r)(-p)}{-r} = \frac{2rp-q^2}{r}\\
q' &= \frac{\alpha\beta^2\gamma}{\gamma\alpha}+\frac{\alpha^2\beta\gamma}{\gamma\beta}+\frac{\alpha\beta\gamma^2}{\alpha\beta} = \alpha\beta\gamma\left(\frac{\beta^2}{\alpha\beta\gamma}+\frac{\alpha^2}{\alpha\beta\gamma}+\frac{\gamma^2}{\alpha\beta\gamma}\right) = \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)\\
&= p^2-2q\\
-r' &= \frac{\alpha^2\beta^2\gamma^2}{\alpha\beta\gamma} = \alpha\beta\gamma = -r
\end{aligned}$$

Thus, we have $p' = \frac{q^2-2rp}{r}$, $q' = p^2-2q$, $r' = r$, and so the equation is $x^3+\frac{q^2-2rp}{r}x^2+(p^2-2q)x+r = 0$ or

$$rx^3+(q^2-2rp)x^2+r(p^2-2q)x+r^2 = 0. \tag{29.1}$$

(2) We want the new roots to be $\dfrac{\alpha}{\beta+\gamma}=-\dfrac{\alpha}{p+\alpha}$, $\dfrac{\beta}{\gamma+\alpha}=-\dfrac{\beta}{p+\beta}$, $\dfrac{\gamma}{\alpha+\beta}=-\dfrac{\beta}{p+\beta}$, so we have

$$\begin{aligned}
p' &= \frac{\alpha}{p+\alpha}+\frac{\beta}{p+\beta}+\frac{\gamma}{p+\gamma}=\frac{\alpha(p+\beta)(p+\gamma)+\beta(p+\alpha)(p+\gamma)+\gamma(p+\alpha)(p+\beta)}{(p+\alpha)(p+\beta)(p+\gamma)}\\
&= \frac{\alpha[p^2+(\beta+\gamma)p+\beta\gamma]+\beta[p^2+(\alpha+\gamma)p+\alpha\gamma]+\gamma[p^2+(\alpha+\beta)p+\alpha\beta]}{p^3+(\alpha+\beta+\gamma)p^2+(\alpha\beta+\alpha\gamma+\beta\gamma)p+\alpha\beta\gamma}\\
&= \frac{(\alpha+\beta+\gamma)p^2+[\alpha\beta+\alpha\gamma+\alpha\beta+\beta\gamma+\alpha\gamma+\beta\gamma]p+3\alpha\beta\gamma}{p^3+(-p)p^2+pq-r}=\frac{(-p)p^2+2pq+3(-r)}{pq-r}=\frac{-p^3+2pq-3r}{pq-r}\\
q' &= \frac{\alpha\beta}{(p+\alpha)(p+\beta)}+\frac{\alpha\gamma}{(p+\alpha)(p+\gamma)}+\frac{\beta\gamma}{(p+\beta)(p+\gamma)}=\frac{\alpha\beta(p+\gamma)+\alpha\gamma(p+\beta)+\beta\gamma(p+\alpha)}{(p+\alpha)(p+\beta)(p+\gamma)}\\
&= \frac{(\alpha\beta+\alpha\gamma+\beta\gamma)p+3\alpha\beta\gamma}{pq-r}=\frac{pq-3r}{pq-r}\\
r' &= \frac{\alpha\beta\gamma}{(p+\alpha)(p+\beta)(p+\gamma)}=\frac{-r}{pq-r}
\end{aligned}$$

Therefore, the transformed equation is $x^3+\dfrac{-p^3+2pq-3r}{pq-r}x^2+\dfrac{pq-3r}{pq-r}x+\dfrac{-r}{pq-r}=0$, or, clearing of fractions,

$$(pq-r)x^3-(p^3-2pq+3r)x^2+(pq-3r)x-r=0 \tag{29.2}$$

Problem 11. If the roots of $x^3+2x^2+3x+4=0$ are α, β, γ, find the equation whose roots are

(1) $\beta^2+\gamma^2$, $\gamma^2+\alpha^2$, $\alpha^2+\beta^2$. (2) $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$.

(3) $\beta\gamma+\dfrac{1}{\alpha}$, $\gamma\alpha+\dfrac{1}{\beta}$, $\alpha\beta+\dfrac{1}{\beta}$. (4) $\dfrac{\alpha}{\beta+\gamma-\alpha}$, $\dfrac{\beta}{\gamma+\alpha-\beta}$, $\dfrac{\gamma}{\alpha+\beta-\gamma}$.

(5) $\alpha\left(\dfrac{1}{\beta}+\dfrac{1}{\gamma}\right)$, $\beta\left(\dfrac{1}{\gamma}+\dfrac{1}{\alpha}\right)$, $\gamma\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)$.

Solution.
(1) Roots: $\beta^2+\gamma^2$, $\gamma^2+\alpha^2$, $\alpha^2+\beta^2$.

$$\begin{aligned}
-p' &= \beta^2+\gamma^2+\gamma^2+\alpha^2+\alpha^2+\beta^2=2(\alpha^2+\beta^2+\gamma^2)=2(p^2-2q)=2p^2-4q\\
q' &= (\beta^2+\gamma^2)(\gamma^2+\alpha^2)+(\beta^2+\gamma^2)(\alpha^2+\beta^2)+(\gamma^2+\alpha^2)(\alpha^2+\beta^2)=3(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)+\alpha^4+\beta^4+\gamma^4\\
&= 3[(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)]+(\alpha^2+\beta^2+\gamma^2)^2-2(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)\\
&= 3[q^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)]+[(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)]^2-2[(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)]\\
&= 3[q^2-2(-r)(-p)]+(p^2-2q)^2-2[q^2-2(-r)(-p)]\\
&= 3[q^2-2pr]+(p^2-2q)^2-2[q^2-2pr]=3q^2-6pr+p^4-4p^2q+4q^2-2q^2+4pr\\
&= p^4-4p^2q+5q^2-2pr\\
-r' &= (\beta^2+\gamma^2)(\gamma^2+\alpha^2)(\alpha^2+\beta^2)=(p^2-2q-\alpha^2)(p^2-2q-\beta^2)(p^2-2q-\gamma^2)\\
&= (p^2-2q)^3+[-\alpha^2-\beta^2-\gamma^2](p^2-2q)^2+[\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2](p^2-2q)-\alpha^2\beta^2\gamma^2\\
&= (p^2-2q)^3-(p^2-2q)(p^2-2q)^2+[q^2-2pr](p^2-2q)-r^2=(q^2-2pr)(p^2-2q)-r^2
\end{aligned}$$

In summary, $p'=4q-2p^2$, $q'=p^4-4p^2q+5q^2-2pr$, $r'=r^2-(q^2-2pr)(p^2-2q)$, and the transformed equation is

$$x^3+(4q-2p^2)x^2+(p^4-4p^2q+5q^2-2pr)x+r^2-(q^2-2pr)(p^2-2q)=0. \tag{29.1}$$

Since the coefficients of the given equation are $p=2$, $q=3$, $r=4$, eq. (1) becomes $x^3+4x^2-3x+2=0$.

(2) Roots: $\alpha(\beta+\gamma) = -\alpha(p+\alpha)$, $\beta(\gamma+\alpha) = -\beta(p+\beta)$, $\gamma(\alpha+\beta) = -\gamma(p+\gamma)$.

$$\begin{aligned}
p' &= \alpha(p+\alpha) + \beta(p+\beta) + \gamma(p+\gamma) = (\alpha+\beta+\gamma)p + \alpha^2+\beta^2+\gamma^2 \\
&= (-p)p + (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma) = -p^2+p^2-2q = -2q \\
q' &= \alpha\beta(p+\alpha)(p+\beta) + \alpha\gamma(p+\alpha)(p+\gamma) + \beta\gamma(p+\beta)(p+\gamma) \\
&= \alpha\beta[p^2+(\alpha+\beta)p+\alpha\beta] + \alpha\gamma[p^2+(\alpha+\gamma)p+\alpha\gamma] + \beta\gamma[p^2+(\beta+\gamma)p+\beta\gamma] \\
&= qp^2 + [\alpha\beta(-p-\gamma) + \alpha\gamma(-p-\beta) + \beta\gamma(-p-\alpha)]p + \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 \\
&= qp^2 + [-qp-3(-r)]p + (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2) \\
&= qp^2 - qp^2 + 3rp + q^2 - 2(-r)(-p) = 3pr + q^2 - 2pr = q^2 + pr \\
r' &= \alpha\beta\gamma[p^3 + (\alpha+\beta+\gamma)p^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)p + \alpha\beta\gamma] = -r[p^3+(-p)p^2+qp-r] = r^2 - pqr
\end{aligned}$$

Thus, the transformed equation is

$$x^3 - 2qx^2 + (q^2+pr)x + r^2 - pqr = 0 \tag{29.2}$$

Using the coefficients $p=2$, $q=3$, $r=4$, the equation becomes $x^3-6x^2+17x-8=0$.

(3) Roots: $\beta\gamma + \dfrac{1}{\alpha} = \dfrac{\alpha\beta\gamma+1}{\alpha} = \dfrac{1-r}{\alpha}$, $\gamma\alpha + \dfrac{1}{\beta} = \dfrac{\alpha\beta\gamma+1}{\beta} = \dfrac{1-r}{\beta}$, $\alpha\beta + \dfrac{1}{\gamma} = \dfrac{\alpha\beta\gamma+1}{\gamma} = \dfrac{1-r}{\gamma}$

$$\begin{aligned}
-p' &= \frac{1-r}{\alpha} + \frac{1-r}{\beta} + \frac{1-r}{\gamma} = \frac{\beta\gamma(1-r)+\alpha\gamma(1-r)+\alpha\beta(1-r)}{\alpha\beta\gamma} = \frac{q(1-r)}{-r} \\
q' &= \frac{(1-r)^2}{\alpha\beta} + \frac{(1-r)^2}{\alpha\gamma} + \frac{(1-r)^2}{\beta\gamma} = \frac{(1-r)^2}{\alpha\beta\gamma}(\alpha+\beta+\gamma) = \frac{(1-r)^2(-p)}{-r} = \frac{(1-r)^2p}{r} \\
-r' &= \frac{(1-r)^3}{\alpha\beta\gamma} = -\frac{(1-r)^3}{r}
\end{aligned}$$

Thus, the transformed equation is $x^3 + \dfrac{(1-r)q}{r}x^2 + \dfrac{(1-r)^2p}{r}x + \dfrac{(1-r)^3}{r} = 0$, or, clearing of fractions,

$$rx^3 + (1-r)qx^2 + (1-r)^2px + (1-r)^3 = 0 \tag{29.3}$$

Using the coefficients $p=2$, $q=3$, $r=4$, the equation becomes $4x^3-9x^2+18x-27=0$.

(4) Roots: $\dfrac{\alpha}{\beta+\gamma-\alpha} = -\dfrac{\alpha}{p+2\alpha}$, $\dfrac{\beta}{\gamma+\alpha-\beta} = -\dfrac{\beta}{p+2\beta}$, $\dfrac{\gamma}{\alpha+\beta-\gamma} = -\dfrac{\gamma}{p+2\gamma}$.

$$\begin{aligned}
p' &= \frac{\alpha}{p+2\alpha} + \frac{\beta}{p+2\beta} + \frac{\gamma}{p+2\gamma} = \frac{\alpha(p+2\beta)(p+2\gamma)+\beta(p+2\alpha)(p+2\gamma)+\gamma(p+2\alpha)(p+2\beta)}{(p+2\alpha)(p+2\beta)(p+2\gamma)} \\
&= \frac{\alpha[p^2+2(\beta+\gamma)p+4\beta\gamma] + \beta[p^2+2(\alpha+\gamma)p+4\alpha\gamma] + \gamma[p^2+2(\alpha+\beta)+4\alpha\beta]}{p^3+2(\alpha+\beta+\gamma)p^2+4(\alpha\beta+\alpha\gamma+\beta\gamma)p+8\alpha\beta\gamma} \\
&= \frac{(-p)p^2+2(\alpha\beta+\alpha\gamma+\alpha\beta+\beta\gamma+\alpha\gamma+\beta\gamma)p+12\alpha\beta\gamma}{p^3+2(-p)p^2+4qp-8r} = \frac{-p^3+4qp-12r}{-p^3+4pq-8r} = \frac{p^3-4pq+12r}{p^3-4pq+8r} \\
q' &= \frac{\alpha}{p+2\alpha}\frac{\beta}{p+2\beta} + \frac{\alpha}{p+2\alpha}\frac{\gamma}{p+2\gamma} + \frac{\beta}{p+2\beta}\frac{\gamma}{p+2\gamma} = \frac{\alpha\beta(p+2\gamma)+\alpha\gamma(p+2\beta)+\beta\gamma(p+2\alpha)}{-p^3+4pq-8r} \\
&= \frac{pq-6r}{-p^3+4pq-8r} = \frac{6r-pq}{p^3-4pq+8r} \\
r' &= \frac{\alpha}{p+2\alpha}\frac{\beta}{p+2\beta}\frac{\gamma}{p+2\gamma} = \frac{\alpha\beta\gamma}{-p^3+4pq-8r} = \frac{-r}{-p^3+4pq-8r} = \frac{r}{p^3-4pq+8r}
\end{aligned}$$

Thus, the transformed equation is $x^3 + \dfrac{p^3-4pq+12r}{p^3-4pq+8r}x^2 + \dfrac{6r-pq}{p^3-4pq+8r}x + \dfrac{r}{p^3-4pq+8r} = 0$, or, clearing of fractions,

$$(p^3-4pq+8r)x^3 + (p^3-4pq+12r)x^2 + (6r-pq)x + r = 0 \tag{29.4}$$

Substituting the values $p = 2$, $q = 3$, $r = 4$, and simplifying, we get $8x3 + 16x^2 + 9x + 2 = 0$.

(5) Roots:

$$\alpha\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)=\alpha\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)-1=\alpha\left(\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}\right)-1=\frac{\alpha q}{-r}-1=\frac{\alpha q+r}{-r}=-\frac{\alpha q+r}{r}$$
$$\beta\left(\frac{1}{\gamma}+\frac{1}{\alpha}\right)=\beta\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)-1=\beta\left(\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}\right)-1=\frac{\beta q}{-r}-1=\frac{\beta q+r}{-r}=-\frac{\beta q+r}{r}$$
$$\gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=\gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)-1=\gamma\left(\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}\right)-1=\frac{\gamma q}{-r}-1=\frac{\gamma q+r}{-r}=-\frac{\gamma q+r}{r}$$

$$\begin{aligned}
p' &= \frac{\alpha q+r}{r}+\frac{\beta q+r}{r}+\frac{\gamma q+r}{r}=\frac{(\alpha+\beta+\gamma)q+3r}{r}=\frac{-pq+3r}{r}\\
q' &= \frac{\alpha q+r}{r}\frac{\beta q+r}{r}+\frac{\alpha q+r}{r}\frac{\gamma q+r}{r}+\frac{\beta q+r}{r}\frac{\gamma q+r}{r}\\
&= \frac{\alpha\beta q^2+(\alpha+\beta)qr+r^2+\alpha\gamma q^2+(\alpha+\gamma)qr+r^2+\beta\gamma q^2+(\beta+\gamma)qr+r^2}{r^2}\\
&= \frac{(\alpha\beta+\alpha\gamma+\beta\gamma)q^2+2(\alpha+\beta+\gamma)qr+3r^2}{r^2}=\frac{q^3+2(-p)qr+3r^2}{r^2}=\frac{q^3-2pqr+3r^2}{r^2}\\
r' &= \frac{\alpha q+r}{r}\frac{\beta q+r}{r}\frac{\gamma q+r}{r}=\frac{\alpha\beta\gamma q^3+(\alpha\beta q^2+\alpha\gamma q^2+\beta\gamma q^2)r+(\alpha q+\beta q+\gamma q)r^2+r^3}{r^3}\\
&= \frac{-rq^3+q^3r-pqr^2+r^3}{r^3}=\frac{r^3-pqr^2}{r^3}=\frac{r-pq}{r}
\end{aligned}$$

Thus, the transformed equation is $x^3+\frac{3r-pq}{r}x^2+\frac{q^3-2pqr+3r^2}{r^2}x+\frac{r-pq}{r}=0$, or, clearing of fractions,

$$r^2x^3+(3r^2-pqr)x^2+(q^3-2pqr+3r^2)x+r^2-pqr=0. \tag{29.5}$$

Substituting the valuse $p = 2$, $q = 3$, $r = 4$, this becomes $16x^3 + 24x^2 + 27x - 8 = 0$.

Chapter 30

Cubic and Biquadratic Equations

Method to transform a given equation into another which lacks some particular power of x:

Set $x = y + k$, where k is a constant chosen to remove a particular power of y.

Theorem: In any quadratic of the form $x^2 + px + q$, the coefficient of x with its sign changed is equal to the sum of the roots, and the constant term is equal to the product of the roots. For if α and β are the roots of the quadratic, then $\boldsymbol{(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta = 0}$.

Cube Roots of Unity. These are the solutions to the equation $x^3 - 1 = 0$. Since $x^3 - 1 = (x-1)(x^2+x+1) = 0$, the equation is equivalent to the two equations $x - 1 = 0$ and $x^2 + x + 1 = 0$, which gives the solutions 1, $\omega \equiv \dfrac{-1+\sqrt{3}i}{2}$, and $\omega^2 = \dfrac{-1-\sqrt{3}i}{2}$. Since there is no x^2 term in this equation, it follows that $1 + \omega + \omega^2 = 0$.

The General Cubic—Cardan's Formula

Perform a translation via $x = y + k$ to eliminate the x^2 power, which means that every cubic can be reduced to the form

$$x^3 + px + q = 0. \tag{30.1}$$

In eq. (1) put

$$x = y + z \tag{30.2}$$

to get

$$y^3 + z^3 + (3yz + p)(y + z) + q = 0. \tag{30.3}$$

We now impose a second condition on the two variables y and z, namely,

$$3yz + p = 0. \tag{30.4}$$

And therefore, eq. (3) becomes

$$y^3 + z^3 + q = 0. \tag{30.5}$$

From eqs. (5) and (4), we have

$$y^3 + z^3 = -q \tag{30.6}$$

$$y^3 z^3 = -\frac{p^3}{27} \tag{30.7}$$

Therefore, by the above quadratic theorem, y^3 and z^3 are the roots of a quadratic equation of the form

$$u^2 + qu - \frac{p^3}{27} = 0. \tag{30.8}$$

Solving eq. (8) and representing the expressions obtained for the roots by A and B respectively, we have

$$y^3 = A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \quad \text{and} \quad z^3 = B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}. \tag{30.9}$$

These eqs. (9) give three values for y and three for z, namely,

$$y = \sqrt[3]{A}, \quad \omega\sqrt[3]{A}, \quad \omega^2\sqrt[3]{A}, \quad \text{and} \quad z = \sqrt[3]{B}, \quad \omega\sqrt[3]{B}, \quad \omega^2\sqrt[3]{B}. \tag{30.10}$$

However, by eq. (4), $yz = -p/3$, and the only pairs of the values of (y, z) in eq. (10) that satisfy this condition are

$$(y, z) = (\sqrt[3]{A}, \sqrt[3]{B}); \quad (\omega\sqrt[3]{A}, \omega^2\sqrt[3]{B}); \quad (\omega^2\sqrt[3]{A}, \omega\sqrt[3]{B}). \tag{30.11}$$

Thus, the three roots of eq. (1) are

$$x_1 = \sqrt[3]{A} + \sqrt[3]{B}, \quad x_2 = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}, \quad x_3 = \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}, \tag{30.12}$$

$$\text{where} \quad A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \quad \text{and} \quad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}. \tag{30.13}$$

Nature of the Roots

When p and q are real, the nature of the roots depends on the value of the *discriminant* $q^2/4 + p^3/27$ as follows:

- If $q^2/4 + p^3/27 > 0$, one root is real, two are imaginary (x_1 is real, x_2 and x_3 are conjugate imaginary.
- If $q^2/4 + p^3/27 = 0$, all the roots are real, and two are equal ($x_1 = -2\sqrt[3]{q/2}$, $x_2 = x_3 = \sqrt[3]{q/2}$).
- If $q^2/4 + p^3/27 < 0$, all the roots are real and unequal. Although the expressions for x_1, x_2, x_3 denote real numbers, they cannot be reduced to a real form by algebraic transformations. Hence, this case is called *irreducible*.

Irreducible Case of Cubic

Without loss of generality, the general cubic can be written as

$$x^3 + ax^2 + bx + c = 0. \tag{30.1}$$

Setting $x = y - a/3$ eliminates the quadratic term and puts it in the form

$$y^3 + py + q = 0 \tag{30.2}$$

with

$$p = b - \frac{a^2}{3} \text{ and } q = c - \frac{ab}{3} + \frac{2a^3}{27}. \tag{30.3}$$

Starting with the expansion of $\cos(3\theta)$ and making use of the addition formulas for the sine and cosine, it is straightforward to derive the trig identity

$$4\cos^3\theta - 3\cos\theta - \cos(3\theta) = 0. \tag{30.4}$$

Now return to eq. (2) to see if we can transform it into this form. Begin by setting $x = t\cos\theta$, which gives

$$t^3\cos^3\theta + pt\cos\theta + q = 0 \tag{30.5}$$

Next, multiply through by $4/t^3$ to get

$$4\cos^3\theta + \frac{4p}{t^2}\cos\theta + \frac{4q}{t^3} = 0. \tag{30.6}$$

Choosing $t = \sqrt{-4p/3}$ then gives

$$4\cos^3\theta - 3\cos\theta - \frac{3q}{p}\sqrt{-\frac{3}{4p}} = 0. \tag{30.7}$$

Now we see that if we choose θ such that

$$\cos(3\theta) = \frac{3q}{p}\sqrt{-\frac{3}{4p}}, \tag{30.8}$$

then the cubic is automatically satisfies, guaranteed by the trig identity eq. (4). Therefore, we set

$$3\theta = \cos^{-1}\left(\frac{3q}{p}\sqrt{-\frac{3}{4p}}\right) + 2\pi k \quad \text{where} \quad k = 0,\ 1,\ 2, \tag{30.9}$$

and the solutions for x are

$$\boxed{x_k = \sqrt{-\frac{4p}{3}}\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3q}{p}\sqrt{-\frac{3}{4p}}\right) + k\frac{2\pi}{3}\right] \quad \text{for} \quad k = 0,\ 1,\ 2}. \tag{30.10}$$

This provides the solution to the cubic when the discriminant $q^2/4 + p^3/27 < 0$.

The General Quartic—Ferrari's Solution

With the aid of the transformation $x = y + k$ every quartic can be reduced to the form

$$x^4 + ax^2 + bx + c = 0. \tag{30.1}$$

With a view to transforming this into a difference of two squares, add and subtract $xu + u^2/4$, where u denotes a constant to be found. This gives

$$\begin{aligned}
x^4 + x^2u + \frac{u^2}{4} - x^2u - \frac{u^2}{4} + ax^2 + bx + c &= 0\\
\left(x^2 + \frac{u}{2}\right)^2 - (u-a)x^2 + bx + c - \frac{u^2}{4} &= 0\\
\left(x^2 + \frac{u}{2}\right)^2 - (u-a)\left(x^2 - \frac{b}{u-a}x\right) + c - \frac{u^2}{4} &= 0\\
\left(x^2 + \frac{u}{2}\right)^2 - (u-a)\left(x - \frac{b}{2(u-a)}\right)^2 + \frac{b^2}{4(u-a)} + c - \frac{u^2}{4} &= 0 \qquad (30.2)
\end{aligned}$$

We require $b^2 = 4(u-a)\left(\frac{u^2}{4} - c\right) = u^3 - au^2 - 4cu + 4ac$, which gives a cubic for u:

$$u^3 - au^2 - 4cu + (4ac - b^2) = 0. \tag{30.3}$$

Let u_1 be a particular solution of this cubic, then eq. (2) reduces to

$$x^2 + \frac{u}{2} = \pm\sqrt{u_1 - a}\left(x - \frac{b}{2(u_1 - a)}\right),$$

which results in two quadratics:

$$x^2 - \sqrt{u_1 - a}\,x + \left(\frac{u_1}{2} + \frac{b}{2\sqrt{u_1 - a}}\right) = 0, \tag{30.4}$$

$$x^2 + \sqrt{u_1 - a}\,x + \left(\frac{u_1}{2} - \frac{b}{2\sqrt{u_1 - a}}\right) = 0. \tag{30.5}$$

30.1 Biquadratic Equation, Example 2 (p. 486)

Solve the equation $x^4 - 4x^3 + x^2 + 4x + 1 = 0$.

Solution. We try the product of two quadratics.

$$x^4 - 4x^3 + x^2 + 4x + 1 = (x^2 + ax + b)(x^2 + cx + d) = x^4 + (c+a)x^3 + (d + ac + b)x^2 + (ad + bc)x + bd$$

This requires $c + a = -4$, $d + ac + b = 1$, $ad + bc = 4$, and $bd = 1$, and we find $a = -3$, $b = -1$, $c = -1$, $d = -1$. Therefore,

$$x^4 - 4x^3 + x^2 + 4x + 1 = 0 \implies (x^2 - 3x - 1)(x^2 - x - 1) = 0,$$

and the solutions are $x = \dfrac{3 \pm \sqrt{13}}{2}, \dfrac{1 \pm \sqrt{5}}{2}$.

30.2 Cubic and Biquadratic Equations, Exercise LXXX (p. 491)

Solve equations 1-10 by the methods of §871 and §874.

Problem 1. $x^3 - 9x - 28 = 0$

Solution. This equation is already in reduced form with $p = -9$ and $q = -28$. Thus,

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = 14 + \sqrt{(14)^2 + \frac{(-9)^3}{27}} = 14 + \sqrt{(14)^2 - 27} = 14 + 13 = 27 \text{ and } B = 14 - 13 = 1.$$

Thus, the solutions are

$$x_1 = \sqrt[3]{A} + \sqrt[3]{B} = 3 + 1 = 4$$

$$x_2 = \omega\sqrt[3]{A} + \sqrt[3]{B}\omega^2 = \omega \cdot 3 + \omega^2 \cdot 1 = \left(\frac{-1+\sqrt{3}i}{2}\right)3 + \frac{-1-\sqrt{3}i}{2} = -2 + \sqrt{3}i$$

$$x_3 = \omega^2\sqrt[3]{A} + \sqrt[3]{B}\omega = \omega^2 \cdot 3 + \omega \cdot 1 = \left(\frac{-1-\sqrt{3}i}{2}\right)3 + \frac{-1+\sqrt{3}i}{2} = -2 - \sqrt{3}i$$

Problem 2. $x^3 - 9x^2 + 9x - 8 = 0$

Solution. Let $x = y + k$. Then we have

$$\begin{aligned} x^3 - 9x^2 + 9x - 8 = 0 &\implies y^3 + 3ky^2 + 3k^2y + k^3 - 9y^2 - 18ky - 9k^2 + 9y + 9k - 8 = 0 \\ &\implies y^3 + (3k-9)y^2 + (3k^2 - 18k + 9)y + k^3 - 9k^2 + 9k - 8 = 0 \end{aligned}$$

Setting $k = 3$ eliminates the y^2 term and we get $y^3 - 18y - 35 = 0$. Thus, $p = -18$ and $q = -35$. The discriminant is

$$\frac{q^2}{4} + \frac{p^3}{27} = \frac{(-35)^2}{4} + \frac{(-18)^3}{27} = \frac{361}{4} = \left(\frac{19}{2}\right)^2 > 0 \quad\implies\quad \text{one real and two complex roots.}$$

We have

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = \frac{35}{2} + \frac{19}{2} = 27 \quad \text{and} \quad A = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = \frac{35}{2} - \frac{19}{2} = 8,$$

which gives

$$\begin{aligned} y_1 &= \sqrt[3]{A} + \sqrt[3]{B} = 3 + 2 = 5, \\ y_2 &= 3\omega + 2\omega^2 = 3\left(\frac{-1+\sqrt{3}i}{2}\right) + 2\left(\frac{-1-\sqrt{3}i}{2}\right) = \frac{-5+\sqrt{3}i}{2}, \\ y_3 &= 3\omega + 2\omega^2 = 3\left(\frac{-1-\sqrt{3}i}{2}\right) + 2\left(\frac{-1+\sqrt{3}i}{2}\right) = \frac{-5-\sqrt{3}i}{2}. \end{aligned}$$

Notice that $y_1 + y_2 + y_3 = 0$, as they must since the cubic lacks a y^2 term. Finally, $x_1 = 8$, $x_2 = (1+\sqrt{3}i)/2$, $x_3 = (1-\sqrt{3}i)/2$.

Problem 3. $x^3 - 3x - 4 = 0$

Solution. Here $p = -3$ and $q = -4$. The discriminant is $\frac{q^2}{4} + \frac{p^3}{27} = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = 4 - 1 = 3 > 0$, so there is one real root and two complex roots. We have

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = 2 + \sqrt{3} \quad \text{and} \quad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = 2 - \sqrt{3}$$

And the solutions are $x_1 = \sqrt[3]{A} + \sqrt[3]{B}$, $x_2 = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}$, and $x_2 = \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}$, where $\omega = (-1+\sqrt{3}i)/2$ and $\omega^2 = (-1-\sqrt{3}i)/2$.

Problem 4. $4x^3 - 7x - 6 = 0$

Solution. $4x^3 - 7x - 6 = 0 \implies x^3 - \frac{7}{4}x - \frac{3}{2} = 0$ so that $p = -\frac{7}{4}$ and $q = -\frac{3}{2}$. The discriminant is

$$\frac{q^2}{4} + \frac{p^3}{27} = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \left(-\frac{3}{4}\right)^2 + \left(-\frac{7}{12}\right)^3 = \frac{629}{1728} > 0 \implies \text{ one real root and two complex roots}$$

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = \frac{3}{4} + \sqrt{\frac{629}{1728}} = \frac{3}{4} + \frac{\sqrt{1887}}{72} \text{ and } B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = \frac{3}{4} - \sqrt{\frac{629}{1728}} = \frac{3}{4} - \frac{\sqrt{1887}}{72}$$

The solutions are $x_1 = \sqrt[3]{A} + \sqrt[3]{B}$, $x_2 = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}$, and $x_2 = \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}$, where $\omega = (-1 + \sqrt{3}i)/2$ and $\omega^2 = (-1 - \sqrt{3}i)/2$.

Problem 5. $x^3 + 3x^2 + 9x - 1 = 0$

Solution. Let $x = y + k$. Then

$$\begin{aligned} x^3 + 3x^2 + 9x - 1 = 0 &\implies y^3 + 3ky^2 + 3k^2y + k^3 + 3y^2 + 6ky + 3k^2 + 9y + 9k - 1 = 0 \\ &\implies y^3 + (3k+3)y^2 + (3k^2 + 6k + 9)y + k^3 + 3k^2 + 9k - 1 = 0 \end{aligned}$$

Setting $k = -1$ eliminates the y^2 term and reduces the cubic to $y^3 + 6y - 8 = 0$. Thus, with $p = 6$ and $q = -8$, the discriminant is

$$\frac{q^2}{4} + \frac{p^3}{27} = (\frac{q}{2})^2 + (\frac{p}{3})^3 = (-4)^2 + (2)^3 = 16 + 8 = 24.$$

And since this is positive, we have one real root and two complex roots. With

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = 4 + \sqrt{24} = 4 + 2\sqrt{6} \quad \text{and} \quad B = 4 - 2\sqrt{6},$$

the solutions are $\sqrt[3]{A} + \sqrt[3]{B}$, $\omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}$, $\omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}$.

Problem 6. $3x^3 - 9x^2 + 14x + 7 = 0$

Solution. To eliminate the x^2 term and transform this cubic into standard form, set $x = y + k$ to get

$$\begin{aligned} 3x^3 - 9x^2 + 14x + 7 = 0 &\implies 3(y^3 + 3ky^2 + 3k^2y + k^3) - 9(y^2 + 2ky + k^2) + 14(y+k) - 7 = 0 \\ &\implies 3y^3 + (9k - 9)y^2 + (9k^2 - 18k + 14)y + 3k^3 - 9k^2 + 14k + 7 = 0 \\ &\implies 3y^3 + 5y + 15 = 0 \text{ after setting } k = 1 \\ &\implies y^3 + \frac{5}{3}y + 5 = 0 \end{aligned}$$

Thus, we have $p = \frac{5}{3}$ and $q = 5$. The discriminant is

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \left(\frac{5}{2}\right)^2 + \left(\frac{5}{9}\right)^3 = \frac{25}{4} + \frac{125}{729} = 25\left(\frac{1}{4} + \frac{5}{729}\right) = \left(\frac{5}{54}\right)^2 749$$

And since this is positive, the given equation has one real root and two complex roots. With

$$A = -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} = -\frac{5}{2} + \frac{5\sqrt{749}}{54} \quad \text{and} \quad B = -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} = -\frac{5}{2} - \frac{5\sqrt{749}}{54},$$

the solutions are $1 + \sqrt[3]{A} + \sqrt[3]{B}$, $1 + \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}$, $1 + \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}$.

Problem 7. $x^4 + x^2 + 6x + 1 = 0$

Solution. This is already in reduced form without an x^3 term. Here $a = 1$, $b = 6$, $c = 1$. This gives for the cubic

$$u^3 - au^2 - 4cu + (4ac - b^2) = 0 \implies u^3 - u^2 - 4u - 32 = 0 \implies (u-4)(u^2 + 3u + 8) = 0,$$

so that $u_1 \equiv 4$ is a particular solution. With this, the quartic splits into two quadratic equations:

$$x^2 + \sqrt{u_1 - a}\,x + \left(\frac{u_1}{2} - \frac{b}{2\sqrt{u_1 - a}}\right) = 0 \implies x^2 + \sqrt{3}\,x + (2 - \sqrt{3}) = 0 \quad \text{and}$$
$$x^2 - \sqrt{u_1 - a}\,x + \left(\frac{u_1}{2} + \frac{b}{2\sqrt{u_1 - a}}\right) = 0 \implies x^2 - \sqrt{3}\,x + (2 + \sqrt{3}) = 0$$

Thus, the solutions are

$$x = \frac{-\sqrt{3} \pm \sqrt{3 - 4(2 - \sqrt{3})}}{2} = \frac{-\sqrt{3} \pm \sqrt{4\sqrt{3} - 5}}{2}$$
$$x = \frac{\sqrt{3} \pm \sqrt{3 - 4(2 + \sqrt{3})}}{2} = \frac{\sqrt{3} \pm i\sqrt{4\sqrt{3} + 5}}{2}$$

Problem 8. $x^4 - 4x^3 + x^2 + 4x + 1 = 0$

Solution. Since $x = 0$ is not a solution, we can divide through by x^2 to get

$$x^2 - 4x + 1 + \frac{4}{x} + \frac{1}{x^2} = 0 \implies \left(x^2 + \frac{1}{x^2}\right) - 4\left(x - \frac{1}{x}\right) + 1 = 0 \implies \left(x - \frac{1}{x}\right)^2 + 2 - 4\left(x - \frac{1}{x}\right) + 1 = 0$$
$$\implies \left(x - \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) + 3 = 0,$$

which gives $x - \dfrac{1}{x} = \dfrac{4 \pm \sqrt{16 - 4(3)}}{2}$ or $x - \dfrac{1}{x} = 3,\ 1$, resulting in two quadratics in x:

$$x^2 - 3x - 1 = 0 \quad \text{and} \quad x^2 - x - 1 = 0.$$

Thus, the solutions are $x = \dfrac{3 \pm \sqrt{13}}{2}$ and $x = \dfrac{1 \pm \sqrt{5}}{2}$.

Problem 9. $x^4 + 12x - 5 = 0$

Solution. Using Ferrari's method, we have $a = 0$, $b = 12$, and $c = -5$, which gives the cubic

$$u^3 + 20u - 144 = 0 \implies (u-4)(u^2 + 4u + 36) = 0$$

Letting $u_1 = 4$, we then get two quadratic equations: $x^2 + 2x - 1 = 0$ and $x^2 - 2x + 5 = 0$. Thus, the solutions are $x = \dfrac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$ and $x = \dfrac{2 \pm \sqrt{4 - 4(5)}}{2} = 1 \pm 2i$.

Problem 10. $x^4 + 8x^3 + 12x^2 - 11x + 2 = 0$

Solution. We solve this using Ferrari's method. Let $x = y + k$, then

$$x^4 + 8x^3 + 12x^2 - 11x + 2 = 0 \implies$$
$$y^4 + 4ky^3 + 6k^2y^2 + 4k^3y + k^4 + 8(y^3 + 3ky^2 + 3k^2y + k^3) + 12(y^2 + 2ky + k^2) - 11(y + k) + 2 = 0 \implies$$
$$y^4 + (4k+8)y^3 + (6k^2 + 24k + 12)y^2 + (4k^3 + 24k^2 + 24k - 11)y + k^4 + 8k^3 + 12k^2 - 11k + 2 = 0$$

and setting $k = -2$ eliminates the y^3 term to give $y^4 - 12y^2 + 5y + 24 = 0$. We have $a = -12$, $b = 5$, $c = 24$, so the cubic equation is

$$u^3 - au^2 - 4cu + (4ac - b^2) = 0 \implies u^3 + 12u^2 - 96u - 1177 = 0 \implies (u+11)(u^2+u-107) = 0.$$

So let $u_1 = -11$. Then the quartic in y reduces to two quadratics:

$$y^2 - \sqrt{u_1 - a}\, y + \left(\frac{u_1}{2} + \frac{b}{2\sqrt{u_1 - a}}\right) = 0 \implies y^2 - y - 3 = 0$$
$$y^2 + \sqrt{u_1 - a}\, y + \left(\frac{u_1}{2} - \frac{b}{2\sqrt{u_1 - a}}\right) = 0 \implies y^2 + y - 8 = 0$$

Thus, the solutions for y are $\dfrac{1 \pm \sqrt{13}}{2}$ and $\dfrac{-1 \pm \sqrt{33}}{2}$. And since $k = -2$, the solutions for x are $\dfrac{-3 \pm \sqrt{13}}{2}$ and $\dfrac{-5 \pm \sqrt{33}}{2}$.

Problem 11. $3x^6 - 2x^5 + 6x^4 - 2x^3 + 6x^2 - 2x + 3 = 0$

Solution. We recognize this as a reciprocal equation, which can be transformed as follows.

$$3x^6 - 2x^5 + 6x^4 - 2x^3 + 6x^2 - 2x + 3 = 0 \implies$$
$$x^3\left(3x^3 - 2x^2 + 6x - 2 + \frac{6}{x} - \frac{2}{x^2} + \frac{3}{x^3}\right) \implies$$
$$x^3\left[3\left(x^3 + \frac{1}{x^3}\right) - 2\left(x^2 + \frac{1}{x^2}\right) + 6\left(x + \frac{1}{x}\right) - 2\right] = 0 \implies$$
$$x^3\left[3\left(x + \frac{1}{x}\right)^3 - 9x^2\frac{1}{x} - 9x\frac{1}{x^2} - 2\left(x + \frac{1}{x}\right)^2 + 4 + 6\left(x + \frac{1}{x}\right) - 2\right] = 0 \implies$$
$$x^3\left[3\left(x + \frac{1}{x}\right)^3 - 2\left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) + 2\right] = 0$$

Let $y = x + \dfrac{1}{x}$, then we have

$$\begin{aligned} 3y^3 - 2y^2 - 3y + 2 = 0 &\implies 3y^3 - 3y - 2y^2 + 2 = 0 \\ &\implies 3y(y^2 - 1) - 2(y^2 - 1) = 0 \\ &\implies (3y - 2)(y - 1)(y + 1) = 0, \end{aligned}$$

which gives $x + \dfrac{1}{x} = \dfrac{2}{3}$, 1, -1, or $3x^2 - 2x + 3 = 0$, $x^2 - x + 1 = 0$, and $x^2 + x + 1 = 0$. Thus, the solutions are

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4 - 4(3)(3)}}{6} &= \frac{1 \pm 2\sqrt{2}i}{3} \\ x &= \frac{1 \pm \sqrt{1 - 4}}{2} &= \frac{1 \pm \sqrt{3}i}{2} \\ x &= \frac{-1 \pm \sqrt{1 - 4}}{2} &= \frac{-1 \pm \sqrt{3}i}{2}. \end{aligned}$$

We make use of the following formulas, which are easily derived. Define $z \equiv x + \dfrac{1}{x}$.

$$x^2 + \frac{1}{x^2} = z^2 - 2, \qquad x^3 + \frac{1}{x^3} = z^3 - 3z, \qquad x^4 + \frac{1}{x^4} = z^4 - 4z^2 + 2$$

Problem 12. $2x^8 - 9x^7 + 18x^6 - 30x^5 + 32x^4 - 30x^3 + 18x^2 - 9x + 2 = 0$

Solution. Since $x = 0$ is not a solution, we can divide through by x^4 to get

$$\begin{aligned}
&2x^8 - 9x^7 + 18x^6 - 30x^5 + 32x^4 - 30x^3 + 18x^2 - 9x + 2 = 0 \implies \\
&2\left(x^4 + \frac{1}{x^4}\right) - 9\left(x^3 + \frac{1}{x^3}\right) + 18\left(x^2 + \frac{1}{x^2}\right) - 30\left(x + \frac{1}{x}\right) + 32 = 0 \implies \\
&2(z^4 - 4z^2 + 2) - 9(z^3 - 3z) + 18(z^2 - 2) - 30z + 32 = 0 \implies \\
&2z^4 - 9z^3 + 10z^2 - 3z = 0 \implies \\
&z(z-1)(z-3)(2z-1) = 0,
\end{aligned}$$

where $z \equiv x + \dfrac{1}{x}$. This gives five equations in x:

$$\begin{aligned}
x + \frac{1}{x} = 0 &\implies x^2 + 1 = 0 &&\implies x = \pm i \\
x + \frac{1}{x} - 1 = 0 &\implies x^2 - x + 1 = 0 &&\implies x = \frac{1 \pm \sqrt{3}i}{2} \\
x + \frac{1}{x} - 3 = 0 &\implies x^2 - 3x + 1 = 0 &&\implies x = \frac{3 \pm \sqrt{5}}{2} \\
2\left(x + \frac{1}{x}\right) - 1 = 0 &\implies 2x^2 - x + 2 = 0 &&\implies x = \frac{1 \pm \sqrt{15}i}{4}
\end{aligned}$$

Problem 13. $6x^7 - x^6 + 2x^5 - 7x^4 - 7x^3 + 2x^2 - x + 6 = 0$

Solution. $6x^7 - x^6 + 2x^5 - 7x^4 - 7x^3 + 2x^2 - x + 6 = 0 \implies (x-1)^2(x+1)(6x^4 + 5x^3 + 13x^2 + 5x + 6) = 0$. Dividing the last factor by x^2 gives

$$\begin{aligned}
\frac{1}{x^2}(6x^4 + 5x^3 + 13x^2 + 5x + 6) = 0 &\implies 6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) + 13 = 0 \\
&\implies 6(z^2 - 2) + 5z + 13 = 0 \\
&\implies 6z^2 + 5z + 1 = 0 \\
&\implies (3z+1)(2z+1) = 0
\end{aligned}$$

So we have $3\left(x + \dfrac{1}{x}\right) + 1 = 0 \implies 3x^2 + x + 3 = 0$ and $2\left(x + \dfrac{1}{x}\right) + 1 = 0 \implies 2x^2 + x + 2 = 0$. Thus, the solutions are $x = 1$, $x = 1$, $x = -1$, $x = \dfrac{-1 \pm \sqrt{35}i}{6}$, and $x = \dfrac{-1 \pm \sqrt{15}i}{4}$.

Problem 14. Find the cubic in z on which, by §875, the solution of $x^7 - 1 = 0$ depends.

Solution.

$$\begin{aligned}
x^7-1=0 &\implies (x-1)(x^6+x^5+x^4+x^3+x^2+x+1)=0\\
&\implies (x-1)x^3\left(x^3+x^2+x+1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right)=0\\
&\implies (x-1)x^3\left[\left(x^3+\frac{1}{x^3}\right)+\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1\right]=0\\
&\implies (x-1)x^3\left[\left(x+\frac{1}{x}\right)^3-3x^2\frac{1}{x}-3x\frac{1}{x^2}+\left(x+\frac{1}{x}\right)^2-2+\left(x+\frac{1}{x}\right)+1\right]=0\\
&\implies (x-1)x^3\left[\left(x+\frac{1}{x}\right)^3-3\left(x+\frac{1}{x}\right)+\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-1\right]=0\\
&\implies (x-1)x^3\left[\left(x+\frac{1}{x}\right)^3+\left(x+\frac{1}{x}\right)^2-2\left(x+\frac{1}{x}\right)-1\right]=0
\end{aligned}$$

Thus, if we let $z=x+\dfrac{1}{x}$, then we need to solve the cubic $z^3+z^2-2z-1=0$.

Problem 15. Find the condition that all the roots of $x^3+3ax^2+3bx+c=0$ be real.

Solution. Let $x=y+k$, then we get

$$\begin{aligned}
x^3+3ax^2+3bx+c=0 &\implies y^3+3ky^2+3k^2y+k^3+3a(y^2+2ky+k^2)+3b(y+k)+c=0\\
&\implies y^3+(3k+3a)y^2+(3k^2+6ak+3b)y+k^3+3ak^2+3bk+c=0
\end{aligned}$$

Setting $k=-a$ eliminates the y^2 term and we get $y^3+(3b-3a^2)y+2a^3-3ab+c=0$, so that $p=3(b-a^2)$ an d$q=2a^3-3ab+c$. Thus, the discriminant is

$$\frac{q^2}{4}+\frac{p^3}{27}=\frac{(2a^3-3ab+c)^2}{4}+(b-a^2)^3$$

The condition that all three roots be real is that the discriminant be less than or equal to zero. Therefore, we must have $\dfrac{(2a^3-3ab+c)^2}{4}+(b-a^2)^3\le 0$.

Problem 16. Write down the trigonometric expressions for the roots of $x^5-1=0$, and of $x^6+1=0$.

Solution. The n nth roots of a complex number are given by

$$\sqrt[n]{r(\cos\theta+i\sin\theta)}=r^{1/n}\left(\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right)\quad\text{where } k=0,1,\ldots,n-1 \tag{30.1}$$

Therefore,

$$\begin{aligned}
x^5-1=0 &\implies x=\sqrt[5]{1(\cos 0+i\sin 0)}=\cos\frac{2k\pi}{5}+i\sin\frac{2k\pi}{5},\ \text{where } k=0,1,\ldots,4\\
x^6+1=0 &\implies x=\sqrt[6]{1(\cos\pi+i\sin\pi)}=\cos\frac{\pi+2k\pi}{6}+i\sin\frac{\pi+2k\pi}{6},\ \text{where } k=0,1,\ldots,5
\end{aligned}$$

Problem 17. Solve the following irreducible cubics.

(1) $x^3-3x-1=0$. **(2)** $x^3-6x-4=0$.

Solution. **(1)** Here $p=-3$ and $q=-1$, so that the discriminant is $(q/2)^2+(p/3)^3=1/4-1=-3/4$, and since this is negative, we have the irreducible case with three real roots. The roots are given by

$$x_k=\sqrt{-\frac{4p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{p}\sqrt{-\frac{3}{4p}}\right)+k\frac{2\pi}{3}\right)=2\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{1}{2}\right)+k\frac{2\pi}{3}\right)=2\cos\left(\frac{1}{3}\frac{\pi}{3}+k\frac{2\pi}{3}\right)$$

Thus, the solutions for eq. (1) are $x_0 = 2\cos(20°)$, $x_1 = 2\cos(140°)$, and $x_2 = 2\cos(260°)$.

(2) Here $p = -6$ and $q = -4$, so that the discriminant is $(q/2)^2 + (p/3)^3 = 4 - 8 = -4$, and since this is negative, we have the irreducible case with three real roots. The roots are given by

$$x_k = \sqrt{-\frac{4p}{3}} \cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{p}\sqrt{-\frac{3}{4p}}\right) + k\frac{2\pi}{3}\right) = 2\sqrt{2}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + k\frac{2\pi}{3}\right) = 2\sqrt{2}\cos\left(\frac{1}{3}\frac{\pi}{4} + k\frac{2\pi}{3}\right)$$

Thus, the solutions for eq. (2) are $x_0 = 2\sqrt{2}\cos(15°)$, $x_1 = 2\sqrt{2}\cos(135°)$, and $x_2 = 2\sqrt{2}\cos(255°)$.

Problem 18. In a sphere whose diameter is $3\sqrt{3}$ a right prism with a square base is inscribed. If the volume of the prism is 27, what is its altitude?

Solution. Let y be the base of the prism and x be the height. Then the volume is $y^2x = 27$, and since it is inscribed in the sphere, we have

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = (\frac{3\sqrt{3}}{2})^2 \implies x^2 + 2y^2 = 27$$

Solving the first equation for y^2 and substituting into the second equation gives

$$x^2 + \frac{54}{x} = 27 \implies x^3 - 27x + 54 = 0$$

Thus, we have the reduced cubic with $p = -27$ and $q = 54$. The discriminant is

$$\frac{q^2}{4} + \frac{p^3}{27} = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \left(\frac{54}{2}\right)^2 + \left(\frac{-27}{3}\right)^3 = (27)^2 - (9)^3 = 0$$

Thus, we have the case of three real roots with two of them being equal.

$$A = -\frac{q}{2} = -27 \text{ and } B = -\frac{q}{2} = -27.$$

The solutions are

$$x_1 = \sqrt[3]{A} + \sqrt{B} = -6, \quad x_2 = x_3 = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B} = -3\left(\frac{-1+\sqrt{3}i}{2}\right) - 3\left(\frac{-1-\sqrt{3}i}{2}\right) = 3$$

Thus, the only acceptable solution for the altitude is $x = 3$, and this is also the side length of the base since $y = \sqrt{27/x} = \sqrt{27/3} = 3$.

Problem 19. The volume of a certain right circular cylinder is 50π and its entire surface area is $105\pi/2$. Find the radius of its base and its altitude.

Solution. Let r be the radius and h be the height. Then we have

$$V = \pi r^2 h = 50\pi \implies r^2 h = 50 \tag{30.1}$$

$$A = 2\pi rh + 2(\pi r^2) = 105\pi/2 \implies 4rh + 4r^2 = 105 \tag{30.2}$$

From eq. (1), we have $h = 50/r^2$. Substituting this into eq. (2) and simplifying gives

$$r^4 - \frac{105}{4}r^2 + 50r = r\left(r^3 - \frac{105}{4}r + 50\right) = 0 \tag{30.3}$$

Since $r = 0$ is not a viable solution, we must solve the cubic. With $p = -105/4$ and $q = 50$, the discriminant is

$$\frac{q^2}{4} + \frac{p^3}{27} = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = (25)^2 - (105/12)^3 < 0, \tag{30.4}$$

which means we have the irreducible case with three real and distinct solutions. From eq. (10) of the irreducible case, we have

$$\begin{aligned} r_k &= \sqrt{-\frac{4p}{3}}\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3q}{p}\sqrt{-\frac{3}{4p}}\right)+k\frac{2\pi}{3}\right] \quad \text{for} \quad k=0,\,1,\,2 \\ &= \sqrt{35}\cos\left[\frac{1}{3}\cos^{-1}\left(-\frac{40}{7\sqrt{35}}\right)+k\frac{2\pi}{3}\right], \end{aligned}$$

which gives $r_0 = \dfrac{\sqrt{345}-5}{4} = 3.393543905$, $r_1 = -\dfrac{5+\sqrt{345}}{4} = -5.893543905251677$, $r_2 = \dfrac{5}{2}$. Of course, only the first and last are acceptable solutions. Thus, either

$$r = 3.393543905 \text{ and } h = 4.34173247 \qquad \text{or} \qquad r = 2.5 \text{ and } h = 8. \tag{30.5}$$

Problem 20. The altitude of a right circular cone is 6 and the radius of its base is 4. In this cone a right circular cylinder is inscribed whose volume is four ninths that of the cone. Find the altitude of the cylinder.

Solution. Let R be the radius of the base and H be the height of the right circular cone. We are given that $R=4$ and $H=6$. The volume of the right circular cone is $V_{\text{RCC}} = \frac{1}{3}\pi R^2 H = \frac{1}{3}\pi(4)^2(6) = 32\pi$. Let r be the radius of the inscribed right circular cylinder and let x be its height. Then by similar triangles, we have

$$\frac{r}{H-x} = \frac{R}{H}. \tag{30.1}$$

The volume of the circular cylinder is given by

$$V_{\text{CC}} = \pi r^2 x = \pi\left(\frac{R}{H}\right)^2 (H-x)^2 x = \pi\left(\frac{4}{6}\right)^2 (6-x)^2 x = \pi\frac{16}{36}(x^3-12x^2+36x).$$

We are given that this volume is 4/9 that of the right circular cone. Therefore,

$$\pi\frac{16}{36}(x^3-12x^2+36x) = \frac{4}{9}32\pi \implies x^3-12x^2+36x-32=0 \implies (x-2)^2(x-8)=0,$$

by synthetic division. Clearly $x=2$ is the only acceptable solution, so the height of the circular cylinder is $x=2$, and from eq. (1) we find the radius is $r=\frac{4}{6}(6-2)=\frac{8}{3}$.

Chapter 31

Determinants and Elimination

Determinants

Theorem.

The determinant of a matrix and its transpose are equal: $\det A^t = \det A$.

Theorem.

If any two adjacent rows (or columns) of a determinant are interchanged, then the determinant merely changes sign.

Corollary.

If two rows (or columns) of a determinant are identical, then the determinant vanishes.

Theorem.

If all the elements of a row (or column) are multiplied by the same number k, then the determinant is multiplied by k.

Corollary.

If the corresponding elements of two rows (or columns) are proportional, the determinant vanishes.

31.1 Determinants and Elimination LXXXI (p. 497)

Expand the following determinants.

Problem 1. $\begin{vmatrix} p & q & r \\ q & p & s \\ r & s & p \end{vmatrix}$.

Solution. $\begin{vmatrix} p & q & r \\ q & p & s \\ r & s & p \end{vmatrix} = p(p^2 - s^2) - q(qp - sr) + r(qs - pr) = p^3 - p(s^2 + q^2 + r^2) + 2qsr.$

Problem 2. $\begin{vmatrix} 1 & x & a \\ 1 & y & b \\ 1 & z & c \end{vmatrix}$.

Solution. $\begin{vmatrix} 1 & x & a \\ 1 & y & b \\ 1 & z & c \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ a & b & c \end{vmatrix} = cy - bz - (cx - az) + bx - ay = (b-c)x + (c-a)y + (a-b)z.$

Problem 3. $\begin{vmatrix} p & -q & r \\ q & p & -s \\ -r & s & p \end{vmatrix}.$

Solution. $\begin{vmatrix} p & -q & r \\ q & p & -s \\ -r & s & p \end{vmatrix} = p(p^2 + s^2) + q(qp - rs) + r(qs + pr) = p(p^2 + q^2 + r^2 + s^2).$

Problem 4. $\begin{vmatrix} 0 & -q & -r \\ q & 0 & -s \\ r & s & 0 \end{vmatrix}.$

Solution. $\begin{vmatrix} 0 & -q & -r \\ q & 0 & -s \\ r & s & 0 \end{vmatrix} = q(rs) - r(qs) = 0.$

Find the values of the following determinants.

Problem 5. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}.$

Solution. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 1(1-6) - 2(3-4) + 3(9-2) = -5 - 2(-1) + 3(7) = -5 + 2 + 21 = 18.$

Problem 6. $\begin{vmatrix} 1 & -3 & 4 \\ 2 & 0 & -5 \\ 3 & -1 & 7 \end{vmatrix}.$

Solution. $\begin{vmatrix} 1 & -3 & 4 \\ 2 & 0 & -5 \\ 3 & -1 & 7 \end{vmatrix} = 1(-5) + 3(14 + 15) + 4(-2) = -5 + 3(29) - 8 = 74.$

Problem 7. $\begin{vmatrix} 8 & 9 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 6 & 1 \\ 4 & 3 & 5 & 0 \end{vmatrix}.$

Solution. $\begin{vmatrix} 8 & 9 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 6 & 1 \\ 4 & 3 & 5 & 0 \end{vmatrix} = 8\begin{vmatrix} 3 & 0 & 0 \\ 1 & 6 & 1 \\ 3 & 5 & 0 \end{vmatrix} - 9\begin{vmatrix} 2 & 0 & 0 \\ 1 & 6 & 1 \\ 4 & 5 & 0 \end{vmatrix} = 8(3)(-5) - 9(2)(-5) = (24 - 18)(-5) = -30.$

Prove the following by expanding the determinants.

Problem 8. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}.$

Solution.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \tag{31.1}$$

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3) \\ &= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \end{aligned} \tag{31.2}$$

$$\begin{aligned} -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} &= -[b_1(a_2c_3 - c_2a_3) - a_1(b_2c_3 - c_2b_3) + c_1(b_2a_3 - a_2b_3) \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \end{aligned} \tag{31.3}$$

And we see that eqs. (1), (2), and (3) are all the same.

Problem 9. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$

Solution.

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \end{aligned}$$

31.2 Determinants and Elimination LXXXIV (p. 511)

Solve the following systems of equations by determinants.

Problem 1. $\begin{cases} 2x + 3y - 5z = 3, \\ x - 2y + z = 0, \\ 3x + y + 3z = 7. \end{cases}$

Solution.

$$\begin{vmatrix} 2 & 3 & -5 \\ 1 & -2 & 1 \\ 3 & 1 & 3 \end{vmatrix} = 2(-7) - 3(0) - 5(7) = -49 \qquad \begin{vmatrix} 3 & 3 & -5 \\ 0 & -2 & 1 \\ 7 & 1 & 3 \end{vmatrix} = 3(-7) - 3(-7) - 5(14) = -70$$

$$\begin{vmatrix} 2 & 3 & -5 \\ 1 & 0 & 1 \\ 3 & 7 & 3 \end{vmatrix} = 2(-7) - 3(0) - 5(7) = -49 \qquad \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 0 \\ 3 & 1 & 7 \end{vmatrix} = 2(-14) - 3(7) + 3(7) = -28$$

Thus, Cramer's rule gives $x = \dfrac{-70}{-49} = \dfrac{10}{7}$, $y = \dfrac{-49}{-49} = 1$, $z = \dfrac{-28}{-49} = \dfrac{4}{7}$.

Problem 2. $\begin{cases} 2x + 4y - 3z = 3, \\ 3x - 8y + 6z = 1, \\ 8x - 2y - 9z = 4. \end{cases}$

Solution.

$$\begin{vmatrix} 2 & 4 & -3 \\ 3 & -8 & 6 \\ 8 & -2 & -9 \end{vmatrix} = (-2)(-3)\begin{vmatrix} 2 & -2 & 1 \\ 3 & 4 & -2 \\ 8 & 1 & 3 \end{vmatrix} = 6[2(14)+2(25)+1(-29)] = 294$$

$$\begin{vmatrix} 3 & 4 & -3 \\ 1 & -8 & 6 \\ 4 & -2 & -9 \end{vmatrix} = 6\begin{vmatrix} 3 & -2 & 1 \\ 1 & 4 & -2 \\ 4 & 1 & 3 \end{vmatrix} = 6[3(14)+2(11)+1(-15)] = 294$$

$$\begin{vmatrix} 2 & 3 & -3 \\ 3 & 1 & 6 \\ 8 & 4 & -9 \end{vmatrix} = -3\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 8 & 4 & 3 \end{vmatrix} = -3[2(11)-3(25)+1(4)] = 147$$

$$\begin{vmatrix} 2 & 4 & 3 \\ 3 & -8 & 1 \\ 8 & -2 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & -2 & 3 \\ 3 & 4 & 1 \\ 8 & 1 & 4 \end{vmatrix} = -2[2(15)+2(4)+3(-29)] = 98$$

Thus, Cramer's rule gives $x = \dfrac{294}{294} = 1$, $y = \dfrac{147}{294} = \dfrac{1}{2}$, $z = \dfrac{98}{294} = \dfrac{1}{3}$.

Problem 3. $\begin{cases} ax + by + cz = d, \\ a^2x + b^2y + c^2z = d^2, \\ a^3x + b^3y + c^3z = d^3. \end{cases}$

Solution. The determinant of the coefficients is

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = a(b^2c^3 - c^2b^3) - b(a^2c^3 - c^2a^3) + c(a^2b^3 - b^2a^3) = abc[(bc^2 - cb^2) - (ac^2 - ca^2) + (ab^2 - ba^2)]$$

$$= abc[bc(c-b) - ac(c-a) + ab(b-a)] \tag{31.1}$$

On the other hand, it's clear that the determinant vanishes if $a = b$, if $a = c$, or if $b = c$. Hence, $(a-b)$, $a-c$, and $b-c$ must all be factors. Now,

$$\begin{aligned} (a-b)(b-c)(c-a) &= (a-b)(bc - ab - c^2 + ac) = abc - a^2b - ac^2 + a^2c - b^2c + ab^2 + bc^2 - abc \\ &= (ab^2 - a^2b) + (bc^2 - b^2c) + (a^2c - ac^2) \\ &= ab(b-a) + bc(c-b) + ac(a-c) \\ &= bc(c-b) - ac(c-a) + ab(b-a) \end{aligned} \tag{31.2}$$

Comparing eqs. (1) and (2), we see that the value of the determinant is $abc(a-b)(b-c)(c-a)$. The evaluation of the other three determinants in Cramer's rule is then just a matter of replacing a, b, or c with d in this expression:

$$\begin{vmatrix} d & b & c \\ d^2 & b^2 & c^2 \\ d^3 & b^3 & c^3 \end{vmatrix} = dbc(d-b)(b-c)(c-d) \implies x = \frac{dbc(d-b)(b-c)(c-d)}{abc(a-b)(b-c)(c-a)} = \frac{d(d-b)(c-d)}{a(a-b)(c-a)}$$

$$\begin{vmatrix} a & d & c \\ a^2 & d^2 & c^2 \\ a^3 & d^3 & c^3 \end{vmatrix} = adc(a-d)db-c)(c-a) \implies y = \frac{adc(a-d)(d-c)(c-a)}{abc(a-b)(b-c)(c-a)} = \frac{d(a-d)(d-c)}{b(a-b)(b-c)}$$

$$\begin{vmatrix} a & b & d \\ a^2 & b^2 & d^2 \\ a^3 & b^3 & d^3 \end{vmatrix} = abd(a-b)(b-d)(d-a) \implies z = \frac{abd(a-b)(b-d)(d-a)}{abc(a-b)(b-c)(c-a)} = \frac{d(b-d)(d-a)}{c(b-c)(c-a)}$$

Problem 4. $\begin{cases} 2x - 4y + 3z + 4t = -3, \\ 3x - 2y + 6z + 5t = -1, \\ 5x + 8y + 9z + 3t = 9, \\ x - 10y - 3z - 7t = 2. \end{cases}$

Solution.

$$\begin{aligned}
\begin{vmatrix} 2 & -4 & 3 & 4 \\ 3 & -2 & 6 & 5 \\ 5 & 8 & 9 & 3 \\ 1 & -10 & -3 & -7 \end{vmatrix} &= -6 \begin{vmatrix} 2 & 2 & 1 & 4 \\ 3 & 1 & 2 & 5 \\ 5 & -4 & 3 & 3 \\ 1 & 5 & -1 & -7 \end{vmatrix} \\
&= -6 \left\{ 2 \begin{vmatrix} 1 & 2 & 5 \\ -4 & 3 & 3 \\ 5 & -1 & -7 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 & 5 \\ 5 & 3 & 3 \\ 1 & -1 & -7 \end{vmatrix} + \begin{vmatrix} 3 & 1 & 5 \\ 5 & -4 & 3 \\ 1 & 5 & -7 \end{vmatrix} - 4 \begin{vmatrix} 3 & 1 & 2 \\ 5 & -4 & 3 \\ 1 & 5 & -1 \end{vmatrix} \right\} \\
&= -12[1(-18) - 2(13) + 5(-11)] + 12[3(-18) - 2(-38) + 5(-8)] \\
&\quad - 6[3(13) - 1(-38) + 5(29)] + 24[3(-11) - 1(-8) + 2(29)] \\
&= -12[-99] + 12[-18] - 6[222] + 24[33] = 432
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} -3 & -4 & 3 & 4 \\ -1 & -2 & 6 & 5 \\ 9 & 8 & 9 & 3 \\ 2 & -10 & -3 & -7 \end{vmatrix} &= -6 \begin{vmatrix} -3 & 2 & 1 & 4 \\ -1 & 1 & 2 & 5 \\ 9 & -4 & 3 & 3 \\ 2 & 5 & -1 & -7 \end{vmatrix} \\
&= -6 \left\{ -3 \begin{vmatrix} 1 & 2 & 5 \\ -4 & 3 & 3 \\ 5 & -1 & -7 \end{vmatrix} - 2 \begin{vmatrix} -1 & 2 & 5 \\ 9 & 3 & 3 \\ 2 & -1 & -7 \end{vmatrix} + \begin{vmatrix} -1 & 1 & 5 \\ 9 & -4 & 3 \\ 2 & 5 & -7 \end{vmatrix} - 4 \begin{vmatrix} -1 & 1 & 2 \\ 9 & -4 & 3 \\ 2 & 5 & -1 \end{vmatrix} \right\} \\
&= 18[1(-18) - 2(13) + 5(-11)] + 12[-1(-18) - 2(-69) + 5(-15)] \\
&\quad - 6[-1(13) - 1(-69) + 5(53)] + 24[-1(-11) - 1(-15) + 2(53)] \\
&= 18[-99] + 12[81] - 6[321] + 24[132] = 432 \implies x = \frac{432}{432} = 1
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} 2 & -3 & 3 & 4 \\ 3 & -1 & 6 & 5 \\ 5 & 9 & 9 & 3 \\ 1 & 2 & -3 & -7 \end{vmatrix} &= 3 \begin{vmatrix} 2 & -3 & 1 & 4 \\ 3 & -1 & 2 & 5 \\ 5 & 9 & 3 & 3 \\ 1 & 2 & -1 & -7 \end{vmatrix} \\
&= 3 \left\{ 2 \begin{vmatrix} -1 & 2 & 5 \\ 9 & 3 & 3 \\ 2 & -1 & -7 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 & 5 \\ 5 & 3 & 3 \\ 1 & -1 & -7 \end{vmatrix} + \begin{vmatrix} 3 & -1 & 5 \\ 5 & 9 & 3 \\ 1 & 2 & -7 \end{vmatrix} - 4 \begin{vmatrix} 3 & -1 & 2 \\ 5 & 9 & 3 \\ 1 & 2 & -1 \end{vmatrix} \right\} \\
&= 3\{2[-1(-18) - 2(-69) + 5(-15)] + 3[3(-18) - 2(-38) + 5(-8)] \\
&\quad + [3(-69) + 1(-38) + 5(1)] - 4[3(-15) + 1(-8) + 2(2)]\} \\
&= 3\{2[81] + 3[-18] + 1[-240] - 4[-51]\} = 3\{72\} = 216 \implies y = \frac{216}{432} = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} 2 & -4 & -3 & 4 \\ 3 & -2 & -1 & 5 \\ 5 & 8 & 9 & 3 \\ 1 & -10 & 2 & -7 \end{vmatrix} &= -2 \begin{vmatrix} 2 & 2 & -3 & 4 \\ 3 & 1 & -1 & 5 \\ 5 & -4 & 9 & 3 \\ 1 & 5 & 2 & -7 \end{vmatrix} \\
&= -2 \left\{ 2 \begin{vmatrix} 1 & -1 & 5 \\ -4 & 9 & 3 \\ 5 & 2 & -7 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 & 5 \\ 5 & 9 & 3 \\ 1 & 2 & -7 \end{vmatrix} - 3 \begin{vmatrix} 3 & 1 & 5 \\ 5 & -4 & 3 \\ 1 & 5 & -7 \end{vmatrix} - 4 \begin{vmatrix} 3 & 1 & -1 \\ 5 & -4 & 9 \\ 1 & 5 & 2 \end{vmatrix} \right\} \\
&= -2\{2[1(-69) + 1(13) + 5(-53)] - 2[3(-69) + 1(-38) + 5(1)] \\
&\quad - 3[3(13) - 1(-38) + 5(29)] - 4[3(-53) - 1(1) - 1(29)]\} \\
&- 2\{2[-321] - 2[-240] - 3[222] - 4[-189]\} = -2\{-72\} = 144 \implies z = \frac{144}{432} = \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} 2 & -4 & 3 & -3 \\ 3 & -2 & 6 & -1 \\ 5 & 8 & 9 & 9 \\ 1 & -10 & -3 & 2 \end{vmatrix} &= -6 \begin{vmatrix} 2 & 2 & 1 & -3 \\ 3 & 1 & 2 & -1 \\ 5 & -4 & 3 & 9 \\ 1 & 5 & -1 & 2 \end{vmatrix} \\
&= -6 \left\{ 2 \begin{vmatrix} 1 & 2 & -1 \\ -4 & 3 & 9 \\ 5 & -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 & -1 \\ 5 & 3 & 9 \\ 1 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 & -1 \\ 5 & -4 & 9 \\ 1 & 5 & 2 \end{vmatrix} + 3 \begin{vmatrix} 3 & 1 & 2 \\ 5 & -4 & 3 \\ 1 & 5 & -1 \end{vmatrix} \right\} \\
&= -6\{2[1(15) - 2(-53) - 1(-11)] - 2[3(15) - 2(1) - 1(-8)] \\
&\qquad + 1[3(-53) - 1(1) - 1(29)] + 3[3(-11) - 1(-8) + 2(29)]\} \\
&= -6\{2[132] - 2[51] + 1[-189] + 3[33]\} = -6\{72\} \implies t = \frac{-432}{432} = -1
\end{aligned}$$

In summary, the solution is $(x, y, z, t) = (1, 1/2, 1/3, -1)$.

Show that the following systems of equations are consistent, and solve them for the rations $x : y : z$.

Problem 5. $\begin{cases} x + 2y - z = 0, \\ 3x - y + 4z = 0, \\ 4x + y + 3z = 0. \end{cases}$

Solution. The solution will be the trivial solution $(x, y, z) = (0, 0, 0)$ unless the determinant of the coefficients is zero. And indeed, we find $\begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 4 \\ 4 & 1 & 3 \end{vmatrix} = 1(-7) - 2(-7) - 1(7) = -7 + 14 - 7 = 0$, so there is also a non-trivial solution, which is not unique. The ratios are given by the minors: $x : y : z = A_1 : A_2 : A_3$. We have

$$A_1 = \begin{vmatrix} -1 & 4 \\ 1 & 3 \end{vmatrix} = -7, \qquad A_2 = -\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 7, \qquad A_3 = \begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix} = 7.$$

Hence, $x : y : z = -1 : 1 : 1$, which means the solution is of the form $(x, y, z) = (-k, k, k)$, where k is an arbitrary constant.

Problem 6. $\begin{cases} a_1 x + b_1 y + (ka_1 + lb_1)z = 0, \\ a_2 x + b_2 y + (ka_2 + lb_2)z = 0, \\ a_3 x + b_3 y + (ka_3 + lb_3)z = 0. \end{cases}$

Solution. It is clear that the determinant of the coefficients vanishes since the third column is merely a linear combination of the first two columns. To find a non-trivial solution, we calculate the ratio of the minors. Now,

$$\begin{aligned}
A_1 &= \begin{vmatrix} b_2 & ka_2 + lb_2 \\ b_3 & ka_3 + lb_3 \end{vmatrix} = b_2(ka_3 + lb_3) - (ka_2 + lb_2)b_3 = k(a_3 b_2 - a_2 b_3) \\
A_2 &= \begin{vmatrix} a_2 & ka_2 + lb_2 \\ a_3 & ka_3 + lb_3 \end{vmatrix} = -a_2(ka_3 + lb_3) + (ka_2 + lb_2)a_3 = l(a_3 b_2 - a_2 b_3) \\
a_3 &= \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_2 b_3 - a_3 b_2 = -(a_3 b_2 - a_2 b_3)
\end{aligned}$$

Therefore, $x : y : z = k(a_3 b_2 - a_2 b_3) : l((a_3 b_2 - a_2 b_3) : -(a_3 b_2 - a_2 b_3) = k : l : -1$ if $a_3 b_2 - a_2 b_3 \neq 0$.

For what values of λ are the following equations consistent?

Problem 7. $\begin{cases} 4x + 3y + z = \lambda x, \\ 3x - 4y + 7z = \lambda y, \\ x + 7y - 6z = \lambda z. \end{cases}$

Solution. We can express this as the matrix equation $\begin{bmatrix} 4-\lambda & 3 & 1 \\ 3 & -4-\lambda & 7 \\ 1 & 7 & -6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This system will have a non-trivial solution if and only if the determinant of the coefficient matrix is zero, which gives us a condition on λ:

$$\begin{aligned}
\begin{vmatrix} 4-\lambda & 3 & 1 \\ 3 & -4-\lambda & 7 \\ 1 & 7 & -6-\lambda \end{vmatrix} &= (4-\lambda)[(\lambda+4)(\lambda+6)-49] - 3[-3\lambda-18-7] + 21 + \lambda + 4 = 0 \\
&\implies -(\lambda+6)(\lambda^2-16) + 49\lambda - 196 + 9\lambda + 75 + 21 + \lambda + 4 = 0 \\
&\implies -\lambda^3 - 6\lambda^2 + 16\lambda + 96 + 59\lambda - 96 = 0 \\
&\implies -\lambda^3 - 6\lambda^2 + 75\lambda = 0 \\
&\implies \lambda(\lambda^2 + 6\lambda - 75) = 0 \\
&\implies \lambda = 0, \frac{-6 \pm \sqrt{36+4(75)}}{2} = \frac{-6 \pm \sqrt{336}}{2} = \frac{-6 \pm 4\sqrt{21}}{2} = -3 \pm 2\sqrt{2}
\end{aligned}$$