Shape Factor of a Randomly Oriented Cylinder

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Abstract

The Dirac delta function *of a function* can be used to derive an exact closed-form expression for the probability density function and cumulative distribution function for the shape factor of a randomly oriented right circular cylinder.

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1 Introduction

This document defines a dimensionless shape factor, which is purely a function of shape and orientation and is independent of mass and material density. The shape factor of some simple shapes are calculated exactly, but the focus is on a right circular cylinder (RCC). Not only is it possible to express the shape factor as a function of the length to diameter (L/D) ratio and orientation, but it is also possible to derive an exact, closed-form expression for the shape factor probability distribution. It is found that the probability density function is not a symmetrical distribution about its mode but rather is highly skewed.

2 Dimensionless Shape Factor

The presented area, A_p, is a function of the dimensions of the cylinder, its shape, and its orientation. The only combination of the mass and material density that has the dimensions of an area is $(\text{mass/density})^{2/3}$. Therefore, the presented area is of the form

$$A_p \equiv \gamma(m/\rho)^{2/3}, \tag{1}$$

where γ is a *dimensionless shape factor* that accounts for both fragment shape and orientation. It is purely a function of geometry and is independent of both mass and density. Since the dimensionless shape factor is purely a function of geometry, it is possible to calculate it for various standard shapes, as shown in Table 1.

Table 1. Dimensionless Shape Factors for Some Common Shapes

Shape	Orientation	Shape Factor, γ
Sphere	Not applicable	$(3/2)^{2/3}(\pi/4)^{1/3} \approx 1.209$
Cube	Face-forward	1
Cube	Edge-forward	$\sqrt{2} \approx 1.414$
Cube	Random^a	$3/2$
Cube	Corner-forward	$\sqrt{3} \approx 1.732$
Cylinder ($L/D = 1$)	Face-forward	$(\pi/4)^{1/3} \approx 0.927$
Cylinder ($L/D = 1$)	Side-forward	$(\pi/4)^{-2/3} \approx 1.175$
Cylinder ($L/D = 1$)	Random^a	$(3/2)(\pi/4)^{1/3} \approx 1.384$

a For random orientation, the rule $A_p = (\text{Surface Area})/4$ was used (see Appendix).

In the remainder of this document, we use the terminology "shape factor" to mean the dimensionless shape factor, as defined by eq. (1).

3 Shape Factor of a Cylinder

The orientation of the cylinder can be parametrized by the angle θ , as shown in Figure 1.

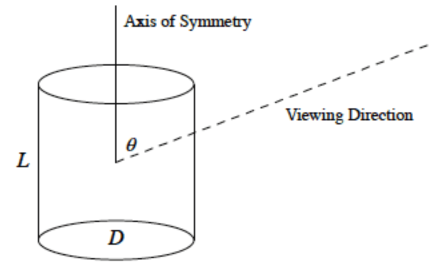

Figure 1. Right Circular Cylinder with Orientation Angle.

When viewed from the side ($\theta = \pi/2$), the projected area is LD. When viewed from the top ($\theta = 0$), the projected area is $\pi D^2/4$. For an arbitrary direction, θ, the projected area is the linear combination of these two faces scaled by the cosine of the projected area surface normal. Thus,

$$A_p = LD\cos(\pi/2 - \theta) + \frac{\pi}{4}D^2|\cos\theta| = D^2\left(\frac{L}{D}\sin\theta + \frac{\pi}{4}|\cos\theta|\right). \tag{2}$$

It is necessary to take the absolute value of the cosine term because $\cos\theta$ is negative when $\pi/2 < \theta \leq \pi$. The volume of the cylinder is given by

$$V = \frac{m}{\rho} = \frac{\pi}{4}D^2L. \tag{3}$$

Substituting these two expressions into eq. (1) and solving for γ , we find that

$$\gamma = \left(\frac{\pi}{4}\frac{L}{D}\right)^{-2/3}\left(\frac{L}{D}\sin\theta + \frac{\pi}{4}|\cos\theta|\right). \tag{4}$$

It is convenient to define the parameters

$$a \equiv \left(\frac{\pi}{4}\frac{L}{D}\right)^{-2/3}\frac{L}{D} \quad \text{and} \quad b \equiv \left(\frac{\pi}{4}\frac{L}{D}\right)^{-2/3}\frac{\pi}{4}. \tag{5}$$

Then the shape factor may be written as

$$\gamma(\theta) = a\sin\theta + b|\cos\theta|, \tag{6}$$

where $0 \leq \theta \leq \pi$. Notice that the presented area of the cylinder has the same value at the viewing angle θ as it has at $\pi - \theta$, so that $\gamma(\theta) = \gamma(\pi - \theta)$. Thus, we could restrict the orientation to $0 \leq \theta \leq \pi/2$ and just perform averages over the top hemisphere of the unit sphere. This would allow us to drop the absolute value signs in eq. (6). Nevertheless, to avoid confusion, we shall use the form expressed in eq. (6) and perform averages over the entire unit sphere.

This expresses the shape factor as an explicit function of the orientation angle, θ. It is also an implicit function of the L/D ratio through the coefficients a and b. A plot of γ as a function of both orientation angle θ and L/D ratio is shown in Figure 2.

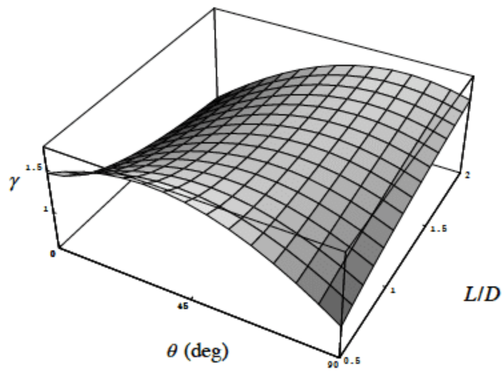

Figure 2. Shape Factor of a Cylinder as a Function of Orientation Angle and L/D Ratio.

From eq. (6), we see that $\gamma(0) = b$ and $\gamma(\pi/2) = a$. The *minimum* value of γ is one of these, depending upon the particular L/D ratio:

$$\gamma_{\min} = \begin{cases} a & \text{if } L/D < \pi/4 \\ b & \text{if } L/D > \pi/4 \end{cases} \qquad \text{or, simply} \quad \gamma_{\min} = \min(a, b) \tag{7}$$

The *maximum* value of γ can be obtained by setting $d\gamma/d\theta = 0$, and this gives

$$\gamma_{\max} = \sqrt{a^2 + b^2} = \left(\frac{\pi}{4}\frac{L}{D}\right)^{-2/3} \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{L}{D}\right)^2}. \tag{8}$$

The maximum value of the shape factor is realized at the orientation angle

$$\theta_{\gamma=\gamma_{\max}} = \tan^{-1}\left(\frac{a}{b}\right) = \tan^{-1}\left(\frac{L/D}{\pi/4}\right). \tag{9}$$

Thus, in general, there are two graphs for the shape factor as a function of orientation angle, depending upon the L/D ratio of the cylinder. These two cases are exemplified in Figures 3 and 4.

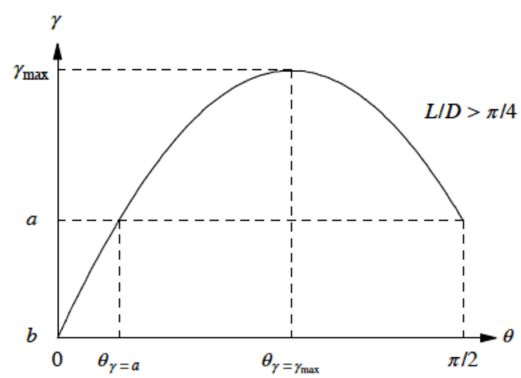

Figure 3. Shape Factor Dependence Upon Orientation for $L/D > \pi/4$.

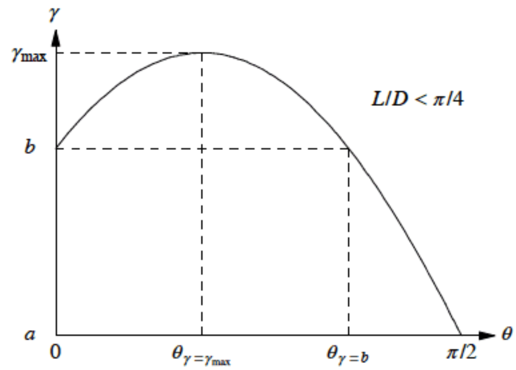

Figure 4. Shape Factor Dependence Upon Orientation for $L/D < \pi/4$.

The angle $\theta_{\gamma=a}$ in Figure 3 and $\theta_{\gamma=b}$ in Figure 4 can be obtained through the use of eq. (6) and we find

$$\theta_{\gamma=a} = \tan^{-1}\left(\frac{a^2-b^2}{2ab}\right) \quad \text{when} \quad L/D > \pi/4 \tag{10}$$

and

$$\theta_{\gamma=a} = \tan^{-1}\left(\frac{2ab}{b^2-a^2}\right) \quad \text{when} \quad L/D < \pi/4. \tag{11}$$

From Figures 3 and 4, we also see that when $\max(a,b) \leq \gamma \leq \gamma_{\max}$, the orientation angle is double-valued, and when $\min(a,b) \leq \gamma < \max(a,b)$, the orientation angle is single-valued.

4 Shape Factor as a Function of Orientation

The minimum shape factor is given by eq. (7), and the maximum shape factor is given by eq. (8). The mean shape factor, when averaged over all orientations equally, is given by

$$\bar{\gamma} = \frac{1}{4\pi}\int \gamma(\theta)d\Omega = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \gamma(\theta)\sin\theta d\theta. \tag{12}$$

Substituting equation (8) and performing the integration, we find[1]

$$\bar{\gamma} = a\frac{\pi}{4} + b\frac{1}{2} = \left(\frac{\pi}{4}\right)^{1/3}\left(\frac{L}{D}\right)^{-2/3}\left(\frac{L}{D}+\frac{1}{2}\right). \tag{13}$$

Calculation shows that $\bar{\gamma}$ is a minimum when $L/D = 1$, where it takes on the value $(3/2)(\pi/4)^{1/3} \approx 1.38395$. If we only know the average shape factor of a fragment, and it exceeds this value, then eq. (13) can be used to find the L/D ratio of an equivalent disk-shaped cylinder. This solution is shown plotted in Figure 5.

[1] This result also follows from the general theorem that the average projected area of a convex solid is one-fourth the total surface area (see Appendix)

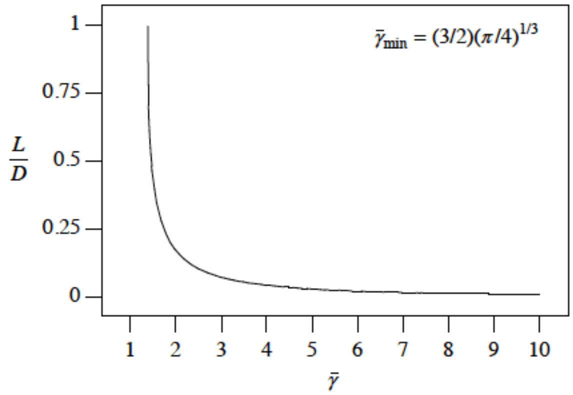

Figure 5. Disk-Shaped Cylinder Derived From the Average Shape Factor.

Here is a simple program that implements this solution of the cubic equation for the L/D ratio:

Listing 1. asf2ld.cpp

```
// asf2ld.cpp: Find the L/D ratio of an RCC with the given average shape factor
#include <iostream>
#include <cmath>
using namespace std;

int main( void ) {

   const double G_BAR_MIN = 1.5 * pow( 0.25 * M_PI, 1. / 3. );
   double gBar, r, alpha, x1, x2, l_dMin, l_dMax;
   while ( cin >> gBar ) {

      if ( gBar > G_BAR_MIN ) {
         r = gBar / G_BAR_MIN;
         alpha = acos( -pow( r, -1.5 ) );
         x1 = +2. * sqrt( r ) * cos( alpha / 3. );
         x2 = -2. * sqrt( r ) * cos( ( alpha + M_PI ) / 3. );
         l_dMin = pow( x1, -3. );
         l_dMax = pow( x2, -3. );
      }
      else if ( gBar == G_BAR_MIN ) l_dMin = l_dMax = 1.;
      else {
         cerr << "The minimum average shape factor for an RCC is " << G_BAR_MIN << endl;
         exit( 1 );
      }
      cout << "Two possible values:" << endl;
      cout << "   L/D = " << l_dMin << endl;
      cout << "   L/D = " << l_dMax << endl;
   }
   return 0;
}
```

The average, minimum, and maximum shape factors are plotted in Figure 6.

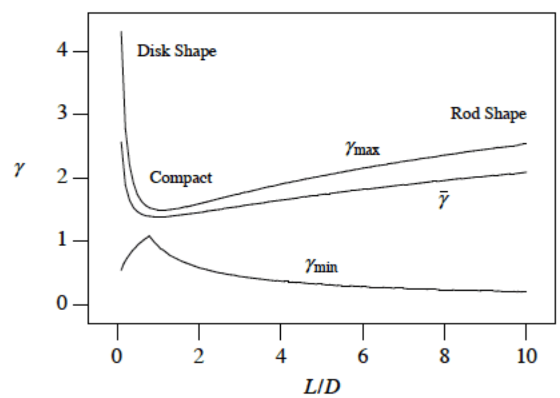

Figure 6. Minimum, Maximum, and Mean Shape Factors as a Function of L/D Ratio.

It is straightforward to establish the following:

- $\bar{\gamma}$ is a minimum when $L/D = 1$.
- $\gamma_{\rm max}$ is a minimum when $L/D = \sqrt{2}\pi/4 \approx 1.111$.
- $\gamma_{\rm min}$ is a maximum when $L/D = \pi/4 \approx 0.785$.
- $\gamma_{\rm max} - \gamma_{\rm min}$ (i.e., the range of γ over all possible orientations) is a minimum when $L/D = \pi/4 \approx 0.785$. Its value at the minimum is $(\sqrt{2}-1)(\pi/4)^{-1/3} \approx 0.449$. This quantity is a measure of the deviation from spherical symmetry, since a sphere has $\gamma_{\rm max} - \gamma_{\rm min} = 0$. For completeness, we can also calculate the variance:

$$\sigma^2 = \frac{1}{4\pi}\int (\gamma(\theta) - \bar{\gamma})^2 d\Omega = \int_0^{\pi/2} (\gamma(\theta) - \bar{\gamma})^2 \sin\theta d\theta, \tag{14}$$

and we find

$$\sigma^2 = \left(\frac{2}{3} - \frac{\pi^2}{16}\right) a^2 + \left(\frac{2}{3} - \frac{\pi}{4}\right) ab + \frac{1}{12} b^2. \tag{15}$$

5 Shape Factor Distribution of a Randomly Oriented Cylinder

It is possible to derive a closed-form expression for the shape factor probability distribution function. We start with the formula for the probability density.

$$f(\gamma) = \int_0^{\pi/2} \delta(a\sin\theta + b\cos\theta - \gamma)\sin\theta d\theta, \tag{16}$$

where $\delta(x)$ is the Dirac delta function, or, setting $x = \cos\theta$,

$$f(\gamma) = \int_0^1 \delta(g(x))dx, \tag{17}$$

where

$$g(x) \equiv a\sqrt{1-x^2} + bx - \gamma. \tag{18}$$

To evaluate this integral, we use the fact that the delta function *of a function* can be expressed in terms of the roots of the function:[2]

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|dg/dx|_{x=x_i}}, \tag{19}$$

where x_i $(i = 1, \ldots, n)$ are the n roots of the equation $g(x) = 0$. The roots of $g(x)$ are found to be

$$x = \frac{b\gamma \pm a\sqrt{a^2 + b^2 - \gamma^2}}{a^2 + b^2}. \tag{20}$$

This can be simplified by introducing the quantities

$$a' \equiv \frac{a}{\gamma_{\rm max}}, \quad b' \equiv \frac{b}{\gamma_{\rm max}}, \quad \text{and} \quad \gamma' \equiv \frac{\gamma}{\gamma_{\rm max}}. \tag{21}$$

where $\gamma_{\rm max}$ is given by eq. (8). The roots can be written as

$$x = b'\gamma' \pm a'\sqrt{1 - \gamma'^2}. \tag{22}$$

Now notice that ${a'}^2 + {b'}^2 = 1$ and $0 < \gamma' \leq 1$ so that we can define angles

$$\alpha \equiv \cos^{-1}(b') = \cos^{-1}(b/\gamma_{\rm max}) \quad \text{and} \quad \beta \equiv \cos^{-1}(\gamma') = \cos^{-1}(\gamma/\gamma_{\rm max}). \tag{23}$$

[2] Philippe Dennery and Andre Krzywicki, *Mathematics for Physicists*, Harper & Row, New York, 1967.

This allows us to express eq. (22) as

$$x = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta = \cos(\alpha \mp \beta). \tag{24}$$

So the two roots are

$$x_1 = \cos(\alpha - \beta) and x_2 = \cos(\alpha + \beta), \tag{25}$$

or, in terms of the orientation angle, θ,

$$\theta_1 = \alpha - \beta \quad \text{and} \quad \theta_2 = \alpha + \beta. \tag{26}$$

Evaluation of the derivative of $g(x)$ is straightforward, and we find

$$g'(x_1) = \frac{\gamma_{\max}\sin(-\beta)}{\sin(\alpha - \beta)} \quad \text{and} \quad g'(x_2) = \frac{\gamma_{\max}\sin\beta}{\sin(\alpha + \beta)} \tag{27}$$

Thus,

$$f(\gamma) = \int_0^1 \delta(g(x))\,dx = \int_0^1 \left(\frac{\delta(x - x_1)}{|g'(x_1)|} + \frac{\delta(x - x_2)}{|g'(x_2)|}\right) dx, \tag{28}$$

and, using eq. (27),

$$f(\gamma) = \int_0^1 \left(\frac{|\sin(\alpha - \beta)|}{|\gamma_{\max}\sin(-\beta)|}\delta(x - x_1) + \frac{|\sin(\alpha + \beta)|}{|\gamma_{\max}\sin\beta|}\delta(x - x_2)\right) dx. \tag{29}$$

In order to evaluate this integral, it is sufficient to know the location of the two roots, x_1 and x_2, or, equivalently, θ_1 and θ_2. For $L/D > \pi/4$, we find that when $b \leq \gamma < a$, then $0 \leq \theta_1 < \theta_{\gamma=a}$, and when $a \leq \gamma \leq \gamma_{\max}$, then $\theta_1 = \theta_{\gamma=a}$ and $\theta_{\gamma=\gamma_{\max}} \leq \theta_2 \leq \pi/2$. Similarly, when $L/D < \pi/4$,we find that when $a \leq \gamma < b$, then $\theta_{\gamma=b} < \theta_2 \leq \pi/2$, and when $b \leq \gamma \leq \gamma_{\max}$, then $0 \leq \theta_1 \leq \theta_{\gamma=\gamma_{\max}}$ and $\theta_2 = \theta_{\gamma=b}$. Thus, only one root lies in the interval where the orientation angle is single-valued and both roots lie in the interval where the orientation angle is double-valued. The evaluation of the integral is now trivial, and the results are as follows.

- $\boldsymbol{L/D > \pi/4}$

$$\text{If } b \leq \gamma < a, \text{ then } f(\gamma) = \frac{\sin(\alpha - \beta)}{\gamma_{\max}\sin\beta}. \tag{30}$$

$$\text{If } a \leq \gamma < \gamma_{\max}, \text{ then } f(\gamma) = \frac{\sin(\alpha - \beta)}{\gamma_{\max}\sin\beta} + \frac{\sin(\alpha + \beta)}{\gamma_{\max}\sin\beta}. \tag{31}$$

- $\boldsymbol{L/D < \pi/4}$

$$\text{If } a \leq \gamma < b, \text{ then } f(\gamma) = \frac{\sin(\alpha + \beta)}{\gamma_{\max}\sin\beta}. \tag{32}$$

$$\text{If } b \leq \gamma < \gamma_{\max}, \text{ then } f(\gamma) = \frac{\sin(\alpha - \beta)}{\gamma_{\max}\sin\beta} + \frac{\sin(\alpha + \beta)}{\gamma_{\max}\sin\beta}. \tag{33}$$

Expressing these results in terms of the parameters a and b gives us the following results.

5.1 Probability Density Function

- $\boldsymbol{L/D > \pi/4}$

$$f(\gamma) = \begin{cases} \dfrac{a\gamma - b\sqrt{\gamma_{\max}^2 - \gamma^2}}{\gamma_{\max}^2\sqrt{\gamma_{\max}^2 - \gamma^2}} & b \leq \gamma < a \qquad 0 \leq \theta < \theta_d \\[2ex] \dfrac{2a\gamma}{\gamma_{\max}^2\sqrt{\gamma_{\max}^2 - \gamma^2}} & a \leq \gamma < \gamma_{\max} \qquad \theta_d \leq \theta < \pi/2 \end{cases} \tag{34}$$

Figure 7 is a plot of this function for a compact cylinder.

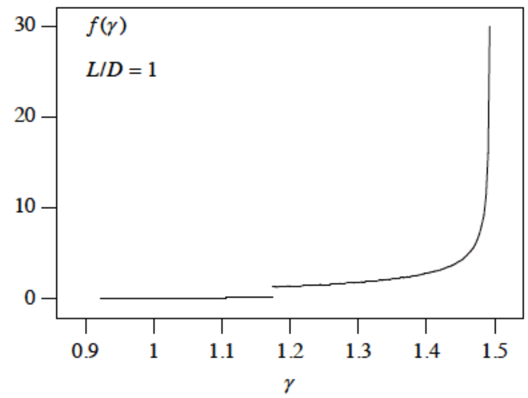

Figure 7. Shape Factor Probability Density Function for a Cylinder with $L/D > \pi/4$.

Notice the discontinuity at $\gamma = (\pi/4)^{-2/3}$, where the shape factor changes from being single-valued to being double-valued.

- $L/D < \pi/4$

$$f(\gamma) = \begin{cases} \dfrac{a\gamma + b\sqrt{\gamma_{\max}^2 - \gamma^2}}{\gamma_{\max}^2\sqrt{\gamma_{\max}^2 - \gamma^2}} & a \le \gamma < b \qquad \theta_d \le \theta < \pi/2 \\ \dfrac{2a\gamma}{\gamma_{\max}^2\sqrt{\gamma_{\max}^2 - \gamma^2}} & b \le \gamma < \gamma_{\max} \qquad 0 \le \theta < \theta_d \end{cases} \tag{35}$$

Figure 8 shows a plot of these functions for a disk-shaped cylinder.

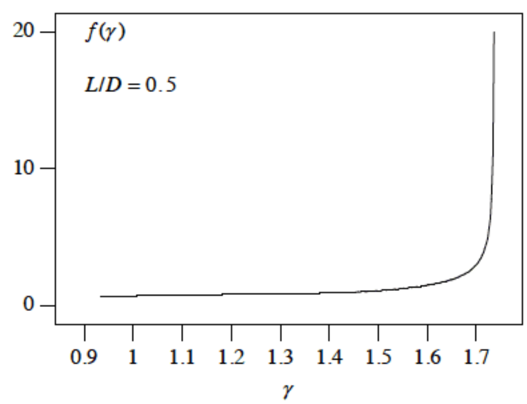

Figure 8. Shape Factor Probability Density Function for a Cylinder with $L/D < \pi/4$.

Notice that the most probable value of the shape factor, (i.e., its *mode*), is $\gamma_{\max}$, regardless of the magnitude of the L/D ratio.

5.2 Cumulative Distribution Function

- $L/D > \pi/4$

$$F(\gamma) = \begin{cases} 1 - \dfrac{b\gamma + a\sqrt{\gamma_{\max}^2 - \gamma^2}}{\gamma_{\max}^2} & b \le \gamma < a \qquad 0 \le \theta < \theta_d \\ 1 - \dfrac{2a\sqrt{\gamma_{\max}^2 - \gamma^2}}{\gamma_{\max}^2} & a \le \gamma < \gamma_{\max} \qquad \theta_d \le \theta < \pi/2 \end{cases} \tag{36}$$

Figure 9 is a plot of the cumulative distribution for a compact cylinder.

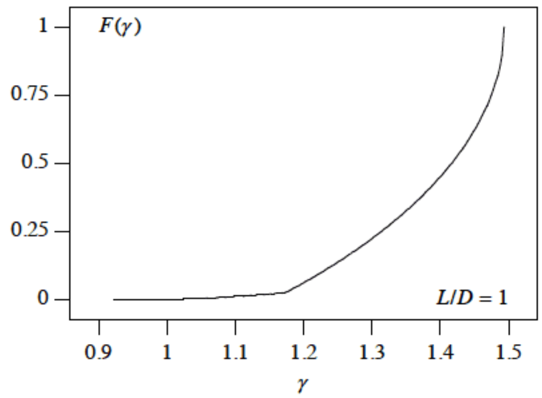

Figure 9. Shape Factor Cumulative Distribution Function for a Cylinder with $L/D > \pi/4$.

- $\boldsymbol{L/D > \pi/4}$

$$F(\gamma) = \begin{cases} \dfrac{b\gamma - a\sqrt{\gamma_{\max}^2 - \gamma^2}}{\gamma_{\max}^2} & a \le \gamma < b \qquad \theta_d \le \theta < \pi/2 \\ 1 - \dfrac{2a\sqrt{\gamma_{\max}^2 - \gamma^2}}{\gamma_{\max}^2} & b \le \gamma < \gamma_{\max} \qquad 0 \le \theta \le \theta_d \end{cases} \tag{37}$$

A plot of this cumulative distribution for a disk-shaped cylinder is shown in Figure 10.

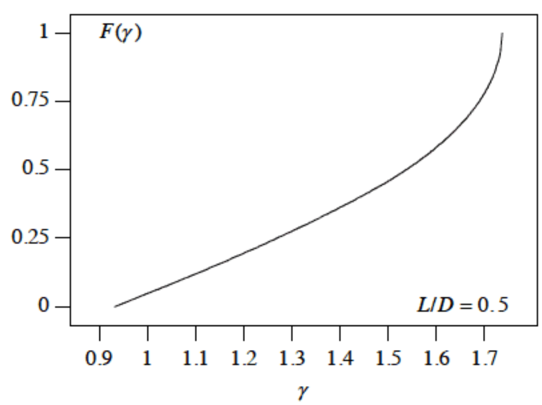

Figure 10. Shape Factor Cumulative Distribution Function for a Cylinder with $L/D < \pi/4$.

5.3 Generating the Random Shape Factor Distribution

The cumulative distribution expressed by eqs. (36) and (37) can be inverted to give γ in terms of the cumulative probability. Then by selecting the cumulative probability from a uniform random distribution on the unit interval, we generate a random distribution of shape factors, where the frequency of occurrence follows the probability density function—a standard technique for generating random numbers.

- ***L/D > π/4***

$$\gamma = \begin{cases} b(1-P) + a\sqrt{1-(1-P)^2} & 0 \le P \le 1 - \dfrac{2ab}{\gamma_{\max}^2} \\ \gamma_{\max}\sqrt{1-(\gamma_{\max}(1-P)/2a)^2} & 1 - \dfrac{2ab}{\gamma_{\max}^2} \le P \le 1 \end{cases} \tag{38}$$

- ***L/D < π/4***

$$\gamma = \begin{cases} bP + a\sqrt{1-P^2} & 0 \le P \le \dfrac{b^2-a^2}{\gamma_{\max}^2} \\ \gamma_{\max}\sqrt{1-(\gamma_{\max}(1-P)/2a)^2} & \dfrac{b^2-a^2}{\gamma_{\max}^2} \le P \le 1 \end{cases} \tag{39}$$

These equations enable us to define an algorithm for generating random shape factors having the same distribution as that corresponding to the uniform random orientation of a cylinder. First, define the following quantities:

$$a \equiv \left(\frac{\pi}{4}\frac{L}{D}\right)^{-2/3}\frac{L}{D}, \quad b \equiv \left(\frac{\pi}{4}\frac{L}{D}\right)^{-2/3}\frac{\pi}{4}, \quad \gamma_{\max} = \sqrt{a^2+b^2}, \quad P_1 \equiv 1 - \frac{2ab}{\gamma_{\max}^2}, \quad P_2 \equiv \frac{b^2-a^2}{\gamma_{\max}^2}. \tag{40}$$

Then the algorithm is:

Algorithm 1 Generate Shape Factor Distribution for a Cylinder Randomly Oriented over the Unit Sphere

procedure GIVEN $a,\ b,\ \gamma_{\max},\ P_1,\ P_2$
  $P \sim U(0,1)$ ▷ Sample from the uniform distribution
  if $a \ge b$ **then** ▷ $a \ge b$ (corresponding to $L/D \ge \pi/4$)
    if $P \le P_1$ **then**
      return $\gamma \leftarrow b(1-P) + a\sqrt{1-(1-P)^2}$
    else
      return $\gamma \leftarrow \gamma_{\max}\sqrt{1-(\gamma_{\max}(1-P)/2a)^2}$
    end if
  else ▷ $a < b$ (corresponding to $L/D < \pi/4$)
    if $P \le P_2$ **then**
      return $\gamma \leftarrow bP + a\sqrt{1-P^2}$
    else
      return $\gamma \leftarrow \gamma_{\max}\sqrt{1-(\gamma_{\max}(1-P)/2a)^2}$
    end if
  end if
end procedure

This algorithm is more than twice as fast in execution time than the considerably simpler shape factor simulation:

1. Generate $(\theta, \phi) \sim$ Surface of the Unit Sphere $\equiv S^2(\theta, \phi)$
2. Return $\gamma = a\sin\theta + b\cos\theta$.

Uniform sampling over the unit sphere can be performed very simply with the aid of Archimedes' theorem.[3]

Archimedes' Theorem

The area of a sphere equals the area of every right circular cylinder circumscribed about the sphere excluding the bases.

[3] M. Shao & N. Badler, *Spherical Sampling by Archimedes' Theorem*, University of Pennsylvania, Technical Report (CIS) 1-1-1996.

The *global* version of Archimedes' theorem states that the area of a sphere is equal to the area of a cylinder circumscribed about the sphere, excluding the bases. The area of a unit sphere is 4π. The area of the circumscribed cylinder is the circumference times the height: $2\pi \times 2 = 4\pi$. The *local* version of the theorem states further that any region on the sphere is equal to the axial projection on the cylinder. This is a very powerful theorem for our purposes since it is much easier to define a sampling strategy on the cylinder, which we can lay out flat and independently sample ϕ and z, and then use Archimedes' theorem to map onto the unit sphere.

Let θ and ϕ be the polar and azimuthal angles, respectively, on the unit sphere, and let ϕ and z be coordinates on the circumscribed cylinder, where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, and $z \in [-1, 1]$.Then the mapping from the cylinder to the sphere

$$[0, 2\pi] \times [-1, 1] \mapsto S^2(\theta, \phi) \quad \text{is simply} \quad \theta = \cos^{-1} z,$$

while the ϕ value remains the same. Thus, the algorithm is

Algorithm 2 Uniform Sampling over the Unit Sphere

```
procedure
    φ ~ U(0, 2π)            ▷ Sample azimuthal angle from the uniform distribution
    z ~ U(−1, 1)            ▷ Sample cylinder height from the uniform dsistribution
    θ ← cos⁻¹ z             ▷ Transform cylinder height to polar angle
    return (θ, φ)
end procedure
```

Table 2 summarize some properties of the shape factor distribution for a randomly oriented cylinder.

Table 2. Shape Factor Parameters for a Randomly Oriented Cylinder

Parameter	Expression	Values when $L/D = 1$
Length	L	L
Diameter	D	D
a	$\left(\frac{\pi}{4}\frac{L}{D}\right)^{-2/3} \frac{L}{D}$	1.17474
b	$\left(\frac{\pi}{4}\frac{L}{D}\right)^{-2/3} \frac{\pi}{4}$	0.922635
Minimum	$\gamma_{\min} = \min(a, b)$	0.922635
Maximum	$\gamma_{\max} = \sqrt{a^2 + b^2}$	1.49374
Mean	$\bar{\gamma} = \frac{\pi}{4}a + \frac{1}{2}b$	1.38395
Variance	$\sigma^2 = \left(\frac{2}{3} - \frac{\pi^2}{16}\right) a^2 + \left(\frac{2}{3} - \frac{\pi}{4}\right) ab + \frac{1}{12}b^2$	0.0109974
Mode	$\hat{\gamma} = \gamma_{\max}$	1.49374
Median	$\hat{\gamma} = \begin{cases} \gamma_{\max}\sqrt{1 - (\gamma_{\max}/4a)^2} & \text{if } \frac{\pi}{4} < \frac{L}{D} < (\sqrt{3}+2)\frac{\pi}{4} \\ \frac{\sqrt{3}a + b}{2} & \text{if } (\sqrt{3}+2)\frac{\pi}{4} < \frac{L}{D} \\ \gamma_{\max}\sqrt{1 - (\gamma_{\max}/4a)^2} & \text{if } \frac{1}{\sqrt{3}}\frac{\pi}{4} < \frac{L}{D} < \frac{\pi}{4} \\ \frac{\sqrt{3}a + b}{2} & \text{if } \frac{L}{D} < \frac{1}{\sqrt{3}}\frac{\pi}{4} \end{cases}$	1.41626

Appendices

Appendix A Mean Presented Are of a Convex Solid

Let $\hat{\boldsymbol{n}}$ be the outward normal on the surface of the solid and let $\hat{\boldsymbol{n}}'$ be a unit vector along an arbitrary direction. The presented area of the convex solid in the direction $\hat{\boldsymbol{n}}$ is given by the integral

$$A_p = \int_{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}'>0} \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' dS, \tag{1}$$

where dS is the element of surface area. The *mean* presented area, averaged over all directions $\hat{\boldsymbol{n}}'$, is given by

$$\bar{A}_p = \frac{1}{4\pi}\int d\Omega' \int_{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}'>0} \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' dS, \tag{2}$$

where the first integral is over all solid angles of a unit sphere. Interchanging the order of integration, we have

$$\bar{A}_p = \int dS \frac{1}{4\pi}\int_{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}'>0} \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' d\Omega'. \tag{3}$$

In the last integral,$\hat{\boldsymbol{n}}$ remains fixed while $\hat{\boldsymbol{n}}'$ varies over all solid angles such that $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' > 0$. By changing coordinates so that $\hat{\boldsymbol{n}}$ is along the polar axis, $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' = \cos\theta$, and this integral is esily evaluated:

$$\int_{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}'>0} \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}}' d\Omega' = \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta\sin\theta d\theta = \pi. \tag{4}$$

Substituting this result into eq. (3) gives

$$\bar{A}_p = \frac{1}{4}\int dS. \tag{5}$$

The integral is the total surface area of the solid, so we have the general result:[4]

> The mean presented area of a convex solid is one-fourth of the total surface area.

It is a well-known fact that a sphere is the shape that minimizes surface area. With the aid of the above result, we can conclude that the minimum mean presented area of *any* convex solid is

$$\bar{A}_p \geq \frac{1}{4}(\text{surface area of a sphere}) = \pi r^2, \tag{6}$$

where r is the sphere radius. Substituting this into the equation $\bar{A}_p = \bar{\gamma}(m/\rho)^{2/3}$, we find that the average shape factor for any convex solid obey the inequality

$$\boxed{\bar{\gamma} \geq (3/2)^{3/2}(\pi/4)^{1/3} \approx 1.20899}. \tag{7}$$

Notice, in particular, that a right circular cylinder obeys this inequality since is has a minimum average shape factor of $(3/2)(\pi/4)^{1/3} \approx 1.38395$.

[4] This theorem has been attributed to Cauchy. It was first brought to the author's attention by Dr. Robert Shnidman, who also outlined this particular derivation.

Appendix B Solving for L/D given the Average Shape Factor

The shape factor of an RCC that has been uniformly averaged over all possible orientations is given by

$$\bar{\gamma} = \left(\frac{\pi}{4}\right)^{1/3}\left(\frac{L}{D}\right)^{-2/3}\left(\frac{L}{D}+\frac{1}{2}\right) = \left(\frac{\pi}{4}\right)^{1/3}\left[\left(\frac{L}{D}\right)^{1/3}+\frac{1}{2}\left(\frac{L}{D}\right)^{-2/3}\right]. \tag{1}$$

By differentiating this with respect to L/D and setting the result to zero, we find an extremum when $L/D = 1$. The second derivative is positive when $L/D = 1$, which tells us that the extremum is in fact the minimum, and its value there is

$$\bar{\gamma}_{\min} = \frac{3}{2}\left(\frac{\pi}{4}\right)^{1/3} \approx 1.38395. \tag{2}$$

To solve eq. (1) for L/D for a given $\bar{\gamma}$, it is convenient to first set $x = (L/D)^{-1/3}$ since this will eliminate the x^2 term in the cubic:

$$x^3 - 3\frac{\bar{\gamma}}{\bar{\gamma}_{\min}}x + 2 = 0. \tag{3}$$

Comparing this to the standard form, $x^3 + ax^2 + bx + c = 0$, we see that $a = 0$, $b = -3\bar{\gamma}/\bar{\gamma}_{\min}$, and $c = 2$. Also, [5]

$$p = -\frac{a^2}{3} + b = -3\frac{\bar{\gamma}}{\bar{\gamma}_{\min}}, \qquad q = 2\left(\frac{a}{3}\right)^3 - \frac{ab}{3} + c = 2, \qquad Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = 1 - \left(\frac{\bar{\gamma}}{\bar{\gamma}_{\min}}\right)^3. \tag{4}$$

Since $\bar{\gamma} \geq \bar{\gamma}_{\min}$, $Q \leq 0$, which means this is the "irreducible case" and it has a trigonometric solution. The solutions can be described as follows. First, let

$$\cos\alpha = -\frac{q}{2\sqrt{-(p/3)^3}} = -(\bar{\gamma}/\bar{\gamma}_{\min})^{-3/2} \qquad \Longrightarrow \qquad \alpha = \cos^{-1}(-(\bar{\gamma}/\bar{\gamma}_{\min})^{-3/2}), \tag{5}$$

and for our case of $\bar{\gamma} \geq \bar{\gamma}_{\min}$, α is in the second quadrant: $\pi/2 < \alpha \leq \pi$. Then the solutions are

$$\begin{aligned}
x_1 &= +2\sqrt{-p/3}\cos(\alpha/3) &&= +2\sqrt{\bar{\gamma}/\bar{\gamma}_{\min}}\cos(\alpha/3),\\
x_2 &= -2\sqrt{-p/3}\cos(\alpha/3+\pi/3)) &&= -2\sqrt{\bar{\gamma}/\bar{\gamma}_{\min}}\cos(\alpha/3+\pi/3),\\
x_3 &= -2\sqrt{-p/3}\cos(\alpha/3-\pi/3) &&= -2\sqrt{\bar{\gamma}/\bar{\gamma}_{\min}}\cos(\alpha/3-\pi/3).
\end{aligned} \tag{6}$$

The only acceptable solutions are positive, of course, and that eliminates x_3. Finally, $L/D = x^{-3}$. Here is a simple program that implements this solution of the cubic equation for the L/D ratio:

Listing 2. asf2ld.cpp

```cpp
// asf2ld.cpp: Find the L/D ratio of an RCC with the given average shape factor
#include <iostream>
#include <cmath>
using namespace std;

int main( void ) {

   const double G_BAR_MIN = 1.5 * pow( 0.25 * M_PI, 1. / 3. );
   double gBar, r, alpha, x1, x2, l_dMin, l_dMax;
   while ( cin >> gBar ) {

      if ( gBar > G_BAR_MIN ) {
         r = gBar / G_BAR_MIN;
         alpha = acos( -pow( r, -1.5 ) );
         x1 = +2. * sqrt( r ) * cos( alpha / 3. );
         x2 = -2. * sqrt( r ) * cos( ( alpha + M_PI ) / 3. );
         l_dMin = pow( x1, -3. );
         l_dMax = pow( x2, -3. );
      }
      else if ( gBar == G_BAR_MIN ) l_dMin = l_dMax = 1.;
      else {
         cerr << "The minimum average shape factor for an RCC is " << G_BAR_MIN << endl;
         exit( 1 );
      }
      cout << "Two possible values:" << endl;
      cout << "   L/D = " << l_dMin << endl;
      cout << "   L/D = " << l_dMax << endl;
   }
   return 0;
}
```

[5] The notation and solution is taken from pp. 17-18 of G. A. Korn & T. M. Korn, *Manual of Mathematics*, McGraw-Hill, New York, 1967.

B.1 Digression on Solving the Cubic Equation

Without loss of generality, the general cubic can be written as

$$x^3 + ax^2 + bx + c = 0.$$

Setting $x = y - a/3$ eliminates the quadratic term and puts it in the form

$$y^3 + py + q = 0 \tag{a}$$

with

$$p = b - \frac{a^2}{3} \quad \text{and} \quad q = c - \frac{ab}{3} + \frac{2a^3}{27}.$$

A useful device for solving the reduced cubic, eq. (a), is to make use of the trigonometric identity[6]

$$\boxed{4\cos^3\theta - 3\cos\theta - \cos(3\theta) = 0,} \tag{b}$$

which is easily derived from the double angle formulas in the expansion of $\cos(3\theta)$, as follows:

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= (\cos^2\theta - 1 + \cos^2\theta)\cos\theta - 2(1 - \cos^2\theta)\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta.
\end{aligned}$$

To transform eq. (a) into this form, we let $x = s\cos\theta$, where s and θ are to be found, which yields

$$s^3\cos^3\theta + ps\cos\theta + q = 0.$$

Multiplying through by $4/s^3$, we have

$$4\cos^3\theta + \frac{4p}{s^2}\cos\theta + \frac{4q}{s^3} = 0.$$

Now we choose

$$\frac{4p}{s^2} = -3 \qquad \Longrightarrow \qquad s = \sqrt{-\frac{4p}{3}}$$

to give

$$4\cos^3\theta - 3\cos\theta - \frac{3q}{p}\sqrt{-\frac{3}{4p}} = 0.$$

Finally, comparing this with eq. (b), we see that if we choose θ such that

$$\cos(3\theta) = \frac{3q}{p}\sqrt{-\frac{3}{4p}},$$

then the cubic is automatically satisfied, guaranteed by the trigonometric identity. Therefore, we set

$$3\theta = \cos^{-1}\left(\frac{3q}{p}\sqrt{-\frac{3}{4p}}\right) + 2\pi k \quad \text{where } k = 0, \pm 1,$$

and hence the solutions for y are

$$\boxed{y_k = \sqrt{-\frac{4p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{p}\sqrt{-\frac{3}{4p}}\right) + k\frac{2\pi}{3}\right) \quad \text{for } k = 0, \pm 1.} \tag{c}$$

[6] Discovered by François Viète (1540-1603) in 1591. See Jörg Bewersdorff, *Galois Theory for Beginners: A Historical Perspective*, Am. Math. Soc., Providence, RI, 2006.