Exact Analytical Solution for a Particle in a Velocity-Dependent Force Field

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Abstract

Introducing an integrating factor to solve a first-order linear differential equation, in the case of a *vector* equation, also serves as a generator for rotations. In other words, the exponential $\exp(J\omega t)$, where J is a matrix for the vector product, serves a dual role: as an integrating factor to analytically solve the differential equation, and at the same time consolidates the rotations to simple sine and cosine functions, which are readily integrable. This technique[1] is explained and then applied to three velocity-dependent forces found in classical mechanics: the Lorentz force, the Coriolis force, and the centrifugal force.

[1] J. M. Yáñez, G. Gutiérrez, F. González-Cataldo, and D. Laroze, "An exact solution for a particle in a velocity-dependent force field," Am. J. Phys. **89**(12), 1103–1112 (2021).

The coordinate-free formula for the counterclockwise rotation of a vector $\boldsymbol{v}$ about the unit vector $\hat{\boldsymbol{n}}$ through the angle θ is (see the Appendix for a derivation of this formula)

$$R_{\hat{\boldsymbol{n}}}(\theta)\boldsymbol{v} = \boldsymbol{v} + \hat{\boldsymbol{n}} \times \boldsymbol{v} \sin\theta + \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{v})(1 - \cos\theta). \tag{1}$$

Let us transform this formula from a vector equation to a matrix equation.[1,2,3] Define the matrix operator $J(\hat{\boldsymbol{n}})$ as follows.

$$J(\hat{\boldsymbol{n}})\boldsymbol{v} \equiv \hat{\boldsymbol{n}}\times\boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ n_1 & n_2 & n_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\boldsymbol{\imath}}(n_2v_3 - n_3v_2) + \hat{\boldsymbol{\jmath}}(n_3v_1 - n_1v_3) + \hat{\boldsymbol{k}}(n_1v_2 - n_2v_1) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

where $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are unit vectors along x, y, and z axes, respectively, so that

$$J(\hat{\boldsymbol{n}}) \equiv \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}.$$

We see that $[J(\hat{\boldsymbol{n}})]_{ij} = -\epsilon_{ijk}n_k$, $\det J = 0$, which implies J has no inverse, $\operatorname{tr} J = 0$, and $J^\dagger = -J$. Also, most importantly for our purposes,

$$\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{v}) = J(\hat{\boldsymbol{n}})(\hat{\boldsymbol{n}} \times \boldsymbol{v}) = J(\hat{\boldsymbol{n}})J(\hat{\boldsymbol{n}})\boldsymbol{v} = J^2(\hat{\boldsymbol{n}})\boldsymbol{v},$$

so that

$$J^3(\hat{\boldsymbol{n}})\boldsymbol{v} = \hat{\boldsymbol{n}} \times [\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{v})] = \hat{\boldsymbol{n}} \times [\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{v}) - \boldsymbol{v}] = -\hat{\boldsymbol{n}} \times \boldsymbol{v} = -J(\hat{\boldsymbol{n}})\boldsymbol{v}.$$

Hence,

$$J^3 = -J, \qquad J^4 = -J^2, \qquad J^5 = -J^3 = J, \qquad J^6 = J^2, \qquad \ldots \tag{2}$$

For completeness,

$$J^2(\hat{\boldsymbol{n}}) \equiv \begin{bmatrix} -(1-n_1^2) & n_1n_2 & n_1n_3 \\ n_1n_2 & -(1-n_2^2) & n_2n_3 \\ n_1n_3 & n_2n_3 & -(1-n_3^2) \end{bmatrix},$$

although we will not need the explicit form of either J or J^2. Now, since the exponential of a matrix is defined by its power series, and making use of eq. (2), we have

$$\begin{aligned} e^{J\theta} &= I + J\theta + \frac{(J\theta)^2}{2!} + \frac{(J\theta)^3}{3!} + \frac{(J\theta)^4}{4!} + \frac{(J\theta)^5}{5!} + \frac{(J\theta)^6}{6!} + \cdots \\ &= I + J\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \mp \cdots\right) + J^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} \mp \cdots\right) \\ &= I + J\sin\theta + J^2(1-\cos\theta) \end{aligned}$$

so that J is the *generator of rotations* and we can express the rotation operator in eq. (1) as

$$R_{\hat{\boldsymbol{n}}}(\theta) = e^{J(\hat{\boldsymbol{n}})\theta} = I + J(\hat{\boldsymbol{n}})\sin\theta + J^2(\hat{\boldsymbol{n}})(1-\cos\theta). \tag{3}$$

Lorentz Force, $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

The equation of motion for a charged particle of mass m and charge q in the presence of an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, is

$$m\frac{d\boldsymbol{v}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

which we can rewrite as

$$\frac{d\boldsymbol{v}}{dt} = \frac{q}{m}\boldsymbol{E} - \frac{qB}{m}\hat{\boldsymbol{b}} \times \boldsymbol{v} = \frac{\omega}{B}\boldsymbol{E} - \omega\hat{\boldsymbol{b}} \times \boldsymbol{v} = \frac{\omega}{B}\boldsymbol{E} - \omega J(\hat{\boldsymbol{b}})\boldsymbol{v}$$

where $\omega \equiv qB/m$, $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$, and $J(\hat{\boldsymbol{b}})\boldsymbol{v} = \hat{\boldsymbol{b}} \times \boldsymbol{v}$, or

$$\frac{d\boldsymbol{v}}{dt} + \omega J(\hat{\boldsymbol{b}})\boldsymbol{v} = \frac{\omega}{B}\boldsymbol{E}.$$

An integrating factor for this first-order linear differential equation is $e^{J(\hat{\boldsymbol{b}})\omega t}$, which gives

$$\frac{d}{dt}\left(e^{J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{v}\right) = \frac{\omega}{B}e^{J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{E}.$$

Integrating,

$$e^{J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{v}(t) = \boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t e^{J(\hat{\boldsymbol{b}})\omega\xi}d\xi\right)\boldsymbol{E},$$

and

$$\boldsymbol{v}(t) = e^{-J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t e^{-J(\hat{\boldsymbol{b}})\omega t}e^{J(\hat{\boldsymbol{b}})\omega\xi}d\xi\right)\boldsymbol{E}.$$

Let $\tau = t - \xi$, $d\tau = -d\xi$, then the solution for the velocity is

$$\boldsymbol{v}(t) = e^{-J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t e^{-J(\hat{\boldsymbol{b}})\omega\tau}d\tau\right)\boldsymbol{E}.$$

Integrating once more, the solution for the position is

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \left(\int_0^t e^{-J(\hat{\boldsymbol{b}})\omega\tau}d\tau\right)\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t\int_0^\tau e^{-J(\hat{\boldsymbol{b}})\omega\xi}d\xi d\tau\right)\boldsymbol{E}.$$

Making use of eq. (3), this can be written as

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \left(\int_0^t R_{\hat{\boldsymbol{b}}}(-\omega\tau)\,d\tau\right)\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t\int_0^\tau R_{\hat{\boldsymbol{b}}}(-\omega\xi)\,d\xi d\tau\right)\boldsymbol{E}.$$

Also from eq. (3),

$$\begin{aligned}\int_0^t R_{\hat{\boldsymbol{b}}}(-\omega\tau)d\tau &= \int_0^t [I - \sin(\omega\tau)J(\hat{\boldsymbol{b}}) + (1-\cos(\omega\tau))J^2(\hat{\boldsymbol{b}})]d\tau \\ &= \left[\tau + \left(\frac{\cos\omega\tau}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\tau - \frac{\sin\omega\tau}{\omega}\right)J^2(\hat{\boldsymbol{b}})\right]_0^t \\ &= t + \left(\frac{\cos\omega t - 1}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(t - \frac{\sin\omega t}{\omega}\right)J^2(\hat{\boldsymbol{b}})\end{aligned}$$

and

$$\begin{aligned}\int_0^t\int_0^\tau R_{\hat{\boldsymbol{b}}}(-\omega\xi)d\xi d\tau &= \int_0^t\left[\tau + \left(\frac{\cos\omega\tau - 1}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\tau - \frac{\sin\omega\tau}{\omega}\right)J^2(\hat{\boldsymbol{b}})\right]d\tau \\ &= \left[\frac{\tau^2}{2} + \left(\frac{\sin\omega\tau}{\omega^2} - \frac{\tau}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\frac{\tau^2}{2} + \frac{\cos\omega\tau}{\omega^2}\right)J^2(\hat{\boldsymbol{b}})\right]_0^t \\ &= \frac{t^2}{2} + \left(\frac{\sin\omega t}{\omega^2} - \frac{t}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\frac{t^2}{2} + \frac{\cos\omega t - 1}{\omega^2}\right)J^2(\hat{\boldsymbol{b}})\end{aligned}$$

so that

$$\begin{aligned}
\boldsymbol{r}(t) &= \boldsymbol{r}_0 + \left(\int_0^t R_{\hat{\boldsymbol{b}}}(-\omega\tau)d\tau\right)\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t\int_0^\tau R_{\hat{\boldsymbol{b}}}(-\omega\xi)d\xi d\tau\right)\boldsymbol{E} \\
&= \boldsymbol{r}_0 + \left[t + \left(\frac{\cos\omega t - 1}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(t - \frac{\sin\omega t}{\omega}\right)J^2(\hat{\boldsymbol{b}})\right]\boldsymbol{v}_0 \\
&\quad + \frac{\omega}{B}\left[\frac{t^2}{2} + \left(\frac{\sin\omega t}{\omega^2} - \frac{t}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\frac{t^2}{2} + \frac{\cos\omega t - 1}{\omega^2}\right)J^2(\hat{\boldsymbol{b}})\right]\boldsymbol{E} \\
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \left(\frac{\cos\omega t - 1}{\omega}\right)\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 + \left(t - \frac{\sin\omega t}{\omega}\right)\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) \\
&\quad + \frac{\omega}{B}\left[\frac{\boldsymbol{E}t^2}{2} + \left(\frac{\sin\omega t}{\omega^2} - \frac{t}{\omega}\right)\hat{\boldsymbol{b}}\times\boldsymbol{E} + \left(\frac{t^2}{2} + \frac{\cos\omega t - 1}{\omega^2}\right)\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right] \\
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{\omega t^2}{2B}\Big(\boldsymbol{E} + \hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\Big) + \left(\frac{1-\cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 - \frac{1}{B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right) \\
&\quad + \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) - \frac{1}{B}\hat{\boldsymbol{b}}\times\boldsymbol{E}\right) \\
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{\omega t^2}{2B}\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\boldsymbol{E}) + \left(\frac{1-\cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 - \frac{1}{B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right) \\
&\quad + \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) - \frac{1}{B}\hat{\boldsymbol{b}}\times\boldsymbol{E}\right)
\end{aligned}$$

Exact analytical solution for the case of the Lorentz force, $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$

$$\begin{aligned}
\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{\omega t^2}{2B}\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\boldsymbol{E}) + \left(\frac{1-\cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 - \frac{1}{B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right) \\
+ \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) - \frac{1}{B}(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right)
\end{aligned}$$

where $\omega \equiv qB/m$ and $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}/B$.

Let us consider a particular case with the electric field along the z-axis and the magnetic field along the x-axis: $\boldsymbol{E} = E\hat{\boldsymbol{k}}$, $\boldsymbol{B} = B\hat{\boldsymbol{\imath}}$, and with the initial conditions that the charged particle is placed at the origin with zero initial velocity.[4] The position of the particle as a function of time is

$$\begin{aligned}
\boldsymbol{r}(t) &= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{\omega t^2}{2B}\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\boldsymbol{E}) + \left(\frac{1-\cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 - \frac{1}{B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right) + \\
&\quad \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) - \frac{1}{B}(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right) \\
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \left(\frac{1-\cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{\imath}}\times\boldsymbol{v}_0 - \frac{E}{B}\hat{\boldsymbol{k}}\right) + \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{\imath}}\times(\hat{\boldsymbol{\imath}}\times\boldsymbol{v}_0) + \frac{E}{B}\hat{\boldsymbol{\jmath}}\right) \\
&= \left(\frac{1-\cos\omega t}{\omega}\right)\frac{E}{B}\hat{\boldsymbol{k}} + \left(t - \frac{\sin\omega t}{\omega}\right)\frac{E}{B}\hat{\boldsymbol{\jmath}}
\end{aligned}$$

Thus we have

$$y = (\omega t - \sin\omega t)\frac{E}{\omega B} \quad \text{and} \quad z = (1 - \cos\omega t)\frac{E}{\omega B}$$

Define $R \equiv \dfrac{E}{\omega B}$, then we have

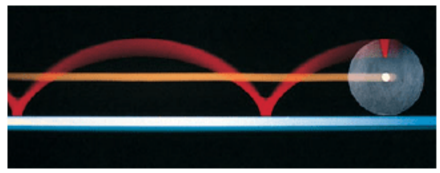
$$y = R\omega t - R\sin\omega t \quad \text{and} \quad z = R - R\cos\omega t$$

We can combine these two equations by solving the first for $\sin\omega t$ and the second for $\cos\omega t$ and making use of the trigonometric identity $\sin^2\omega t + \cos^2\omega t = 1$ to get

$$(y - R\omega t)^2 + (z - R)^2 = R^2.$$

This is the equation of a circle with center at $(0, R\omega t, R)$, moving along the y-axis at a speed of $v = \omega R$. As shown in the following figure, it follows the path of a cycloid.

Figure 11-2 A time-exposure photograph of a rolling disk. Small lights have been attached to the disk, one at its center and one at its edge. The latter traces out a curve called a *cycloid*.



Richard Megna/Fundamental Photographs

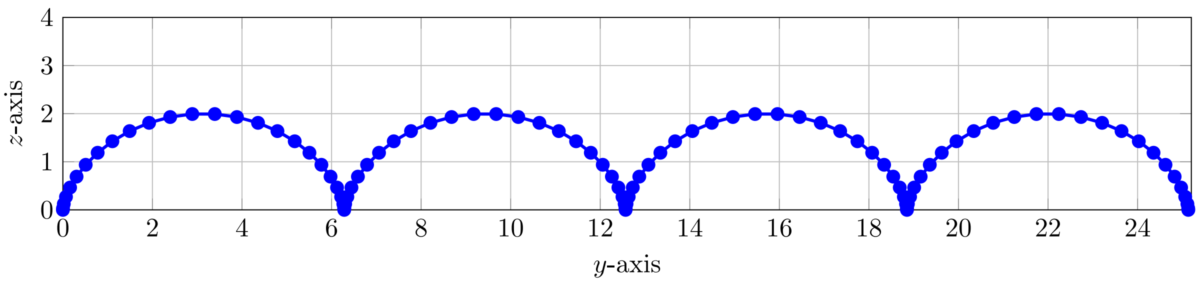


Figure 1. Trajectory of a charged particle with the electric field along the z-axis (up) and the magnetic field along the x-axis (out of the page). The charged particle begins at the origin with zero initial velocity and moves along a cycloid in the y-z plane.

The particle trajectory was generated by the following program, which makes use of the C++ Vector class `Vector.h`.

Listing 1. lorentz.cpp

```
// lorentz.cpp: Solution to equation of motion for Lorentz force

#include "Vector.h"
#include <iostream>
#include <cmath>
using namespace va;   // vector algebra namespace
using namespace std;

int main( void ) {

   const Vector ihat( 1., 0., 0. );   // unit vector along x-axis
   const Vector jhat( 0., 1., 0. );   // unit vector along y-axis
   const Vector khat( 0., 0., 1. );   // unit vector along z-axis

   const double M     = 1.;
   const double Q     = 1.;
   const double EE    = 1.;
   const double BB    = 1.;
   const double OMEGA = Q * BB / M;
   const double T     = 8. * M_PI / ( OMEGA );
   double t, theta;

   const Vector r0( 0., 0., 0. );
   const Vector v0( 0., 0., 0. );
   const Vector E( 0., 0., EE );   // electric field along z-axis
   const Vector B( BB, 0., 0. );   // magnetic field along x-axis
   const Vector b = unit( B );     // unit vector along magnetic field

   Vector r;
   cout << "t" << "\t" << "y" << "\t" << "z" << endl;
   for ( t = 0; t <= T; t += 0.01 * T ) {

      theta = OMEGA * t;
      r = r0 + v0 * t + 0.5 * ( OMEGA * t * t / BB ) * b * ( b * E ) +
          ( ( 1. - cos( theta ) ) / OMEGA ) * ( -( b ^ v0 ) - ( b ^ ( b ^ E ) ) / BB ) +
          ( t - sin( theta ) / OMEGA ) * ( ( b ^ ( b ^ v0 ) ) - ( b ^ E ) / BB );
      cout << t << "\t" << r * jhat << "\t" << r * khat << endl;
   }
   return EXIT_SUCCESS;
}
```

Coriolis Force, $\boldsymbol{F} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega}\times\boldsymbol{v}$

The equation of motion for a particle of mass m in the presence of gravity while in a non-inertial frame of reference with rotational velocity $\boldsymbol{\Omega}$ is

$$m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega}\times\boldsymbol{v},$$

which we can rewrite as

$$\frac{d\boldsymbol{v}}{dt} + 2\Omega\,J(\hat{\boldsymbol{\omega}})\boldsymbol{v} = \boldsymbol{g},$$

where $\hat{\boldsymbol{\omega}} = \boldsymbol{\Omega}/\Omega$ and $J(\hat{\boldsymbol{\omega}})\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{v}$. An integrating factor is $e^{2J(\hat{\boldsymbol{\omega}})\Omega t}$, which gives

$$\frac{d}{dt}\left(e^{2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{v}\right) = e^{2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{g}$$

and integrates to

$$e^{2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{v}(t) = \boldsymbol{v}_0 + \left(\int_0^t e^{2J(\hat{\boldsymbol{b}})\Omega\xi}d\xi\right)\boldsymbol{g}$$

or

$$\boldsymbol{v}(t) = e^{-2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{v}_0 + \left(\int_0^t e^{-2J(\hat{\boldsymbol{\omega}})\Omega t}e^{2J(\hat{\boldsymbol{b}})\Omega\xi}d\xi\right)\boldsymbol{g}.$$

Changing variables with $\tau = t-\xi$, $d\tau = -d\xi$, we have

$$\boldsymbol{v}(t) = e^{-2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{v}_0 + \left(\int_0^t e^{-2J(\hat{\boldsymbol{\omega}})\Omega\tau}d\tau\right)\boldsymbol{g},$$

and the solution for the position vector is

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{r}_0 + \left(\int_0^t e^{-2J(\hat{\boldsymbol{\omega}})\Omega\tau}d\tau\right)\boldsymbol{v}_0 + \left(\int_0^t\int_0^\tau e^{-2J(\hat{\boldsymbol{\omega}})\Omega\xi}d\xi d\tau\right)\boldsymbol{g}\\ &= \boldsymbol{r}_0 + \left(\int_0^t R_{\hat{\boldsymbol{\omega}}}(-2\Omega\tau)\,d\tau\right)\boldsymbol{v}_0 + \left(\int_0^t\int_0^\tau R_{\hat{\boldsymbol{\omega}}}(-2\Omega\xi)\,d\xi d\tau\right)\boldsymbol{g}.\end{aligned}$$

Evaluating the integrals, we have

$$\begin{aligned}\int_0^t R_{\hat{\boldsymbol{\omega}}}(-2\Omega\tau)d\tau &= \int_0^t[I - \sin(2\Omega\tau)J(\hat{\boldsymbol{\omega}}) + (1-\cos(2\Omega\tau))J^2(\hat{\boldsymbol{\omega}})]d\tau\\ &= \left[\tau + \left(\frac{\cos 2\Omega\tau}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(\tau - \frac{\sin 2\Omega\tau}{2\Omega}\right)J^2(\hat{\boldsymbol{\omega}})\right]_0^t\\ &= t + \left(\frac{\cos 2\Omega t - 1}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(t - \frac{\sin 2\Omega t}{2\Omega}\right)J^2(\hat{\boldsymbol{\omega}})\\ \int_0^t\int_0^\tau R_{\hat{\boldsymbol{\omega}}}(-2\Omega\xi)d\xi d\tau &= \int_0^t\left[\tau + \left(\frac{\cos 2\Omega\tau - 1}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(\tau - \frac{\sin 2\Omega\tau}{2\Omega}\right)J^2(\hat{\boldsymbol{\omega}})\right]d\tau\\ &= \left[\frac{\tau^2}{2} + \left(\frac{\sin 2\Omega\tau}{4\Omega^2} - \frac{\tau}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(\frac{\tau^2}{2} + \frac{\cos 2\Omega\tau}{4\Omega^2}\right)J^2(\hat{\boldsymbol{\omega}})\right]_0^t\\ &= \frac{t^2}{2} + \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2}\right)J^2(\hat{\boldsymbol{\omega}})\end{aligned}$$

Thus,

$$\begin{aligned}
\boldsymbol{r}(t) = \boldsymbol{r}_0 &+ \left[t + \left(\frac{\cos 2\Omega t - 1}{2\Omega}\right) J(\hat{\boldsymbol{\omega}}) + \left(t - \frac{\sin 2\Omega t}{2\Omega}\right) J^2(\hat{\boldsymbol{\omega}})\right] \boldsymbol{v}_0 \\
&+ \left[\frac{t^2}{2} + \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega}\right) J(\hat{\boldsymbol{\omega}}) + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2}\right) J^2(\hat{\boldsymbol{\omega}})\right] \boldsymbol{g} \\
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \left(\frac{\cos 2\Omega t - 1}{2\Omega}\right) \hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 + \left(t - \frac{\sin 2\Omega t}{2\Omega}\right) \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0) \\
&+ \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega}\right) \hat{\boldsymbol{\omega}} \times \boldsymbol{g} + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2}\right) \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g}) \\
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{1}{2}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})]t^2 - \left(\frac{1 - \cos 2\Omega t}{2\Omega}\right) \left(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 + \frac{1}{2\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})\right) \\
&- \left(t - \frac{\sin 2\Omega t}{2\Omega}\right) \left(\frac{1}{2\Omega}\hat{\boldsymbol{\omega}} \times \boldsymbol{g} - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right)
\end{aligned}$$

Exact analytical solution for the case of the Coriolis force, $\boldsymbol{F} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega} \times \boldsymbol{v}$

$$\begin{aligned}
\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t &+ \frac{1}{2}\boldsymbol{g}t^2 + \frac{1}{2}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})t^2 \\
&- \left(\frac{1 - \cos 2\Omega t}{2\Omega}\right) \left(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 + \frac{1}{2\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})\right) \\
&- \left(t - \frac{\sin 2\Omega t}{2\Omega}\right) \left(\frac{1}{2\Omega}\hat{\boldsymbol{\omega}} \times \boldsymbol{g} - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right)
\end{aligned}$$

This is an exact solution for all values of Ω, but in the case when $\Omega \ll 1$, we may expand this in powers of Ω to compare to previous approximations. We have

$$\begin{aligned}
\left(\frac{1 - \cos 2\Omega t}{2\Omega}\right) &= \frac{1}{2\Omega}\left(\frac{4\Omega^2}{2!}t^2 - \frac{16\Omega^4}{4!}t^4 \pm \cdots\right) = \Omega t^2 - \frac{1}{3}\Omega^3 t^4 \pm \cdots \\
\left(t - \frac{\sin 2\Omega t}{2\Omega}\right) &= t - \frac{1}{2\Omega}\left(2\Omega t - \frac{8\Omega^3}{3!}t^3 \pm \cdots\right) = \frac{2}{3}\Omega^2 t^3 \mp \cdots
\end{aligned}$$

and we find, to first order in Ω,

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \Omega\left(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 t^2 + \frac{1}{3}\hat{\boldsymbol{\omega}} \times \boldsymbol{g}t^3\right) + \mathcal{O}(\Omega^2) \tag{4}$$

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \boldsymbol{\Omega} \times \boldsymbol{v}_0 t^2 - \frac{1}{3}\boldsymbol{\Omega} \times \boldsymbol{g}t^3 + \mathcal{O}(\Omega^2), \tag{5}$$

which agrees with the approximation in *Mechanics* by Landau and Lifshitz.[5] In particular, we verify that $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2$ when $\boldsymbol{\Omega} \to 0$.

The paths of a particle under the force of gravity and deviated by the Coriolis force are shown in Figure 2.

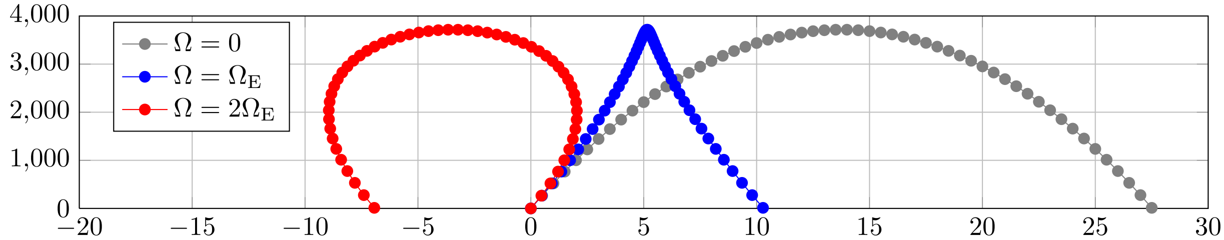


Figure 2. Trajectories of a particle under the force of gravity alone (solid gray) and under gravity and deviated by the Coriolis force (colored circles). The z axis (vertical) corresponds to the radial direction of the Earth. The positive y axis (horizontal) corresponds to East. The same initial conditions lead to a qualitatively different trajectory (red) when the angular speed of the Earth, Ω_{E}, is doubled. Initial velocity, $\boldsymbol{v}_0 = (0, 0.5, 270)$ m/s, and latitude $= 30°$ in all three cases.

These particle trajectories were generated by the following program, which makes use of the C++ Vector class `Vector.h`.

Listing 2. coriolis.cpp

```
// coriolis.cpp

#include "Vector.h"
#include <iostream>
using namespace va;
using namespace std;

int main( void ) {

   const Vector ihat( 1., 0., 0. );   // points south
   const Vector jhat( 0., 1., 0. );   // points east
   const Vector khat( 0., 0., 1. );   // points up (radially from the earth)

   const double VX          = 0., VY = 0.5, VZ = 260., H = 0.;
   const double ALPHA       = atan( VZ / VY );
   const double V0          = sqrt( VX * VX + VY * VY + VZ * VZ );
   const double OMEGA_EARTH = 7.29e-5, G = 9.81, LAT = 30. * D2R;
   const double OMEGA       = OMEGA_EARTH;
   const double TWO_OMEGA   = 2. * OMEGA_EARTH;
   const double COS_LAT     = cos( LAT ), SIN_LAT = sin( LAT );
   const double COS_ALPHA   = cos( ALPHA ), SIN_ALPHA = sin( ALPHA );
   const Vector omega       = Vector( -COS_LAT, 0., SIN_LAT );   // unit vector
   const Vector g           = -G * khat;
   const Vector r0          = Vector( 0., 0., H );
   const Vector v0          = Vector( 0., V0 * COS_ALPHA, V0 * SIN_ALPHA );
   double T;
   if ( H == 0. )
      T  = 2. * V0 * SIN_ALPHA / ( G - ( 2. * OMEGA ) * COS_LAT * V0 * COS_ALPHA );
   else
      T = sqrt( 2. * H / G );

   double f1 = 0.5 * ( 1. - cos( 2. * OMEGA * T ) ) / OMEGA;
   double f2 = T - 0.5 * sin( 2. * OMEGA * T ) / OMEGA;

   Vector w1 = omega ^ g;
   Vector w2 = omega ^ w1;
   Vector g1 = ( omega ^ g ) / ( 2. * OMEGA );
   Vector g2 = omega ^ g1;
   Vector v1 = omega ^ v0;
   Vector v2 = omega ^ v1;

   Vector r, r1, r2, r_prime, g1_prime, g2_prime;
   double t2, f1_prime, f2_prime;
   cout << "t" << "\t" << "y0" << "\t" << "y1" << "\t" << "y2" << "\t" << "z" << endl;
   for ( double t = 0.; t <= T; t += 1. ) {

      t2 = t * t;
      f1 = 0.5 * ( 1. - cos( 2. * OMEGA * t ) ) / OMEGA;
      f2 = t - 0.5 * sin( 2. * OMEGA * t ) / OMEGA;
      f1_prime = 0.5 * ( 1. - cos( 2. * TWO_OMEGA * t ) ) / TWO_OMEGA;
      f2_prime = t - 0.5 * sin( 2. * TWO_OMEGA * t ) / TWO_OMEGA;
      g1_prime = ( omega ^ g ) / ( 2. * TWO_OMEGA );
      g2_prime = omega ^ g1_prime;
      r1 = r0 + v0 * t + 0.5 * g * t2;
      r2 = 0.5 * w2 * t2;
      r = r1 + r2 - f1 * ( v1 + g2 ) - f2 * ( g1 - v2 );
      r_prime = r1 + r2 - f1_prime * ( v1 + g2_prime ) - f2_prime * ( g1_prime - v2 );
      cout << t << "\t"           // elapsed time
           << r1 * jhat << "\t"         // distance east with OMEGA = 0
           << r * jhat << "\t"          // distance east
           << r_prime * jhat << "\t"    // distance east with OMEGA = 2. * OMEGA_EARTH
           << r1 * khat << endl;        // height
   }
   return EXIT_SUCCESS;
}
```

Coriolis + Centrifugal Force, $\boldsymbol{F} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega}\times\boldsymbol{v} - m\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$

The rate of change of any vector in an inertial frame of reference (*fixed frame*) is equal to the rate of change of the vector in the *rotating frame* plus the vector product of the angular velocity, $\boldsymbol{\Omega}$, with the given vector.[6] Thus we have the operator equation:

$$\left(\frac{d}{dt}\right)_{\text{fixed}} = \left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega}\times$$

Applied to the position vector, we have

$$\begin{aligned}
\dot{\boldsymbol{r}}_{\text{fixed}} &= \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega}\times\right]\boldsymbol{r} = \dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r} \\
\ddot{\boldsymbol{r}}_{\text{fixed}} &= \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega}\times\right](\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r}) \\
&= \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega}\times\right]\dot{\boldsymbol{r}} + \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega}\times\right](\boldsymbol{\Omega}\times\boldsymbol{r}) \\
&= \ddot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\dot{\boldsymbol{r}} + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r} + \boldsymbol{\Omega}\times\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) \\
&= \ddot{\boldsymbol{r}} + \underbrace{2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}} + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r}}_{\text{Coriolis force}} + \underbrace{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})}_{\text{Centrifugal force}}
\end{aligned}$$

The total force is

$$\boldsymbol{F} = m\boldsymbol{a}_{\text{fixed}} = m\boldsymbol{a}_{\text{rotating}} + 2m\boldsymbol{\Omega}\times\boldsymbol{v}_{\text{rotating}} + m\dot{\boldsymbol{\Omega}}\times\boldsymbol{r} + m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$$

For an observer in the rotating frame, the effective force is

$$\boldsymbol{F}_{\text{eff}} \equiv m\boldsymbol{a}_{\text{rotating}} = m\boldsymbol{g} - 2m\boldsymbol{\Omega}\times\dot{\boldsymbol{r}} - m\dot{\boldsymbol{\Omega}}\times\boldsymbol{r} - m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$$

Thus, the equation of motion in the *rotating frame* is

$$\begin{aligned}
\ddot{\boldsymbol{r}} + 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}} + \dot{\boldsymbol{\Omega}}\times\boldsymbol{r} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) &= \boldsymbol{g} \\
\ddot{\boldsymbol{r}} + 2\Omega J(\hat{\boldsymbol{\omega}})\dot{\boldsymbol{r}} + [\dot{\Omega}J(\hat{\boldsymbol{\omega}}) + \Omega^2 J^2(\hat{\boldsymbol{\omega}})]\boldsymbol{r} &= \boldsymbol{g}
\end{aligned}$$

We rewrite this as[3]

$$\ddot{\boldsymbol{r}} + 2A(t)\dot{\boldsymbol{r}} + B(t)\boldsymbol{r} = \boldsymbol{g} \tag{6}$$

where $A(t)$ and $B(t)$ are the matrices $A(t) \equiv \Omega J(\hat{\boldsymbol{\omega}})$ and $B(t) \equiv \dot{\Omega}J(\hat{\boldsymbol{\omega}}) + \Omega^2 J^2(\hat{\boldsymbol{\omega}})$. Let $\boldsymbol{r} = V(t)\,\boldsymbol{y}$, then

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \dot{V}\boldsymbol{y} + V\dot{\boldsymbol{y}} \\
\ddot{\boldsymbol{r}} &= \ddot{V}\boldsymbol{y} + 2\dot{V}\dot{\boldsymbol{y}} + V\ddot{\boldsymbol{y}},
\end{aligned}$$

which transforms eq. (6) into

$$\begin{aligned}
\ddot{V}\boldsymbol{y} + 2\dot{V}\dot{\boldsymbol{y}} + V\ddot{\boldsymbol{y}} + 2A(\dot{V}\boldsymbol{y} + V\dot{\boldsymbol{y}}) + BV\boldsymbol{y} &= \boldsymbol{g} \\
V\ddot{\boldsymbol{y}} + 2(\dot{V} + AV)\dot{\boldsymbol{y}} + (\ddot{V} + 2A\dot{V} + BV)\boldsymbol{y} &= \boldsymbol{g}
\end{aligned} \tag{7}$$

And now we choose V to eliminate the $\dot{\boldsymbol{y}}$ term:

$$\dot{V} + AV = 0 \quad\Longrightarrow\quad \dot{V} = -AV \quad\Longrightarrow\quad V = e^{-At}.$$

Also

$$\dot{V} = -AV \quad\Longrightarrow\quad \ddot{V} = -\dot{A}V - A\dot{V} = -\dot{A}V + A^2V,$$

so that eq. (7) becomes

$$\begin{aligned}V\ddot{\boldsymbol{y}} + (-\dot{A}V + A^2V - 2A^2V + BV)\boldsymbol{y} &= \boldsymbol{g}\\ V\ddot{\boldsymbol{y}} + (B - A^2 - \dot{A})V\boldsymbol{y} &= \boldsymbol{g}\end{aligned} \tag{8}$$

Now, using the expressions for A and B, we find that the coefficient of $\boldsymbol{y}$ also vanishes:

$$B - A^2 - \dot{A} = \dot{\Omega}J(\hat{\boldsymbol{\omega}}) + \Omega^2J^2(\hat{\boldsymbol{\omega}}) - \Omega^2J^2(\hat{\boldsymbol{\omega}}) - \dot{\Omega}J(\hat{\boldsymbol{\omega}}) = 0$$

Thus, eq. (8) reduces to simply

$$V\ddot{\boldsymbol{y}} = \boldsymbol{g}$$

and therefore we have

$$\ddot{\boldsymbol{y}} = V^{-1}\boldsymbol{g} = e^{At}\boldsymbol{g} = e^{J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{g} = [I + J(\hat{\boldsymbol{\omega}})\sin\Omega t + J^2(\hat{\boldsymbol{\omega}})(1-\cos\Omega t)]\boldsymbol{g}$$

For convenience, let's write J for $J(\hat{\boldsymbol{\omega}})$. Starting with

$$\ddot{\boldsymbol{y}} = [I + \sin\Omega t J + (1-\cos\Omega t)J^2]\boldsymbol{g}$$

and integrating over time twice, we get

$$\begin{aligned}\dot{\boldsymbol{y}}(t) &= \dot{\boldsymbol{y}}(0) + \left(\int_0^t [I + \sin\Omega\tau J + (1-\cos\Omega\tau)J^2]d\tau\right)\boldsymbol{g}\\ &= \dot{\boldsymbol{y}}(0) + \boldsymbol{g}t + \frac{1-\cos\Omega t}{\Omega}J\boldsymbol{g} + \left(t - \frac{\sin\Omega t}{\Omega}\right)J^2\boldsymbol{g}\\ \boldsymbol{y}(t) &= \boldsymbol{y}(0) + \dot{\boldsymbol{y}}(0)t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{t - \Omega^{-1}\sin\Omega t}{\Omega}J\boldsymbol{g} + \frac{\Omega\frac{1}{2}t^2 + \Omega^{-1}(\cos\Omega t - 1)}{\Omega}J^2\boldsymbol{g}\\ &= \boldsymbol{y}(0) + \dot{\boldsymbol{y}}(0)t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{\Omega t - \sin\Omega t}{\Omega^2}J\boldsymbol{g} + \frac{\Omega^2\frac{1}{2}t^2 + \cos\Omega t - 1}{\Omega^2}J^2\boldsymbol{g}\end{aligned}$$

Now, $\boldsymbol{r}_0 = \boldsymbol{y}(0)$ and $\boldsymbol{v}_0 \equiv \dot{r}_0 = \dot{\boldsymbol{y}}(0) - \Omega J\boldsymbol{y}(0)$, so that we can make the replacements

$$\begin{aligned}\boldsymbol{y}(0) &\implies \boldsymbol{r}_0\\ \dot{\boldsymbol{y}}(0) &\implies \boldsymbol{v}_0 + \Omega J\boldsymbol{r}_0\end{aligned}$$

Also, $\boldsymbol{r}(t)$ and $\boldsymbol{y}(t)$ are related by the transformation $\boldsymbol{r}(t) = e^{-J\Omega t}\boldsymbol{y}(t)$. Thus, we have

$$\begin{aligned}\boldsymbol{r}(t) &= e^{-J\Omega t}\left[\boldsymbol{r}_0 + \boldsymbol{v}_0t + \Omega J\boldsymbol{r}_0t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{\Omega t - \sin\Omega t}{\Omega^2}J\boldsymbol{g} + \frac{\frac{1}{2}\Omega^2t^2 + \cos\Omega t - 1}{\Omega^2}J^2\boldsymbol{g}\right]\\ &= e^{-J\Omega t}\left\{\boldsymbol{r}_0 + \boldsymbol{v}_0t + \frac{1}{2}\boldsymbol{g}t^2 - \frac{1}{\Omega^2}\underbrace{\left[I + \sin\Omega tJ + (1-\cos\Omega t)J^2\right]}_{e^{J\Omega t}}\boldsymbol{g} + \Omega J\boldsymbol{r}_0t + \frac{1}{\Omega^2}\left[I + \Omega tJ + \frac{1}{2}\Omega^2t^2J^2\right]\boldsymbol{g}\right\}\\ &= -\frac{\boldsymbol{g}}{\Omega^2} + \left[I - \sin\Omega tJ + (1-\cos\Omega t)J^2\right]\left\{(\boldsymbol{r}_0 + \boldsymbol{v}_0t + \frac{1}{2}\boldsymbol{g}t^2) + \Omega J\boldsymbol{r}_0t + \frac{1}{\Omega^2}(I + \Omega tJ + \frac{1}{2}\Omega^2t^2J^2)\boldsymbol{g}\right\}\\ &= -\frac{\boldsymbol{g}}{\Omega^2} + (\boldsymbol{r}_0 + \boldsymbol{v}_0t + \frac{1}{2}\boldsymbol{g}t^2) - \sin\Omega tJ(\boldsymbol{r}_0 + \boldsymbol{v}_0t + \frac{1}{2}\boldsymbol{g}t^2) + (1-\cos\Omega t)J^2(\boldsymbol{r}_0 + \boldsymbol{v}_0t + \frac{1}{2}\boldsymbol{g}t^2)\\ &\quad + \Omega J\boldsymbol{r}_0t - \Omega\sin\Omega tJ^2\boldsymbol{r}_0t - (1-\cos\Omega t)\Omega J\boldsymbol{r}_0t + \frac{1}{\Omega^2}(I + \Omega tJ + \frac{1}{2}\Omega^2t^2J^2)\boldsymbol{g}\\ &\quad - \sin\Omega t\frac{1}{\Omega^2}(J + \Omega tJ^2 - \frac{1}{2}\Omega^2t^2J)\boldsymbol{g} + (1-\cos\Omega t)\frac{1}{\Omega^2}(J^2 - \Omega tJ - \frac{1}{2}\Omega^2t^2J^2)\boldsymbol{g}\end{aligned}$$

where we used $J^3 = -J$ and $J^4 = -J^2$. Collecting terms, we have

$$
\begin{aligned}
\boldsymbol{r}(t) = &-\frac{\boldsymbol{g}}{\Omega^2} + \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{\boldsymbol{g}}{\Omega^2} \\
&+ J\left[-\sin\Omega t(\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2) + \Omega\boldsymbol{r}_0 t - \Omega\boldsymbol{r}_0 t + \cos\Omega t\,\Omega\boldsymbol{r}_0 t + \frac{\boldsymbol{g}t}{\Omega} - \frac{\sin\Omega t}{\Omega^2}\boldsymbol{g} + \frac{1}{2}\sin\Omega t\,\boldsymbol{g}t^2 - \frac{\boldsymbol{g}t}{\Omega} + \frac{\cos\Omega t}{\Omega}\boldsymbol{g}t\right] \\
&+ J^2\left[(1-\cos\Omega t)(\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2) - \Omega\sin\Omega t\,\boldsymbol{r}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \frac{\Omega t\sin\Omega t}{\Omega^2}\boldsymbol{g} + \frac{1-\cos\Omega t}{\Omega^2}\boldsymbol{g} - (1-\cos\omega t)\frac{1}{2}\boldsymbol{g}t^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + J\left[-\sin\Omega t\,(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) + \Omega\cos\Omega t\,\boldsymbol{r}_0 t - \frac{\sin\Omega t}{\Omega^2}\boldsymbol{g} + \frac{\cos\Omega t}{\Omega}\boldsymbol{g}t\right] \\
&\quad + J^2\left[(1-\cos\Omega t)(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) - \Omega\sin\Omega t\,\boldsymbol{r}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{1-\cos\Omega t - \Omega t\sin\Omega t}{\Omega^2}\boldsymbol{g}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + J\left[-\sin\Omega t\,(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) + \Omega\cos\Omega t\,\boldsymbol{r}_0 t + \frac{\Omega t\cos\Omega t - \sin\Omega t}{\Omega^2}\boldsymbol{g}\right] \\
&\quad + J^2\left[(1-\cos\Omega t)(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) - \Omega\sin\Omega t\,\boldsymbol{r}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{1-\cos\Omega t - \Omega t\sin\Omega t}{\Omega^2}\boldsymbol{g}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + J\left[-\sin\Omega t\,(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) + \Omega t\cos\Omega t\,\boldsymbol{r}_0 + \frac{\Omega t\cos\Omega t - \sin\Omega t}{\Omega^2}\boldsymbol{g}\right] \\
&\quad + J^2\left[\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \cos\Omega t(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) - \Omega t\sin\Omega t\,\boldsymbol{r}_0 + \frac{\boldsymbol{g}}{\Omega^2} - \frac{\cos\Omega t + \Omega t\sin\Omega t}{\Omega^2}\boldsymbol{g}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{\boldsymbol{g}}{\Omega^2})] - \sin\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) - \cos\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0)] \\
&\quad - \Omega t\sin\Omega t\frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0) - \Omega t\cos\Omega t\frac{1}{\Omega}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)] + \Omega t\cos\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) - \Omega t\sin\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0)] \\
&\quad - \sin\Omega t(\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) - \cos\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2})] + \Omega t\cos\Omega t(\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) - \Omega t\sin\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2})]
\end{aligned}
$$

Introducing $\boldsymbol{r}_0^* \equiv \boldsymbol{r}_0 + \boldsymbol{g}/\Omega^2$, we can write this as

$$
\begin{aligned}
\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 &+ \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0^* + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2)] - \sin\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) - \cos\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*)] \\
&- \Omega t\sin\Omega t\frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0) - \Omega t\cos\Omega t\frac{1}{\Omega}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)] + \Omega t\cos\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) - \Omega t\sin\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*)]
\end{aligned}
$$

Exact analytical solution for the case of both the Coriolis and the Centrifugal force
$\boldsymbol{F} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega} \times \boldsymbol{v} - m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$

$$
\begin{aligned}
\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 &+ \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0^* + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2)] - \sin\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) - \cos\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*)] \\
&- \Omega t\sin\Omega t\left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) + \frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right] + \Omega t\cos\Omega t\left[(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) - \frac{1}{\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right]
\end{aligned}
$$

Again, this result is exact. And again, let us expand in powers of Ω to compare to previous approximations.

$$\begin{aligned}\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \frac{\boldsymbol{g}}{\Omega^2} + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2)] \\ - \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \cdots\right)[\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \frac{\boldsymbol{g}}{\Omega^2})] \\ - \left(1 - \frac{(\Omega t)^2}{2!} \pm \cdots\right)[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \frac{\boldsymbol{g}}{\Omega^2}))] \\ - \Omega t \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \cdots\right)\left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \frac{\boldsymbol{g}}{\Omega^2})) + \frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right] \\ + \Omega t \left(1 - \frac{(\Omega t)^2}{2!} \pm \cdots\right)\left[(\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \frac{\boldsymbol{g}}{\Omega^2})) - \frac{1}{\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right]\end{aligned}$$

$$\begin{aligned}\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2)] + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) \\ - \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \cdots\right)[\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0 + \hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}] \\ - \left(1 - \frac{(\Omega t)^2}{2!} + \frac{(\Omega t)^4}{4!} \mp \cdots\right)[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2})] \\ - \Omega t \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \cdots\right)\left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) + \frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right] \\ + \Omega t \left(1 - \frac{(\Omega t)^2}{2!} \pm \cdots\right)\left[(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) + (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) - \frac{1}{\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right]\end{aligned}$$

$$\begin{aligned}\boldsymbol{r}(t) = \ & \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2)] + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) - \Omega[\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0 + \hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}]t + \frac{(\Omega t)^3}{3!}\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2} \\ & - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) + \Omega^2[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2})]\frac{t^2}{2} - \Omega^4[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2})]\frac{t^4}{24} \\ & - \Omega^2\left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) + \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) + \frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right]t^2 + \Omega^4[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2})]\frac{t^4}{6} \\ & + \Omega\left[(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) + (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) - \frac{1}{\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right]t - \Omega^3\left[(\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) - \frac{1}{\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right]\frac{t^3}{2} + \mathcal{O}(\Omega^3)\end{aligned}$$

Thus, the second-order approximation reduces to

$$\begin{aligned}\boldsymbol{r}(t) = \ & \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \Omega\left(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 t^2 + \frac{1}{3}\hat{\boldsymbol{\omega}} \times \boldsymbol{g}t^3\right) \\ & + \Omega^2\left(-\frac{1}{2}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0)t^2 + \frac{1}{2}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)t^3 + \frac{1}{8}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})t^4\right) + \mathcal{O}(\Omega^3)\end{aligned} \tag{9}$$

$$\begin{aligned}\boldsymbol{r}(t) = \ & \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \boldsymbol{\Omega} \times \boldsymbol{v}_0 t^2 - \frac{1}{3}\boldsymbol{\Omega} \times \boldsymbol{g}t^3 \\ & - \frac{1}{2}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_0)t^2 + \frac{1}{2}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{v}_0)t^3 + \frac{1}{8}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{g})t^4 + \mathcal{O}(\Omega^3)\end{aligned} \tag{10}$$

We see that if we neglect terms of $\mathcal{O}(\Omega^2)$ and higher that it reduces to that found for the Coriolis force, eqs. (4) and (5). We also verify that $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2$ when $\Omega \to 0$.

Let's compute the deflection of a particle falling freely under the Earth's gravity from a height h and at a northern latitude λ. We take the z-axis to be vertically outward from the surface of the earth, the x-axis to point south, and the y-axis to point east, to form a right-handed coordinate system, as shown in Figure 3.

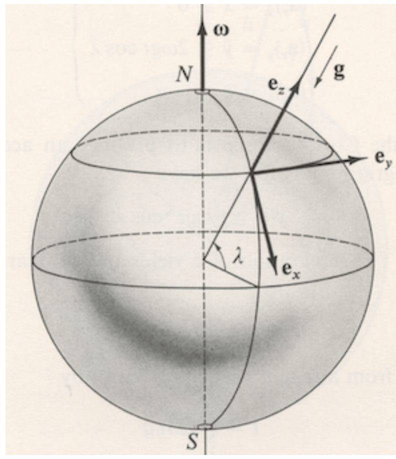


Figure 3. Earth-frame coordinate system reproduced from J. B. Marion, *Classical Dynamics of Particles and Systems*.[5] Replace $\boldsymbol{\omega} \to \hat{\boldsymbol{\omega}}$, $\hat{\boldsymbol{e}}_x \to \hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{e}}_y \to \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{e}}_z \to \hat{\boldsymbol{k}}$ to be consistent with the notation used here.

We have $\hat{\boldsymbol{\omega}} = -\cos\lambda\hat{\boldsymbol{\imath}} + \sin\lambda\hat{\boldsymbol{k}}$, $\boldsymbol{r}_0 = h\hat{\boldsymbol{k}}$, $\boldsymbol{v}_0 = 0$, and $\boldsymbol{g} = -g\hat{\boldsymbol{k}}$, so that

$$
\begin{aligned}
\boldsymbol{r}(t) &= h\hat{\boldsymbol{k}} - \frac{1}{2}gt^2\hat{\boldsymbol{k}} + \Omega\frac{1}{3}(\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})gt^3 - \Omega^2\frac{1}{2}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})]ht^2 - \Omega^2\frac{1}{8}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})]gt^4 \\
&= (h - \frac{1}{2}gt^2)\hat{\boldsymbol{k}} + \Omega\frac{1}{3}(\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})gt^3 - \Omega^2[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})](\frac{1}{2}ht^2 + \frac{1}{8}gt^4)
\end{aligned}
$$

The total time of fall is $t = \sqrt{2h/g}$, so that $h - \frac{1}{2}gt^2 = 0$, and $\frac{1}{2}ht^2 + \frac{1}{8}gt^4 = \frac{h^2}{g} + \frac{1}{2}\frac{h^2}{g} = \frac{3}{2}\frac{h^2}{g}$ Therefore,

$$
\boldsymbol{r} = \Omega\frac{1}{3}(\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})g\left(\frac{2h}{g}\right)^{3/2} - \Omega^2\frac{3}{2}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})]\frac{h^2}{g} \tag{11}
$$

Now

$$
\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & 0 & 1 \end{vmatrix} = \cos\lambda\hat{\boldsymbol{\jmath}}
$$

and

$$
\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}}) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & \cos\lambda & 0 \end{vmatrix} = -\sin\lambda\cos\lambda\hat{\boldsymbol{\imath}} - \cos^2\lambda\hat{\boldsymbol{k}}
$$

so that eq. (11) becomes

$$
\begin{aligned}
\boldsymbol{r} &= \frac{1}{3}\sqrt{\frac{8h^3}{g}}\Omega\cos\lambda\hat{\boldsymbol{\jmath}} + \frac{3}{2}\frac{h^2}{g}\Omega^2(\sin\lambda\cos\lambda\hat{\boldsymbol{\imath}} + \cos^2\lambda\hat{\boldsymbol{k}}) \\
&= \frac{3}{2}\frac{h^2\Omega^2}{g}\sin\lambda\cos\lambda\hat{\boldsymbol{\imath}} + \frac{1}{3}\sqrt{\frac{8h^3}{g}}\Omega\cos\lambda\hat{\boldsymbol{\jmath}} + \frac{3}{2}\frac{h^2\Omega^2}{g}\cos^2\lambda\hat{\boldsymbol{k}}
\end{aligned}
$$

For a height $h = 100$ m, latitude $\lambda = 45°$, and $\Omega = 7.29 \times 10^{-5}$ rad/s for the earth's rotational velocity, we get a (first-order) easterly deflection (along $\hat{\boldsymbol{\jmath}}$) of 1.55 cm, and a (second-order) southerly deflection (along $\hat{\boldsymbol{\imath}}$) of 4×10^{-4} cm.

The following program uses the exact formula to compute the position of the particle when it hits the ground and also the approximate values cited above.

Listing 3. centrifugal.cpp

```
// centrifugal.cpp

#include "Vector.h"
#include <iostream>
#include <cstdlib>
#include <cmath>
using namespace va;   // vector algebra namespace
using namespace std;

int main( void ) {

   Vector ihat( 1., 0., 0. ), jhat( 0., 1., 0. ), khat( 0., 0., 1. );   // unit vectors for cartesian coordinate system

   const double OMEGA  = 7.29e-5;     // Earth's rotational velocity
   const double G      = 9.81;        // acceleration due to gravity (m/s^2)
   const double H      = 100.;        // height (m)
   const double LAT    = 45. * D2R;   // northern latitude (deg converted to rad)
   const double OMEGA2 = OMEGA * OMEGA;
   const double T      = sqrt( 2. * H / G );   // time to fall
   const double T2      = T * T;
   const double T3      = T * T2;
   const double T4      = T2 * T2;
   const double THETA   = OMEGA * T;
   const double COS     = cos( THETA );
   const double SIN     = sin( THETA );
   const double COS_LAT = cos( LAT );
   const double SIN_LAT = sin( LAT );

   const Vector omega   = Vector( -COS_LAT, 0., SIN_LAT );   // unit vector for Earth's rotation velocity
   const Vector g       = -G * khat;
   const Vector r0      = Vector( 0., 0., H );
   const Vector r0_star = r0 + g / OMEGA2;
   const Vector v0      = Vector( 0., 0., 0. );

   const Vector R1 = r0 + v0 * T + 0.5 * g * T2;
   const Vector R2 = R1 + g / OMEGA2;
   const Vector R3 = omega ^ R2;
   const Vector R4 = omega ^ R3;
   const Vector R5 = omega ^ r0_star;
   const Vector R6 = omega ^ R5;

   const Vector W1 = omega ^ g;
   const Vector W2 = omega ^ W1;
   const Vector R0 = omega ^ ( omega ^ r0 );
   const Vector V1 = omega ^ v0;
   const Vector V2 = omega ^ V1;

   Vector r_exact  = R1 + R4 - SIN * R5 - COS * R6 - THETA * SIN * ( R6 + V1 / OMEGA ) + THETA * COS * ( R5 - V2 / OMEGA );
   Vector r_approx = R1 - OMEGA * ( V1 * T2 + W1 * T3 / 3. ) + OMEGA2 * ( -0.5 * R0 * T2 + 0.5 * V2 * T3 + 0.125 * W2 * T4 );

   cout.precision(6);
   cout << "r_exact (cm)  = " << std::fixed << r_exact * 100. << endl;    // output in cm
   cout << "r_approx (cm) = " << std::fixed << r_approx * 100. << endl;   // output in cm

   return EXIT_SUCCESS;
}
```

The output from running `./centrifugal` is

```
r_exact (cm)  = 0.000400 1.551678 0.000400
r_approx (cm) = 0.000406 1.551679 0.000406
```

References

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5. L. Landau and M. Lifshitz, *Mechanics* (Butterworth-Heinemann, Oxford,1970), Vol. 1.

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Appendix: Rotation of a Vector

We want to derive a formula for the counterclockwise rotation, through the angle θ, of an arbitrary vector $\boldsymbol{a}$ about the unit vector $\hat{\boldsymbol{u}}$. We start with the identity

$$\boldsymbol{a} = \underbrace{\boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}}_{\boldsymbol{a}_\perp} + \underbrace{(\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}}_{\boldsymbol{a}_\parallel},$$

where the third term on the right is the component of $\boldsymbol{a}$ that is parallel to $\hat{\boldsymbol{u}}$ and so will remain unchanged after a rotation about $\hat{\boldsymbol{u}}$:

$$\boldsymbol{a}'_\parallel = \boldsymbol{a}_\parallel.$$

The first two terms form the component of $\boldsymbol{a}$ that is perpendicular to $\hat{\boldsymbol{u}}$ and will be rotated into

$$\boldsymbol{a}'_\perp = [\boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] \cos\theta + \hat{\boldsymbol{u}} \times [\boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] \sin\theta.$$

Hence,

$$\begin{aligned}
\boldsymbol{a}' &\equiv R_{\hat{\boldsymbol{u}}}(\theta)\, \boldsymbol{a} \\
&= \boldsymbol{a}'_\perp + \boldsymbol{a}'_\parallel \\
&= [\boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] \cos\theta + \hat{\boldsymbol{u}} \times [\boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] \sin\theta + (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}} \\
&= [\boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] \cos\theta + \hat{\boldsymbol{u}} \times \boldsymbol{a} \sin\theta + (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}.
\end{aligned}$$

Now,

$$\hat{\boldsymbol{u}} \times (\boldsymbol{a} \times \hat{\boldsymbol{u}}) = \boldsymbol{a}(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}) - \hat{\boldsymbol{u}}(\hat{\boldsymbol{u}} \cdot \boldsymbol{a}) = \boldsymbol{a} - (\boldsymbol{a} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}},$$

and therefore

$$\begin{aligned}
\boldsymbol{a}' &= \hat{\boldsymbol{u}} \times (\boldsymbol{a} \times \hat{\boldsymbol{u}}) \cos\theta + \hat{\boldsymbol{u}} \times \boldsymbol{a} \sin\theta + \boldsymbol{a} - \hat{\boldsymbol{u}} \times (\boldsymbol{a} \times \hat{\boldsymbol{u}}) \\
&= -\hat{\boldsymbol{u}} \times (\hat{\boldsymbol{u}} \times \boldsymbol{a}) \cos\theta + \hat{\boldsymbol{u}} \times \boldsymbol{a} \sin\theta + \boldsymbol{a} + \hat{\boldsymbol{u}} \times (\hat{\boldsymbol{u}} \times \boldsymbol{a}) \\
&= \boldsymbol{a} + \hat{\boldsymbol{u}} \times \boldsymbol{a}\, \sin\theta + \hat{\boldsymbol{u}} \times (\hat{\boldsymbol{u}} \times \boldsymbol{a})\, (1 - \cos\theta).
\end{aligned}$$

Thus, we have

Rotation of a Vector

$$R_{\hat{\boldsymbol{u}}}(\theta)\, \boldsymbol{a} = \boldsymbol{a} + \hat{\boldsymbol{u}} \times \boldsymbol{a}\, \sin\theta + \hat{\boldsymbol{u}} \times (\hat{\boldsymbol{u}} \times \boldsymbol{a})\, (1 - \cos\theta)$$