Exact Analytical Solution for a Particle in a Velocity-Dependent Force Field

Richard Saucier

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Abstract

Introducing an integrating factor to solve a first-order linear differential equation, in the case of a *vector* equation, also serves as a generator for rotations. In other words, the exponential $\exp(J\omega t)$, where J is a matrix for the vector product, serves a dual role: as an integrating factor to analytically solve the differential equation, and at the same time consolidates the rotations to simple sine and cosine functions, which are readily integrable. This technique[1] is explained and then applied to three velocity-dependent forces found in classical mechanics: the Lorentz force, the Coriolis force, and the centrifugal force.

[1] J. M. Yáñez, G. Gutiérrez, F. González-Cataldo, and D. Laroze, "An exact solution for a particle in a velocity-dependent force field," Am. J. Phys. **89**(12), 1103–1112 (2021).

The coordinate-free formula for the counterclockwise rotation of a vector $\boldsymbol{v}$ about the unit vector $\hat{\boldsymbol{n}}$ through the angle θ is

$$R_{\hat{\boldsymbol{n}}}(\theta)\boldsymbol{v} = \boldsymbol{v} + \hat{\boldsymbol{n}} \times \boldsymbol{v}\sin\theta + \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{v})(1 - \cos\theta). \tag{1}$$

Let us transform this formula from a vector equation to a matrix equation.[1,2,3] Define the matrix operator $J(\hat{\boldsymbol{n}})$ as follows.

$$J(\hat{\boldsymbol{n}})\boldsymbol{v} \equiv \hat{\boldsymbol{n}} \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ n_1 & n_2 & n_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\boldsymbol{i}}(n_2 v_3 - n_3 v_2) + \hat{\boldsymbol{j}}(n_3 v_1 - n_1 v_3) + \hat{\boldsymbol{k}}(n_1 v_2 - n_2 v_1) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

where $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, and $\hat{\boldsymbol{k}}$ are unit vectors along x, y, and z axes, respectively, so that

$$J(\hat{\boldsymbol{n}}) \equiv \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}.$$

We see that $[J(\hat{\boldsymbol{n}})]_{ij} = -\epsilon_{ijk}n_k$, $\det J = 0$, which implies J has no inverse, $\operatorname{tr} J = 0$, and $J^\dagger = -J$. Also, most importantly for our purposes,

$$\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{v}) = J(\hat{\boldsymbol{n}})(\hat{\boldsymbol{n}} \times \boldsymbol{v}) = J(\hat{\boldsymbol{n}})J(\hat{\boldsymbol{n}})\boldsymbol{v} = J^2(\hat{\boldsymbol{n}})\boldsymbol{v},$$

so that

$$J^3(\hat{\boldsymbol{n}})\boldsymbol{v} = \hat{\boldsymbol{n}} \times [\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{v})] = \hat{\boldsymbol{n}} \times [\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{v}) - \boldsymbol{v}] = -\hat{\boldsymbol{n}} \times \boldsymbol{v} = -J(\hat{\boldsymbol{n}})\boldsymbol{v}.$$

Hence,

$$J^3 = -J, \qquad J^4 = -J^2, \qquad J^5 = -J^3 = J, \qquad J^6 = J^2, \qquad \dots \tag{2}$$

For completeness,

$$J^2(\hat{\boldsymbol{n}}) \equiv \begin{bmatrix} -(1 - n_1^2) & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & -(1 - n_2^2) & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & -(1 - n_3^2) \end{bmatrix},$$

although we will not need the explicit form of either J or J^2. Now, since the exponential of a matrix is defined by its power series, and making use of eq. (2), we have

$$\begin{aligned} e^{J\theta} &= I + J\theta + \frac{(J\theta)^2}{2!} + \frac{(J\theta)^3}{3!} + \frac{(J\theta)^4}{4!} + \frac{(J\theta)^5}{5!} + \frac{(J\theta)^6}{6!} + \cdots \\ &= I + J\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \mp \cdots\right) + J^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} \mp \cdots\right) \\ &= I + J\sin\theta + J^2(1 - \cos\theta) \end{aligned}$$

so that J is the *generator of rotations* and we can express the rotation operator in eq. (1) as

$$R_{\hat{\boldsymbol{n}}}(\theta) = e^{J(\hat{\boldsymbol{n}})\theta} = I + J(\hat{\boldsymbol{n}})\sin\theta + J^2(\hat{\boldsymbol{n}})(1 - \cos\theta). \tag{3}$$

Lorentz Force, $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

The equation of motion for a charged particle of mass m and charge q in the presence of an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, is

$$m\frac{d\boldsymbol{v}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

which we can rewrite as

$$\frac{d\boldsymbol{v}}{dt} = \frac{q}{m}\boldsymbol{E} - \frac{qB}{m}\hat{\boldsymbol{b}} \times \boldsymbol{v} = \frac{\omega}{B}\boldsymbol{E} - \omega\hat{\boldsymbol{b}} \times \boldsymbol{v} = \frac{\omega}{B}\boldsymbol{E} - \omega J(\hat{\boldsymbol{b}})\boldsymbol{v}$$

where $\omega \equiv qB/m$, $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$, and $J(\hat{\boldsymbol{b}})\boldsymbol{v} = \hat{\boldsymbol{b}} \times \boldsymbol{v}$, or

$$\frac{d\boldsymbol{v}}{dt} + \omega J(\hat{\boldsymbol{b}})\boldsymbol{v} = \frac{\omega}{B}\boldsymbol{E}.$$

An integrating factor for this first-order linear differential equation is $e^{J(\hat{\boldsymbol{b}})\omega t}$, which gives

$$\frac{d}{dt}\left(e^{J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{v}\right) = \frac{\omega}{B}e^{J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{E}.$$

Integrating,

$$e^{J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{v}(t) = \boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t e^{J(\hat{\boldsymbol{b}})\omega\xi}d\xi\right)\boldsymbol{E},$$

and

$$\boldsymbol{v}(t) = e^{-J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t e^{-J(\hat{\boldsymbol{b}})\omega t}e^{J(\hat{\boldsymbol{b}})\omega\xi}d\xi\right)\boldsymbol{E}.$$

Let $\tau = t - \xi$, $d\tau = -d\xi$, then the solution for the velocity is

$$\boldsymbol{v}(t) = e^{-J(\hat{\boldsymbol{b}})\omega t}\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t e^{-J(\hat{\boldsymbol{b}})\omega\tau}d\tau\right)\boldsymbol{E}.$$

Integrating once more, the solution for the position is

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \left(\int_0^t e^{-J(\hat{\boldsymbol{b}})\omega\tau}d\tau\right)\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t \int_0^\tau e^{-J(\hat{\boldsymbol{b}})\omega\xi}d\xi d\tau\right)\boldsymbol{E}.$$

Making use of eq. (3), this can be written as

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \left(\int_0^t R_{\hat{\boldsymbol{b}}}(-\omega\tau)\,d\tau\right)\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t \int_0^\tau R_{\hat{\boldsymbol{b}}}(-\omega\xi)\,d\xi d\tau\right)\boldsymbol{E}.$$

Also from eq. (3),

$$\int_0^t R_{\hat{\boldsymbol{b}}}(-\omega\tau)d\tau = \int_0^t [I - \sin(\omega\tau)J(\hat{\boldsymbol{b}}) + (1 - \cos(\omega\tau))J^2(\hat{\boldsymbol{b}})]d\tau$$

$$= \left[\tau + \left(\frac{\cos\omega\tau}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\tau - \frac{\sin\omega\tau}{\omega}\right)J^2(\hat{\boldsymbol{b}})\right]_0^t$$

$$= t + \left(\frac{\cos\omega t - 1}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(t - \frac{\sin\omega t}{\omega}\right)J^2(\hat{\boldsymbol{b}})$$

and

$$\int_0^t \int_0^\tau R_{\hat{\boldsymbol{b}}}(-\omega\xi)d\xi d\tau = \int_0^t \left[\tau + \left(\frac{\cos\omega\tau - 1}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\tau - \frac{\sin\omega\tau}{\omega}\right)J^2(\hat{\boldsymbol{b}})\right]d\tau$$

$$= \left[\frac{\tau^2}{2} + \left(\frac{\sin\omega\tau}{\omega^2} - \frac{\tau}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\frac{\tau^2}{2} + \frac{\cos\omega\tau}{\omega^2}\right)J^2(\hat{\boldsymbol{b}})\right]_0^t$$

$$= \frac{t^2}{2} + \left(\frac{\sin\omega t}{\omega^2} - \frac{t}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\frac{t^2}{2} + \frac{\cos\omega t - 1}{\omega^2}\right)J^2(\hat{\boldsymbol{b}})$$

so that

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \left(\int_0^t R_{\hat{\boldsymbol{b}}}(-\omega\tau)d\tau\right)\boldsymbol{v}_0 + \frac{\omega}{B}\left(\int_0^t\int_0^\tau R_{\hat{\boldsymbol{b}}}(-\omega\xi)d\xi d\tau\right)\boldsymbol{E}$$

$$= \boldsymbol{r}_0 + \left[t + \left(\frac{\cos\omega t - 1}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(t - \frac{\sin\omega t}{\omega}\right)J^2(\hat{\boldsymbol{b}})\right]\boldsymbol{v}_0$$

$$+ \frac{\omega}{B}\left[\frac{t^2}{2} + \left(\frac{\sin\omega t}{\omega^2} - \frac{t}{\omega}\right)J(\hat{\boldsymbol{b}}) + \left(\frac{t^2}{2} + \frac{\cos\omega t - 1}{\omega^2}\right)J^2(\hat{\boldsymbol{b}})\right]\boldsymbol{E}$$

$$= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \left(\frac{\cos\omega t - 1}{\omega}\right)\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 + \left(t - \frac{\sin\omega t}{\omega}\right)\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0)$$

$$+ \frac{\omega}{B}\left[\frac{\boldsymbol{E}t^2}{2} + \left(\frac{\sin\omega t}{\omega^2} - \frac{t}{\omega}\right)\hat{\boldsymbol{b}}\times\boldsymbol{E} + \left(\frac{t^2}{2} + \frac{\cos\omega t - 1}{\omega^2}\right)\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right]$$

$$= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{\omega t^2}{2B}\left(\boldsymbol{E} + \hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right) + \left(\frac{1 - \cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 - \frac{1}{B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right)$$

$$+ \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) - \frac{1}{B}\hat{\boldsymbol{b}}\times\boldsymbol{E}\right)$$

$$= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{\omega t^2}{2B}\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\boldsymbol{E}) + \left(\frac{1 - \cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 - \frac{1}{B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right)$$

$$+ \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) - \frac{1}{B}\hat{\boldsymbol{b}}\times\boldsymbol{E}\right)$$

Exact analytical solution for the case of the Lorentz force, $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{\omega t^2}{2B}\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\boldsymbol{E}) + \left(\frac{1 - \cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 - \frac{1}{B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right)$$

$$+ \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) - \frac{1}{B}(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right)$$

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where $\omega \equiv qB/m$ and $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}/B$.

Let us consider a particular case with the electric field along the z-axis and the magnetic field along the x-axis: $\boldsymbol{E} = E\hat{\boldsymbol{k}}$, $\boldsymbol{B} = B\hat{\boldsymbol{\imath}}$, and with the initial conditions that the charged particle is placed at the origin with zero initial velocity.[4] The position of the particle as a function of time is

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{\omega t^2}{2B}\hat{\boldsymbol{b}}(\hat{\boldsymbol{b}}\cdot\boldsymbol{E}) + \left(\frac{1 - \cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{b}}\times\boldsymbol{v}_0 - \frac{1}{B}\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right) +$$

$$\left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{b}}\times(\hat{\boldsymbol{b}}\times\boldsymbol{v}_0) - \frac{1}{B}(\hat{\boldsymbol{b}}\times\boldsymbol{E})\right)$$

$$= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \left(\frac{1 - \cos\omega t}{\omega}\right)\left(-\hat{\boldsymbol{\imath}}\times\boldsymbol{v}_0 - \frac{E}{B}\hat{\boldsymbol{k}}\right) + \left(t - \frac{\sin\omega t}{\omega}\right)\left(\hat{\boldsymbol{\imath}}\times(\hat{\boldsymbol{\imath}}\times\boldsymbol{v}_0) + \frac{E}{B}\hat{\boldsymbol{\jmath}}\right)$$

$$= \left(\frac{1 - \cos\omega t}{\omega}\right)\frac{E}{B}\hat{\boldsymbol{k}} + \left(t - \frac{\sin\omega t}{\omega}\right)\frac{E}{B}\hat{\boldsymbol{\jmath}}$$

Thus we have

$$y = (\omega t - \sin\omega t)\frac{E}{\omega B} \quad\text{and}\quad z = (1 - \cos\omega t)\frac{E}{\omega B}$$

Define $R \equiv \dfrac{E}{\omega B}$, then we have

$$y = R\omega t - R\sin\omega t \quad\text{and}\quad z = R - R\cos\omega t$$

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We can combine these two equations by solving the first for $\sin \omega t$ and the second for $\cos \omega t$ and making use of the trigonometric identity $\sin^2 \omega t + \cos^2 \omega t = 1$ to get

$$(y - R\omega t)^2 + (z - R)^2 = R^2.$$

This is the equation of a circle with center at $(0, R\omega t, R)$, moving along the y-axis at a speed of $v = \omega R$. As shown in the following figure, it follows the path of a cycloid.

Figure 11-2 A time-exposure photograph of a rolling disk. Small lights have been attached to the disk, one at its center and one at its edge. The latter traces out a curve called a *cycloid*.
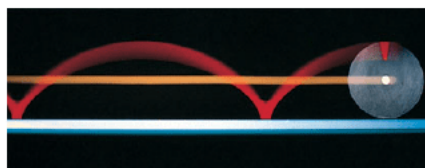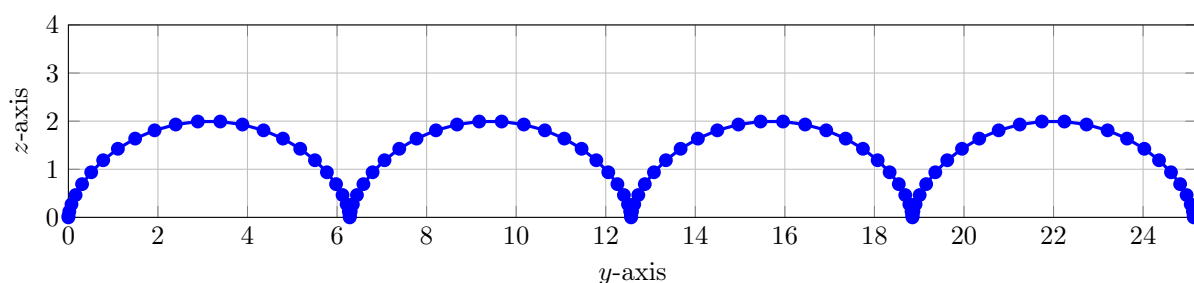
Richard Megna/Fundamental Photographs

Figure 1. Trajectory of a charged particle with the electric field along the z-axis (up) and the magnetic field along the x-axis (out of the page). The charged particle begins at the origin with zero initial velocity and moves along a cycloid in the y-z plane.

The particle trajectory was generated by the following program, which makes use of the C++ Vector class `Vector.h`.

Listing 1. lorentz.cpp

```cpp
// lorentz.cpp: Solution to equation of motion for Lorentz force

#include "Vector.h"
#include <iostream>
#include <cmath>
using namespace va;     // vector algebra namespace
using namespace std;

int main( void ) {

    const Vector ihat( 1., 0., 0. );   // unit vector along x-axis
    const Vector jhat( 0., 1., 0. );   // unit vector along y-axis
    const Vector khat( 0., 0., 1. );   // unit vector along z-axis

    const double M     = 1.;
    const double Q     = 1.;
    const double EE    = 1.;
    const double BB    = 1.;
    const double OMEGA = Q * BB / M;
    const double T     = 8. * M_PI / ( OMEGA );
    double t, theta;

    const Vector r0( 0., 0., 0. );
    const Vector v0( 0., 0., 0. );
    const Vector E( 0., 0., EE );   // electric field along z-axis
    const Vector B( BB, 0., 0. );   // magnetic field along x-axis
    const Vector b = unit( B );     // unit vector along magnetic field

    Vector r;
    cout << "t" << "\t" << "y" << "\t" << "z" << endl;
    for ( t = 0; t <= T; t += 0.01 * T ) {

        theta = OMEGA * t;
        r = r0 + v0 * t + 0.5 * ( OMEGA * t * t / BB ) * b * ( b * E ) +
            ( ( 1. - cos( theta ) ) / OMEGA ) * ( -( b ^ v0 ) - ( b ^ ( b ^ E ) ) / BB ) +
            ( t - sin( theta ) / OMEGA ) * ( ( b ^ ( b ^ v0 ) ) - ( b ^ E ) / BB );
        cout << t << "\t" << r * jhat << "\t" << r * khat << endl;
    }
    return EXIT_SUCCESS;
}
```

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Coriolis Force, $\boldsymbol{F} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega} \times \boldsymbol{v}$

The equation of motion for a particle of mass m in the presence of gravity while in a non-inertial frame of reference with rotational velocity $\boldsymbol{\Omega}$ is

$$m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega} \times \boldsymbol{v},$$

which we can rewrite as

$$\frac{d\boldsymbol{v}}{dt} + 2\Omega\,J(\hat{\boldsymbol{\omega}})\boldsymbol{v} = \boldsymbol{g},$$

where $\hat{\boldsymbol{\omega}} = \boldsymbol{\Omega}/\Omega$ and $J(\hat{\boldsymbol{\omega}})\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v}$. An integrating factor is $e^{2J(\hat{\boldsymbol{\omega}})\Omega t}$, which gives

$$\frac{d}{dt}\left(e^{2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{v}\right) = e^{2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{g}$$

and integrates to

$$e^{2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{v}(t) = \boldsymbol{v}_0 + \left(\int_0^t e^{2J(\hat{\boldsymbol{b}})\Omega\xi}d\xi\right)\boldsymbol{g}$$

or

$$\boldsymbol{v}(t) = e^{-2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{v}_0 + \left(\int_0^t e^{-2J(\hat{\boldsymbol{\omega}})\Omega t}e^{2J(\hat{\boldsymbol{b}})\Omega\xi}d\xi\right)\boldsymbol{g}.$$

Changing variables with $\tau = t - \xi$, $d\tau = -d\xi$, we have

$$\boldsymbol{v}(t) = e^{-2J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{v}_0 + \left(\int_0^t e^{-2J(\hat{\boldsymbol{\omega}})\Omega\tau}d\tau\right)\boldsymbol{g},$$

and the solution for the position vector is

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \left(\int_0^t e^{-2J(\hat{\boldsymbol{\omega}})\Omega\tau}d\tau\right)\boldsymbol{v}_0 + \left(\int_0^t \int_0^\tau e^{-2J(\hat{\boldsymbol{\omega}})\Omega\xi}d\xi d\tau\right)\boldsymbol{g}$$

$$= \boldsymbol{r}_0 + \left(\int_0^t R_{\hat{\boldsymbol{\omega}}}(-2\Omega\tau)\,d\tau\right)\boldsymbol{v}_0 + \left(\int_0^t \int_0^\tau R_{\hat{\boldsymbol{\omega}}}(-2\Omega\xi)\,d\xi d\tau\right)\boldsymbol{g}.$$

Evaluating the integrals, we have

$$\int_0^t R_{\hat{\boldsymbol{\omega}}}(-2\Omega\tau)d\tau = \int_0^t [I - \sin(2\Omega\tau)J(\hat{\boldsymbol{\omega}}) + (1 - \cos(2\Omega\tau))J^2(\hat{\boldsymbol{\omega}})]d\tau$$

$$= \left[\tau + \left(\frac{\cos 2\Omega\tau}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(\tau - \frac{\sin 2\Omega\tau}{2\Omega}\right)J^2(\hat{\boldsymbol{\omega}})\right]_0^t$$

$$= t + \left(\frac{\cos 2\Omega t - 1}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(t - \frac{\sin 2\Omega t}{2\Omega}\right)J^2(\hat{\boldsymbol{\omega}})$$

$$\int_0^t \int_0^\tau R_{\hat{\boldsymbol{\omega}}}(-2\Omega\xi)d\xi d\tau = \int_0^t \left[\tau + \left(\frac{\cos 2\Omega\tau - 1}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(\tau - \frac{\sin 2\Omega\tau}{2\Omega}\right)J^2(\hat{\boldsymbol{\omega}})\right]d\tau$$

$$= \left[\frac{\tau^2}{2} + \left(\frac{\sin 2\Omega\tau}{4\Omega^2} - \frac{\tau}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(\frac{\tau^2}{2} + \frac{\cos 2\Omega\tau}{4\Omega^2}\right)J^2(\hat{\boldsymbol{\omega}})\right]_0^t$$

$$= \frac{t^2}{2} + \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega}\right)J(\hat{\boldsymbol{\omega}}) + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2}\right)J^2(\hat{\boldsymbol{\omega}})$$

Thus,

$$\begin{aligned}
\boldsymbol{r}(t) = \boldsymbol{r}_0 &+ \left[t + \left(\frac{\cos 2\Omega t - 1}{2\Omega}\right) J(\hat{\boldsymbol{\omega}}) + \left(t - \frac{\sin 2\Omega t}{2\Omega}\right) J^2(\hat{\boldsymbol{\omega}})\right] \boldsymbol{v}_0 \\
&+ \left[\frac{t^2}{2} + \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega}\right) J(\hat{\boldsymbol{\omega}}) + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2}\right) J^2(\hat{\boldsymbol{\omega}})\right] \boldsymbol{g} \\
= \boldsymbol{r}_0 &+ \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \left(\frac{\cos 2\Omega t - 1}{2\Omega}\right) \hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 + \left(t - \frac{\sin 2\Omega t}{2\Omega}\right) \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0) \\
&+ \left(\frac{\sin 2\Omega t}{4\Omega^2} - \frac{t}{2\Omega}\right) \hat{\boldsymbol{\omega}} \times \boldsymbol{g} + \left(\frac{t^2}{2} + \frac{\cos 2\Omega t - 1}{4\Omega^2}\right) \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g}) \\
= \boldsymbol{r}_0 &+ \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{1}{2}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})]t^2 - \left(\frac{1 - \cos 2\Omega t}{2\Omega}\right)\left(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 + \frac{1}{2\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})\right) \\
&- \left(t - \frac{\sin 2\Omega t}{2\Omega}\right)\left(\frac{1}{2\Omega}\hat{\boldsymbol{\omega}} \times \boldsymbol{g} - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right)
\end{aligned}$$

Exact analytical solution for the case of the Coriolis force, $\boldsymbol{F} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega} \times \boldsymbol{v}$

$$\begin{aligned}
\boldsymbol{r}(t) = \boldsymbol{r}_0 &+ \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{1}{2}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})t^2 \\
&- \left(\frac{1 - \cos 2\Omega t}{2\Omega}\right)\left(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 + \frac{1}{2\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{g})\right) \\
&- \left(t - \frac{\sin 2\Omega t}{2\Omega}\right)\left(\frac{1}{2\Omega}\hat{\boldsymbol{\omega}} \times \boldsymbol{g} - \hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right)
\end{aligned}$$

This is an exact solution for all values of Ω, but in the case when $\Omega \ll 1$, we may expand this in powers of Ω to compare to previous approximations. We have

$$\left(\frac{1 - \cos 2\Omega t}{2\Omega}\right) = \frac{1}{2\Omega}\left(\frac{4\Omega^2}{2!}t^2 - \frac{16\Omega^4}{4!}t^4 \pm \cdots\right) = \Omega t^2 - \frac{1}{3}\Omega^3 t^4 \pm \cdots$$

$$\left(t - \frac{\sin 2\Omega t}{2\Omega}\right) = t - \frac{1}{2\Omega}\left(2\Omega t - \frac{8\Omega^3}{3!}t^3 \pm \cdots\right) = \frac{2}{3}\Omega^2 t^3 \mp \cdots$$

and we find, to first order in Ω,

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \Omega\left(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0 t^2 + \frac{1}{3}\hat{\boldsymbol{\omega}} \times \boldsymbol{g}t^3\right) + \mathcal{O}(\Omega^2) \tag{4}$$

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \boldsymbol{\Omega} \times \boldsymbol{v}_0 t^2 - \frac{1}{3}\boldsymbol{\Omega} \times \boldsymbol{g}t^3 + \mathcal{O}(\Omega^2), \tag{5}$$

which agrees with the approximation in *Mechanics* by Landau and Lifshitz.[5] In particular, we verify that $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2$ when $\boldsymbol{\Omega} \to 0$.

The paths of a particle under the force of gravity and deviated by the Coriolis force are shown in Figure 2.
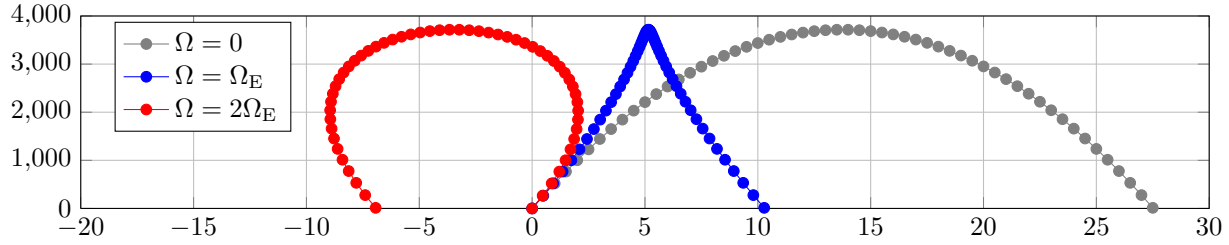
Figure 2. Trajectories of a particle under the force of gravity alone (solid gray) and under gravity and deviated by the Coriolis force (colored circles). The z axis (vertical) corresponds to the radial direction of the Earth. The positive y axis (horizontal) corresponds to East. The same initial conditions lead to a qualitatively different trajectory (red) when the angular speed of the Earth, Ω_{E}, is doubled. Initial velocity, $\boldsymbol{v}_0 = (0, 0.5, 270)$ m/s, and latitude $= 30°$ in all three cases.

These particle trajectories were generated by the following program, which makes use of the C++ Vector class `Vector.h`.

Listing 2. coriolis.cpp

```cpp
// coriolis.cpp

#include "Vector.h"
#include <iostream>
using namespace va;
using namespace std;

int main( void ) {

    const Vector ihat( 1., 0., 0. );    // points south
    const Vector jhat( 0., 1., 0. );    // points east
    const Vector khat( 0., 0., 1. );    // points up (radially from the earth)

    const double VX           = 0., VY = 0.5, VZ = 260., H = 0.;
    const double ALPHA        = atan( VZ / VY );
    const double V0           = sqrt( VX * VX + VY * VY + VZ * VZ );
    const double OMEGA_EARTH   = 7.29e-5, G = 9.81, LAT = 30. * D2R;
    const double OMEGA         = OMEGA_EARTH;
    const double TWO_OMEGA     = 2. * OMEGA_EARTH;
    const double COS_LAT       = cos( LAT ), SIN_LAT = sin( LAT );
    const double COS_ALPHA     = cos( ALPHA ), SIN_ALPHA = sin( ALPHA );
    const Vector omega         = Vector( -COS_LAT, 0., SIN_LAT );    // unit vector
    const Vector g             = -G * khat;
    const Vector r0            = Vector( 0., 0., H );
    const Vector v0            = Vector( 0., V0 * COS_ALPHA, V0 * SIN_ALPHA );
    double T;
    if ( H == 0. )
        T = 2. * V0 * SIN_ALPHA / ( G - ( 2. * OMEGA ) * COS_LAT * V0 * COS_ALPHA );
    else
        T = sqrt( 2. * H / G );

    double f1 = 0.5 * ( 1. - cos( 2. * OMEGA * T ) ) / OMEGA;
    double f2 = T - 0.5 * sin( 2. * OMEGA * T ) / OMEGA;

    Vector w1 = omega ^ g;
    Vector w2 = omega ^ w1;
    Vector g1 = ( omega ^ g ) / ( 2. * OMEGA );
    Vector g2 = omega ^ g1;
    Vector v1 = omega ^ v0;
    Vector v2 = omega ^ v1;

    Vector r, r1, r2, r_prime, g1_prime, g2_prime;
    double t2, f1_prime, f2_prime;
    cout << "t" << "\t" << "y0" << "\t" << "y1" << "\t" << "y2" << "\t" << "z" << endl;
    for ( double t = 0.; t <= T; t += 1. ) {

        t2 = t * t;
        f1 = 0.5 * ( 1. - cos( 2. * OMEGA * t ) ) / OMEGA;
        f2 = t - 0.5 * sin( 2. * OMEGA * t ) / OMEGA;
        f1_prime = 0.5 * ( 1. - cos( 2. * TWO_OMEGA * t ) ) / TWO_OMEGA;
        f2_prime = t - 0.5 * sin( 2. * TWO_OMEGA * t ) / TWO_OMEGA;
        g1_prime = ( omega ^ g ) / ( 2. * TWO_OMEGA );
        g2_prime = omega ^ g1_prime;
        r1 = r0 + v0 * t + 0.5 * g * t2;
        r2 = 0.5 * w2 * t2;
        r = r1 + r2 - f1 * ( v1 + g2 ) - f2 * ( g1 - v2 );
        r_prime = r1 + r2 - f1_prime * ( v1 + g2_prime ) - f2_prime * ( g1_prime - v2 );
        cout << t << "\t"                 // elapsed time
            << r1 * jhat << "\t"          // distance east with OMEGA = 0
            << r * jhat << "\t"           // distance east
            << r_prime * jhat << "\t"     // distance east with OMEGA = 2. * OMEGA_EARTH
            << r1 * khat << endl;         // height
    }
    return EXIT_SUCCESS;
}
```

Coriolis + Centrifugal Force, $\boldsymbol{F} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega} \times \boldsymbol{v} - m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$

The rate of change of any vector in an inertial frame of reference (*fixed frame*) is equal to the rate of change of the vector in the *rotating frame* plus the vector product of the angular velocity, $\boldsymbol{\Omega}$, with the given vector.[6] Thus we have the operator equation:

$$\left(\frac{d}{dt}\right)_{\text{fixed}} = \left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega} \times$$

Applied to the position vector, we have

$$\dot{\boldsymbol{r}}_{\text{fixed}} = \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega} \times\right] \boldsymbol{r} = \dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r}$$

$$\ddot{\boldsymbol{r}}_{\text{fixed}} = \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega} \times\right] (\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \boldsymbol{r})$$

$$= \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega} \times\right] \dot{\boldsymbol{r}} + \left[\left(\frac{d}{dt}\right)_{\text{rotating}} + \boldsymbol{\Omega} \times\right] (\boldsymbol{\Omega} \times \boldsymbol{r})$$

$$= \ddot{\boldsymbol{r}} + \boldsymbol{\Omega} \times \dot{\boldsymbol{r}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r} + \boldsymbol{\Omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

$$= \ddot{\boldsymbol{r}} + \underbrace{2\boldsymbol{\Omega} \times \dot{\boldsymbol{r}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}}_{\text{Coriolis force}} + \underbrace{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})}_{\text{Centrifugal force}}$$

The total force is

$$\boldsymbol{F} = m\boldsymbol{a}_{\text{fixed}} = m\boldsymbol{a}_{\text{rotating}} + 2m\boldsymbol{\Omega} \times \boldsymbol{v}_{\text{rotating}} + m\dot{\boldsymbol{\Omega}} \times \boldsymbol{r} + m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

For an observer in the rotating frame, the effective force is

$$\boldsymbol{F}_{\text{eff}} \equiv m\boldsymbol{a}_{\text{rotating}} = m\boldsymbol{g} - 2m\boldsymbol{\Omega} \times \dot{\boldsymbol{r}} - m\dot{\boldsymbol{\Omega}} \times \boldsymbol{r} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

Thus, the equation of motion in the *rotating frame* is

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{r}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{g}$$

$$\ddot{\boldsymbol{r}} + 2\Omega J(\hat{\boldsymbol{\omega}})\dot{\boldsymbol{r}} + [\dot{\Omega}J(\hat{\boldsymbol{\omega}}) + \Omega^2 J^2(\hat{\boldsymbol{\omega}})]\boldsymbol{r} = \boldsymbol{g}$$

We rewrite this as[3]

$$\ddot{\boldsymbol{r}} + 2A(t)\dot{\boldsymbol{r}} + B(t)\boldsymbol{r} = \boldsymbol{g} \tag{6}$$

where $A(t)$ and $B(t)$ are the matrices $A(t) \equiv \Omega J(\hat{\boldsymbol{\omega}})$ and $B(t) \equiv \dot{\Omega}J(\hat{\boldsymbol{\omega}}) + \Omega^2 J^2(\hat{\boldsymbol{\omega}})$. Let $\boldsymbol{r} = V(t)\,\boldsymbol{y}$, then

$$\dot{\boldsymbol{r}} = \dot{V}\boldsymbol{y} + V\dot{\boldsymbol{y}}$$

$$\ddot{\boldsymbol{r}} = \ddot{V}\boldsymbol{y} + 2\dot{V}\dot{\boldsymbol{y}} + V\ddot{\boldsymbol{y}},$$

which transforms eq. (6) into

$$\ddot{V}\boldsymbol{y} + 2\dot{V}\dot{\boldsymbol{y}} + V\ddot{\boldsymbol{y}} + 2A(\dot{V}\boldsymbol{y} + V\dot{\boldsymbol{y}}) + BV\boldsymbol{y} = \boldsymbol{g}$$

$$V\ddot{\boldsymbol{y}} + 2(\dot{V} + AV)\dot{\boldsymbol{y}} + (\ddot{V} + 2A\dot{V} + BV)\boldsymbol{y} = \boldsymbol{g} \tag{7}$$

And now we choose V to eliminate the $\dot{\boldsymbol{y}}$ term:

$$\dot{V} + AV = 0 \quad \Longrightarrow \quad \dot{V} = -AV \quad \Longrightarrow \quad V = e^{-At}.$$

Also

$$\dot{V} = -AV \quad \Longrightarrow \quad \ddot{V} = -\dot{A}V - A\dot{V} = -\dot{A}V + A^2V,$$

so that eq. (7) becomes

$$V\ddot{\boldsymbol{y}} + (-\dot{A}V + A^2V - 2A^2V + BV)\boldsymbol{y} = \boldsymbol{g}$$

$$V\ddot{\boldsymbol{y}} + (B - A^2 - \dot{A})V\boldsymbol{y} = \boldsymbol{g} \tag{8}$$

Now, using the expressions for A and B, we find that the coefficient of $\boldsymbol{y}$ also vanishes:

$$B - A^2 - \dot{A} = \dot{\Omega}J(\hat{\boldsymbol{\omega}}) + \Omega^2 J^2(\hat{\boldsymbol{\omega}}) - \Omega^2 J^2(\hat{\boldsymbol{\omega}}) - \dot{\Omega}J(\hat{\boldsymbol{\omega}}) = 0$$

Thus, eq. (8) reduces to simply

$$V\ddot{\boldsymbol{y}} = \boldsymbol{g}$$

and therefore we have

$$\ddot{\boldsymbol{y}} = V^{-1}\boldsymbol{g} = e^{At}\boldsymbol{g} = e^{J(\hat{\boldsymbol{\omega}})\Omega t}\boldsymbol{g} = [I + J(\hat{\boldsymbol{\omega}})\sin\Omega t + J^2(\hat{\boldsymbol{\omega}})(1 - \cos\Omega t)]\boldsymbol{g}$$

For convenience, let's write J for $J(\hat{\boldsymbol{\omega}})$. Starting with

$$\ddot{\boldsymbol{y}} = [I + \sin\Omega t J + (1 - \cos\Omega t)J^2]\boldsymbol{g}$$

and integrating over time twice, we get

$$\dot{\boldsymbol{y}}(t) = \dot{\boldsymbol{y}}(0) + \left(\int_0^t [I + \sin\Omega\tau J + (1 - \cos\Omega\tau)J^2]d\tau\right)\boldsymbol{g}$$

$$= \dot{\boldsymbol{y}}(0) + \boldsymbol{g}t + \frac{1 - \cos\Omega t}{\Omega}J\boldsymbol{g} + \left(t - \frac{\sin\Omega t}{\Omega}\right)J^2\boldsymbol{g}$$

$$\boldsymbol{y}(t) = \boldsymbol{y}(0) + \dot{\boldsymbol{y}}(0)t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{t - \Omega^{-1}\sin\Omega t}{\Omega}J\boldsymbol{g} + \frac{\Omega\frac{1}{2}t^2 + \Omega^{-1}(\cos\Omega t - 1)}{\Omega}J^2\boldsymbol{g}$$

$$= \boldsymbol{y}(0) + \dot{\boldsymbol{y}}(0)t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{\Omega t - \sin\Omega t}{\Omega^2}J\boldsymbol{g} + \frac{\Omega^2\frac{1}{2}t^2 + \cos\Omega t - 1}{\Omega^2}J^2\boldsymbol{g}$$

Now, $\boldsymbol{r}_0 = \boldsymbol{y}(0)$ and $\boldsymbol{v}_0 \equiv \dot{\boldsymbol{r}}_0 = \dot{\boldsymbol{y}}(0) - \Omega J\boldsymbol{y}(0)$, so that we can make the replacements

$$\boldsymbol{y}(0) \implies \boldsymbol{r}_0$$

$$\dot{\boldsymbol{y}}(0) \implies \boldsymbol{v}_0 + \Omega J\boldsymbol{r}_0$$

Also, $\boldsymbol{r}(t)$ and $\boldsymbol{y}(t)$ are related by the transformation $\boldsymbol{r}(t) = e^{-J\Omega t}\boldsymbol{y}(t)$. Thus, we have

$$\boldsymbol{r}(t) = e^{-J\Omega t}\left[\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \Omega J\boldsymbol{r}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{\Omega t - \sin\Omega t}{\Omega^2}J\boldsymbol{g} + \frac{\frac{1}{2}\Omega^2 t^2 + \cos\Omega t - 1}{\Omega^2}J^2\boldsymbol{g}\right]$$

$$= e^{-J\Omega t}\left\{\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \frac{1}{\Omega^2}\underbrace{[I + \sin\Omega t J + (1 - \cos\Omega t)J^2]}_{e^{J\Omega t}}\boldsymbol{g} + \Omega J\boldsymbol{r}_0 t + \frac{1}{\Omega^2}\left[I + \Omega t J + \frac{1}{2}\Omega^2 t^2 J^2\right]\boldsymbol{g}\right\}$$

$$= -\frac{\boldsymbol{g}}{\Omega^2} + \left[I - \sin\Omega t J + (1 - \cos\Omega t)J^2\right]\left\{(\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2) + \Omega J\boldsymbol{r}_0 t + \frac{1}{\Omega^2}(I + \Omega t J + \frac{1}{2}\Omega^2 t^2 J^2)\boldsymbol{g}\right\}$$

$$= -\frac{\boldsymbol{g}}{\Omega^2} + (\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2) - \sin\Omega t J(\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2) + (1 - \cos\Omega t)J^2(\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2)$$

$$+ \Omega J\boldsymbol{r}_0 t - \Omega\sin\Omega t J^2\boldsymbol{r}_0 t - (1 - \cos\Omega t)\Omega J\boldsymbol{r}_0 t + \frac{1}{\Omega^2}(I + \Omega t J + \frac{1}{2}\Omega^2 t^2 J^2)\boldsymbol{g}$$

$$- \sin\Omega t \frac{1}{\Omega^2}(J + \Omega t J^2 - \frac{1}{2}\Omega^2 t^2 J)\boldsymbol{g} + (1 - \cos\Omega t)\frac{1}{\Omega^2}(J^2 - \Omega t J - \frac{1}{2}\Omega^2 t^2 J^2)\boldsymbol{g}$$

where we used $J^3 = -J$ and $J^4 = -J^2$. Collecting terms, we have

$$\boldsymbol{r}(t) = -\frac{\boldsymbol{g}}{\Omega^2} + \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{\boldsymbol{g}}{\Omega^2}$$

$$+ J\left[-\sin\Omega t(\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2) + \Omega\boldsymbol{r}_0 t - \Omega\boldsymbol{r}_0 t + \cos\Omega t\,\Omega\boldsymbol{r}_0 t + \frac{\boldsymbol{g}t}{\Omega} - \frac{\sin\Omega t}{\Omega^2}\boldsymbol{g} + \frac{1}{2}\sin\Omega t\,\boldsymbol{g}t^2 - \frac{\boldsymbol{g}t}{\Omega} + \frac{\cos\Omega t}{\Omega}\boldsymbol{g}t \right]$$

$$+ J^2\left[(1 - \cos\Omega t)(\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2) - \Omega\sin\Omega t\,\boldsymbol{r}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \frac{\Omega t\sin\Omega t}{\Omega^2}\boldsymbol{g} + \frac{1 - \cos\Omega t}{\Omega^2}\boldsymbol{g} - (1 - \cos\omega t)\frac{1}{2}\boldsymbol{g}t^2 \right]$$

$$= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + J\left[-\sin\Omega t\,(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) + \Omega\cos\Omega t\,\boldsymbol{r}_0 t - \frac{\sin\Omega t}{\Omega^2}\boldsymbol{g} + \frac{\cos\Omega t}{\Omega}\boldsymbol{g}t \right]$$

$$+ J^2\left[(1 - \cos\Omega t)(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) - \Omega\sin\Omega t\,\boldsymbol{r}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{1 - \cos\Omega t - \Omega t\sin\Omega t}{\Omega^2}\boldsymbol{g} \right]$$

$$= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + J\left[-\sin\Omega t\,(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) + \Omega\cos\Omega t\,\boldsymbol{r}_0 t + \frac{\Omega t\cos\Omega t - \sin\Omega t}{\Omega^2}\boldsymbol{g} \right]$$

$$+ J^2\left[(1 - \cos\Omega t)(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) - \Omega\sin\Omega t\,\boldsymbol{r}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{1 - \cos\Omega t - \Omega t\sin\Omega t}{\Omega^2}\boldsymbol{g} \right]$$

$$= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + J\left[-\sin\Omega t\,(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) + \Omega t\cos\Omega t\,\boldsymbol{r}_0 + \frac{\Omega t\cos\Omega t - \sin\Omega t}{\Omega^2}\boldsymbol{g} \right]$$

$$+ J^2\left[\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 - \cos\Omega t(\boldsymbol{r}_0 + \boldsymbol{v}_0 t) - \Omega t\sin\Omega t\,\boldsymbol{r}_0 + \frac{\boldsymbol{g}}{\Omega^2} - \frac{\cos\Omega t + \Omega t\sin\Omega t}{\Omega^2}\boldsymbol{g} \right]$$

$$= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \frac{\boldsymbol{g}}{\Omega^2})] - \sin\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) - \cos\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0)]$$

$$- \Omega t\sin\Omega t\frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0) - \Omega t\cos\Omega t\frac{1}{\Omega}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)] + \Omega t\cos\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0) - \Omega t\sin\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0)]$$

$$- \sin\Omega t(\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) - \cos\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2})] + \Omega t\cos\Omega t(\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2}) - \Omega t\sin\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \frac{\boldsymbol{g}}{\Omega^2})]$$

Introducing $\boldsymbol{r}_0^* \equiv \boldsymbol{r}_0 + \boldsymbol{g}/\Omega^2$, we can write this as

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0^* + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2)] - \sin\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) - \cos\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*)]$$

$$- \Omega t\sin\Omega t\frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0) - \Omega t\cos\Omega t\frac{1}{\Omega}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)] + \Omega t\cos\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) - \Omega t\sin\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*)]$$

Exact analytical solution for the case of both the Coriolis and the Centrifugal force
$$\boldsymbol{F} = m\boldsymbol{g} - 2m\,\boldsymbol{\Omega} \times \boldsymbol{v} - m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2 + \hat{\boldsymbol{\omega}} \times [\hat{\boldsymbol{\omega}} \times (\boldsymbol{r}_0^* + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2)] - \sin\Omega t(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) - \cos\Omega t[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*)]$$

$$- \Omega t\sin\Omega t\left[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) + \frac{1}{\Omega}(\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right] + \Omega t\cos\Omega t\left[(\hat{\boldsymbol{\omega}} \times \boldsymbol{r}_0^*) - \frac{1}{\Omega}\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \boldsymbol{v}_0)\right]$$

Again, this result is exact. And again, let us expand in powers of Ω to compare to previous approximations.

$$r(t) = r_0 + v_0 t + \frac{1}{2}gt^2 + \hat{\omega} \times [\hat{\omega} \times (r_0 + \frac{g}{\Omega^2} + v_0 t + \frac{1}{2}gt^2)]$$
$$- \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \cdots \right)[\hat{\omega} \times (r_0 + \frac{g}{\Omega^2})]$$
$$- \left(1 - \frac{(\Omega t)^2}{2!} \pm \cdots \right)[\hat{\omega} \times (\hat{\omega} \times (r_0 + \frac{g}{\Omega^2}))]$$
$$- \Omega t \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \cdots \right)\left[\hat{\omega} \times (\hat{\omega} \times (r_0 + \frac{g}{\Omega^2})) + \frac{1}{\Omega}(\hat{\omega} \times v_0)\right]$$
$$+ \Omega t \left(1 - \frac{(\Omega t)^2}{2!} \pm \cdots \right)\left[(\hat{\omega} \times (r_0 + \frac{g}{\Omega^2})) - \frac{1}{\Omega}\hat{\omega} \times (\hat{\omega} \times v_0)\right]$$

$$r(t) = r_0 + v_0 t + \frac{1}{2}gt^2 + \hat{\omega} \times [\hat{\omega} \times (r_0 + v_0 t + \frac{1}{2}gt^2)] + \hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2})$$
$$- \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \cdots \right)[\hat{\omega} \times r_0 + \hat{\omega} \times \frac{g}{\Omega^2}]$$
$$- \left(1 - \frac{(\Omega t)^2}{2!} + \frac{(\Omega t)^4}{4!} \mp \cdots \right)[\hat{\omega} \times (\hat{\omega} \times r_0) + \hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2})]$$
$$- \Omega t \left(\Omega t - \frac{(\Omega t)^3}{3!} \pm \cdots \right)\left[\hat{\omega} \times (\hat{\omega} \times r_0) + \hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2}) + \frac{1}{\Omega}(\hat{\omega} \times v_0)\right]$$
$$+ \Omega t \left(1 - \frac{(\Omega t)^2}{2!} \pm \cdots \right)\left[(\hat{\omega} \times r_0) + (\hat{\omega} \times \frac{g}{\Omega^2}) - \frac{1}{\Omega}\hat{\omega} \times (\hat{\omega} \times v_0)\right]$$

$$r(t) = r_0 + v_0 t + \frac{1}{2}gt^2 + \hat{\omega} \times [\hat{\omega} \times (r_0 + v_0 t + \frac{1}{2}gt^2)] + \hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2}) - \Omega[\hat{\omega} \times r_0 + \hat{\omega} \times \frac{g}{\Omega^2}]t + \frac{(\Omega t)^3}{3!}\hat{\omega} \times \frac{g}{\Omega^2}$$
$$- \hat{\omega} \times (\hat{\omega} \times r_0) - \hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2}) + \Omega^2[\hat{\omega} \times (\hat{\omega} \times r_0) + \hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2})]\frac{t^2}{2} - \Omega^4[\hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2})]\frac{t^4}{24}$$
$$- \Omega^2\left[\hat{\omega} \times (\hat{\omega} \times r_0) + \hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2}) + \frac{1}{\Omega}(\hat{\omega} \times v_0)\right]t^2 + \Omega^4[\hat{\omega} \times (\hat{\omega} \times \frac{g}{\Omega^2})]\frac{t^4}{6}$$
$$+ \Omega\left[(\hat{\omega} \times r_0) + (\hat{\omega} \times \frac{g}{\Omega^2}) - \frac{1}{\Omega}\hat{\omega} \times (\hat{\omega} \times v_0)\right]t - \Omega^3\left[(\hat{\omega} \times \frac{g}{\Omega^2}) - \frac{1}{\Omega}\hat{\omega} \times (\hat{\omega} \times v_0)\right]\frac{t^3}{2} + \mathcal{O}(\Omega^3)$$

Thus, the second-order approximation reduces to

$$\begin{aligned} r(t) = & \; r_0 + v_0 t + \frac{1}{2}gt^2 - \Omega\left(\hat{\omega} \times v_0 t^2 + \frac{1}{3}\hat{\omega} \times gt^3\right) \\ & + \Omega^2\left(-\frac{1}{2}\hat{\omega} \times (\hat{\omega} \times r_0)t^2 + \frac{1}{2}\hat{\omega} \times (\hat{\omega} \times v_0)t^3 + \frac{1}{8}\hat{\omega} \times (\hat{\omega} \times g)t^4\right) + \mathcal{O}(\Omega^3) \end{aligned} \tag{9}$$

$$\begin{aligned} r(t) = & \; r_0 + v_0 t + \frac{1}{2}gt^2 - \Omega \times v_0 t^2 - \frac{1}{3}\Omega \times gt^3 \\ & - \frac{1}{2}\Omega \times (\Omega \times r_0)t^2 + \frac{1}{2}\Omega \times (\Omega \times v_0)t^3 + \frac{1}{8}\Omega \times (\Omega \times g)t^4 + \mathcal{O}(\Omega^3) \end{aligned} \tag{10}$$

We see that if we neglect terms of $\mathcal{O}(\Omega^2)$ and higher that it reduces to that found for the Coriolis force, eqs. (4) and (5). We also verify that $r(t) = r_0 + v_0 t + \frac{1}{2}gt^2$ when $\Omega \to 0$.

Let's compute the deflection of a particle falling freely under the Earth's gravity from a height h and at a northern latitude λ. We take the z-axis to be vertically outward from the surface of the earth, the x-axis to point south, and the y-axis to point east, to form a right-handed coordinate system, as shown in Figure 3.
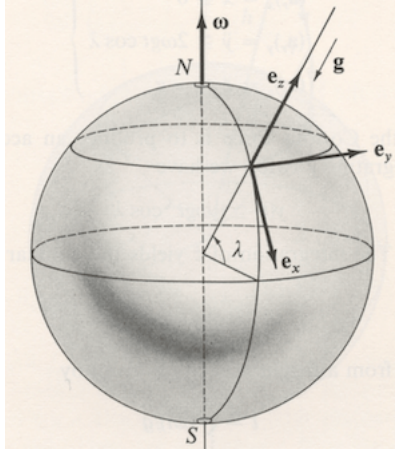

Figure 3. Earth-frame coordinate system reproduced from J. B. Marion, *Classical Dynamics of Particles and Systems*.[5] Replace $\boldsymbol{\omega} \to \hat{\boldsymbol{\omega}}$, $\hat{e}_x \to \hat{\boldsymbol{\imath}}$, $\hat{e}_y \to \hat{\boldsymbol{\jmath}}$, and $\hat{e}_z \to \hat{\boldsymbol{k}}$ to be consistent with the notation used here.

We have $\hat{\boldsymbol{\omega}} = -\cos\lambda\,\hat{\boldsymbol{\imath}} + \sin\lambda\,\hat{\boldsymbol{k}}$, $\boldsymbol{r}_0 = h\hat{\boldsymbol{k}}$, $\boldsymbol{v}_0 = 0$, and $\boldsymbol{g} = -g\hat{\boldsymbol{k}}$, so that

$$\boldsymbol{r}(t) = h\hat{\boldsymbol{k}} - \frac{1}{2}gt^2\hat{\boldsymbol{k}} + \Omega\frac{1}{3}(\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})gt^3 - \Omega^2\frac{1}{2}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})]ht^2 - \Omega^2\frac{1}{8}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})]gt^4$$

$$= (h - \frac{1}{2}gt^2)\hat{\boldsymbol{k}} + \Omega\frac{1}{3}(\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})gt^3 - \Omega^2[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})](\frac{1}{2}ht^2 + \frac{1}{8}gt^4)$$

The total time of fall is $t = \sqrt{2h/g}$, so that $h - \frac{1}{2}gt^2 = 0$, and $\frac{1}{2}ht^2 + \frac{1}{8}gt^4 = \frac{h^2}{g} + \frac{1}{2}\frac{h^2}{g} = \frac{3}{2}\frac{h^2}{g}$ Therefore,

$$\boldsymbol{r} = \Omega\frac{1}{3}(\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})g\left(\frac{2h}{g}\right)^{3/2} - \Omega^2\frac{3}{2}[\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}})]\frac{h^2}{g} \tag{11}$$

Now

$$\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & 0 & 1 \end{vmatrix} = \cos\lambda\,\hat{\boldsymbol{\jmath}}$$

and

$$\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{k}}) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & \cos\lambda & 0 \end{vmatrix} = -\sin\lambda\cos\lambda\,\hat{\boldsymbol{\imath}} - \cos^2\lambda\,\hat{\boldsymbol{k}}$$

so that eq. (11) becomes

$$\boldsymbol{r} = \frac{1}{3}\sqrt{\frac{8h^3}{g}}\Omega\cos\lambda\,\hat{\boldsymbol{\jmath}} + \frac{3}{2}\frac{h^2}{g}\Omega^2(\sin\lambda\cos\lambda\,\hat{\boldsymbol{\imath}} + \cos^2\lambda\,\hat{\boldsymbol{k}})$$

$$= \frac{3}{2}\frac{h^2\Omega^2}{g}\sin\lambda\cos\lambda\,\hat{\boldsymbol{\imath}} + \frac{1}{3}\sqrt{\frac{8h^3}{g}}\Omega\cos\lambda\,\hat{\boldsymbol{\jmath}} + \frac{3}{2}\frac{h^2\Omega^2}{g}\cos^2\lambda\,\hat{\boldsymbol{k}}$$

For a height $h = 100$ m, latitude $\lambda = 45°$, and $\Omega = 7.29 \times 10^{-5}$ rad/s for the earth's rotational velocity, we get a (first-order) easterly deflection (along $\hat{\boldsymbol{\jmath}}$) of 1.55 cm, and a (second-order) southerly deflection (along $\hat{\boldsymbol{\imath}}$) of 4×10^{-4} cm.

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The following program uses the exact formula to compute the position of the particle when it hits the ground and also the approximate values cited above.

Listing 3. centrifugal.cpp

```cpp
// centrifugal.cpp

#include "Vector.h"
#include <iostream>
#include <cstdlib>
#include <cmath>
using namespace va;    // vector algebra namespace
using namespace std;

int main( void ) {

    Vector ihat( 1., 0., 0. ), jhat( 0., 1., 0. ), khat( 0., 0., 1. );   // unit vectors for cartesian coordinate system

    const double OMEGA  = 7.29e-5;     // Earth's rotational velocity
    const double G      = 9.81;        // acceleration due to gravity (m/s^2)
    const double H      = 100.;        // height (m)
    const double LAT    = 45. * D2R;   // northern latitude (deg converted to rad)
    const double OMEGA2 = OMEGA * OMEGA;
    const double T      = sqrt( 2. * H / G );   // time to fall
    const double T2     = T * T;
    const double T3     = T * T2;
    const double T4     = T2 * T2;
    const double THETA  = OMEGA * T;
    const double COS    = cos( THETA );
    const double SIN    = sin( THETA );
    const double COS_LAT = cos( LAT );
    const double SIN_LAT = sin( LAT );

    const Vector omega   = Vector( -COS_LAT, 0., SIN_LAT );   // unit vector for Earth's rotation velocity
    const Vector g       = -G * khat;
    const Vector r0      = Vector( 0., 0., H );
    const Vector r0_star = r0 + g / OMEGA2;
    const Vector v0      = Vector( 0., 0., 0. );

    const Vector R1 = r0 + v0 * T + 0.5 * g * T2;
    const Vector R2 = R1 + g / OMEGA2;
    const Vector R3 = omega ^ R2;
    const Vector R4 = omega ^ R3;
    const Vector R5 = omega ^ r0_star;
    const Vector R6 = omega ^ R5;

    const Vector W1 = omega ^ g;
    const Vector W2 = omega ^ W1;
    const Vector R0 = omega ^ ( omega ^ r0 );
    const Vector V1 = omega ^ v0;
    const Vector V2 = omega ^ V1;

    Vector r_exact  = R1 + R4 - SIN * R5 - COS * R6 - THETA * SIN * ( R6 + V1 / OMEGA ) + THETA * COS * ( R5 - V2 / OMEGA );
    Vector r_approx = R1 - OMEGA * ( V1 * T2 + W1 * T3 / 3. ) + OMEGA2 * ( -0.5 * R0 * T2 + 0.5 * V2 * T3 + 0.125 * W2 * T4 );

    cout.precision(6);
    cout << "r_exact (cm)  = " << std::fixed << r_exact * 100. << endl;    // output in cm
    cout << "r_approx (cm) = " << std::fixed << r_approx * 100. << endl;   // output in cm

    return EXIT_SUCCESS;
}
```

The output from running `./centrifugal` is

```
r_exact (cm)  = 0.000400 1.551678 0.000400
r_approx (cm) = 0.000406 1.551679 0.000406
```

References

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