ON THE TRAJECTORIES OF THE $3x + 1$ PROBLEM

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ABSTRACT. This paper studies certain trajectories of the Collatz function. I show that if for each odd number n, $n \sim 3n + 2$ then every positive integer $n \in \mathbb{N} \setminus 2^{\mathbb{N}}$ has the representation

$$n = \left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i}\right) / 3^{k+1}$$

where $0 \leq a_0 \leq a_1 \leq \cdots \leq a_{k+1}$. As a consequence, in order to prove Collatz Conjecture I illustrate that it is sufficient to prove $n \sim 3n + 2$ for any odd $n \in \mathbb{N} \setminus 2^{\mathbb{N}}$. This is the main result of the paper.

1. Introduction

Some problems in mathematics are easy to state but take very complex tools to prove and often it takes new tools to be developed. The *Collatz Conjecture* is either one of this type or else it might be a conspiracy theory to slow down the field of mathematics (a joke that spread across Yale University). The *Collatz Conjecture* is a well known unsolved mathematical problem that concerns the recursive behavior of the function

$$T(n) = \begin{cases} \frac{n}{2} & \textit{if } n \equiv 0 (\text{mod} 2) \\ 3n + 1 & \textit{if } n \equiv 1 (\text{mod} 2) \end{cases}$$

over the set of integers $\mathbb{Z}$. In this paper I focus on the Collatz function over the positive whole numbers $\mathbb{N}$. The problem is most commonly referred to as the *"$3x + 1$" problem.* The history and exact origin of the problem is somewhat vague. Some early history on the problem is discussed by Jeffrey C. Lagarias at `http://www.cecm.sfu.ca/organics/papers/lagarias/paper/html/node1.html`. Lagarias gives 197 different documentations on the topic in his annotated bibliographies [10] and [11]. Dr. Lothar Collatz is credited for the discovery of the problem during his career as a student. Dr. Lothar Collatz even asserts himself that he was the first to study this problem in his letter `http://www.cecm.sfu.ca/organics/papers/lagarias/paper/html/letter.html`.

The problem states that $\forall n \in \mathbb{N}$ there is a value $k \in \mathbb{N}$ such that $T^k(n) = 1$ were T^k is the kth application of the map T. That is $T^k : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by the rule

$$(n, k) \mapsto \underbrace{T \circ T \circ T \cdots \circ T}_{k-times}$$

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Example 1.1. *For the value $n = 11$ we can view the iterations with arrows (to indicate direction) as followed:*

$$11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to \cdots \to 1$$

We can also visualize this backwards by reversing the operations of the map and traces our steps in the reverse direction. Doing so for this example we have the following:

$$\begin{aligned} 1 \to \cdots \to \frac{2^4 - 1}{3} \to \frac{2^5 - 2}{3} \to \frac{2^6 - 2^2}{3} \to \frac{2^7 - 2^3}{3} \to \frac{2^7 - 2^3 - 3}{3^2} \\ \to \frac{2^8 - 2^4 - 2 \cdot 3}{3^2} \to \frac{2^9 - 2^5 - 2^2 \cdot 3}{3^2} \to \frac{2^9 - 2^5 - 2^2 \cdot 3 - 3^2}{3^3} \\ \to \frac{2^10 - 2^6 - 2^3 \cdot 3 - 2 \cdot 3^2}{3^3} \to \frac{2^{10} - 2^6 - 2^3 \cdot 3 - 2 \cdot 3^2 - 3^3}{3^4} = 11 \end{aligned}$$

It has been verified that all natural numbers $n < 87 \cdot 2^{60}$ iterate to the value 1 under Collatz function. A neat discussion about the empirical results and the record-holders are discussed by Tomás Oliveria e Silvia at his home page `http://sweet.ua.pt/tos/3x+1.html`, and by Eric Roosendaal `http://www.ericr.nl/wondrous/index.html`. A complete list of the record holders can be found here `http://www.ericr.nl/wondrous/pathrecs.html` which was accomplished by the yoyo@home project in 2017. In [10] and [11] Lagarias also mentioned the empirical evidence of the problem given by Oliveirá e Silva. An interesting study on properties of divergent trajectories of the Collatz function (if one exist) can be found here `http://www.csun.edu/~vcmth02i/Collatz.pdf`. A similar result of this paper is given by Charles C. Cadogan in his works [5] and [4], which can also be found in Lagarias's bibliography [11]. Some more interesting discussions and findings that relate to this work can be found in [6, 9, 12, 13, 14].

2. Terminology

In order to prove the main result of the paper the following definitions are needed.

Definition 2.1. *let $m \in \mathbb{N}$. Write $m \equiv \mathcal{R}$ if and only if the number m has the representation*

$$m = \left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i} \right) / 3^{k+1}$$

where (a_i) is a monotonically increasing sequence of positive integers for $i \leq k+1$.

Definition 2.2. *The Trajectory or Forward Orbit of a positive integer n is the set*

$$O^+(n) = \{n, T(n), T^2(n), \cdots\}$$

were $T : \mathbb{N} \to \mathbb{N}$ is the Collatz function defined by $n \to \frac{n}{2}$ if n is even and $n \to 3n+1$ if n is odd, and T^k is the function $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$(n, k) \mapsto \underbrace{(T \circ T \circ \cdots \circ T)}_{k-times}$$

Definition 2.3. *Given two integers n_1 and n_2 define the relation $n_1 \sim n_2$ if and only if the two trajectories $O^+(n_1)$ and $O^+(n_2)$ coalesce, i.e. $O^+(n_1) \cap O^+(n_2) \neq \emptyset$.*

3. Backwards Iterations and Integer Representations

As shown by [1, 2, 3, 7, 15], and discussed in [8] we have that

$$\forall m \in \mathbb{N} \setminus 2^{\mathbb{N}}, 1 \in O^+(m) \Longleftrightarrow m = \left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i} \right) /3^{k+1}$$

for some sequence (a_j) where $0 \leq a_0 \leq a_1 \leq a_2 \leq \cdots a_{k+1}$. If m has this representation given above then m cannot be a power a 2 because any element of the set $2^{\mathbb{N}} = \{2, 2^2, 2^3, \cdots\}$ cannot be represented in this form. This can easily be verified by observing that the numbers that can be written in the form $\left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i}\right)/3^{k+1}$ are exactly those numbers obtained by iterating the functions $n \to 2n$ and $n \to \frac{n-1}{3}$ in succession, beginning at the number $n = 1$, were as each function is applied at least once and the function $n \to \frac{n-1}{3}$ is never applied twice in succession. The numbers in the set $2^{\mathbb{N}}$ can only be obtained by iterating the function $n \to 2n$, beginning at $n = 1$, so that the function $n \to \frac{n-1}{3}$ is never applied. This shows that no integer in the set $2^{\mathbb{N}}$ has the representation $\left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i}\right)/3^{k+1}$. The next lemma shows that if a number n has this representation then the number $2n$ and $\frac{2n-1}{3}$ must also have the same representation.

Lemma 3.1. *(Closure of $\mathcal{R}$) If $n \equiv \mathcal{R}$ then $2n \equiv \mathcal{R}$ and if $n \equiv \mathcal{R}$ then $\frac{2n-1}{3} \equiv \mathcal{R}$.*

Proof. (Direct proof) First suppose $n \equiv \mathcal{R}$. Then there is a sequence (a_j) so that

$$n = \left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i} \right) /3^{k+1}$$

where $0 \leq a_0 \leq a_1 \leq a_2 \leq \cdots a_{k+1}$. Write

$$2n = 2\left(\left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i} \right) /3^{k+1} \right) = \left(2^{a_{k+1}+1} - \sum_{i=0}^{k} 2^{a_i+1} 3^{k-i} \right) /3^{k+1}$$

Define the sequence (b_j) by $b_j = a_{j+1} + 1$ for each $0 \leq j \leq k$ so that (b_j) is also positive and increasing. Then we have

$$2n = \left(2^{b_j} - \sum_{i=0}^{k} 2^{b_j} 3^{k-i} \right) /3^{k+1} \equiv \mathcal{R}$$

Therefore $2n \equiv \mathcal{R}$. Now suppose that $n \equiv \mathcal{R}$ then by the above $2n \equiv \mathcal{R}$. Write

$$2n = \left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i} \right) /3^{k+1}$$

were as $1 \leq a_0 \leq a_1 \leq \cdots a_{k+1}$. Then

$$\frac{2n-1}{3} = \frac{\left(2^{a_{k+1}} - \sum_{i=0}^{k} 2^{a_i} 3^{k-i}\right)/3^{k+1} - 1}{3}$$

Define the sequence (b_j) by

$$b_j = \begin{cases} 0 & \text{if } j = 0 \\ a_{j-1} & \text{if } 1 \leq j \leq k+1 \end{cases}$$

Then

$$\frac{2n-1}{3} = \left(2^{b_{k+1}} - \sum_{i=0}^{k+1} 2^{b_i} 3^{k+1-i}\right) / 3^{k+2}$$

and therefore $\frac{2n-1}{3} \equiv \mathcal{R}$ □

Lemma 3.2. *Let $a \in \mathbb{N}$. Then*

$$3^{\frac{a}{2}+1} + 2 < 2^a + 1$$

for all $a \geq 8$.

Proof. (By induction) The proof follows by induction. Let

$$S = \{1, 2, 3, \cdots, 7\} \cup \{a \in \mathbb{N} : 3^{\frac{a}{2}+1} + 2 \leq 2^a + 1\}$$

The value $a = 8$ is in S since

$$245 = 3^5 + 2 < 2^8 + 1 = 257$$

Now if $a \geq 8 \in S$ then

$$\begin{aligned} 3^{\frac{a}{2}+1} + 2 \leq 2^a + 1 &\Rightarrow 3^{\frac{1}{2}}(3^{\frac{a}{2}+1} + 2) \leq 2(2^a + 1) \\ &\Rightarrow 3^{\frac{1}{2}}(3^{\frac{1}{2}^{(a+2)}} + 2) \leq 2(2^a + 1) \\ &\Rightarrow 3^{\frac{1}{2}^{(a+3)}} + 23^{\frac{1}{2}} \leq 2^{a+1} + 2 \\ &\Rightarrow 3^{\frac{1}{2}^{(a+3)}} + 3 \leq 2^{a+1} + 2 \\ &\Rightarrow 3^{\frac{1}{2}^{(a+3)}} + 2 \leq 2^{a+1} + 1 \\ &\Rightarrow 3^{\frac{a+1}{2}+1} + 2 \leq 2^{a+1} + 1 \end{aligned}$$

Therefore $a + 1 \in S$ so that $S = \mathbb{N}$. □

Lemma 3.3. *Let $n \in 2\mathbb{N} = \{2n : n \in \mathbb{N}\}$ and write n is its conical form*

$$n = 2^{\epsilon} \prod_{i=1}^{k} p_{\beta_i}^{\alpha_i}$$

Then there exist a value $k \in \mathbb{N}$ so that

$$T^k(n+1) = \begin{cases} 3^{\frac{\epsilon}{2}} \left(\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i}\right) + 1 & \textit{if } \epsilon \equiv 0 (\bmod 2) \\ 3^{\lfloor \frac{\epsilon}{2} \rfloor + 1} \left(\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i}\right) + 2 & \textit{if } \epsilon \equiv 1 (\bmod 2) \end{cases}$$

Proof. Let $n \in \mathbb{N}$ and write $n = 2^{\epsilon} \prod_{i=1}^{k} p_{\beta_i}^{\alpha_i}$ for some $\epsilon \in \mathbb{N} \cup \{0\}$. First, assume $\epsilon \equiv 0 (\bmod 2)$ and take $k = 3\frac{\epsilon}{2}$. Then the claim is that $T^k(n+1) = 3^{\frac{\epsilon}{2}} \left(\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i}\right) + 1$. Write $n + 1 = 2^{\epsilon} \prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 1$. Define the sequence (a_i) by the rule

$$a_{i+1} = \begin{cases} a_i + 1 & \text{if } i \equiv 0 (\bmod 2) \\ a_i + 2 & \text{if } i \equiv 1 (\bmod 2) \end{cases}$$

were as $a_1 = 1$. Now since $n+1$ must be odd by computing T^k consecutively we find

$$
\begin{aligned}
n+1 = 2^{\epsilon}\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 1 \Rightarrow T(n+1) &= 32^{\epsilon+2}\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 2^2 \\
\Rightarrow T^3(n+1) &= 3^2 2^{\epsilon}\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 1 \\
\Rightarrow T^4(n+1) &= 3^3 2^{\epsilon+6}\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 2^2 \\
\Rightarrow T^6(n+1) &= 3^4 2^{\epsilon}\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 1 \\
&\vdots \\
\Rightarrow T^{a_i}(n+1) &= 3^{i}\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 1
\end{aligned}
$$

Take $i = \frac{\epsilon}{2}$ and $a_i = k$ then $T^k = 3^{\frac{\epsilon}{2}}\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 1$ as desired.
Now assume $\epsilon \equiv 1(\bmod 2)$. Set $i = \lfloor \frac{\epsilon}{2} \rfloor$ and $k = a_i + 1$ then by analogy of the result above we have

$$T^k(n+1) = T^{a_i+1}(n+1) = 3^{\lfloor \frac{\epsilon}{2} \rfloor + 1}\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} + 2$$

which is the desired result. □

Lemma 3.4. *Assume that $a \in \mathbb{N}$. If a is even, then there exist a value k such that $T^k(2^a+1) = 3^{\frac{a}{2}} + 1$. If a is a odd, then there exist a value k such that $T^k(2^a+1) = 3^{\lfloor \frac{a}{2} \rfloor + 1} + 2$.*

Proof. (Corollary to Lemma 3.3) Let $n = 2^a$ for $a \in \mathbb{N}$. Then in regards to Lemma 3.3 we have $\prod_{i=1}^{k} p_{\beta_i}^{\alpha_i} = 1$. Therefore there is a value $k \in \mathbb{N}$ so that

$$T^k(n+1) = \begin{cases} 3^{\frac{a}{2}} + 1 & \textit{if } a \equiv 0(\mathrm{mod} 2) \\ 3^{\lfloor \frac{a}{2} \rfloor + 1} + 2 & \textit{if } a \equiv 1(\mathrm{mod} 2) \end{cases}$$

□

Theorem 3.1. *If for all $n \in \mathbb{N} \setminus 2^{\mathbb{N}}$ $n \sim 3n+2$, then for all $n \in \mathbb{N} \setminus 2^{\mathbb{N}}$ it follows that $n \equiv \mathcal{R}$*

Proof. (By induction) First suppose for all $n \in \mathbb{N} \setminus 2^{\mathbb{N}}$ that $n \sim 3n+2$. Let $X = \{x_1, x_2, \cdots\} = \mathbb{N} \setminus 2^{\mathbb{N}}$ were as $x_1 < x_2 < x_3 < \cdots$ and let S be the set defined by $S = \{n \in \mathbb{N} : x_n \equiv \mathcal{R}\}$. Now $1 \in S$ since $x_1 = \inf(\mathbb{N} \setminus 2^{\mathbb{N}}) = 3$ and $3 = \frac{2^5 - 2^{2^2} - 1}{3^2} \equiv \mathcal{R}$. Now Suppose $k \in S$ for $1 \leq k \leq n$. The proof that $n+1 \in S$ is broken into three cases.

In the first case suppose x_n is not one less than a power of 2 and x_n is even. Then it follows that x_{n+1} is odd. Since x_{n+1} is odd there exist a value $t \geq 2 \in \mathbb{N}$ such that $x_{n+1} = 2t + 1$. Write

$$x_{n+1} = 2t + 1 = \frac{2(3t+2) - 1}{3}$$

By the inductive hypothesis we know $n \equiv \mathcal{R}$. Also under the assumption $t \sim 3t + 2$ for each value t it follows that $3t + 2 \equiv \mathcal{R}$ or else a power of 2. If $3t + 2 \equiv \mathcal{R}$ then by direct application of Lemma 3.1 it follows that $\frac{2(3t+2)-1}{3} \equiv \mathcal{R}$. Otherwise, if $3t + 2 = 2^a$ for some positive integer a then $\frac{2(3t+2)-1}{3} = \frac{2^{a+1}-1}{3} \equiv \mathcal{R}$. Therefore in either case $x_{n+1} \equiv \mathcal{R}$.

In the second case, suppose x_n is not one less than a power of 2 and that x_n is odd. Then it follows that x_{n+1} is even. Since x_{n+1} is even there is a value t such that $x_{n+1} = 2t$. Notice that t cannot be a power of 2 since $x_{n+1} = 2t = x_n + 1 \neq (2^a - 1) + 1 = 2^a$. From this it follows that t cannot be a power of 2. Therefore, since $t = \frac{x_{n+1}}{2} = \frac{x_n+1}{2} < x_n$ it follows that t is an element of the inductive set S, and hence $t \equiv \mathcal{R}$. By Lemma 3.1 it follows that $2t \equiv \mathcal{R}$. Therefore, $x_{n+1} \equiv \mathcal{R}$.

In the third and final case, suppose that x_n is exactly one less than a power of 2. That is, suppose $x_n = 2^a - 1$ for some positive integer $a \geq 8$. Then it follows that $x_{n+1} = 2^a + 1$. Now if a is even then by direct application of Lemma 3.4 there exist a value k such that $T^k(2^a + 1) = 3^{\frac{a}{2}} + 1$. However by Lemma 2 we have the inequality

$$3^{\frac{a}{2}} + 1 < 3^{\frac{a}{2}+1} + 2 < 2^a + 1 = x_{n+1}$$

whenever $a \geq 8$. Therefore $x_{n+1} = 2^a + 1$ iterates to the number $3^{\frac{a}{2}} + 1$ and this number is either a power of 2 or or it is an elemental of the inductive set S, in either case we have $x_{n+1} \equiv \mathcal{R}$.

Now if a is odd then by direct application of Lemma 3.4 there exist a value k such that $T^k(2^a + 1) = 3^{\lfloor \frac{a}{2} \rfloor} + 2$. By Lemma 3.2 we have the inequality

$$3^{\lfloor \frac{a}{2} \rfloor + 1} + 2 < 3^{\frac{a}{2}+1} + 2 < 2^a + 1 = x_{n+1}$$

whenever $a \geq 8$. Therefore $x_{n+1} = 2^a + 1$ iterates to the number $3^{\lfloor \frac{a}{2} \rfloor} + 2$ and this number is either a power of 2 or it is an elemental of the inductive set S, in either case we have $x_{n+1} \equiv \mathcal{R}$. The separate cases $a < 8$ can be checked and verified by strait forward computation.

Now in all three we found that $x_{n+1} \equiv \mathcal{R}$. Since there are no more cases it follows that $S = \mathbb{N}$. This competes the proof. $\square$

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