ON THE DISTRIBUTION OF THE FIRST POINT OF COALESCENCE FOR SOME COLLATZ TRAJECTORIES

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Abstract. This paper is a numerical evaluation of of some trajectories of the Collatz function. Specifically, I assess the coalescence points of each integer $n \equiv 0(\text{mod}2)$ and $n \equiv 2(\text{mod}3)$ through an explicit algorithm which has been developed to test on any two different modulus classes. From the data discovered I illustrate that these points lie close to some specific type of linear Diophantine equations. Afterwards, I show that the first point of coalescence of the integers n and $3n + 2$ appear to tend to an expected value of $4/5n$. However, when the algorithm was pushed to its peak estimation it has been discovered that the expected value starts to decrease away from the estimation of $4/5n$. From the evidence gathered, I show that the first point of coalescence of the integers n and $3n + 2$ seem to appear eradicate from a "step by step" point of view but from a topological point of view seem to be localized around specific lines.

1. Introduction

This paper is a numerical evaluation of the paper *Integer Representations and Trajectories of the $3x + 1$ problem* regarding *Collatz conjecture*. In his prior work, Burson established a conditional proof of the Collatz conjecture under the assumption that that integers n and $3n + 2$ eventually cycle together. More precisely stated, Burson showed that in order to prove the Collatz conjecture one only needs to show that the residues $[0]_3, [2]_3 \in \mathbb{Z}_3$ share the same cycle. For the proof of these result refer back to Burson [4], as it will not be presented here. On the other hand I will provide the necessary definitions that are needed to grasp the material presented herein. I will give a well defined description of coalesce point of two whole numbers that are related by this specific relation and then model its distribution through a Python algorithm and present the results using matlab graphing utilities. The numerical assessment consist of localizing the first point of coalescence of the trajectories $O^+(n)$, and $O^+(3n+2)$ for values of n in the interval $[1, 100]$, $[1, 500]$, $[1, 1000]$, $[1, 10000]$, and $[1, 100000]$. Do to algorithmic data structures and time complexity the interval $[1, 100000]$ is not exceeded in the computational process. The algorithm's that have been used in this work are not provided here, but rather only the results are discussed, along with some additional capabilities that the algorithm is built with. The algorithm is publicly available and can be accessed at `https://github.com/rsb29592/Collatz-Trajectories`.

The evidence gathered indicates that first coalescence point of the trajectories $O^+(n)$ and $O^+(3n + 2)$ are closely related to the integer solutions of the functions

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$f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ defined by the rules

$$f(x) = \frac{3}{2}^{\epsilon} x, \quad g(x) = \frac{3}{2}^{\epsilon'} x$$

This is left as an observation and will not be further investigated in this work. Even if a plausible explanation can be developed to explain why the coalescence points seem to be localized around these equations the existence of such an approximation will demand much more effort and may even be be indeterminable. It is also apparent that these results concerning the trajectories can be generalized in the sense of the relationship established between the input data tested. That is, if $a, b \in \mathbb{N}$ then instead of considering when the equality $O^+(n) = O^+(3n+2)$ holds we can instead consider when the equality holds $O^+(n) = O^+(an+b)$. Although, this appears to be a much more difficult question than that has already been presented the algorithm that has been used in this work has been extended to incorporated such a scenario in case it leads to further results. However, since the results of any other coalescence point is not apparently needed at this point of the investigation, the results that were found for any other trajectories will not be discussed in this work.

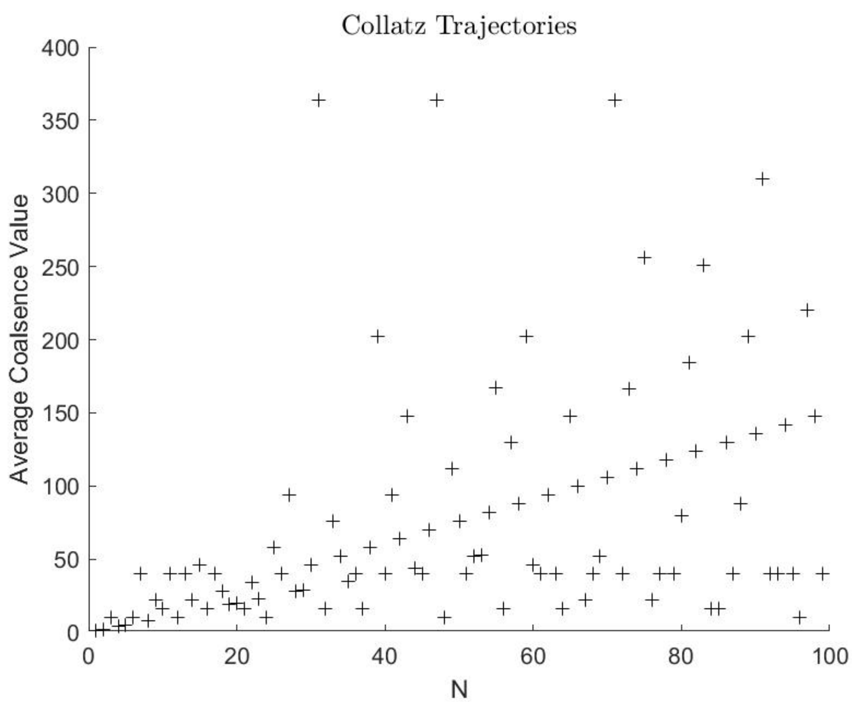

Fig 1: Assessment of the of the first point of coalescence for n and $3n + 2$ for $n \in [1, 100]$.

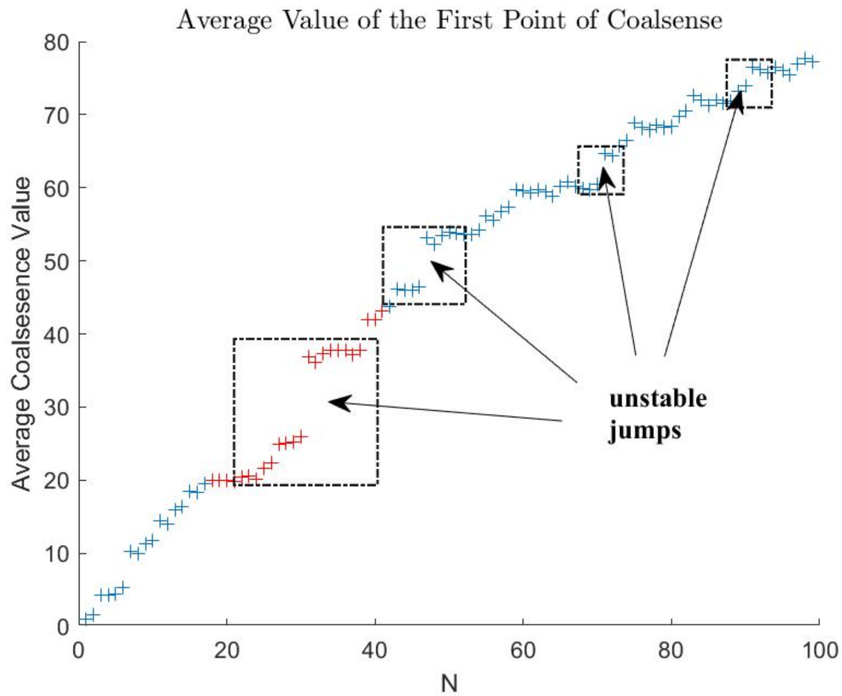

Fig 2: Assessment of the average value of the first point of coalescence for n and $3n+2$ for $n \in [1, 100]$.

1. Notation

In this section I provide the notation and terminology that is presented throughout the work. A detailed explanation of some of these definitions can also be found in [4, 1, 2].

Definition 1. *The Trajectory or Forward Orbit of a positive integer n is the set*

$$O^+(n) = \{n, T(n), T^2(n), \cdots\}$$

were $T : \mathbb{N} \to \mathbb{N}$ is the Collatz function defined by $n \mapsto \frac{n}{2}$ if n is even and $n \mapsto 3n+1$ if n is odd, and T^k is the function $T^k : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by the rule

$$(n, k) \mapsto \underbrace{(T \circ T \circ \cdots \circ T)}_{k-times} \circ (n)$$

Definition 2. *The height $\mathcal{H}(n)$ is defined as the least value k such that the Collatz function $T^k(n) = 1$. If such a value does not exist then we denote $k = \infty$ to mean precisely this.*

Definition 3. *Given two integers n_1 and n_2 define the relation $n_1 \sim n_2$ if and only if the two trajectories $O^+(n_1)$ and $O^+(n_2)$ coalesce. More precisely stated, $n_1 \sim n_2$ if and only if $O^+(n_1) \cap O^+(n_2) \neq \emptyset$.*

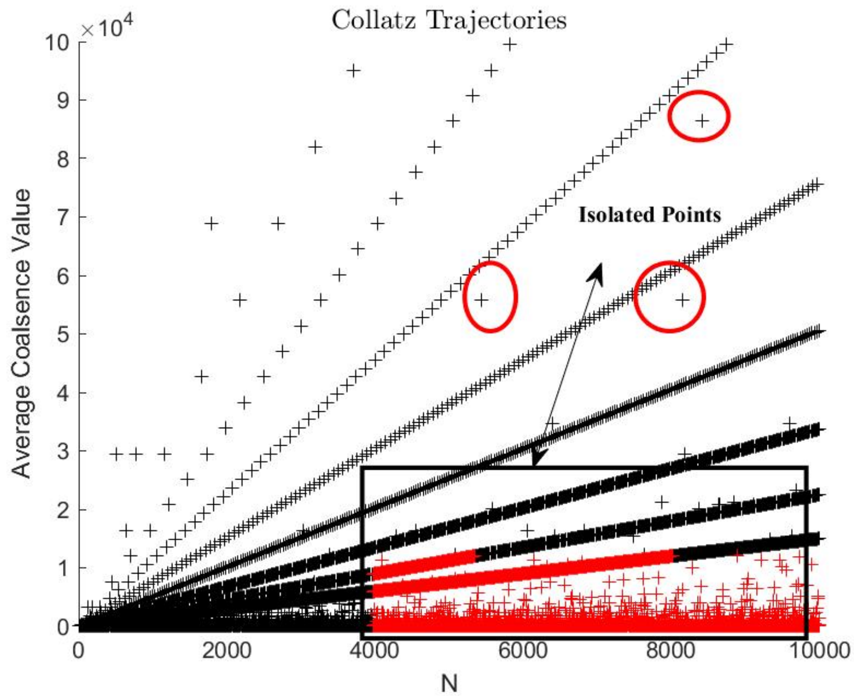

Fig 1: Assessment of the of the first point of coalescence for n and $3n+2$ for $n \in [1, 100]$.

Definition 4. *Given two integers n, m that coalescence under the Collatz map, that is $n \sim m$, the Merging Height $k^{\sim}$ of n or m is defined as*

$$k^{\sim} := \inf\{k_1, k_2 \in \mathbb{N} : T^{k_1}(n) = T^{k_2}(m)\}$$

Definition 5. *Given two integers n, m that coalescence under the Collatz map, that is $n \sim m$, the Coalescence Point $C_p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ of two numbers whole n and m is defined by the rule $(n, m) \mapsto T^{k^{\sim}}(n)$ or $(n, m) \mapsto T^{k^{\sim}}(m)$ since they are equivalent.*

2. 3. The first point of coalescence for integers 0(mod3), and 2(mod3)

In this section I illustrate the empirical results that have been obtained. First, as discussed prior, the algorithm that has been used in this work has been developed to handle a much more broader scenario that need be. That is if $a \in \mathbb{N}$ and $b \in \mathbb{N}$ then the algorithm is designed to compute the first point of coalescence of the to integers n and $an + b$ up to any value than can be specific by the user. By letting $a = 3$ and $b = 2$ then one obtains the important trajectory relationship shown in [4]. After discussing these result of the coalescence point, I will endow into a much more serious investigation. That is, I will demonstrate that coalescence point can be approximated to be near specific regions or lines in $\mathbb{R}^2$. The algorithm designed for this work has also been developed to compute the expected values of the first point

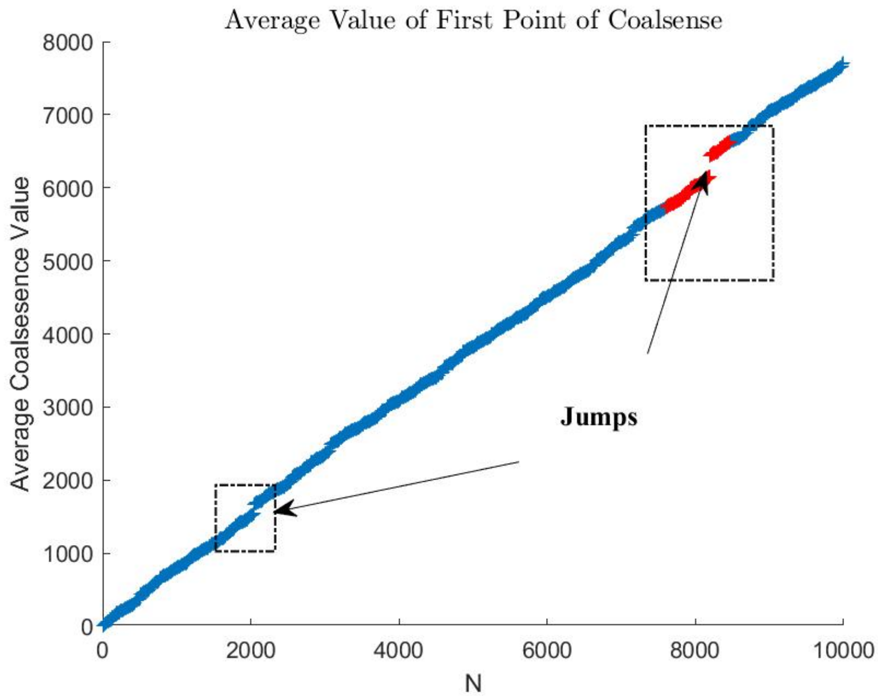

Fig 2: Assessment of the average value of the first point of coalescence for n and $3n+2$ for $n \in [1, 1000]$.

of coalescence for the relation specified by the user. The results for the expected value are illustrated on the same intervals as the coalescence point.

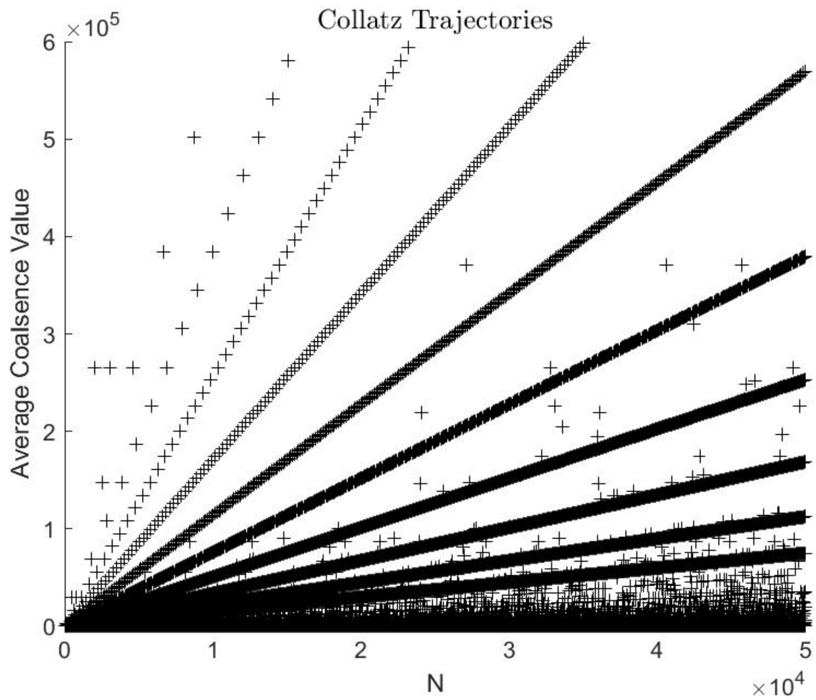

Fig 1: Assessment of the of the first point of coalescence for n and $3n+2$ for $n \in [1, 50000]$.

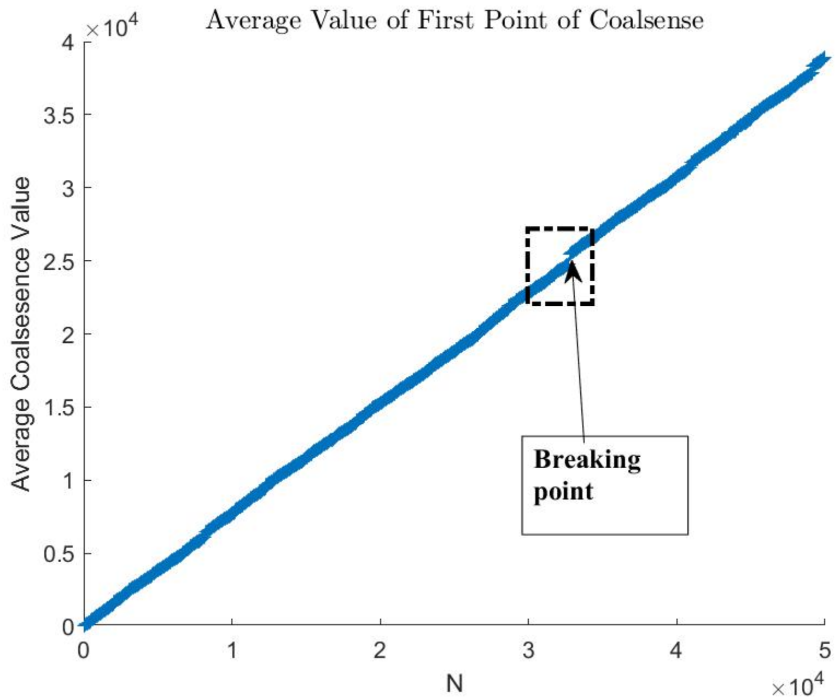

Fig 2: Assessment of the average value of the first point of coalescence for n and $3n+2$ for $n \in [1, 50000]$.

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