Solução de equação de diferenças

1

$$2y[k+1] - y[k] = 3x[k] \tag{1}$$
$$\tag{2}$$

Use:

$$x[k] = u[k] \tag{3}$$
$$y[0] = 0 \tag{4}$$
$$\mathcal{Z}\left\{a^k u[k]\right\} = \frac{z}{z-a} \tag{5}$$

Solução:

$$\mathcal{Z}\left\{2y[k+1] - y[k]\right\} = \mathcal{Z}\left\{3x[k]\right\} \tag{6}$$
$$2\mathcal{Z}\left\{y[k+1]\right\} - \mathcal{Z}\left\{y[k]\right\} = 3\mathcal{Z}\left\{x[k]\right\} \tag{7}$$

$$\mathcal{Z}\left\{y[k]\right\} = Y(z) \tag{8}$$
$$\mathcal{Z}\left\{x[k]\right\} = X(z) \tag{9}$$
$$\mathcal{Z}\left\{y[k+1]\right\} = z\,Y(z) \tag{10}$$

Assim, de (7)

$$2zY(z) - Y(z) = 3X(z) \tag{11}$$
$$(2z-1)Y(z) = 3X(z) \tag{12}$$
$$\frac{Y(z)}{X(z)} = \frac{3}{2z-1} \tag{13}$$

Como $x[k] = u[k]$, então $X(z) = \mathcal{Z}\{u[k]\} = z/(z-1)$. Então:

$$Y(z) = \frac{3X(z)}{2z-1} = \frac{3z}{(z-1)(2z-1)} \tag{14}$$

$$= \frac{3z}{2(z-1)(z-0.5)} = \frac{1.5z}{(z-1)(z-0.5)} \tag{15}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{1.5}{(z-1)(z-0.5)} \tag{16}$$

$$\frac{Y(z)}{z} = R(z) = \frac{A_1}{z-1} + \frac{A_2}{z-0.5} \tag{17}$$

$$A_1 = R(z)(z-1)|_{z=1} \tag{18}$$

$$A_2 = R(z)(z-0.5)|_{z=0.5} \tag{19}$$

$$A_1 = \frac{1.5z}{\cancel{(z-1)}(z-0.5)}\cancel{(z-1)}\Big|_{z=1} \tag{20}$$

$$A_1 = \frac{1.5 \cdot 1}{1-0.5} = 3 \tag{21}$$

$$A_2 = \frac{1.5z}{\cancel{(z-0.5)}(z-1)}\cancel{(z-0.5)}\Big|_{z=0.5} \tag{22}$$

$$A_2 = \frac{1.5 \cdot 0.5}{0.5-1} = -1.5 \tag{23}$$

$$\frac{Y(z)}{z} = \frac{3}{z-1} - \frac{1.5}{z-0.5} \tag{24}$$

$$Y(z) = \frac{3z}{z-1} - \frac{1.5z}{z-0.5} \tag{25}$$

$$y[k] = \mathcal{Z}^{-1}\{Y(z)\} \tag{26}$$

$$y[k] = \mathcal{Z}^{-1}\left\{\frac{3z}{z-1} - \frac{1.5z}{z-0.5}\right\} \tag{27}$$

$$= 3\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} - 1.5\mathcal{Z}^{-1}\left\{\frac{z}{z-0.5}\right\} \tag{28}$$

$$= 3u[k] - 1.5(0.5)^k u[k] \tag{29}$$

2

$$10y[k+2] = 9y[k] - y[k+1] + 5x[k+1] + x[k] \tag{30}$$
$$\tag{31}$$

Use condições iniciais nulas e:

$$x[k] = \left(\frac{1}{2}\right)^k u[k] \tag{32}$$
$$\mathcal{Z}\left\{a^k u[k]\right\} = \frac{z}{z-a} \tag{33}$$

$$\mathcal{Z}\left\{10y[k+2]\right\} = \mathcal{Z}\left\{9y[k] - y[k+1] + 5x[k+1] + x[k]\right\} \tag{34}$$
$$10\mathcal{Z}\left\{y[k+2]\right\} = 9\mathcal{Z}\left\{y[k]\right\} - \mathcal{Z}\left\{y[k+1]\right\} + 5\mathcal{Z}\left\{x[k+1]\right\} + \mathcal{Z}\left\{x[k]\right\} \tag{35}$$

No entanto:

$$\mathcal{Z}\left\{y[k]\right\} = Y(z) \tag{36}$$
$$\mathcal{Z}\left\{x[k]\right\} = X(z) = \frac{z}{z-0.5} \tag{37}$$
$$\mathcal{Z}\left\{y[k+1]\right\} = z\,Y(z) \tag{38}$$
$$\mathcal{Z}\left\{y[k+2]\right\} = z^2\,Y(z) \tag{39}$$
$$\mathcal{Z}\left\{x[k+1]\right\} = z\,X(z) \tag{40}$$

Assim, de (35)

$$10z^2Y(z) = 9Y(z) - zY(z) + 5zX(z) + X(z) \tag{41}$$
$$\frac{Y(z)}{X(z)} = \frac{5z+1}{10z^2+z-9} \tag{42}$$

Agora:

$$Y(z) = \frac{5z+1}{10z^2+z-9}X(z) \tag{43}$$

$$= \frac{5z+1}{10z^2+z-9}\,\frac{z}{z-0.5} \tag{44}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{5z+1}{(10z^2+z-9)(z-0.5)} = \frac{5z+1}{10(z+1)(z-0.5)(z-0.9)} \tag{45}$$

Assim:

$$\frac{Y(z)}{z} = R(z) = \frac{A_1}{z+1} + \frac{A_2}{z-0.5} + \frac{A_3}{z-0.9} \tag{46}$$

$$A_1 = R(z)(z+1)|_{z=-1} \tag{47}$$

$$A_2 = R(z)(z-0.5)|_{z=0.5} \tag{48}$$

$$A_3 = R(z)(z-0.9)|_{z=0.9} \tag{49}$$

$$A_1 = \frac{5z+1}{10\cancel{(z+1)}(z-0.5)(z-0.9)}\cancel{(z+1)}\Bigg|_{z=-1} \tag{50}$$

$$A_1 = \frac{-5+1}{10(-1-0.5)(-1-0.9)} \approx -0.1403 \tag{51}$$

$$A_2 = \frac{5z+1}{(z+1)\cancel{(z-0.5)}(z-0.9)}\cancel{(z-0.5)}\Bigg|_{z=0.5} \tag{52}$$

$$A_2 = \frac{5\cdot 0.5+1}{(0.5+1)(0.5-0.9)} \approx -0.5833 \tag{53}$$

$$A_3 = \frac{5z+1}{(z+1)(z-0.5)\cancel{(z-0.9)}}\cancel{(z-0.9)}\Bigg|_{z=0.9} \tag{54}$$

$$A_3 = \frac{5\cdot 0.9+1}{(0.9+1)(0.9-0.5)} \approx 0.7237 \tag{55}$$

$$\frac{Y(z)}{z} = \frac{-0.1403}{z+1} - \frac{0.5833}{z-0.5} + \frac{0.7237}{z-0.9} \tag{56}$$

$$Y(z) = \frac{-0.1403z}{z+1} - \frac{0.5833z}{z-0.5} + \frac{0.7237z}{z-0.9} \tag{57}$$

$$y[k] = \mathcal{Z}^{-1}\left\{Y(z)\right\} \tag{58}$$

$$y[k] = \mathcal{Z}^{-1}\left\{\frac{-0.1403z}{z+1} - \frac{0.5833z}{z-0.5} + \frac{0.7237z}{z-0.9}\right\} \tag{59}$$

$$\begin{aligned} &= -0.1403\mathcal{Z}^{-1}\left\{\frac{z}{z+1}\right\} - 0.5833\mathcal{Z}^{-1}\left\{\frac{z}{z-0.5}\right\} \\ &\quad + 0.7237\mathcal{Z}^{-1}\left\{\frac{z}{z-0.9}\right\} \end{aligned} \tag{60}$$

$$= \left(0.7237(0.9)^k - -0.5833(0.5)^k - 0.1403(-1)^k\right)u[k] \tag{61}$$