Projeto por alocação de polos

25 de outubro de 2022

1 Problema

Projete um controlador com ação integral para o sistema

$$G(z) = \frac{B(z)}{A(z)} = \frac{(z+1)(z-0.8)}{z(z-0.9)} \tag{1}$$

De modo que $\xi = 0.5$, $\omega_n = 2$. Utilize o período de amostragem $T = 0.1$.

2 Solução

Controlador:

$$C(z) = \frac{N(z)}{R(z)} \tag{2}$$

onde N e R são polinômios a se determinar.

Suponha que $n = \text{grau}(N)$ e $r = \text{grau}(R)$. Para o controlador ser causal devemos ter $r \geq n$.

Em malha fechada:

$$\begin{aligned} s &= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} && (3)\\ &= -1 \pm j2\sqrt{0.75} \approx -1 \pm j1.73 && (4)\\ z &= \exp(sT) \approx 0.81 \pm j0.15 && (5)\\ M(z) &= (z - 0.81 + j0.15)(z - 0.81 - j0.15)M'(z) && (6)\\ &= (z^2 - 1.78z + 0.82)M'(z) && (7) \end{aligned}$$

Equação de malha:

$$\begin{aligned} 1 + C(z)G(z) &= 1 + C(z)\,\frac{B(z)}{A(z)} && (8)\\ &= 1 + C(z)\,\frac{(z+1)(z-0.8)}{z(z-0.9)} && (9) \end{aligned}$$

Proposta:

$$C(z) = \frac{N(z)}{(z-0.8)(z-1)R'(z)} \tag{10}$$

Então:

$$1 + C(z)G(z) = 1 + \frac{N(z)}{\cancel{(z-0.8)}(z-1)R'(z)} \frac{(z+1)\cancel{(z-0.8)}}{z(z-0.9)} \tag{11}$$

$$= \frac{z(z-1)(z-0.9)R'(z) + (z+1)S(z)}{z(z-0.9)(z-1)R'(z)} \tag{12}$$

Polinômio de malha:

$$z(z-1)(z-0.9)R'(z) + (z+1)N(z) = M(z) \tag{13}$$

$$z(z-1)(z-0.9)R'(z) + (z+1)N(z) = (z^2 - 1.78z + 0.82)M'(z) \tag{14}$$

Supondo grau$(R') = r' \geq 0$ e grau$(M') = m' \geq 0$:

$$\underbrace{z(z-1)(z-0.9)R'(z)}_{3+r'} + \underbrace{(z+1)N(z)}_{1+n} = \underbrace{(z^2 - 1.78z + 0.82)M'(z)}_{2+m'} \tag{15}$$

Para o controlador ser causal, devemos ter $r' + 2 \geq n$. Na melhor hipótese $n = r' + 2$. Então:

$$r' + 3 = m' + 2 \tag{16}$$

$$r' = m' - 1 \geq 0 \tag{17}$$

Logo $m' \geq 1$. A situação mais simples (menores graus possíveis) é $m' = 1$ e $r' = 0$.

$m' = 1 \Rightarrow$ mais um pólo em malha fechada. Escolhe-se $z = 0$. Assim:

$$M'(z) = z \tag{18}$$

$$M(z) = (z^2 - 1.78z + 0.82)z \tag{19}$$

$r' = 0 \Rightarrow n = r' + 2 = 2$. Então:

$$R'(z) = k_1 \tag{20}$$

$$N(z) = k_2 z^2 + k_3 z + k_4 \tag{21}$$

Voltando à equação de malha:

$$z(z-1)(z-0.9)R'(z) + (z+1)N(z) = (z^2 - 1.78z + 0.82)M'(z) \tag{22}$$
$$z(z-1)(z-0.9)k_1 + (z+1)(k_2 z^2 + k_3 z + k_4) = (z^2 - 1.78z + 0.82)z \tag{23}$$

Então:

$$k_1 + k_2 = 1 \tag{24}$$
$$-1.9k1 + k_2 + k_3 = -1.78 \tag{25}$$
$$0.9k_1 + k_3 + k_4 = 0.82 \tag{26}$$
$$k_4 = 0 \tag{27}$$

Resolvendo obtém-se: $k_1 = 0.95$, $k_2 = 0.05$, $k_3 = -0.03$, $k_4 = 0$. Logo:

$$C(z) = \frac{0.05z^2 - 0.03z}{0.95(z-1)(z-0.8)} \tag{28}$$
$$= \frac{0.055z^2 - 0.034z}{(z-1)(z-0.8)}. \tag{29}$$

Implementação:

$$C(z) = \frac{0.055z^2 - 0.034z}{(z-1)(z-0.8)} \tag{30}$$
$$= \frac{0.055 - 0.034z^{-1}}{1 - 1.8z^{-1} + 0.8z^{-2}} \tag{31}$$
$$\Rightarrow u[k] = 1.8u[k-1] - 0.8u[k-2] + 0.55e[k] - 0.034e[k-1] \tag{32}$$

```
import·numpy·as·np
import·sympy·as·sp
from matplotlib import pyplot as plt
```

```
z = sp.symbols('z')
k1,k2,k3,k4 = sp.symbols(['k1','k2','k3','k4'])
S = k2*z**2+k3*z+k4
R = k1*(z-1)*(z-0.8)
B = (z+1)*(z-0.8)
A = (z-0.9)*z
```

```
xi =0.5
wn = 2
s = -xi*wn+1j*wn*np.sqrt(1-xi**2)
T = 0.1
z1 = np.exp(s*T)
np.poly([z1,np.conj(z1)])
```

```
array([ 1.        , -1.78259751,  0.81873075])
```

```
M=sp.expand(sp.numer(sp.simplify(1+(B*S/(A*R)))))
M=sp.collect(M,z)
print(M)
MM = sp.collect(sp.expand((z**2-1.78*z+0.82)*z),z)
print(MM)
```

```
k4 + z**3*(k1 + k2) + z**2*(-1.9*k1 + k2 + k3) + z*(0.9*k1 + k3 + k4)
z**3 - 1.78*z**2 + 0.82*z
```

```
eq=[M.coeff(z,n=p)-MM.coeff(z,n=p) for p in range(4)]
sol=sp.solve(eq)
sol
```

```
{k1: 0.947368421052632,
 k2: 0.0526315789473684,
 k3: -0.0326315789473684,
 k4: 0.0}
```

```
C=S.subs(sol)/R.subs(sol)
C.normal()
```

$$\frac{0.0555555555555556z^2 - 0.0344444444444444z}{(z-1)(z-0.8)}$$