Transformada $\mathcal{Z}$

Eq diferenças lineares invariantes

$$x[k] \longrightarrow X(z) \qquad z \in \mathbb{C}$$

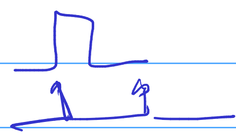
$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

$$X(z) = \sum_{k=0}^{\infty} x[k]\, z^{-k}$$

Surge do seguinte:

$$x(t) \Rightarrow \text{amostrar} \Rightarrow \tilde{x}(t) = x(t) \cdot \Pi(t)$$

$\Pi(t)$ = trem de impulsos



$$\Pi(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Substituindo tudo:

$$\tilde{X}(s) = \int_0^\infty \tilde{x}(t)\, e^{-st}\, dt$$

$$\tilde{x}(t) = x(t) \cdot \Pi(t) =$$
$$= x(t) \sum_k \delta(t - kT)$$

$$\tilde{X}(s) = \int_0^\infty x(t) \sum_k \delta(t - kT)\, e^{-sT}\, dt$$

$$\tilde{X}(s) = \int_0^{\infty} \sum_k x(t)\,\delta(t-kT)\,e^{-st}\,dt$$

$$= \sum_k \int_0^{\infty} \underline{x(t)}\,\delta(t-kT)\,\underline{e^{-st}}\,dt$$

$$t - kT = 0 \Rightarrow t = kT$$

$$= \sum_k x(kT)\,e^{-skT}$$

$$x(kT) \hat{=} x[k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,(e^{sT})^{-k}$$

$$e^{sT} \leftarrow z$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,z^{-k}$$

Para sinais causais, $x[k] = 0$, $k < 0$

Transf. Z unilateral

$$\mathcal{Z}\{x[k]\} = X(z) = \sum_{k=0}^{\infty} x[k]\,z^{-k}$$

$$\mathcal{Z}\{a^k u[k]\} = \frac{z}{z-a}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2$$

$$\mathcal{Z}\{a^k u[k]\} = \sum_{k=0}^{\infty} a^k \overset{1}{u[k]}\,z^{-k}$$

$$= \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} (a z^{-1})^k$$

$$\mathcal{Z}\{a^k u[k]\} = \sum_{k=0}^{\infty} (az^{-1})^k$$

$$= 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

PG de razão $q = az^{-1}$

Limite da PG $= \dfrac{a_0}{1-q}$

$$\mathcal{Z}\{a^k u[k]\} = \frac{1}{1 - az^{-1}} \cdot \frac{z}{z} = \frac{z}{z-a}$$

Só vale se $|q| < 1$

$$|az^{-1}| < 1 \Rightarrow |z| > |a|$$

$$\mathcal{Z}\{x[k+1]\} = zX(z) -$$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)$$

$$\mathcal{Z}\{x[k+1]\} = \sum_{k=0}^{\infty} x[k+1]\, z^{-k}$$

$$\mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]\, z^{-k}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$\mathcal{Z}\{x[k+1]\} = \sum_{k=0}^{\infty} x[k+1]\, z^{-k}$$

$$= x[1] + x[2]\, z^{-1} + x[3]\, z^{-2} + x[4]\, z^{-3} + \cdots$$

$$\times\ z^{-1}$$

$$z^{-1}\, \mathcal{Z}\{x[k+1]\} = x[1]\, z^{-1} + x[2]\, z^{-2} + \cdots$$

$$x[0] \qquad\qquad x[0]$$

$$z^{-1}\, \mathcal{Z}\{x[k+1]\} + x[0] = \mathcal{Z}\{x[k]\}$$

$$z^{-1}\, \mathcal{Z}\{x[k+1]\} + x[0] = X(z) \qquad (\cdot\, z)$$

$$\mathcal{Z}\{x[k+1]\} + z\, x[0] = z\, X(z)$$

$$\mathcal{Z}\{x[k+1]\} = z\, X(z) - z\, \underline{\underline{x[0]}}$$

$$\mathcal{Z}\{x[k+2]\} = ?$$

$$x[k] \Longleftrightarrow X(z)$$

$$x[k+1] = v[k] \Longleftrightarrow V(z)$$

$$\mathcal{Z}\{x[k+2]\} = \mathcal{Z}\{v[k+1]\}$$

$$= z\, V(z) - z\, v[0]$$

$$V(z) = \mathcal{Z}\{v[k]\} = \mathcal{Z}\{x[k+1]\}$$

$$= zX(z) - zx[0]$$

$$\mathcal{Z}\{x[k+2]\} = z\left(zX(z) - zx[0]\right) - zv[0]$$

$$= z^2X(z) - z^2x[0] - zv[0]$$

$$v[k] = x[k+1] \Rightarrow v[0] = x[1]$$

$$\Rightarrow \mathcal{Z}\{x[k+2]\} = z^2X(z) - z^2x[0] - zx[1]$$

$$\mathcal{Z}\{x[k+n]\} = z^nX(z) - z^nx[0] - z^{n-1}x[1] - z^{n-2}x[2] - \dots - z^1x[n-1]$$

$$\mathcal{Z}\{x[k+4]\} = z^4X(z) - z^4x[0] - z^3x[1] - z^2x[2] - z^1x[3]$$

$$y[k+2] = y[k+1] + y[k]$$

$y[0] = 0$
$y[1] = 1$

$$\mathcal{Z}\{y[k+2]\} = \mathcal{Z}\{y[k+1] + y[k]\}$$

$$\mathcal{Z}\{y[k]\} = Y(z)$$

$$\mathcal{Z}\{y[k+1]\} = zY(z) - \cancel{z\,y[0]}$$

$$\mathcal{Z}\{y[k+2]\} = z^2Y(z) - \cancel{z^2 y[0]} - z\,y[1]$$

$$= z^2 Y(z) - z$$

$$z^2 Y(z) - z = zY(z) + Y(z)$$

$$z^2 Y(z) - zY(z) - Y(z) = z$$

$$Y(z)(z^2 - z - 1) = z$$

$$a^k u[k] \Leftrightarrow \frac{z}{z-a}$$

$$Y(z) = \frac{z}{z^2 - z - 1}$$

$$\frac{Y(z)}{z} = \frac{1}{z^2-z-1} = R(z)$$

$$z^2 - z - 1 = 0 \Rightarrow z = \frac{1 \pm \sqrt{1^2 - 4\cdot(-1)}}{2}$$

$$(z-p_1)(z-p_2) \qquad z = \frac{1 \pm \sqrt{5}}{2} \begin{cases} p_1 \\ p_2 \end{cases}$$

$$\frac{Y(z)}{z} = \frac{A}{z-p_1} + \frac{B}{z-p_2} = R(z)$$

$$A = R(z)(z-p_1)\Big|_{z=p_1}$$

$$= \frac{1}{\cancel{(z-p_1)}(z-p_2)} \cdot \cancel{(z-p_1)}\Big|_{z=p_1}$$

$$= \frac{1}{p_1-p_2} = \frac{1}{\frac{\cancel{1}+\sqrt{5}}{2} - \left(\frac{\cancel{1}-\sqrt{5}}{2}\right)} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$B = R(z)(z-p_2)\Big|_{z=p_2}$$

$$= \frac{1}{(z-p_1)\cancel{(z-p_2)}} \cancel{(z-p_2)}\Big|_{z=p_2} = \frac{1}{p_2-p_1} = \frac{-\sqrt{5}}{5}$$

$$\frac{Y(z)}{z} = \frac{\sqrt{5}/5}{z-p_1} - \frac{\sqrt{5}/5}{z-p_2}$$

$$Y(z) = \frac{\left(\frac{\sqrt{5}}{5}\right)z}{z-p_1} - \frac{\left(\frac{\sqrt{5}}{5}\right)z}{z-p_2}$$

$$y[k] = \left(\frac{\sqrt{5}}{5}\right)p_1^k u[k] - \left(\frac{\sqrt{5}}{5}\right)p_2^k u[k]$$

$$= \frac{\sqrt{5}}{5}\left(p_1^k - p_2^k\right)u[k]$$

$$= \frac{\sqrt{5}}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]u[k]$$