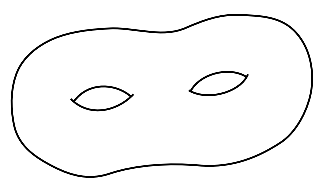
An important question in the theory of 3–dimensional topology is to classify and understand 3–dimensional spaces. Cutting spaces into simple pieces along surfaces, understanding the pieces, and then understanding how they're put together is one way to study a given 3–dimensional space. Two examples of this decomposition are Heegaard splittings, which cut a closed manifold into two handlebodies, and Dehn surgery, which decomposes a manifold into a solid torus and a knot exterior. Studying which manifolds can be obtained by Dehn surgery on which knots has historically been the source of many powerful techniques in low dimensional topology.

My research focuses on when a knot in a handlebody has a nontrivial surgery again yielding a handlebody. My objective is to classify such knots. To get started, I have constructed an infinite family of counterexamples to a conjecture concerning the form of these knots. My counterexamples have given insight into some strategies for a classification theorem, and they also yield many interesting examples of the interplay between the Heegaard structure of a 3–manifold and Dehn surgeries on knots in the manifold.

Dehn surgery on knots in handlebodies

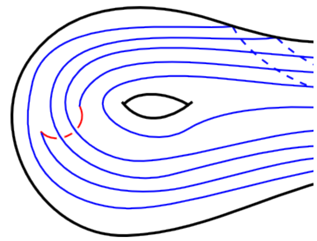


A 3–manifold is a space which is locally homeomorphic to $\mathbb{R}^3$; locally, it looks like the space we inhabit. A handlebody is a special type of 3–manifold obtained from a ball by attaching "handles." The boundary of a handlebody is a surface of genus g, and we also call g the genus of the handlebody. A handlebody of genus 2 is pictured at right. Handlebodies play the role of the simple pieces into which a manifold is cut in the theory of Heegaard splittings [8].

Dehn surgery on a knot K at slope α consists of removing a neighborhood of K from the ambient 3–manifold and gluing back in a solid torus so that the curve α bounds a disk in the resulting manifold. Here α, the surgery slope, is a curve in the boundary of a neighborhood of K. Dehn surgery is fundamental in 3–manifold topology because every 3–manifold can be obtained by Dehn surgery along a link in S^3 [9].

Given the importance of Dehn surgery and handlebody decompositions of manifolds, it's a natural question to ask when a knot in a handlebody has a surgery again yielding a handlebody. Call a knot in a handlebody H which has a nontrivial handlebody surgery an H–knot. Gabai [6] and Berge [2] both investigated the case of knots in solid tori (handlebodies of genus $g = 1$). Gabai completely classified surgeries on knots in solid tori and also gave an algorithm to determine when such a knot has a handlebody surgery. Berge gave a complete list of knots in solid tori with solid tori surgeries.

Both Berge and Gabai recognized that knots in solid tori with solid tori surgeries must be 1–bridge. This means that the knot is isotopic to a curve of the form $\alpha \cup \beta$, where α lies in the boundary of the handlebody and β is properly embedded, unknotted arc. Pictured is part of a handlebody together with a 1–bridge knot. The short, semicircular arc (red) is β.



Bridge number is a measure of the complexity of a knot, so this result says that knots which are simple in one way (they have surgeries yielding a handlebody) are simple in another way (they're 1–bridge). The fact that H–knots are 1–bridge when $g = 1$ played an important role in the proof that it is possible to recognize these knots.

Wu [10] looked at surgery on 1–bridge knots in handlebodies. Based on his investigations he conjectured that H–knots in handlebodies of genus $g > 1$ should also be 1–bridge. However, we have shown [3]

Theorem 1 (B.)**.** *Up to homeomorphism, there are infinitely many pairs* (H, K) *where* H *is a handlebody of genus* 2*,* K *is an* H*–knot, and* K *is not* 1*–bridge in* H*.*

Objectives and significance

My research objective is to classify H–knots, an important class of knots which have resisted a simple description for over two decades. Any knot K in a handlebody H which is isotopic into ∂H is an H–knot, but these are boring examples. Wu [11] has given an algorithm to determine whether a 1–bridge knot in a handlebody H is an H–knot (and find the appropriate surgery slopes). It remains to classify H–knots that are not 1–bridge.

Studying which manifolds can be obtained from surgery on certain knots has given rise to many of the important techniques in 3–manifold topology. For example, Gabai's proof that the unknot is the only knot which has a surgery yielding $S^2 \times S^1$ [5] emphasized the use of taut foliations and introduced the concept of thin position. Thin position played an important role in the proof that S^3 cannot be obtained by nontrivial surgery on any nontrivial knot [7]. Other important ingredients in this theorem are the techniques developed in the proof of the cyclic surgery theorem [4], which concerns the slope at which a knot can have a surgery yielding a lens space.

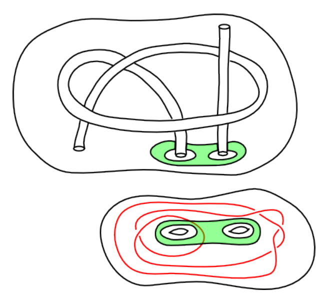
Given this history of questions in Dehn surgery leading to both strong results as well as new techniques, I believe that a classification of H–knots would be a deep theory and a powerful tool to use in all areas of 3–manifold topology.

Current and future work

There are many questions surrounding the classification of H–knots. Here are a few that I am working on.

Show that a large class of H–knots aren't 1–bridge

The simplest knot given in Theorem 1 is pictured at right, where the handlebody is obtained from the two pictured handlebodies by gluing along the pictured 3–punctured sphere.



Although we focus on a particular family of knots in the above theorem, there is a more general construction which gives knots which seem to be n–bridge for $n > 1$. Techniques similar to the ones used in the proof of Theorem 1 will be used to show this.

Develop techniques to detect higher bridge number

It is possible to define higher bridge numbers in a similar manner to 1–bridge; an n–bridge knot in a handlebody is isotopic to a curve which lies on the boundary of the handlebody except for n unknotted arcs in the interior. Although we have shown that H–knots need not be 1–bridge, perhaps their bridge number is bounded. Examining the knots considered above in relation to the surface P discussed below, we conjecture:

Conjecture 1. *There are H–knots with arbitrarily high bridge number.*

Part of the proof of Theorem 1 involves a characterization of 1–bridge knots in handlebodies. Unfortunately, its straightforward generalization to higher bridge number is false. Yokota [12] has given criteria that can be used to compute lower bounds for bridge numbers of knots in handlebodies, but these bounds are difficult to compute in practice. We intend to use extra information about the H–knot, such as the surfaces described below, to give bounds on the bridge numbers of H–knots.

Examine surfaces in the handlebodies in the complement of the knots

A crucial fact used in the proof of Theorem 1 is the existence of a 3–punctured sphere properly embedded in the handlebody which is incompressible and ∂–incompressible in the complement of the knot. In fact, 3–punctured spheres which are incompressible and ∂–incompressible in the complement of the knot exist in all the knots we have constructed, but we have not shown that similar such surfaces must exist. For example, given an H–knot which is not 1–bridge, must there exist an incompressible, non-boundary parallel surface which is incompressible in the complement of the knot? Can the surface be taken to be planar? Do such surfaces also exist in H–knots which *are* 1–bridge, for example the H–knots given by Berge [1]?

Determine an appropriate measure of complexity for H–knots

Gabai and Berge showed that H–knots must be 1–bridge when H is a solid torus, and this fact plays a crucial role in their classification. If the philosophy that "knots which admit simple surgeries should be simple" is accurate, then Theorem 1 shows that we need a new notion of simplicity for H–knots when the genus of H is greater than one. What is this notion, and is the above philosophy even correct?

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