Spatial Analysis and Modeling

05 - Spatial Regression

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Recap Linear Regression

Linear Regression - Notation

linear relationship between a dependent variable y_i (at location i) and a set of explanatory variables x_{ih}, for h = 1, … , k subject to random error

$$y_i = \Sigma_h x_{ih} \beta_h + e_i$$

e_i is a random error term, with $E[e_i] = 0$, i.e., no systematic error

Linear Regression - Notation (continued)

in matrix notation, a $n \times 1$ column vector y and a $n \times k$ vector X, with a $k \times 1$ coefficient vector β and a $n \times 1$ random error vector e

$$y = X\beta + e$$

$$E[e] = 0$$

Conditional Expectation

under a set of regularity conditions (to follow) the conditional expectation of y given X is linear in X

$$E[y|X] = E[X\beta|X]+E[e|X] = X\beta + 0$$

in other words, what would y be on average if we knew X

Marginal Effect

the effect of a change in X on y

in a linear regression the marginal effect equals the regression coefficient (this is not the case in a non linear regression)

$$E[y|\Delta X] = \Delta X \beta$$

Selected Regularity Conditions

X non-stochastic (or if stochastic, with bounds on second moment) - the only randomness follows from the dependent variable y, any randomness in X is inconsequential

error term independent identically distributed (i.i.d), i.e., $Var[e_i]=\sigma^2$ or $E[ee'] = \sigma^2 I$ = spherical error term

x_i and e_i uncorrelated for all i, i.e., signal (X) and noise (e) are not related

Ordinary Least Squares (OLS) Regression

under set of regularity conditions, yields the best (smallest variance) unbiased estimator = Gauss-Markov theorem

$$b = (X'X)^{-1} X'y$$

$$E[b] = E[(X'X)^{-1}X'(X\beta)] + E[(X'X)^{-1}X'e] = \beta$$

(since $E[X'e] = 0$)

Predicted Value

value of y_i given x_i using the estimates b

$y_{ip} = \Sigma_h x_{ih} b_h$

Residual

difference between observed and predicted

$u_i = y_i - y_{ip}$

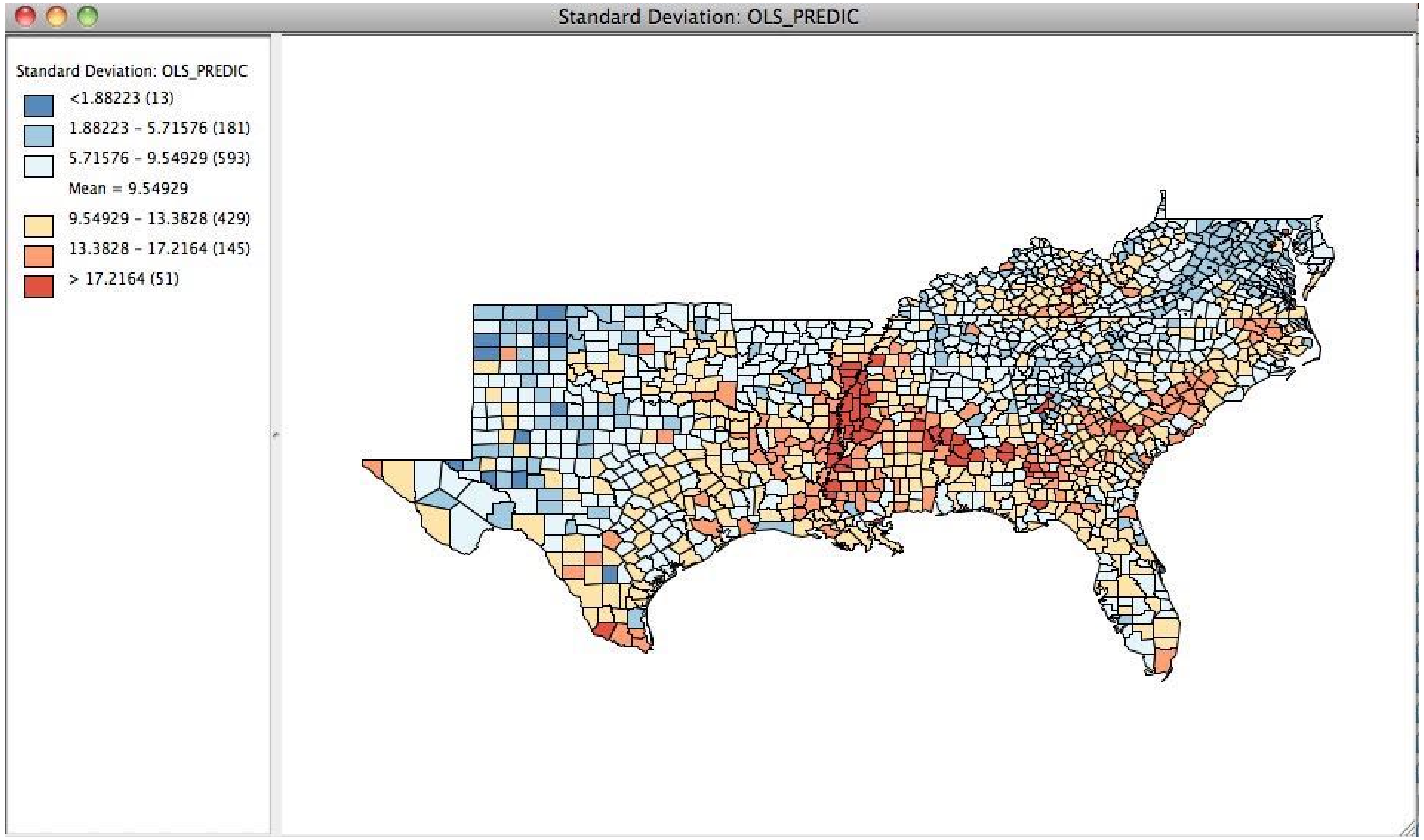
for regression with constant term avg[u_i] = 0
residual is NOT the same as the error term (e_i)

Visual Diagnostics

Predicted Value Map

shows spatial distribution of model prediction

a form of smoothing, i.e., what the model suggests y should be, given the X at each location



Predicted Value map
1990 county homicide rates (standard deviational map)

Residual Map

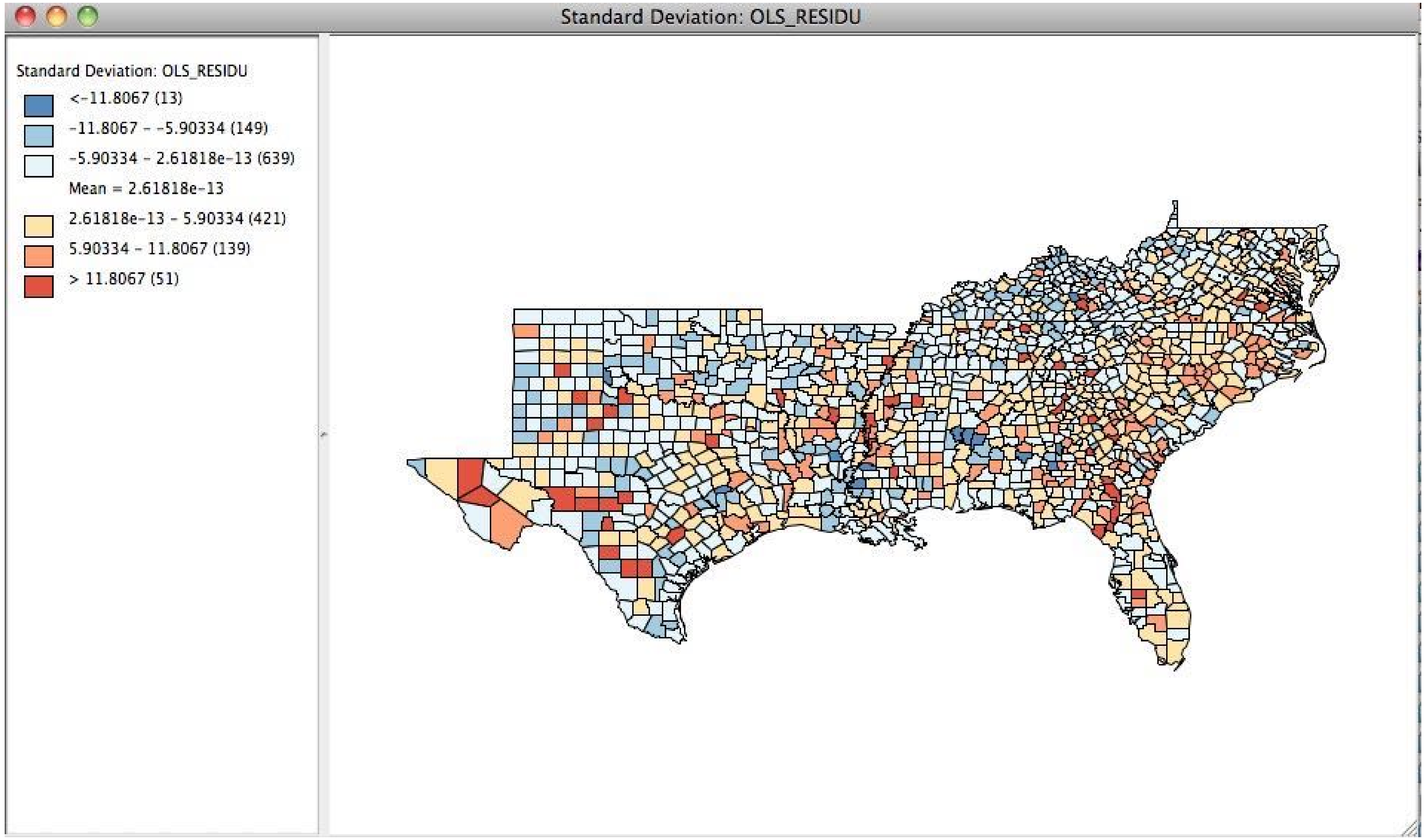
high values (red) = model under-predicts ($y > y_p$)

low values (blue) = model over-predicts ($y < y_p$)

note extremes = poor fit of model

note spatial patterns, but visual inspection
can be misleading

need for formal diagnostics



Residual map - 1990 county homicide rates
(standard deviational map)

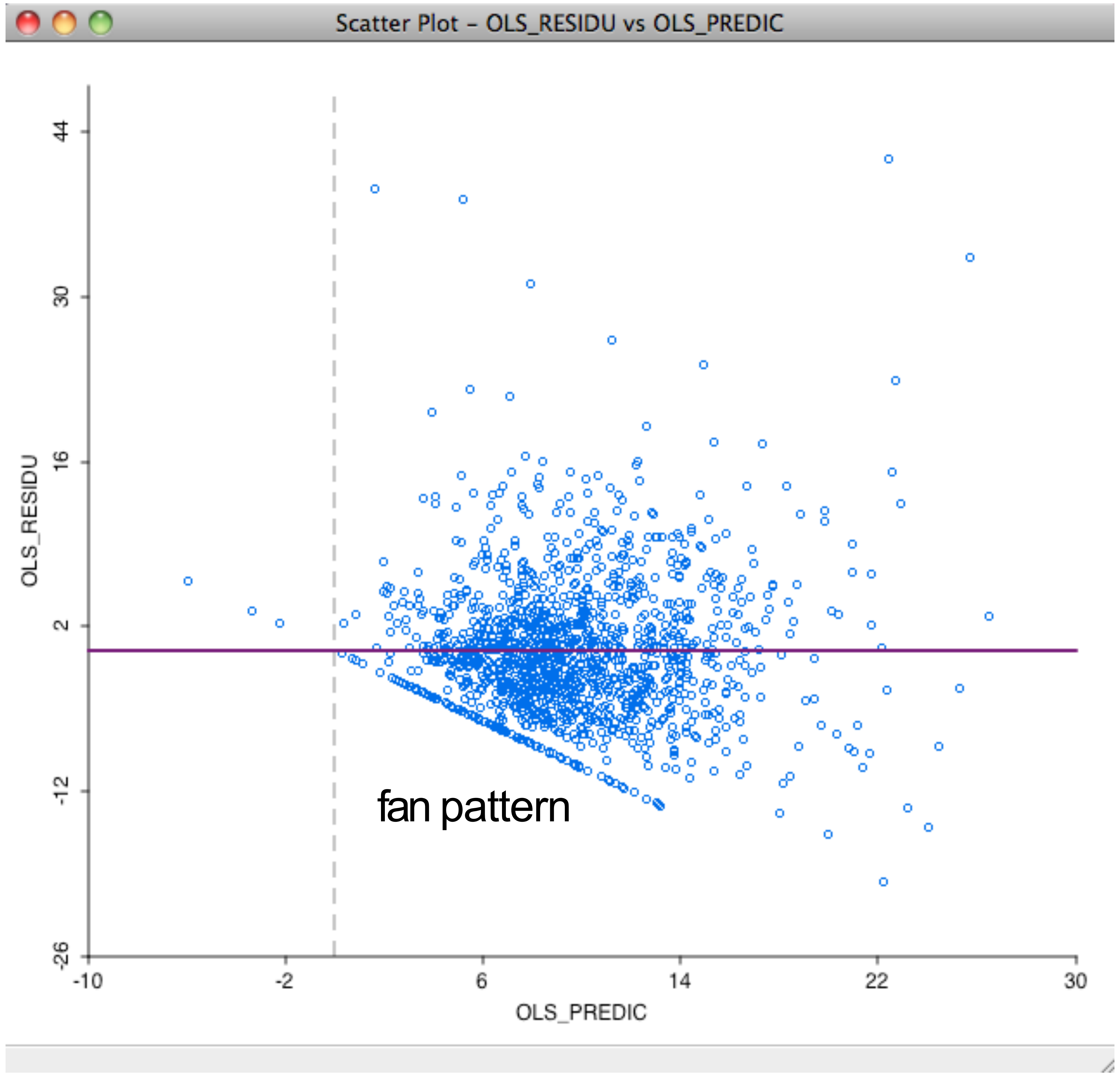
Diagnostic Plots

use scatter plot function

plot residuals (y-axis) vs predicted values (x-axis) as a visual diagnostic for heteroskedasticity

pattern should be more or less within the same range

“fan” or “flares” suggest heteroskedasticity, i.e., non-constant error variance



Residual Plot - Heteroskedasticity

Caution

visual inspection of plots and maps can be misleading

use plots and maps to suggest additional variables for model

no substitute for formal specification diagnostics

Specification Tests

```
REGRESSION DIAGNOSTICS
MULTICOLLINEARITY CONDITION NUMBER   30.863223
TEST ON NORMALITY OF ERRORS
TEST                     DF           VALUE            PROB
Jarque-Bera               2            2833.409        0.0000000



DIAGNOSTICS FOR HETEROSKEDASTICITY
RANDOM COEFFICIENTS
TEST                     DF           VALUE            PROB
Breusch-Pagan test        5            515.0796        0.0000000
Koenker-Bassett test      5            124.2749        0.0000000
SPECIFICATION ROBUST TEST
TEST                     DF           VALUE            PROB
White                    20            242.8053        0.0000000
========================= END OF REPORT ========================
```

Regression Report - Diagnostics

Multicollinearity

condition number

based on eigenvalues of X’X

“classic” rule of thumb = values > 30 suggest a problem

in practice, not taken too literally

in example, slightly over 30

Normality

crucial assumption for exact inference

in practice, less crucial, since asymptotics (large sample theory) yield similar properties

Bera-Jarque (1981) test is based on third moment (skewness - asymmetry) and fourth moment (kurtosis - thick tails) of residuals distributed as $\chi 2$ with two degrees of freedom

in example: 2833, thus highly non-normal

Random Coefficients

a special form of heteroskedasticity

measure of fit of regression of squared residuals on squared explanatory variables

Breusch-Pagan (1979) test and a robustified version (for non-normality) Koenker-Bassett (1982) test

in example: both reject null very strongly

Heteroskedasticity

White (1980) test against heteroskedasticity of unknown form

measure of fit of regression of squared residuals on a polynomial in the explanatory variables

robust to many forms of misspecification

in example: rejects the null soundly

Caveats

in large data sets (as in the south example), lack of normality is not a real problem

heteroskedasticity is hard to distinguish from spatial autocorrelation (and vice versa)

many types of misspecification (missing variables, nonlinearity) may lead to the rejection of a given null hypothesis, even when unrelated to that null hypothesis

Spatial Covariance

Structure of Covariance

$Cov[y_i, y_j] = E[y_i . y_j] - E[y_i]E[y_j] \neq 0$

which pairs y_i, y_j have non-zero covariance

if the non-zero covariances show a spatial pattern, then it is a “spatial” covariance

need a way to embed spatial structure

How to Embed Spatial Structure

need to respect location, distance, network structure

need to respect Tobler’s law (distance decay)

variance-covariance matrix must be positive definite

Three Approaches

direct representation

non-parametric function

spatial process model

Direct Representation

covariance is a specified function of distance

$$\mathrm{Cov}[y_i, y_j] = f(\theta, d_{ij})$$

f = function, e.g., negative exponential
θ = parameter vector
d_{ij} = distance metric

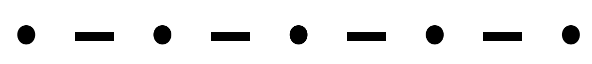
Requirements for Distance Function

distance decay: as d ↑ Cov ↓

covariance symmetric (distance has to be symmetric)

variance-covariance matrix must be positive definite for all parameter values

Illustration

observations on a line 5 points 

0	1	2	3	4
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

Negative Exponential Distance Function

`Cov[yi,yj] = exp(-0.2 x dij)`

compute covariance for each distance pair

$d_{01} = 1 \rightarrow Cov(y_0,y_1) = e^{-0.2} = 0.82$

0	0.82	0.67	0.55	0.45
0.82	0	0.82	0.67	0.55
0.67	0.82	0	0.82	0.67
0.55	0.67	0.82	0	0.82
0.45	0.55	0.67	0.82	0

Properties

smooth decay of covariance with distance

potential issues for d = 0, sometimes requires ad hoc adjustments, e.g., for 1/d

scale sensitive: magnitude of covariance depends on distance metric

for d too large, function may become zero

e.g., distance in meters vs km

covariance matrix not necessarily positive definite

Non-Parametric Function

spatial autocorrelation is an unspecified function of distance

$\rho_{ij} = g(d_{ij})$

g is a general, unspecified functional form how to fit the function?

e.g., use kernel estimator

Nonparametric Kernel Estimator

Hall-Patil (1994)

$\rho(d)=\Sigma_i\Sigma_j K(d_{ij}/h)(z_i.z_j)/\Sigma_i\Sigma_j K(d_{ij}/h)$

$z_i.z_j$ is the covariance between i and j

K is the kernel function (many choices), with h as the bandwidth

results depends critically on h, less so on K

Spline Nonparametric Covariance Function

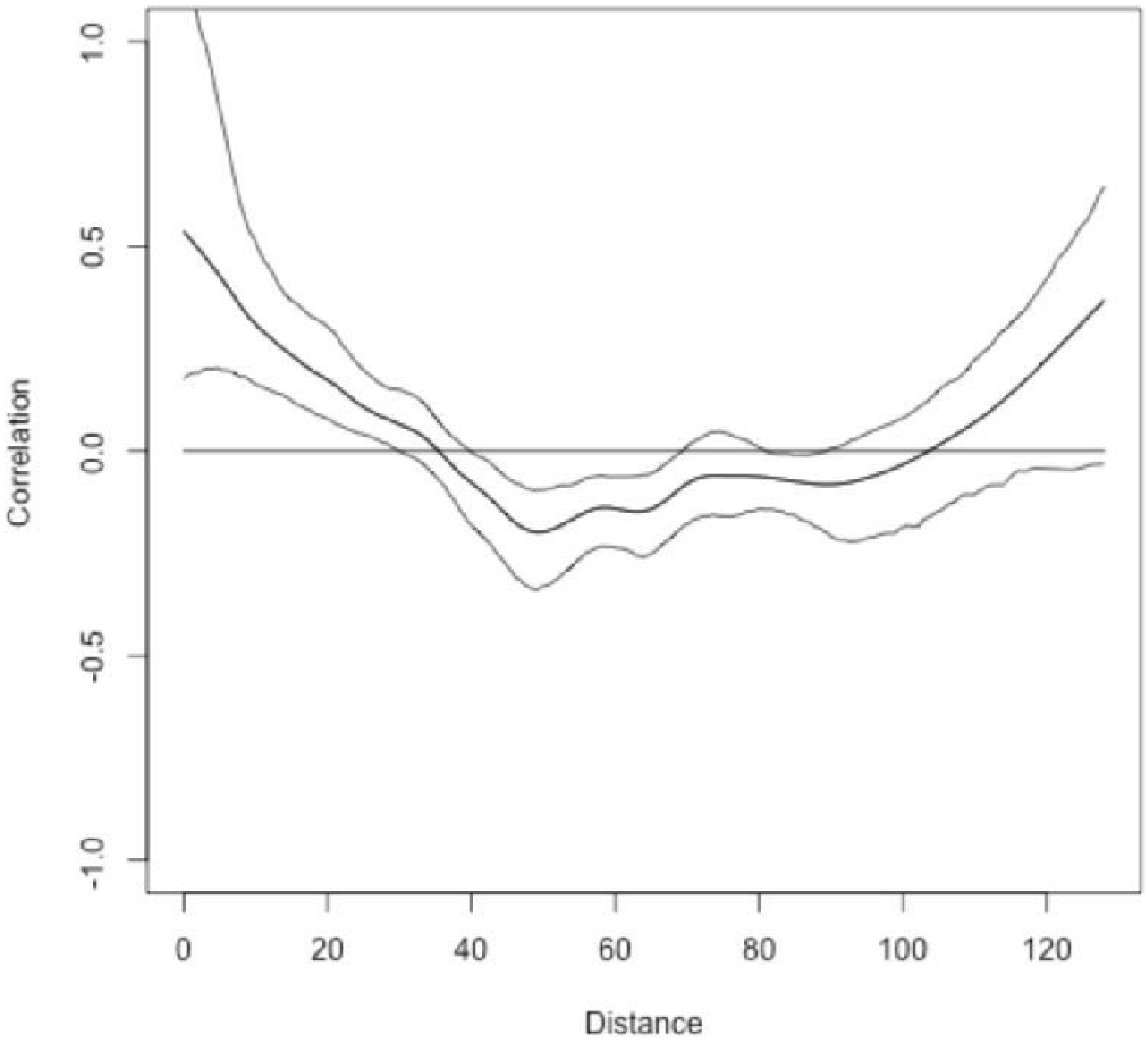
use cubic B-spline kernel

$$K(u) = \frac{1}{2}\exp\left(-\frac{|u|}{\sqrt{2}}\right)\sin\left(-\frac{|u|}{\sqrt{2}} + \frac{\pi}{4}\right),$$

use bootstrap envelope for inference

resample tuple (x, y, z) with replacement

confidence interval around spline kernel



spline spatial covariance function with bootstrap envelope

Interpretation

range of significant spatial autocorrelation

alternative to specifying an explicit distance function

sensitive to kernel fit may

violate Tobler's law

possible non-positive definite covariance matrix

Spatial Process Model

spatial stochastic process or spatial random field

{ Z(s): s ∈ D }

s ∈ Rd: location (e.g., lat-lon)

D ∈ Rd: index set = possible locations Z(s): random variable at location s

covariance structure follows from the specification of the process, i.e., how a random variable at a location depends on random variables at other locations

Simultaneous vs Conditional Processes

simultaneous = model for the complete pattern

values at all locations are explained (as functions of exogenous variables)

conditional = model for prediction

values at unobserved locations are predicted by a model fit on observed locations

alternative use: a spatial prior for parameters in a Bayesian hierarchical model

Prediction

in simultaneous models

$y = f(x)$

no y on right hand side (reduced form)

in conditional models

$y_i = f(y_{j^*})$

with y_{j^*} as values at other locations

Spatial Autoregressive Process

Spatial Autoregressive Process (SAR)

analog to time series setup

example, first order autoregressive process $y_t = \rho y_{t-1} + u_t$

by consecutive substitution, this becomes a moving average process in the error terms

$$y_t = \rho(\rho y_{t-2} + u_{t-1}) + u_t = \rho^2 y_{t-2} + \rho u_{t-1} + u_t.$$

$$y_t = u_t + \rho u_{t-1} + \rho^2 u_{t-2} + \ldots.$$

Covariance Structure

follows from the specification of the process
in the time series case

$$\mathrm{Var}[y_t] = \sigma^2 + \rho^2\sigma^2 + \rho^4\sigma^2 + \cdots = \frac{\sigma^2}{1-\rho^2}$$

$$\mathrm{E}[y_t y_{t-1}] = \mathrm{E}[(\rho y_{t-1} + u_t)y_{t-1}] = \rho\mathrm{Var}[y_{t-1}] + 0 = \frac{\rho\sigma^2}{1-\rho^2}.$$

$$\mathrm{E}[y_t y_{t-s}] = \frac{\rho^s\sigma^2}{1-\rho^2}.$$

Fundamental Difference = Feedback

substitution approach breaks down

y_i is present on the right hand side

$$\begin{aligned} y_i &= \rho[\rho(y_{i-2} + y_i) + u_{i-1}] + \rho[\rho(y_i + y_{i+2}) + u_{i+1}] + u_i \\ &= \rho^2(2y_i + y_{i-2} + y_{i+2}) + \rho u_{i-1} + \rho u_{i+1} + u_i \end{aligned}$$

alternative: write full system of simultaneous equations using a spatial weights matrix

$$\mathbf{y} = \rho \mathbf{W} \mathbf{y} + \mathbf{u},$$

SAR Process Covariance Matrix

follows from the reduced form

$$\mathbf{y} \quad = \quad (\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{u}.$$

$$\begin{aligned}
\mathrm{Var}[\mathbf{y}] &= \mathrm{E}[\mathbf{y}\mathbf{y}'] = \mathrm{E}[(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{u}\mathbf{u}'(\mathbf{I} - \rho\mathbf{W}')^{-1}] \\
&= (\mathbf{I} - \rho\mathbf{W})^{-1}\mathrm{E}[\mathbf{u}\mathbf{u}'](\mathbf{I} - \rho\mathbf{W}')^{-1} \\
&= (\mathbf{I} - \rho\mathbf{W})^{-1}[\sigma^2\mathbf{I}](\mathbf{I} - \rho\mathbf{W}')^{-1} \\
&= \sigma^2(\mathbf{I} - \rho\mathbf{W})^{-1}(\mathbf{I} - \rho\mathbf{W}')^{-1},
\end{aligned}$$

$$\mathrm{Var}[\mathbf{y}] = \sigma^2[(\mathbf{I} - \rho\mathbf{W})'(\mathbf{I} - \rho\mathbf{W})]^{-1}$$

Example: Unconnected Line

ρ = 0.2, unit variance

$$\mathbf{W_L} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\mathrm{Cov}_L = \begin{bmatrix} 1.1386 & 0.4753 & 0.1488 & 0.0413 & 0.0101 \\ 0.4753 & 1.2874 & 0.5165 & 0.1589 & 0.0413 \\ 0.1488 & 0.5165 & 1.2975 & 0.5165 & 0.1488 \\ 0.0413 & 0.1589 & 0.5165 & 1.2874 & 0.4753 \\ 0.0101 & 0.0413 & 0.1488 & 0.4753 & 1.1386 \end{bmatrix}.$$

Example: connected line (circle, torus)

ρ = 0.2, unit variance

$$\mathbf{W_C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\mathrm{Cov}_C = \begin{bmatrix} 1.3047 & 0.5318 & 0.2048 & 0.2048 & 0.5318 \\ 0.5318 & 1.3047 & 0.5318 & 0.2048 & 0.2048 \\ 0.2048 & 0.5318 & 1.3047 & 0.5318 & 0.2048 \\ 0.2048 & 0.2048 & 0.5318 & 1.3047 & 0.5318 \\ 0.5318 & 0.2048 & 0.2048 & 0.5318 & 1.3047 \end{bmatrix}.$$

Properties of SAR Covariance

Depends on weights matrix (e.g., connected vs unconnected) but DIFFERENT from the weights matrix structure

Every location covaries with every other location

Covariance decreases with order of contiguity, BUT same order of contiguity does not yield same covariance

Heteroskedastic when non-constant number of neighbors (not in torus case)

Strange covariances in torus structure

Spatially Lagged Variables

Specifying Spatial Dependence

In time series analysis, the concept of a shift y_{t-k} is observation shifted by k periods

For regular lattices, shift north, south, east, west: $y_{i-1,j}$, $y_{i+1,j}$, $y_{i,j-1}$, $y_{i,j+1}$ spatial shift of $y_{i,j}$

No analog for irregular spatial layouts

Instead, the notion of a spatially lagged variable (Anselin 1988)

Spatial Lag

Weighted average of neighboring values

Neighbors defined by spatial weights (w_{ij})

$y_{iL} = w_{i1}.y_1 + w_{i2}.y_2 + ... + w_{iN}.y = \Sigma_j \, w_{ij}.y_j$

In practice: very few neighbors (weights are sparse)

Spatial Lag in Matrix Notation

Spatial weights matrix times the vector of observations

$$y_L = Wy$$

Wy as such is often used as symbol for a spatially lagged dependent variable

Spatial Lag vs Window Average

Similar to a window average, the spatial lag is a smoother

Lag **Wy** has smaller variance than original variable y

spatial lag is NOT a window average since $\mathbf{w_{ii} = 0}$
observation at “center” of window is not included

Spatially Lagged Variables in a Regression

spatially lagged dependent variable
Wy spatial (autoregressive) lag model

spatially lagged explanatory variables
WX spatial cross-regressive model or SLX model

spatially lagged error terms
We spatial (autoregressive) error model

Spatial Lag Model

Motivation

Explicit model for spatial interaction = substantive spatial dependence

Peer-effects, etc.

Equilibrium outcome of spatial interaction process, a spatial reaction function (Brueckner 2003)

Non-behavioral motivation = data issue (scale)

Identification Issues

Inverse problem

Different processes can yield the same pattern

Reflection problem (Manski 1993)

Parameter identification in spatial/social interaction models

New economic geography critique (Gibbons and Overman 2012)

Difficulty with interpretation of causal effects

Types of Social Interaction

Interaction effects among individual agents = **endogenous effects**

Exogenous group characteristics = **contextual effects**

Observed or unobserved characteristics that agents have in common = **correlated effects**

Mixed Regressive-Spatial Autoregressive

$$y = \rho Wy + X\beta + u$$

Wy = spatial autoregressive (spatial lag)

X = regressive

ρ = spatial autoregressive coefficient

Spatial Filter

Remove effect of spatial autocorrelation

$$y - \rho W y = X\beta + u$$

$$(I - \rho W) y = X\beta + u$$

$(I - \rho W)$ is **spatial filter**

Effect of Spatial Filter

Similar to detrending

Deals with scale problems, i.e., non-behavioral motivation for including spatial lag term

Spatial filter still requires estimate of ρ

Spatial Multiplier

Derived from reduced form

What is the change in y as a result of the change in X

$$E[y|\Delta X] = (I-\rho W)^{-1}(\Delta X)\beta = [I+\rho W+\rho^2 W^2 + ...](\Delta X)\beta$$

effect is more than (ΔX) β ‹ multiplier

Direct and Indirect Effects

Total effect of a change in the exogenous variable

$(I-\rho W)^{-1} (\Delta X)\beta$

direct effect

$(\Delta X)\beta$

indirect effect

$[(I - \rho W)^{-1} - I] (\Delta X)\beta$

$[\rho W + \rho^2 W^2 + ...] (\Delta X)\beta$

Applications of Spatial Multiplier

Policy analysis

Effect of a change in a policy variable x at i extends beyond i to its neighbors, neighbors of neighbors, etc.

Simulate the spatial imprint of a policy change by solving the reduced form for a change in X

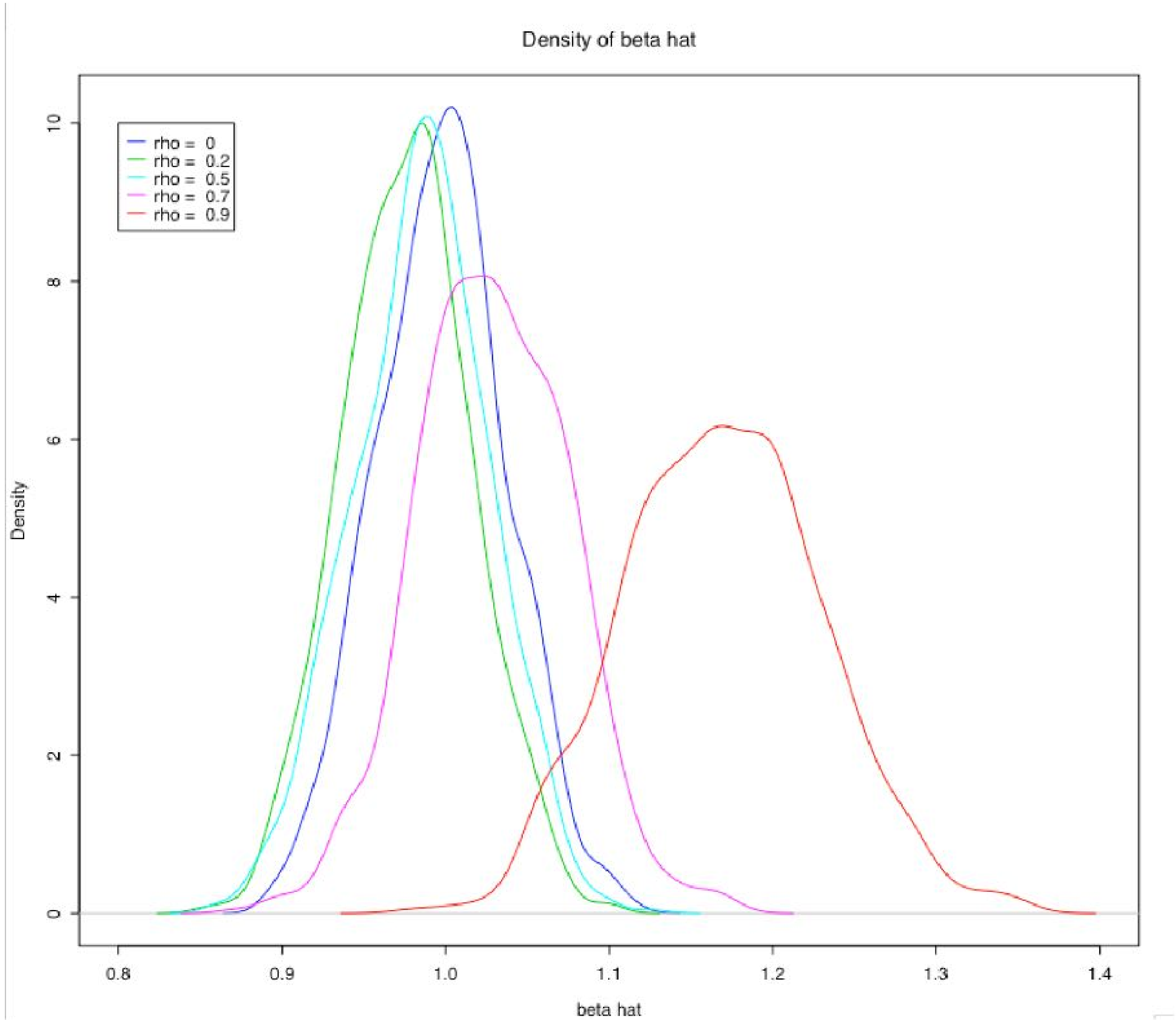
Effects of Ignoring a Spatial Lag

= ignoring substantive spatial interaction

omitted variable problem

OLS biased and inconsistent

Potentially: wrong estimate, wrong sign, wrong standard error, wrong significance, wrong fit



effect of ignoring spatial lag on OLS estimate

OLS vs. SAR

Consider the following linear regression of Obama's margin of victory **(y)** on county-level socio-economic attributes **(X)**:

$\mathbf{y} = \mathbf{X}\beta + s.$

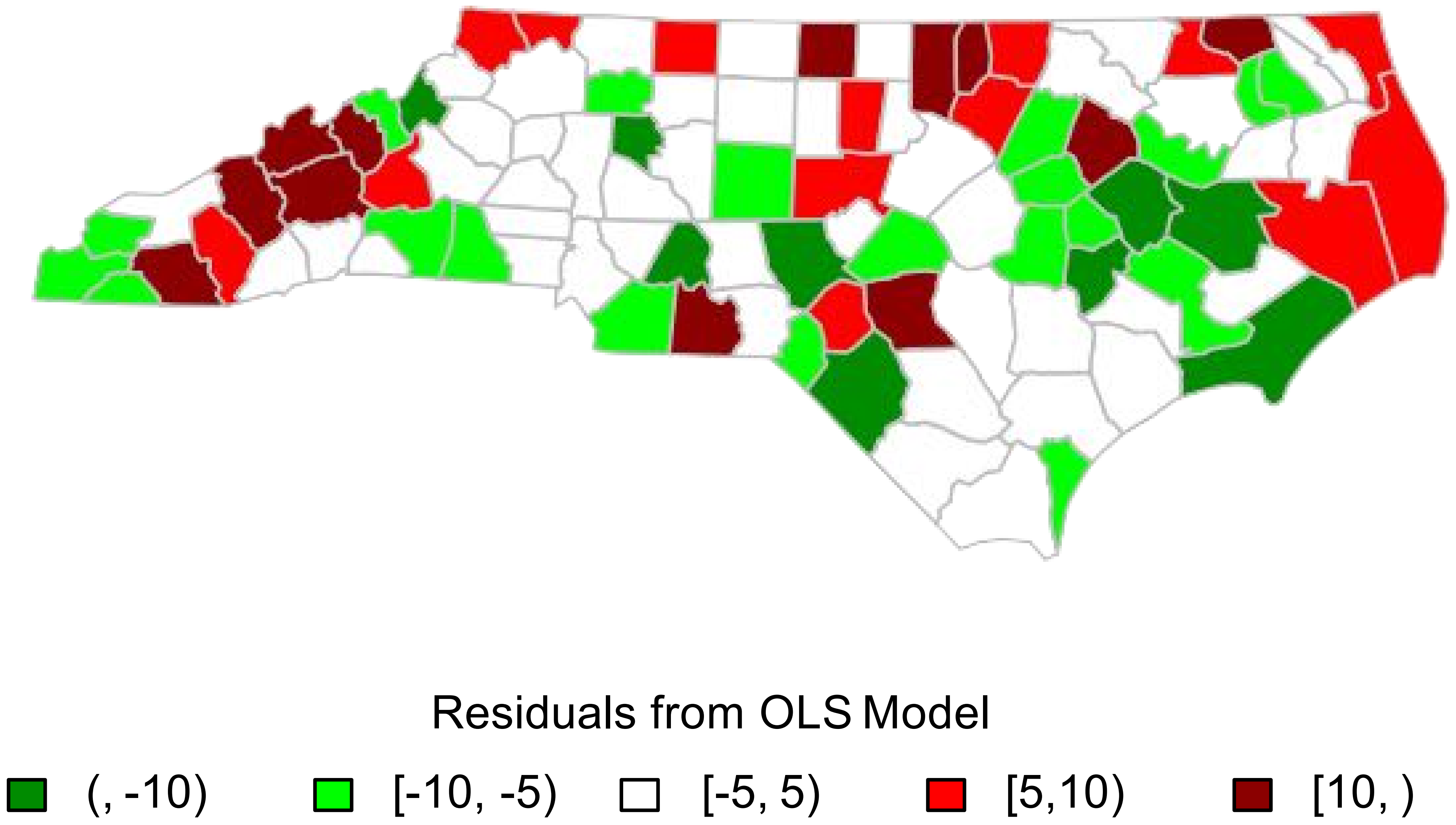
	OLS
(Intercept)	-35.58
	(6.23)***
Percent non-white	1.09
	(0.06)***
Percent college-educated	1.65
	(0.15)***
Veterans	-2.6e-4
	(1.2e-4)*
Median income	-7e-4
	(1.6e-4)***
AIC	729.2
N	100
Moran's *I* Residuals	0.25***

*p ≤ .05, **p ≤ .01, ***p ≤ .001

The Moran's I statistic shows a significant amount of spatial autocorrelation in the residuals.

OLS Residuals

Below is a map of residuals from a linear regression of Obama’s margin of victory on county-level socio-economic attributes.



OLS vs. SAR

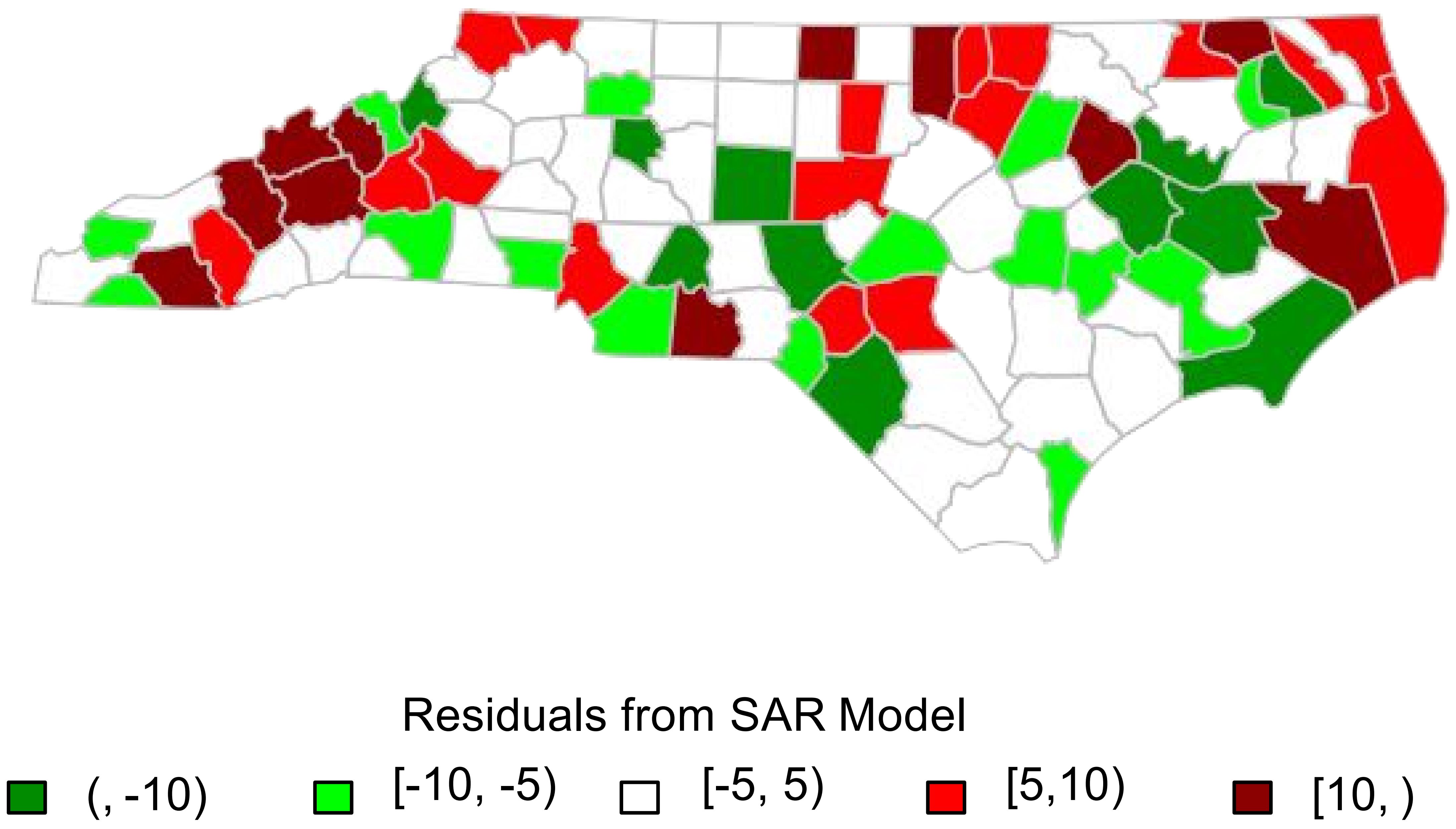
And the same model estimated by SAR: $y = \rho W y + X\beta + u$.

	OLS	SAR
(Intercept)	-35.58	-28.40
	(6.23)***	(7.05)***
Percent non-white	1.09	0.98
	(0.06)***	(0.08)***
Percent college-educated	1.65	1.62
	(0.15)***	(0.14)***
Veterans	-2.6e-4	-1.8e-4
	(1.2e-4)*	(1e-4)
Median income	-7e-4	-7.8e-4
	(1.6e-4)***	(1.6e-4)***
Lagged Obama margin (ρ)		0.16
		(0.08)*
AIC	729.2	727.09
N	100	100
Moran's I Residuals	0.25***	0.15**
*p ≤ .05, **p ≤ .01, ***p ≤ .001		

The ρ coefficient is positive and significant, indicating spatial autocorrelation in the dependent variable. But Moran's I indicates that residuals remain clustered.

SAR Residuals

Below is a map of <u>residuals</u> from the SAR model.

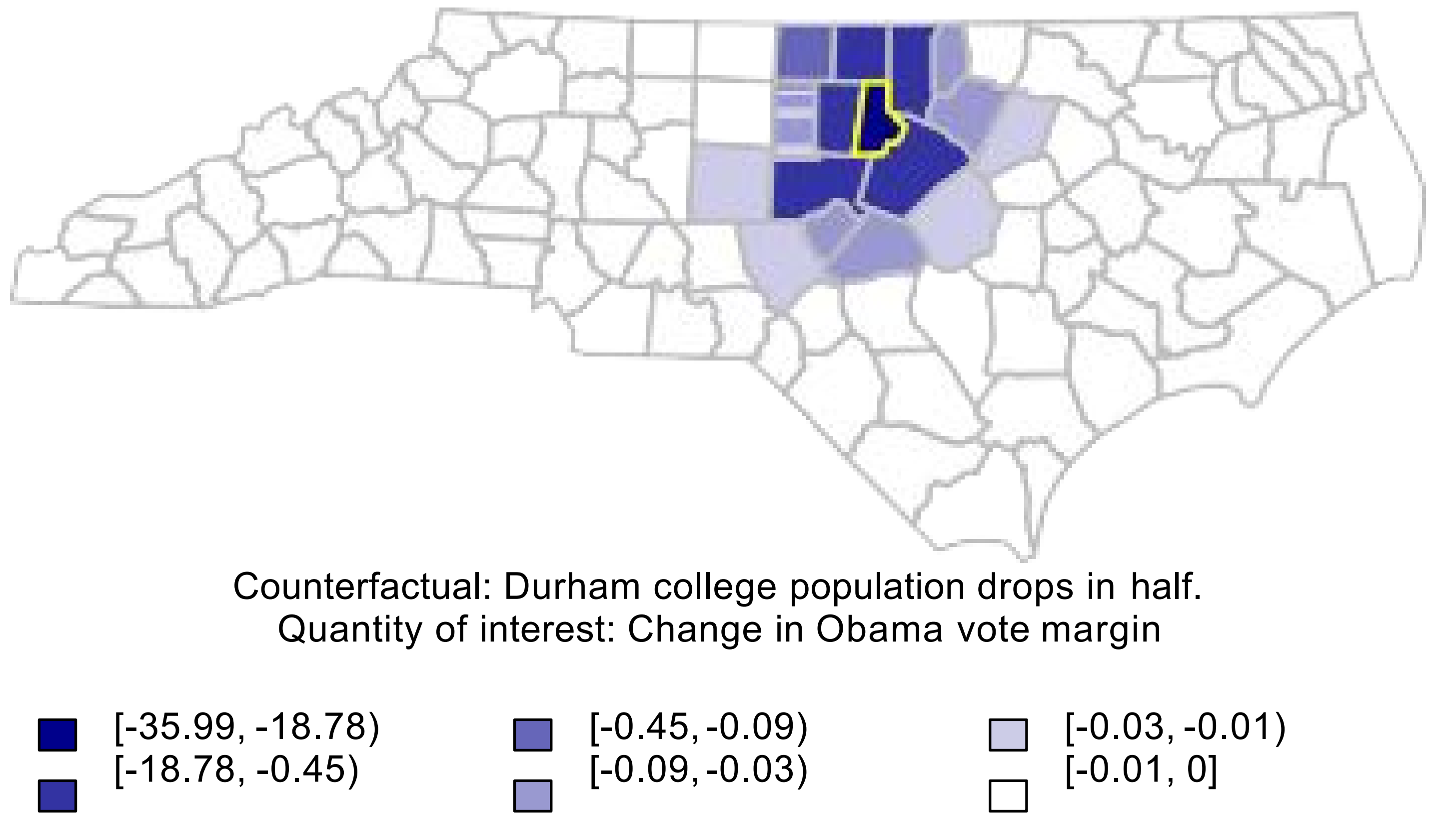


SAR Equilibrium Effects

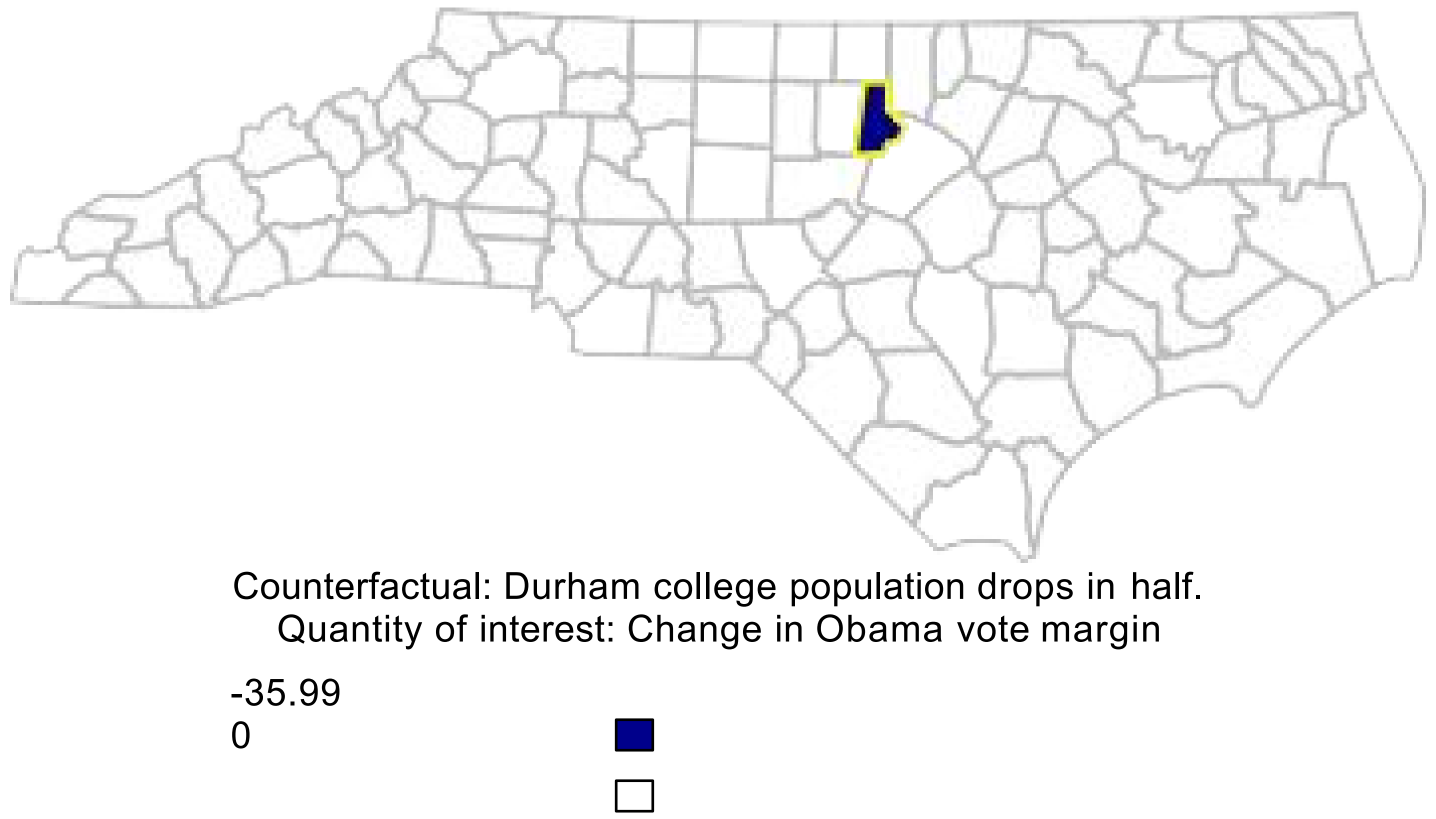
<u>Counterfactual</u>: A 50% decline in Durham's college-educated population.

Below are the equilibrium effects (change in Obama's county vote margin) associated with this counterfactual.

SAR



OLS



Spatial Error Model

Motivation

spatial pattern in error term due to omitted random factors = nuisance spatial dependence

mismatch spatial scale process with spatial scale observations (administrative units as “markets”)

no substantive interpretation

problem of efficiency of the estimates

Non-Spherical Error Variance

Due to spatial autocorrelation, error covariances are non-zero

Off-diagonal elements are non-zero

$E[uu'] = \Sigma \neq \sigma^2 I$

Spatial structure in the covariance $E[u_i u_j] \neq 0$, for $i \neq j$

Spatial Autoregressive Error Model

SAR error

$y = X\beta + u$ with $u = \lambda W u + e$

covariance matrix $\Sigma = \sigma^2 [(I - \lambda W)'(I - \lambda W)]^{-1}$

but inverse covariance matrix does not contain inverse terms:

$\Sigma^{-1} = (1/\sigma^2) [(I - \lambda W)'(I - \lambda W)]$

Reduced Form

$$y = X\beta + (I - \lambda W)^{-1}e$$

No substantive spatial multiplier effect

Effect of spatial autocorrelation is on error variance, used in kriging (spatial prediction)

SAR Errors and Heteroskedasticity

Variance $\Sigma = \sigma^2 [(I - \lambda W)'(I - \lambda W)]^{-1}$ has non-constant diagonal terms - depends on number of neighbors

This induces heteroskedasticity in u, even with homoskedastic errors e

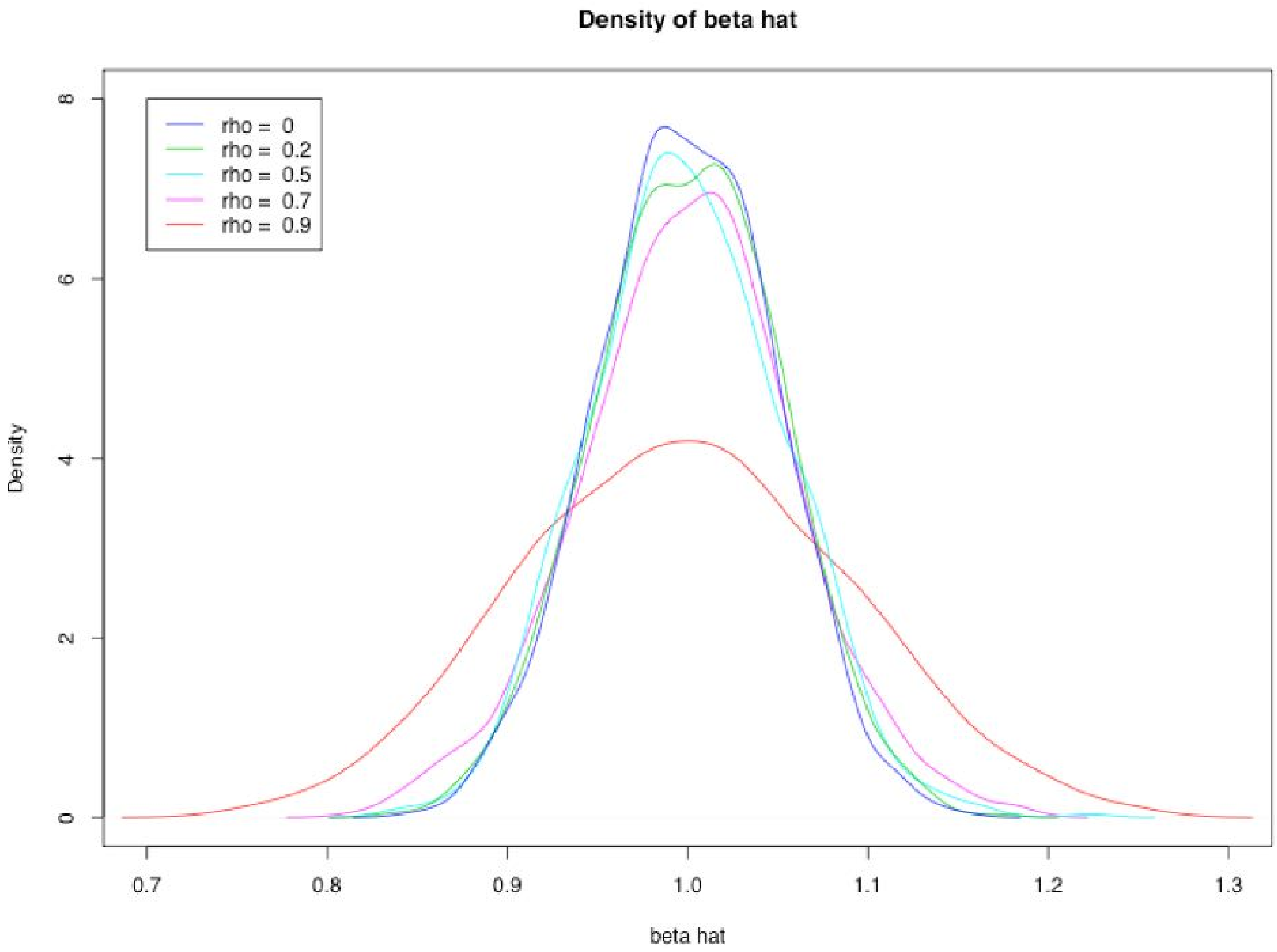
Difficult to disentangle true heteroskedasticity from induced heteroskedasticity

Effects of Ignoring SAR Errors

Problem of efficiency

OLS remains unbiased but inefficient

Potentially: correct estimate, wrong standard error, wrong significance, wrong fit



effect of ignoring SAR errors on OLS estimate

Spatially Lagged Error Model

Use of the spatial error model may be motivated by **omitted variable bias**.

Suppose that y is explained entirely by two explanatory variables x and z , where x, z ~ $N(0, \mathbf{I}_n)$ and are independent.

$$y = x\beta + z\theta$$

If z is not observed, the vector $z\theta$ is nested into the error term u.

$$y = x\beta + u$$

Examples of latent variable z : culture, social capital, neighborhood prestige.

SEM Estimates

Let's run the model: $\mathbf{y} = \mathbf{X}\beta + \lambda\mathbf{W}\mathbf{u} + e$.

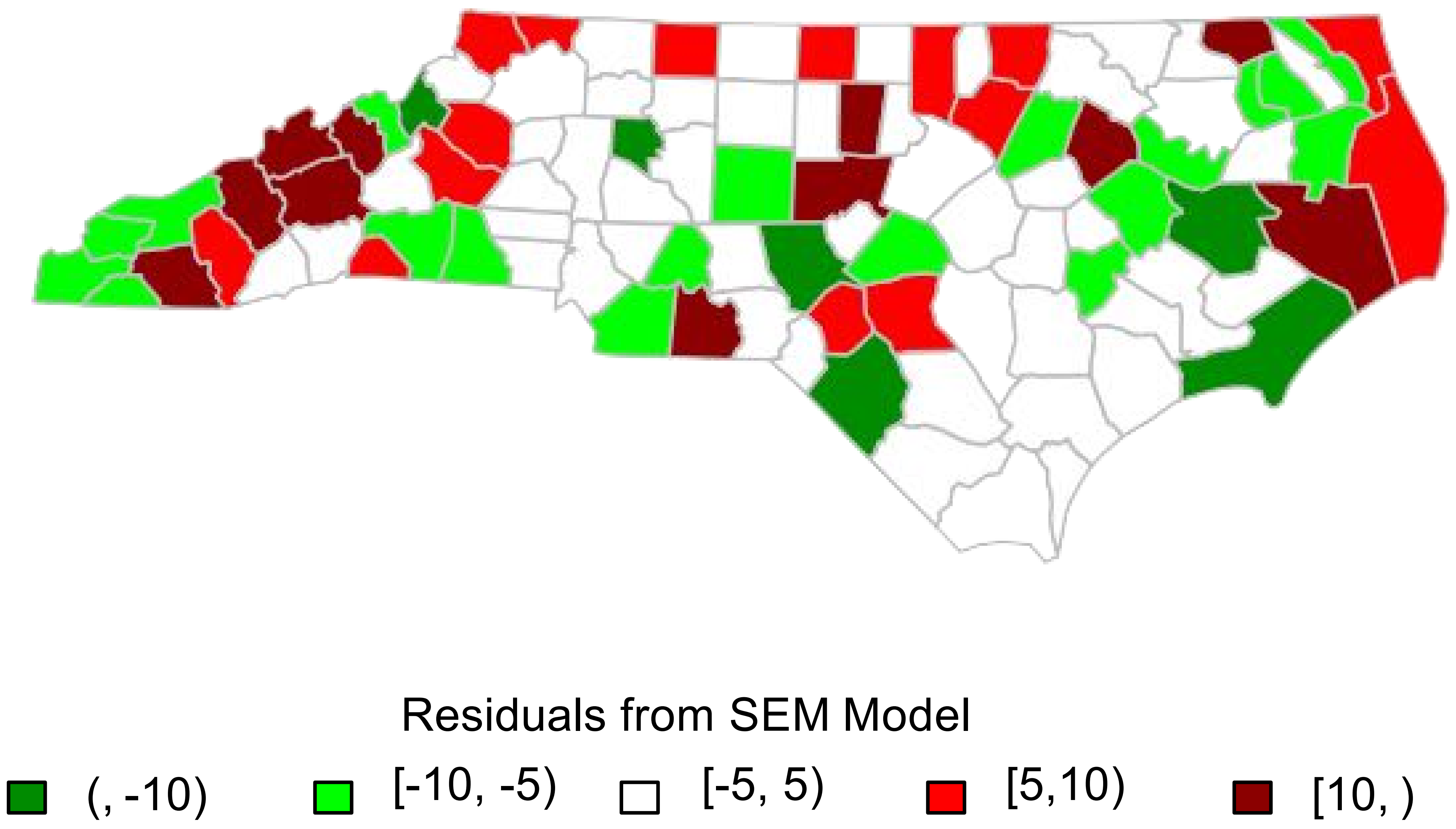
	OLS	SAR	SEM
(Intercept)	-35.58	-28.40	-38.67
	(6.23)***	(7.05)***	(7.34)***
Percent non-white	1.09	0.98	1.16
	(0.06)***	(0.08)***	(0.07)***
Percent college-educated	1.65	1.62	1.44
	(0.15)***	(0.14)***	(0.13)***
Veterans	-2.6e-4	-1.8e-4	-1.5e-4
	(1.2e-4)*	(1e-4)	(1e-4)
Median income	-7e-4	-7.8e-4	-5.9e-4
	(1.6e-4)***	(1.6e-4)***	(1.6e-4)***
Lagged Obama margin (ρ)		0.16	
		(0.08)*	
Lagged error (λ)			0.53
			(0.11)***
AIC	729.2	727.09	715.74
N	100	100	100
Moran's I Residuals	0.25***	0.15**	-0.003

$^*p \leq .05$, $^{**}p \leq .01$, $^{***}p \leq .001$

The λ coefficient indicates strong spatial dependence in the errors.

SEM Residuals

Below is a map of <u>residuals</u> from the SEM model.



Spatial Durbin Model

Spatial Durbin Model

Like the SEM, the Spatial Durbin Model can be motivated by concern over **omitted variables**.

Recall the DGP for the SEM:

$$y = X\beta + (I - \lambda W)^{-1}u$$

Now suppose that **X** and **u** are correlated

One way to account for this correlation would be to conceive of **u** as a linear combination of **X** and an error term **v** that is independent of **X**.

$$u = X\gamma + v$$

$$v \sim N(0, \sigma^2 I)$$

where the scalar parameter γ and σ^2 govern the strength of the relationship between **X** and $\mathbf{z = (I - \lambda W)^{-1}}$

Spatial Durbin Model

Substituting this expression for **u**, we have the following DGP:

$$y = X\beta + (I_n - \lambda W)^{-1}(\gamma X + v)$$

$$y = X\beta + (I_n - \lambda W)^{-1}\gamma X + (I_n - \lambda W)^{-1}v$$

$$(I_n - \lambda W)y = (I_n - \lambda W)X\beta + \gamma X + v$$

$$y = \lambda Wy + X(\beta + \gamma) + WX(-\lambda\beta) + v$$

This is the **Spatial Durbin Model (SDM)**, which includes a spatial lag of the dependent variable **y**, as well as the explanatory variables **X**

Spatial Durbin Model

The Spatial Durbin Model can also be motivated by concern over **spatial heterogeneity**.

Recall the vector of intercepts **a**:

$$a = (I_n - \lambda W)^{-1} s$$

Now suppose that **X** and **s** are correlated.

As before, let's restate s as a linear combination of **X** and random noise **v**.

$$a = X\gamma + v$$

Substituting this back into the SEM yields the same expression of SDM as before:

$$y = \lambda W y + X(\beta + \gamma) + WX(-\lambda\beta) + v$$

SDM Estimates

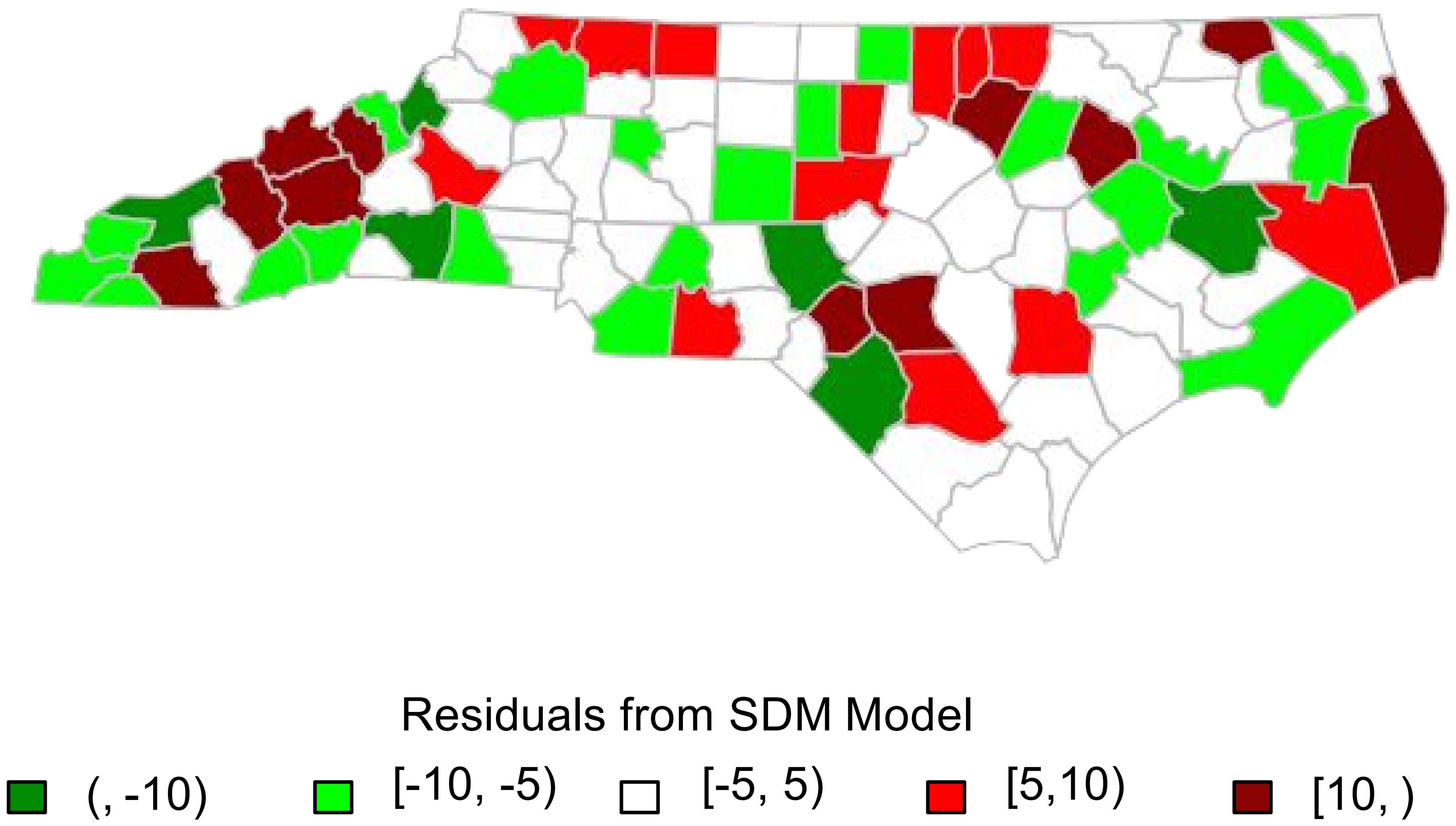
	OLS	SAR	SEM	SDM
(Intercept)	-35.58 (6.23)***	-28.40 (7.05)***	-38.67 (7.34)***	-26.22 (9.66)***
Percent non-white	1.09 (0.06)***	0.98 (0.08)***	1.16 (0.07)***	1.23 (0.92)***
.	.	.	.	.
Lagged Obama margin (ρ)		0.16 (0.08)*		0.42 (0.12)**
Lagged error (λ)			0.53 (0.11)***	
Lagged non-white ($\theta_{\text{non-white}}$)				-0.59 (0.17)***
.				.
AIC	729.2	727.09	715.74	714.22
N	100	100	100	100
Moran's *I* Residuals	0.25***	0.15**	-0.003	0.003

*p ≤ .05, **p ≤ .01, ***p ≤ .001

The SDM results in a slightly better fit

SDM Residuals

Below is a map of residuals from the SDM model.



SLX Model

Motivation for SLX Model

No spatially lagged dependent variable
deal with endogeneity in **WX** if needed

A model for local spillovers

No effect from **X** beyond first order neighbors

No need for spatial econometric estimators

Extensions: Spatial Durbin Error Model (SDEM)

The SDEM model contains spatial dependence in both the explanatory variables and the errors.

$$y = \iota_n\alpha + X\beta + WX\gamma + (I - \rho W)^{-1}s$$

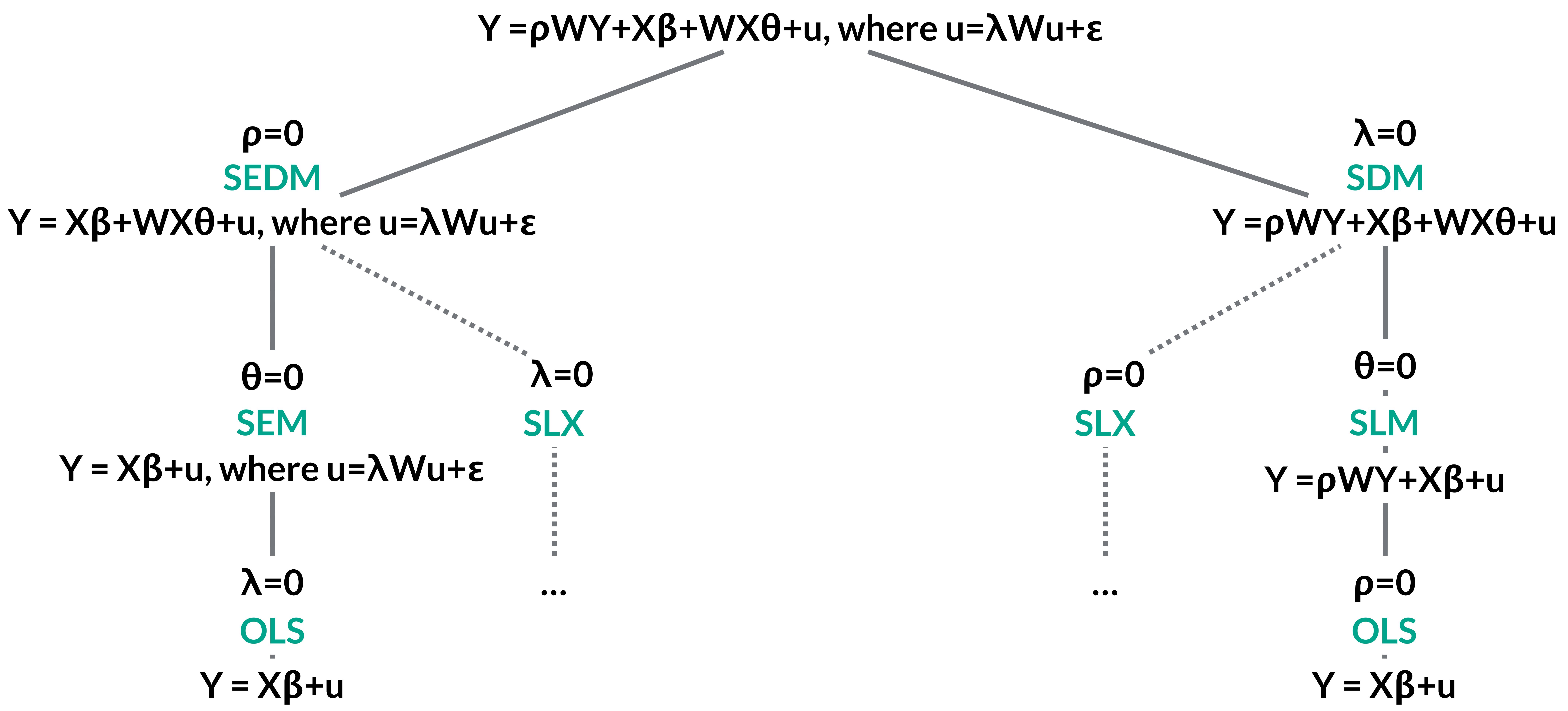
$$s \sim N(0, \sigma^2 I_n)$$

Direct impacts correspond to the β parameters; indirect impacts correspond to the γ parameters

The model can be generalized to incorporate two weights matrices without affecting interpretation of parameters:

$$y = \iota_n\alpha + X\beta + W_1X\gamma + (I_n - \rho W_2)^{-1}s$$

Nested Models



Specification tests

- Reject the null hypothesis against a fully specified alternative model, such as a spatial lag model or a spatial SAR error model
- Based on maximum likelihood (ML) principles
- Examples
  - Wald test (= asymptotic t-test)
  - Likelihood Ratio test (LR)
  - Lagrange Multiplier test (LM)
    - test on significance of slope (gradient) of likelihood function (score)
    - only requires estimation of null model
    - based on OLS residuals

GWR

Geographically Weighted Regression (GWR)

A key assumption that we have made in the models examined thus far is that the structure of the model remains constant over the study area (no local variations in the parameter estimates).

If we are interested in accounting for potential **spatial heterogeneity** in parameter estimates, we can use a Geographically Weighted Regression (GWR) model (Fotheringham et al., 2002).

GWR permits the parameter estimates to vary locally, similar to a parameter drift for a time series model.

GWR has been used primarily for exploratory data analysis, rather than hypothesis testing.

Geographically Weighted Regression (GWR)

GWR rewrites the linear model in a slightly different form:

$$y_i = X\beta_i + s$$

where **i** is the location at which the local parameters are to be estimated.
Parameter estimates are solved using a weighting scheme:

$$\beta_i = (X^j W_i X)^{-1} X^j W_i y$$

where the weight **w_{ij}** for the **j** observation is calculated with a Gaussian function.

$$w_{ij} = e^{\left(\frac{-d_{ij}}{h}\right)^2}$$

where **$d_{i,j}$** is the Euclidean distance between the location of observation **i** and location **j** , and **h** is the bandwidth. Bandwidth may be user-defined or selected by minimization of root mean square prediction error.

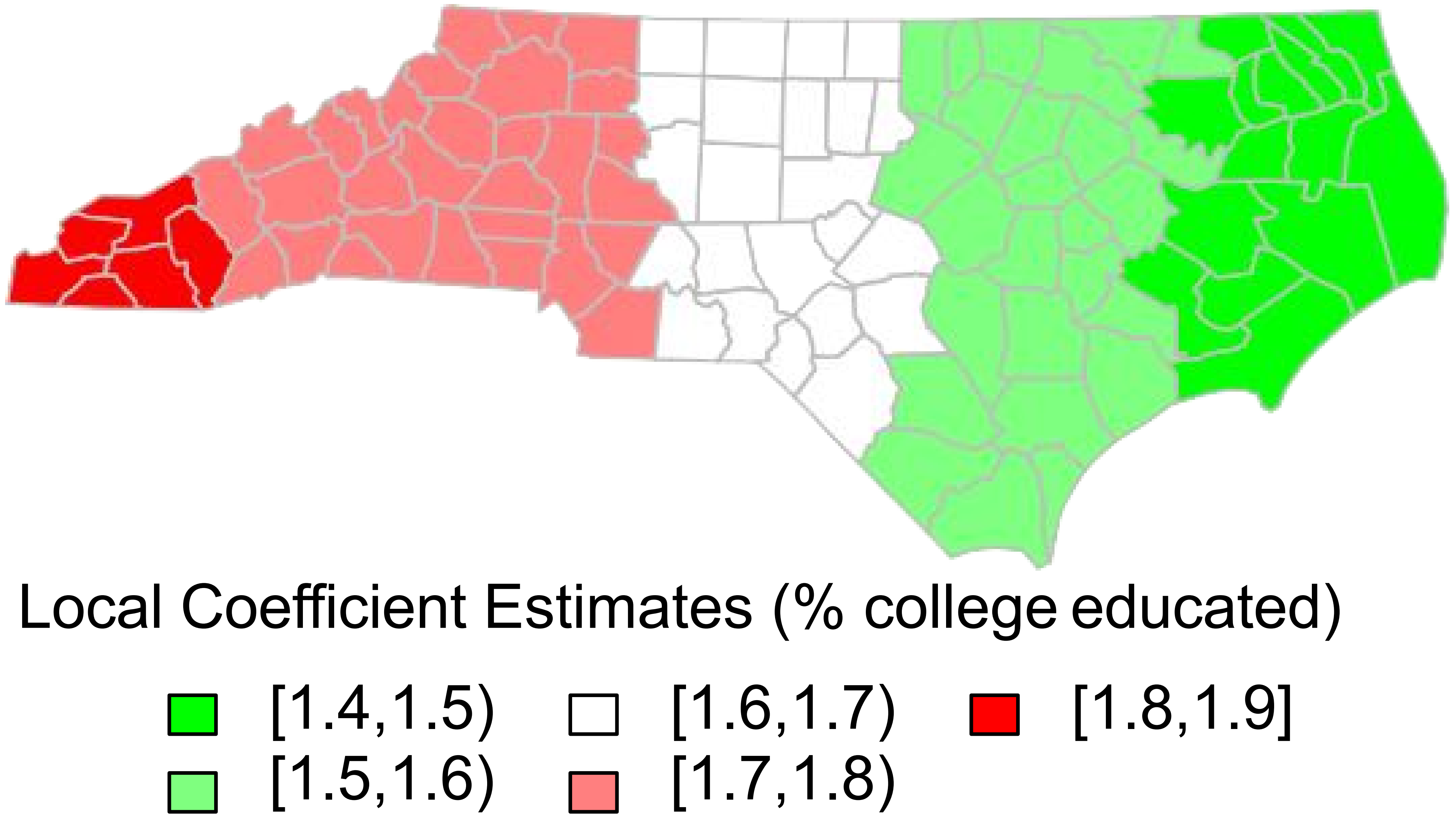
GWR Estimates

Let's try running the same election model as before with GWR:

	Geographically Weighted Regression				
	Global	Min	Median	Max	S.E.
(Intercept)	-35.58	-55.42	-37.65	-24.81	(8.64)
Percent non-white	1.09	0.99	1.12	1.25	(0.06)
Percent college-educated	1.65	1.44	1.63	1.83	(0.11)
Veterans	-3e-4	-3e-4	2.6e-4	-8e-5	(6e-5)
Median income	-7e-4	-1e-3	-9e-4	-3e-4	(2e-4)
Bandwidth	245131.2				
N	100				
Moran's *I* Residuals	0.218				
Moran's *I* Std. Deviate	3.645***				
	'$p \leq .1$, *$p \leq .05$, **$p \leq .01$, ***$p \leq .001$				

GWR Local Coefficient Estimates

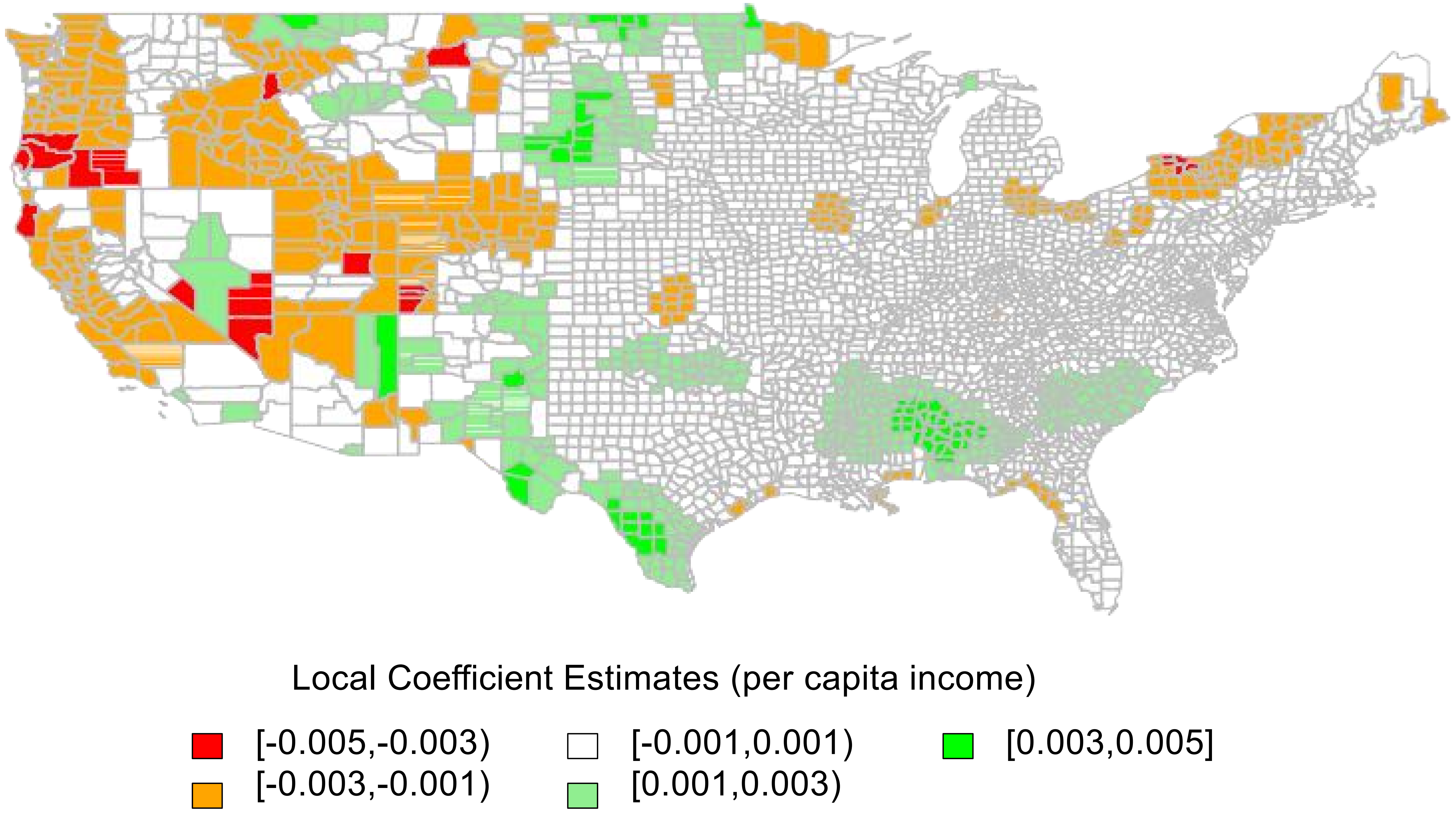
Below is a map of <u>local coefficients</u>. The relationship between college education and Obama's victory margin is largest in **red** areas, and smallest in **green** areas.



GWR Local Coefficient Estimates

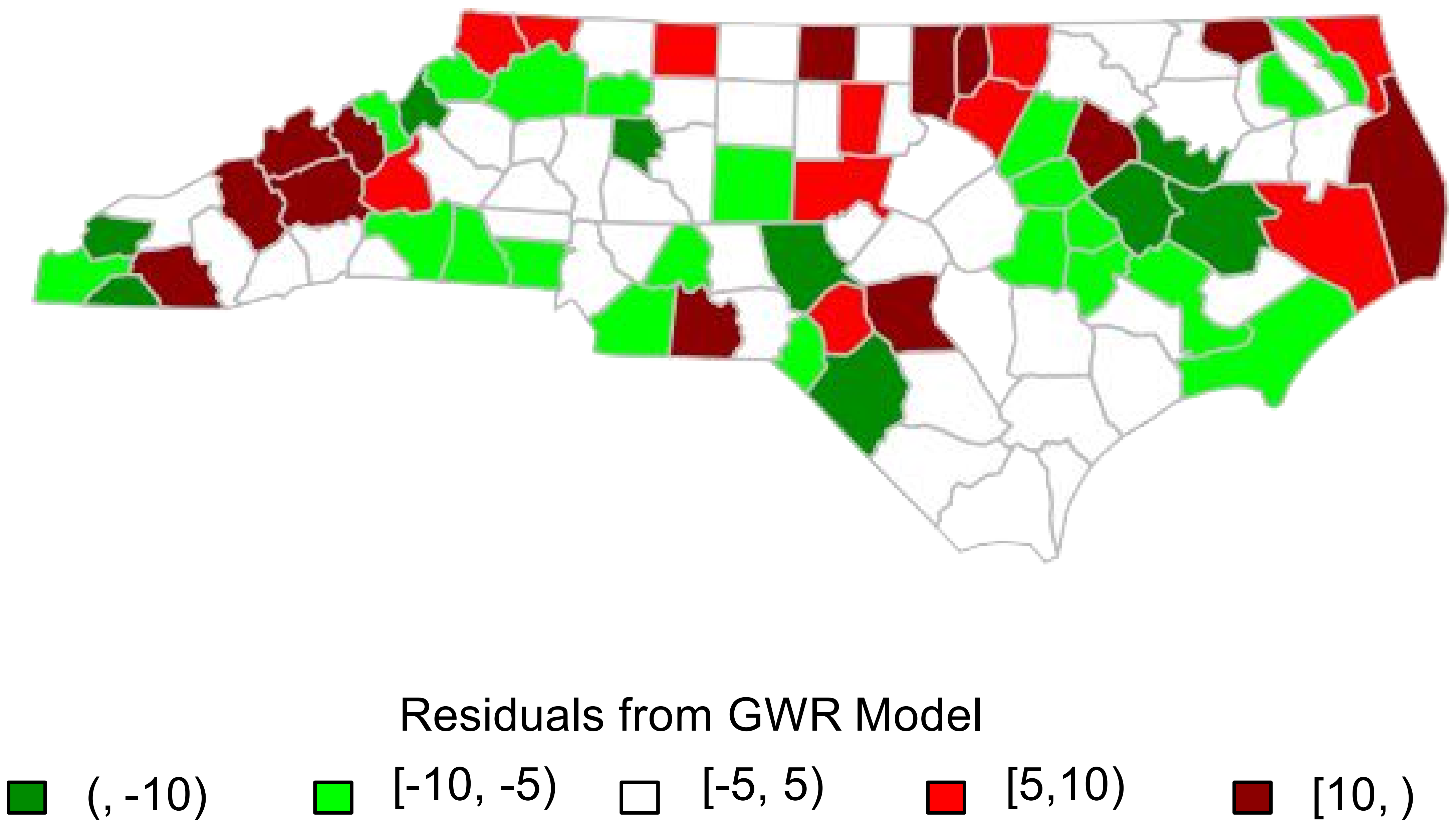
A more interesting example…

The relationship b/w per capita income and Bush's victory margin is negative in **red** areas, and positive in **green** areas.



GWR Residuals

Below is a map of <u>residuals</u> from the GWR model.



Resources

- Spatial Regression (Spring 2017) class
  - Luc Anselin (these slides are heavily based on his material)
  - https://spatial.uchicago.edu/directory/luc-anselin-phd
  - https://www.youtube.com/playlist?list=PLzREt6r1Nenkk7x197-CKPFZ0BuAOCRGT

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