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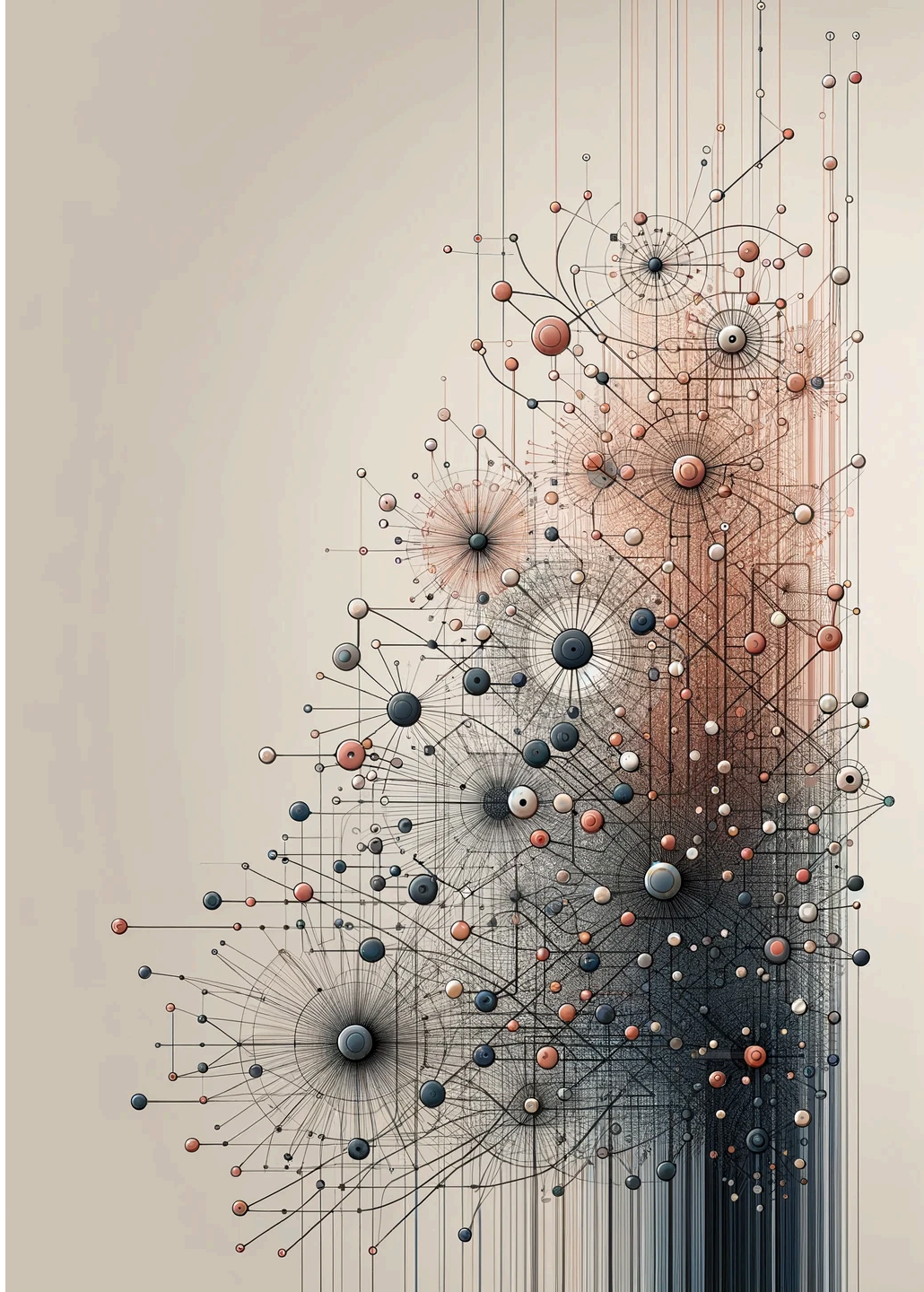
Analisi e Visualizzazione delle Reti Complesse

NS08 - Network Models

Prof. Rossano Schifanella



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Outline

- Random networks
- Small-world networks
- The configuration model
- Preferential attachment
- Other preferential models

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Recap on structural characteristics of real networks

Features of real networks

- **small-world property**
  - Most real-world networks have **short paths**
- **high clustering coefficient**
  - The clustering coefficient of a node is the fraction of pairs of the node's neighbors that are connected to each other:

$$C(i) = \frac{\tau(i)}{k_i(k_i-1)/2} = \frac{2\tau(i)}{k_i(k_i-1)}$$

  - where $\tau(i)$ is the number of triangles involving i. Note that in this definition, the clustering coefficient is undefined if $k_i < 2$, i.e., a node must have at least degree 2 to have any triangles.
  - NetworkX assumes $C = 0$ if $k = 0$ or $k = 1$
  - Many networks have high clustering coefficients
  - Other networks, e.g., bipartite and tree-like networks, have low clustering coefficient

Features of real networks

- **scale-free**
  - distribution of node degrees follows a power-law distribution
  - the degree of a node goes from small to very large valuesofeten spanning several order of magnitude (heavy-tailed)
  - **presence of hubs**
    - nodes with high degree
  - **high heterogeneity parameter**
    - a measure of how broad the degree distribution is $\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle^2}$

Models

Model:

A set of instructions to build networks

Goal:

Find models that generate networks with the same characteristics as real-world networks

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Random networks

Random networks

Simple idea

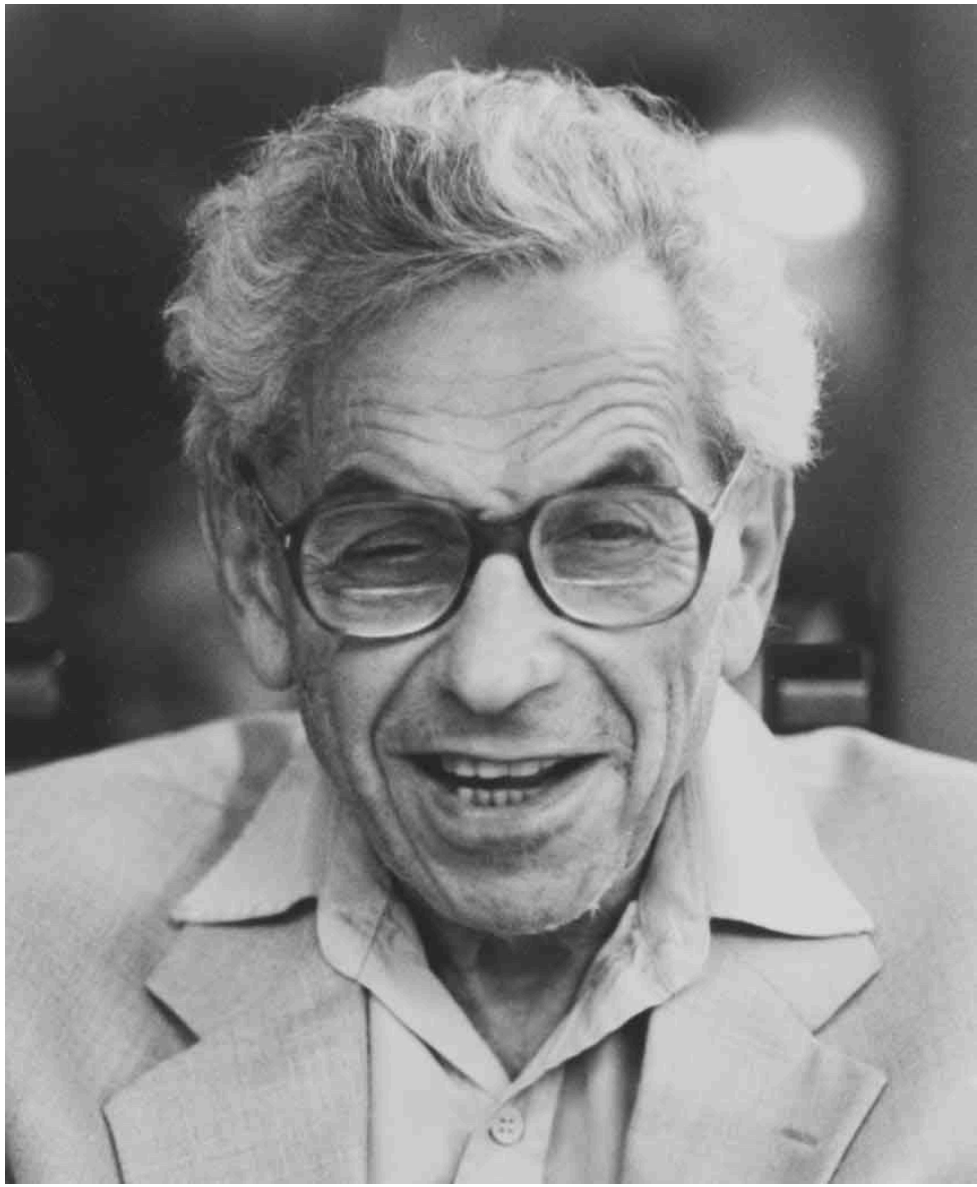
Placing links at random between pairs of nodes

Paul Erdős (1913-1996)

Famous for the Erdős–Rényi random model

We present in these slides an equivalent formulation: the Gilbert's model

Main difference: version by Erdős and Rényi the number of links of the network is fixed, whereas in the model by Gilbert it is variable



Random networks

Albert Gilbert's random network model

$G(n, p)$ where n is the number of nodes and $0 < p < 1$ is the probability that an edge occurs.

Algorithm:

1. Start with n nodes and zero links
2. Go over all pairs of nodes $n = \binom{N}{2}$; for each pair of nodes i and j, generate a random number r between 0 and 1
   - If $r < p \Rightarrow$ **i and j get connected**
   - If $r > p \Rightarrow$ **i and j remain disconnected**

The probability of obtaining any one particular random graph with m edges is $p^m(1-p)^{n-m}$

Random networks: evolution

Let us focus on the **connected components**

- With $p = 0$ **no links**: N components with one node each
- With $p = 1$ **all links are there**: one component (complete network) with N nodes

Question:

What happens as we add links to the network?

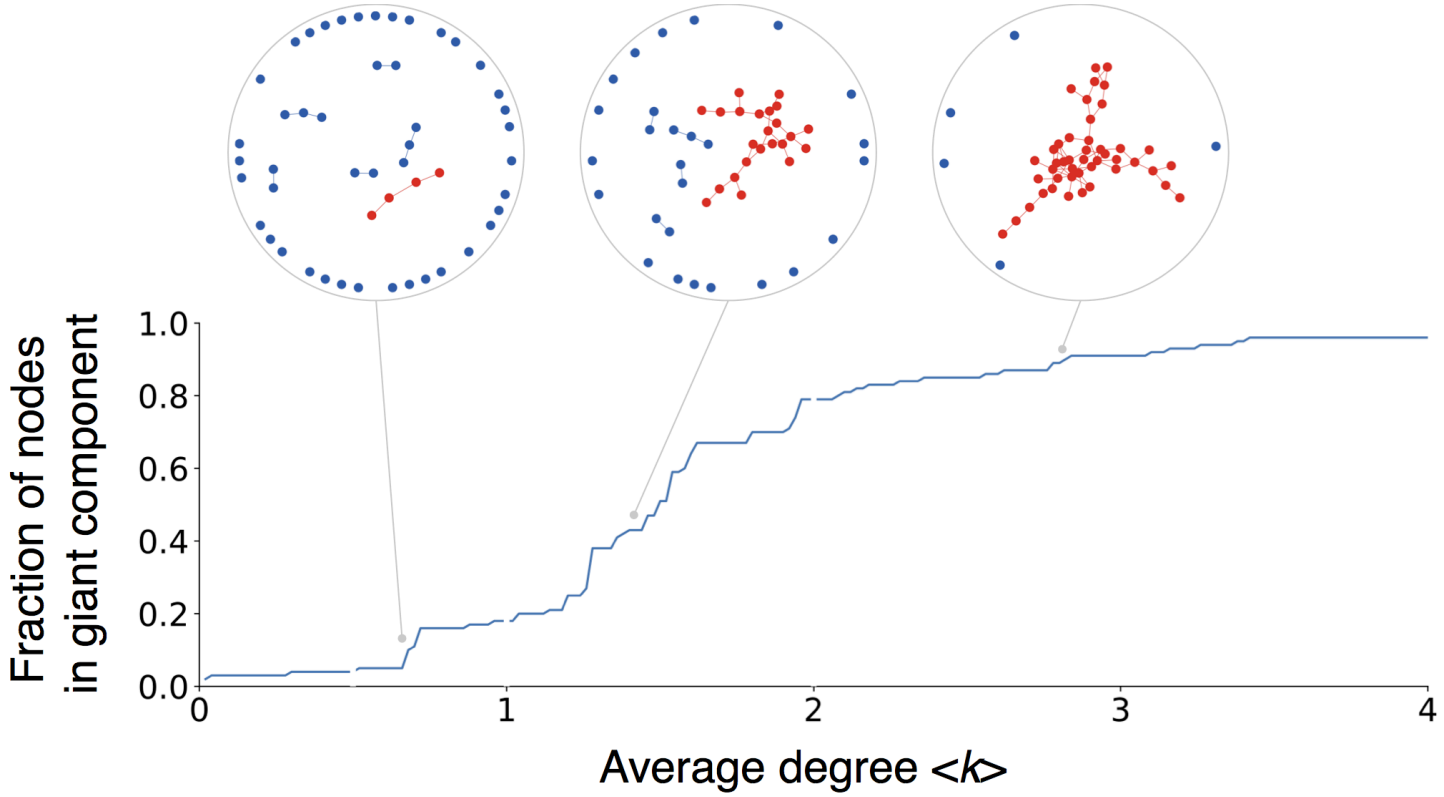
Naïve expectation:

The size of the largest component grows smoothly with the number of links

Wrong expectation:

There is an **abrupt increase** for a given value of the link probability p

Random networks: evolution



Around $\langle k \rangle = 1$ a giant component grows very fast at the expense of the other, smaller components.

[Example in NetLogo]

Random networks: number of links, density, average degree

- Equivalence with the tossing of a biased coin, which yields heads with probability p
- Number of independent trials (tosses): t
- Number of heads after t trials: h
- Special cases:
  - $p = 0$ **the coin never yields heads** $\Rightarrow h = 0$
  - $p = 1$ **the coin always yields heads** $\Rightarrow h = t$
  - $p = \frac{1}{2}$ **the coin yields heads (about) half of the times** $\Rightarrow h \approx \frac{t}{2}$
- General rule:
  - $h \approx p \cdot t$

Random networks: number of links, density, average degree

Expected number of links $\langle L \rangle$ of a random network with N nodes:

Number of heads with probability of yielding heads equal to p and the number of trials t equal to the number of all node pairs of the network:

$$t = \frac{N(N-1)}{2} \rightarrow \langle L \rangle = p\binom{N}{2} = p\frac{N(N-1)}{2}$$

Expected density of links d of a random network with N nodes:

$$d = \frac{\langle L \rangle}{\frac{N(N-1)}{2}} = \frac{p\frac{N(N-1)}{2}}{\frac{N(N-1)}{2}} = p$$

Real-world networks are **sparse**.

For random networks to be better models of real networks, p **must be very small**.

Random networks: number of links, density, average degree

Expected average degree $\langle k \rangle$ of a random network with N nodes:

Number of heads with probability of yielding heads equal to p and the number of trials t equal to the number of potential neighbors of a node:

$$t = N - 1 \rightarrow \langle k \rangle = p(N - 1)$$

Alternatively:

$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N - 1)$$

Random networks: degree distribution

Question: What is the probability that a node has k neighbors?

Back to coin tossing problem: What is the probability that a coin that yields heads with probability p results in k heads out of $N-1$ (independent) trials?

Binomial distribution:

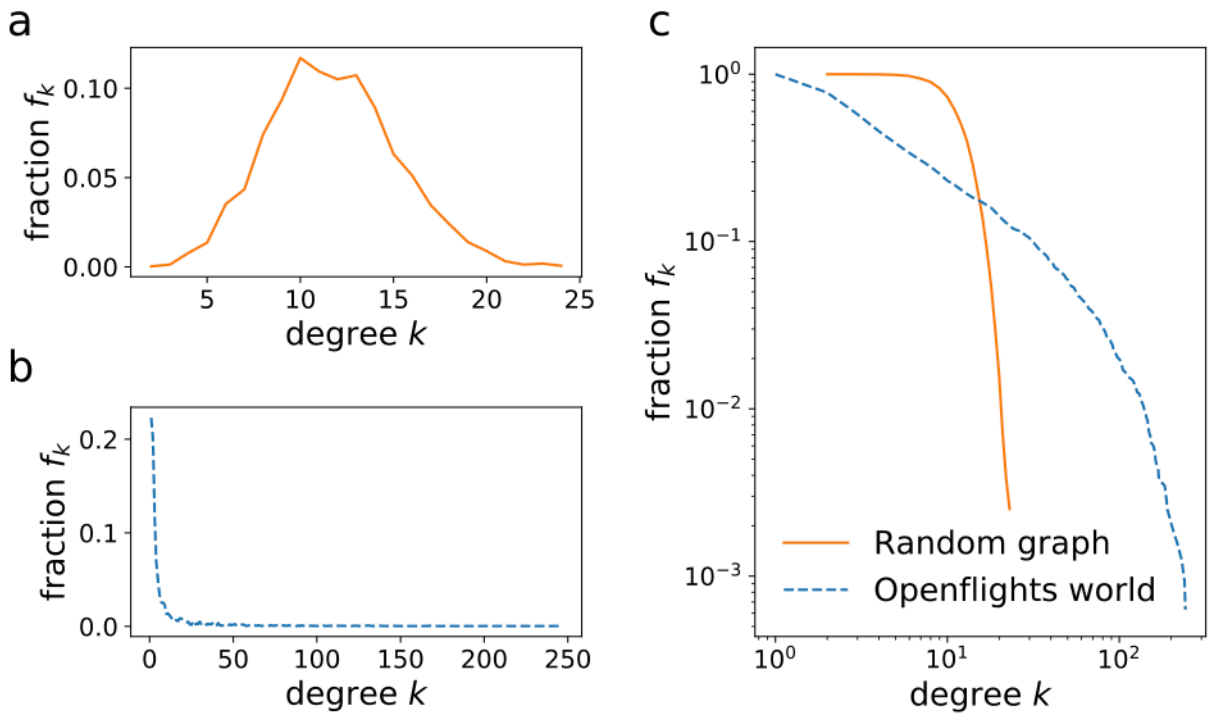
$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

For small p and large N the binomial distribution is well approximated by a bell-shaped curve

$\Rightarrow$ **Most degree values are concentrated around the peak**, so the **average degree is a good descriptor of the distribution**

Random networks: degree distribution

The degree distribution of random networks is **very different** from the broad distributions of most real-world networks!



Random networks: small-world property

Question: How many nodes are there (on average) d steps away from any node?

Premise: Since nodes have approximately the same degree, let us assume they all have the same degree k

- At distance $d = 1$ there are k nodes
- At distance $d = 2$ there are $k(k–1)$ nodes
- At distance d there are $k(k–1)^{d–1}$ nodes
- If k is not too small, the total number of nodes within a distance d from a given node is approximately:

$$N_d \sim k(k-1)^{d-1} \sim k^d$$

Random networks: small-world property

Question: how many steps does it take to cover the whole network?

$$N \sim k^{d_{max}}$$

$$log(N) \sim d_{max} log(k)$$

$$d_{max} \sim \frac{log(N)}{log(k)}$$

The diameter of the network **grows like the logarithm** of the network size

Example: N = 7,000,000,000, k =150 (Dunbar's number)

$$d_{max} = 4.52$$

Random networks: clustering coefficient

The clustering coefficient of a node i can be interpreted as the probability that two neighbors of i are connected

$$C_i = \frac{\text{number of pairs of connected neighbors of i}}{\text{number of pairs of neighbors of i}}$$

Question: what is the probability that two neighbors of a node are connected?

Answer: since links are placed independently of each other, it is the probability p that any two nodes of the graph are connected:

$$C_i = p = \frac{\langle k \rangle}{N-1} \sim \frac{\langle k \rangle}{N}$$

Since $\langle k \rangle$ is a small number, the average clustering coefficient of random networks with realistic values for $\langle k \rangle$ and N is much smaller than the ones observed in real-world networks

Random networks: summary

- Links are placed at **random**, independently of each other
- **Distances between pairs of nodes are short** (small-world property): **good!**
- The **average clustering coefficient is much lower** than on real networks of the same size and average degree: **bad!**
- The nodes have approximately the same degree; there are **no hubs**: **bad!**

Conclusion: the random network **is not a good model** for many real-world networks

In NetworkX:

```
G = nx.gnm_random_graph(N,L) # Erdős–Rényi random graph
G = nx.gnp_random_graph(N,p) # Gilbert random graph
```

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Small-world networks

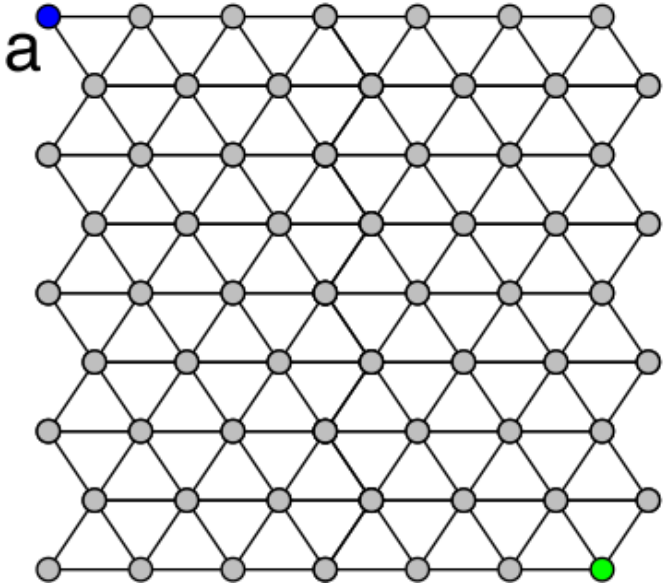
Small-world networks

Goal: building networks with the small-world property and high clustering coefficient

Solution: interpolating between a regular lattice (high clustering) and a random network (small-world property)

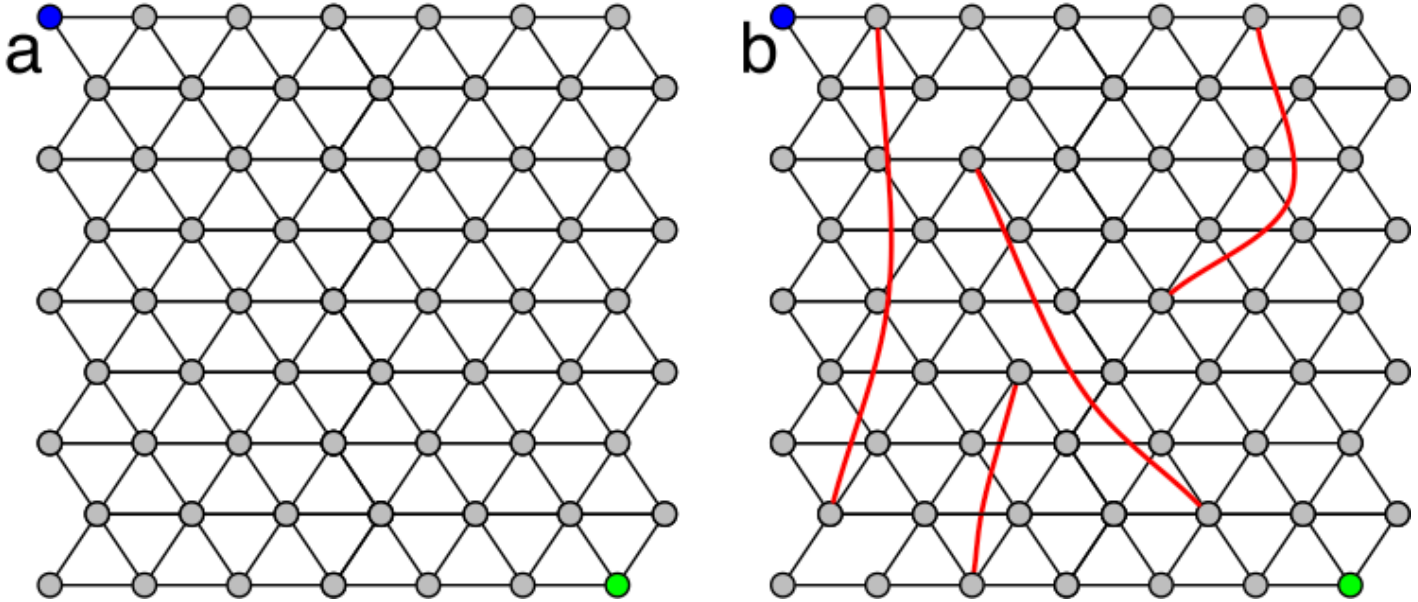
The clustering coefficient of a lattice is high:

- The internal nodes have $k = 6$ neighbors, 6 pairs of which are connected
- $C = \frac{6}{\frac{(6*5)}{2}} = \frac{6}{15} = \frac{2}{5} = 0.4$
- Most nodes are internal, so the average clustering coefficient of the network is close to 0.4!



Small-world networks

Large average shortest path length: going from one node to another can take a large number of steps, which grows rapidly with the size of the network

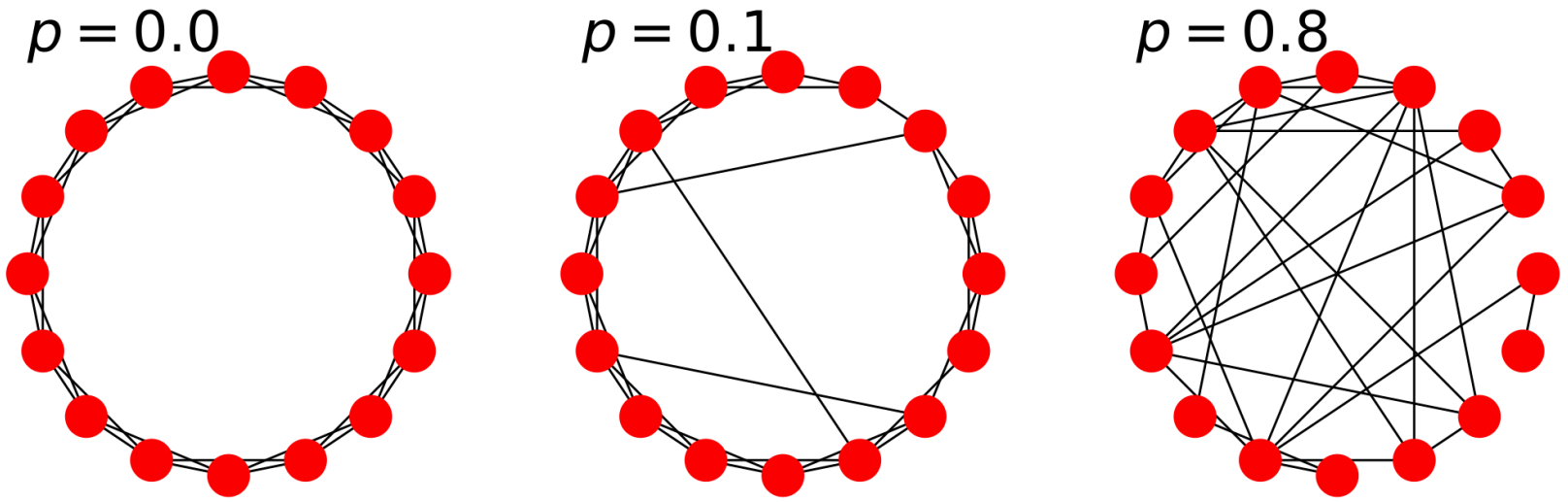


Solution: shortcuts!

The Watts-Strogatz model

N nodes form a **regular ring lattice** with degree k

With probability p, **each link is rewired randomly**

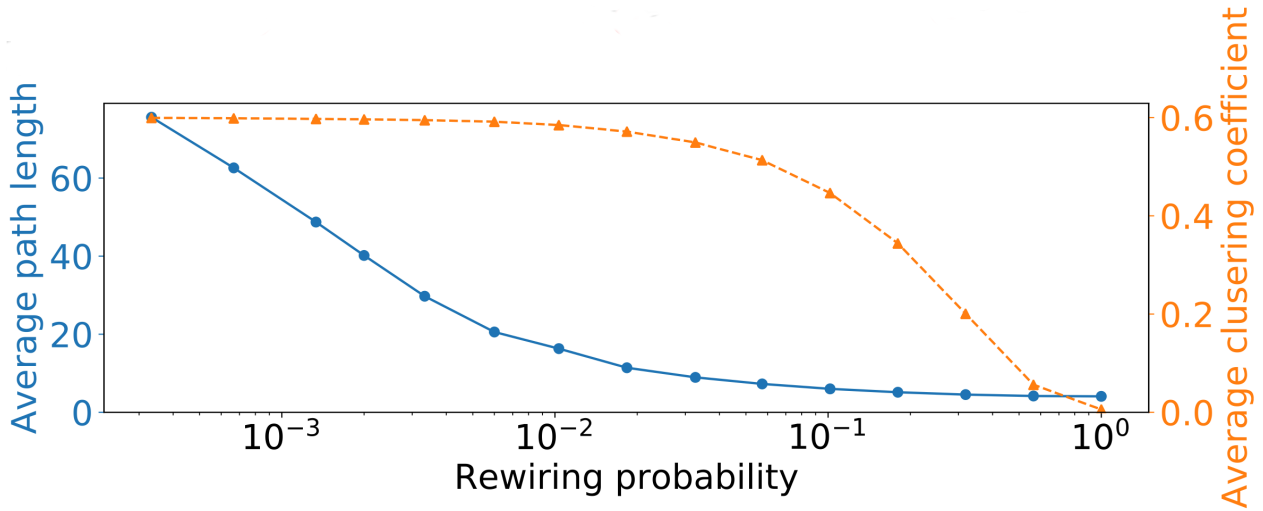


[Collective dynamics of ‘small-world’ networks]

The Watts-Strogatz model

The expected number of rewired links is $pL = pN\frac{k}{2}$

- If $p = 0$, no links are rewired: no change
- If p is small, few links are rewired: the average clustering coefficient stays approximately the same because very few triangles are destroyed, but distances shrink considerably
- If $p = 1$, all links are rewired: the network becomes a random network



The Watts-Strogatz model

Distances become short already for low values of p;

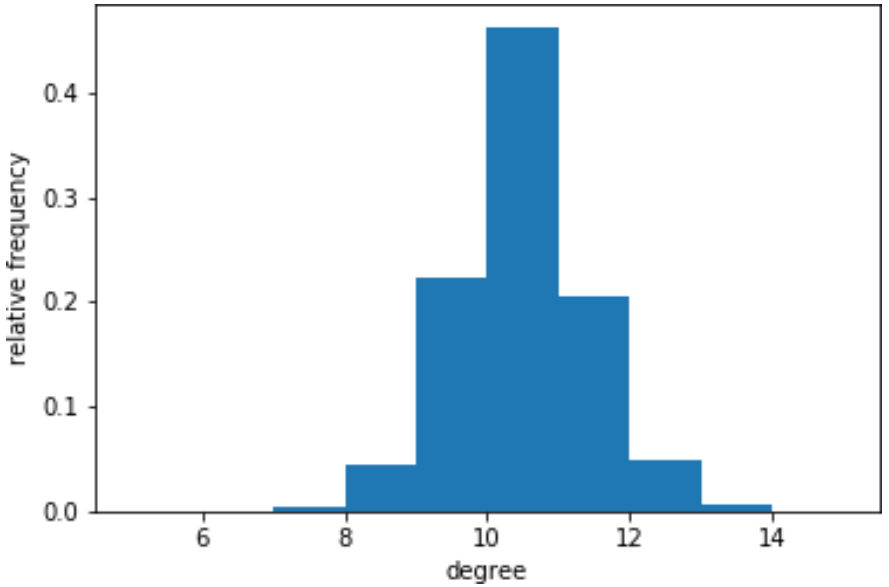
The average clustering coefficient stays high up to large values of p.

There is a range of values of p where the **average path length is short** and **the clustering coefficient is high**!

[Example in NetLogo]

The Watts-Strogatz model: degree distribution

- The degree distribution is peaked as most nodes have the same degree: **no hubs**!
- The Watts-Strogatz model fails to reproduce the broad degree distributions observed in many real-world networks
  - Example if Figure: N = 10000, k = 10, p = 0.1



The Watts-Strogatz model: summary

- A regular lattice whose links are randomly rewired, with some probability p
- There is a range of values of the rewiring probability p for which distances between pairs of nodes are short **(small-world property)** and the **average clustering coefficient** is high: **good!**
- The nodes have approximately the same degree, **there are no hubs**: **bad!**

In NetworkX:

```
G = nx.watts_strogatz_graph(N,k,p)
```

The Configuration Model

The Configuration Model

Problem: is it possible to build networks with a predefinite degree distribution?

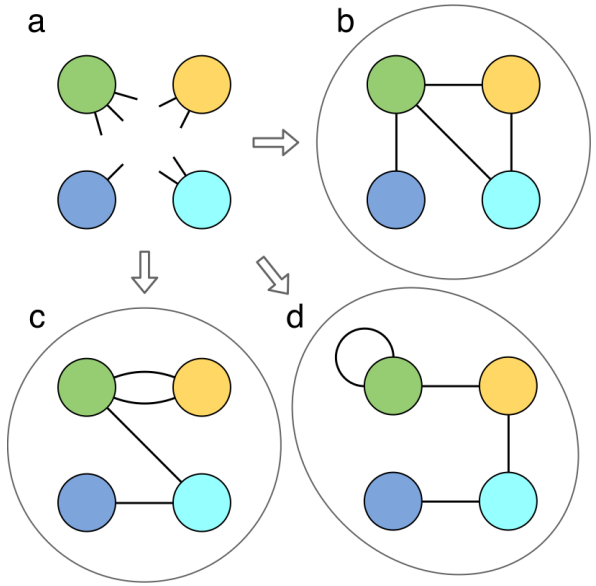
Solution: the **configuration model**

Focus: build networks with a predefinite degree sequence

Degree sequence: $(k_1, k_2, \ldots, k_{N-1}, k_N)$

Caveat: many degree sequences can be extracted from the same distribution

Principle: assign a degree to each node (e.g., from the desired distribution or a real network), place as many **stubs** on each node as the degree of the node, and **attach pairs of stubs at random**



The Configuration Model

Degree-preserving randomization: generate randomized versions of a given network with the same degree sequence, using the configuration model

Why: useful to see whether **a specific property** of the original network **is determined by its degree distribution alone**

- If the property is maintained in the randomized configurations, then the degree distribution is the main driver
- If the property is lost in the randomized configurations, other factors must be responsible

In NetworkX:

```
G = nx.configuration_model(D)     # network with degree sequence D
```

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Preferential models

Network growth

Note: real-world networks are **dynamic**!

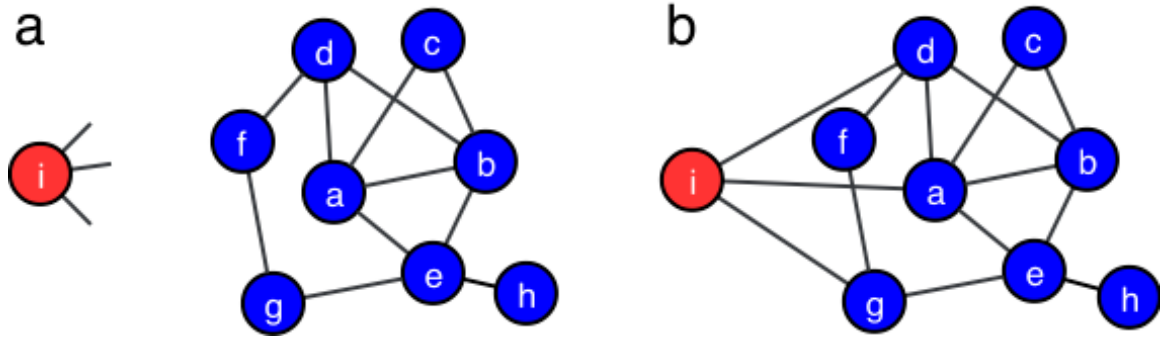
Examples:

- The Web in 1991 had a single node, today there are trillions
- Citation networks of scientific articles and collaboration networks of scientists keep growing due to the publication of new papers
- The collaboration network of actors keeps growing due to the release of new movies
- The protein interaction network has been growing for over 4 billion years, from a few genes to over 20,000

Network growth

General procedure:

1. A new node comes with a given number of **stubs**, indicating the number of future neighbors of the node (degree)
2. The **stubs are attached** to some of the old nodes, according to some rule



Preferential attachment

Note: Nodes prefer to link to the more connected nodes

Examples:

- Our knowledge of the Web is biased towards **popular pages**, which are highly linked, so it is more likely that our website points to highly linked Web sites
- Scientists are more familiar with **highly cited papers** (which are often the most important ones), so they will tend to cite them more often than poorly cited ones in their own papers
- The more movies an actor makes, the more popular they get and the higher the chances of being cast in a new movie

Which model?

Our network model should have the following features:

- **Growth: The number of nodes grows** in time after adding new nodes.
  - The models considered so far are static.
- **Preferential attachment: new nodes tend to be connected to the more connected nodes**.
  - The models considered so far set links among pairs of random nodes, regardless of their degree

> "For to every one who has will more be given, and he will have abundance; but from him who has not, even what he has will be taken away" — Gospel of Matthew 25:29

Take-home message: the rich get richer, and the poor get poorer

[Example in NetLogo]

Polya's urn model

Start: an urn contains X white and Y black balls

Process: a ball is drawn from the urn and put back in with another ball of the same color

Example: if we first pick a white ball, there will be $X + 1$ white and Y black balls in the urn; white will become more likely to be picked than black in the future

- **Preferential attachment used to explain heavy-tail distributions of many quantities:**
  - the number of species per genus of flowering plants,
  - the number of (distinct) words in a text,
  - the populations of cities,
  - individual wealth,
  - scientific production,
  - citation statistics, firm size, and many others!

The Barabási-Albert model

Procedure:

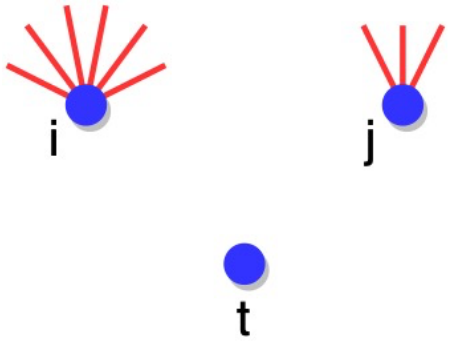
- Start with a group of m_0 nodes, usually fully connected (clique)
- At each step a new node i is added to the system, and sets m links with older nodes ($m \leq m_0$)
- The probability that the new node i chooses an older node j as a neighbor is **proportional to the degree k_j of j**:

$$\prod(i \leftrightarrow j) = \frac{k_j}{\sum_l k_l}$$

- The procedure ends when the given number N of nodes is reached

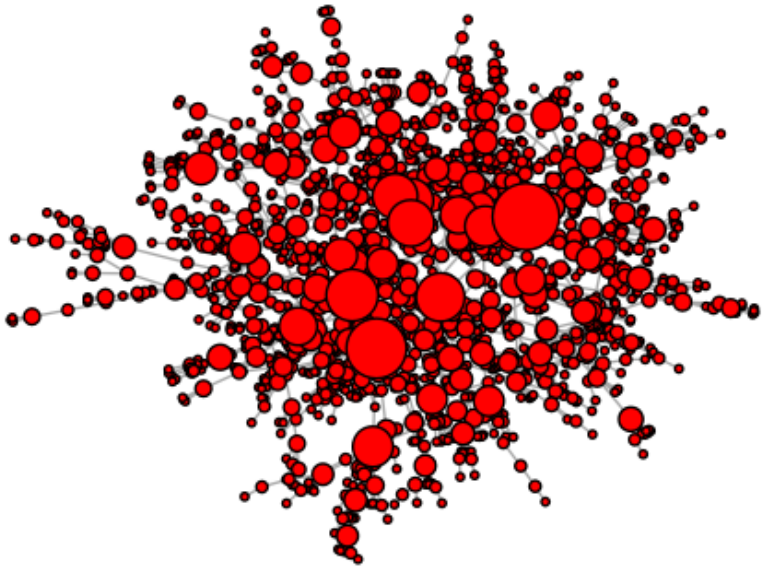
The Barabási-Albert model

Example: if t has to choose between node i, with degree 6, and node j, with degree 3, the probability of choosing i is twice the probability of choosing j



The Barabási-Albert model

- **Rich-gets-richer phenomenon:** due to preferential attachment, the more connected nodes have higher chances to acquire new links, which gives them a bigger and bigger advantage over the other nodes in the future
- **This is how hubs are generated**



In NetworkX;

```
G = nx.barabasi_albert_graph(N, m) # BA model network
```

The Barabási-Albert model

Hubs are the oldest nodes: they get the initial links and acquire an advantage over the other nodes, which increases via preferential attachment

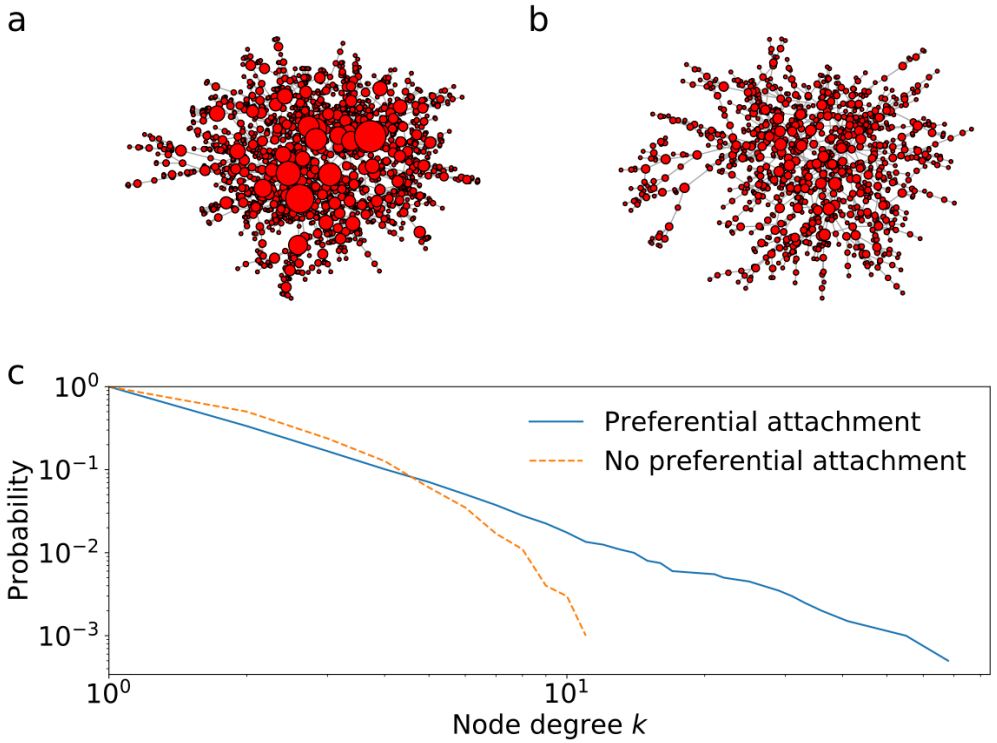
Question: if old nodes have an advantage over newer nodes anyway, do we need preferential attachment at all? Can we explain the existence of hubs just because of growth?

Alternative model: each new node chooses its neighbors at random, not with probability proportional to their degree

The Barabási-Albert model

Conclusion: growth + random attachment does not generate **hubs**.

Preferential attachment is necessary!



Other preferential models

- **The Barabási-Albert model uses linear preferential attachment:** the linking probability is proportional to the degree
- **Question:** what happens if the linking probability is proportional to a power of the degree?
- **Non-linear preferential attachment!**

Tutorial on network models:

04-network-models.ipynb in the course GitHub repo.

Picking nodes in Python

Question: how do we pick nodes with a given probability?

Answer: use the random module

Example: picking list elements (e.g., nodes) with the same probability, i.e., at random:

```
import random
nodes  = [1 ,2, 3, 4]
selected_node = random.choice(nodes)
```

Picking nodes in Python

Question: what if nodes have to be picked with different probabilities?

Answer: we need to provide a list whose elements are the weights associated with the nodes

Note: weights are used to calculate probabilities, they do not have to be integers or add up to one

Example: picking nodes with probability proportional to their degrees, as in preferential attachment:

```
import random
nodes  = [1 ,2, 3, 4]
degrees = [3, 1, 2, 2]
selected_node = random.choice(nodes, degrees)
```

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Final remarks and a case study

Final Remarks

- Watts-Strogatz model explains **short paths** and **high clustering**, but fails to get how individuals actually find short paths from only a local perspective
  - Kleinberg's decentralized search (coming soon)
- Barabasi-Albert model generates **scale-free networks**, i.e., degree distribution following power law with exponent $2 \leq \alpha \leq 3$
  - analytical analysis of rich get richer process (coming soon)
- Are random network models realistic? Why do we use them?

Other preferential models

Non-linear preferential attachment

Procedure:

- Start with a group of m_0 nodes, usually fully connected (clique)
- At each step a new node i is added to the system, and sets m links with older nodes ($m \leq m_0$)
- The probability that the new node i chooses an older node j as neighbor is **proportional to the power α of the degree k_j of j**:

$$\prod(i \leftrightarrow j) = \frac{k_j^\alpha}{\sum_l k_l^\alpha}$$

- The procedure ends when the given number N of nodes is reached

Non-linear preferential attachment

$$\prod(i \leftrightarrow j) = \frac{k_j^\alpha}{\sum_l k_l^\alpha}$$

For $\alpha = 1$ we recover the linear preferential attachment (BA model)

Question: what happens when $\alpha \neq 1$?

Answer: it depends on whether $\alpha > 1$ or $\alpha < 1$

Non-linear preferential attachment

- For $\alpha < 1$, the link probability does not grow fast enough with degree, so the advantage of high-degree nodes over the others is not as big.
  - As a result, the degree distribution does not have a heavy tail: **the hubs disappear!**
- If $\alpha > 1$, high-degree nodes accumulate new links much faster than low-degree nodes.
  - As a consequence, one of the nodes will end up being connected to a fraction of all other nodes.
  - For $\alpha > 2$, a single node may be connected to all other nodes (**winner-takes-all effect**), all other nodes having low degree
- **Conclusion: non-linear preferential attachment fails to generate hubs**
  - Linear preferential attachment is the only way to go
- **Problem: strict proportionality of linking probability to degree appears unrealistic**

Limits of preferential attachment

- It yields a **fixed pattern for the degree distribution**: the slope is the same for any choice of the model parameters.
  - Degree distributions in real-world networks could decay faster or more slowly
- **The hubs are the oldest nodes**: new nodes cannot overcome their degree
- **It does not create many triangles**: the average clustering coefficient is much lower than in many real-world networks
- **Nodes and links are only added**: in real networks they can also be deleted
- Since each node is attached to older nodes, **the network consists of a single connected component**.
  - Many real-world networks have **multiple components**

Extensions of the BA model: Attractiveness model

- **Pitfall of preferential attachment**: What happens if a node has no neighbors (degree zero)?
  - **It will never get connections from other nodes!**
- **No problem for standard initial condition**: the initial subgraph is complete (clique), so every node has nonzero degree
- **What if the network is directed and the linking probability is proportional to the in-degree?**
  - Bad, as each new node has in-degree zero, so it will never be linked by future nodes!

Extensions of the BA model: Attractiveness model

Procedure:

- Start with a group of m_0 nodes, usually fully connected (clique)
- At each step a new node i is added to the system, and sets m links with some of the older nodes $(m < m_0)$
- The probability that the new node i chooses an older node j as neighbor is proportional to the sum of the degree k_j of j and an attractiveness A, indicating the intrinsic appeal of j:

$$\prod(i \leftrightarrow j) = \frac{A + k_j}{\sum_l (A + k_l)}$$

Extensions of the BA model: Attractiveness model

$$\prod(i \leftrightarrow j) = \frac{A + k_j}{\sum_l (A + k_l)}$$

- For A = 0 we recover the BA model
- For every value of A we get networks with heavy-tailed degree distributions
- The pattern of the distribution **changes with A**, so it is possible to match distributions of real-world networks, **unlike the BA model**

Extensions of the BA model: Fitness model

Pitfall of preferential attachment: the hubs are the oldest nodes. **Unrealistic!**

Examples:

- In the Web, new pages can overrun old pages (e.g., Google!)
- In science, new papers can be more successful than (many) old papers

Reason: each node has its own individual appeal

Extensions of the BA model: Fitness model

Procedure:

- Start with a group of m_0 nodes, usually fully connected (clique)
- At each step a new node i is added to the system, and sets m links with some of the older nodes $(m \leq m_0)$
- The probability that the new node i chooses an older node j as neighbor is proportional to the product of the degree k_j of j with a fitness η_j, indicating the intrinsic appeal of j:

$$\prod(i \leftrightarrow j) = \frac{\eta_j \cdot k_j}{\sum_l (\eta_l \cdot k_l)}$$

Extensions of the BA model: Fitness model

The fitness values are extracted from a distribution $\rho(\eta)$ and assigned to each new node

Difference with attractiveness model

- The fitness enters as a **factor** in the link probability, not as a summand
- The fitness is **characteristic of each node**, it is not a constant

Extensions of the BA model: Fitness model

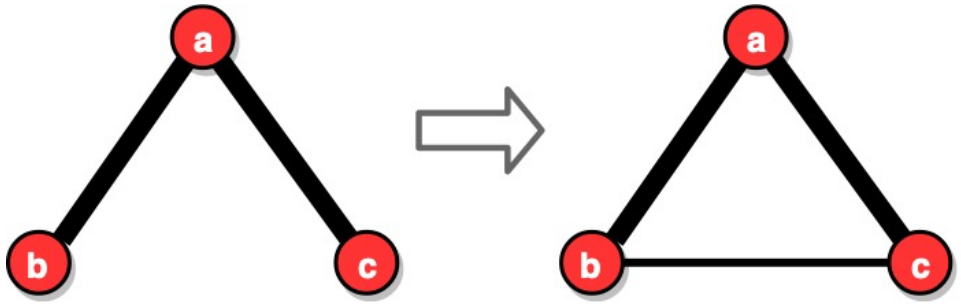
Results:

- If the fitness distribution $\rho(\eta)$ has finite support, i.e., the fitness is distributed over a finite range of values, the degree distribution of the network is heavy-tailed
- If the fitness distribution $\rho(\eta)$ has infinite support, i.e., the fitness is distributed over an infinite range of values, the node with the largest fitness attracts a fraction of all links **(monopoly)**
- Nodes with large fitness can acquire a large degree even if they are introduced late in the system **(good!)**

Extensions of the BA model: Random walk model

- **Pitfall of preferential attachment**: the BA model does not generate many triangles. Why?
- To close a triangle we need to set a link between two neighboring nodes, whereas in the BA model links are set based on degree, regardless of whether the future neighbors have common neighbors

Solution: introduce a mechanism for triadic closure in the model!

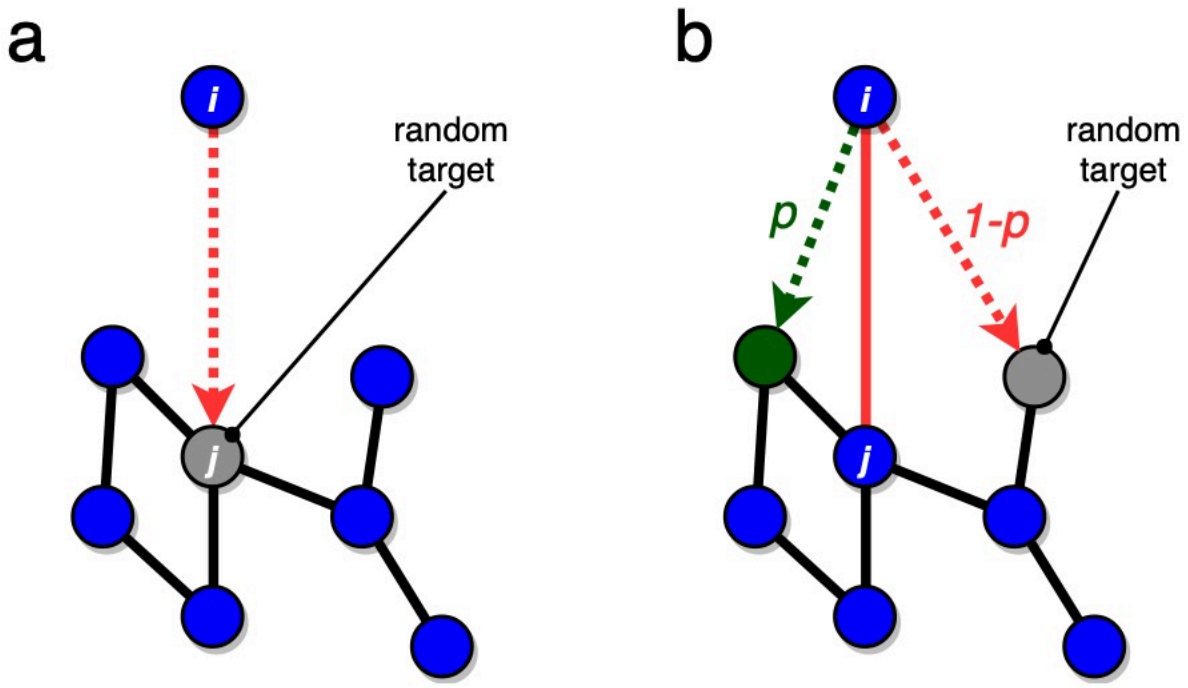


Extensions of the BA model: Random walk model

Procedure:

- Start with a group of m_0 nodes, usually fully connected (clique)
- At each step a new node i is added to the system, and sets m links with some of the older nodes $(1 < m \leq m_0)$
- The first link targets a randomly chosen node j
- From the second link onwards:
  - With probability p the link is set with a neighbor of j, chosen at random
  - With probability $1-p$ the link is set with a randomly chosen node

Extensions of the BA model: Random walk model



Extensions of the BA model: Random walk model

Results:

- The **degree distribution is heavy-tailed**
- The **average clustering coefficient is much higher than in BA networks**
  - the larger, the greater the probability p of triadic closure
- When the triadic closure probability p is sufficiently high that many triangles are formed $(p \approx 1)$ the network has **community structure**
  - i.e., it is made of cohesive groups of nodes, loosely connected to each other

Extensions of the BA model: Random walk model

Question:

- If links are set at random, as it seems, how can the model generate hubs?

Answer:

- Choosing **a random node and a random neighbor of the node amounts to choosing a link at random**
- The probability that the endpoint(s) of a randomly selected link have a given degree is **proportional to the degree**

Extensions of the BA model: Random walk model

Conclusion:

The triadic closure mechanism of the random walk model **induces effective preferential attachment**

Take-home messages:

- Preferential attachment can be induced by simple mechanisms based on random choices;
- **it is not necessary to require the knowledge of the degree of the nodes, nor a strict expression of the link probability**

Extensions of the BA model: Copy model

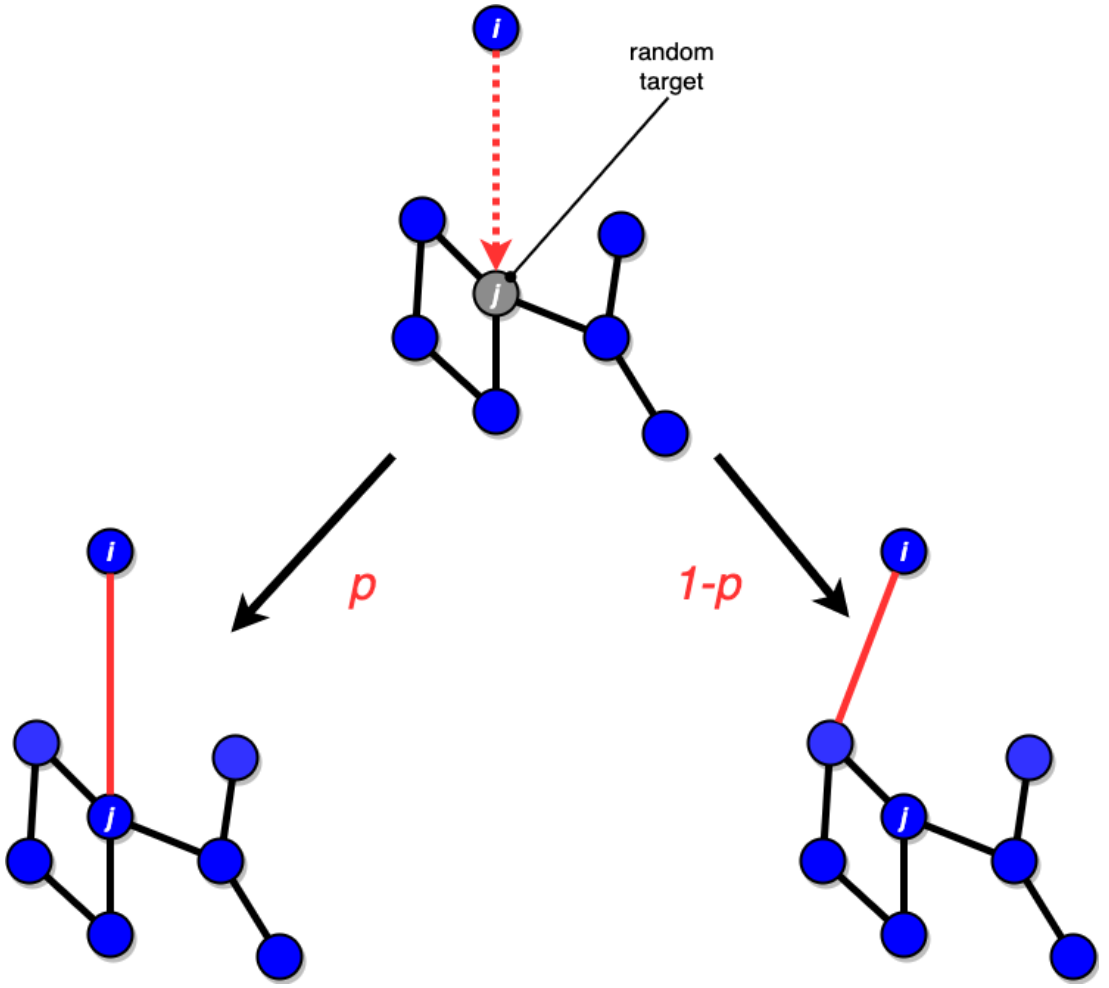
Motivation: the authors of a new web page tend to copy the hyperlinks of other pages on the same topic

Steps:

- **Growth:** at each time step a new node i is added to the network
- **Target selection:** a node j is selected at random
- **Random connection:** with probability p the new node is connected to j
- **Link copying:** with probability $1-p$ the new node is connected to a neighbor of j, chosen at random

Copy model

Difference from random walk model:
here we do not link to the target and to its neighbor (no triadic closure)!



Extensions of the BA model: Copy model

Relevance to real systems:

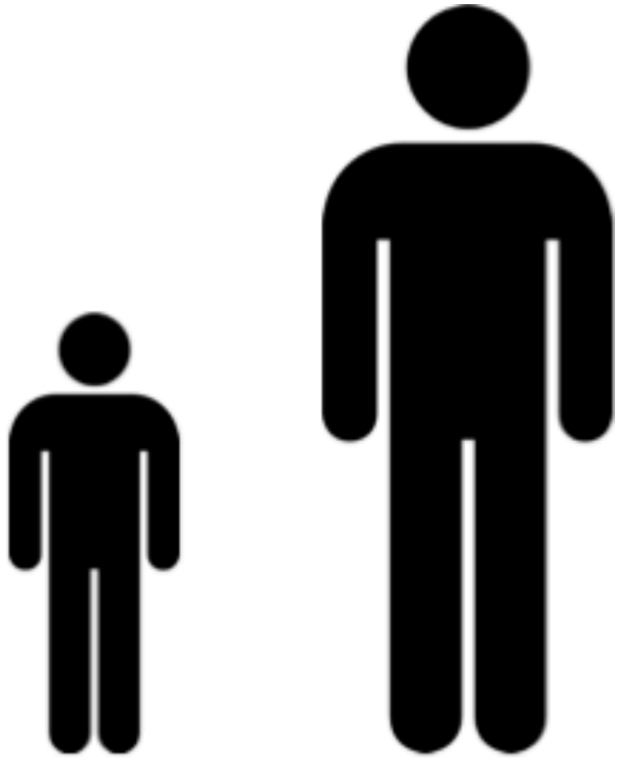
- **Social Networks:** The more acquaintances an individual has, the higher is the chance that she will be introduced to new individuals by her existing acquaintances (we "copy" the friends of our friends)
- **Citation Networks:** Authors decide what to read and cite by "copying" references from the papers they have read. Consequently papers with more citations are more likely to be studied and cited again
- **Gene duplication:** Responsible for the emergence of new genes in a cell, can be mapped into the copying model

Extensions of the BA model: Rank model

Pitfall of preferential attachment: BA model implies that nodes have a perception of how important other nodes are, i.e., how large is their degree

Objection: in the real world there is no such perception of the absolute value of things, it is far easier to perceive the relative value

Solution: ranking



Extensions of the BA model: Rank model

Procedure:

- Nodes are ranked based on a property of interest (e.g., age, degree). The rank of node i is R_i
- Start with a group of m_0 nodes, usually fully connected (clique)
- At each step a new node i is added to the system, and sets m links with some of the older nodes $(m \leq m_0)$
- The probability that the new node i chooses an older node j as neighbor is **proportional to a power of the rank** of j:

$$\prod(i \leftrightarrow j) = \frac{R_j^{-\alpha}}{\sum_l R_l^{-\alpha}}$$

Extensions of the BA model: Rank model

$$\prod(i \leftrightarrow j) = \frac{R_j^{-\alpha}}{\sum_l R_l^{-\alpha}}$$

Remark: highly-ranked nodes (those with low values of R) have high probabilities of being linked, much higher than poorly-ranked nodes

Result: the model generates networks with hubs for **any value** of the exponent α and **any property** used to rank the nodes

UNIVERSITÀ DI TORINO

Reading material

References

[ns1] **Chapter 5 (Network Models)**

Q&A