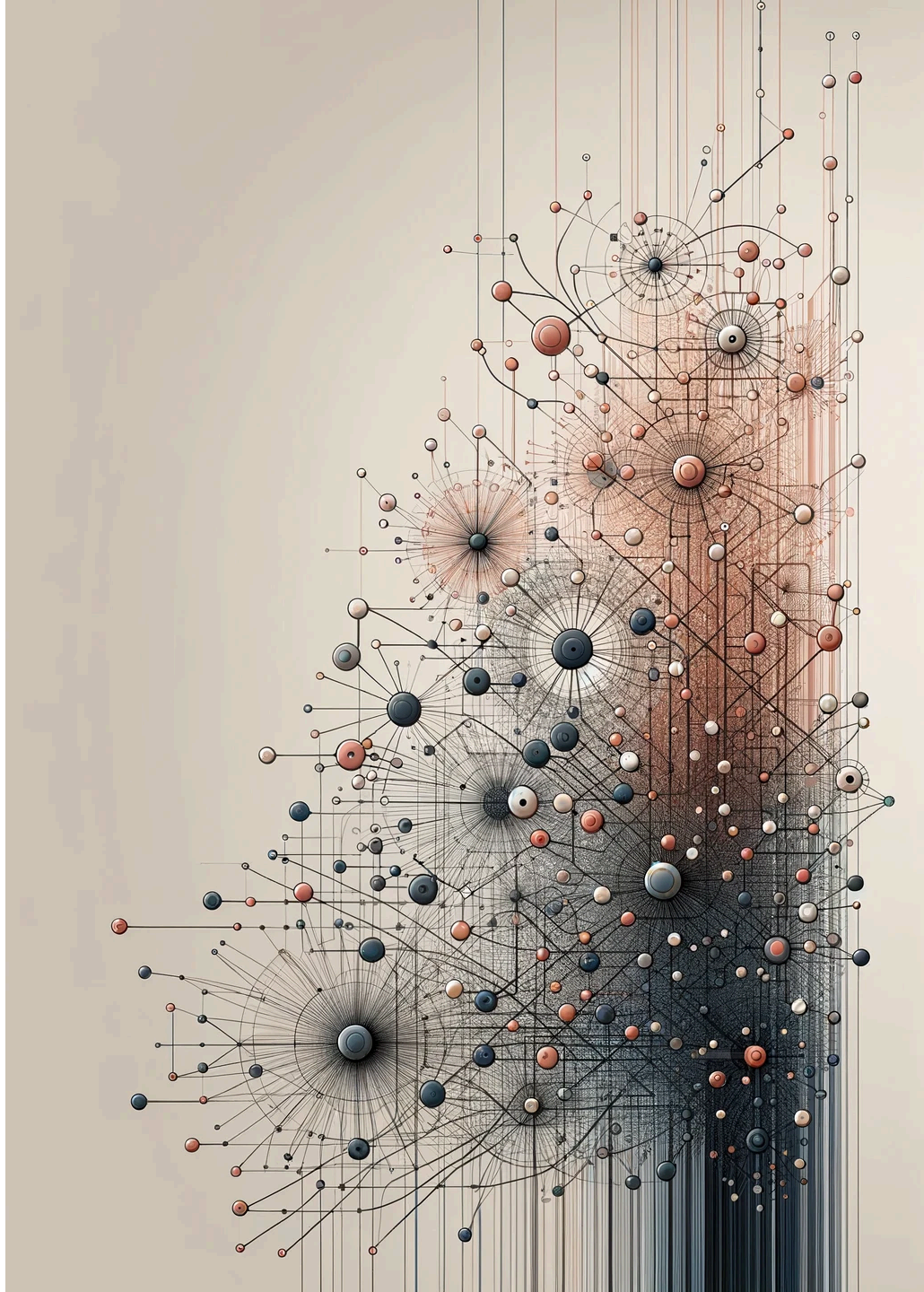
UNIVERSITÀ DI TORINO

Analisi e Visualizzazione delle Reti Complesse

NS20 - Link Analysis and Web Search

Prof. Rossano Schifanella



UNIVERSITÀ DI TORINO

Agenda

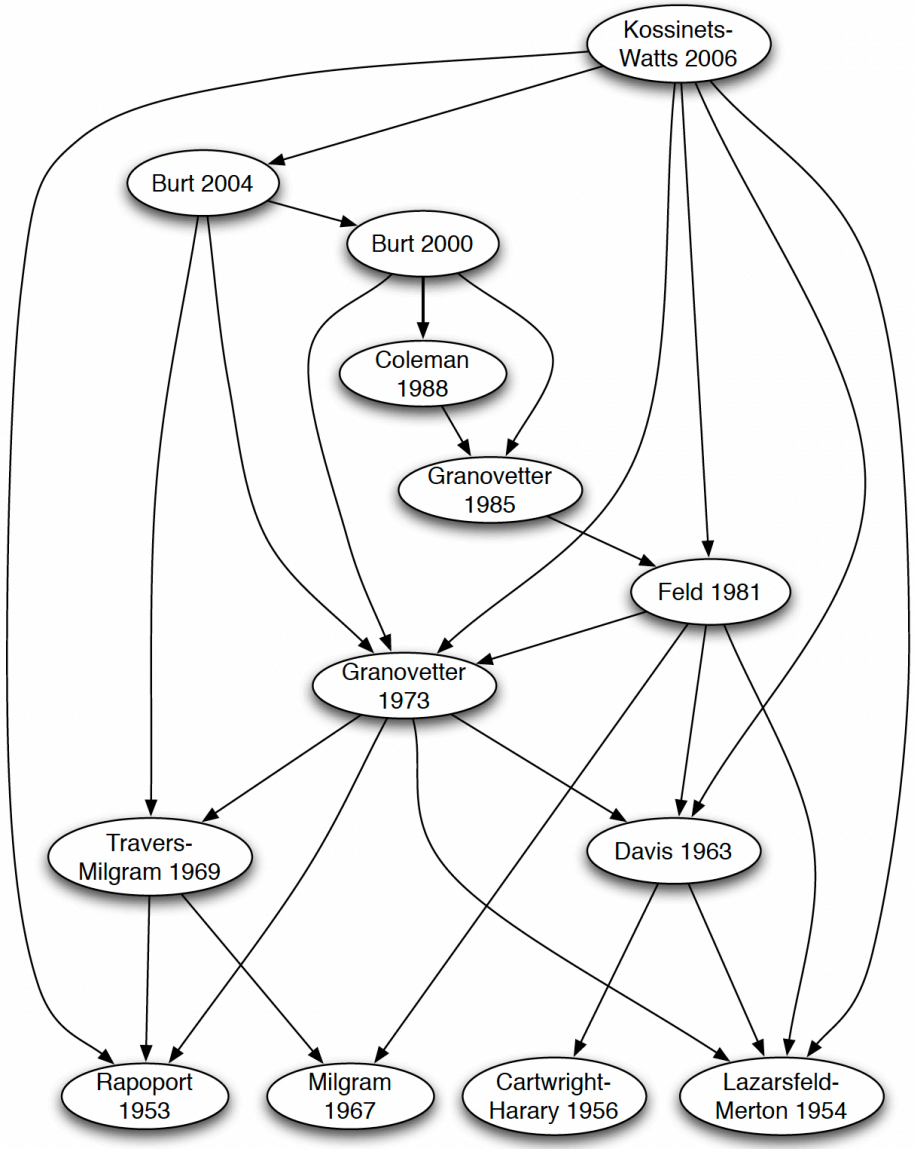
- The Structure of the Web
- Searching the Web
- Link Analysis
  - HITS: Hubs and Authorities (+ Spectral Analysis)
  - Page Rank (+ Spectral Analysis)
  - Random Walks and PR
- Practical implications
  - Modern Web search
  - Link Analysis beyond the Web

Directed and weighted networks

- Many real-world networks are **directed** and/or **weighted**.
  - **Food webs**: (directed) links go from prey to predator and (weights) represent amount of prey/energy consumed
  - **Web and Wikipedia**: links go from source to target page and may have weights representing clicks
  - **Social Media** (as Twitter): links may be weighted by numbers of interactions, such as retweets, mentions, likes, comments, etc.
  - **Citation Networks**: citations from one scientific paper to another
  - **Communication Networks** (including Who-Talks-to-Whom networks)

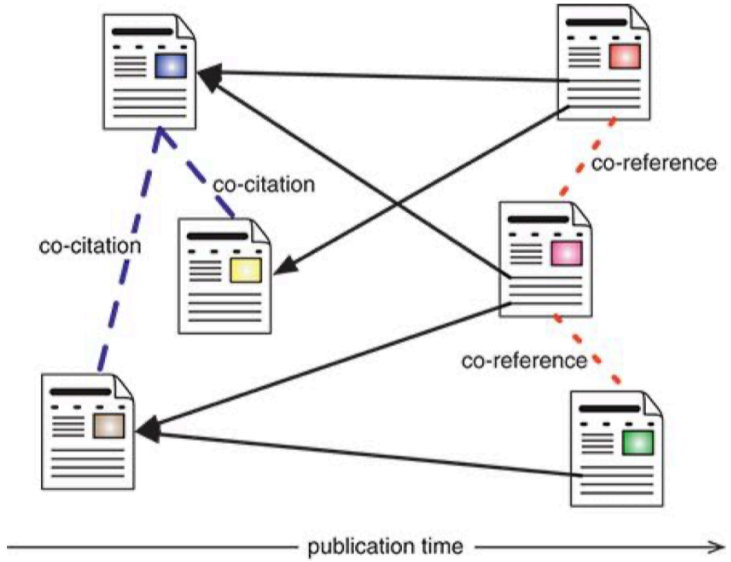
Ex: Citation Networks

- Nodes are research articles and links represent citations of references
  - Links tend to point strictly backward in time
  - There is an "arrow of time" that returns a sense of flow from present to past



Bibliographic coupling

- Semantic similarity measures: **co-citation** and **co-reference**
  - aka bibliographic coupling
- **Co-citation network**: shared predecessors (cited by same papers)
- **Co-reference network**: shared successors (citing same papers)



The World Wide Web

- Created by **Tim Barners Lee** in 1989-1991
  - Tim Barners Lee when asked: "Did you invent the internet?"

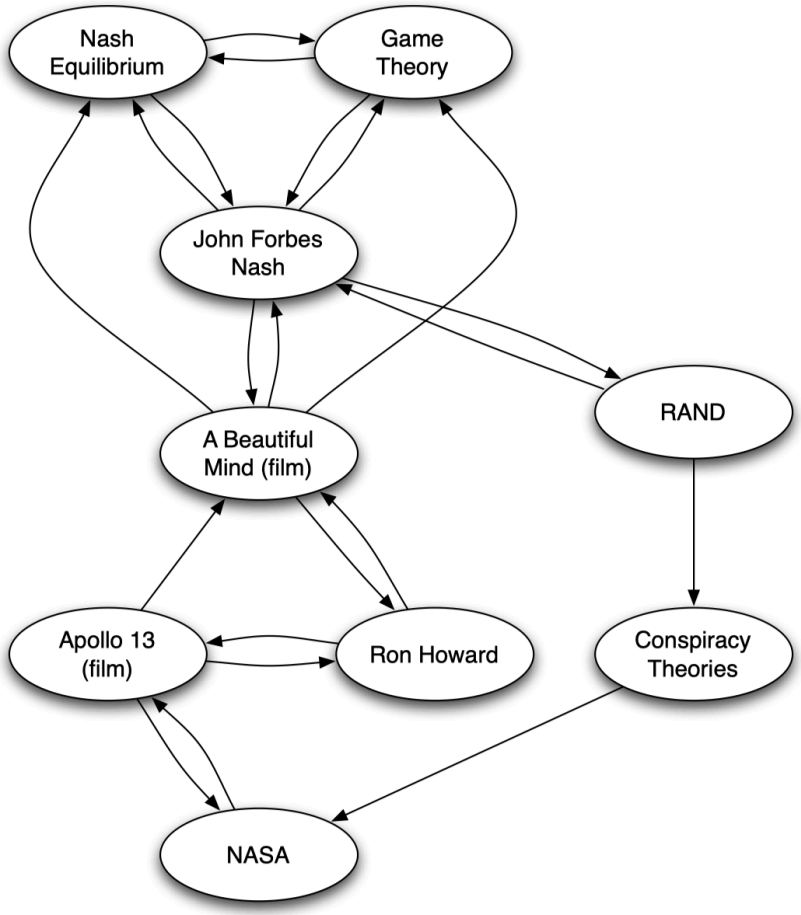
> No, no, no!
>
> When I was doing the WWW, most of the bits I needed were already done.
>
> Vint Cerf and people he worked with had figured out the Internet Protocol, and also the Transmission Control Protocol.
>
> Paul Mockapetris and friends had figured out the Domain Name System.
>
> People had already used TCP/IP and DNS to make email, and other cool things. So I could email other people who maybe would like to help work on making the WWW.
>
> I didn't invent the hypertext link either. The idea of jumping from one document to another had been thought about lots of people, including Vanevar Bush in 1945, and by Ted Nelson (who actually invented the word hypertext). Bush did it before computers really existed. Ted thought of a system but didn't use the internet. Doug Engelbart in the 1960's made a great system just like WWW except that it just ran on one [big] computer, as the internet hadn't been invented yet. Lots of hypertext systems had been made which just worked on one computer, and didn't link all the way across the world.
>
> I just had to take the hypertext idea and connect it to the TCP and DNS ideas and -- ta-da! -- the World Wide Web.

The World Wide Web

- A way to exchange documents via the Internet
  - **Web page**: public
  - **Links** are hyperlinks
  - **Hypertext**: organization of information using a network metaphor
  - **Browser**: connects to the Internet
- The Web as a **directed graph**
  - Which is the informative power of the underlying graph structure?
    - understand better the logical relationships expressed by its links
    - break its structure into smaller units
    - identify important pages as a step in organizing results of the Web

Wikipedia

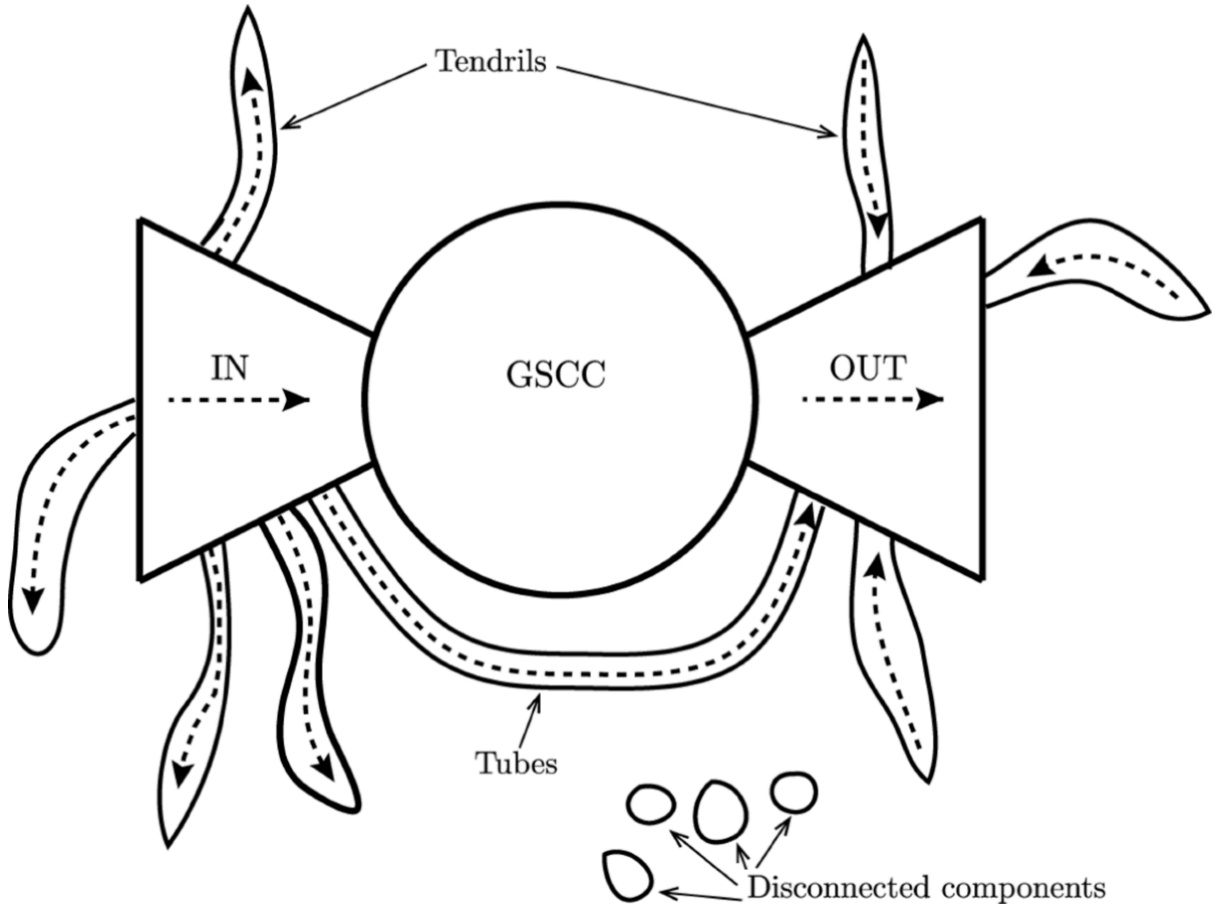
- A subnetwork of the Web
- T. B. Lee was influenced by the pioneering work of **Vannevar Bush**
- **memex**: technological device that implements the idea of associative memory
- The gigantic brain in the "As we may think" paper (1945)
- The cross-references among a set of wikipedia pages is very much related to such original intuition



The Structure of the Web

- Broder et al. (1999) used Altavista to build a map of the Web dividing the graph into a few pieces
  - Reference: Graph structure in the web, WWW2000.
- They crawled the Web from large companies sites
  - 203 million URLs
  - 1,466 million links
- They found a largest connected component (or Giant component), containing a significant fraction of all the nodes

The Bow-Tie Structure of the Web



UNIVERSITÀ DI TORINO

Components and reachability

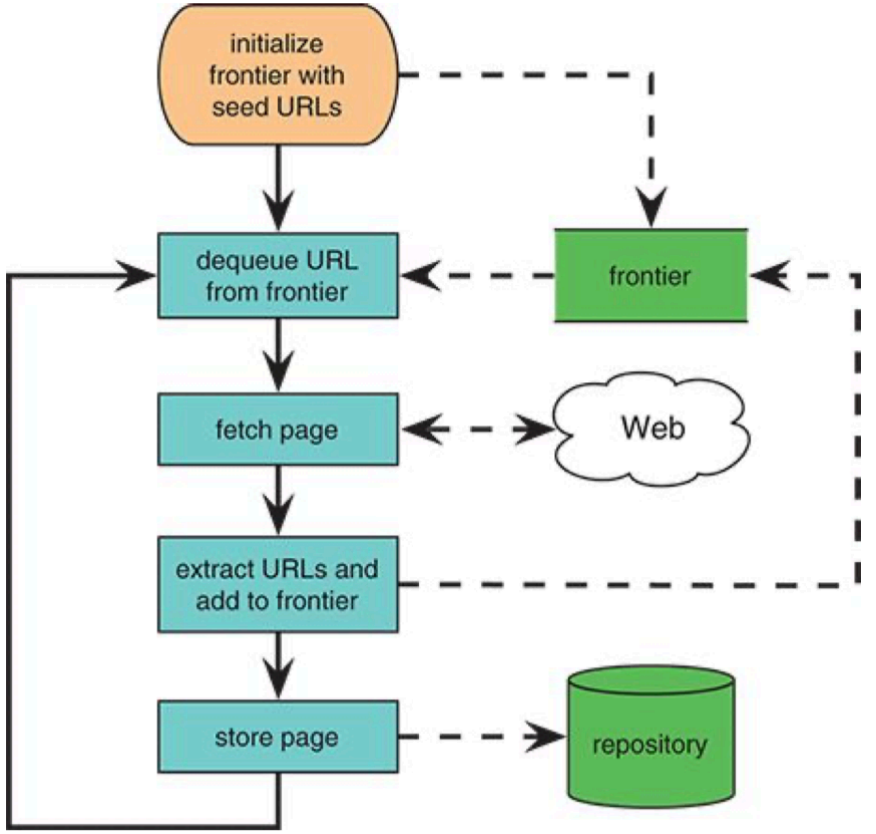
- **GSCC (Giant Strongest Connected Component, aka the Core)**:
  - the largest connected subgraph s.t. I can find paths leading from a node to another and back
- **IN (In Component)**:
  - from every node in IN there is a path leading to SCC
- **OUT (Out Component)**:
  - nodes in the SCC are connected to nodes in OUT, no way back
- **Tendrils**:
  - nodes reachable from IN or nodes that reach OUT
- **Tubes**:
  - nodes reached from IN and reaching OUT not connected to SCC
- **Weakly Connected Component** = SCC + IN + OUT + Tendrils + Tubes
- **The Web** = Smaller disconnected components + WCC

So what?

- **Web crawlers**: programs that automatically download Web pages
  - The primary application of crawlers are **search engines** (nowadays also generative AI).
  - Crawlers collect information, scattered across billions of pages served by millions of servers around the globe, to a central location where it can be analyzed and mined.
  - A search engine takes the information collected by a crawler and creates an **index** to efficiently retrieve pages that contain keywords and phrases queried by users.
  - The Web changes rapidly, so search engines use crawlers to stay fresh information as pages and links are added, deleted, moved, and updated.

Web Crawler

- The basic concept of crawler is simple: a **breadth-first traversal** of the Web graph
  - Start from a set of high-quality **seed pages**.
  - The queue of unvisited URLs is called **frontier**.
  - Fetch pages dequeued from the frontier, extract links and add them to the frontier.
  - Heuristics in an attempt to prioritize links in the frontier that are likely to lead to quality content
- **Devil in the details**: complications due to scalability, page revisit scheduling, spider traps (when meaningless URLs are generated automatically by servers), canonical URLs, robust HTML parsing, and the ethics of dealing with servers.

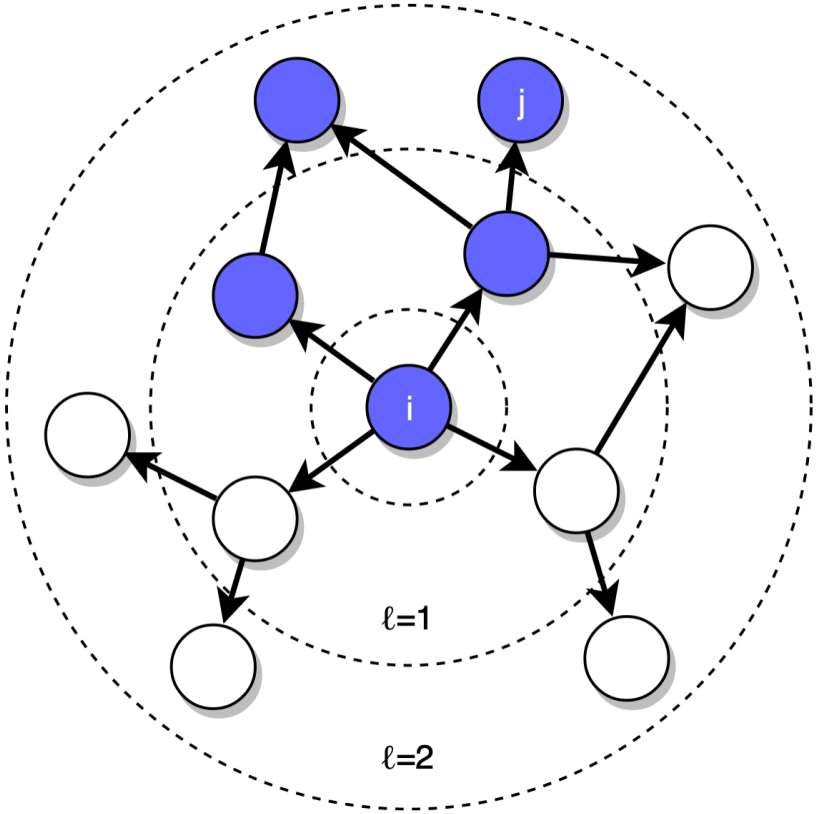


Web degree distributions

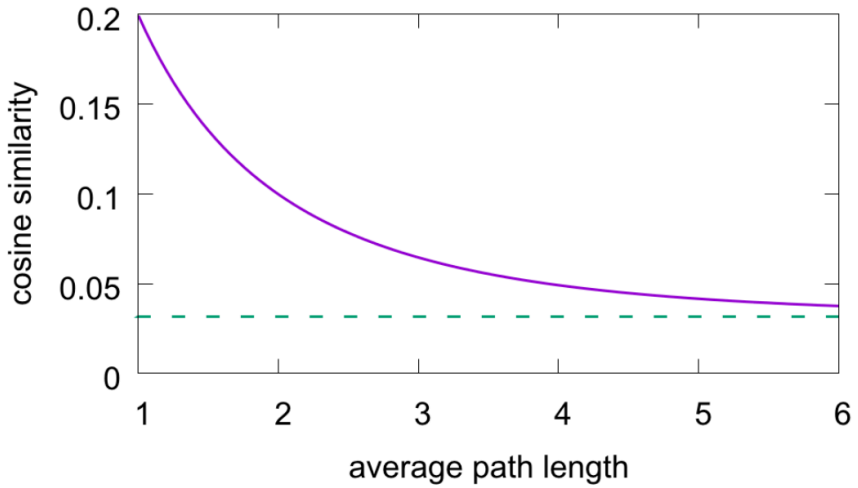
- **cumulative in-degree distribution**, which arises from collective behavior of authors.
- Heavy tail with **huge hubs**: $\langle k_{in} \rangle \approx 10, \quad \kappa \approx 40$
  - heterogeneity parameter κ is large!
- **out-degree** is **less meaningful** as it depends on individual content providers, can be gamed
  - e.g., link farms for spamdexing
- Short paths in the core: $N \approx 1.8 \times 10^9, \quad \langle \ell \rangle \approx 13$
- Hubs leads to the emergence of the **small world phenomenon**.

Topical locality

- **Homophily** for the Web: pages about related topics tend to link to each other
- **Links are not random**
- We can think of similarity based on **textual content** or **categorical topics**



Topic locality and topic drift



- **Topic drift**: as we browse away from a starting page, the similarity decreases
- Topical locality makes Web browsing possible as we stay on topic
- Crawlers use BFS because of topical locality, to target good-quality pages linked by other good-quality pages

PageRank and HITS

- **Centrality measures** for nodes in directed networks.
- Sergey Brin and Larry Page introduced PageRank in 1998 as a key ingredient of Google.
- Jon Kleinberg introduced HITS in 1999.
- Both are based on eigenvector centrality and designed for web information retrieval.
- In NetworkX:

```
PR_dict = nx.pagerank(D)     # D must be a DiGraph
H_dict, A_dict = nx.hits(G)   # G should be a DiGraph
```

Reading material:

- Page, Lawrence; Brin, Sergey; Motwani, Rajeev and Winograd, Terry, The PageRank citation ranking: Bringing order to the Web. 1999
- Jon Kleinberg, Authoritative sources in a hyperlinked environment Journal of the ACM 46 (5): 604-32, 1999. doi:10.1145/324133.324140.
- A. Langville and C. Meyer, "A survey of eigenvector methods of web information retrieval.", SIAM Review, vol. 47, No. 1

Searching the Web

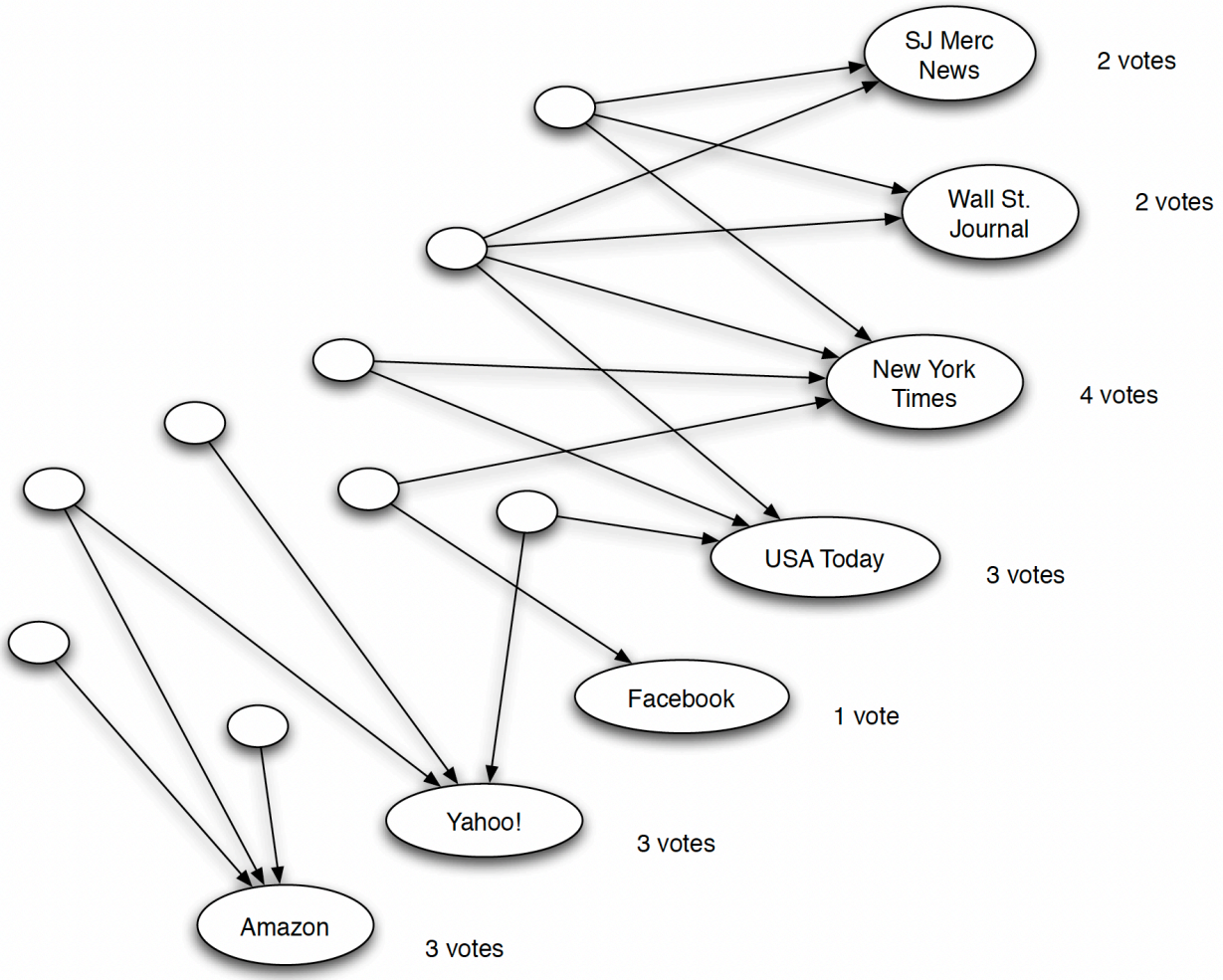
- **Search Engine**
  - **problem:** how to rank (web) pages related to a given topic
- **Information Retrieval**
  - automated strategies to search in libraries, scientific papers, repositories, and so on
  - in response to **keywords-based queries**
- List of keywords is "inexpressive" (e.g., polysemy, synonymy)
- **Diversity**: given a topic we find pages created by a large variety of authors
  - e.g., experts, novices, children, conspiracy theorists
- Pages are **dynamic** and always changing
- **Filters**: abundance of information, what is important?
- Can the structure of the Web, dominated by links, help us to find such filters?
  - first attempt: count words in documents
  - can we do better?

HITS: Link Analysis using Hubs and Authorities

- Disclaimer: the term "Hub" is used slightly differently here w.r.t. previous lectures
- Information contained "between" pages can be used as well
- Count **in-links**:
  - select documents on a given topic
  - **in-links are a measure of authority** of a page on such a topic: it is an implicit endorsement from the community of web pages' authors
  - It is hard for search engines to automatically assess the intent of each link. In aggregate, we assume they mean endorsement.

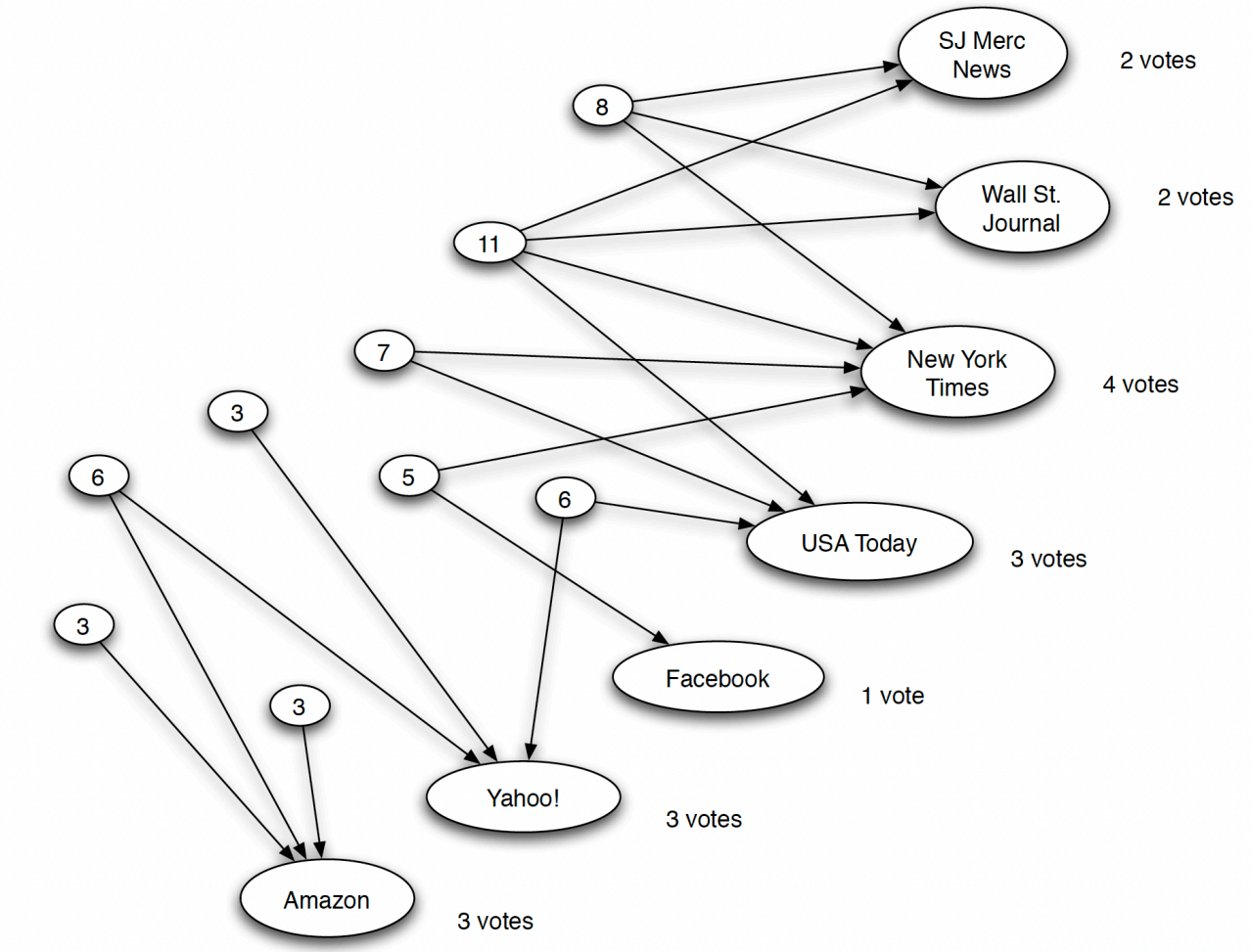
Finding lists

- query: "newspapers"
- we could have intuitively valid authority values with **in-degrees**
  - however, we get a mix of prominent newspapers along with pages that are going to receive a lot of in-links no matter what the query is
- **Lists**: pages that provide many different out-links to other pages
  - among the pages casting votes, a few of them voted for many of the pages that received a lot of votes



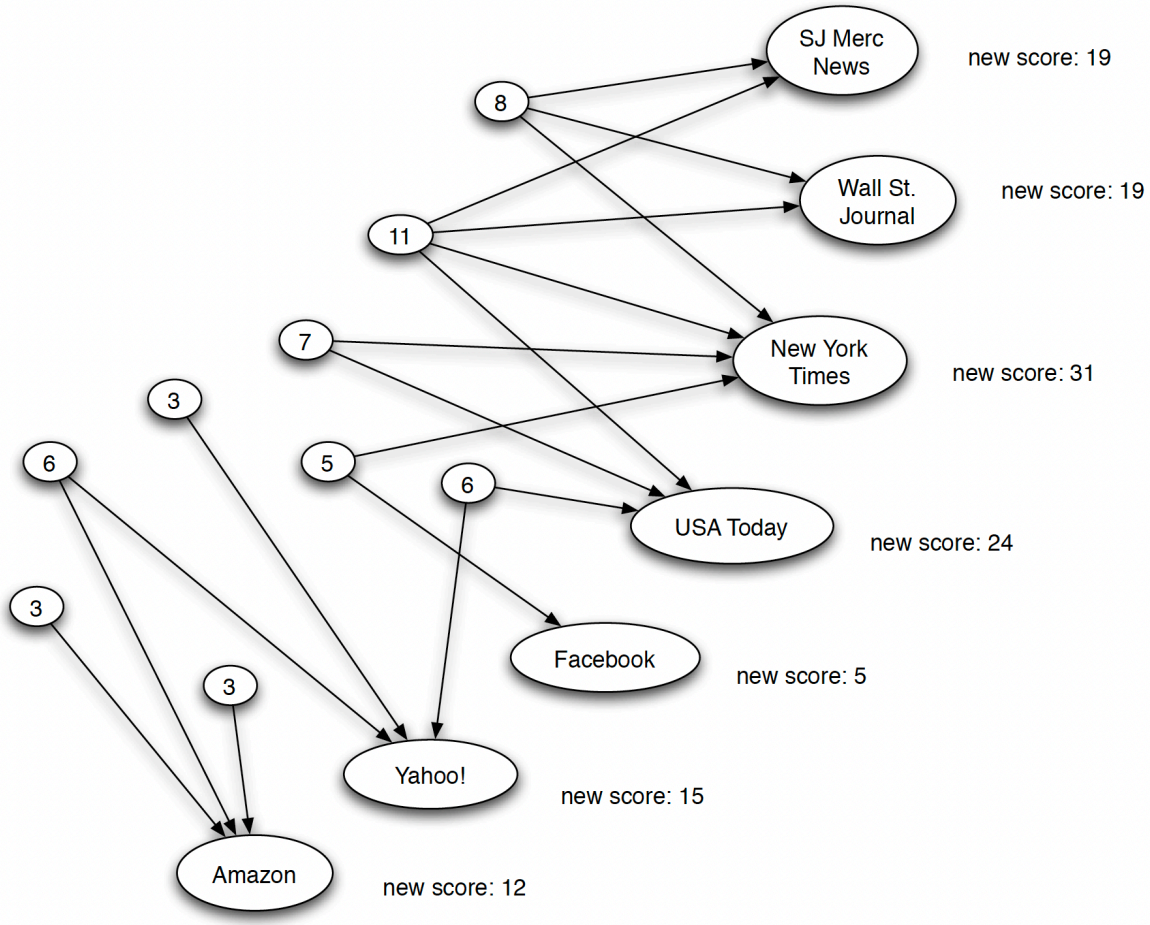
List values

- Hub value for a list page: **the sum of in-links received by all pages it linked to**
  - in other words: the sum of the votes received by all pages that it voted for
- Assumptions:
  - list pages have a better sense for where the good results are
  - authorities are often **competitors**!



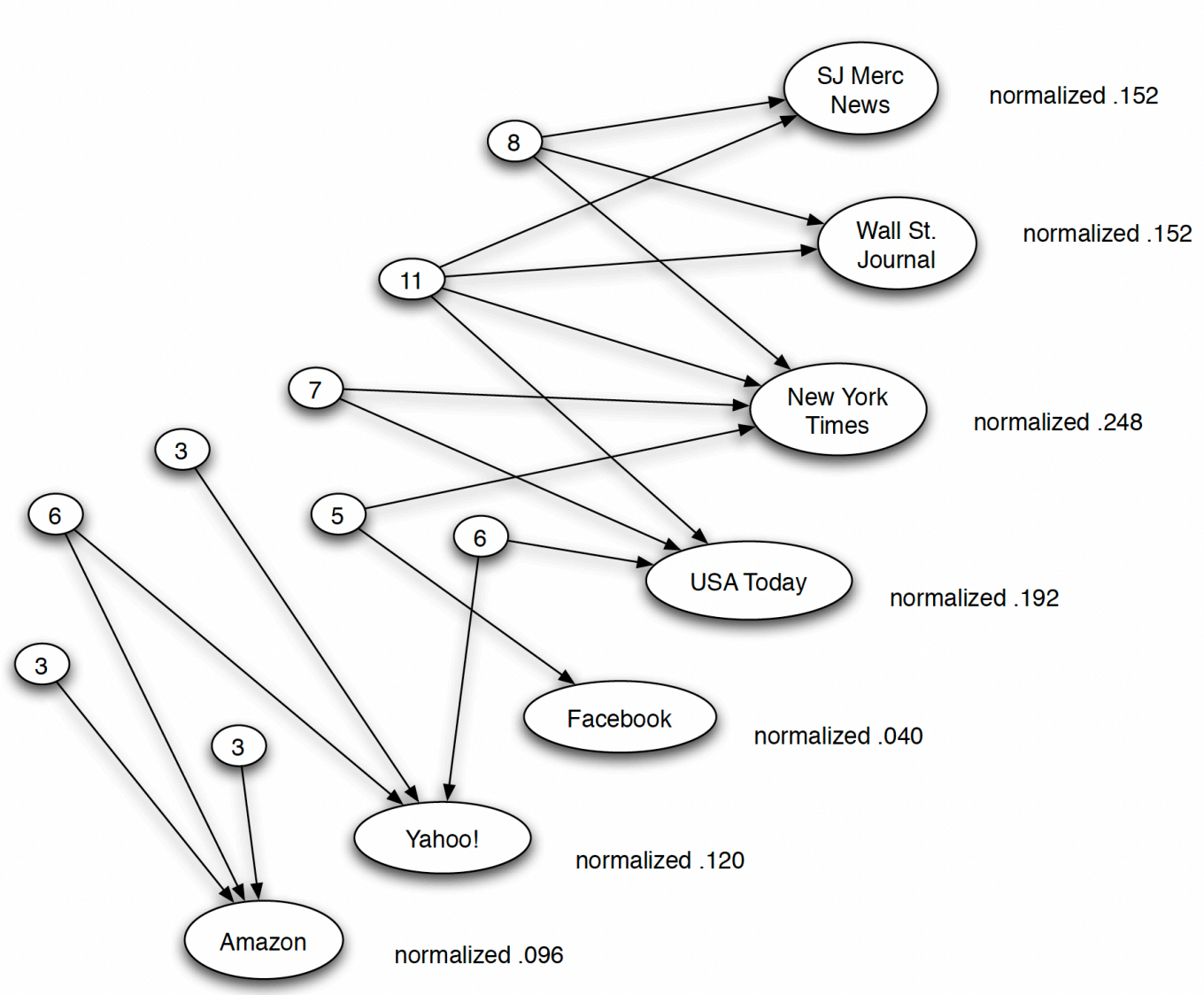
The Principle of Repeated Improvement

- If we think that high-scoring list pages (hubs) have a better sense of where good results are:
  - we can **weight links from hubs more heavily**
- Recalculate
  - the authority score is not just the number of in-links, it is the sum of the hub values of the in-link pages
- **Why stop here?** we can refine values at both sides



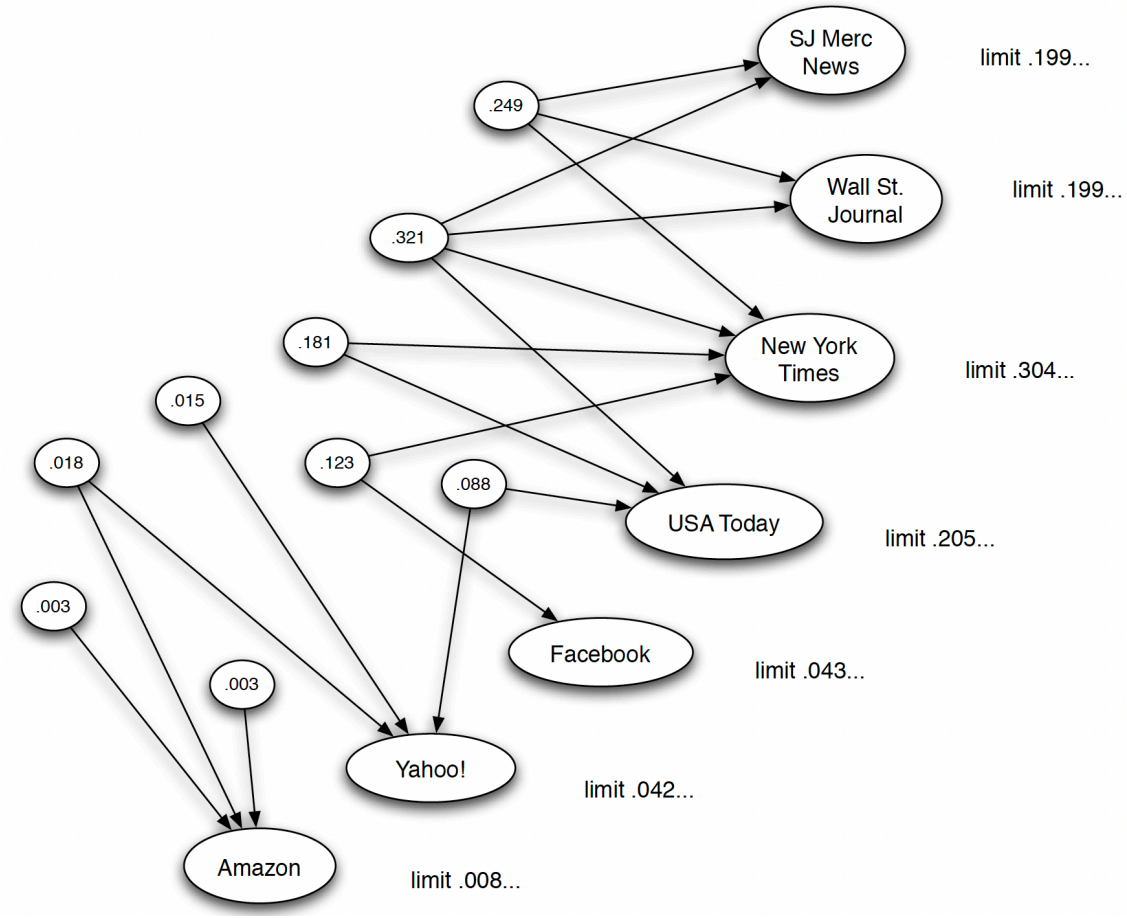
Hubs and Authorities

- Each page p has an **authority score** $auth(p)$ and **hub score** $hub(p)$ initialized to 1
- **Authority Update Rule**: for each page p, update $auth(p)$ to be the sum of the hub scores of all pages that point to it.
- **Hub Update Rule**: for each page p, update $hub(p)$ to be the sum of the authority scores of all pages that it points to.
- At step k, we performed k hub-authority updates. Each step involves:
  - First apply the **Authority Update Rule** to the current set of scores.
  - Then apply the **Hub Update Rule** to the resulting set of scores.
- Getting high values: **normalize hub and authority values** by divinding down the scores by the sum of the scores of authoritys/hubs.


SJ Merc News
normalized .152
8
Wall St. Journal
normalized .152
11
New York Times
normalized .248
7
3
6
5
6
USA Today
normalized .192
3
Facebook
normalized .040
3
Yahoo!
normalized .120
Amazon
normalized .096

Stabilization

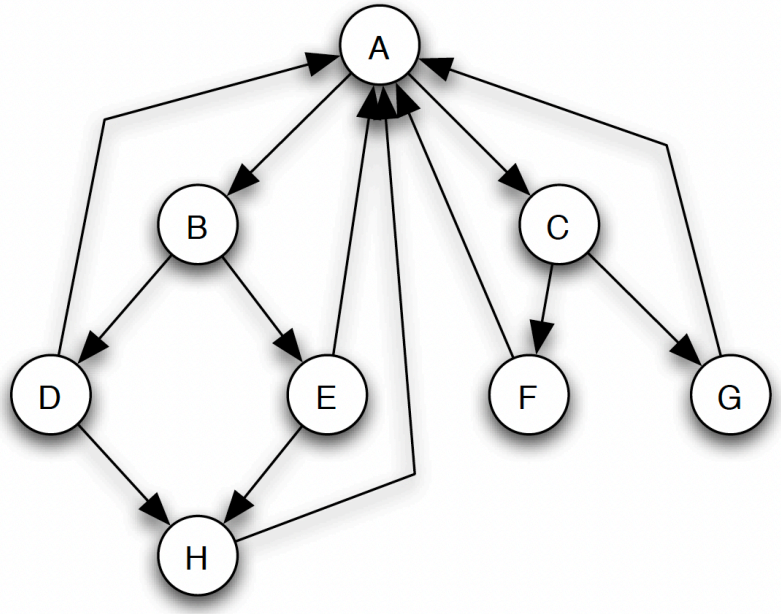
- It can be proved that normalized values **converge** when $k \to \infty$
  - **(optional proof)** see Section 14.6 Advanced Material: Spectral Analysis, Random Walks, and Web Search
- **Initial values are not important**
- Limiting values for hubs and authorities are **properties of the links structure**.
- Different form of game theoretical concept of equilibrium
  - **authority score** is proportional to the hub scores of the pages that point to you
  - **hub score** is proportional to the authority scores of the pages you point to.



Page Rank

- **Endorsement** viewed as passing directly from one important node to another
  - in other words, a page is important if it is cited by other important pages
  - endorsements received by in-links and passed across outgoing links
- Basic definition:
  - **Initilization**: Init all the pages p to a $PR(p) = \frac{1}{n}$, where n is the number of pages
  - $\boldsymbol{k^{th}}$ **step**: we perform a sequence of k updates to the PageRank values, using the following rule for each update:
  - **Basic PageRank Update Rule**:
    - each page divides its current PageRank equally across its out-going links, and passes these equal shares to the pages it points to.
    - if a page has no out-going links, it passes all its current PageRank to itself.
    - each page updates its new PageRank to be the sum of the shares it receives.

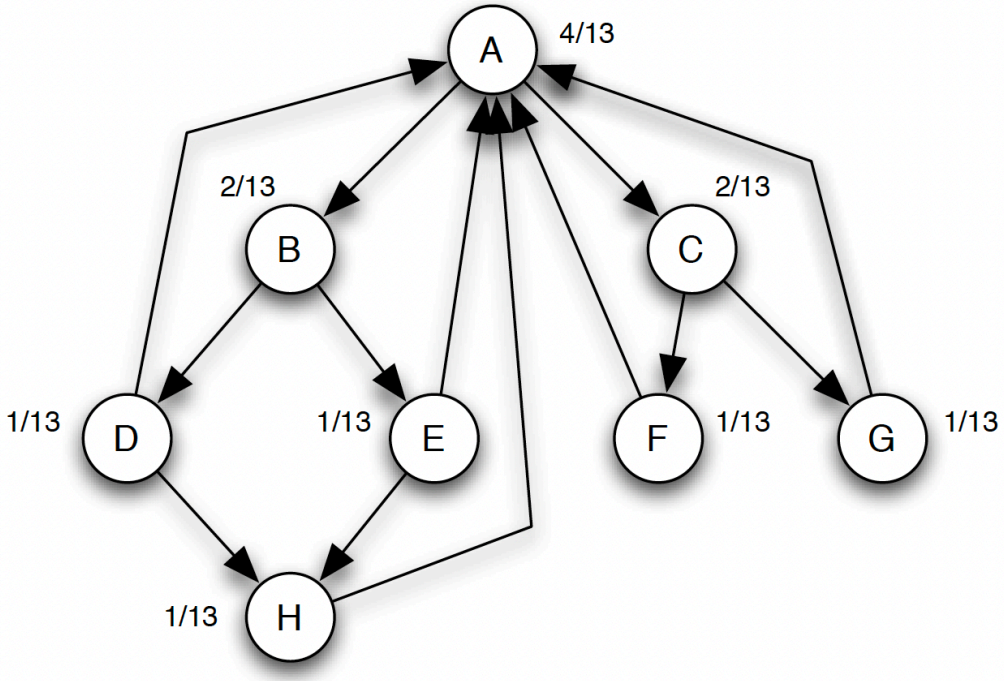
Example



Step	A	B	C	D	E	F	G	H
1	1/2	1/16	1/16	1/16	1/16	1/16	1/16	1/8
2	3/16	1/4	1/4	1/32	1/32	1/32	1/32	1/16

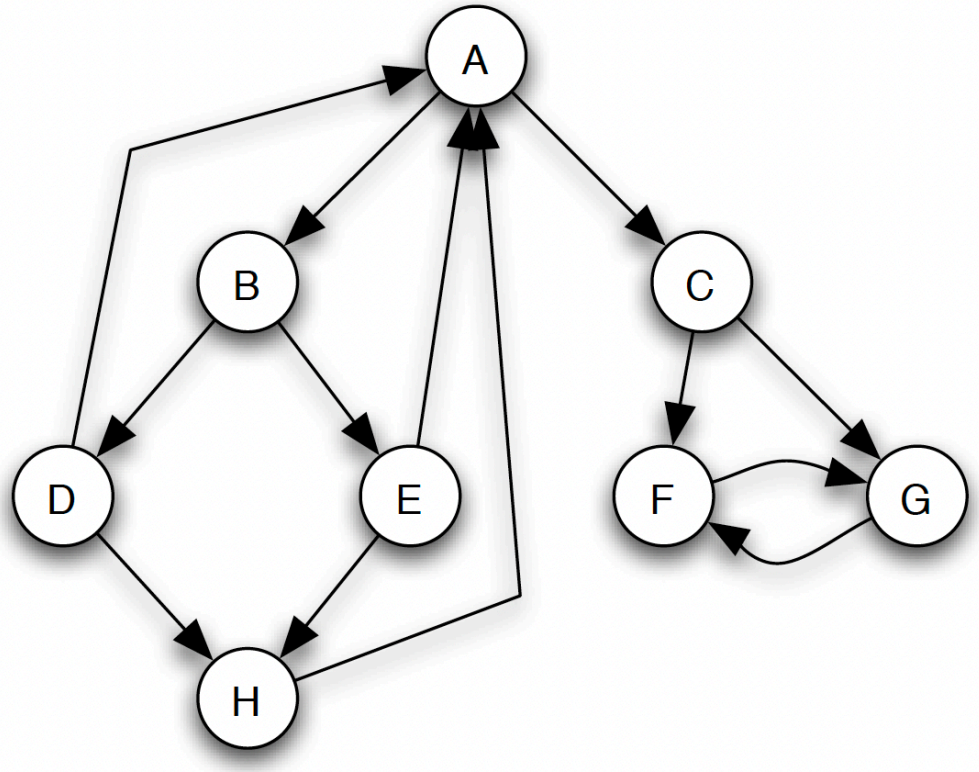
PageRank and stabilization

- PR values of all the nodes **converge** when $k \to \infty$
  - **but for some degenerate cases**.
- **Equilibrium**: if we apply our PR update rule, then our limiting **values do not change**.
- One can prove that if the network is **strongly connected**, then there is a **single set of equilibrium values** (when they exist)
  - **(optional proof)**: see Section 14.6 Advanced Material: Spectral Analysis, Random Walks, and Web Search



Scaling the definition of PageRank

- Which are these degenerate cases?
- **Problem:** in some networks some nodes receive all the PR values of the the network
  - **example on the right**: we converge to PR values of $\frac{1}{2}$ for each of F and G, and 0 for all other nodes
- Why?
  - We do not have path back to some other nodes



- **Solution:** let's force this "fluid" to stream back to other nodes "sometimes":
- Select a **scaling factor** (aka **damping factor**) s : $s \in [0, 1]$
- Get a portion s of PR values from in-links and then add $\frac{(1-s)}{n}$
  - This means that the total PR in the network has shrunk from 1 to s. We divide the residual $1 - s$ units of PR equally over all nodes, giving $\frac{(1-s)}{n}$ to each.
  - Called *Scaled PageRank Update Rule*
- Now we have convergence for $k \to \infty$
- **Note**: typically $s \in [0.8, 0.9]$

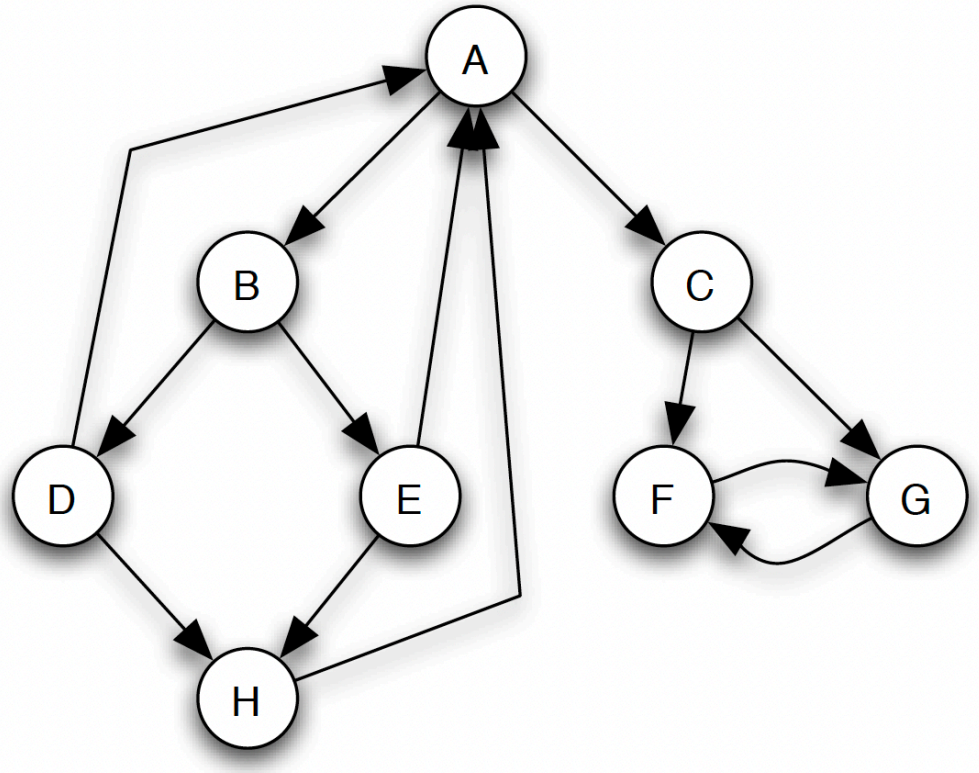
UNIVERSITÀ DI TORINO

Random walks: An equivalent definition of PageRank

- Randomly browsing a network of Web pages, navigating each page with equal probability following links.
  - start by choosing a page at random
  - pick a random out-going link from their current page, and follow it to where it leads
- Follow links for a sequence of length k
- **Claim:** the probability of being at a page x after k steps of this random walk is precisely the PR of x after k applications of the Basic PageRank Update Rule.
- **Additional intuition:** $PR(x)$ is the limiting probability that a random walk across hyperlinks will end up at x as we sum the walk for larger and larger number of steps

Leakage

- The leakage of F and G has a natural interpretation: when the surfer reaches F or G, then **it is stuck forever**.
- **Solution**:
  - with probability s, the random walker clicks on an hyperlink in the page.
  - with probability $1 - s$, it jumps to a randomly selected node.



Practical implications (also beyond the Web)

Modern Web search

- Google today use PR as one of the many features of their ranking framework
  - original Page&Brin's paper The Anatomy of a Large-Scale Hypertextual Web Search Engine.
  - e.g., Hilltop (an extension of HITS) has been probably used for a while.
- **anchor texts**
- **clicking behavior**
- and much more (and who knows what they are actually using!)

SEO vs Google

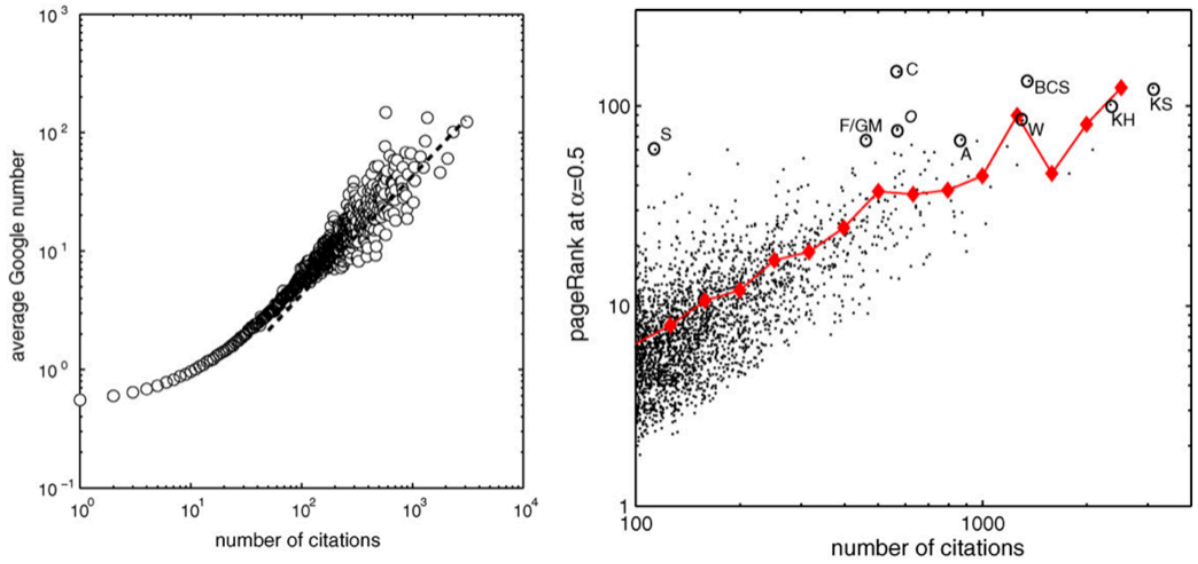
- SEO: Search Engine Optimization
- SEO companies: reverse engineering of search engine's ranking functions
- SE companies: define new measures
  - ... in loop!
- **Feedback effects**: perfect results are "moving targets"
- It is a game theoretic principle.
  - you should always expect the world to react to what you do.

UNIVERSITÀ DI TORINO

Applications beyond the Web

- Citation Analysis
- Link Analysis of U.S. Supreme Court Citations

Page Rank and citation analysis



- **Reference**: Finding Scientific Gems with Google Page Ranks (2007).
- **Dataset**: collection of scientific papers with their references
- positive correlation between number of citations and average PR values
- BUT outliers are papers with a limited number of citations but highly influential anyhow

Google rank	Google # ($\times 10^{-4}$)	Cite rank	# cites	Publication				Title	Author(s)
1	4.65	54	574	PRL	10	531	1963	Unitary symmetry and leptonic...	N. Cabibbo
2	4.29	5	1364	PR	108	1175	1957	Theory of superconductivity	J. Bardeen, L. Cooper, and J. Schrieffer
3	3.81	1	3227	PR	140	A1133	1965	Self-consistent equations...	W. Kohn and L.J. Sham
4	3.17	2	2460	PR	136	B864	1964	Inhomogeneous electron gas	P. Hohenberg and W. Kohn
5	2.65	6	1306	PRL	19	1264	1967	A model of leptons	S. Weinberg
6	2.48	55	568	PR	65	117	1944	Crystal statistics I	L. Onsager
7	2.43	56	568	RMP	15	1	1943	Stochastic problems in...	S. Chandrasekhar
8	2.23	95	462	PR	109	193	1958	Theory of the Fermi interaction	R.P. Feynman and M. Gell-Mann
9	2.15	17	871	PR	109	1492	1958	Absence of diffusion in...	P.W. Anderson
10	2.13	1853	114	PR	34	1293	1929	The theory of complex spectra	J.C. Slater

Pros: PR helps to find "gems" in networks!

Cons: Indicators can change our behaviors

Introduction to spectral analysis

- **Pre-requisites**:
  - linear algebra
  - vector and matrix multiplication
- Eigenvalues/eigenvectors calculation to study the structure of networks
  - **spectral analysis**
- Limiting values in the **Hubs/Authorities** and **PageRank** algorithms are coordinates in the eigenvectors for given eigenvalues in matrices derived from our graphs.

Definitions

Adjacency Matrix M

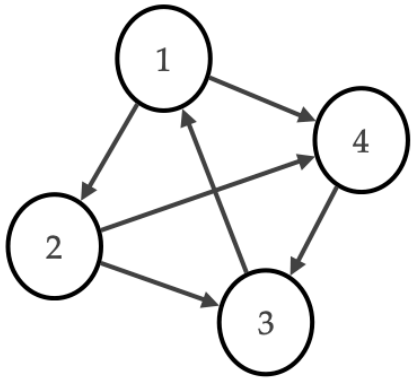
$1 \dots n$ nodes

$M : n \times n$

$$M_{ij} = \begin{cases} 1, & \texttt{if } (i,j) \texttt{ is an edge} \\ 0, & \texttt{otherwise} \end{cases}$$

The adjacency matrix is not necessarily efficient for computational representations

- but conceptually very useful
- for practical use, consider adjacency lists or edge lists instead



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Update rules of Hubs and Authorities

$\mathbf{h}$, $\mathbf{a}$: n-dimensional vectors (respectively, hub and authorities values)

Hub Update Rule

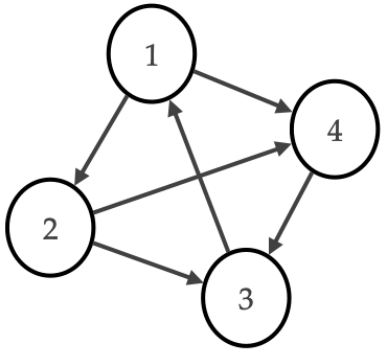
$$h_i \leftarrow \sum_{j=1}^{n} M_{ij} a_j = M_{i1} a_1 + M_{i1} a_2 + \ldots + M_{in} a_n$$

$$\mathbf{h} \leftarrow \mathbf{M} \cdot \mathbf{a}$$

Authority Update Rule

$$a_i \leftarrow \sum_{j=1}^{n} M_{ji} h_j$$

$$\mathbf{a} \leftarrow \mathbf{M^T} \cdot \mathbf{h}$$



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

$\mathbf{M}$ $\mathbf{a}$ $\mathbf{h}$

UNIVERSITÀ DI TORINO

Understanding the k-step hub-authority computation

initilization: $\mathbf{h}^{<\mathbf{0}>} = \underbrace{(1, 1, \ldots, 1)}_{n}$

after k applications of the rule: $\mathbf{a}^{<\mathbf{k}>}, \mathbf{h}^{<\mathbf{k}>}$

UNIVERSITÀ DI TORINO

Understanding the k-step hub-authority computation

First step

$$\mathbf{a}^{<1>} = \mathbf{M}^{\mathbf{T}}\mathbf{h}^{<0>}$$
$$\mathbf{h}^{<1>} = \mathbf{M}\mathbf{a}^{<1>} = \mathbf{M}\mathbf{M}^{\mathbf{T}}\mathbf{h}^{<0>}$$

Second step

$$\mathbf{a}^{<2>} = \mathbf{M}^{\mathbf{T}}\mathbf{h}^{<1>} = (\mathbf{M}^{\mathbf{T}}\mathbf{M})\mathbf{M}^{\mathbf{T}}\mathbf{h}^{<0>}$$
$$\mathbf{h}^{<2>} = \mathbf{M}\mathbf{a}^{<2>} = (\mathbf{M}\mathbf{M}^{\mathbf{T}})(\mathbf{M}\mathbf{M}^{\mathbf{T}})\mathbf{h}^{<0>} = (\mathbf{M}\mathbf{M}^{\mathbf{T}})^{\mathbf{2}}\mathbf{h}^{<0>}$$

k^{th} step

$$\mathbf{a}^{<\mathbf{k}>} = (\mathbf{M}^{\mathbf{T}}\mathbf{M})^{\mathbf{k-1}}\mathbf{M}^{\mathbf{T}}\mathbf{h}^{<0>}$$
$$\mathbf{h}^{<\mathbf{k}>} = (\mathbf{M}\mathbf{M}^{\mathbf{T}})^{\mathbf{k}}\mathbf{h}^{<0>}$$

$\mathbf{a}$, $\mathbf{h}$ vectors: multiplication of an initial vector $\mathbf{h}^{<0>}$ by larger powers of $\mathbf{M}^{\mathbf{T}}\mathbf{M}$ and $\mathbf{M}\mathbf{M}^{\mathbf{T}}$.

Multiplications and Eigenvectors (optional)

normalization: we can find constants c and d s.t. $\frac{\mathbf{h}^{<\mathbf{k}>}}{c^k}$ and $\frac{\mathbf{a}^{<\mathbf{k}>}}{d^k}$.

We want to prove that they converge for $k \to \infty$

Focus on the hub vectors:

If $\frac{\mathbf{h}^{<\mathbf{k}>}}{c^k} = \frac{(\mathbf{M}\mathbf{M}^\mathbf{T})^k \cdot \mathbf{h}^{<\mathbf{0}>}}{c^k}$ converges to a limit $\mathbf{h}^{<*>}$, then I can expect that:

$$c \cdot \mathbf{h}^{<*>} = (\mathbf{M}\mathbf{M}^\mathbf{T}) \cdot \mathbf{h}^{<*>}$$

Hence, we need to prove that the sequence of $\frac{\mathbf{h}^{<\mathbf{k}>}}{c^k}$ converges to the eigenvector of $\mathbf{M}\mathbf{M}^\mathbf{T}$

Eigenvectors and square matrices (optional)

- Observe that $\mathbf{M}\mathbf{M}^{\mathbf{T}}$ is a symmetric matrix
- Fact 1: "Any symmetric matrix $n \times n$ has a set of n eigenvectors $\mathbf{z_1}, \cdots, \mathbf{z_n}$ that are orthogonal and all unit vectors - that is they form a basis for the space $\mathbb{R}^n \implies \mathbf{z_i} \cdot \mathbf{z_j} = 0$ and $\mathbf{z_i} \cdot \mathbf{z_i} = 1$
- That means that for our symmetric $\mathbf{M}\mathbf{M}^{\mathbf{T}}$ we can find:
  - n mutual orthogonal eigenvectors: $\mathbf{z_1}, \cdots, \mathbf{z_n}$ (the spectrum of $\mathbf{M}\mathbf{M}^{\mathbf{T}}$)
  - n corresponding eigenvalues: $c_1, \cdots, c_n$
- Let's sort eigenvectors s.t. corresponding eigenvalues: $c_1 \geq c_2 \geq \cdots \geq c_n$
- Assume (for now): $c_1 > c_2$

Eigenvectors and square matrices (optional)

- Let's consider $\mathbf{x} \in \mathbb{R}^n$

$$\begin{aligned}(\mathbf{M}\mathbf{M}^\mathbf{T})\mathbf{x} &= (\mathbf{M}\mathbf{M}^\mathbf{T})(p_1\mathbf{z_1} + p_2\mathbf{z_2} + \ldots + p_n\mathbf{z_n}) \\ &= p_1(\mathbf{M}\mathbf{M}^\mathbf{T})\mathbf{z_1} + p_2(\mathbf{M}\mathbf{M}^\mathbf{T})\mathbf{z_2} + \ldots + p_n(\mathbf{M}\mathbf{M}^\mathbf{T})\mathbf{z_n} \\ &= p_1c_1\mathbf{z_1} + p_2c_2\mathbf{z_2} + \ldots + p_nc_n\mathbf{z_n}\end{aligned}$$

- We will it use this equation to analyze multiplication by larger powers of $(\mathbf{M}\mathbf{M}^\mathbf{T})$

$$(\mathbf{M}\mathbf{M}^\mathbf{T})^\mathbf{k}\mathbf{x} = p_1c_1^k\mathbf{z_1} + p_2c_2^k\mathbf{z_2} + \ldots + p_nc_n^k\mathbf{z_n}$$

UNIVERSITÀ DI TORINO

Convergence of the Hub-Authority computation (optional)

vector of hub scores at step k:

$$\mathbf{h}^{<\mathbf{k}>} = (\mathbf{M}\mathbf{M}^{\mathbf{T}})^k \cdot \mathbf{h}^{<\mathbf{0}>}$$

$$\mathbf{h}^{<\mathbf{0}>} = q_1\mathbf{z_1} + q_2\mathbf{z_2} + \ldots + q_n\mathbf{z_n}$$

$$\mathbf{h}^{<\mathbf{k}>} = c_1^k q_1\mathbf{z_1} + c_2^k q_2\mathbf{z_2} + \ldots + c_n^k q_n\mathbf{z_n}$$

Let's divide both sides by c_1^k:

$$\frac{\mathbf{h}^{<\mathbf{k}>}}{c_1^k} = \frac{c_1^k q_1\mathbf{z_1}}{c_1^k} + \frac{c_2^k q_2\mathbf{z_2}}{c_1^k} + \ldots + \frac{c_n^k q_n\mathbf{z_n}}{c_1^k}$$

assumption: $c_1 > c_2 \Rightarrow \lim_{k\to\infty}\left(\frac{c_2}{c_1}\right)^k = 0$

Then:

$$lim_{k\to\infty}\frac{\mathbf{h}^{<\mathbf{k}>}}{c_1^k} = q_1\mathbf{z_1}$$

Wrapping up (optional)

- (i) a limit in the direction of $\mathbf{z_1}$ is reached regardless of initial values of $\mathbf{h}^{<\mathbf{0}>}$:

let's suppose that $\mathbf{h}^{<\mathbf{0}>} = \mathbf{x}$ and that is a positive vector:

$$\mathbf{x} = p_1\mathbf{z_1} + p_2\mathbf{z_2} + \ldots + p_n\mathbf{z_n} \Rightarrow (\mathbf{M}\mathbf{M}^\mathbf{T})^\mathbf{k}\mathbf{x} = c_1^k p_1\mathbf{z_1} + c_2^k p_2\mathbf{z_2} + \ldots + c_n^k p_n\mathbf{z_n}$$

$$lim_{k\to\infty} \frac{\mathbf{h}^{<\mathbf{k}>}}{c_1^k} = p_1\mathbf{z_1}$$

- (ii) coefficient $p_!$ (or q_1) must be $\neq 0$: assuring that $p_1\mathbf{z_1}$ (or $q_1\mathbf{z_1}$) are non zero vectors, in the direction of $\mathbf{z_1}$

Wrapping up (optional)

- (iii) relax assumption: $c_1 > c_2$

in general we can have $l > 1$ eigenvalues s.t. $c_1 = c_2 = \ldots = c_l$ until we find that $c_1 > c_{l+1}$

$$\frac{\mathbf{h}^{<\mathbf{k}>}}{c_1^k} = \frac{c_1^k q_1 \mathbf{z_1} + c_2^k q_2 \mathbf{z_2} + \ldots + c_l^k q_l \mathbf{z_l}}{c_1^k} + \frac{c_{l+1} n^k q_{l+1} \mathbf{z_{l+1}} + \ldots\ldots + c_n^k q_n \mathbf{z_n}}{c_1^k}$$

$$= q_1 \mathbf{z_1} + q_2 \mathbf{z_2} + \ldots + q_l \mathbf{z_l} + 0$$

with $k \to \infty$ is still a convergence

- (iv) authority values: the argument is very similar to hub values (multiplication by $\mathbf{M^T M}$)

Spectral Analysis of Page Rank

At step 0 (init):

$\forall i : r_i = \frac{1}{n}$; $\texttt{n: \# pages}\ r_i = PR(i)$

At step k:

$\forall i : r_i = \sum_{j=1}^{n} M_{ji} \frac{r_j}{k_j^{\texttt{out}}}$ (basic PR update rule)

$\forall i : r_i = s \cdot \sum_{j=1}^{n} M_{ji} \frac{r_j}{k_j^{\texttt{out}}} + (1 - s) \cdot \frac{1}{n}$ (scaled PR update rule)

Using matrix notation

$\mathbf{N}$: Matrix derived from M

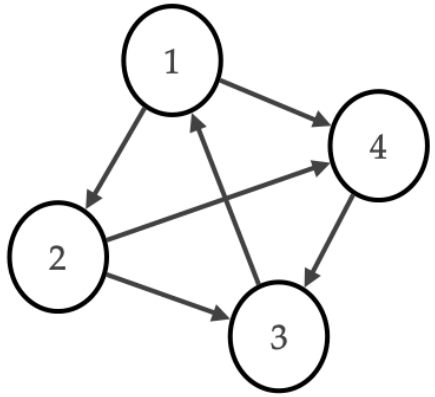
nodes: $1, \ldots, n$

$\mathbf{N} : n \times n$

$$N_{ij} = \begin{cases} \frac{1}{k_i^{\text{out}}}, & \texttt{if } (i,j) \\ 1, & \texttt{if } (i == j) \texttt{ and } k_i^{\text{out}} = 0 \\ 0, & \texttt{otherwise} \end{cases}$$

$\boldsymbol{N_{ij}}$: the share of i's PR that j should get in one update step

If i has no outgoing links, then we define $N_{ii} = 1$



$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

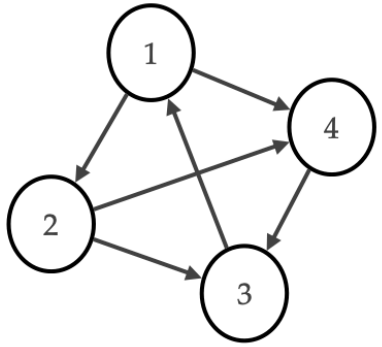
Update rule (basic and scaled)

1.Basic update rule:

$$\forall i : r_i = \sum_{j=1}^{n} N_{ji} r_j \leftarrow N_{1i} r_1 + N_{2i} r_2 + \ldots + N_{1n} r_n \mathbf{r} \leftarrow \mathbf{N^T} \cdot \mathbf{r}$$

2.Scaled update rule (factor s):

$$\widetilde{N}_{ij} = s \cdot N_{ij} + (1 - s) \cdot \frac{1}{n}$$



$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

UNIVERSITÀ DI TORINO

3.Application of scaled update rule:

$$\forall i : r_i = \sum_{j=1}^{n} \widetilde{N}_{ji} r_j \leftarrow \widetilde{N}_{1i} r_1 + \widetilde{N}_{2i} r_2 + \ldots + \widetilde{N}_{1n} r_n \mathbf{r} \leftarrow \widetilde{\mathbf{N}}^{\mathbf{T}} \cdot \mathbf{r}$$

Repeated improvement (optional)

$\mathbf{r}^{<\mathbf{0}>} = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$, initial PR vector

$\mathbf{r}^{<\mathbf{k}>} = \left(\widetilde{\mathbf{N}}^{\mathbf{T}}\right)^{k} \cdot \mathbf{r}^{<\mathbf{0}>}$

Limiting vector $r^{<*>}$ satisfies $\widetilde{\mathbf{N}}^{\mathbf{T}} \cdot \mathbf{r}^{<*>} = 1 \cdot \mathbf{r}^{<*>}$

$\mathbf{r}^{<*>}$ should be an eigenvector of $\widetilde{\mathbf{N}}^{\mathbf{T}}$ with corresponding eigenvalue of 1

Problem: $\widetilde{\mathbf{N}}^{\mathbf{T}}$ is not symmetric

- this means that eigenvalues can be complex numbers and eigenvectors have no relationships to one another

Convergence of the scaled PR update rule (optional)

$\forall i, j : \widetilde{N}_{ij} > 0$

Perron's theorem

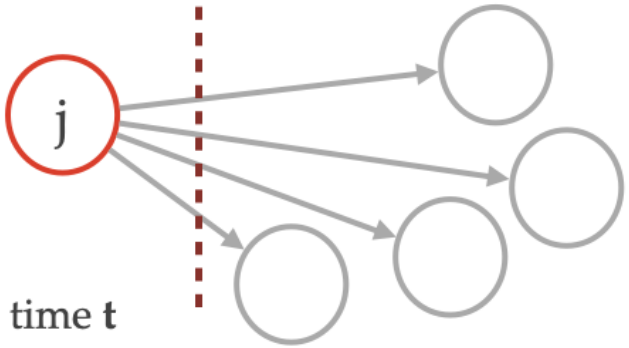
Matrix $\mathbf{P}$ (with entries > 0)

i) $\mathbf{P}$ has an eigenvalue $c > 0$ s.t. $c > c' \; \forall c'$ (with c' another eigenvalue)

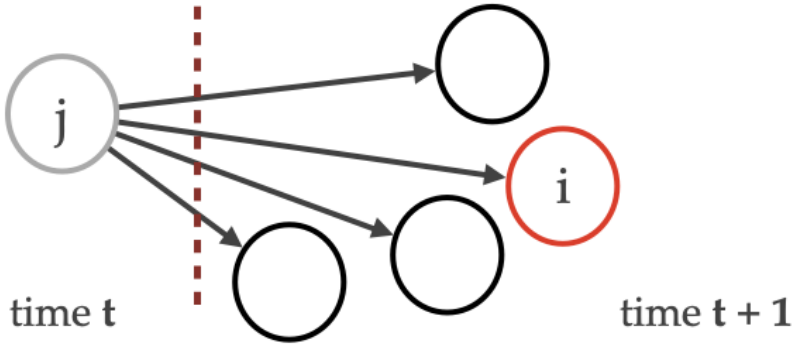
ii) Exists an eigenvector $\mathbf{y}$ with real positive values corresponding to c, and $\mathbf{y}$ is unique (up to a multiplication constant)

iii) if $c = 1$, then for any starting vector $\mathbf{x} \neq \mathbf{0}$ with non negative coordinates, the sequence of vectors $p^k \mathbf{x}$ converges to a vector in the direction of $\mathbf{y}$ ($k \to \infty$)

Formulation of the PageRank using Random Walks



Formulation of the PageRank using Random Walks



Which is the probability of being at node i at time $t+1$?

$b_1, b_2, \ldots, b_n$: the probabilities of being at node i in a given step.

$b_i \leftarrow \sum_{j=1}^{n} M_{ji} \frac{b_j}{k_j^{out}}$: the probability of being at node j in the following step

Using matrix $\mathbf{N}$: $b_i \leftarrow N_{1i}b_1 + N_{2i}b_2 + \ldots + N_{1n}b_n \Rightarrow \mathbf{b} \leftarrow \mathbf{N^T} \cdot \mathbf{b}$

claim: PR of page i is exactly the probability of being at node i after k step.

A scaled version of the random walk

For a given probability s: the walker follows a random outgoing edge

With prob $(1-s)$: the walker is teleported uniformly at random to another node

$$b_i \leftarrow s \cdot \sum_{j=1}^{n} M_{ji} \frac{b_j}{k_j^{out}} + \frac{(1-s)}{n}$$

Using matrix:

$$\widetilde{N} : b_i \leftarrow \widetilde{N}_{1i} b_1 + \widetilde{N}_{2i} b_2 + \ldots + \widetilde{N}_{1n} b_n \Rightarrow \mathbf{b} \leftarrow \widetilde{\mathbf{N}}^{\mathbf{T}} \cdot \mathbf{b}$$

claim: PR is equivalent to the scaled version of random walks.

UNIVERSITÀ DI TORINO

Reading material

[ns2] **Chapter 13** The Structure of the Web

[ns2] **Chapter 14 (14.1-14.5)** Link Analysis and Web Search

[ns1] **Chapter 4** [Directions and Weights]

UNIVERSITÀ DI TORINO

Q&A