# Analisi e Visualizzazione delle Reti Complesse

## NS09 - Preferential Attachment Models

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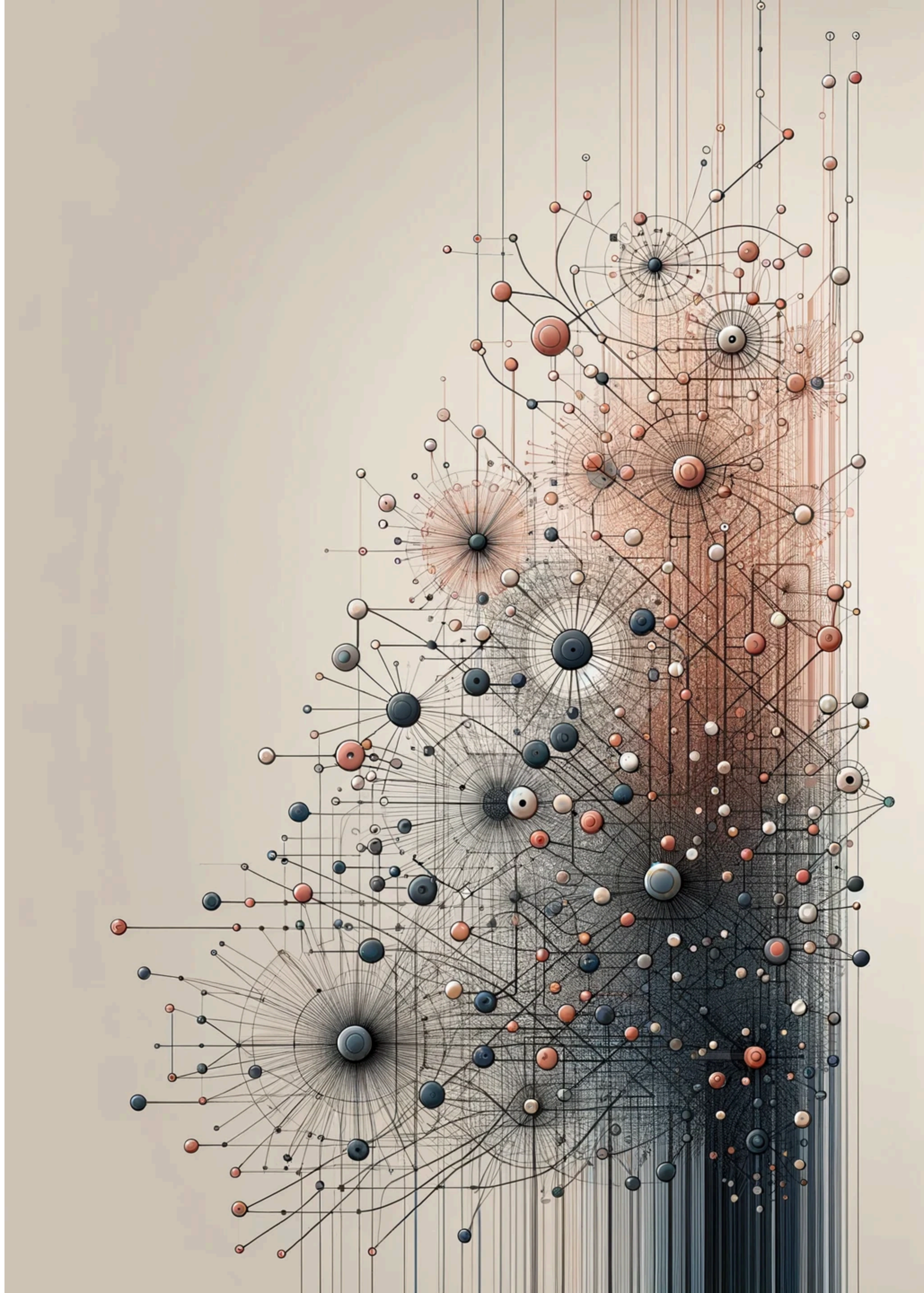


# Outline

- Preferential attachment
- Barabási-Albert model
- Other preferential models
  - Attractiveness model
  - Fitness model
  - Random walk model
  - Copy model

# Network growth

**Note:** real-world networks are **dynamic**!

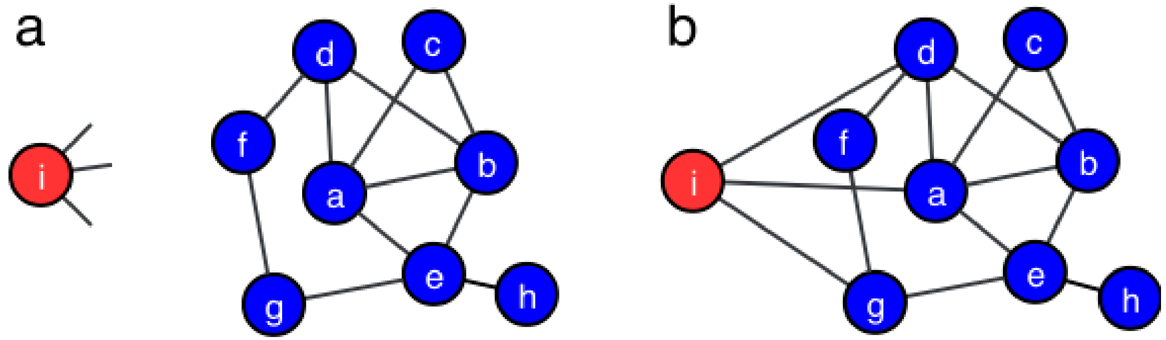
Examples:

- The Web in 1991 had a single node, today there are trillions
- Citation networks of scientific articles and collaboration networks of scientists keep growing due to the publication of new papers
- The collaboration network of actors keeps growing due to the release of new movies
- The protein interaction network has been growing for over 4 billion years, from a few genes to over 20,000

## Network growth

**General procedure:**

1. A new node comes with a given number of **stubs**, indicating the number of future neighbors of the node (degree)
2. The **stubs are attached** to some of the old nodes, according to some rule



## Preferential attachment

**Note: Nodes prefer to link to the more connected nodes**

**Examples:**

- Our knowledge of the Web is biased towards **popular pages**, which are highly linked, so it is more likely that our website points to highly linked Web sites
- Scientists are more familiar with **highly cited papers** (which are often the most important ones), so they will tend to cite them more often than poorly cited ones in their own papers
- The more movies an actor makes, the more popular they get and the higher the chances of being cast in a new movie

# Which model?

Our network model should have the following features:

- **Growth: The number of nodes grows** in time after adding new nodes.
  - The models considered so far are static.
- **Preferential attachment: new nodes tend to be connected to the more connected nodes**.
  - The models considered so far set links among pairs of random nodes, regardless of their degree

> "For to every one who has will more be given, and he will have abundance; but from him who has not, even what he has will be taken away" — Gospel of Matthew 25:29

**Take-home message: the rich get richer, and the poor get poorer**

[Example in NetLogo]

## Polya's urn model

**Start:** an urn contains $X$ white and $Y$ black balls

**Process:** a ball is drawn from the urn and put back in with another ball of the same color

**Example:** if we first pick a white ball, there will be $X + 1$ white and $Y$ black balls in the urn; white will become more likely to be picked than black in the future

- **Preferential attachment used to explain heavy-tail distributions of many quantities:**
  - the number of species per genus of flowering plants,
  - the number of (distinct) words in a text,
  - the populations of cities,
  - individual wealth,
  - scientific production,
  - citation statistics, firm size, and many others!

# The Barabási-Albert model

**Procedure:**

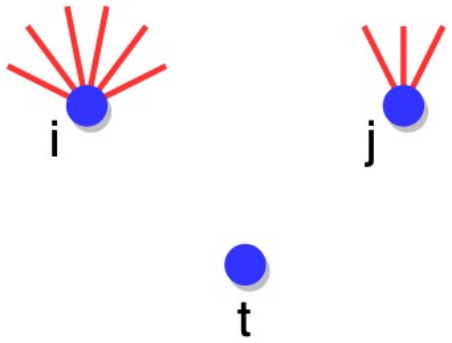
- Start with a group of $m_0$ nodes, usually fully connected (clique)
- At each step a new node $i$ is added to the system, and sets $m$ links with older nodes ($m \leq m_0$)
- The probability that the new node $i$ chooses an older node $j$ as a neighbor is **proportional to the degree $k_j$ of $j$:**

$$\prod(i \leftrightarrow j) = \frac{k_j}{\sum_l k_l}$$

- The procedure ends when the given number $N$ of nodes is reached

## The Barabási-Albert model

**Example:** if $t$ has to choose between node $i$, with degree 6, and node $j$, with degree 3, the probability of choosing $i$ is twice the probability of choosing $j$

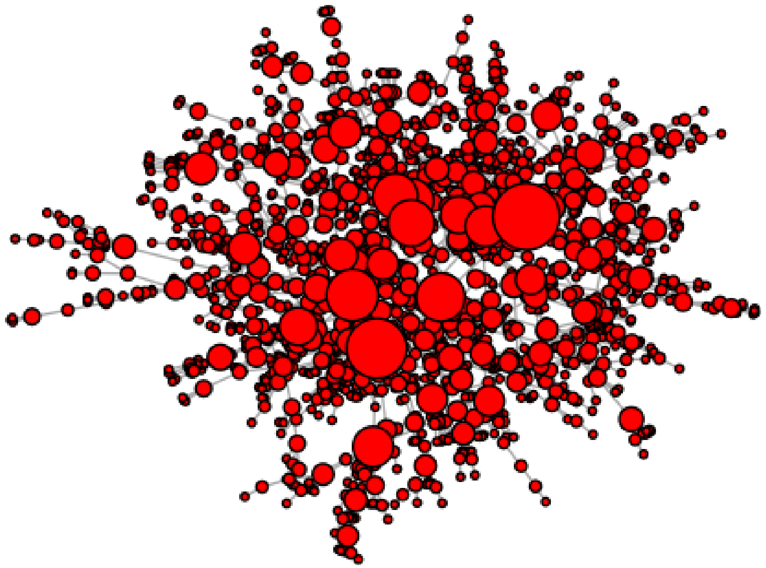


## The Barabási-Albert model

- **Rich-gets-richer phenomenon:** due to preferential attachment, the more connected nodes have higher chances to acquire new links, which gives them a bigger and bigger advantage over the other nodes in the future
- **This is how hubs are generated**

In NetworkX:

```
G = nx.barabasi_albert_graph(N, m) # BA model network
```



## The Barabási-Albert model

**Hubs are the oldest nodes:** they get the initial links and acquire an advantage over the other nodes, which increases via preferential attachment

**Question:** if old nodes have an advantage over newer nodes anyway, do we need preferential attachment at all? Can we explain the existence of hubs just because of growth?

**Alternative model:** each new node chooses its neighbors at random, not with probability proportional to their degree

# Key properties of the Barabási-Albert model

The Barabási-Albert model generates networks with three key characteristics:

1. **Scale-free (power-law) degree distribution**
    - Follows $P(k) \sim k^{-3}$
    - Creates hub nodes with very high degree
    - Heavy-tailed distribution unlike random or Watts-Strogatz networks
2. **Small-world property**
    - Average path length scales logarithmically with network size: $\ell \sim \ln N / \ln \ln N$
    - Even shorter paths than random networks of same size and density
    - Hubs act as "super-connectors" creating shortcuts through the network
    - Efficient information transfer despite preferential growth

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## Key properties of the Barabási-Albert model

3. **Low clustering**
    - Clustering coefficient $C \sim N^{-0.75}$ (decreases with network size)
    - Much lower than real social networks and Watts-Strogatz model
    - Approaches zero for large networks

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# Low clustering in the Barabási-Albert model

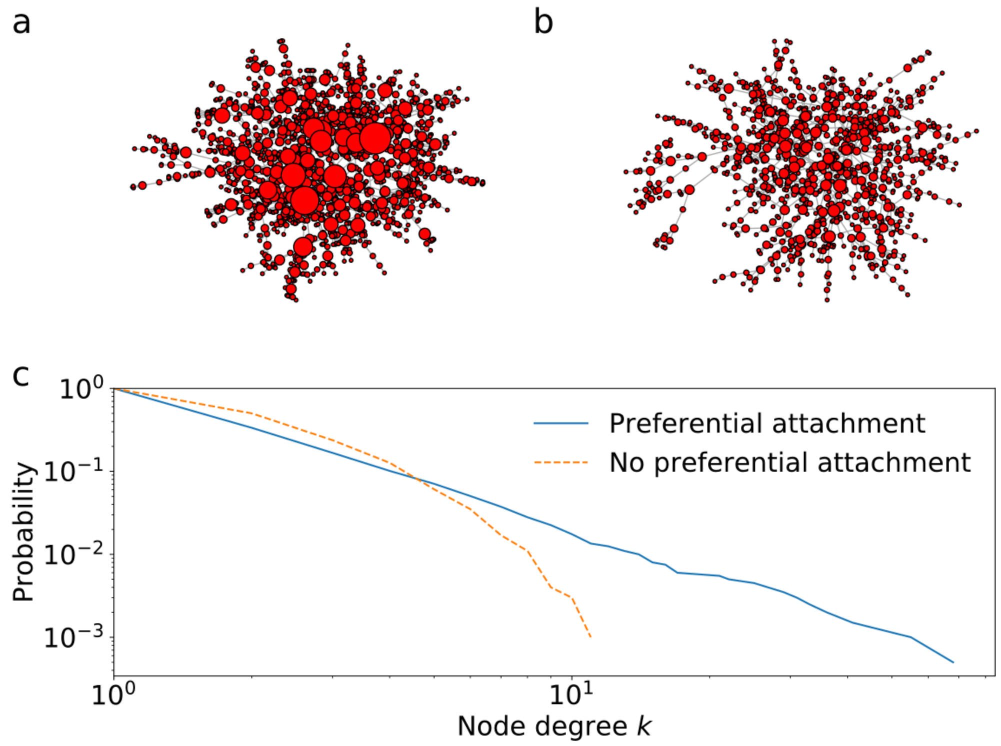
**Why does the BA model generate networks with low clustering?**

- **Tree-like growth:** New nodes typically connect to existing nodes that aren't connected to each other
    - This creates a tree-like structure without many triangles
- **No mechanism for triangle formation:** The model has no explicit mechanism to close triangles
    - Preferential attachment only considers degree, not network proximity
- **Hub-centric connections:** When new nodes connect to hubs, these connections rarely form triangles
    - The neighbors of a hub are typically not connected to each other

# The Barabási-Albert model

**Conclusion: growth + random attachment does not** generate **hubs**.

**Preferential attachment is necessary!**



# Final Remarks

## Network models: summary comparison

| | | | |
|---|---|---|---|
| **High clustering** | ❌ Low | ✅ High | ❌ Low |
| **Small-world** | ✅ Yes | ✅ Yes | ✅ Yes |
| **Scale-free** | ❌ No | ❌ No | ✅ Yes |

### Comparison of models:

- **Random networks:** Generate short paths but lack clustering and hubs
- **Watts-Strogatz:** Captures both high clustering and short paths, but lacks hubs
- **Barabási-Albert:** Creates scale-free networks with hubs and short paths, but low clustering

# Other preferential models

# Non-linear preferential attachment

**Procedure:**

- Start with a group of $m_0$ nodes, usually fully connected (clique)
- At each step a new node $i$ is added to the system, and sets $m$ links with older nodes ($m \leq m_0$)
- The probability that the new node $i$ chooses an older node $j$ as neighbor is **proportional to the power $\alpha$ of the degree $k_j$ of** $j$:

$$\prod(i \leftrightarrow j) = \frac{k_j^\alpha}{\sum_l k_l^\alpha}$$

- The procedure ends when the given number $N$ of nodes is reached

## Non-linear preferential attachment

$$\prod(i \leftrightarrow j) = \frac{k_j^\alpha}{\sum_l k_l^\alpha}$$

For $\alpha = 1$ we recover the linear preferential attachment (BA model)

**Question:** what happens when $\alpha \neq 1$?

**Answer:** it depends on whether $\alpha > 1$ or $\alpha < 1$

# Non-linear preferential attachment

- For $\alpha < 1$, the link probability does not grow fast enough with degree, so the advantage of high-degree nodes over the others is not as big.
    - As a result, the degree distribution does not have a heavy tail: **the hubs disappear!**
- If $\alpha > 1$, high-degree nodes accumulate new links much faster than low-degree nodes.
    - As a consequence, one of the nodes will end up being connected to a fraction of all other nodes.
    - For $\alpha > 2$, a single node may be connected to all other nodes (**winner-takes-all effect**), all other nodes having low degree
- **Conclusion: non-linear preferential attachment fails to generate hubs**
    - Linear preferential attachment is the only way to go
- **Problem: strict proportionality of linking probability to degree appears unrealistic**

## Limits of preferential attachment

- It yields a **fixed pattern for the degree distribution**: the slope is the same for any choice of the model parameters.
  - Degree distributions in real-world networks could decay faster or more slowly
- **The hubs are the oldest nodes**: new nodes cannot overcome their degree
- **It does not create many triangles**: the average clustering coefficient is much lower than in many real-world networks
- **Nodes and links are only added**: in real networks they can also be deleted
- Since each node is attached to older nodes, **the network consists of a single connected component**.
  - Many real-world networks have **multiple components**

## Extensions of the BA model: Attractiveness model

- **Pitfall of preferential attachment**: What happens if a node has no neighbors (degree zero)?
    - **It will never get connections from other nodes!**
- **No problem for standard initial condition**: the initial subgraph is complete (clique), so every node has nonzero degree
- **What if the network is directed and the linking probability is proportional to the in-degree?**
    - Bad, as each new node has in-degree zero, so it will never be linked by future nodes!

# Extensions of the BA model: Attractiveness model

**Procedure:**

- Start with a group of $m_0$ nodes, usually fully connected (clique)
- At each step a new node $i$ is added to the system, and sets $m$ links with some of the older nodes $(m < m_0)$
- The probability that the new node $i$ chooses an older node $j$ as neighbor is proportional to the sum of the degree $k_j$ of $j$ and an attractiveness $A$, indicating the intrinsic appeal of $j$:

$$\prod(i \leftrightarrow j) = \frac{A + k_j}{\sum_l (A + k_l)}$$

**Key features:**

- **The attractiveness parameter $A$:**
    - Works as a "degree offset" or initial attractiveness
    - Ensures every node has a non-zero chance of receiving links, solving the "zero-degree problem"
    - Makes degree differences less important when $A$ is large (more uniform attachment)
- **Mathematical behavior:**
    - For very large $A \gg \langle k \rangle$: Approaches uniform random attachment
    - For $A = 0$: Recovers the original BA model
    - For $0 < A < \infty$: Generates power-law degree distributions with exponent $\gamma = 3 + A/m$
- **Applications:**
    - Models citation networks where new papers have intrinsic value but no citations
    - Web page networks where new pages have visibility before gaining links
    - Social networks where individuals have baseline popularity

**Consequences and insights:**

- **Flexibility in fitting real data:**
  - Can fit networks with power-law exponents $\gamma > 3$ (unlike standard BA where $\gamma = 3$)
  - The attractiveness parameter provides a tuning knob for the degree distribution
- **Network evolution dynamics:**
  - Reduces "first-mover advantage" - newer nodes can still gain significant connectivity
  - Creates more balanced competition between nodes of different ages
  - Emergence of hubs is less dramatic than in the standard BA model
- **Variations:**
  - Class-based attractiveness: Different node types have different baseline attractiveness
  - Time-dependent attractiveness: A node's intrinsic appeal can decay or grow over time

# Extensions of the BA model: Fitness model

**Pitfall of preferential attachment:** the hubs are the oldest nodes. **Unrealistic!**

**Examples:**

- In the Web, new pages can overrun old pages (e.g., Google!)
- In science, new papers can be more successful than (many) old papers

**Reason:** each node has its own individual appeal

# Extensions of the BA model: Fitness model

**Procedure:**

- Start with a group of $m_0$ nodes, usually fully connected (clique)
- At each step a new node $i$ is added to the system, and sets $m$ links with some of the older nodes $(m \leq m_0)$
- The probability that the new node $i$ chooses an older node $j$ as neighbor is proportional to the product of the degree $k_j$ of $j$ with a fitness $\eta_j$, indicating the intrinsic appeal of $j$:

$$\prod(i \leftrightarrow j) = \frac{\eta_j \cdot k_j}{\sum_l (\eta_l \cdot k_l)}$$

## Key features:

- **Node-specific fitness values $\eta_j$:**
  - Represents intrinsic quality or attractiveness of each node
  - Usually drawn from some probability distribution $\rho(\eta)$
  - Allows young nodes with high fitness to overcome age disadvantage
  - Creates a "meritocratic" component to network growth
- **Mathematical formulation:**
  - Multiplicative effect (unlike additive effect in Attractiveness model)
  - Results in "preferential attachment with fitness" where both connectivity and quality matter
- **Analytical results:**
  - The model can produce power-law degree distributions with a fitness-dependent exponent
  - Can lead to "winner-takes-all" effects when fitness distribution has certain properties
  - Introduces the concept of dynamic monopoly: a high-fitness node can eventually capture a finite fraction of all links

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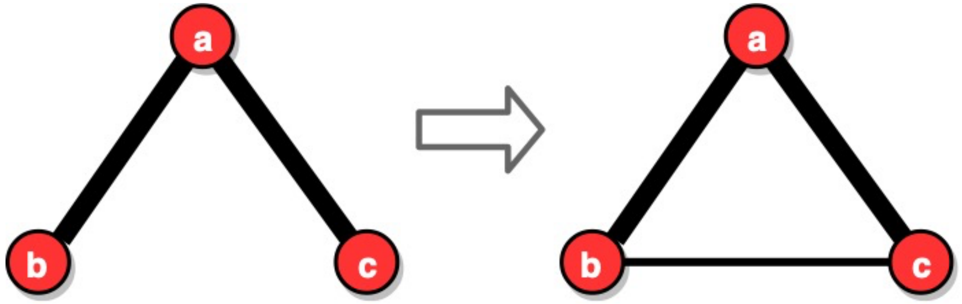
## Consequences and real-world applications:

- **Breaking the age barrier:**
  - Allows young nodes with high fitness to become hubs ("Google effect")
  - Creates more realistic scenarios where quality matters alongside age
- **Fitness distribution effects:**
  - Uniform fitness: More equal competition, but still power-law distribution
  - Heavy-tailed fitness: Extreme domination by a few super-fit nodes
  - Bounded fitness: More balanced competition with power-law degree distribution
- **Real-world interpretations of fitness:**
  - Scientific papers: Intrinsic quality/importance
  - Websites: Usefulness/relevance to users
  - Social networks: Inherent sociability/popularity of individuals
  - Business networks: Innovation capability of companies

# Extensions of the BA model: Random walk model

- **Pitfall of preferential attachment**: the BA model does not generate many triangles. Why?
- To close a triangle we need to set a link between two neighboring nodes, whereas in the BA model links are set based on degree, regardless of whether the future neighbors have common neighbors

**Solution**: introduce a mechanism for triadic closure in the model!



# Extensions of the BA model: Random walk model

**Procedure:**

- Start with a group of $m_0$ nodes, usually fully connected (clique)
- At each step a new node $i$ is added to the system, and sets $m$ links with some of the older nodes $(1 < m \leq m_0)$
- The first link targets a randomly chosen node $j$
- From the second link onwards:
  - With probability $p$ the link is set with a neighbor of $j$, chosen at random
  - With probability $1{-}p$ the link is set with a randomly chosen node

**Key features:**

- **Triadic closure mechanism:**
  - Creates triangles in the network, unlike standard BA model
  - Parameter $p$ controls the strength of triadic closure tendency
  - Mimics real-world connection formation through mutual friends/references
- **Mathematical behavior:**
  - Effective preferential attachment emerges naturally
  - Higher $p$ values lead to higher clustering coefficients
  - Still maintains scale-free degree distribution despite different mechanism
- **Network properties:**
  - High clustering coefficient (increases with $p$)
  - Community structure emerges for high $p$ values
  - Power-law degree distribution (implicit preferential attachment)
  - Small-world characteristics

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## Mechanism and implications:

- **Implicit preferential attachment:**
  - High-degree nodes have more neighbors, making them more likely endpoints in a random walk
  - This creates an effective preferential attachment without explicit degree calculations
  - Demonstrates how simple local rules can lead to complex global structures
- **Community formation:**
  - As $p$ approaches 1, dense local clusters form naturally
  - These communities are connected by fewer inter-community links
  - Models how social networks develop community structure
- **Applications:**
  - Social networks: People meet through existing friends
  - Academic networks: Researchers find papers through references of papers they've read
  - Information networks: Content discovery through related content
  - Geographic networks: Spatial proximity influences connection patterns

# Extensions of the BA model: Copy model

**Motivation:** the authors of a new web page tend to copy the hyperlinks of other pages on the same topic

**Steps:**

- **Growth:** at each time step a new node $i$ is added to the network
- **Target selection:** a node $j$ is selected at random
- **Random connection:** with probability $p$ the new node is connected to $j$
- **Link copying:** with probability $1-p$ the new node is connected to a neighbor of $j$, chosen at random

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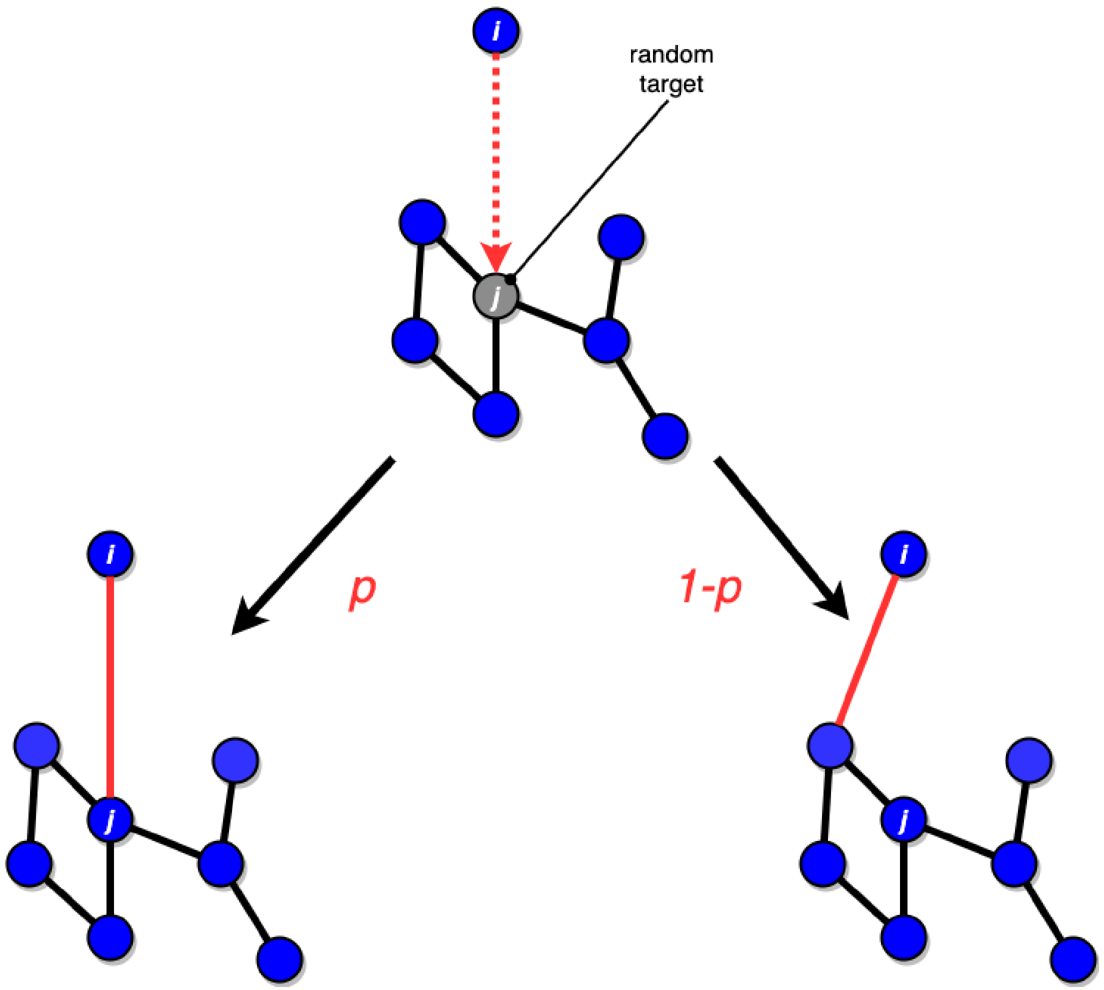
## Copy model

**Difference from random walk model:**

here we do not link to the target and to its neighbor (no triadic closure)!



**Key features:**

- **Link copying mechanism:**
  - Models how new entities often imitate existing ones
  - Parameter $p$ controls balance between innovation (new links) and imitation (copied links)
  - Creates implicit preferential attachment through copying
- **Mathematical properties:**
  - Generates power-law degree distribution with exponent $\gamma = (2 - p)/(1 - p)$
  - Tunable exponent based on copying probability $p$
  - Creates higher clustering than BA model but lower than Random Walk model
- **Network characteristics:**
  - Scale-free degree distribution
  - Moderate clustering coefficient
  - Community structure emerges based on copying patterns
  - Short average path lengths (small-world property)

**Applications and variations:**

- **Web page creation:**
  - New websites often copy links from existing sites with similar content
  - Explains why web graph has both scale-free structure and community organization
- **Biological networks:**
  - Gene duplication: When genes duplicate, the new gene initially interacts with the same proteins
- **Content networks:**
  - Social media content sharing with partial copying of formats/references
  - Academic citation patterns where authors reference similar sources as existing papers
- **Variations:**
  - Partial copy model: Only a fraction of links are eligible for copying
  - Weighted copy model: Copying probability depends on link strength
  - Aging copy model: Older nodes are less likely to be selected as targets

# Comparison of Preferential Attachment Extensions

| Model | Key Mechanism | Degree Distribution | Clustering |
|---|---|---|---|
| **BA (original)** | Linear PA | Power-law ($\gamma = 3$) | Low |
| **Attractiveness** | Degree offset | Power-law ($\gamma > 3$) | Low |
| **Fitness** | Node quality | Power-law (variable $\gamma$) | Low |
| **Random Walk** | Triadic closure | Power-law | High |
| **Copy** | Link imitation | Power-law (tunable $\gamma$) | Medium |

## Extensions of BA Model: Comparative Analysis

### Which model best replicates real-world networks?

- **Random Walk Model:**
    - **Best for social networks** where triangle formation is crucial
    - High clustering coefficient matches observed patterns in human networks
    - Emergent community structure aligns with social group formation
    - Naturally explains how preferential attachment emerges from simple processes
- **Fitness Model:**
    - **Best for meritocratic networks** like scientific citations or technological adoption
    - Captures the phenomenon of late-arriving high-quality nodes becoming hubs
    - Explains competitions between entities of different intrinsic quality
    - More realistic than pure age-dependent attachment models

## Extensions of BA Model: Comparative Analysis

- **Copy Model:**
  - **Best for information networks** like the Web or citation networks
  - Effectively models how new content builds on existing content
  - Good balance between clustering and scale-free structure
  - Well-suited for biological networks formed through duplication processes
- **Attractiveness Model:**
  - **Best for networks with baseline visibility** requirements
  - Provides more realistic power-law exponents for many empirical networks
  - Helps explain how new nodes gain initial connections
  - Simpler mathematically than other extensions

# Extensions of BA Model: Conclusion

**Real-world network modeling requires:**

1. **Hybrid approaches** combining multiple mechanisms
   - For example: Random Walk + Fitness provides both clustering and quality-based growth
2. **Context-specific models** tailored to specific domains
   - No single model fits all real-world network types
3. **Additional mechanisms** beyond these basic extensions
   - Node deletion, edge rewiring, aging effects, and geographical constraints
4. **Empirical validation** against multiple network metrics
   - Degree distribution alone is insufficient; clustering, path lengths, assortativity, and community structure must be considered

The most successful models incorporate both **structural mechanisms** (how nodes connect) and **node-specific attributes** (what makes some nodes special).

# Reading material

References

[ns1] **Chapter 5 (Network Models)**

# Q&A