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Analisi e Visualizzazione delle Reti Complesse

NS16-17 - Games and Traffic Networks

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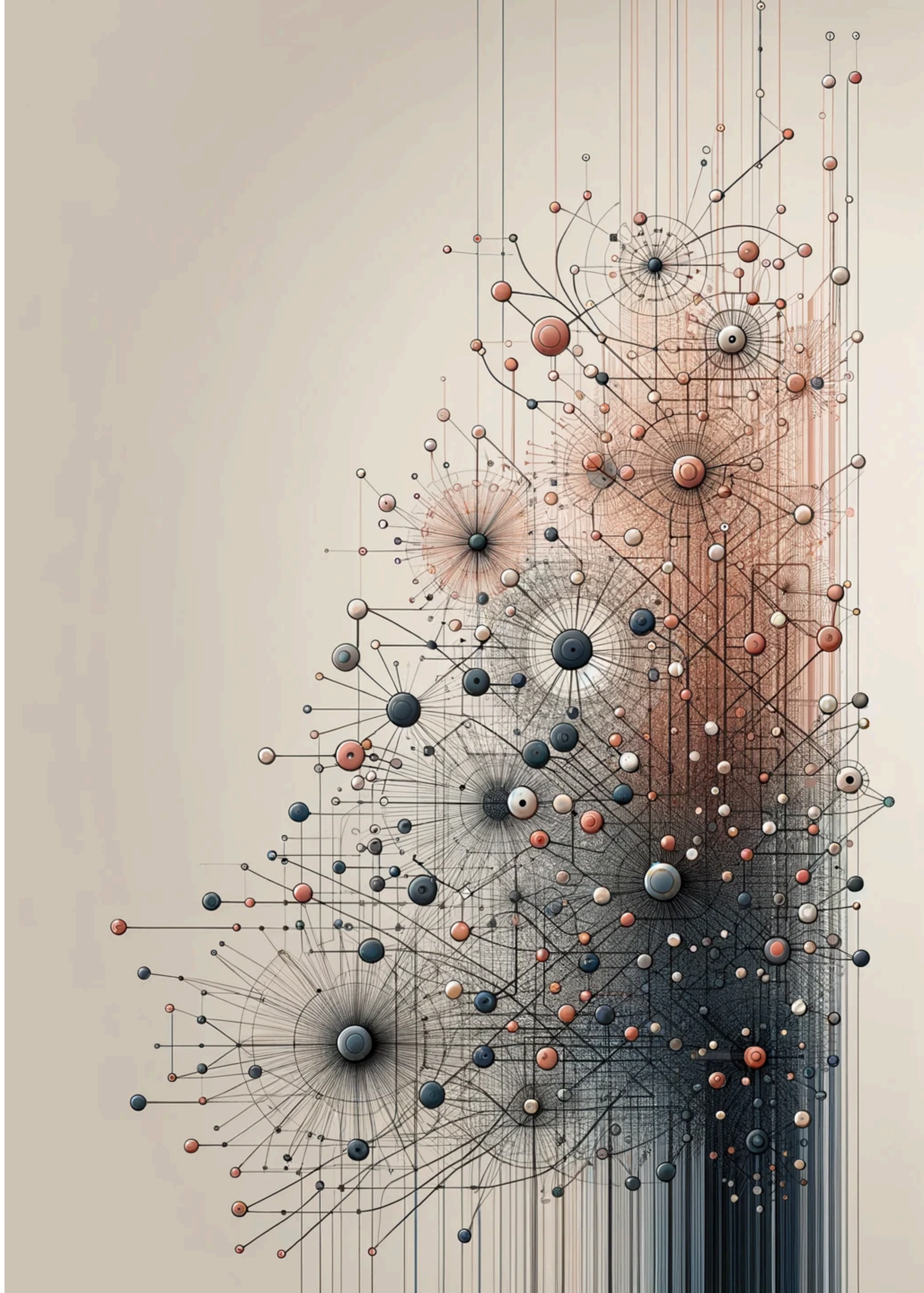

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Agenda

- What is a Game?
- Reasoning about Behavior in a Game
- Best Responses and Dominant Strategies
- Nash Equilibrium
- Multiple Equilibria: Coordination Games
- Multiple Equilibria: The Hawk-Dove Game
- Mixed Strategies
- Mixed Strategies: Examples and Empirical Analysis
- Pareto-Optimality and Social Optimality

What is a Game?

- Complex networks describe the **interactions** between a set of items.
  - characterized by **intrinsic interdependence**
- Each decision maker has an **individual satisfaction** to maximize (e.g., a profit) and its **strategy depends also on other people's choices**.
- Game Theory gives us a simplified framework to understand how individual strategies can create an intrinsic interdependence in the behaviors of participants to a complex system

Basic Ingredients

1. **Players**
2. **Strategies**: set of options for each player
3. **Payoff**: the outcome for each selected strategy

That is summarized in a **payoff matrix**.

Assumptions:

- Everything that a player cares of is in the payoff matrix
  - e.g., the two players are solely concerned with their own payoff
- **Everything** about the structure of the game **is known**.
- Players are **rational**.

Example

- Two students have two large pieces of work due the next day: an **exam** and a **presentation**.
- **Assumptions**:
  - they cannot study for the exam AND prepare the presentation
  - they cannot communicate with each other
- **Exam**:
  - if one studies: gets 92 points
  - if one does not study: 80
- **Presentation**:
  - if one or (xor) the other prepare it: 92 for both
  - if neither of them prepare it: 84
  - if both of them prepare it: 100
- **Final vote**: average on the exam and presentation scores

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

Definition

- **strictly dominant strategy**: when a player has a strategy that is strictly better than all other options regardless of what the other player does
  - we should expect that the player will play it

Example

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

- **Strict dominant strategy** for both players: **Exam**
  - both players get a payoff of 88
- **counterintuitive**: (P,P) would have been better off for both!
- **explanation**: if your partner decides to prepare the Presentation, you would be tempted anyhow to try the Exam since its your dominant strategy (payoff 92)

The Prisoner's dilemma

		Suspect 2 NC	Suspect 2 C
Suspect 1	NC	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

- **Two suspects** have been apprehended by the police and interrogated.
- The police suspect the two individuals are responsible for the robbery, but there is **no evidence**.
- They both resisted arrest and can be charged with that lesser crime (1-year sentence)
- Suspects are asked to confess.
- Possibile strategie **Confess (C)** or **Not Confess (NC)**.
- The payoff matrix shows the penalties (the larger the better).
- Strictly dominant strategy for both: **Confess**.

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Changing the payoff: different outcome

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$98, 98$	$94, 96$
	Exam	$96, 94$	$92, 92$

- It arises only when payoffs are designed in a certain way
- simple changes versus more benign outcomes
  - e.g., an easier exam: you will get 96 if you don't study
- The the strict dominant strategy for both becomes Presentation!

Formalization: Best Responses

- **Players**: 1, 2 (it can be generalized for more players)
- **Strategies**: S, T (we can have more strategies)
- $P_1(S,T)$: payoff for P_1 playing S given T (fixed) played by P_2
- S is a **best response** for P_1: $\forall S' : P_1(S,T) \geq P_1(S',T)$
- S is a **strict best response** for P_1: $\forall S' : P_1(S,T) > P_1(S',T)$
- For P_2 we have **symmetrical definitions**

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Formalization: Dominant Strategies

- **Dominant Strategy**: a P_1's strategy that is **best response** to every strategy of P_2
- **Strictly Dominant Strategy**: a P_1's strategy that is **strict best response** to every strategy of P_2

What if only one player has a strictly dominant strategy?

		Firm 2: *Low-Priced*	Firm 2: *Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

- New example:
    - two firms planning to produce and market a new product
- Two market segments:
    - people who would buy a **low-priced** version of the product (60%)
    - people who would buy a **upscale** version (40%)
- **Firm 1 is a much more popular brand**, when the two firms **directly compete** in a market segment, Firm 1 gets **80%** of the sales and Firm 2 gets **20%** of the sales.

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What if only one player has a strictly dominant strategy?

		Firm 2 Low-Priced	Firm 2 Upscale
Firm 1	*Low-Priced*	.48, .12	.60, .40
Firm 1	*Upscale*	.40, .60	.32, .08

- **Strictly dominant strategy for Firm 1: Low-Priced**
- **No dominant strategy for Firm 2!**
- Firm 2 can confidently predict that Firm 1 will play Low-Priced
  - Firm 1 has a strict dominat strategy and it wants to maximize its profit
- **Firm 2 will play Upscale**
  - Firm 2 is subordinate to Firm 1: **its best strategy is to stay away from Firm 1 market segment**
- Note that players move simultaneously, they have common knowledge of the game

What if none has a (strict) dominant strategy?

		Firm 2 A	Firm 2 B	Firm 2 C
Firm 1	A	4, 4	0, 2	0, 2
	B	0, 0	1, 1	0, 2
	C	0, 0	0, 2	1, 1

- **Three-Client Game**: two firms (three clients: A, B, C)
- If the two firms approach the same client, then the client will give half its business to each.
- Firm 1 is too small to attract business on its own, so if it approaches one client while Firm 2 approaches a different one, then Firm 1 gets a payoff of 0.
- If Firm 2 approaches client B or C on its own, it will get their full business. However, A is a larger client, and will only do business with the firms if both approach A.
- Because A is a larger client, doing business with it is worth 8 (and hence 4 to each firm if it's split), while doing business with B or C is worth 2 (and hence 1 to each firm if it's split).

		Firm 2		
		A	B	C
Firm 1	A	$4, 4$	$0, 2$	$0, 2$
	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

- **Neither firm has a dominant strategy!**
- For Firm 1, A is a strict best response to strategy A by Firm 2, B is a strict best response to B, and C is a strict best response to C.
- For Firm 2, A is a strict best response to strategy A by Firm 1, C is a strict best response to B, and B is a strict best response to C

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Nash Equilibrium

- Even when there are no dominant strategies, we should expect players to use strategies that are best responses to each other.
- Suppose that Player 1 chooses a strategy S and Player 2 chooses a strategy T. We say that this **pair of strategies (S, T)** is a **Nash equilibrium** if **S is a best response to T**, and **T is a best response to S**.
- The idea is that if the players choose strategies that are best responses to each other, then **no player has an incentive to deviate to an alternative strategy**
    - concept of **equilibrium**
- In the Three-Client Game:
    - **(A,A) forms a Nash equilibrium**
    - No other pair of strategies are best responses to each other

Multiple Equilibria: Coordination Games

		Your Partner: *PowerPoint*	Your Partner: *Keynote*
You	*PowerPoint*	$1,1$	$0,0$
	Keynote	$0,0$	$1,1$

- Example:
  - Suppose you and a partner are each preparing slides for a joint project presentation;
  - you can't reach your partner by phone
  - you have to decide whether to prepare your half of the slides in PowerPoint or in Keynote.
  - Either would be fine, but it will be much easier if you use the same software.
- Players need to coordinate with no communication
- **Two different Nash Equilibria** (PowerPoint,PowerPoint) (Keynote,Keynote)

What to do?

- Thomas Schelling's idea of **focal points**:
- look for **natural reasons** to focus on one of the Nash equilibrium
  - possibly **outside the payoff structure** of the game
- (social) conventions can help
  - example of drivers approaching each other
- Try to embed in the payoff matrix the intrinsic features that help you to select an equilibrium **(Unbalanced Coordination Game)**

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	2, 2

The Hawk-Dove Game: another example of multiple equilibria

		Animal 2	
		D	H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

- Players engage in a kind of **anti-coordination** activity
- Two animals are engaged in a contest to decide how a piece of food will be divided between them
- Each animal can choose to behave **aggressively** (Hawk strategy) or **passively** (Dove strategy)
- If both behave passively, they **divide** the food evenly
- If one behaves aggressively while the other behaves passively, then **the aggressor gets most of the food**
- If both animals behave aggressively, then they **destroy the food**

The Hawk-Dove Game

		Animal 2	
		D	H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

- This game has two Nash equilibria: (D, H) and (H, D)
- Without knowing more about the animals we cannot predict which of these equilibria will be played
- Suppose we substitute two countries for the two animals
    - We would need to know more about the countries to predict which equilibrium will be played

Mixed strategies

- There are games which have **no Nash equilibria at all** with pure strategies
- To address this, we can enlarge the set of strategies to include **randomization**

Matching Pennies game

		Player 2: H	Player 2: T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- Two people hold a penny, and simultaneously choose whether to show **heads** (H) or **tails** (T)
- Player 1 loses his penny to player 2 if they match, and wins player 2's penny if they don't match
- Example of **zero-sum games** (payoffs of the players sum to zero in every outcome)
- This game has **no Nash equilibrium with pure strategies** - for any choice of strategies, one player always has an incentive to switch

Understanding Mixed Strategies

- Instead of choosing H or T directly, each player selects a **probability** with which they will play each option:
  - P_1 chooses H with probability p (and T with probability $1 - p$)
  - P_2 chooses H with probability q (and T with probability $1 - q$)
- This means each player **randomizes** between the given strategies
- **Pure strategies** are special cases of mixed strategies:
  - $p = 0$ means P_1 always plays T
  - $p = 1$ means P_1 always plays H
- A mixed strategy creates **uncertainty** for the opponent, making their best response harder to determine

Calculating Payoffs for Mixed Strategies

		Player 2 H	Player 2 T
Player 1	H	$-1, +1$	$+1, -1$
Player 1	T	$+1, -1$	$-1, +1$

- With mixed strategies, payoffs become **expected values**
- Let's calculate from P_1's perspective:
- If P_1 plays **H**, the expected payoff is:
  - With probability q, P_2 plays H: payoff = -1
  - With probability $(1 - q)$, P_2 plays T: payoff = +1
  - Expected payoff = $(-1) \cdot q + (+1) \cdot (1 - q) = 1 - 2q$

Calculating Payoffs for Mixed Strategies

Player 2

		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- If P_1 plays **T**, the expected payoff is:
  - With probability q, P_2 plays H: payoff = +1
  - With probability $(1-q)$, P_2 plays T: payoff = -1
  - Expected payoff = $(+1) \cdot q + (-1) \cdot (1-q) = 2q - 1$

Finding the Mixed-Strategy Nash Equilibrium

- In a Nash equilibrium with mixed strategies, each player must be **indifferent** between their pure strategies
- For P_1 to be willing to randomize between H and T, both must yield the same expected payoff:
  - Expected payoff of H = Expected payoff of T
  - $1 - 2q = 2q - 1$
  - $2 = 4q$
  - $q = \frac{1}{2}$
- Similarly, for P_2 to be willing to randomize, P_1 must play $p = \frac{1}{2}$
- **Why?** If one strategy gave higher expected payoff, a rational player would choose it 100% of the time (not mix)
- Therefore, the only possible mixed-strategy Nash equilibrium is when $p = q = \frac{1}{2}$

The Indifference Principle Explained

- The "indifference principle" is key to mixed-strategy equilibria:
  - For a player to be willing to randomize between strategies, they must be **precisely indifferent** between them
- Visualizing the principle in Matching Pennies:
  - If P_2 plays H more than 50% of the time $(q > \frac{1}{2})$:
    - P_1's best response is to play T (100%)
  - If P_2 plays H less than 50% of the time $(q < \frac{1}{2})$:
    - P_1's best response is to play H (100%)
  - Only at $q = \frac{1}{2}$ is P_1 willing to randomize
- **Key insight**: Each player's mixed strategy is designed to make the other player indifferent
- This creates an **un-exploitable strategy** - the opponent cannot gain an advantage regardless of what they choose
- Nash proved that every finite game **has at least one Nash equilibrium**, possibly requiring mixed strategies and he won the Nobel price for it

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Optimalities: Introduction

- So far we've focused on **Nash Equilibria** - states where no player wants to change their strategy
- But are these equilibria **good outcomes** for the players?
- Two important concepts help us evaluate the quality of game outcomes:
  - **Pareto Optimality**: Is it possible to improve anyone's situation without hurting others?
  - **Social Optimality**: Does the outcome maximize the total welfare of all players?
- These concepts help us understand the gap between:
  - What happens through individual decision-making
  - What would be best for players collectively

Pareto Optimality: Definition

- An outcome is **Pareto-optimal** if:
  - No other outcome exists where **someone can be made better off** without making **anyone else worse off**
  - Any change that benefits one player must harm at least one other player
- Intuitively: All opportunities for "win-win" improvements have been exhausted
- **Key insight**: Many different outcomes can be Pareto-optimal, with very different distributions of payoffs
- Pareto optimality says nothing about fairness—extremely unequal outcomes can still be Pareto-optimal!

Pareto Optimality: Example

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

Let's analyze all four outcomes in our exam-presentation game:

Strategy	Pareto-optimal?	Reason
(E,E)	No	Both players could improve by switching to (P,P)
(E,P)	Yes	Any improvement for P2 would harm P1
(P,E)	Yes	Any improvement for P1 would harm P2
(P,P)	Yes	Both achieve maximum payoff - ideal outcome

- The Nash Equilibrium (E,E) is **not** Pareto-optimal here - a common phenomenon!
- This creates the **efficiency paradox**: individually rational decisions lead to collectively suboptimal results

Social Optimality: Definition

- A **stronger concept** than Pareto optimality
- An outcome is **socially optimal** if it **maximizes the sum of all players' payoffs**
- Also called "social welfare maximizer" or "utilitarian optimum"
- **Mathematical definition**: For payoffs p_i of each player i, we maximize $\sum_i p_i$
- Important relationships:
  - Every socially optimal outcome must be Pareto-optimal (why?)
  - Not every Pareto-optimal outcome is socially optimal
  - Nash Equilibria often fail to be socially optimal

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Real-World Implications

- The tension between Nash Equilibria and optimal outcomes explains many real-world problems:
    - **Climate change**: Countries acting in self-interest (Nash Eq.) vs. globally optimal emission reductions
    - **Vaccination**: Individual decision vs. herd immunity benefits (social optimum)
    - **Traffic congestion**: Individual route selection vs. optimal traffic distribution
- Solutions often require:
    - **Binding agreements** or contracts
    - **Regulation** or external enforcement
    - **Mechanism design**: Changing the game structure to align individual incentives with socially optimal outcomes

Networks and Game Theory

- Nodes connected with many other nodes: if one agent has to select, for some given purpose, one (or some) connection out of your choices, then you need a strategy
- It is likely that agents will select the strategy that leads to the highest payoff
- A multi agents system: each agent will evaluate payoffs according their and everyone's else strategies
- **Traffic network:** individuals need to evaluate routes in the presence of congestion
  - congestion is the result of the decisions made by themselves and everyone else
- **Models for network traffic may lead to unexpected results**

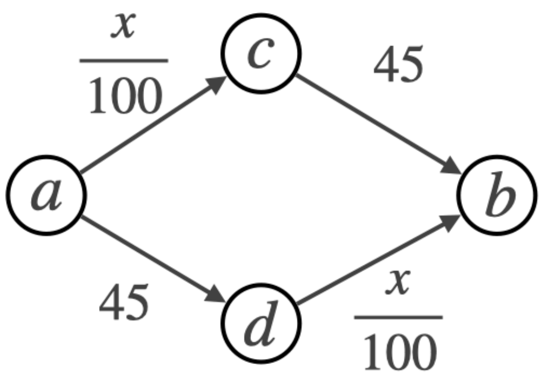
Traffic at Equilibrium

Agenda

- Traffic at Equilibrium
- Braess's Paradox
- The Social Cost of Traffic at Equilibrium

Transportation network model

- **Directed graph**
  - edges are highways
  - nodes are exits (you can get on or off a particular highway)
- **Assumption:** everyone wants to drive from a to b
- **Weights:** travel time
  - fixed
  - depending of the traffic x
- Suppose we have $x = 4000$ cars
- **The traffic game**:
  - players: drivers
  - each player's has 2 possible strategies that are the two routes from a to b
- **payoff:** the negative of a player's travel time (the faster the better)

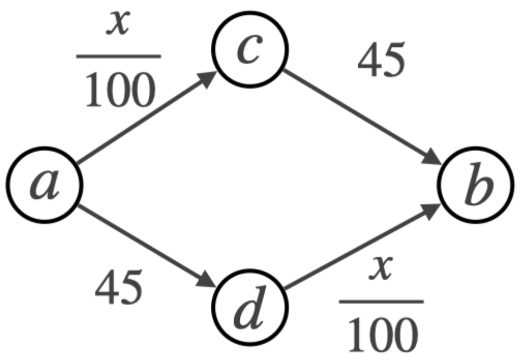


Games with more than 2 players

- As in the 2 players game:
  - the payoff of each player depends on the strategies chosen by all
  - **Nash equilibrium:** a list of strategies (one for each player), so that each one is a best response to all the others
  - The concept of **dominant strategies**, **mixed strategies**, **Nash equilibrium with mixed strategies**: they all have direct parallels

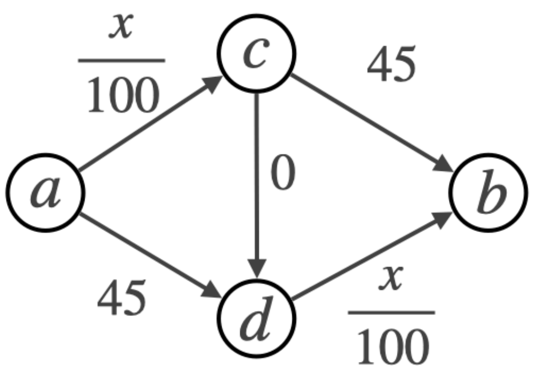
Understanding Equilibrium Traffic

- Let's analyze the travel times on each route:
  - **Path 1** $(a \to c \to b)$: $T_1(x) = \frac{x}{100} + 45$ minutes (with x drivers)
  - **Path 2** $(a \to d \to b)$: $T_2(y) = 45 + \frac{y}{100}$ minutes (with y drivers)
  - Total drivers: $x + y = 4000$
- Individual drivers choose the route with **minimum travel time** for themselves
  - They don't consider the effect of their choice on others
- **Nash equilibrium occurs when** $T_1(x) = T_2(y)$, as:
  - If one route were faster, drivers would switch to it
  - This switching continues until travel times equalize
- At equilibrium: $x = y = 2000$ drivers on each route
  - Travel time per driver: $45 + \frac{2000}{100} = 65$ minutes



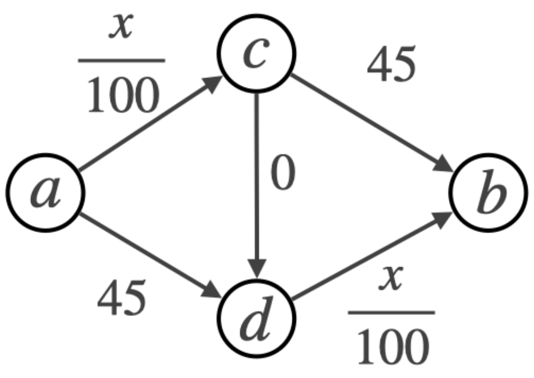
Braess's Paradox: The Setup

- Original network has two paths from a to b:
  - Path 1 $(a \to c \to b)$ with travel time $T_1(x) = \frac{x}{100} + 45$
  - Path 2 $(a \to d \to b)$ with travel time $T_2(y) = 45 + \frac{y}{100}$
- At equilibrium: $x = y = 2000$ and travel time = 65 minutes
- **City planners decide**: "Let's add a new super-fast road from c to d with zero travel time!"
  - Intuition: This should improve traffic flow and reduce travel times
  - But paradoxically, this makes everyone worse off



Braess's Paradox: The Analysis

- After adding the $c \to d$ link, a new path emerges:
  - Path 3: $a \to c \to d \to b$
  - Travel time: $T_3(z) = \frac{z}{100} + 0 + \frac{z}{100} = \frac{2z}{100}$
- Let's examine what happens if all drivers take this new path:
  - With $z = 4000$: $T_3 = \frac{2 \cdot 4000}{100} = 80$ minutes
- **New equilibrium**: All 4000 drivers use Path 3
  - Why? Because switching to either original path would take 85 minutes
  - If one driver switches to Path 1 (a→c→b):
    - Travel time = $T_{a\to c}(4000) + T_{c\to b}(1) = \frac{4000}{100} + 45 = 85$ minutes
  - If one driver switches to Path 2 (a→d→b):
    - Travel time = $T_{a\to d}(1) + T_{d\to b}(4000) = 45 + \frac{4000}{100} = 85$ minutes
- Result: Everyone's travel time increased from 65 to 80 minutes!
  - The "improvement" harmed everyone



Why Braess's Paradox Occurs

- The paradox occurs because drivers make **individually rational decisions** without considering the collective impact
- The new shortcut creates a "**selfish routing trap**":
  - It's always in each individual's interest to use the shortcut
  - But when everyone does so, the system becomes worse for everyone
- The problem is **non-cooperative behavior**:
  - If drivers could coordinate and agree to use only the original routes (50% each), everyone would have 65-minute travel times
  - But without coordination, they're stuck in the 80-minute equilibrium
- This illustrates a key lesson: **Network improvements can backfire** when users act selfishly
  - Similar to the Prisoner's Dilemma but in a network context

Braess's Paradox in Real Cities: New York

- **42nd Street Closure** (1990):
  - During Earth Day, New York City closed 42nd Street, a major crosstown artery
  - **Expected outcome**: Severe traffic congestion in midtown Manhattan
  - **Actual result**: Traffic flow *improved* throughout the area
  - The New York Times headline: "What if They Closed 42nd Street and Nobody Noticed?"
- **Why it happened**:
  - The street closure eliminated a "shortcut" that was actually creating bottlenecks
  - Drivers distributed more evenly across the grid system
  - Some drivers switched to alternate modes of transportation
- Similar effects were observed during the 1973 engineers' strike when traffic controllers shut off many traffic lights
- [Ref] https://www.nytimes.com/1990/12/25/health/what-if-they-closed-42d-street-and-nobody-noticed.html

Braess's Paradox: Seoul and Stuttgart

- **Cheonggyecheon Stream Restoration (Seoul, 2003)**:
  - A major **six-lane elevated highway** was completely removed
  - Replaced with a 3.6-mile-long urban park and restored stream
  - **Expected**: Traffic gridlock throughout central Seoul
  - **Result**: Travel times decreased by up to 14%, despite removing road capacity
  - 15.1% of drivers shifted to public transportation
- **Stuttgart, Germany (1969)**:
  - A new street was built in the center to improve traffic flow
  - **Result**: Traffic conditions immediately worsened
  - When the street was later closed, traffic flow improved
  - One of the earliest documented real-world examples of the paradox

Measuring the Paradox: Research Findings

- A 2008 study examined 246 road networks in major cities:
  - In approximately **50%** of networks, removing select roads would improve overall traffic flow
  - The maximum improvement could be up to 20%
- London Congestion Charge (introduced 2003):
  - Reduced vehicle numbers by 18% within the charging zone
  - Traffic speeds increased by 30%, far exceeding predictions
  - Demonstrated how pricing mechanisms can overcome the paradox
- These examples show that **counterintuitive solutions** may be optimal for traffic management
  - Sometimes *less infrastructure* leads to better outcomes
  - Read more in this paper

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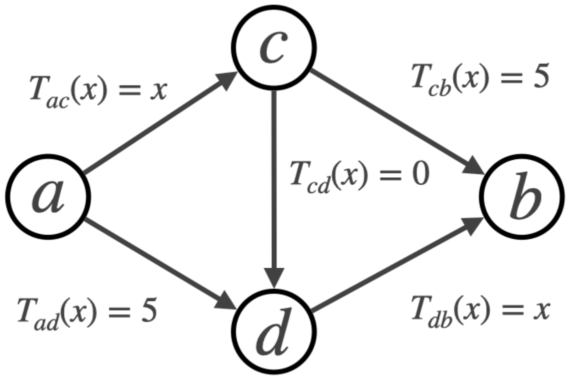
Social Cost of Traffic at Equilibrium

Travel time function

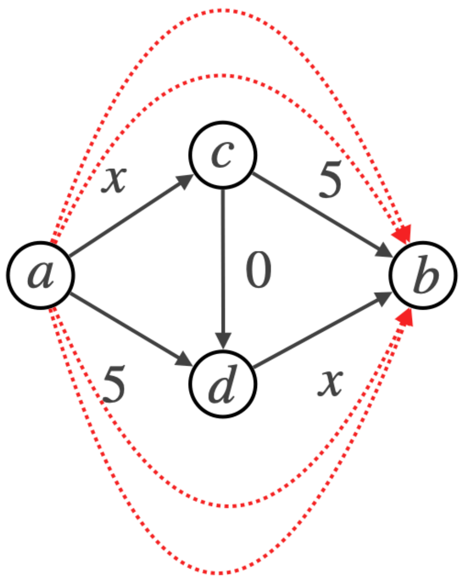
- We want to quantify how far from optimal is traffic at equilibrium
- Each edge e has a travel-time function $T_e(x)$
- **Assumption:** linear in the amount of traffic

$$T_e(x) = a_e x + b_e \quad \texttt{with} \quad a_e, b_e > 0$$



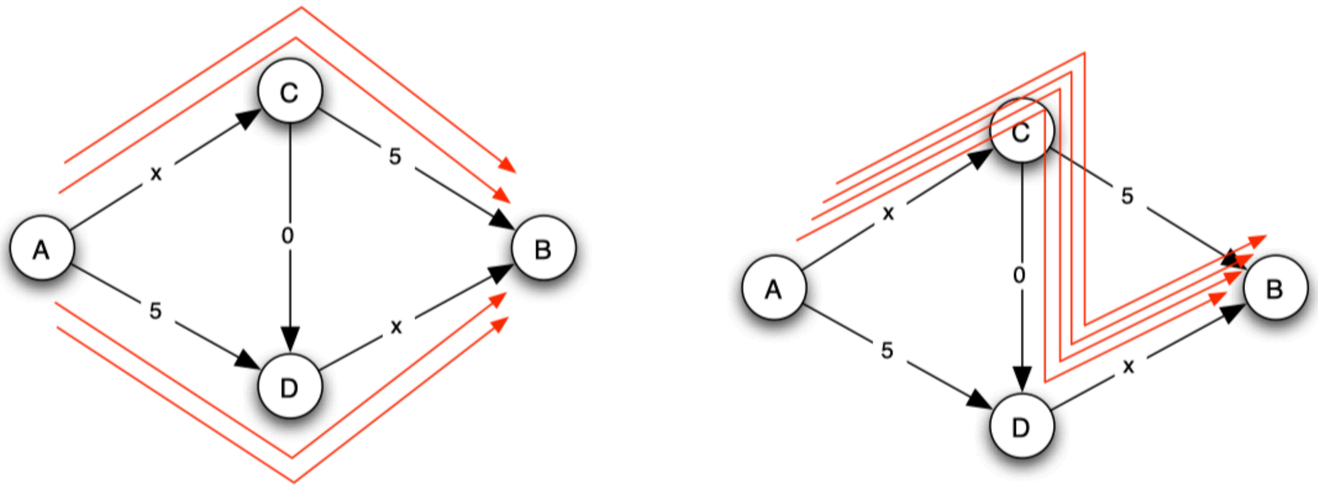
Traffic pattern

- A traffic pattern: a choice of a path by each driver
- **Social cost of a traffic pattern $\mathcal{P}$:** the sum of the travel times incurred by all drivers when they use this traffic pattern
  - Ex: 4 drivers, each starting from a and with destination b
- When a traffic pattern achieves the minimum possible cost: **socially optimal**
- **socially optimal traffic patterns are social welfare maximizers in this traffic game**



Nash equilibrium

- **The unique Nash equilibrium in this game has a larger social cost**.
- Is there always an equilibrium traffic pattern?
- Does it always exists an equilibrium traffic pattern whose social cost is less than the social optimum?



(a) *The social optimum.* (b) *The Nash equilibrium.*

Finding a traffic pattern at equilibrium

- To prove that an equilibrium exists, let's use a procedure that looks for one:

1. Start from any traffic pattern
2. If it is an equilibrium, stop
3. Else, there is at least one driver whose best response is some **alternate path providing a strictly lower travel time**
4. pick one of these drivers and have her **switch to this alternate path**, then go to step 2.

- This procedure is called a **best-response dynamics**
  - We need to show that **best-response dynamics will eventually stop**.

Does a best-response dynamics always stop?

- **No.** In a zero sum game it will run forever because it lacks of an equilibrium (with pure strategies)
- In principle, even in the traffic game we can have a best-response dynamics that run forever **if we do not have an equilibrium**.
- **We will prove that in our traffic game the procedure stops**, proving consequently that:
  - equilibria exist
  - an equilibrium can be reached by a simple process in which drivers constantly update what they are doing according to their best response

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Progress Measure

- To check if the best-response dynamics will eventually stop, we need a progress measure to track the process and to assess how far we are from the process to stop
- Is the social cost of the current traffic pattern a good progress measure?
    - **Answer: No**. In fact, some best-response updates by drivers can make the social cost better, but others can make it worse
    - The social cost of the current traffic pattern can oscillate, and the relationship with our progress toward an equilibrium is not clear
- The alternate quantity must strictly decrease with each best-response update

Potential Energy

- As a good progress measure, let's introduce the **potential energy** of an edge e:

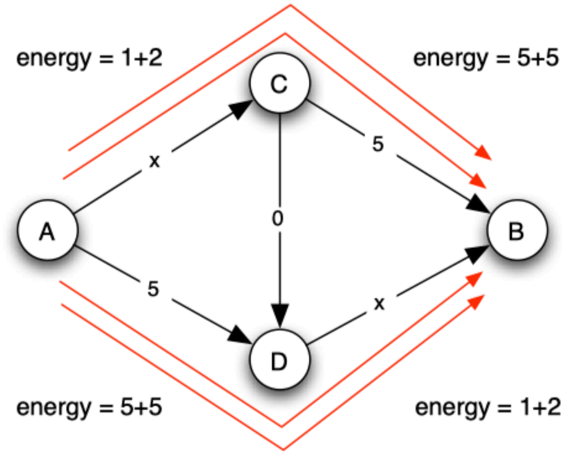
$$\texttt{Energy}(e) = T_e(1) + T_e(2) + \cdots + T_e(x)$$

- if an edge e has no driver on it:
    - $\texttt{Energy}(e) = 0$
- The potential energy of a traffic pattern $\mathcal{P}$ is the sum of all the potential energies of all the edges, with the current number of drivers in this traffic pattern:

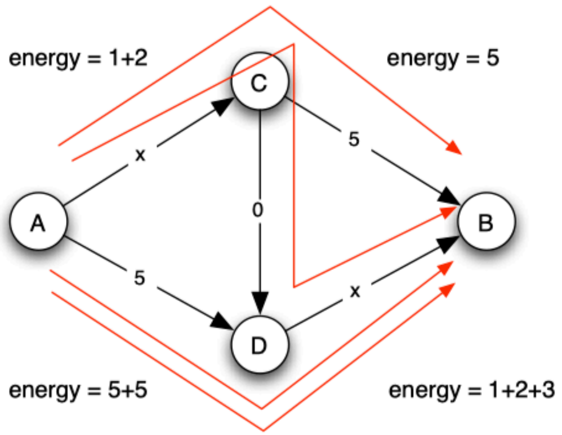
$$\texttt{Energy}(\mathcal{P}) = \sum_{e_i \in \mathcal{P}} \texttt{Energy}(e_i)$$

- $\texttt{Energy}(e) \neq xT_e(x)$
    - It is a sort of **cumulative** quantity: we imagine **drivers crossing the edge one by one**, and each driver only experiences the delay caused by themselves and the drivers crossing the edge in front of them.

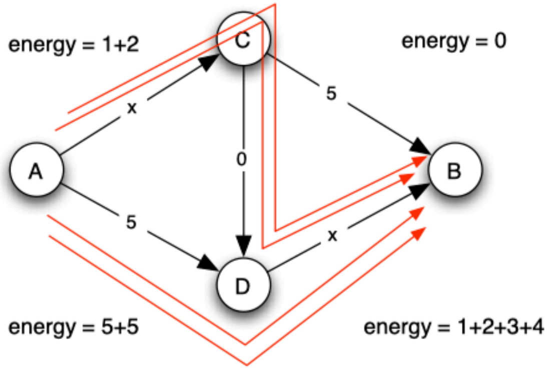
Example of best-response dynamics



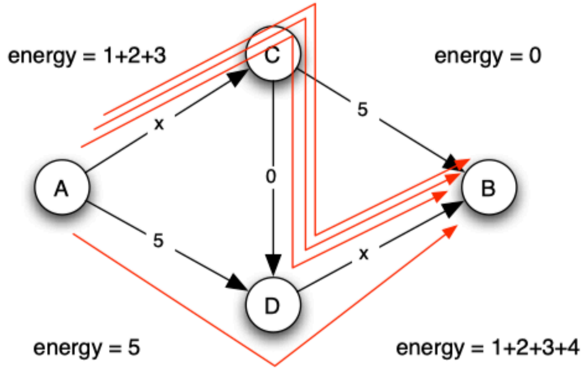
(a) *The initial traffic pattern. (Potential energy is 26.)*



(b) *After one step of best-response dynamics. (Potential energy is 24.)*

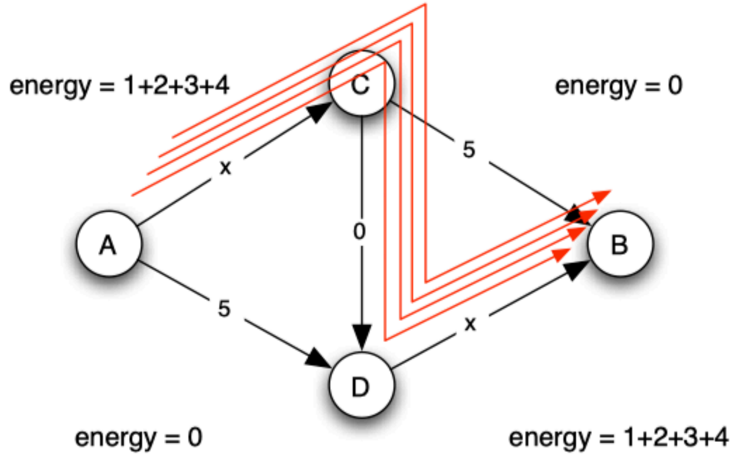


(c) *After two steps. (Potential energy is 23.)*



(d) *After three steps. (Potential energy is 21.)*

Example of best-response dynamics



(e) *After four steps: Equilibrium is reached. (Potential energy is 20.)*

- It's like a **two-steps** process:
    - first drivers abandons his current path, temporarily leaving the system; then, the driver returns to the system by adopting a new path. This **first step releases potential energy** as the driver leaves the system, and **the second step adds potential energy** as he re-joins.
    - What is the **net change**?

Does the best-response dynamics stop?

- If we prove that the best-response dynamics will stop, then we have proved that an equilibrium always exist
- That is equivalent to say:
  - If we prove that the potential energy strictly decreases at each step, then we have proved that best-response dynamics stops
- Observe in our example that potential energy always decreases at every step:
  - when a driver abandons one path in favor of another, **the change in potential energy is exactly the improvement in the driver's travel time**
  - and a driver always change to a better travel time due to the nature of the best-response dynamics
- **Is this true for any network and any best-response by a driver?**

Let's recall that the potential energy of edge e with x drivers is:

$$\texttt{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x)$$

When one of these drivers leaves his current path, it drops to:

$$= T_e(1) + T_e(2) + \ldots + T_e(x-1)$$

Summing up

- $\texttt{Energy}(p)$ decreases according to all the travel times that the driver was experiencing on every edges in path p: $\sum_{e \in p} T_e(x)$
- It is like that, abandoning path p for the new path p', **the driver releases a potential energy that is equal to** $\sum_{e \in p} T_e(x)$

By the same reasoning, for every edge e' in the new path p', before the new driver adopts it, we have this potential energy:

$$\texttt{Energy}(e') = T_{e'}(1) + T_{e'}(2) + \ldots + T_{e'}(x-1)$$

When one of the new driver joins it increases to:

$$= T_{e'}(1) + T_{e'}(2) + \ldots + T_{e'}(x-1) + T_{e'}(x)$$

Summing up, $\texttt{Energy}(p')$ increases according to all the travel times that the new driver is experiencing on every edges in path p': $\sum_{e' \in p'} T_{e'}(x)$

The net change in potential energy is the driver new travel time minus their old travel time

$$\Delta E = \sum_{e' \in p'} T_{e'}(x) - \sum_{e \in p} T_e(x)$$

ΔE must be negative, because driver must have an incentive to change path (the new strategy must be a best response) $\implies$ **the potential energy strictly decreases throughout the process**.

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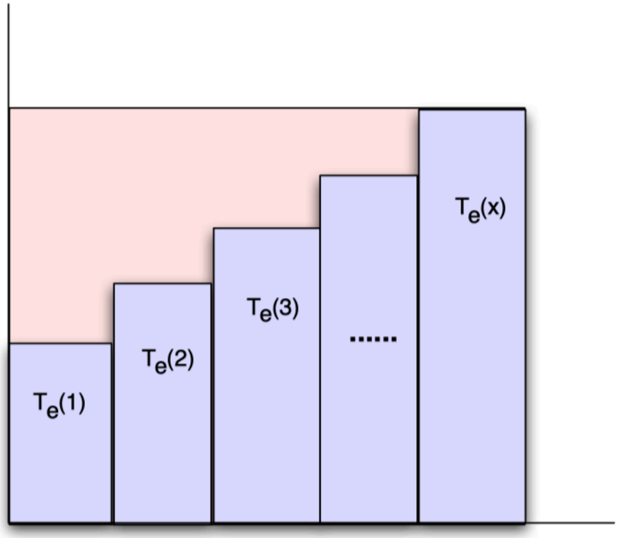
Comparing Equilibrium traffic to the Social Optimum

- We proved that an equilibrium traffic pattern always exists.
- How can we compare the travel time at equilibrium to that of a social optimum?
- Let's look for a relationship between the potential energy of an edge and the total travel time of all the drivers crossing the edge.
- Then we can sum up these quantities for all the edges in the traffic patterns and compare travel times at equilibrium and at social optimum.

Relating Potential Energy to Travel Time for a Single Edge



The potential energy is the area under the shaded rectangles; it is always at least half the total travel time, which is the area inside the enclosing rectangle (red).

$$
\begin{aligned}
\texttt{Energy}(e) &= T_e(1) + T_e(2) + \ldots + T_e(x) \\
&= a_e(1 + 2 + \ldots + x) + b_e x \\
&= \frac{a_e x(x+1)}{2} + b_e x \\
&= x\left(\frac{a_e(x+1)}{2} + b_e\right) \\
&\geq \frac{1}{2} x(a_e x + b_e) \\
&= \frac{1}{2} x T_e(x) \\
&= \frac{1}{2}\texttt{TTT}(e)
\end{aligned}
$$

Where $\texttt{TTT}(e) = xT_e(x)$ is the **total travel time** on edge e.

So we have:

$$
\texttt{Energy}(e) \geq \frac{1}{2}\texttt{TTT}(e)
$$

Wrapping up

We have that $\texttt{Energy}(e) \leq \texttt{TTT}(e)$ and $\texttt{Energy}(e) \geq \frac{1}{2}\texttt{TTT}(e) \implies \frac{1}{2}\texttt{TTT}(e) \leq \texttt{Energy}(e) \leq \texttt{TTT}(e)$

Moreover, if $\mathcal{P}$ is a traffic pattern, recall that:

$$\texttt{Energy}(\mathcal{P}) = \sum_{e_i \in \mathcal{P}} \texttt{Energy}(e)$$

Recall also that the **social cost ($\texttt{SC}(\mathcal{P})$)** of traffic pattern $\mathcal{P}$ is the sum of the travel times incurred by all drivers when they use this traffic pattern:

$$\texttt{SC}(\mathcal{P}) = \sum_{e \in \mathcal{P}} \texttt{TTT}(e)$$

Finally, recall that the potential energy decreases as best-response dynamics moves from $\mathcal{P}$ to $\mathcal{P}'$: $\texttt{Energy}(\mathcal{P}') \leq \texttt{Energy}(\mathcal{P})$

The Price of Anarchy: Bounding Equilibrium Cost

- We want to answer a fundamental question: **How inefficient is selfish routing?**
  - How much worse is the equilibrium traffic pattern compared to the optimal pattern?
- **Step 1:** We know the following relationships:
  - $\frac{1}{2}\texttt{TTT}(e) \leq \texttt{Energy}(e) \leq \texttt{TTT}(e)$ for any edge e
  - The potential energy at equilibrium is minimal: $\texttt{Energy}(\mathcal{P}') \leq \texttt{Energy}(\mathcal{P})$
- **Step 2:** Let's connect social cost to potential energy:
  - At equilibrium $\mathcal{P}'$: $\texttt{SC}(\mathcal{P}') = \sum_{e' \in \mathcal{P}'} \texttt{TTT}(e')$
  - Since $\texttt{TTT}(e) \leq 2 \cdot \texttt{Energy}(e)$ for any edge:

$$\texttt{SC}(\mathcal{P}') = \sum_{e' \in \mathcal{P}'} \texttt{TTT}(e') \leq \sum_{e' \in \mathcal{P}'} 2 \cdot \texttt{Energy}(e') = 2 \cdot \texttt{Energy}(\mathcal{P}')$$

Proving the Bound Step by Step

- **Step 3:** Connect the equilibrium energy to the optimal energy:
  - At equilibrium, energy is minimized: $\texttt{Energy}(\mathcal{P}') \leq \texttt{Energy}(\mathcal{P})$
- **Step 4:** Connect the optimal energy to the optimal social cost:
  - For the socially optimal pattern $\mathcal{P}$: $\texttt{Energy}(\mathcal{P}) \leq \texttt{SC}(\mathcal{P})$
  - This follows from $\texttt{Energy}(e) \leq \texttt{TTT}(e)$ for any edge
- **Step 5:** Putting it all together:

$$\texttt{SC}(\mathcal{P}') \leq 2 \cdot \texttt{Energy}(\mathcal{P}') \leq 2 \cdot \texttt{Energy}(\mathcal{P}) \leq 2 \cdot \texttt{SC}(\mathcal{P})$$

- **Key result:** The social cost at Nash equilibrium is **at most twice** the optimal social cost
  - This is known as the "**Price of Anarchy**" = 2

Understanding the Price of Anarchy

- The Price of Anarchy tells us the **worst-case ratio** between:
  - The cost of the Nash equilibrium (selfish routing)
  - The cost of the social optimum (centralized routing)
- **Practical implications**:
  - Even with completely selfish drivers, the total travel time cannot be more than double the optimal
  - This is a **tight bound** for linear cost functions - there are networks where this bound is actually reached
  - For real road networks, the inefficiency is often much less than 2×
- **Policy relevance**:
  - Shows the maximum benefit possible from perfect traffic management
  - Helps evaluate whether infrastructure investments or regulations are worthwhile
  - Better bounds exist: for specific network types, the Price of Anarchy can be as low as 4/3

Conclusions

- We found that:
  - in the traffic game we can always find a traffic pattern at equilibrium
  - the social cost of the traffic pattern at equilibrium is at most twice the socially optimal cost (we found a bound!)
- It is also possible to find a better bound: traffic pattern social cost at equilibrium is no more than 4/3 times as large than socially optimal traffic pattern
  - Reading material: Anshelevich et. al, The price of stability for network design with fair cost allocation, 2004, at Foundations of Computer Science, 1975., 16th Annual Symposium on 38(4):295- 304

Reading material

[ns2] **Chapter 6 (6.1-6.9)** Games

[ns2] **Chapter 8 (8.1 - 8.3)** Modeling Network Traffic using Game Theory

Q&A