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# Analisi e Visualizzazione delle Reti Complesse

## NS08 - Network Models

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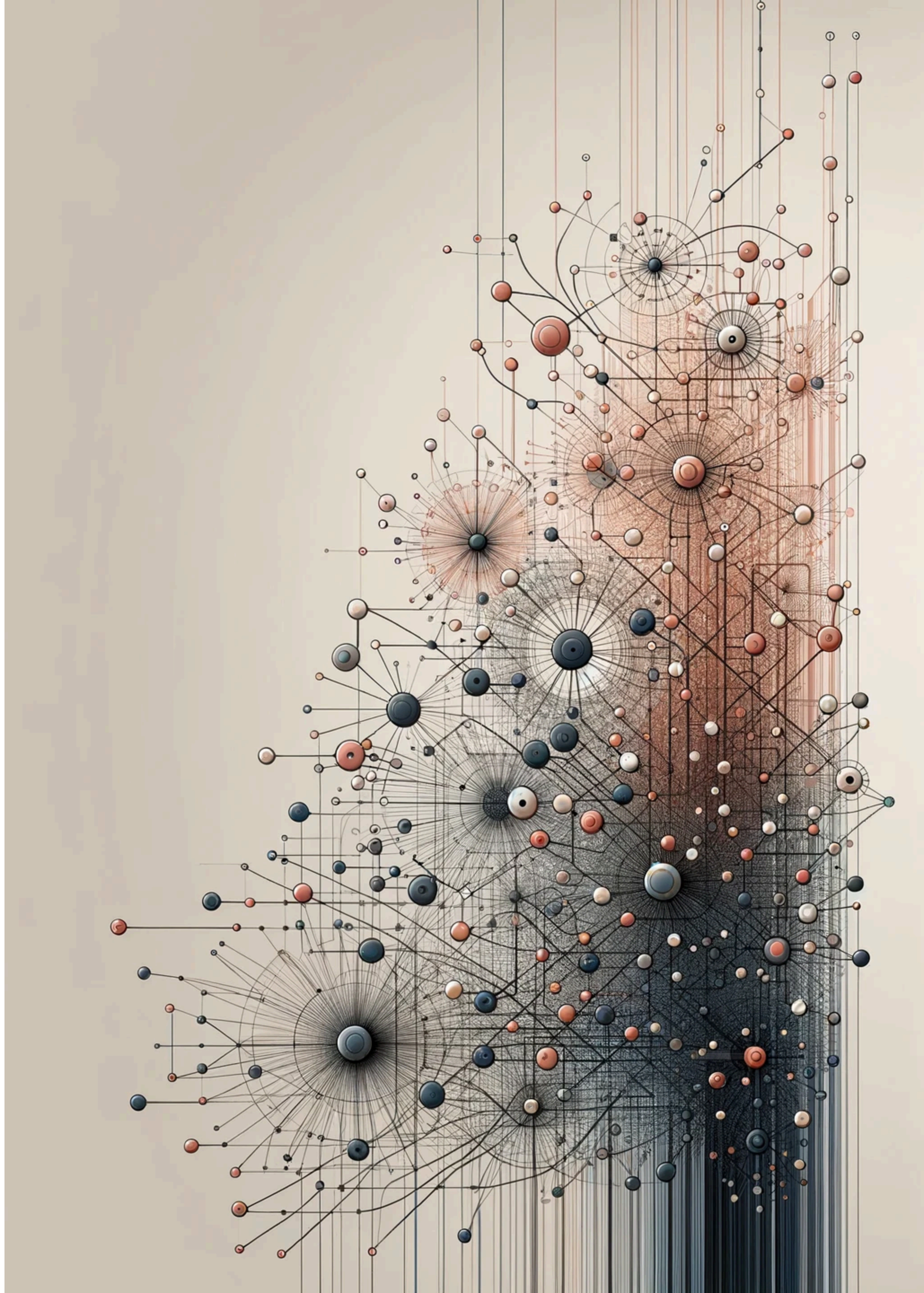


# Outline

- Random networks
- Small-world networks
- The configuration model

# Recap on structural characteristics of real networks

# Features of real networks

- **small-world property**
  - Most real-world networks have **short paths** between any pair of nodes
  - The average shortest path length scales logarithmically with network size: $\langle l \rangle \sim \log N$
  - Typically characterized by a "six degrees of separation" phenomenon
  - Examples:
    - Social networks: any two people are connected through a short chain of acquaintances
    - Brain networks: information can travel efficiently between distant brain regions
    - Internet: data can reach any destination in a few hops
  - Implications:
    - Efficient information/disease spread
    - Fast communication across the network
    - Network is "navigable" with local information
  - Co-exists with clustering in real networks (unlike random networks)

## Features of real networks

- **high clustering coefficient**
    - The clustering coefficient of a node is the fraction of pairs of the node's neighbors that are connected to each other:

$$C(i) = \frac{\tau(i)}{k_i(k_i - 1)/2} = \frac{2\tau(i)}{k_i(k_i - 1)}$$

    - where $\tau(i)$ is the number of triangles involving $i$. Note that in this definition, the clustering coefficient is undefined if $k_i < 2$, i.e., a node must have at least degree 2 to have any triangles.
    - NetworkX assumes $C = 0$ if $k = 0$ or $k = 1$
    - Many networks have high clustering coefficients
    - Other networks, e.g., bipartite and tree-like networks, have low clustering coefficient

# Features of real networks

- **scale-free**
    - A network is **scale-free** if its **degree distribution** follows a **power-law**: $P(k) \sim k^{-\gamma}$
    - Where $k$ is the node degree and $\gamma$ is the exponent (typically $2 < \gamma < 3$ for real networks)
    - A power-law distribution $P(k) \sim k^{-\gamma}$, when plotted on a **log-log scale**, the distribution appears as a **straight line**
        - Mathematically, taking the logarithm of both sides: $\log P(k) = -\gamma \log k + \log C$ (where $C$ is a constant)
        - This is equivalent to the equation of a straight line $y = mx + b$ where $y = \log P(k)$, $x = \log k$, $m = -\gamma$ (the slope is the negative exponent) and $b = \log C$ (y-intercept)
    - In a power-law distribution:
        - There are **many nodes with few connections**
        - There are **few nodes with many connections** (hubs)
        - The distribution has a **heavy tail**

- The term "scale-free" refers to the **lack of a characteristic scale** in the degree distribution
  - No "typical" node with which to characterize the network
  - Degrees span several orders of magnitude
  - The power-law distribution remains the same at different scales.

## Characteristics of scale-free networks

- **Presence of hubs**: nodes with abnormally high degree
  - Hubs can have orders of magnitude more connections than average nodes
  - Hubs often play critical roles in the network's function
- **Heavy-tailed degree distribution**
  - The probability of finding extremely high-degree nodes is not negligible
  - No sharp cutoff in the maximum degree
- **High heterogeneity parameter** $\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle^2}$
  - In scale-free networks, $\kappa$ can be very large or even diverge
  - In random networks, $\kappa \approx 1 + \frac{1}{\langle k \rangle}$

## Properties of scale-free networks

- **Robustness against random failures**
  - Random removal of nodes is unlikely to affect hubs
  - Network connectivity remains largely intact
- **Vulnerability to targeted attacks**
  - Removing hubs can quickly fragment the network
  - Critical for understanding network resilience
- **Ultra-small world property**
  - Average path length scales as $\sim \ln \ln N$ (where $N$ is the number of nodes)
  - Even shorter paths than in random networks ($\sim \ln N$)

# Models

**Model**:

A set of instructions to build networks

**Goal:**

Find models that generate networks with the same characteristics as real-world networks

**Why build models?**

To understand the fundamental processes that create real networks

# The importance of network models

- **Explanation**: Models help us understand the underlying mechanisms that generate network structures we observe
  - Why do real networks have hubs?
  - Why do social networks form dense communities?
  - What causes the small-world property?
- **Prediction**: Models let us forecast how networks might evolve or respond to changes
  - How will removing certain nodes affect connectivity?
  - How quickly will information spread?
  - How resilient is the network to failures or attacks?
- **Benchmark**: Models provide reference points to compare real networks against
  - How different is our observed network from random?
  - Which features are significant vs. expected by chance?

## Scientific value of network models

- **Parsimony**: Good models capture complex network properties using simple rules
    - Example: Preferential attachment explains scale-free degree distributions with a single mechanism
- **Generative understanding**: Models reveal how microscopic rules lead to macroscopic properties
    - Example: Simple rewiring rules in Watts-Strogatz model create small-world networks
- **Counterfactuals**: Models let us explore "what if" scenarios
    - Example: How would the Internet evolve with different connection rules?
- **Abstraction**: Models strip away details to reveal fundamental principles
    - Example: Despite their differences, citation networks and the Web share similar growth patterns

# Random networks

# Random networks

**Simple idea**

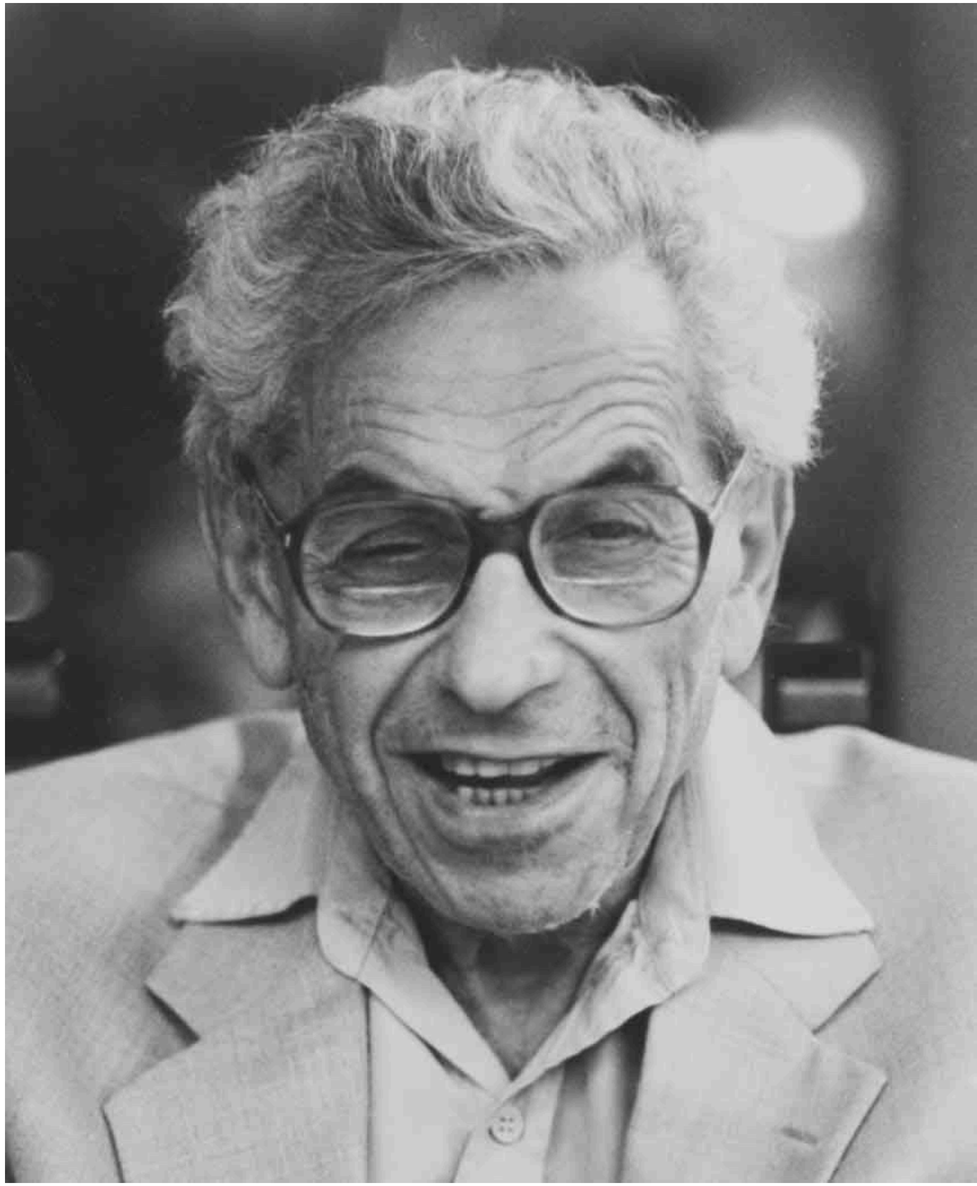
Placing links at random between pairs of nodes

**Paul Erdős (1913-1996)**

Famous for the Erdős–Rényi random model

We present in these slides an equivalent formulation: the Gilbert's model

Main difference: version by Erdős and Rényi the number of links of the network is fixed, whereas in the model by Gilbert it is variable



# Random networks

**Gilbert's random network model**

$G(n,p)$ where $n$ is the number of nodes and $0 < p < 1$ is the probability that an edge occurs.

**Algorithm:**

1. Start with $n$ nodes and zero links
2. Go over all pairs of nodes $\binom{n}{2}$; for each pair of nodes $i$ and $j$, generate a random number $r$ between 0 and 1

- If $r < p \Rightarrow \quad$ **$i$ and $j$ get connected**
- If $r > p \Rightarrow \quad$ **$i$ and $j$ remain disconnected**

The probability of obtaining any one particular random graph with $m$ edges is $p^m(1-p)^{\binom{n}{2}-m}$

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# Random networks: evolution

Let us focus on the **connected components**

- With $p = 0$ **no links**: N components with one node each
- With $p = 1$ **all links are there**: one component (complete network) with $N$ nodes

**Question:**

What happens as we add links to the network?

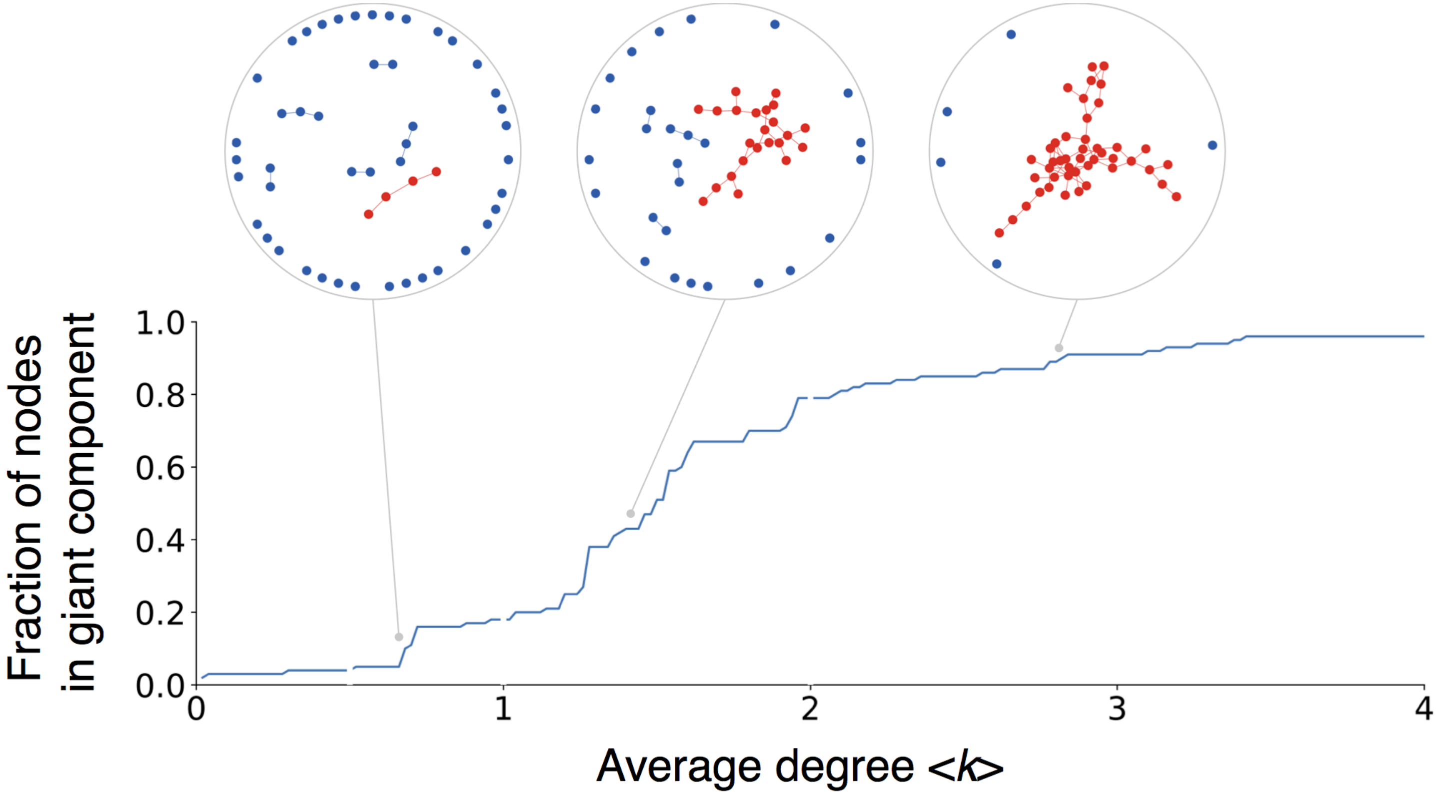
**Naïve expectation:**

The size of the largest component grows smoothly with the number of links

**Wrong expectation:**

There is an **abrupt increase** for a given value of the link probability $p$

# Random networks: evolution




Around $\langle k \rangle = 1$ a giant component grows very fast at the expense of the other, smaller components.

[Example in NetLogo]

## Random networks: number of links, density, average degree

- Equivalence with the tossing of a biased coin, which yields heads with probability p
- Number of independent trials (tosses): $t$
- Number of heads after $t$ trials: $h$
- Special cases:
  - $p = 0$ **the coin never yields heads** $\Rightarrow h = 0$
  - $p = 1$ **the coin always yields heads** $\Rightarrow h = t$
  - $p = \frac{1}{2}$ **the coin yields heads (about) half of the times** $\Rightarrow h \approx \frac{t}{2}$
- General rule:
  - $h \approx p \cdot t$

# Random networks: number of links, density, average degree

**Expected number of links $\langle L \rangle$ of a random network with $N$ nodes:**

Number of heads with probability of yielding heads equal to $p$ and the number of trials $t$ equal to the number of all node pairs of the network:

$$t = \frac{N(N-1)}{2} \rightarrow \langle L \rangle = p\binom{N}{2} = p\frac{N(N-1)}{2}$$

**Expected density of links $d$ of a random network with $N$ nodes:**

$$d = \frac{\langle L \rangle}{\frac{N(N-1)}{2}} = \frac{p\frac{N(N-1)}{2}}{\frac{N(N-1)}{2}} = p$$

Real-world networks are **sparse**.

For random networks to be better models of real networks, $p$ **must be very small**.

## Random networks: number of links, density, average degree

**Expected average degree $\langle k \rangle$ of a random network with $N$ nodes:**

Let's derive this step by step:

1. For any node in a random network, it has $N-1$ potential neighbors (all other nodes)
2. For each potential connection, the probability of a link existing is $p$
3. Therefore, the expected number of links per node is:

$$\langle k \rangle = p \cdot (N-1)$$

**Alternative derivation:**

1. The expected total number of links in the network is $\langle L\rangle = p\frac{N(N-1)}{2}$
2. The average degree is twice the number of links divided by the number of nodes:

$$\langle k\rangle = \frac{2\langle L\rangle}{N} = \frac{2 \cdot p\frac{N(N-1)}{2}}{N} = p(N-1)$$

**Example:** For a random network with $N = 1000$ nodes and link probability $p = 0.01$:

- Expected average degree is $\langle k\rangle = 0.01 \times 999 \approx 10$ links per node

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## Random networks: degree distribution

**Question:** What is the probability that a node has $k$ neighbors?

**Back to coin tossing problem:** What is the probability that a coin that yields heads with probability $p$ results in $k$ heads out of $N$–$1$ (independent) trials?

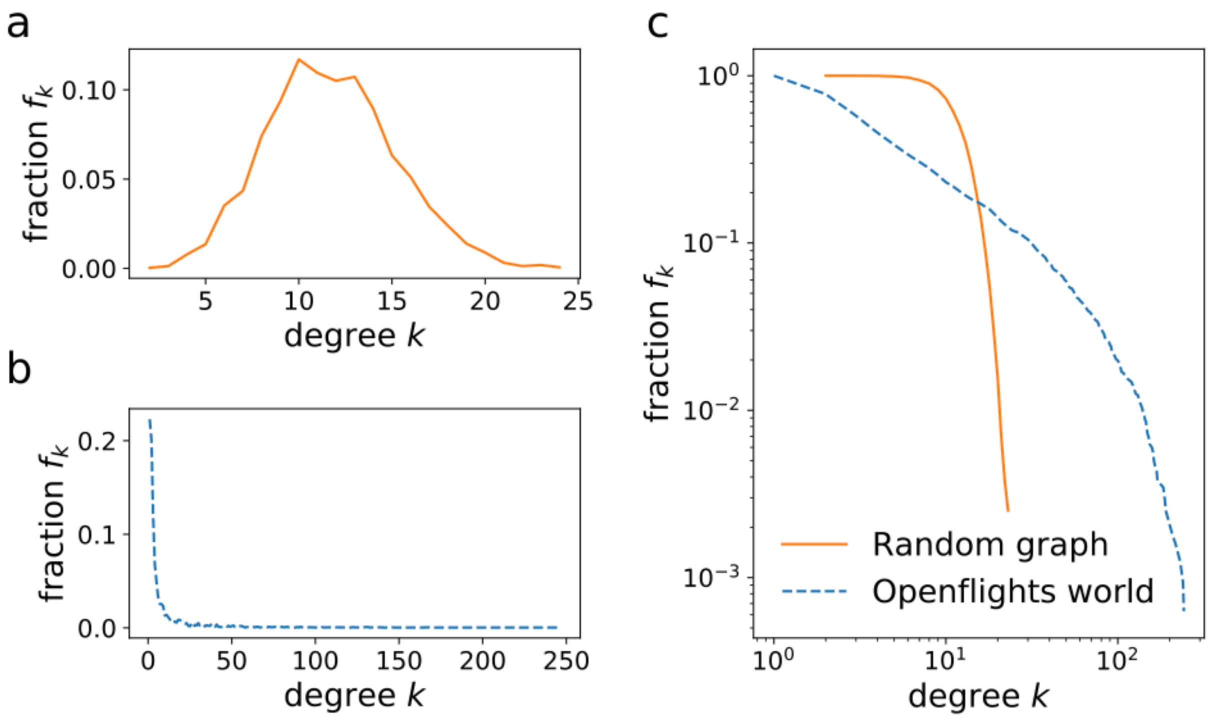
**Binomial distribution:**

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

For small $p$ and large $N$ the binomial distribution is well approximated by a bell-shaped curve $\Rightarrow$ **Most degree values are concentrated around the peak**, so the **average degree is a good descriptor of the distribution**

# Random networks: degree distribution

The degree distribution of random networks is **very different** from the broad distributions of most real-world networks!



# Random networks: small-world property

**Question:** How many nodes are there (on average) $d$ steps away from any node?

**Premise:** Since nodes have approximately the same degree, let us assume they all have the same degree $k$

- At distance $d = 1$ there are $k$ nodes
- At distance $d = 2$ there are $k(k–1)$ nodes
- At distance $d$ there are $k(k–1)^{d–1}$ nodes
- If $k$ is not too small, the total number of nodes within a distance $d$ from a given node is approximately:

$$N_d \sim k(k-1)^{d-1} \sim k^d$$

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## Random networks: small-world property

**Question:** how many steps does it take to cover the whole network?

$$N \sim k^{d_{max}}$$
$$log(N) \sim d_{max} log(k)$$
$$d_{max} \sim \frac{log(N)}{log(k)}$$

The diameter of the network **grows like the logarithm** of the network size

Example: N = 7,000,000,000, k =150 (Dunbar's number)

$$d_{max} = 4.52$$

# Random networks: clustering coefficient

The clustering coefficient of a node $i$ can be interpreted as the probability that two neighbors of $i$ are connected

$$C_i = \frac{\text{number of pairs of connected neighbors of i}}{\text{number of pairs of neighbors of i}}$$

**Question:** what is the probability that two neighbors of a node are connected?

**Answer:** since links are placed independently of each other, it is the probability $p$ that any two nodes of the graph are connected:

$$C_i = p = \frac{\langle k \rangle}{N-1} \sim \frac{\langle k \rangle}{N}$$

Since $\langle k \rangle$ is a small number, the average clustering coefficient of random networks with realistic values for $\langle k \rangle$ and $N$ is much smaller than the ones observed in real-world networks

## Random networks: summary

- Links are placed at **random**, independently of each other
- **Distances between pairs of nodes are short** (small-world property): **good!**
- The **average clustering coefficient is much lower** than on real networks of the same size and average degree: **bad!**
- The nodes have approximately the same degree; there are **no hubs**: **bad!**

**Conclusion:** the random network **is not a good model** for many real-world networks

In NetworkX:

```
G = nx.gnm_random_graph(N,L) # Erdős–Rényi random graph
G = nx.gnp_random_graph(N,p) # Gilbert random graph
```

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# Why random network models are important

Despite their limitations in capturing real network properties, random networks remain valuable:

- **Null models**: Serve as baseline for comparison
  - "Is this network property significant or expected by chance?"
  - Statistical testing against randomized models
- **Mathematical tractability**:
  - Have analytical solutions for many properties
  - Allow rigorous proofs and predictions
- **Historical significance**:
  - First systematic approach to model networks mathematically
  - Foundation for more complex models

# Small-world networks

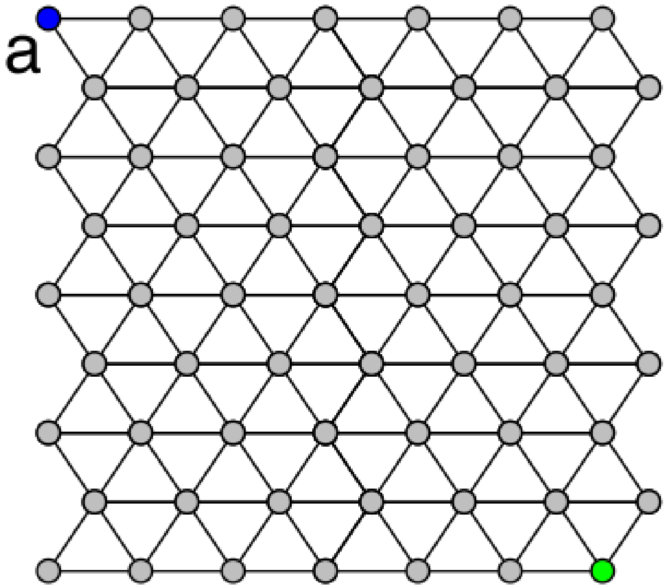
## Small-world networks

**Problem:** Real networks often have two seemingly contradictory properties:

- **High clustering** (many triangles, like in social networks)
- **Short average path lengths** (small-world property)

**Goal:** Create a model that captures both properties simultaneously

**Approach:** Start with a highly clustered structure and introduce shortcuts

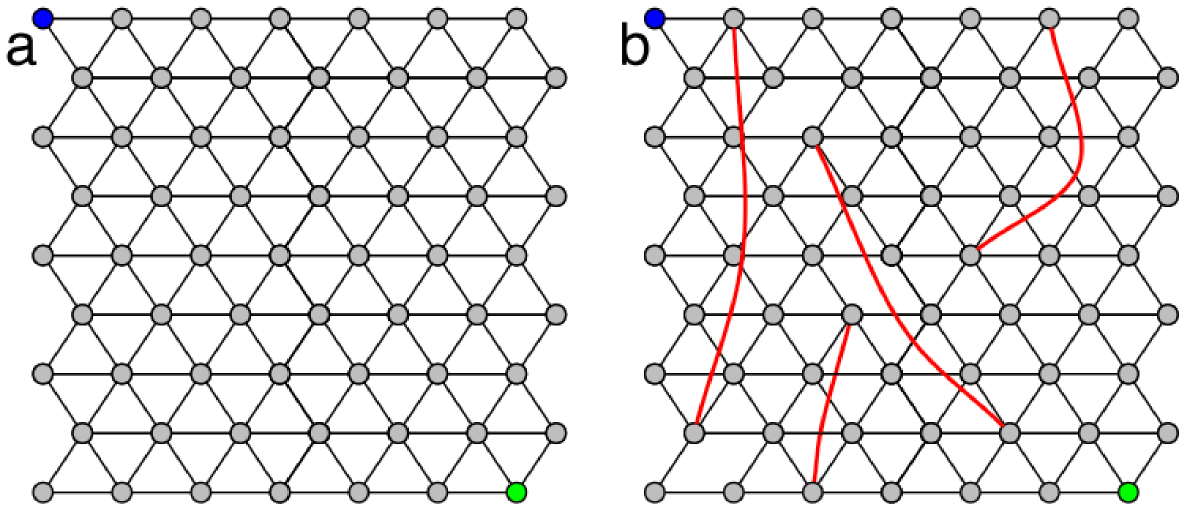


## Small-world networks

**Regular lattices have high clustering but long paths:**

- In a regular lattice, neighbors of a node tend to be connected (high clustering)
- But getting from one side to another requires many steps (long paths)
- This doesn't match real-world networks where paths are much shorter

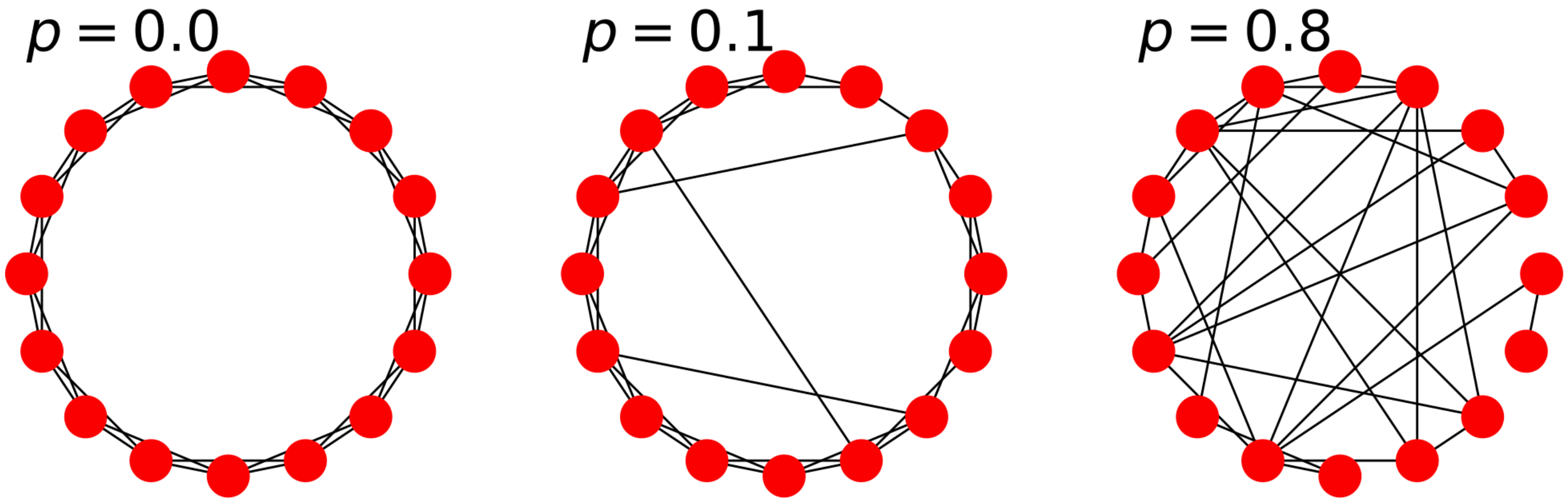
**The key insight:** Just a few random "shortcuts" can dramatically reduce path lengths while preserving most of the clustering



# The Watts-Strogatz model

**Model procedure:**

1. Start with a **regular ring lattice** of N nodes, each connected to k nearest neighbors
2. For each link, with probability p, **rewire one end to a randomly chosen node**
3. Continue until all links have been considered for rewiring
4. The final network is the Watts-Strogatz model with parameters N, k, and p

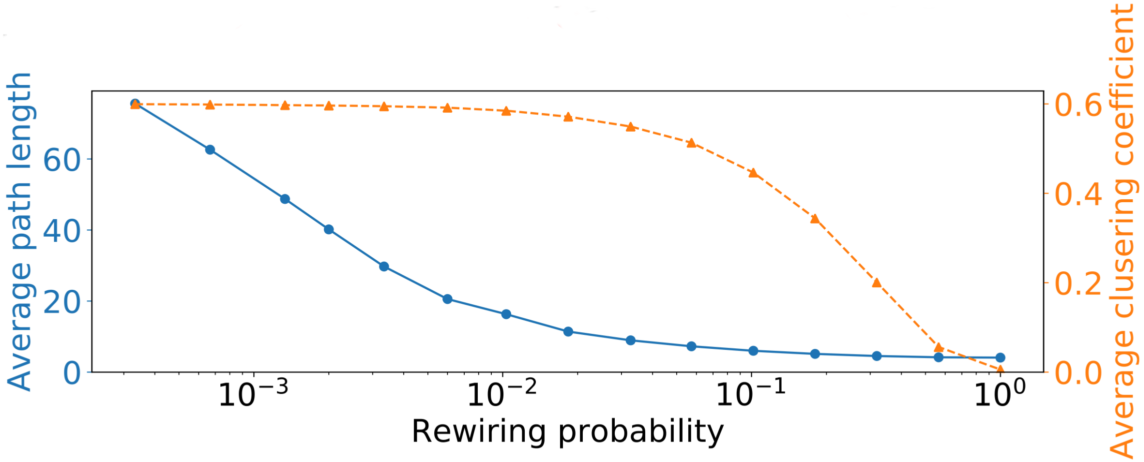


# The Watts-Strogatz model: Effect of rewiring probability

**The rewiring parameter p controls the network structure:**

- **p = 0**: Pure regular lattice (high clustering, long paths)
- **0 < p < 0.1**: Small-world network (high clustering, short paths)
- **p = 1**: Random network (low clustering, short paths)

**Key finding:** Even a small p (≈0.01-0.1) creates sufficient shortcuts to drastically reduce path lengths while maintaining most triangles



# The Watts-Strogatz model: Small-world region

**The "sweet spot"** for representing real-world networks:

- For a narrow range of p (typically $0.01 < p < 0.1$):
  - Average path length $L(p) \approx L(1) << L(0)$
  - Clustering coefficient $C(p) \approx C(0) >> C(1)$

**This means:** Networks in this region have both:

- Short paths like random networks
- High clustering like regular lattices

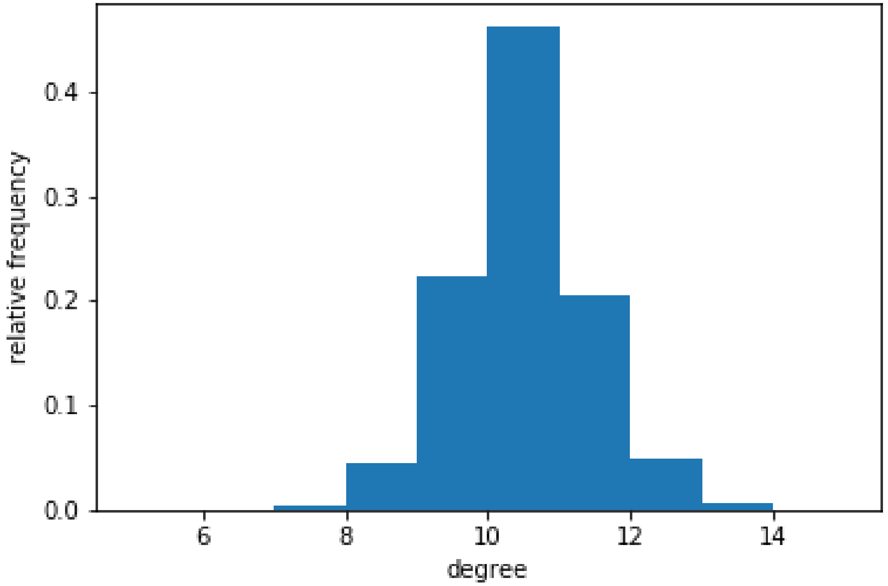
This combination matches many real-world networks like social networks, power grids, and neural networks.

[Example in NetLogo]

# The Watts-Strogatz model: degree distribution

**Limitations in representing real networks:**

- The degree distribution remains narrowly distributed around k
- Most nodes have similar degrees (k or slightly different)
- **No hubs** are generated, unlike many real-world networks
- Example: N = 10000, k = 10, p = 0.1



# The Watts-Strogatz model: summary

**Strengths:**

- Successfully models the coexistence of **high clustering** and **short paths**
- Explains the "small-world phenomenon" observed in many real networks
- Only needs a few random links to create efficient paths

**Limitations:**

- Fails to generate hubs and scale-free properties
- Starts with an artificial regular structure
- Assumes uniform rewiring across all nodes

In NetworkX:

```
G = nx.watts_strogatz_graph(N, k, p)
```

# The Configuration Model

## The Configuration Model

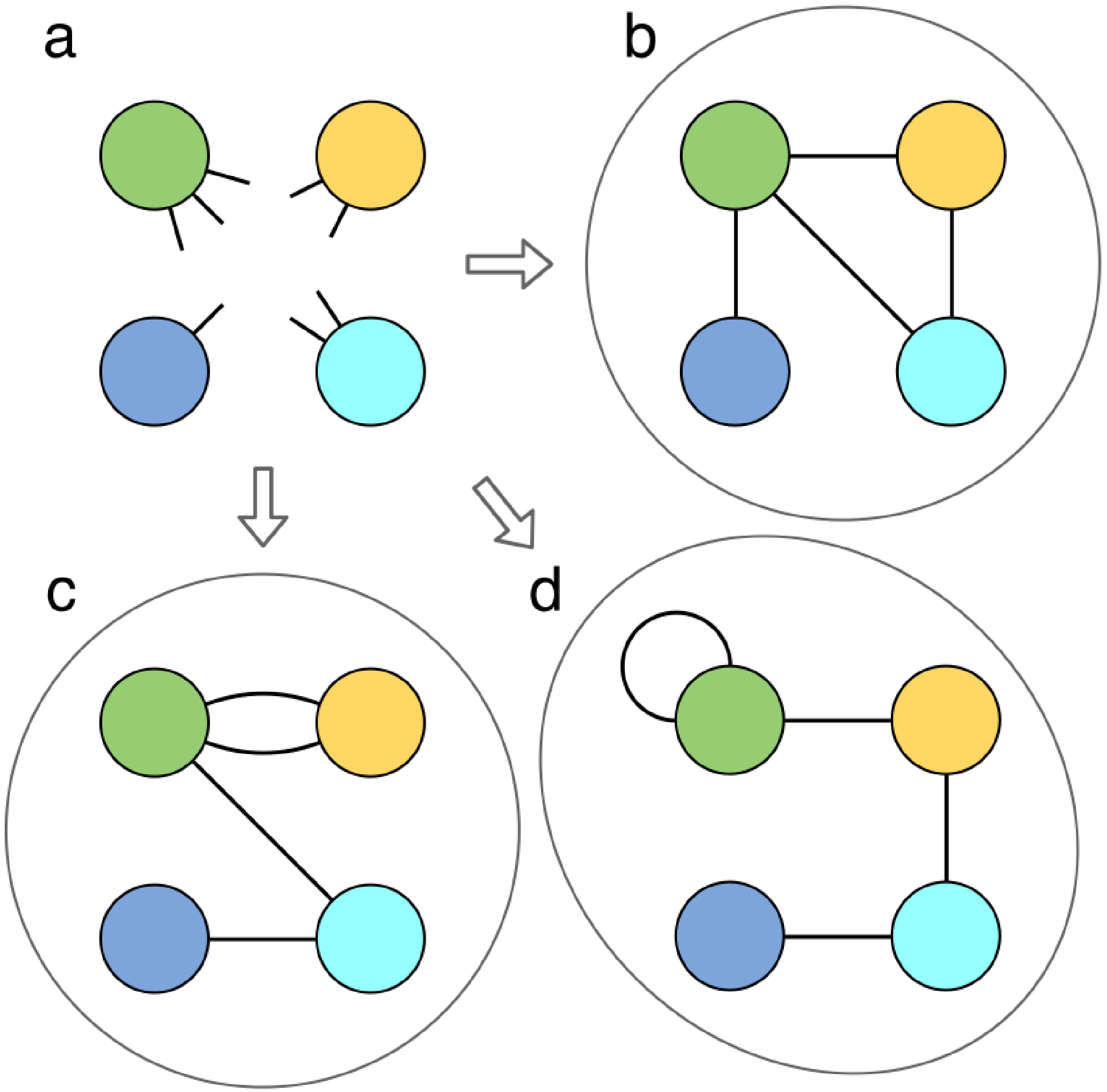
**Problem:** How can we build networks with a specific degree distribution?

**Solution:** The **configuration model** allows us to generate random networks with a prescribed degree sequence

**Core concept:** Instead of building networks with random connections (like Erdős-Rényi) or preferential attachment, we can directly control the degree of each node

**Application areas:**

- Creating null models to test hypotheses about network structure
- Studying the effect of degree distribution on network properties
- Generating synthetic networks with realistic degree distributions


a
b
c
d

# The Configuration Model: Algorithm

**Input:** A degree sequence $(k_1, k_2, \ldots, k_N)$ where $k_i$ is the desired degree of node $i$

**Procedure:**

1. Create $N$ nodes, assign $k_i$ "stubs" (half-edges) to each node $i$
2. Randomly pair stubs to form complete edges
3. Continue until all stubs are connected

**Important constraint:** The sum of degrees must be even: $\sum_{i=1}^{N} k_i = 2m$ (where $m$ is the number of edges)

**Note:** In practice, self-loops and multi-edges may be created and are typically removed or avoided through rewiring

# The Configuration Model: Properties

**Random mixing:** The model creates random connections while preserving the degree sequence

**Clustering coefficient:** Generally low (similar to random networks)

**Path lengths:** Similar to random networks, typically exhibiting the small-world property

**Degree correlations:** No inherent degree correlations (assortativity ≈ 0)

- However, constraints to avoid self-loops can introduce some disassortativity

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**In NetworkX:**

```
# From a degree sequence
degree_sequence = [3, 3, 3, 3, 4, 4, 5, 5]
G = nx.configuration_model(degree_sequence)

# From an existing graph's degree sequence
H = nx.Graph()  # some existing graph
G = nx.configuration_model(dict(H.degree()).values())
```

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## Degree-Preserving Randomization

**Goal:** Generate randomized versions of an existing network while preserving its degree sequence

**Procedure:**

1. Start with the original network
2. Repeatedly select two edges at random (e.g., A-B and C-D)
3. Rewire them (A-D and B-C) if no self-loops or multiple edges would result
4. Repeat many times until the network is sufficiently randomized

**Applications (two of many):**

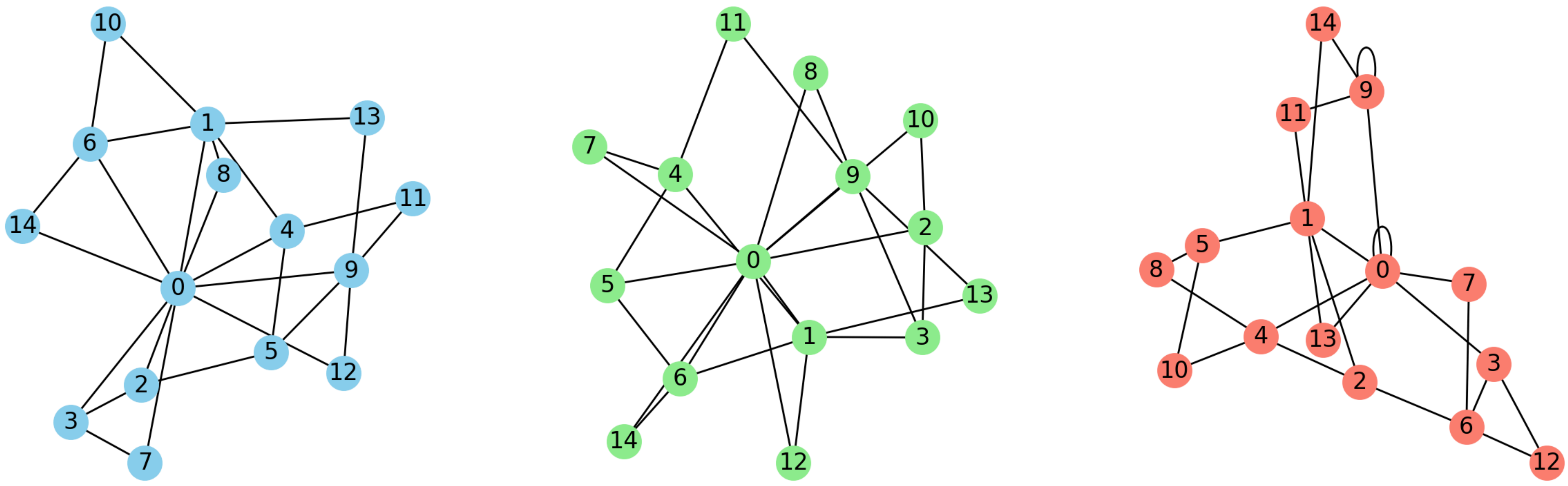
- **Statistical testing of network properties:**
  - Compare properties of real network vs. randomized versions
  - If property values differ significantly, they cannot be explained by degree sequence alone
  - Example: If clustering in real network is much higher than in randomized versions, it suggests meaningful triangles beyond what's expected by chance
- **Motif significance analysis:**
  - Generate many randomized networks preserving degree sequence
  - Count occurrences of subgraphs (motifs) in real and randomized networks
  - Calculate z-scores to identify statistically over/under-represented motifs
  - Common in biological networks to identify functional circuit patterns

- **Null models for community detection:**
    - Provides baseline expectation for modularity-based algorithms
    - Helps determine if observed community structure is statistically significant

**In NetworkX:**

```
G_rand = nx.double_edge_swap(G, nswap=1000)  # Rewire 1000 edge pairs
```

Original Graph | Double Edge Swap | Configuration Model Graph



# The Configuration Model: Limitations

**Self-loops and multiple edges:**

- The basic model can create self-loops (edges connecting a node to itself)
- It can also create multiple edges between the same node pairs

**Realizability:**

- Not all degree sequences can be realized as simple graphs

**Structure:**

- No community structure, clustering, or other mesoscale patterns
- Used primarily as a null model against which to compare real networks

## Reading material

References

[ns1] **Chapter 5 (Network Models)**

# Q&A