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Analisi e Visualizzazione delle Reti Complesse

NS16-17 - Games and Traffic Networks

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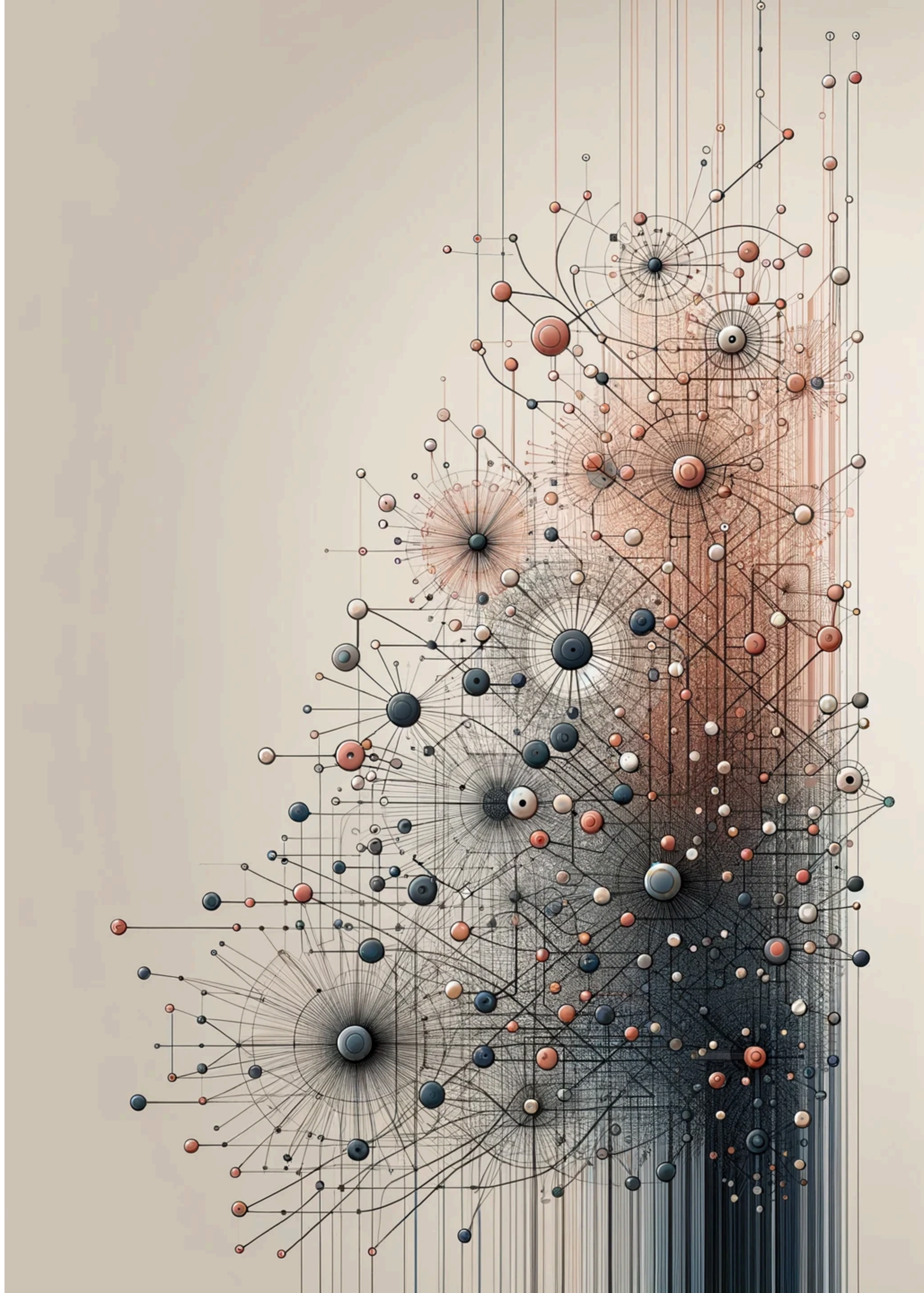

Agenda

- What is a Game?
- Reasoning about Behavior in a Game
- Best Responses and Dominant Strategies
- Nash Equilibrium
- Multiple Equilibria: Coordination Games
- Multiple Equilibria: The Hawk-Dove Game
- Mixed Strategies
- Mixed Strategies: Examples and Empirical Analysis
- Pareto-Optimality and Social Optimality

What is a Game?

- Complex networks describe the **interactions** between a set of items.
    - characterized by **intrinsic interdependence**
- Each decision maker has an **individual satisfaction** to maximize (e.g., a profit) and its **strategy depends also on other people's choices**.
- Game Theory gives us a simplified framework to understand how individual strategies can create an intrinsic interdependence in the behaviors of participants to a complex system

Basic Ingredients

1. **Players**
2. **Strategies**: set of options for each player
3. **Payoff**: the outcome for each selected strategy

That is summarized in a **payoff matrix**.

Assumptions:

- Everything that a player cares of is in the payoff matrix
  - e.g., the two players are solely concerned with their own payoff
- **Everything** about the structure of the game **is known**.
- Players are **rational**.

Example

- Two students have two large pieces of work due the next day: an **exam** and a **presentation**.
- **Assumptions**:
  - they cannot study for the exam AND prepare the presentation
  - they cannot communicate with each other
- **Exam**:
  - if one studies: gets 92 points
  - if one does not study: 80
- **Presentation**:
  - if one or (xor) the other prepare it: 92 for both
  - if neither of them prepare it: 84
  - if both of them prepare it: 100
- **Final vote**: average on the exam and presentation scores

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

Definition

- **strictly dominant strategy**: when a player has a strategy that is strictly better than all other options regardless of what the other player does
  - we should expect that the player will play it

Example

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	$90, 90$	$86, 92$
	Exam	$92, 86$	$88, 88$

- **Strict dominant strategy** for both players: **Exam**
  - both players get a payoff of 88
- **counterintuitive**: (P,P) would have been better off for both!
- **explanation**: if your partner decides to prepare the Presentation, you would be tempted anyhow to try the Exam since its your dominant strategy (payoff 92)

The Prisoner's dilemma

		Suspect 2	
		NC	C
Suspect 1	NC	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

- **Two suspects** have been apprehended by the police and interrogated.
- The police suspect the two individuals are responsible for the robbery, but there is **no evidence**.
- They both resisted arrest and can be charged with that lesser crime (1-year sentence)
- Suspects are asked to confess.
- Possibile strategie **Confess (C)** or **Not Confess (NC)**.
- The payoff matrix shows the penalties (the larger the better).
- Strictly dominant strategy for both: **Confess**.

Changing the payoff: different outcome

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	$98, 98$	$94, 96$
	Exam	$96, 94$	$92, 92$

- It arises only when payoffs are designed in a certain way
- simple changes versus more benign outcomes
  - e.g., an easier exam: you will get 96 if you don't study
- The the strict dominant strategy for both becomes Presentation!

Formalization: Best Responses

- **Players**: 1, 2 (it can be generalized for more players)
- **Strategies**: S, T (we can have more strategies)
- $P_1(S,T)$: payoff for P_1 playing S given T (fixed) played by P_2
- S is a **best response** for P_1: $\forall S' : P_1(S,T) \geq P_1(S',T)$
- S is a **strict best response** for P_1: $\forall S' : P_1(S,T) > P_1(S',T)$
- For P_2 we have **symmetrical definitions**

Formalization: Dominant Strategies

- **Dominant Strategy**: a P_1's strategy that is **best response** to every strategy of P_2
- **Strictly Dominant Strategy**: a P_1's strategy that is **strict best response** to every strategy of P_2

What if only one player has a strictly dominant strategy?

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

- New example:
  - two firms planning to produce and market a new product
- Two market segments:
  - people who would buy a **low-priced** version of the product (60%)
  - people who would buy a **upscale** version (40%)
- **Firm 1 is a much more popular brand**, when the two firms **directly compete** in a market segment, Firm 1 gets **80%** of the sales and Firm 2 gets **20%** of the sales.

What if only one player has a strictly dominant strategy?

		Firm 2	
		Low-Priced	*Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

- **Strictly dominant strategy for Firm 1: Low-Priced**
- **No dominant strategy for Firm 2!**
- Firm 2 can confidently predict that Firm 1 will play Low-Priced
  - Firm 1 has a strict dominat strategy and it wants to maximize its profit
- **Firm 2 will play Upscale**
  - Firm 2 is subordinate to Firm 1: **its best strategy is to stay away from Firm 1 market segment**
- Note that players move simultaneously, they have common knowledge of the game

What if none has a (strict) dominant strategy?

		Firm 2		
		A	B	C
Firm 1	A	$4,4$	$0,2$	$0,2$
	B	$0,0$	$1,1$	$0,2$
	C	$0,0$	$0,2$	$1,1$

- **Three-Client Game**: two firms (three clients: A, B, C)
- If the two firms approach the same client, then the client will give half its business to each.
- Firm 1 is too small to attract business on its own, so if it approaches one client while Firm 2 approaches a different one, then Firm 1 gets a payoff of 0.
- If Firm 2 approaches client B or C on its own, it will get their full business. However, A is a larger client, and will only do business with the firms if both approach A.
- Because A is a larger client, doing business with it is worth 8 (and hence 4 to each firm if it's split), while doing business with B or C is worth 2 (and hence 1 to each firm if it's split).

1

		Firm 2		
		A	B	C
Firm 1	A	$4, 4$	$0, 2$	$0, 2$
	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

- **Neither firm has a dominant strategy!**
- For Firm 1, A is a strict best response to strategy A by Firm 2, B is a strict best response to B, and C is a strict best response to C.
- For Firm 2, A is a strict best response to strategy A by Firm 1, C is a strict best response to B, and B is a strict best response to C

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Nash Equilibrium

- Even when there are no dominant strategies, we should expect players to use strategies that are best responses to each other.
- Suppose that Player 1 chooses a strategy S and Player 2 chooses a strategy T. We say that this **pair of strategies (S, T)** is a **Nash equilibrium** if **S is a best response to T**, and **T is a best response to S**.
- The idea is that if the players choose strategies that are best responses to each other, then **no player has an incentive to deviate to an alternative strategy**
    - concept of **equilibrium**
- In the Three-Client Game:
    - **(A,A) forms a Nash equilibrium**
    - No other pair of strategies are best responses to each other

Multiple Equilibria: Coordination Games

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

- Example:
  - Suppose you and a partner are each preparing slides for a joint project presentation;
  - you can't reach your partner by phone
  - you have to decide whether to prepare your half of the slides in PowerPoint or in Keynote.
  - Either would be fine, but it will be much easier if you use the same software.
- Players need to coordinate with no communication
- **Two different Nash Equilibria** (PowerPoint,PowerPoint) (Keynote,Keynote)

What to do?

- Thomas Schelling's idea of **focal points**:
- look for **natural reasons** to focus on one of the Nash equilibrium
  - possibly **outside the payoff structure** of the game
- (social) conventions can help
  - example of drivers approaching each other
- Try to embed in the payoff matrix the intrinsic features that help you to select an equilibrium **(Unbalanced Coordination Game)**

		Your Partner: *PowerPoint*	Your Partner: *Keynote*
You	*PowerPoint*	1, 1	0, 0
You	*Keynote*	0, 0	2, 2

The Hawk-Dove Game: another example of multiple equilibria

		Animal 2	
		D	H
Animal 1	D	$3,3$	$1,5$
	H	$5,1$	$0,0$

- Players engage in a kind of **anti-coordination** activity
- Two animals are engaged in a contest to decide how a piece of food will be divided between them
- Each animal can choose to behave **aggressively** (Hawk strategy) or **passively** (Dove strategy)
- If both behave passively, they **divide** the food evenly
- If one behaves aggressively while the other behaves passively, then **the aggressor gets most of the food**
- If both animals behave aggressively, then they **destroy the food**

The Hawk-Dove Game

		Animal 2	
		D	H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

- This game has two Nash equilibria: (D, H) and (H, D)
- Without knowing more about the animals we cannot predict which of these equilibria will be played
- Suppose we substitute two countries for the two animals
    - We would need to know more about the countries to predict which equilibrium will be played

Mixed strategies

- There are games which have **no Nash equilibria at all**
- Enlarging the set of strategies to include the possibility of **randomization**
- The simplest class of these games are called **attack-defense** games

Matching Pennies game

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

- Two people hold a penny, and simultaneously choose whether to show **heads** (H) or **tails** (T)
- Player 1 loses his penny to player 2 if they match, and wins player 2's penny if they don't match
- Example of **zero-sum games** (payoffs of the players sum to zero in every outcome)

Mixed Strategies

- The simplest way to introduce **randomized behavior** is to say that each player is not actually choosing H or T directly, but rather is choosing a probability with which she will play H.
- Strategies are **probabilities** between $[0, 1]$
  - P_1 chooses H with probability p (and T with probability $1 - p$)
  - P_2 chooses H with probability q (and T with probability $1 - q$)
- That means that each player chooses a **mixing** between the given strategies
- We can recollect the **pure strategies** when:

$$p = 0 \implies P_1 \quad \text{is playing} \quad T$$

$$p = 1 \implies P_1 \quad \text{is playing} \quad H$$

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Payoffs for Mixed Strategies

		Player 2 H	Player 2 T
Player 1	H	$-1, +1$	$+1, -1$
Player 1	T	$+1, -1$	$-1, +1$

- Payoffs are **random**. How to compare them?
- P_1 point of view:
- expected payoff of pure strategy Head

$$-1 \cdot q + 1 \cdot (1 - q) = 1 - 2q$$

- expected payoff of pure strategy Tail

$$1 \cdot q + (-1) \cdot (1 - q) = 2q - 1$$

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Equilibrium with mixed strategies

- We define a Nash equilibrium for the mixed strategy version just as we did for the pure-strategy version: **it is a pair of strategies (now probabilities) so that each is a best response to the other**.
- In any Nash equilibrium for the mixed-strategy version of Matching Pennies, we must have

$$1 - 2q = 2q - 1$$

- That implies $q = \frac{1}{2}$
- This is simmetric for P_2, obtaining $p = \frac{1}{2}$
- $1 - 2q \neq 2q - 1$ would have been impossible because $1 - 2q$ and $2q - 1$ are, respectively, the expected payoff of pure strategies Head and Tail. If these payoffs are not equal, then one would be strictly greater than the other, contradicting the principle that we have no Nash Equilibria!
- **The pair of strategies $p = \frac{1}{2}$ and $q = \frac{1}{2}$ is the only possibility for a Nash equilibrium.**

Interpretation of the "indifference principle"

- If P_1 believes that P_2 will choose Head more than half of the times, then s/he will win more than half of the times simply choosing Tail
- Symmetric reasoning applies for P_2 as well.
- The choice is un-exploitable for P_1.
- **indifference principle**: the choice of p and q are un-exploitable for the other player to decide their strategies
- Nash main result (deserving a Nobel prize): he proved that every such game **has at least one mixed-strategy equilibrium**

Optimalities

- We have Nash Equilibria such that each player's strategy is a best response to the other player's strategy
- Players are optimizing **individually**
- This does not mean that the **players will necessarily reach an outcome that is in any sense good or the best**
- It is possible to classify outcomes not just by their strategic or equilibrium properties, but also by whether they are good for ourselves and the others

Pareto Optimality

- First defintion from Vilfredo Pareto
  - A choice of strategies — one by each player — is **Pareto-optimal** if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.
- In other words, Pareto optimality is a situation where **no action or allocation is available that makes one individual better off without making another worse off**.
- A binding agreement to actually play the superior pairs of strategies is usually needed.

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

- (Exam, Exam) is the Nash equilibrium
- (Exam, Exam) is not Pareto optimal, since the outcome in which you both prepare for the presentation is strictly better for both of you
- The other pairs of strategies are all Pareto Optima
    - In fact, for (Exam, Presentation) and (Presentation, Exam) for example, although one of you is doing badly, there is no alternate choice of strategies in which everyone is doing at least as well
- Players have the incentive to change their strategy, unless they have a binding agreement

Social Optimality

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

- Stronger definition
- A choice of strategies, one by each players, is a social welfare maximizer (or social optimum) if it **maximizes the sum of the players' payoffs**.
- (Presentation, Presentation) is a Social Optimum (and also a Pareto Optimum).
- Outcomes that are socially optimal must also be Pareto-optimal.
- A Pareto-optimal outcome is not necessarily a socially optimum.
- A Nash Equilibrium is often not a social optimum.

Networks and Game Theory

- Nodes connected with many other nodes: if one agent has to select, for some given purpose, one (or some) connection out of your choices, then you need a strategy
- It is likely that agents will select the strategy that leads to the highest payoff
- A multi agents system: each agent will evaluate payoffs according their and everyone's else strategies
- **Traffic network:** individuals need to evaluate routes in the presence of congestion
  - congestion is the result of the decisions made by themselves and everyone else
- **Models for network traffic may lead to unexpected results**

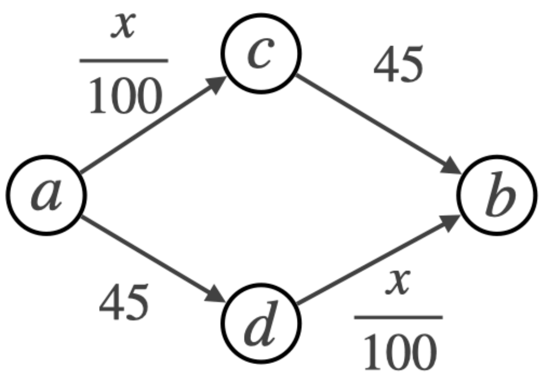
Traffic at Equilibrium

Agenda

- Traffic at Equilibrium
- Braess's Paradox
- The Social Cost of Traffic at Equilibrium

Transportation network model

- **Directed graph**
  - edges are highways
  - nodes are exits (you can get on or off a particular highway)
- **Assumption:** everyone wants to drive from a to b
- **Weights:** travel time
  - fixed
  - depending of the traffic x
- Suppose we have $x = 4000$ cars
- **The traffic game**:
  - players: drivers
  - each player's has 2 possible strategies that are the two routes from a to b
- **payoff:** the negative of a player's travel time (the faster the better)

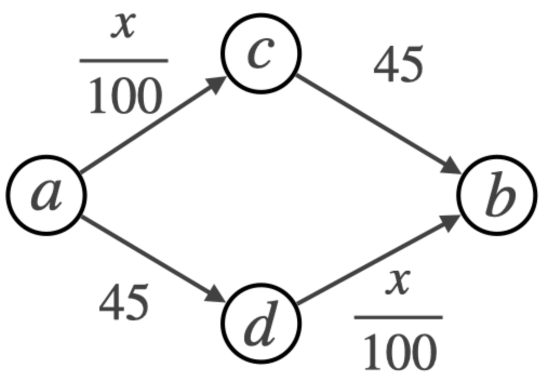


Games with more than 2 players

- As in the 2 players game:
  - the payoff of each player depends on the strategies chosen by all
  - **Nash equilibrium:** a list of strategies (one for each player), so that each one is a best response to all the others
  - The concept of **dominant strategies**, **mixed strategies**, **Nash equilibrium with mixed strategies**: they all have direct parallels

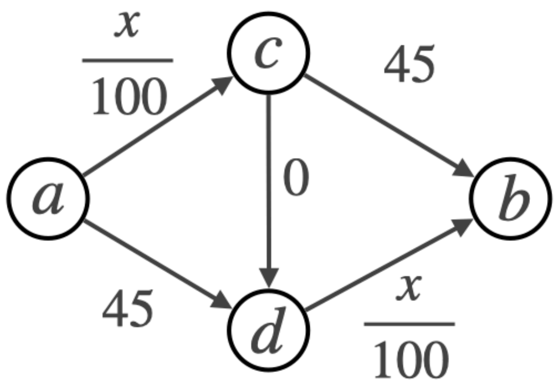
Equilibrium traffic

- **No dominant strategy** in a traffic game
  - either route has the potential to be the best choice for the player if all the other players are using the other route!
- **We have Nash equilibria**: any list of strategies in which the drivers balance themselves evenly between the two routes
  - with an even balance, no driver has an incentive to switch over to the other route
- Can we have a Nash equilibrium without an even balance?
  - Let's suppose we have x drivers using the upper route $1 - x$ drivers using the lower route
  - if $x \neq 1 - x$ then one route is slower than the other, and any driver in the slower route will have an incentive to switch to the other route!



Braess's Paradox

- Small changes can lead to counterintuitive results
- Let's build a new super fast highway from c to d
  - keep it simple: (c,d) travel time is 0
- New (and unique) Nash equilibrium: every driver uses the route through c and d, leading to **worse travel times**



The Braess's paradox

- Introduced by the German mathematician Daniel Braess (1968)
- Even if the NE route takes a longer travel time (80 mins), switching from it will take take 85 mins!
- The new highway acts like a **vortex** that attracts all the drivers into it - to the detriment of all
- There is no way, given self-interested behavior by the driver, to get back to the even balance solution that was better for everyone
- Like many counterintuitive anomalies:
  - it needs the right combination of conditions to actually pop up in real life
  - models $\neq$ reality!
  - however, it can explain some empirical observation in real transportation networks

Some observations

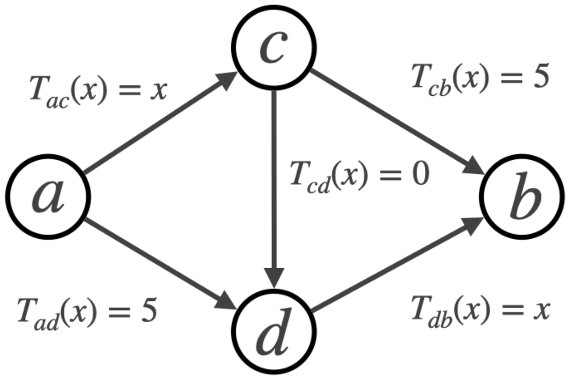
- After all, no paradox at all: very similar to the prisoner's dilemma
    - Single strategy game would imply the best outcome, adding a new strategy make it worse
- **Intuition:** upgrading is always a good thing (or also: having more strategies could only improve things)
- **Experience:** not always true
- A starting point for a large body of work on game-theoretic analysis of network traffic
- How much larger can the equilibrium travel time be after the addition of an edge, relative to what it was before?
- How can we design networks to prevent bad equilibria from arising?
- Reading material: Tim Roughgarden. Selfish Routing and the Price of Anarchy. MIT Press, 2005.

The Social Cost of Traffic at Equilibrium

Travel time function

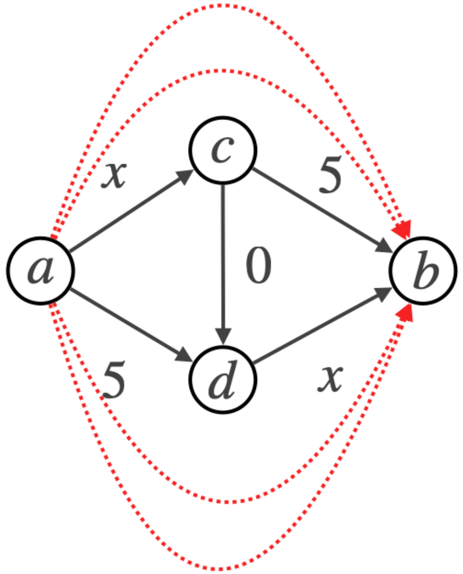
- We want to quantify how far from optimal is traffic at equilibrium
- Each edge e has a travel-time function $T_e(x)$
- **Assumption:** linear in the amount of traffic

$$T_e(x) = a_e x + b_e \quad \texttt{with} \quad a_e, b_e > 0$$



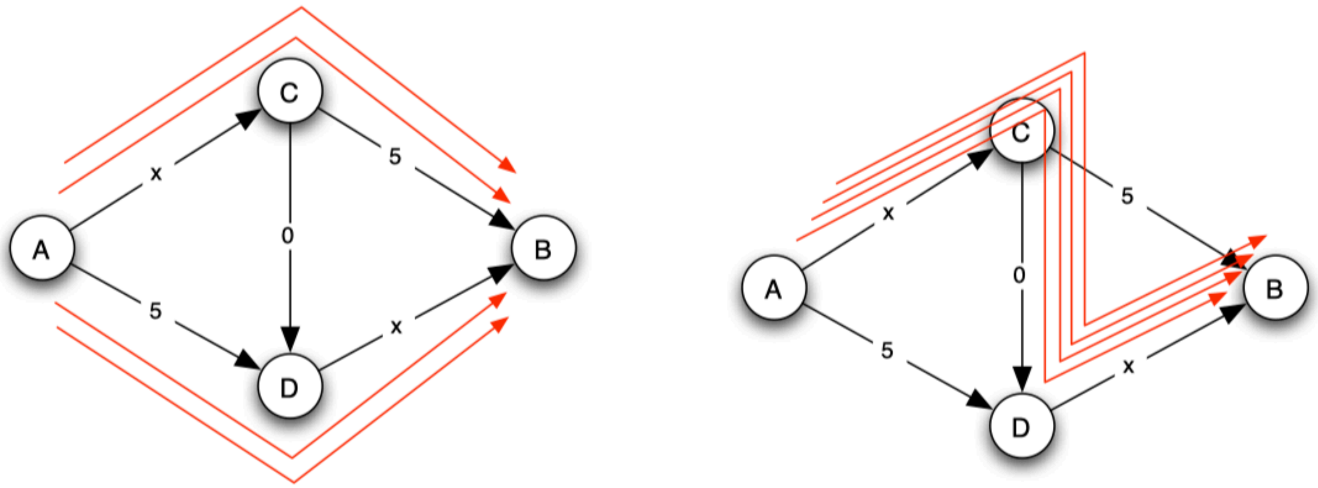
Traffic pattern

- A traffic pattern: a choice of a path by each driver
- **Social cost of a traffic pattern z:** the sum of the travel times incurred by all drivers when they use this traffic pattern
    - Ex: 4 drivers, each starting from a and with destination b
- When a traffic pattern achieves the minimum possible cost: **socially optimal**
- **socially optimal traffic patterns are social welfare maximizers in this traffic game**



Nash equilibrium

- **The unique Nash equilibrium in this game has a larger social cost**.
- Is there always an equilibrium traffic pattern?
- Does it always exists an equilibrium traffic pattern whose social cost is less than the social optimum?



(a) *The social optimum.* (b) *The Nash equilibrium.*

Finding a traffic pattern at equilibrium

- To prove that an equilibrium exists, let's use a procedure that looks for one:

1. Start from any traffic pattern
2. If it is an equilibrium, stop
3. Else, there is at least one driver whose best response is some **alternate path providing a strictly lower travel time**
4. pick one of these drivers and have her **switch to this alternate path**, then go to step 2.

- This procedure is called a **best-response dynamics**
    - We need to show that **best-response dynamics will eventually stop**.

Does a best-response dynamics always stop?

- **No.** In a zero sum game it will run forever because it lacks of an equilibrium (with pure strategies)
- In principle, even in the traffic game we can have a best-response dynamics that run forever **if we do not have an equilibrium**.
- **We will prove that in our traffic game the procedure stops**, proving consequently that:
    - equilibria exist
    - an equilibrium can be reached by a simple process in which drivers constantly update what they are doing according to their best response

Progress Measure

- To check if the best-response dynamics will eventually stop, we need a progress measure to track the process and to assess how far we are from the process to stop
- Is the social cost of the current traffic pattern a good progress measure?
    - **Answer: No**. In fact, some best-response updates by drivers can make the social cost better, but others can make it worse
    - The social cost of the current traffic pattern can oscillate, and the relationship with our progress toward an equilibrium is not clear
- The alternate quantity must strictly decrease with each best-response update

Potential Energy

- As a good progress measure, let's introduce the **potential energy** of an edge e:

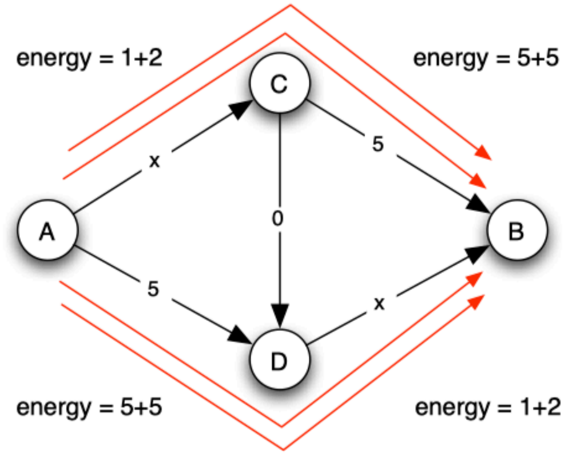
$$\texttt{Energy}(e) = T_e(1) + T_e(2) + \cdots + T_e(x)$$

- if an edge e has no driver on it:
    - $\texttt{Energy}(e) = 0$
- The potential energy of a traffic pattern z is the sum of all the potential energies of all the edges, with the current number of drivers in this traffic pattern:

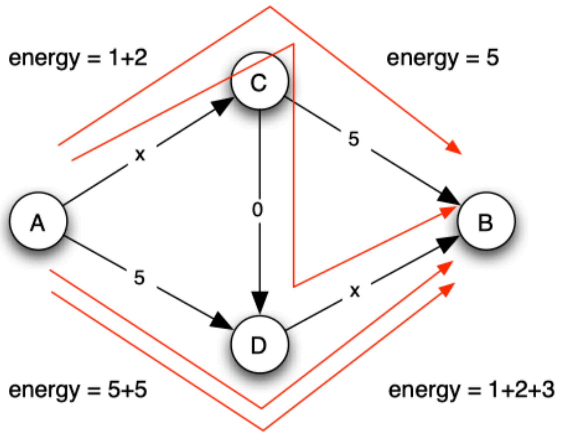
$$\texttt{Energy}(z) = \sum_{e_i \in z} \texttt{Energy}(e_i)$$

- $\texttt{Energy}(e) \neq xT_e(x)$
    - It is a sort of **cumulative** quantity: we imagine **drivers crossing the edge one by one**, and each driver only experiences the delay caused by themselves and the drivers crossing the edge in front of them.

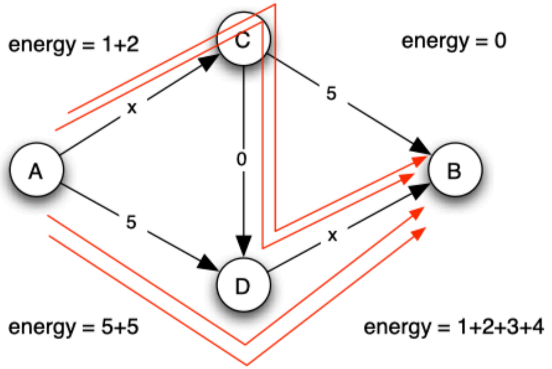
Example of best-response dynamics



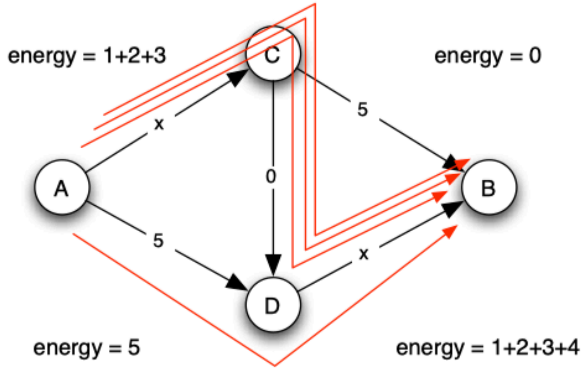
(a) *The initial traffic pattern. (Potential energy is 26.)*



(b) *After one step of best-response dynamics. (Potential energy is 24.)*



(c) *After two steps. (Potential energy is 23.)*



(d) *After three steps. (Potential energy is 21.)*

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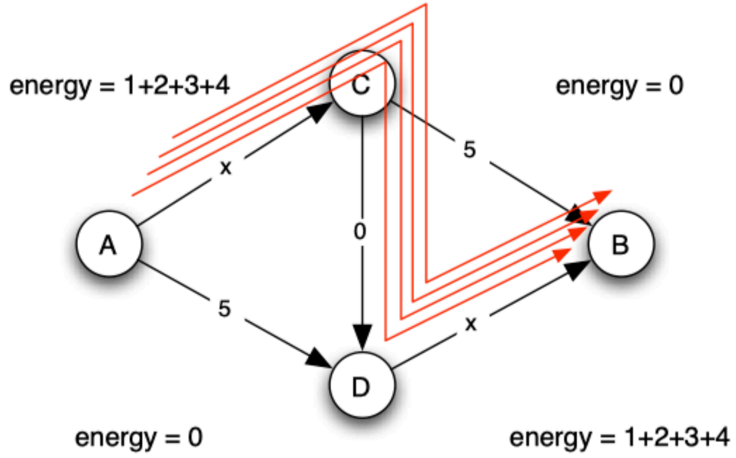
Example of best-response dynamics



(e) *After four steps: Equilibrium is reached. (Potential energy is 20.)*

- It's like a **two-steps** process:
  - first drivers abandons his current path, temporarily leaving the system; then, the driver returns to the system by adopting a new path. This **first step releases potential energy** as the driver leaves the system, and **the second step adds potential energy** as he re-joins.
  - What is the **net change**?

Does the best-response dynamics stop?

- If we prove that the best-response dynamics will stop, then we have proved that an equilibrium always exist
- That is equivalent to say:
    - If we prove that the potential energy strictly decreases at each step, then we have proved that best-response dynamics stops
- Observe in our example that potential energy always decreases at every step:
    - when a driver abandons one path in favor of another, **the change in potential energy is exactly the improvement in the driver's travel time**
    - and a driver always change to a better travel time due to the nature of the best-response dynamics
- **Is this true for any network and any best-response by a driver?**

Let's recall that the potential energy of edge e with x drivers is:

$$\texttt{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x-1) + T_e(x)$$

When one of these drivers leaves his current path, it drops to:

$$= T_e(1) + T_e(2) + \ldots + T_e(x-1)$$

Summing up

- $\texttt{Energy}(z)$ decreases according to all the travel times that the driver was experiencing on every edges in path z: $\sum_{e \in z} T_e(x)$
- It is like that, abandoning path z for the new path z', **the driver releases a potential energy that is equal to** $\sum_{e \in z} T_e(x)$

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By the same reasoning, for every edge e' in the new path z', before the new driver adopts it, we have this potential energy:

$$\texttt{Energy}(e') = T_{e'}(1) + T_{e'}(2) + \ldots + T_{e'}(x-1)$$

When one of the new driver joins it increases to:

$$= T_{e'}(1) + T_{e'}(2) + \ldots + T_{e'}(x-1) + T_{e'}(x)$$

Summing up, $\texttt{Energy}(z')$ increases according to all the travel times that the new driver is experiencing on every edges in path z': $\sum_{e' \in z'} T_{e'}(x)$

The net change in potential energy is the driver new travel time minus their old travel time

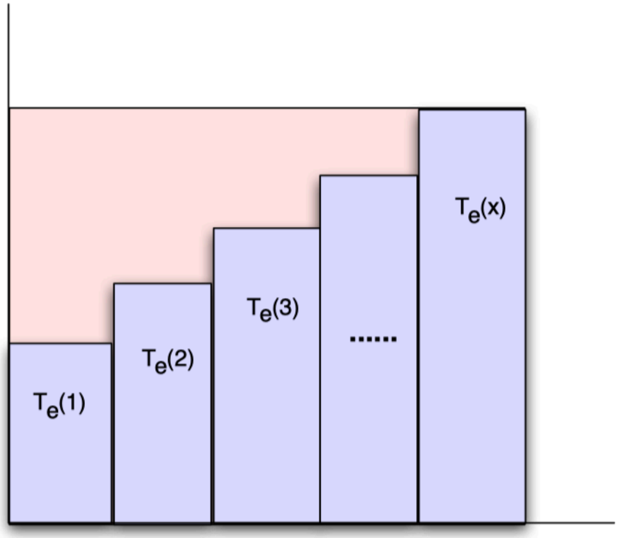
$$\Delta E = \sum_{e' \in z'} T_{e'}(x) - \sum_{e \in z} T_e(x)$$

ΔE must be negative, because driver must have an incentive to change path (the new strategy must be a best response) $\implies$ **the potential energy strictly decreases throughout the process**.

Comparing Equilibrium traffic to the Social Optimum

- We proved that an equilibrium traffic pattern always exists.
- How can we compare the travel time at equilibrium to that of a social optimum?
- Let's look for a relationship between the potential energy of an edge and the total travel time of all the drivers crossing the edge.
- Then we can sum up these quantities for all the edges in the traffic patterns and compare travel times at equilibrium and at social optimum.

Relating Potential Energy to Travel Time for a Single Edge



The potential energy is the area under the shaded rectangles; it is always at least half the total travel time, which is the area inside the enclosing rectangle (red).

$$
\begin{aligned}
\texttt{Energy}(e) &= T_e(1) + T_e(2) + \ldots + T_e(x) \\
&= a_e(1 + 2 + \ldots + x) + b_e x \\
&= \frac{a_e x(x+1)}{2} + b_e x \\
&= x\left(\frac{a_e(x+1)}{2} + b_e\right) \\
&\geq \frac{1}{2} x(a_e x + b_e) \\
&= \frac{1}{2} x T_e(x) \\
&= \frac{1}{2}\texttt{TTT}(e)
\end{aligned}
$$

Where $\texttt{TTT}(e) = xT_e(x)$ is the **total travel time** on edge e.

So we have:

$$
\texttt{Energy}(e) \geq \frac{1}{2}\texttt{TTT}(e)
$$

Wrapping up

We have that $\texttt{Energy}(e) \leq \texttt{TTT}(e)$ and $\texttt{Energy}(e) \geq \frac{1}{2}\texttt{TTT}(e) \implies \frac{1}{2}\texttt{TTT}(e) \leq \texttt{Energy}(e) \leq \texttt{TTT}(e)$

Moreover, if z is a traffic pattern, recall that:

$$\texttt{Energy}(z) = \sum_{e_i \in z} \texttt{Energy}(e)$$

Recall also that the **social cost ($\texttt{SC}(z)$)** of traffic pattern z is the sum of the travel times incurred by all drivers when they use this traffic pattern:

$$\texttt{SC}(z) = \sum_{e \in z} \texttt{TTT}(e)$$

Finally, recall that the potential energy decreases as best-response dynamics moves from z to z': $\texttt{Energy}(z') \leq \texttt{Energy}(z)$

Travel time at equilibrium and at social optimality

If z is the traffic pattern at social optimality, and z' is the traffic pattern at the end of the best-response dynamics (i.e., at equilibrium), we have that:

$$\texttt{Energy}(z') \leq \texttt{Energy}(z)$$

Moreover, we have:

$$\texttt{SC}(z') = \sum_{e' \in z'} \texttt{TTT}(e') \leq \sum_{e' \in z'} 2 \cdot \texttt{Energy}(e') = 2 \cdot \sum_{e' \in z'} \texttt{Energy}(e') = 2 \cdot \texttt{Energy}(z')$$

and

$$\texttt{Energy}(z) = \sum_{e \in z} \texttt{Energy}(e) \leq \sum_{e \in z} \texttt{TTT}(e) = \texttt{SC}(z)$$

then

$$\texttt{SC}(z') \leq 2 \cdot \texttt{Energy}(z') \leq 2 \cdot \texttt{Energy}(z) \leq 2 \cdot \texttt{SC}(z)$$

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Conclusions

- We found that:
  - in the traffic game we can always find a traffic pattern at equilibrium
  - the social cost of the traffic pattern at equilibrium is at most twice the socially optimal cost (we found a bound!)
- It is also possible to find a better bound: traffic pattern social cost at equilibrium is no more than 4/3 times as large than socially optimal traffic pattern
  - Reading material: Anshelevich et. al, The price of stability for network design with fair cost allocation, 2004, at Foundations of Computer Science, 1975., 16th Annual Symposium on 38(4):295- 304

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https://www.nytimes.com/1990/12/25/health/what-if-they-closed-42d-street-and-nobody-noticed.html

What if They Closed 42d Street and Nobody Noticed?

By Gina Kolata

Dec. 25, 1990

ON Earth Day this year, New York City's Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."

But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed.

To mathematicians, this may be a real-world example of Braess's paradox, a statistical theorem that holds that when a network of streets is already jammed with vehicles, adding a new street can make traffic flow even more slowly. Seeking Out a New Street

The reason is that in crowded conditions, drivers will pile into a new street, clogging both it and the streets that provide access to it. By the same token, removing a major thoroughfare may actually ease congestion on the streets that normally provide access to it.

Cheonggyecheon Restoration Project in Seoul

- As part of this project, a six-lane highway was removed to create waterways and recreation parks.
- Despite expectations that traffic congestion would worsen, the change actually decreased traffic speed in the area.
- Reading material: [Cheonggyecheon Restoration Project]

Modeling urban street patterns

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(Dated: April 2, 2008)

Urban street patterns form planar networks whose empirical properties cannot be accounted for by simple models such as regular grids or Voronoi tesselations. Striking statistical regularities across different cities have been recently empirically found, suggesting that a general and details-independent mechanism may be in action. We propose a simple model based on a local optimization process combined with ideas previously proposed in studies of leaf pattern formation. The statistical properties of this model are in good agreement with the observed empirical patterns. Our results thus suggests that in the absence of a global design strategy, the evolution of many different transportation networks indeed follow a simple universal mechanism.

PACS numbers: 89.75.-k, 89.75.Kd, 89.65.Lm

- https://arxiv.org/pdf/0708.4360.pdf

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Reading material

[ns2] **Chapter 6 (6.1-6.9)** Games

[ns2] **Chapter 8 (8.1 - 8.3)** Modeling Network Traffic using Game Theory

Q&A