Calculus Review

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First Semester

1. Limits
2. Derivatives
3. Derivatives II
4. Curve Sketching
5. Differentials and Antiderivatives
6. Definite Integration
7. Logarithmic and Exponential Functions

Second Semester

8. Exponential and Logistic Growth, Inverse Trig
9. Area, Volume, Arc Length, Surface Area
10. Advanced Integration Methods
11. Convergence Tests
12. Convergence and Error Analysis
13. Polynomials and Series
14. Parametric, Polar, Vectors

**: not on AP test, but tested in class

1 Limits

1.1 Format

$$\lim_{x \to c} f(x) = L$$

1.2 Properties

1. Constant:

   $$\lim_{x \to c} b = b \text{ , where b is a real number}$$

2. Scalar multiple:

   $$\lim_{x \to c} b(f(x)) = b\left(\lim_{x \to c} f(x)\right) = bL$$

3. Sum and difference:

   $$\lim_{x \to c}(f \pm g) = \lim_{x \to c} f \pm \lim_{x \to c} g$$

4. Product:

   $$\lim_{x \to c}(f \cdot g) = \lim_{x \to c} f \cdot \lim_{x \to c} g$$

5. Quotient:

   $$\lim_{x \to c} \frac{f}{g} = \lim_{x \to c} f \div \lim_{x \to c} g \quad \text{if} \quad g \neq 0$$

6. Power:

   $$\lim_{x \to c}[f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n \quad \text{if n is a positive integer}$$

1.3 Solving

Employ these methods if direct evaluation yields an indeterminate form (usually $\frac{0}{0}$)

1. Factor and cancel
2. Rationalization: if radical binomial, multiply top and bottom by conjugate of binomial

   $$\text{Example: } \lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \cdot \left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

3. Fraction in fraction: multiply top and bottom by common denominator of "little fractions"

   $$\text{Example: } \lim_{x \to 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} \cdot \left(\frac{4(x+4)}{4(x+4)}\right)$$

1.4 Trig Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

1.5 Continuity

- <u>Definition of continuity</u>:

$$\lim_{x\to c} f(x) = f(c)$$

- <u>Properties</u>: if f and g are continuous at $x = c$, then all of the following are continuous at $x = c$:

  1. bf, where b is a real number
  2. $f \pm g$
  3. fg
  4. f/g, for $g \neq 0$
  5. $f(g(x))$

1.6 Intermediate value theorem (IVT)

If f is continuous on $[a, b]$ and if k is any number between $f(a)$ and $f(b)$, then there exists a c on $[a, b]$ such that $f(c) = k$

2 Derivatives

2.1 Definition of derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

2.2 Derivative at a point

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

2.3 When derivatives fail

- At a sharp turn/point/cusp (left derivative $\neq$ right derivative)
- At any discontinuity (hole, asymptote, break/jump)
- At a vertical tangent

2.4 Power rule

$$\frac{d}{dx} x^n = nx^{n-1}$$

2.5 Trig derivatives

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

2.6 Rate of change

- Tangent line: $y - y_1 = m(x - x_1)$
- Instantaneous rate of change: the derivative; $f'(x)$
- Average rate of change: secant line approximation between two points:

$$f_{avg} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

- Normal line: perpendicular to tangent line at point of tangency

3 Derivatives II

3.1 Product and quotient rule

- Product rule:

$$\frac{d}{dx}\left(f \cdot g\right) = f'g + g'f$$

- Quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - g'f}{g^2}$$

3.2 Trig derivatives

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

3.3 Chain rule

$$\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$$

3.4 Implicit vs. explicit differentiation

- Explicit: given that y is a function of x

Example: $y = 2x + 4 \implies \dfrac{dy}{dx} = 2$

- Implicit: y is taken to be a function of x

Example: $\dfrac{d}{dx}(x^2 + y^2 = 4) \implies 2x + 2y\dfrac{dy}{dx} = 0 \implies \dfrac{dy}{dx} = -\dfrac{x}{y}$

4 Curve Sketching

4.1 Mean value theorem (MVT)

If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists a value c on (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \implies \text{(instantaneous derivative at c)} = \text{(avg. rate of change over } (a, b))$$

4.2 Rolle's theorem

Basically, MVT when avg. rate of change = 0 on (a, b):
If f is continuous on $[a, b]$, differentiable on (a, b), AND if $f(a) = f(b)$, then there exists a value c on (a, b) such that $f'(c) = 0$

4.3 Curve sketching

4.3.1 Interpretation

- f': slope
- f'': concavity (up: $\cup$, down: $\cap$)

4.3.2 Definitions

- Critical numbers: x-values where $f' = 0$ OR undefined; called extrema if f' changes sign (max/min)
- Absolute extrema: absolute max/min; check when $f' = 0$, undefined, and check relevant endpoints
- Point of inflection: where f'' changes sign

4.3.3 Tests

- First derivative test: find where $f' = 0$ or undefined, do number line test to see where f' changes sign
  - ☐ f' goes $(+) \to (-) \implies$ relative maximum
  - ☐ f' goes $(-) \to (+) \implies$ relative minimum
- Second derivative test: for any specific point, if $f' = 0$ at the point and also...
  - ☐ $f'' < 0 \implies$ relative maximum
  - ☐ $f'' > 0 \implies$ relative minimum
- "Concavity test": solve for when $f'' = 0$ or undefined; if f'' changes sign for number line test, then point of inflection
- Candidates test: test all critical numbers and endpoints; compare $f(x)$ values to find absolute max/min

5 Differentials and Antiderivatives

5.1 Local linearization

Use tangent line to approximate value of a function

$$\frac{dy}{dx} = f'(x) \implies y_2 = y_1 + f'(x)(x_2 - x_1)$$

$$(x_2, y_2) = \text{ point in question; example: } \sqrt{3.3} \implies x_2 = 3.3,\ y_2 = \sqrt{3.3},\ f(x) = \sqrt{x}$$

$$(x_1, y_1) = \text{ chosen approximation point: for } \sqrt{3.3},\ \text{choose } x_1 = 4,\ y_1 = \sqrt{4} = 2$$

5.2 Error propagation

Using $dy = f'(x)dx$, y error $= \pm dy$, typically with knowledge of x error dx

Percent error:

$$\% \text{ error} = \frac{\text{error}}{\text{total}} = \frac{dy}{y}$$

5.3 Indefinite integration

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$$

$$\int f'(x)dx = f(x) + c$$

5.4 Integration properties

$$\int (f \pm g)dx = \int f dx \pm \int g dx$$

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx$$

5.5 Power rule for integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

5.6 Trig integrals

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int \csc^2 x dx = -\cot x + c$$

$$\int \sec x \tan x dx = \sec x + c$$

$$\int \csc x \cot x dx = -\csc x + c$$

5.7 Basic differential equations

Goal: find the function whose derivative is given

Format for basic problem: $\frac{dy}{dx} = f'(x)$

5.8 Euler's method

Goal: approximate a value over iterations given a step size dx

Formula: $y_2 = y_1 + f'(x_1)dx$

Example: Given $f(0) = 1$, use Euler's method with a step size of 0.5 to approximate $f(1)$ for $\frac{dy}{dx} = xy^2$

Use $y_2 = y_1 + xy^2 \cdot dx$

1. $f(0.5) \approx f(0) + f'(0) \cdot dx = 1 + (0)(1)^2 \cdot (0.5) = 1 \implies$ use $(0.5, 1)$ for the next step
2. $f(1) \approx f(0.5) + f'(0.5) \cdot dx = 1 + (0.5)(1)^2 \cdot (0.5) = 1.25$

Thus, $f(1) \approx 1.25$ by the methodology above (a smaller step size and a smaller distance between the given and approximated point $\implies$ a better estimate)

5.9 U-substitution

"The chain rule backwards" $\implies$ substitute x and dx in the integral for an easier to use variable, u (and du)

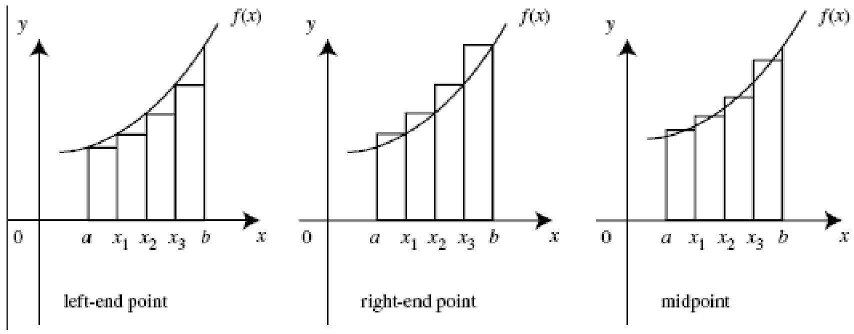
Example: $\int (2x+1)^2 dx = \frac{1}{2} \int u^2 du$ where $u = 2x + 1 \implies du = 2dx$

6 Definite Integration

6.1 Riemann sums

Use area of n rectangles to approximate an integral

Depending on graph, can have under-estimate or over-estimate



6.2 Area by limit definition

Procedure:

- Set up right Riemann sum with n rectangles
- Width of each rectangle $= \frac{\text{interval}}{n} = \frac{b-a}{n}$

Summation formulas:

$$\sum_{i=1}^{n} C = Cn \text{ where } C \text{ is a constant}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

6.3 First fundamental theorem of calculus

If f is a continuous function on $[a, b]$ and if F is the antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

6.4 Properties of definite integrals

$$\int_a^b (f \pm g)dx = \int_a^b f dx \pm \int_a^b g dx$$

$$\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f dx$$

$$\int_a^a f dx = 0$$

$$\int_a^b f dx = -\int_b^a f dx$$

$$\int_a^b f dx + \int_b^c f dx = \int_a^c f dx$$

6.5 MVT for integrals

If f is continuous on $[a, b]$, then there exists a c on $[a, b]$ such that:

$$\int_a^b f(x)dx = f(c)(b-a) \implies \text{(area of } f \text{ from } a \text{ to } b\text{)} = \text{(area of "perfect" rectangle)}$$

6.6 Average value

$$\text{Average value} = \frac{\text{integral}}{\text{interval}} = \frac{\int_a^b f(x)dx}{b-a}$$

Notice that the avg. value $= f(c)$ by comparison to MVT for integrals

6.7 True u-substitution

Change x, dx, AND limits of integration into u-values

Can use u-variables for entire definite integration, or return to x-variables for evaluating step

Example:

$$\int_0^2 (2x+1)^2 dx = \frac{1}{2}\int_1^5 u^2 dx \text{ since } u = 2x+1 \implies u(0) = 1, u(2) = 5$$

Two options for evaluation:

$$\frac{1}{3}u^3 \Big|_1^5$$

$$\frac{1}{3}(2x+1)^3 \Big|_0^2$$

Both yield the correct answer.

6.8 Second fundamental theorem of calculus

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_a^u f(t)dt = f(u)\cdot u'$$

6.9 Trapezoidal rule

$$A = \frac{1}{2}\left(\frac{b-a}{n}\right)\left(f(x_0) + 2\left(f(x_1) + f(x_2) + ... + f(x_{n-1})\right) + f(x_n)\right)$$

Done by repeatedly applying $A_{trapezoid} = \dfrac{1}{2}h(b_1 + b_2)$

7 Logarithmic and Exponential Functions

7.1 Derivative of $\ln x$

$$\ln x = \int_1^x \frac{1}{t} dt$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

7.2 Log properties

$$\ln x^y = y \ln x$$

$$\ln(xy) = \ln x + \ln y$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln 1 = 0, \;\; \ln e = 1, \;\; \ln e^n = n$$

7.3 Logarithmic differentiation

Take ln of both sides and use log properties to simplify derivative

Example: find y' for $y = \dfrac{(x-2)^2}{x\sqrt{x^2+1}} \implies \ln y = 2\ln(x-2) - \ln x - \dfrac{1}{2}\ln(x^2+1)$

7.4 Log rule for integration

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int \frac{u'}{u} = \ln|u| + c$$

7.5 Trig integrals

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \tan x dx = -\ln|\cos x| + c$$

$$\int \cot x dx = \ln|\sin x| + c$$

$$\int \sec x dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x dx = -\ln|\csc x + \cot x| + c$$

7.6 Derivative of an inverse

The inverse of a function is also a function if:

- <u>One-to-one</u>: $f(a) = f(b) \implies a = b$; i.e. vertical line test
- <u>Monotonic</u>: one-directional; f is always increasing/decreasing

<u>Derivative of an inverse</u>: the derivative of f at (b, a) is the reciprocal of the derivative of f^{-1} at (a, b)

$$\implies \frac{d}{dx} f^{-1}(a, b) = \frac{1}{f'(b, a)}$$

7.7 Derivatives and integrals of e^u

$$\frac{d}{dx} e^u = u' \cdot e^u$$

$$\int e^u du = e^u + c$$

7.8 Derivatives and integrals of a^u and $\log_a u$

$$\frac{d}{dx} \log_a u = \frac{u'}{u \ln a}$$

$$\frac{d}{dx} a^u = u' \cdot a^u \ln a$$

$$\int a^u du = \frac{a^u}{\ln a}$$

8 Exponential and Logistic Growth, Inverse Trig

8.1 Exponential growth and decay

Law of exponential growth: $\frac{dy}{dt} = ky \implies y = Ce^{kt}$

8.2 Logistic growth

Logistic differential equation: $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$, with k = growth rate and L = carrying capacity

Alternative formula: $\dfrac{dy}{dt} = \dfrac{k}{L}y(L-y)$

Logistic function:

$$y = \frac{L}{1 + be^{-kt}}$$

8.3 Derivatives of inverse trig functions

$$\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arccos u = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}\operatorname{arccot} u = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}\operatorname{arcsec} u = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}\operatorname{arccsc} u = \frac{-u'}{|u|\sqrt{u^2-1}}$$

8.4 Inverse trig integrals

$$\int \frac{du}{1+u^2} = \arctan u + c$$

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}|u| + c$$

9 Area, Volume, Arc Length, Surface Area

9.1 Area between curves

Subtract (top - bottom) or (right - left)
Example: $\int_a^b \Big(f(x) - g(x) \Big) dx$

9.2 Volume by cross-section

- Square: $V = s^2 h$, where $h = dx$ or dy
- Rectangle: $V = lwh$
- Equilateral triangle: $V = \frac{\sqrt{3}}{4} b^2 h$
- General triangle: $V = \frac{1}{2} bh \cdot H$, where $H = dx$ or dy

9.3 Volume by discs, washers, or shells

- Discs: $V = \pi r^2 h$
- Washers: $V = \pi(R^2 - r^2)h$
- Shells: $V = 2\pi rhw$

9.4 Arc Length

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

9.5 Surface Area

$$SA = 2\pi \int_a^b r \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

10 Advanced Integration Methods

10.1 Integration by parts

$$\int u dv = uv - \int v du$$

u: derivative gets simpler; usually power function

dv: easily integrable

Integration by parts table: for polynomials

Alternating +/-	u and its derivatives	dv and its antiderivatives
+		
-		
+		

10.2 Powers of trig

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

10.3 Trig substitution**

$$\sqrt{a^2-u^2} \rightarrow \text{ use } \sin\theta = \frac{u}{a}$$

$$\sqrt{a^2+u^2} \rightarrow \text{ use } \tan\theta = \frac{u}{a}$$

$$\sqrt{u^2-a^2} \rightarrow \text{ use } \sec\theta = \frac{u}{a}$$

10.4 Partial fractions (roots with multiplicity 1)

Example:

$$\frac{7}{x^2-3x-10} = \frac{7}{(x-5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2} \text{ ; solve for } A, B$$

10.5 L'Hopital's rule

$$\text{If } \lim_{x\to c} \frac{f}{g} = \text{ indeterminate, then evaluate } \lim_{x\to c} \frac{f'}{g'}$$

10.6 Improper integrals

If $\int_a^b f(x)dx$ has infinite limits or if f has a discontinuity on $[a,b]$

Can evaluate the integral with infinite limits if certain function limits exist (i.e. the integral converges)

11 Convergence Tests

11.1 N^{th} term test

If terms of series do not approach zero, then the series will diverge.

Convergence	Divergence
N/A	$\lim\limits_{n \to \infty} a_n \neq 0$

11.2 Geometric series test

Form: $\sum\limits_{n=0}^{\infty} a(r)^n$

Convergence	Divergence
$\lvert r \rvert < 1$	$\lvert r \rvert \geq 1$

Sum of geometric series: $S = \dfrac{a_0}{1-r}$

11.3 Telescoping series test**

Form: $\sum\limits_{n=1}^{\infty} (b_n - b_{n+1})$; typically found when denominator of rational factors nicely

Convergence	Divergence
Expand terms to see what cancels	N/A

11.4 Integral test

If f is positive, continuous, and decreasing for $x \geq 1$ and if $a_n = f(x)$, then, for $\sum\limits_{n=1}^{\infty} a_n$:

Convergence	Divergence
$\int\limits_1^{\infty} f(x)dx$ converges	$\int\limits_1^{\infty} f(x)dx$ diverges

*Use if integral is easy to take

11.5 P-series test

Form: $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$

Convergence	Divergence
$p > 1$	$0 \leq p \leq 1$

11.6 Direct comparison test

Consider $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$

Convergence	Divergence
If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges	If $b_n \leq a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

*Use with mixture of exponential, power, factorial

11.7 Limit comparison test

Consider $\sum a_n$ and $\sum b_n$ with $a_n, b_n > 0$, and $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = L > 0$

Convergence	Divergence
$\sum b_n$ converges $\implies \sum a_n$ converges	$\sum b_n$ diverges $\implies \sum a_n$ diverges

11.8 Alternating series test

$\sum(-1)^n a_n$ converges if (*only works for convergence*):

1. $\lim\limits_{n\to\infty} a_n = 0$ (terms $\to 0$)
2. $|a_{n+1}| \leq |a_n|$ (terms decreasing)

11.9 Ratio test

1. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \implies$ converges
2. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \implies$ diverges
3. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \implies$ test is inconclusive

11.10 Root test**

1. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1 \implies$ converges
2. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1 \implies$ diverges
3. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1 \implies$ test is inconclusive

12 Convergence and Error Analysis

S = actual sum for convergent series

S_n = sum of first n terms

R_n = error in calculating only up to the n^{th} term

12.1 Error using integral test

$$R_n \leq \int_n^\infty f(x)dx$$

$$S_n \leq S \leq S_n + R_n$$

12.2 Error using alternating series test

$$|R_n| \leq a_{n+1}$$

$$S_n - R_n \leq S \leq S_n + R_n$$

12.3 Absolute vs. conditional convergence

<u>Conditional convergence</u>: $\sum a_n$ converges but $\sum |a_n|$ diverges

<u>Absolute convergence</u>: $\sum a_n$ and $\sum |a_n|$ converge

$\sum |a_n|$ converges $\implies \sum a_n$ converges

13 Polynomials and Series

13.1 Tayor and Maclaurin polynomials

Interval of convergence: x-values/times where approximation and graph are the same; use ratio test

1. MacLaurin polynomial: $(c = 0)$

$$p_n(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n$$

2. Taylor polynomial:

$$p_n(x) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

13.2 LaGrange error and Taylor's theorem

Taylor's theorem:

$$R_n \leq \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \right|$$

Lagrange error:

$R_n = |f(x) - p(x)|$ where $p(x)$ is the polynomial estimate to n terms

13.3 Power series

Formula:

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_n(x-c)^n$$

Convergence: use convergence criteria for ratio test to determine interval; also need to check if endpoints converge

13.4 Geometric power series

Get series in form $\frac{a_0}{1-r}$, then change to summation form $\sum_{n=0}^{\infty} a(r)^n$

13.5 Taylor and MacLaurin series

Taylor series: when f has derivatives of all orders at $x = c$ (MacLaurin if $x = 0$)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

13.6 Need-to-know series

1. $\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$
2. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
3. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
4. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

14 Parametric, polar, vector

14.1 Parametric

14.1.1 Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

14.1.2 Arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

14.1.3 Surface area

$$S = \int_a^b 2\pi r \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

14.2 Polar

14.2.1 Polar conversions

$$\begin{array}{c|c} x = r\cos\theta & r = \sqrt{x^2+y^2} \\ y = r\sin\theta & \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{array}$$

14.2.2 Tangent lines

<u>Horizontal tangent</u>: $\frac{dy}{d\theta} = 0$

<u>Vertical tangent</u>: $\frac{dx}{d\theta} = 0$

14.2.3 Area

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta$$

14.3 Vector

<u>Speed</u>: $|\mathbf{v}| = \sqrt{a^2+b^2}$ where $\mathbf{v} = < a, b > = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$

<u>Distance traveled</u> $= \int |\mathbf{v}| = \int$ (speed)