

STATS 200 Study Guide

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Abstract

The following is a summary of the major concepts from the Stanford course STATS 200: *Introduction to Statistical Inference*. These notes were derived from both course lectures and information from the John Rice *Mathematical Statistics and Data Analysis* (3rd ed.) text. Broadly, the course focuses on major statistical tests and results, as well as the highlights from large sample theory.

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Part I

Pre-Midterm

1 Chapter 1: Probability

1.1 Probability Measure

1.1.1 Axioms

1. $P(\Omega) = 1$
2. $A \subset \Omega \implies P(A) \geq 0$
3. A_1, A_2 disjoint $\implies P(A_1 \cup A_2) = P(A_1) + P(A_2)$

1.1.2 Properties

1. $P(A^C) = 1 - P(A)$
2. $P(\emptyset) = 0$
3. $A \subset B \implies P(A) \leq P(B)$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1.2 Law of Total Probability

Let B_1, \dots, B_n be disjoint with $\bigcup B_i = \Omega$ and $P(B_i) > 0$. Then, $\forall i$:

$$P(A) = \sum_{i=1}^n P(A \mid B_i)P(B_i)$$

1.3 Bayes' Theorem

Let B_1, \dots, B_n be disjoint with $\bigcup B_i = \Omega$ and $P(B_i) > 0$. Then, $\forall i$:

$$P(B_j \mid A) = \frac{P(A \mid B_j)P(B_j)}{\sum_i P(A \mid B_i)P(B_i)}$$

1.4 Independence

- Pairwise independent: any two are independent
- Mutually independent: all are independent
 $MI \implies PI$

3 Chapter 3: Joint Distributions

3.1 Theorem: Functional Independence

$$X \perp Y \implies g(X) \perp h(Y) \text{ for any } g, h$$

3.2 Joint Frequency

$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(x, y) dy dx$$

3.3 Marginal Frequency

$$F_X(x) = \int_{-\infty}^x \int_{-\infty}^{\infty} f(x, y) dy dx$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

3.4 Conditional Frequency

$$f_{Y|X}(y | x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$\implies f_{XY}(x, y) = f_{Y|X}(y | x) \cdot f_X(x)$$

$$\implies f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y | x) f_X(x) dx$$

3.5 Multinomial

$$p(x_1, \dots, x_r) = \binom{n}{x_1, \dots, x_r} p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$$

$$\begin{cases} \sum x_i = n \\ \sum p_i = 1 \end{cases}$$

4 Chapter 4: EVs

4.1 Definitions

4.1.1 Covariance

Def:

- $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

Variance property:

- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

4.1.2 Correlation coefficient

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

4.1.3 Conditional expectation

$$E(Y | X = x) = \begin{cases} \sum_y y p_{Y|X}(y | x) & \text{if discrete} \\ \int y f_{Y|X}(y | x) dy & \text{if cts} \end{cases}$$

4.1.4 Moment generating function

$$M(t) = \begin{cases} \sum_x e^{tx} p(x) & \text{if discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if cts} \end{cases}$$

4.1.5 r^{th} moment

$$\mu_r = E(X^r)$$

4.2 Theorems

4.2.1 Markov inequality

$$P(X \geq t) \leq \frac{E(X)}{t}$$

4.2.2 Chebyshev inequality

$$P(|X - \mu| > t) \leq \frac{\sigma^2}{t^2}$$

$$P(|\bar{X}_n - \mu| > k\sigma) \leq 1/k^2$$

4.2.3 Moment generating function theorems

- $M^{(r)}(0) = E(X^r)$
- $Y = a + bX \implies M_Y = e^{at} M_X(bt)$
- $Z = X + Y, X \perp Y \implies M_Z = M_Y M_X$

5 Chapter 5: Limit Theorems

5.1 Definitions

5.1.1 Convergence in probability

$$\lim_{n \rightarrow \infty} P(|Z_n - \alpha| > \epsilon) = 0 \text{ for some } \alpha, \text{ any } \epsilon > 0$$

5.1.2 Almost sure convergence

$\forall \epsilon > 0$, $|Z_n - \alpha| > \epsilon$ only a finite number of times with $P = 1$

Summary: beyond some point in the sequence, the difference is always less than ϵ , but the location of that point is random.

5.2 Theorems

5.2.1 WLLN: weak law of large numbers

Let $\{X_i\}$ be sequence of iid RVs with $E(X_i) = \mu$, $\text{Var}(X_i) = \sigma^2$.

Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Then, $\forall \epsilon > 0$:

$$\lim_{n \rightarrow \infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$$

Summary: $\bar{X}_n \xrightarrow{ip} \mu$

5.2.2 SLLN: strong law of large numbers

$$\bar{X}_n \xrightarrow{as} \mu$$

5.2.3 Continuity theorem

Let F_n be sequence of cdfs with mgfs M_n .

Let F be cdf with mgf M .

$$M_n(t) \rightarrow M(t) \quad \forall t \text{ in an open interval containing } 0$$

$$\implies F_n \rightarrow F \text{ where } F \text{ cts}$$

5.2.4 CLT: central limit theorem

Let $\{X_i\}$ be sequence of iid RVs with $\mu = 0$, $\text{Var} = \sigma^2$, common cdf F , mgf M defined about 0.

Let $S_n = \sum_{i=1}^n X_i$.

$$\implies \lim_{n \rightarrow \infty} P\left(\frac{S_n}{\sigma\sqrt{n}} \leq x\right) = \Phi(x)$$

$$\implies P\left(\frac{\bar{X}_n - E(X)}{\sigma/\sqrt{n}} \leq z\right) \rightarrow \Phi(z)$$

6 Chapter 6: Derivations from Normal

6.1 χ^2

6.1.1 χ_1^2

Let $Z \sim \mathcal{N}(0, 1)$.

$$\implies U = Z^2 \sim \chi_1^2$$

$$\left(\frac{X - \mu}{\sigma} \right) \sim \mathcal{N}(0, 1) \implies \left(\frac{X - \mu}{\sigma} \right)^2 \sim \chi_1^2$$

Summary: square of normal RV is chi-squared, $df = 1$.

6.1.2 χ_n^2

Let $\{U_i\}_{i=1}^n$ iid χ_1^2 .

$$\implies V = \sum_{i=1}^n U_i \sim \chi_n^2$$

Summary: sum of n chi-squared RVs is χ_n^2 .

6.2 t

Definition:

Let $Z \sim \mathcal{N}(0, 1)$, $U \sim \chi_n^2$, $Z \perp U$.

$$\implies \frac{Z}{\sqrt{U/n}} \sim t_n$$

Summary: t_n is normal RV divided by a scaled chi-squared with $df = n$

Density:

$$f(t) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi} \Gamma(n/2)} \left(1 + \frac{t^2}{n} \right)^{-\frac{n+1}{2}}$$

6.3 F

Definition:

Let U, V be iid χ^2 with $df = m, n$ respectively

$$\implies W = \frac{U/m}{V/n} \sim F_{m,n}$$

Summary: F with $df = m, n$ found by dividing two chi-squared RVs divided by their dfs.

Density:

$$f(w) = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(m/2) \Gamma(n/2)} \left(\frac{m}{n} \right)^{m/2} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n} w \right)^{-\frac{(m+n)}{2}}$$

6.4 Sample Statistics

6.4.1 Definitions

Let $\{X_i\}_{i=1}^n$ be iid sample from $\mathcal{N}(\mu, \sigma^2)$.

Sample mean:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \quad \mathbb{E}(\bar{X}) = \mu, \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

Sample variance:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

6.4.2 Theorems

- \bar{X}, S^2 independently distributed
- $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$
- $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$

7 Chapter 7: Sampling

- $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$
- $s_{\bar{X}}^2 = \frac{s^2}{n} (1 - \frac{n}{N})$

8 Chapter 8: Estimation and Fitting

8.1 MoME: Method of Moments

8.1.1 Definitions

- k^{th} moment:

$$\mu_k = E(X^k)$$

- k^{th} sample moment:
If X_1, \dots, X_n iid RVs, then

$$\hat{\mu}_k = \frac{1}{n} \sum_{i=1}^n X_i^k$$

8.1.2 MoME

- (1) Find low order moments; express moments in terms of parameters
- (2) Find parameters in terms of moments
- (3) Insert sample moments into expressions in (2)

8.2 MLE: Maximum Likelihood

8.2.1 Method of MLE

- $L(\theta) = f(\underline{x} \mid \theta)$
- $\ell(\theta) = \sum \ln[f(x_i \mid \theta)]$
- MLE maximizes ℓ

8.2.2 Large sample theory

- If f smooth, MLE from iid sample is consistent
- $I(\theta) = -E(\ell'')$

8.2.3 MLE asymptotically unbiased

- **Theorem:**
If f smooth, then $\sqrt{nI(\theta_0)}(\hat{\theta} - \theta_0) \sim \mathcal{N}(0, 1)$
Summary: mle $\sim \mathcal{N}$ with $\mu = \theta_0$, asymptotic variance
- **Asymptotic variance:**

$$\text{Var}(\theta_0) = \frac{1}{nI(\theta_0)} \approx -\frac{1}{E(\ell'')}$$

8.2.4 CI for MLE

$$CI = \hat{\theta} \pm z_{\alpha/2} \cdot \sqrt{\text{Var}(\theta_0)}$$

8.3 Bayes

8.3.1 Finding the posterior

$$f_{\Theta|X}(\theta \mid x) = \frac{f_{X,\Theta}(x, \theta)}{f_X(x)} = \frac{f_{X|\Theta}(x \mid \theta)f_{\Theta}(\theta)}{\int f_{X|\Theta}(x \mid \theta)f_{\Theta}(\theta) d\theta}$$

8.3.2 Bayesian paradigm

posterior \propto likelihood \cdot prior

8.4 Consistent Estimate

Let $\hat{\theta}_n$ be an estimate of θ based on sample n .

Then, $\hat{\theta}_n$ consistent in probability if $\hat{\theta}_n \xrightarrow{ip} \theta$ as $n \rightarrow \infty$:

$$\forall \epsilon > 0, P(|\hat{\theta}_n - \theta| > \epsilon) \rightarrow 0 \text{ as } n \rightarrow \infty$$

8.5 Efficiency, CRLB

8.5.1 Efficiency

$$\text{eff}(\hat{\theta}, \tilde{\theta}) = \frac{\text{Var}(\hat{\theta})}{\text{Var}(\tilde{\theta})}$$

8.5.2 Cramer-Rao Inequality: CRLB

Let $\{X_i\}_{i=1}^n$ be iid with $f(x | \theta)$.

Let $T = t(X_1, \dots, X_n)$ be unbiased estimator of θ . Then,

$$\text{Var}(T) \geq \frac{1}{nI(\theta)}$$

- If $\text{Var}(T) =$ asymptotic variance, then efficient.
- MLE is asymptotically efficient.

8.6 Sufficiency

8.6.1 Sufficient statistic

$T(\underline{X})$ sufficient for θ if conditional distribution of \underline{X} given $T = t$ does not depend on $\theta \forall t$
 $\implies T$ is a sufficient statistic

8.6.2 Factorization theorem

T sufficient for $\theta \iff f(x | \theta) = g(T, \theta) \cdot h(x)$

8.6.3 Exponential family

$$f(x | \theta) = e^{c(\theta)T(x) + d(\theta) + S(x)}$$

- T sufficient for $\theta \implies \text{MLE} = f(T)$

8.6.4 Rao-Blackwell theorem

Let $\hat{\theta}$ be an estimator of θ with $E(\hat{\theta}^2)$ finite $\forall \theta$.

Suppose T is sufficient for θ , $\tilde{\theta} = E(\hat{\theta} | T)$.

Then, $\forall \theta$:

$$E(\tilde{\theta} - \theta)^2 \leq E(\hat{\theta} - \theta)^2$$

9 Chapter 9: Hypothesis Testing, Goodness of Fit

9.1 Likelihood Ratio

$$LR = \frac{P(x | X_0)}{P(x | H_1)} \cdot \frac{P(H_0)}{P(H_1)}$$
$$\implies \text{reject } H_0 \text{ if } LR < c$$

9.2 Neyman-Pearson Paradigm

9.2.1 Neyman-Pearson lemma

Suppose H_0, H_1 are *simple* hypotheses where test rejects H_0 when $LR < c$ with significance level α . Then, any other test with significance level $\leq \alpha$ has power $\leq LR$ test.

9.2.2 UMP: Uniformly most powerful test

If H_1 composite, test that is most powerful \forall simple alternatives in H_1 is uniformly most powerful (UMP)

9.3 Confidence Intervals

- **Confidence interval:**

$$P(\theta_0 \in C(X) \mid \theta = \theta_0) = 1 - \alpha$$

- **Acceptance region**

$$A(\theta_0) = \{X \mid \theta_0 \in C(X)\}$$

9.4 GLRT

9.4.1 Testing

$$\Lambda = \frac{\max_{\theta \in \omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$$
$$\implies \text{reject } H_0 \text{ if } \Lambda < c$$

9.4.2 GLRT Distribution Theorem

Under smoothness of pdfs, null distribution of $-2 \ln \Lambda \sim \chi_{df}^2$ with $df = \dim(\Omega) - \dim(\omega_0)$ as $n \rightarrow \infty$.

9.5 Multinomial Distribution

- **Hypothesis:** $\begin{cases} H_0 : p = p(\theta), \theta \in \omega_0 \\ H_1 : \text{cell probabilities free} \end{cases}$
- **Distribution:**

$$\chi_{m-k-1}^2 = \sum_{i=1}^m \frac{[x_i - np_i(\hat{\theta})]^2}{np_i(\hat{\theta})}$$

$df = \text{cells} - \text{num of estimated params} - 1$

9.6 Poisson Dispersion Test

- **Hypothesis:** $\begin{cases} H_0 : \text{Counts } x_1, \dots, x_n \text{ Poisson with common } \lambda \\ H_1 : \text{Poisson with different rates} \end{cases}$
- **Result:**

$$-2 \ln \Lambda = 2 \sum_{i=1}^n x_i \ln \left(\frac{x_i}{\bar{x}} \right) \approx \frac{1}{\bar{x}} \sum_{i=1}^n (x_i - \bar{x})^2 \sim \chi_{n-1}^2$$

9.7 Hanging Rootograms

- Hanging histogram: n_j observed counts vs \hat{n}_j predicted counts
 - variability not same across cells
- Hanging rootogram: $\sqrt{n_j} - \sqrt{\hat{n}_j}$
 - appx same variability
- Hanging chi-gram: $\frac{n_j - \hat{n}_j}{\sqrt{\hat{n}_j}}$
 - variance ≈ 1

9.8 Probability Plot

Plot of $F(X_{(k)})$ vs. $\frac{k}{n+1}$ OR plot of $X_{(k)}$ vs. $F^{-1}(\frac{k}{n+1})$

9.9 Tests for Normality

9.9.1 Coefficient of skewness

$$b_1 = \frac{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^3}{s^3}$$

9.9.2 Coefficient of kurtosis

$$b_2 = \frac{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^4}{s^4}$$

9.9.3 Variance-stabilizing transformation

$$\text{Var}(Y) \approx \sigma^2(\mu)[f'(\mu)]^2$$

10 Chapter 10: Summarizing Data

10.1 ecdf

10.1.1 Definition

Suppose X_1, \dots, X_n sample/batch of iid numbers.

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, x]}(X_i)$$

10.1.2 Distribution

$$nF_n(x) \sim \text{Binom}(n, F(x))$$

- $E[F_n(x)] = F(x)$
- $\text{Var}[F_n(x)] = \frac{1}{n} F(x)[1 - F(x)]$

10.2 Survival Analysis

10.2.1 Survival function

$$S(t) = P(T > t) = 1 - F(t)$$

10.2.2 Hazard function

$$h(t) = \frac{f(t)}{1 - F(t)} = -\frac{d}{dt} \ln[1 - F(t)] = -\frac{d}{dt} \ln S(t)$$

10.3 QQ Plot

10.3.1 Definition

Plot quantiles of one distribution against vs. another where the quantiles are $x_p = F^{-1}(p)$

10.3.2 Common transformations

For control F and treatment G :

1. *Linear*: $y_p = x_p + h \implies G(y) = F(y - h)$
2. *Multiplicative*: $y_p = cx_p \implies G(y) = F(y/c)$

10.4 Kernel Density Estimate

Let w_h be a non-negative, symmetric weight function centered at 0 with $\int w = 1$.

Then, the kernel density estimate is:

$$f_h(x) = \frac{1}{n} \sum_{i=1}^n w_h(X - X_i)$$

- Represents a superposition of hills centered on the observations
- $h = \text{bandwidth}$: smoothness & bin width

10.5 Location

10.5.1 M estimates

- Sample mean minimizes negative log-likelihood, or the least squares estimate:

$$\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma} \right)^2$$

- Sample median minimizes:

$$\sum_{i=1}^n \left| \frac{X_i - \mu}{\sigma} \right|$$

- M-estimate minimizes:

$$\sum_{i=1}^n \Psi \left(\frac{X_i - \mu}{\sigma} \right)^2$$

Part II

Post-Midterm

11 Chapter 11: Comparing Two Samples

11.1 Two Independent Samples

11.1.1 Parametric: normal

1. Overview

- Treatment: X_1, \dots, X_n iid $\mathcal{N}(\mu_X, \sigma^2)$
- Control: Y_1, \dots, Y_n iid $\mathcal{N}(\mu_Y, \sigma^2)$
- Pooled sample variance:

$$s_p^2 = \frac{(n-1)s_X^2 + (m-1)s_Y^2}{m+n-2} = s_{\bar{X}-\bar{Y}}^2$$

- **Thm:** distribution of difference

$$t = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{s_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{m+n-2}$$

2. Hypothesis testing

- Hypothesis: $H_0 : \mu_X = \mu_Y$
- Test statistic:

$$t = \frac{\bar{X} - \bar{Y}}{s_{\bar{X}-\bar{Y}}} \sim t_{m+n-2}$$

3. Power: power of rejecting H_0 when it is false

- Factors that affect power:
 - 1) Real difference, $\Delta = |\mu_X - \mu_Y|$: large diff \rightarrow greater power
 - 2) α : $\alpha \uparrow \implies$ power \uparrow
 - 3) σ : $\sigma \downarrow \implies$ power \uparrow
 - 4) Sample sizes n, m : $nm \uparrow \implies$ power \uparrow
- Numerical power:

$$1 - \Phi \left[z(\alpha/2) - \frac{\Delta}{\sigma} \sqrt{\frac{n}{2}} \right] + \Phi \left[-z(\alpha/2) - \frac{\Delta}{\sigma} \sqrt{\frac{n}{2}} \right]$$

11.1.2 Nonparametric: Mann-Whitney

1. Overview

- H_0 : no treatment effect
- U : sum of wins and ties in relevant set
- T : total sum of ranks in set
- Procedure:
 - (1) Group all $m + n$ observations together, rank in order of increasing size
 - (2) Calculate some of ranks of observations from control group
 - (3) Reject H_0 if sum is too extreme

2. Distribution version

- $X_1, \dots, X_n \sim F$ control group
- $Y_1, \dots, Y_m \sim G$ experimental group
- $H_0 : F = G$
- **Thm:** for T_Y as rank sum of Y :

$$E(T_Y) = \frac{m(m+n+1)}{2}$$

$$\text{Var}(T_Y) = \frac{mn(m+n+1)}{12}$$

3. Rank-sum version

- Mann-Whitney test statistic:

$$U_Y = T_Y - \frac{m(m+1)}{2}$$

- **Thm:** under $H_0 : F = G$:

$$E(U_Y) = \frac{mn}{2}$$

$$\text{Var}(U_Y) = \frac{mn(m+n+1)}{12}$$

- For m, n both > 10 :

$$\frac{U_Y - E(U_Y)}{\sqrt{\text{Var}(U_Y)}} \sim \mathcal{N}(0, 1)$$

11.1.3 Bayesian approach

1. Assumptions

- X_i iid \mathcal{N} , mean μ_X , precision ξ
- Y_j iid \mathcal{N} , mean μ_Y , precision ξ

2. Procedure

- (1) Assign prior to (μ_X, μ_Y, ξ)
- (2) Posterior \propto prior \times likelihood; normalize
- (3) Find marginal joint distribution by integrating out ξ
- (4) Find marginal for $\mu_X - \mu_Y$

3. Approximate result: use improper priors

- Final posterior:

$$f_{post}(\mu_X, \mu_Y, \xi) \propto \xi^{\frac{n+m}{2}-1} \exp\left(-\frac{\xi}{2}[(n-1)s_X^2 + (m-1)s_Y^2]\right) \cdot \exp\left(-\frac{n\xi}{2}(\mu_X - \bar{x})^2\right) \cdot \exp\left(-\frac{m\xi}{2}(\mu_Y - \bar{y})^2\right)$$

- Distributions:

$$\mu_X - \mu_Y \sim \mathcal{N}(\bar{X} - \bar{Y}, \sigma^2)$$

$$\sigma^2 = \xi^{-1}(n^{-1} + m^{-1})$$

- Distribution of marginal posterior of $\mu_X - \mu_Y$:

$$\frac{\Delta - (\bar{X} - \bar{Y})}{s_{\bar{X} - \bar{Y}}} \sim t_{m+n-2}$$

4. Bayes vs. frequentist

- Frequentist:

- $\bar{X} - \bar{Y}, s_p$ random
- $\Delta = \mu_X - \mu_Y$ fixed

- Bayes:

- $\bar{X} - \bar{Y}, s_p$ fixed
- $\Delta = \mu_X - \mu_Y$ random
- Statements about Δ from data

11.2 Paired Samples

11.2.1 Overview

1. Assumptions

- Pairs (X_i, Y_i) , $i = 1, \dots, n$
- Different pairs iid, but $\text{Cov}(X_i, Y_i) = \sigma_{XY}$
- $D_i = X_i - Y_i$

2. Population

- $E(D) = \mu_X - \mu_Y$
- $\text{Var}(D) = \sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY} = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$

3. Estimates

- $E(\bar{D}) = \mu_X - \mu_Y$
- $\text{Var}(\bar{D}) = \frac{1}{n}(\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y)$

4. Simplification: if $\sigma_X = \sigma_Y = \sigma$

- $\text{Var}(\bar{D}) = \frac{2\sigma^2(1-\rho)}{n}$
- $\text{Var}(\bar{D}_\perp) = \frac{2\sigma^2}{n}$
- $\text{efficiency} = \frac{\text{Var}(\bar{D})}{\text{Var}(\bar{D}_\perp)} = 1 - \rho$

11.2.2 Parametric: normal/ t -test

1. Assumptions

- $X_i - Y_i$ sample from \mathcal{N} , $D = X - Y$
- $E(D_i) = \mu_X - \mu_Y = \mu_D$
- $\text{Var}(D_i) = \sigma_D^2$

2. Inference: σ_D unknown; $H_0 : \mu_D = 0$; ok for large n by CLT

$$t = \frac{\bar{D} - \mu_D}{s_{\bar{D}}} \sim t_{n-1}$$

11.2.3 Nonparametric: Signed-Rank Test

1. Procedure

- (1) Calculate differences D_i , find $|D_i|$, rank $|D_i|$
- (2) Restore signs of D_i to ranks to create signed ranks
- (3) Calculate W_+ = sum of positive ranks as test statistic

2. Test

- $H_0 : D_i$ distribution symmetric about 0
- **Thm:** under H_0 ,

$$E(W_+) = \frac{n(n+1)}{4}$$

$$\text{Var}(W_+) = \frac{n(n+1)(2n+1)}{24}$$

11.3 Experimental Design

- **Bonferroni method:** for multiple hypothesis testing, test each at α/n to achieve overall error of α

12 Chapter 12: ANOVA (F)

12.1 One-Way ANOVA

- **One-way layout:** independent measurements made under each of several treatments
- Sources of variability:
 1. Within samples
 2. Between samples

12.1.1 Normal theory: F -test

1. Setup

- I = number of groups/treatments
- J = sample size
- $Y_{ij} = j^{th}$ observation of i^{th} treatment

2. Model: $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$

- Variables:
 - μ = overall/total mean
 - α_i = differential effect of i^{th} treatment
 - ϵ_{ij} = random error in j^{th} observation of i^{th} treatment
- Assumptions:
 - ϵ_{ij} iid $\mathcal{N}(0, \sigma^2)$
 - α_i normalized

3. Sum of squares

- Notation:
 - $\bar{Y}_{i.} = \frac{1}{J} \sum_j Y_{ij}$
 - $\bar{Y}_{..} = \frac{1}{IJ} \sum_i \sum_j Y_{ij}$
- Equation: $SS_{TOT} = SS_W + SS_B$
 - Total sum of squares: $SS_{TOT} = \sum_i \sum_j (Y_{ij} - \bar{Y}_{..})^2$
 - Sum of squares within: $SS_W = \sum_i \sum_j (Y_{ij} - \bar{Y}_{i.})^2$
 - Sum of squares between: $SS_B = J \sum_i (Y_{i.} - \bar{Y}_{..})^2$

4. Expected value theorems

- **Thm:** expected SS
Let X_i be independent random variable with $E(X_i) = \mu_i$, $\text{Var}(X_i) = \sigma^2$. Then,

$$E(X_i - \bar{X})^2 = (\mu_i - \bar{\mu})^2 + \frac{n-1}{n} \sigma^2$$

where $\bar{\mu} = \frac{1}{n} \sum_i \mu_i$

- **Thm:** expected value of SS_W , SS_B

$$E(SS_W) = I(J-1)\sigma^2$$

$$E(SS_B) = J \sum_{i=1}^I \alpha_i^2 + (I-1)\sigma^2$$

5. Variance rules & theorems

- Key observations:
 - (1) SS_W can estimate σ^2 : $s_p^2 = \frac{SS_W}{I(J-1)}$
 - (2) If all $\alpha_i = 0$, $\frac{SS_W}{I(J-1)} \approx \frac{SS_B}{I-1}$; if some $\neq 0$, then SS_B inflated \implies motivation for test

- **Thm:** *distribution of SS*

If ϵ_{ij} iid $\mathcal{N}(0, \sigma^2)$:

$$\frac{SS_W}{\sigma^2} \sim \chi_{I(J-1)}^2$$

If also all $\alpha_i = 0$:

$$\frac{SS_B}{\sigma^2} \sim \chi_{I-1}^2$$

with $\frac{SS_W}{\sigma^2} \perp \frac{SS_B}{\sigma^2}$

6. Test

- Test statistic: if H_0 true, $F \approx 1$

$$H_0 : \alpha_1 = \alpha_2 = \dots = \alpha_I = 0$$

$$F = \frac{SS_B/(I-1)}{SS_W/[I(J-1)]} \sim F_{I-1, I(J-1)}$$

7. Test with different number of observations: non-constant J_i

(1) *The identity*

$$\sum_i \sum_j (Y_{ij} - \bar{Y}_{..})^2 = \sum_i \sum_j (Y_{ij} - \bar{Y}_{i.})^2 + \sum_i J_i (\bar{Y}_{i.} - \bar{Y}_{..})^2$$

(2) *Expected values*

$$E(SS_W) = \sigma^2 \sum_i (J_i - 1)$$

$$E(SS_B) = \sum_{i=1}^I J_i \alpha_i^2 + (I-1)\sigma^2$$

8. Summary

- The model: $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$
- Assumptions:
 - (1) $\epsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$
 - F -test approximately valid for large enough samples even if non-normal
 - (2) σ^2 CONSTANT
 - F -test not strongly affected by diff σ^2 as long as equal number of obs per group
 - (3) ϵ_{ij} independent
 - Most important!!

9. Tukey's method of multiple comparisons

- One-way anova: testing fact of difference, not measurement of difference or specific difference pairs
- Tukey method: compare pairs/groups of treatment means via t -test
- **Tukey test:** construct CIs for differences of all pairs of means such that intervals simultaneously have some set coverage probability; can use duality of CI/hypothesis testing to determine differences
- *Assumptions:*
 - Sample sizes are equal (NOT required for Bonferroni)
 - $\epsilon \sim \mathcal{N}$ with constant σ^2

12.1.2 Nonparametric one-way: Kruskal-Wallis

1. Setup

- *Assumptions*: independent observations, no necessary functional form
- *Variables*:
 - R_{ij} = rank of Y_{ij} in pooled sample
 - $\bar{R}_{i.} = \frac{1}{J_i} \sum_{j=1}^{J_i} R_{ij}$: average rank in i^{th} group
 - $\bar{R}_{..} = \frac{N+1}{2}$
 - $SS_B = \sum_i J_i (\bar{R}_{i.} - \bar{R}_{..})^2$

2. Test statistic

$$K = \frac{12}{N(N+1)} SS_B = \frac{12}{N(N+1)} \left(\sum_{i=1}^I J_i \bar{R}_{i.}^2 \right) - 3(N+1) \approx \chi_{I-1}^2$$

12.2 Two-Way ANOVA

- **Two-way anova**: experimental design involving two factors, each at 2+ levels
- *Assumptions*:
 - If I levels of f_1 and J levels of f_2 , IJ combos
 - K independent observations taken from each combination (I, J)

12.2.1 Normal theory, 2-way

1. Assumptions

- $K > 1$ observations per cell
- Balanced: equal observations per cell
- $Y_{ijk} = k^{th}$ observation in cell (i, j)
- ϵ_{ijk} iid $\mathcal{N}(0, \sigma^2)$

2. Model

- *The model*: $Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk}$
- *Constraints*:
 - *Row differential*: $\sum_i \alpha_i = 0$
 - *Column differential*: $\sum_j \beta_j = 0$
 - *Residual*: $\sum_i \delta_{ij} = \sum_j \delta_{ij} = 0$

3. MLEs

- *Log-likelihood*:

$$\ell = -\frac{IJK}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K (Y_{ijk} - \mu - \alpha_i - \beta_j - \delta_{ij})^2$$

- *MLEs*:

$$\hat{\mu} = \bar{Y}_{...}$$

$$\hat{\alpha}_i = \bar{Y}_{i..} - \bar{Y}_{...}$$

$$\hat{\beta}_j = \bar{Y}_{.j.} - \bar{Y}_{...}$$

$$\hat{\delta}_{ij} = \bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}$$

4. **SS**: $SS_{TOT} = SS_A + SS_B + SS_{AB} + SS_E$

$$SS_A = JK \sum_{i=1}^I (\bar{Y}_{i..} - \bar{Y}_{...})^2$$

$$SS_B = IK \sum_{j=1}^J (\bar{Y}_{.j.} - \bar{Y}_{...})^2$$

$$SS_{AB} = K \sum_{i=1}^I \sum_{j=1}^J (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$$

$$SS_E = \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K (Y_{ijk} - \bar{Y}_{ij.})^2$$

$$SS_{TOT} = \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K (Y_{ijk} - \bar{Y}_{...})^2$$

5. Expectations

$$E(SS_A) = (I-1)\sigma^2 + JK \sum_{i=1}^I \alpha_i^2$$

$$E(SS_B) = (J-1)\sigma^2 + IK \sum_{j=1}^J \beta_j^2$$

$$E(SS_{AB}) = (I-1)(J-1)\sigma^2 + K \sum_{i=1}^I \sum_{j=1}^J \delta_{ij}^2$$

$$E(SS_E) = IJ(K-1)\sigma^2$$

6. Distributions of SS

- (1) $\frac{SS_E}{\sigma^2} \sim \chi_{IJ(K-1)}^2$
- (2) Under H_A : $\alpha_i = 0$ for all i : $\frac{SS_A}{\sigma^2} \sim \chi_{I-1}^2$
- (3) Under H_B : $\beta_j = 0$ for all j : $\frac{SS_B}{\sigma^2} \sim \chi_{J-1}^2$
- (4) Under H_{AB} : $\delta_{ij} = 0$ for all i, j : $\frac{SS_{AB}}{\sigma^2} \sim \chi_{(I-1)(J-1)}^2$
- (5) SS are independently distributed

7. The test

- Compare relevant SS to SS_E
- F = ratio of MS where $MS = SS/df$; reject when $F \gg 1$
- Example: *Interaction test*

$$F = \frac{SS_{AB}/[(I-1)(J-1)]}{SS_E/[IJ(K-1)]} = \frac{MS_{AB}}{MS_E}$$

12.2.2 Nonparametric: Friedman's test

- **Assumptions**: none on distribution: only according to ranks
- **Procedure**:

- (1) Within each of the J blocks, rank the observations
- (2) H_0 : no effect due to I treatments
- (3) Relevant variable: $SS_A = J \sum_{i=1}^I (\bar{R}_{i..} - \bar{R}_{...})^2$
- (4) Test statistic approximation:

$$Q = \frac{12J}{I(I+1)} SS_A \sim \chi_{I-1}^2$$

13 Chapter 13: Analysis of Categorical Data (χ^2)

- **Categorical data:** in counts from categories of two-way tables (contingency table)

13.1 Fisher's Exact Test

- *Test statistic:* N_{11} ; hypergeometric under H_0
- *Probability:*

$$P(N_{11} = n_{11}) = \frac{\binom{n_{1.}}{n_{11}} \binom{n_{2.}}{n_{21}}}{\binom{n_{..}}{n_{.1}}}$$

13.2 Chi-Square Test of Homogeneity

1. Setup

- Independent observations from J multinomial distributions, each of which has I cells/categories
- *Test idea:* are all cell probabilities homogeneous/equal (**goodness of fit test**)
- π_{ij} = probability of i^{th} category in j^{th} multinomial

2. Test

- $H_0 : \pi_{i1} = \pi_{i2} = \dots = \pi_{iJ}$ for all i
- n_{ij} = count in i^{th} category in j^{th} multinomial

3. Thm: MLE of π 's

- Under H_0 , mle's of parameters π_i are:

$$\hat{\pi}_i = \frac{n_{i.}}{n_{..}}$$

- $n_{i.}$ = total responses in i^{th} category
- $n_{..}$ = grand total responses

- For j^{th} multinomial, expected count in i^{th} category:

$$E_{ij} = \frac{n_{i.} n_{.j}}{n_{..}}$$

$$O_{ij} = n_{ij}$$

4. χ^2 -statistic

$$X^2 = \sum_{i=1}^I \sum_{j=1}^J \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \sim \chi_{(I-1)(J-1)}^2$$

13.3 Chi-Square Test of Independence

1. Setup

- Sample size n cross-classified in table with I rows, J columns contingency table
- π_{ij} = joint distribution of n_{ij}
- *Marginal probabilities:*

$$\pi_{i.} = \sum_{j=1}^J \pi_{ij} \quad \pi_{.j} = \sum_{i=1}^I \pi_{ij}$$

2. Test

$$H_0 : \pi_{ij} = \pi_{i.}\pi_{.j}$$

3. Thm: MLEs

$$H_0 : \hat{\pi}_{ij} = \hat{\pi}_{i.} + \hat{\pi}_{.j} = \left(\frac{n_{i.}}{n}\right)\left(\frac{n_{.j}}{n}\right)$$

$$H_1 : \hat{\pi}_{ij} = \frac{n_{ij}}{n}$$

13.4 Matched Pairs: McNemar's Test

1. Test: off-diagonal probabilities are equal

$$H_0 : \pi_{12} = \pi_{21}$$

2. MLEs: under H_0 :

$$\hat{\pi}_{11} = \frac{n_{11}}{n}$$

$$\hat{\pi}_{22} = \frac{n_{22}}{n}$$

$$\hat{\pi}_{12} = \hat{\pi}_{21} = \frac{n_{12} + n_{21}}{n}$$

3. Test statistic

$$X^2 = \frac{(n_{12} - n_{21})^2}{n_{12} + n_{21}} \sim \chi_1^2$$

13.5 Odds Ratio

1. Definitions

- Odds:

$$\text{odds}(A) = \frac{P(A)}{1 - P(A)}$$

- Odds ratio: influence of X on D :

$$\Delta = \frac{\text{odds}(D | X)}{\text{odds}(D | X^C)} = \frac{\pi_{11}\pi_{00}}{\pi_{10}\pi_{01}} = \frac{\text{product of diag probs}}{\text{product of off-diag probs}}$$

2. Sampling methods

- (1) Random sample from entire population:

- If D rare, need large n to guarantee enough D

- (2) Prospective study: fixed number of X , X^C sampled; compare incidence of D in the groups

- Can compare & estimate $P(D | X)$, $P(D | X^C)$ and odds ratio
- Individual probabilities π_{ij} cannot be estimated because marginal counts fixed

- (3) Retrospective study: fixed number of D , D^C sampled; compare incidence of X in the groups

- Can directly estimate $P(X | D)$, $P(X | D^C)$
- Can't estimate $P(D | X)$, $P(D | X^C)$ since marginal counts fixed
- Same odds ratio Δ
- Estimate: $\hat{\Delta} = \frac{n_{00}n_{11}}{n_{10}n_{01}}$