Notes on WCZ

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1 Thermal evolution

Conservation of energy relates the instantaneous luminosity profile $L(m)$ to the rate of change of specific entropy s in the planet:

$$\frac{dL}{dm} = -T\frac{\delta s}{\delta t}. \tag{1}$$

Integrating over the mass of the planet and solving for δt yields

$$\delta t = -\frac{1}{L_{\rm int}}\int_0^M T\,\delta s\,dm \tag{2}$$

where $L_{\rm int}$ is the intrinsic luminosity $4\pi R^2\sigma_{\rm SB}T_{\rm int}^4$.

For our assumed ideal mixture of light elements (H/He) and heavy elements (H_2O) this entropy difference is simply

$$\delta s = (1-Z)\delta s_{HHe} + Z\delta s_Z \tag{3}$$

where in practice δs_Z is computed in terms of the change in internal energy and density:

$$T\,\delta s_Z = \delta u_Z + P\delta\left(\rho_Z^{-1}\right). \tag{4}$$

2 Temperature jump across the radiative layer

Letting $P_{\rm top}$ and $T_{\rm top}$ denote the pressure and temperature at the top of the stable (radiative) layer, the temperature within the radiative layer is given by

$$T(P) = T_{\rm top} + \int_{P_{\rm top}}^{P}\left(\frac{dT}{dP}\right)_{\rm rad} dP. \tag{5}$$

The radiative zone is thin under the relatively opaque conditions typical of water condensation zones in the ice giants (Leconte et al. 2017, Friedson & Gonzales 2017). As a result our discretized model does not spatially resolve the radiative layer. Instead we treat the layer as a discontinuous increase in T and water vapor mole fraction $x_{\rm vap}$. Here we derive the magnitude of the temperature jump in the limit of a thin radiative zone.

Over a thin layer, the radiative temperature gradient

$$\left(\frac{dT}{dP}\right)_{\rm rad} = \frac{T}{P}\nabla_{\rm rad} = \frac{T}{P} \times \frac{3}{16}\frac{\kappa_R P}{g}\frac{T_{\rm int}^4}{T^4} \tag{6}$$

is nearly constant. In this case the integral in (5) simplifies to

$$T_{\rm base} \equiv T(P+\Delta P) = T_{\rm top} + \left(\frac{dT}{dP}\right)_{\rm rad} \Delta P. \tag{7}$$

Just as in the moist troposphere above, the local water vapor mole fraction $x_{\rm vap}$ follows the saturation value $x_{\rm vap}^{\rm sat}$ everywhere within the radiative layer, i.e.,

$$x_{\rm vap}(P,T) = x_{\rm vap}^{\rm sat}(P,T) = \frac{e_s(T)}{P}, \qquad P < P_{\rm base}. \tag{8}$$

We denote by $P_{\rm base}$ and $T_{\rm base}$ the pressure and temperature at the base of the radiative zone, which is set by the condition that $x_{\rm vap}$ has reached the deep value $x_{\rm vap}^{\rm deep}$:

$$x_{\rm vap}^{\rm sat}(P_{\rm base}, T_{\rm base}) = \frac{e_s(T_{\rm base})}{P_{\rm base}} = x_{\rm vap}^{\rm deep} \tag{9}$$

$$\implies \Delta P \equiv P_{\rm base} - P_{\rm top} = \frac{e_s(T_{\rm base})}{x_{\rm vap}^{\rm deep}} - P_{\rm top}. \tag{10}$$

(Here e_s is the H_2O saturation vapor pressure, calculated from XXXX relation [describe what the method from thermodynamics.py actually does].) Deeper pressures $P > P_{\rm base}$ are sub-saturated and hence no further condensation of H_2O takes place. ΔP gives the extent, in pressure coordinates, of the radiative layer. Combining (10) and (5) yields

$$T_{\rm base} = T_{\rm top} + \left(\frac{dT}{dP}\right)_{\rm rad}\left(\frac{e_s(T_{\rm base})}{x_{\rm vap}^{\rm deep}} - P_{\rm top}\right) \tag{11}$$

which we numerically solve for $T_{\rm base}$. Deeper temperatures are then obtained by integrating the dry adiabat $\nabla_{\rm ad}$:

$$T(P > P_{\rm base}) = T_{\rm base} + \int_{P_{\rm base}}^{P} \left(\frac{dT}{dP}\right)_{\rm ad} dP. \tag{12}$$