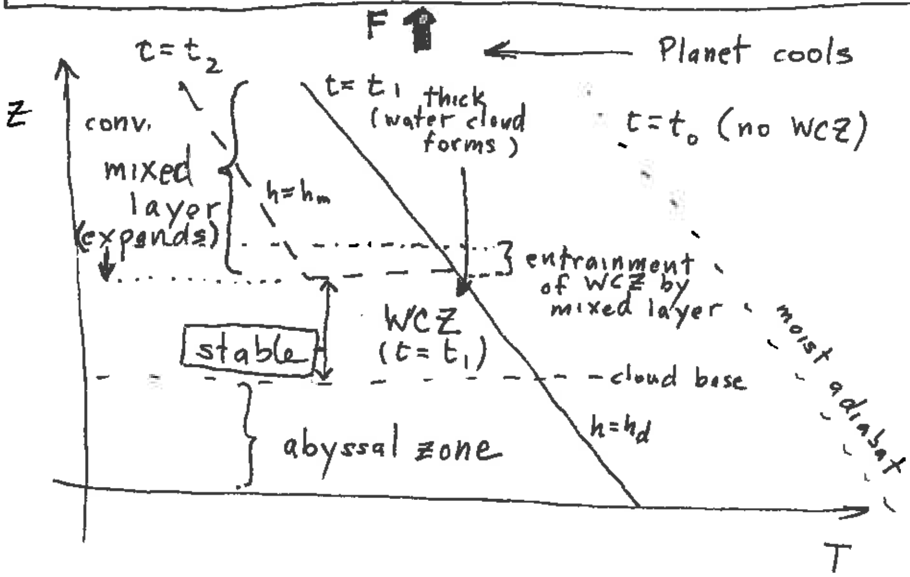
EVOLUTION OF WATER CONDENSATION ZONE (WCZ) IN COOLING ICE GIANT – AJF 11-6-17



ASSUMPTIONS

1) Planet eventually cools enough for (sufficiently) thick water cloud (WCZ) to form at depth
2) Above cloud base, atm remains saturated
3) Rapid rainout of condensate
4) Ideal gas EOS

- Define moist static energy $\boxed{h \equiv C_p T + \Phi - L \ln(1-q)}$
  Φ = geopotential/unit mass = gz, L = latent heat, q = specific humidity
  (The $\ln(1-q)$ is correct also for larger q; usually latent heat term written simply as $+Lq$ because $q \ll 1$)
  See A.E. Gill (1982) "Atmosphere-Ocean Dynamics", p. 83

- Use h because constant during pseudoadiabatic process, h is constant along moist adiabat

- Conservation of energy in mixed layer
  → $h = h_m(t)$, constant with height in mixed layer

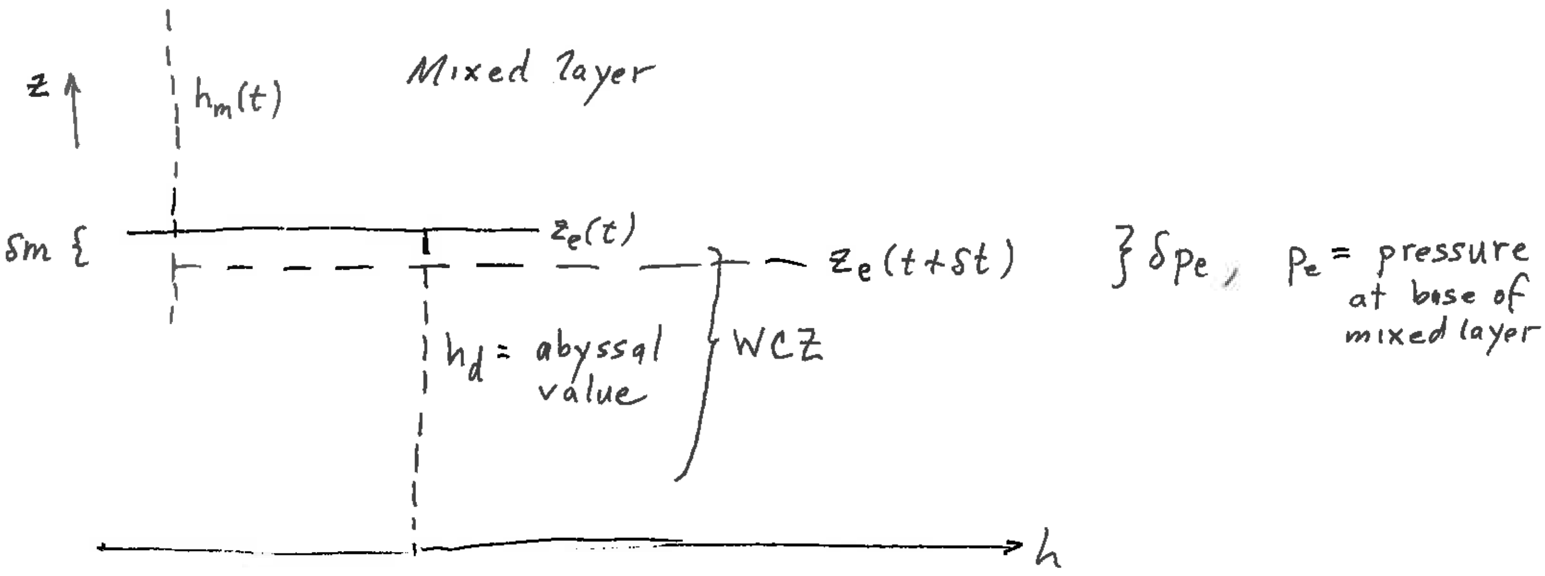
$$\frac{\partial \rho h_m}{\partial t} = -\nabla \cdot \left(\vec{F}_{rad} + \vec{F}_{conv}\right) \quad (1) \quad \text{(static atmosphere)}$$

→ Integrate (1) over mixed layer from mixed-layer base $= z_e$ (<0) to top $z = 0$:

$$\int_{+z_e(t)}^{0} dz' \frac{\partial}{\partial t}(\rho h_m) = -\int_{+z_e}^{0} dz' \frac{d}{dz'}\left(F_{rad} + F_{conv}\right) = -F(0) + F(+z_e)$$

→ If z_e were fixed, then $\frac{\partial}{\partial t}$(total MSE in column) $= F(0) - F(z_e)$, but z_e not fixed and there is a contribution from entrainment

Entrainment of stable zone by mixed layer:



→ At time t, Total MSE of mixed layer $= \int_0^{m(z_e)} dm'\, h_m$

where $dm' = \rho\, dz'$ = element of mass in column

→ At ~~tt~~ $t+\delta t$, Total MSE $= \int_0^{m(z_e)} dm'\, h_m + \int_{m(z_e)}^{m(z_e)+\delta m} dm'\, h_d = m h_m + \delta m\, h_d$

where h_d = MSE/unit mass in WCZ, which = initial h_d of abyssal layer at time of WCZ formation

→ During δt, get $\delta h_m = \dfrac{m h_m + \delta m\, h_d}{m + \delta m} - h_m$ (mixing of small amount of $h_d > h_m$ into mixed layer)

→ since $\delta m / m \ll 1$, $\delta h_m \simeq \dfrac{\delta m}{m}(h_d - h_m) = \dfrac{\delta p_e}{p_e - p_0}(h_d - h_m)$ $\left\{ \text{since } \delta m = \delta p / g \right\}$

and $\dfrac{\delta h_m}{\delta t} \Rightarrow \left.\dfrac{\partial h_m}{\partial t}\right|_{\text{entrainment}} = \dfrac{1}{p_e - p_0}\dfrac{\delta p_e}{\delta t}(h_d - h_m) =$ "entrainment flux"

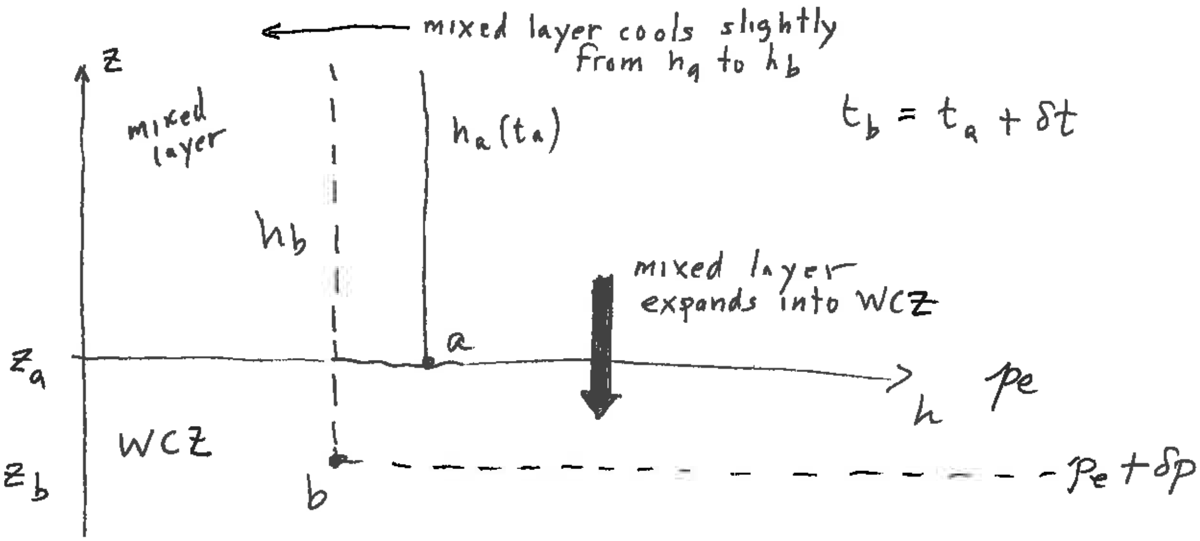
- So, energy conservation with entrainment: (using hydrostatic eqn and $\delta p = -\rho g \delta z$)

$$\underbrace{\frac{p_e - p_0}{g}\frac{\partial h_m}{\partial t}}_{\text{change in column-integr. MSE per unit time}} = -\mathbf{F}(z=0) + F(p_e) + \frac{\omega_e}{g}\underbrace{(h_d - h_m)}_{\text{entrainment heating}} \qquad \boxed{\omega_e \equiv \frac{dp_e}{dt}} \quad (2)$$

Rad. + conv. fluxes at boundaries

- $F =$ radiative + convective heat flux
  - At top of mixed layer (e.g. 10 bars) $F \simeq F_{conv} = \sigma T_i^4$,
    T_i = intrinsic temperature
  - At base of mixed layer, $F \simeq F_{rad} \simeq -K_{rad} \vec{\nabla} T$ (radiative diffusion)
    since WCZ is stable below z_e (defn of z_e). Generally,
    $F_{conv} >> F_{rad}(z_e)$ and $F(p_e) - F(0) \simeq -F_{conv}$

- Want to solve for $h_m(t)$ after WCZ formation, but first need model for $\underline{w_e} \equiv \frac{dp_e}{dt}$, the entrainment rate:



$$h_a = C_p T_a + g z_a - L \ln(1 - q_a)$$
$$h_b = C_p T_b + g z_b - L \ln(1 - q_b)$$

$q \equiv$ saturated spec. humidity

$$h_b - h_a = c_p (T_b - T_a) + \underbrace{g(z_b - z_a)}_{= -\frac{RT_a}{P_a}\delta P} - \underbrace{L \ln \frac{1-q_b}{1-q_a}}_{\text{expand } q_b \text{ about } q_a \text{ to 1st order}}$$

$$q_b \simeq q_a + \left.\frac{\partial q}{\partial T}\right|_{P_a} (T_b - T_a) + P_a \left.\frac{\partial q}{\partial P}\right|_{T_a} \frac{\delta P}{P_a}$$

Plug q_b into expression $-L \ln \frac{1-q_b}{1-q_a} \rightarrow$

$$\simeq \frac{L}{1-q_a} \left\{ \left.\frac{\partial q}{\partial T}\right|_{P_a} (T_b - T_a) + P_a \left.\frac{\partial q}{\partial P}\right|_{T_a} \frac{\delta P}{P_a} \right\}$$

$$\rightarrow h_b - h_a \simeq c_p (T_b - T_a) - \frac{R T_a}{P_a} \delta P + \frac{L}{1-q_a} P_a \left.\frac{\partial q}{\partial P}\right|_{T_a} \frac{\delta P}{P_a} + \frac{L}{1-q_a} \left.\frac{\partial q}{\partial T}\right|_{P_a} (T_b - T_a)$$

- Solve for $T_b - T_a$:

$$T_b - T_a = \frac{h_b - h_a}{c_p (1 + Q_{oT})} + \frac{\left(R T_a - \frac{L}{1-q_a} P_a \left.\frac{\partial q}{\partial P}\right|_{T_a} \right) \frac{\delta P}{P_a}}{c_p (1 + Q_{oT})}$$

with $\boxed{Q_{oT} \equiv \frac{L}{(1-q_a) c_p} \left.\frac{\partial q}{\partial T}\right|_{P_a}}$

- Use this result for $T_b - T_a$ in $q_b - q_a = \left.\frac{\partial q}{\partial T}\right|_{P_a} (T_b - T_a) + P_a \left.\frac{\partial q}{\partial P}\right|_{T_a} \frac{\delta P}{P_a}$

Result:

$$q_b = q_a + \left.\frac{\partial q}{\partial T}\right|_{P_a} \frac{h_b - h_a}{c_p (1 + Q_{oT})} + \left\{ \frac{1}{1 + Q_{oT}} P_a \left.\frac{\partial q}{\partial P}\right|_{T_a} + \frac{R}{c_p} \frac{T_a (\partial q / \partial T)_{P_a}}{1 + Q_{oT}} \right\} \frac{\delta P}{P_a} \qquad (3)$$

- From F & G 2017, Eq (2), WCZ stable where $\alpha < 0$ (or less than some small negative number if we include possibility of convective penetration)

$\rightarrow$ WCZ stable for $\alpha < \alpha_{crit} \lesssim 0$, where

$$\alpha \equiv 1 + \xi \frac{q_s L}{R_W T_0}$$

- Because entrainment leads to cooling and drying of WCZ air, level where $\alpha = \alpha_c$ descends during entrainment. Take level where $\alpha = \alpha_c$ to define bottom of mixed layer

Then, by definition, δP is change in pressure level for the level $\alpha = \alpha_c$ during time interval δt due to entrainment, etc. $\rightarrow$

$$\alpha_b = 1 + \xi \frac{q_b L}{R_W T} \qquad \alpha_a = 1 + \xi \frac{q_a L}{R_W T}, \qquad \text{but } \alpha_b = \alpha_a = \alpha_{crit} \text{ at base of mixed layer}$$

$$\rightarrow q_b = q_a \quad \text{and} \quad \left.\frac{\partial q}{\partial T}\right|_{P_a} \frac{h_b - h_a}{c_p(1+Q_{0T})} + \left\{ \begin{matrix} \text{see} \\ (3) \end{matrix} \right\} \frac{\delta P}{P_a} = 0$$

- Solve for $\omega_e \equiv \left.\frac{\delta P}{\delta t}\right|_{P=P_e} \Rightarrow$

$$\boxed{\omega_e = -P_a \frac{1}{B_1} \left.\frac{\partial q}{\partial T}\right|_{P_a} \frac{1}{c_p} \frac{\partial h_m}{\partial t}} \quad (4)$$

where $B_1 \equiv P_a \left.\frac{\partial q}{\partial P}\right|_{T_a} + \frac{R}{c_p} T_a \left.\frac{\partial q}{\partial T}\right|_{P_a}$

- We can substitute for $\frac{\partial h_m}{\partial t}$ using (2) and find,

$$\boxed{\begin{gathered} \omega_e = -\chi \frac{g}{c_p T} \left\{ F_{rad}(p_e) - F_{conv}(0) \right\} \qquad (5) \\ \text{with} \quad \chi \equiv \frac{P_e}{P_e - P_0} \frac{1}{B_1 C_1} T_e \left.\frac{\partial q}{\partial T}\right|_{P_e}, \quad C_1 = 1 + \frac{P_e}{P_e - P_0} \frac{1}{B_1} \frac{1}{c_p} \left.\frac{\partial q}{\partial T}\right|_{P_e} (h_d - h_m) \end{gathered}}$$

- Having (5) for ω_e closes the system, which can be summarized as follows:

$$\frac{dh_m}{dt} \cong \frac{g}{P_e - P_o}\left\{F_{rad}(P_e) - F_{conv}(P_o)\right\} + \frac{\omega_e}{P_e - P_o}(h_d - h_m)$$

$$\frac{dP_e}{dt} = \omega_e(h_m, P_e)$$

$$\omega_e = -\chi(P_e, h_m)\frac{g}{C_p T_e}\left\{F_{rad}(P_e) - F_{conv}(P_o)\right\},$$

$$\chi(P_e, h_m) = \frac{P_e}{P_e - P_o}\,\frac{1}{B_1 C_1}\,T_e\left.\frac{\partial q}{\partial T}\right|_{P_e}$$

$$B_1 \equiv P_a\left.\frac{\partial q}{\partial P}\right|_{T_a} + \frac{R}{C_p}T_a\left.\frac{\partial q}{\partial T}\right|_{P_a}$$

$$C_1 \equiv 1 + \frac{P_e}{P_e - P_o}\,\frac{1}{B_1}\,\frac{1}{C_p}\left.\frac{\partial q}{\partial T}\right|_{P_e}(h_d - h_m)$$

Comments:

- The term C_1 in the denominator of χ represents the buffering effect that entrainment heating has on the overall rad-conv cooling of the mixed layer (C_1 increases as $h_d - h_m$ increases during the evolution)

- $F_{conv}(P_o)$ is implicitly a function of h_m, since $h_m(P_o) \simeq C_p T_o$ if we identify $z = 0$ with P_o-level. Fortney et al. (2011) links T_o to T_{eff} of planet

- Since $\omega_e = -\rho g \frac{dz_e}{dt}$, we have $\frac{dz_e}{dt} \simeq -\chi\frac{\kappa F_{conv}}{P_e}$, $\kappa \equiv \frac{R}{C_p} \sim \frac{1}{3}$

Scale height of α is $H_\alpha \simeq \frac{\xi y(\Gamma_s/T)(b/T - 1)}{1 + \xi y}$

→

(Comments)

$$H_\alpha \simeq \frac{+\xi y\,(\Gamma_s/T)\left(\frac{b}{T}-1\right)}{1+\xi y}\,, \qquad \Gamma_s = \text{moist adiabat lapse rate}$$

$$y \equiv \frac{qL}{R_w T}$$

$$b = T\,\frac{\partial \ln e_s}{\partial \ln T}\,, \quad e_s = \text{sat. vap. pressure}$$

When $T_{eff} \simeq 60$ K, $p_e \simeq 400$ bar, $T \sim 450$K, $F_{conv} \sim 100$ erg cm^{-2}s^{-1}

$q_{deep} \simeq 0.25 \quad y \simeq 2.6 \quad b \simeq 4700$K $\quad \Gamma_s \sim 6\cdot 10^{-6}$ K cm^{-1}

$\xi = -0.87$ (see FG17) (Appendix A as well)

$\longrightarrow H_\alpha \sim 4.5\cdot 10^6$ cm

time to erode WCZ $\sim H_\alpha / dz_e/dt \sim \dfrac{H_\alpha P}{\chi\,\kappa\,F_{conv}} \sim \dfrac{2\cdot 10^6}{\chi}$ yrs

which is comparable to a cooling timescale for the mixed layer due to radiation to space (FG17)