Notes on WCZ

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1 Thermal evolution

Conservation of energy relates the instantaneous luminosity profile $L(m)$ to the rate of change of specific entropy s in the planet:

$$\frac{dL}{dm} = -T\frac{\delta s}{\delta t}.$$ (1)

Integrating over the mass of the planet and solving for δt yields

$$\delta t = -\frac{1}{L_{\text{int}}} \int_0^M T\,\delta s\,dm$$ (2)

where L_{int} is the intrinsic luminosity $4\pi R^2 \sigma_{\text{SB}} T_{\text{int}}^4$.

For our assumed ideal mixture of light elements (H/He) and heavy elements (H_2O) this entropy difference is simply

$$\delta s = (1 - Z)\delta s_{HHe} + Z\delta s_Z$$ (3)

where in practice δs_Z is computed in terms of the change in internal energy and density:

$$T\,\delta s_Z = \delta u_Z + P\delta\left(\rho_Z^{-1}\right).$$ (4)

2 Temperature jump across the radiative layer

Letting P_{top} and T_{top} denote the pressure and temperature at the top of the stable (radiative) layer, the temperature within the radiative layer is given by

$$T(P) = T_{\text{top}} + \int_{P_{\text{top}}}^P \left(\frac{dT}{dP}\right)_{\text{rad}} dP.$$ (5)

The radiative zone is thin under the relatively opaque conditions typical of water condensation zones in the ice giants (Leconte et al. 2017, Friedson & Gonzales 2017). As a result our discretized model does not spatially resolve the radiative layer. Instead we treat the layer as a discontinuous increase in T and water vapor mole fraction x_{vap}. Here we derive the magnitude of the temperature jump in the limit of a thin radiative zone.

Over a thin layer, the radiative temperature gradient

$$\left(\frac{dT}{dP}\right)_{\text{rad}} = \frac{T}{P}\nabla_{\text{rad}} = \frac{T}{P} \times \frac{3}{16}\frac{\kappa_R P}{g}\frac{T_{\text{int}}^4}{T^4} \tag{6}$$

is nearly constant. In this case the integral in (5) simplifies to

$$T_{\text{base}} \equiv T(P + \Delta P) = T_{\text{top}} + \left(\frac{dT}{dP}\right)_{\text{rad}}\Delta P. \tag{7}$$

Just as in the moist troposphere above, the local water vapor mole fraction x_{vap} follows the saturation value $x_{\text{vap}}^{\text{sat}}$ everywhere within the radiative layer, i.e.,

$$x_{\text{vap}}(P,T) = x_{\text{vap}}^{\text{sat}}(P,T) = \frac{e_s(T)}{P}, \qquad P < P_{\text{base}}. \tag{8}$$

We denote by P_{base} and T_{base} the pressure and temperature at the base of the radiative zone, which is set by the condition that x_{vap} has reached the deep value $x_{\text{vap}}^{\text{deep}}$:

$$x_{\text{vap}}^{\text{sat}}(P_{\text{base}}, T_{\text{base}}) = \frac{e_s(T_{\text{base}})}{P_{\text{base}}} = x_{\text{vap}}^{\text{deep}} \tag{9}$$

$$\implies \Delta P \equiv P_{\text{base}} - P_{\text{top}} = \frac{e_s(T_{\text{base}})}{x_{\text{vap}}^{\text{deep}}} - P_{\text{top}}. \tag{10}$$

(Here e_s is the H_2O saturation vapor pressure, calculated from XXXX relation [describe what the method from thermodynamics.py actually does].) Deeper pressures $P > P_{\text{base}}$ are sub-saturated and hence no further condensation of H_2O takes place. ΔP gives the extent, in pressure coordinates, of the radiative layer. Combining (10) and (5) yields

$$T_{\text{base}} = T_{\text{top}} + \left(\frac{dT}{dP}\right)_{\text{rad}}\left(\frac{e_s(T_{\text{base}})}{x_{\text{vap}}^{\text{deep}}} - P_{\text{top}}\right) \tag{11}$$

which we numerically solve for T_{base}. Deeper temperatures are then obtained by integrating the dry adiabat ∇_{ad}:

$$T(P > P_{\text{base}}) = T_{\text{base}} + \int_{P_{\text{base}}}^{P}\left(\frac{dT}{dP}\right)_{\text{ad}}dP. \tag{12}$$