

## THE CONSTITUTION OF JUPITER AND SATURN

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**Abstract.** The equilibrium configurations of rotating bodies are reviewed in relation to the equation of state of planetary matter. Previous analyses of the constitution of Jupiter and Saturn are examined critically and a new analysis is made. This involves an exhaustive discussion of the thermal and mechanical properties of hydrogen and helium. On the basis of a stated set of assumptions it is found that Jupiter and Saturn are composed of hydrogen in excess of 78 per cent and 63 per cent by weight. The consequences of relaxing the assumptions are tested but definite error limits are not proposed.

## I. INTRODUCTION

The composition of the planets Jupiter and Saturn poses a problem which is of interest from several points of view. In the first place, if one is concerned with the chemical make-up of the matter presently tied up in all the planets and one can ascertain the composition of Jupiter and Saturn, then one has already analyzed more than 90 per cent of the total mass of the planets. Secondly, the compositions of these planets are of interest because of the marked dissimilarity between these bodies and the terrestrial planets. Particularly striking are the low mean densities of Jupiter and Saturn, although both their masses are orders of magnitude larger than those of the terrestrial planets. It is likely that there would be agreement among most cosmogonists that, of all the planets, Jupiter and Saturn should come closest to possessing a primordial composition. Consequently, analysis of the composition of these planets sheds some light on the nature of the primordial composition.

Previous models of these planets have been proposed by Jeffreys (1924), Wildt (1934, 1938, 1947), Brown (1950), Ramsey (1951), Ramsey and Miles (1952), the writer (1951), and Fessenkov and Massevic (1951). Jeffreys' models assumed that Jupiter and Saturn had deep gaseous atmospheres of negligible mass and were shown to be in serious conflict with the equation of state of gases. This fact has been pointed out by Wright (1927), Wildt (1934), and Peek (1937). Wildt's 1934 and 1938 models were constructed in accordance with the prevailing ideas regarding the composition of stellar material and, hence, were bound to subordinate the role of hydrogen. It may be noted, however, that Critchfield (1942)

showed that Wildt's 1938 models could be interpreted, in the light of the equation of state of metallic hydrogen, as being composed mainly of hydrogen. All the models which have been proposed in the past decade have stemmed in large measure from the suggestion of Wildt (1938) and Kothari (1938) that the existence of metallic hydrogen was of importance to the elucidation of the composition of Jupiter and Saturn. With a single exception they have also been influenced by the fact that a detailed formulary of the equation of state of hydrogen was put forth by Kronig, de Boer and Korringa in 1946. The exception is the paper of Brown (1950) which is somewhat lacking in interest because he assumed an equation of state which was inadequate from the standpoint of information then current. Fessenkov and Massevic seem to have been the most cautious in their acceptance of the results of Kronig *et al.* The writer finds that this caution was justified since their equation of state of molecular hydrogen is shown below to be erroneous.

This paper is the result of approximately eight years of part-time effort by the writer on the separate—but related—problems of pure hydrogen planets and the structure of the two greatest planets. It is apparent from the historical introduction that hydrogen has occupied a position of increasing emphasis in speculations about the constitution of the giant planets and it can be stated that it has been clear from the publication of Wildt's 1947 paper onward that hydrogen is a major chemical constituent of Jupiter and Saturn. The purpose of this paper is to quantify this conclusion but at the same time to illuminate the possible errors. In this respect, it is believed

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to differ from the papers of the past decade, including the writer's thesis, which have arrived at quantitatively similar conclusions but, in the writer's opinion, have failed to arrive at a realistic assessment of the confidence that may be placed in their results. It is believed that the results of this paper are superior to any previously determined. However, the reader will come to feel that he is left with a distinct sense of uncertainty as to the composition and structure of the giant planets which he would not derive from a study of Ramsey's work. In fact, it is a purpose of this paper to counteract the sense of finality on which Ramsey chose to terminate his work.

In a sense, this work was begun in 1949 with a study of pure hydrogen model planets and the first results were given in the writer's unpublished thesis (Yale 1951). It was concluded at that time that the major planets were not composed solely of hydrogen, a conclusion reached simultaneously and independently by Ramsey. It was realized, moreover, that further progress would depend upon better knowledge of the equations of state of solid hydrogen and helium. In 1952 the writer was fortunate enough to obtain the aid of the National Science Foundation to further this work. Its completion has been delayed partly by the fact that it had to be done on a part-time basis and partly by the drastic reorientation following upon the attainment of accurate experimental data by J. W. Stewart beginning in 1954 and continuing. His enthusiastic response to astrophysical exigencies has been a major factor shaping the ultimate form of this work.

The first part of this paper deals with the theory of the figure assumed by a rotating mass of matter whose pressure-density relation is prescribed. Much of the necessary theory is available in papers by Darwin (1899) and de Sitter (1924), but these investigators were concerned with determining the figure of a planet when the density distribution along some radius vector is known. This paper of necessity must take a more basic point of view and determine the density distribution from the compressibility of the planetary material; this has necessitated some amplification of the work of Darwin and de Sitter. Next, some general integral theorems relevant to the internal structure of planets in hydrostatic equilibrium are set down. Several of these, given previously by other workers, are gathered here for convenience. In some instances, theorems

developed by previous writers have been modified slightly to account for the fact that the planets may be rotating. In addition, there are two new inequalities which relate the surface density of a planet to the higher terms in its external gravitational potential. These inequalities substantially sharpen a result previously obtained by Jeffreys. In Part III the experimental data on the physical properties of solid hydrogen and helium are collected. There is also a rather detailed discussion of the equations of state of cold solid hydrogen and helium. This discussion and the experimental data show that the equation of state of solid molecular hydrogen cannot conform even approximately to the predictions of Kronig, de Boer and Korrington, which were based on the assumption that the intermolecular interaction was given by a Lennard-Jones potential curve. The theoretical equation of state of the hypothetical metallic phase has also been completely reworked. Here the writer has felt it necessary to return to the original method of Wigner and Huntington and to repeat their work, using a refinement in the theory of cohesion of the monovalent metals given by Wigner subsequent to his study, in collaboration with Huntington (1935), of the hypothetical metallic phase of hydrogen. The resultant change in the calculated pressure-density relation is small, but the theoretical cohesive energy of the metal is increased numerically by about one-third. In addition, attempts are made to estimate theoretically some of the thermal properties of both phases, in particular the thermal expansivity and the thermal conductivity. The nature of the adiabatic curve of the fluid is discussed at some length and it is argued that this relation is essentially "ideal" in the  $p, T$  plane, though definitely non-ideal in the  $p, V$  or  $V, T$  planes. This section concludes with a discussion of the melting curve of hydrogen. Part IV contains the details of models which fit, within limits, the known physical features of Jupiter and Saturn. Part V deals with the abundance problem. Here are given the arguments which lead to the conclusion that Jupiter and Saturn are predominantly composed of hydrogen. In Part VI the difficult problem of the internal temperatures of Jupiter and Saturn is considered in the light of the theoretical estimates in Part III. Part VII is a brief résumé of the principal conclusions of the paper. In addition, the writer ventures some suggestions for future work. Finally, in an appendix, a mass-radius diagram of

cold planets of pure hydrogen is exhibited and some of its implications with regard to cold gravitating bodies are briefly stated.

## II. THE FIGURE OF A ROTATING PLANET

The theory of the figure of a rotating planet correct through terms of the order of the square of the ellipticity has been investigated by several authors, notably Airy (1826), Callandreau (1889), Darwin (1899), and de Sitter (1924). However, in all this work the emphasis has been on the determination of the figure when the density distribution along some radius of the planet is known. It is necessary to recast the theory into a form suitable for solving the problem of the figure of a planet when only the equation of state of the material is known.

A surface of equal density will be assumed to be nearly an ellipsoid of revolution whose equation is

$$r = s \left[ 1 - \frac{4\epsilon^2}{45} - \frac{2}{3} \left( \epsilon + \frac{23\epsilon^2}{42} + \frac{4\kappa}{7} \right) P_2(\cos \phi) + \frac{8}{35} \left( \frac{3}{2} \epsilon^2 + 4\kappa \right) P_4(\cos \phi) \right]. \quad (2.1)$$

Here  $r$  is the length of the radius vector from the center of the planet,  $\phi$  the colatitude,  $s$  the mean radius,  $\epsilon$  the ellipticity, and  $\kappa$  measures the departure from an ellipsoid. The mean radius above is defined as the radius of a sphere of the same volume as that contained within the surface of equal density.

The gravitational potential,  $V$ , at any point  $(r, \phi)$  within the planet correct through the second order terms is then given by the expression

$$-\frac{3V}{4\pi G} = \frac{S_0}{r} - \frac{2}{5} \frac{S_2 P_2}{r^3} + \frac{12}{35} \frac{S_4 P_4}{r^5} + \frac{3}{2} T_0 - \frac{2}{5} r^2 T_2 P_2 + \frac{32}{105} r^4 T_4 P_4 + \frac{\omega^2 r^2}{4\pi G} (1 - P_2) \quad (2.2)$$

with

$$S_0 = \int_0^s \rho(s) d(s^3),$$

$$S_2 = \int_0^s \rho(s) d \left[ s^5 \left( \epsilon + \frac{1}{6} \epsilon^2 + \frac{7}{4} \kappa \right) \right],$$

$$S_4 = \int_0^s \rho(s) d \left[ s^7 \left( \epsilon^2 + \frac{8}{9} \kappa \right) \right],$$

$$T_0 = \int_s^{R_1} \rho(s) d \left[ s^2 \left( 1 - \frac{4}{45} \epsilon^2 \right) \right],$$

$$T_2 = \int_s^{R_1} \rho(s) d \left[ \epsilon + \frac{9}{14} \epsilon^2 + \frac{4}{7} \kappa \right],$$

$$T_4 = \int_s^{R_1} \rho(s) d \left( \frac{\kappa}{s^2} \right).$$

In these last expressions  $\rho(s)$  is the density on the surface of parameter  $s$  and  $R_1$  is the value of  $s$  corresponding to the surface of the planet.

The preceding expression for the gravitational potential is essentially the same as Darwin's (1899) equation (11) except for the term  $T_0$ . This term was dropped by Darwin because it is not needed for studying the figure of a planet of predetermined density distribution. It is also to be noted that the independent variable has been taken as the mean radius rather than the equatorial radius employed by Darwin, and that we have measured the departure from a true ellipsoid by  $\kappa$  rather than the parameter  $f$  used for the same purpose by Darwin. The two parameters are related by  $f = -4\kappa$ .

The assumption of hydrostatic equilibrium implies that the equipotential surfaces are surfaces of equal density. Inserting the expression for  $r$  into the potential equation and equating the coefficients of the linearly independent angular factors to zero leads, after a fair amount of manipulation, to three integral equations

$$-\frac{3V(s)}{4\pi G} = \frac{S_0}{s} \left( 1 + \frac{8\epsilon^2}{45} \right) - \frac{4}{25} \epsilon \frac{S_2}{s^3} + \frac{3}{2} T_0 + \frac{8}{75} \epsilon s^2 T_2 + \frac{\omega^2 s^2}{4\pi G} \left( 1 + \frac{4}{15} \epsilon \right), \quad (2.3)$$

$$\frac{S_0}{s} \left( \frac{2}{3} \epsilon + \frac{31}{63} \epsilon^2 + \frac{8}{21} \kappa \right) - \frac{2}{5} \frac{S_2}{s^3} \left( 1 + \frac{4}{7} \epsilon \right) - \frac{2}{5} s^2 T_2 \left( 1 - \frac{8}{21} \epsilon \right) = \frac{\omega^2 s^2}{4\pi G} \left( 1 + \frac{20}{21} \epsilon \right), \quad (2.4)$$

$$\frac{S_0}{s} (3\epsilon^2 - 8\kappa) - 6\epsilon \frac{S_2}{s^3} + 3 \frac{S_4}{s^3} + \frac{8}{3} s^4 T_4 = 0. \quad (2.5)$$

Now if hydrostatic equilibrium holds, the Bernoullian integral is independent of the parameter

$s$ , i.e.,

$$V(s) + \int \frac{dp}{\rho} = \text{const.} \quad (2.6)$$

where  $p$  is the pressure. In differential form this last expression becomes

$$\frac{1}{\rho} \frac{dp}{ds} = - \frac{dV}{ds}. \quad (2.7)$$

The non-linearity of equations (2.3), (2.4) and (2.5) clearly suggests solution by successive approximations. Accordingly, in the equation (2.3) for  $V(s)$  we drop the second-order terms obtaining

$$V(s) = - \frac{4\pi G}{3} \left[ \frac{1}{s} \int_0^s \rho(s) ds^3 + \frac{3}{2} \int_s^{R_1} \rho(s) ds^2 + \frac{\omega^2 s^2}{4\pi G} \right]. \quad (2.8)$$

Hence, to this order of approximation the differential form of the equation of hydrostatic equilibrium becomes

$$\frac{dp}{ds} = - \left[ \frac{GM(s)}{s^2} - \frac{2}{3} \omega^2 s \right] \rho, \quad (2.9)$$

wherein we have set  $M(s)$  equal to the mass contained within the surface of parameter  $s$

$$M(s) = \frac{4\pi}{3} \int_0^s \rho(s) ds^3.$$

Equation (2.9) has been previously obtained by Ramsey (1951) who averaged the centrifugal force over a surface of constant density. In order to build model planets of a material whose equation of state is known and whose density will be correct to the first order in  $\omega^2$  one can formally treat the problem as if the planet were spherical by using the extra term  $\frac{2}{3}\omega^2 s$  in the value of the local gravitational acceleration. Moreover, if the density distribution is known to the first order in  $\omega^2$ , the quantities  $\epsilon$  and  $\kappa$  can be calculated correctly to the second order. This last statement can easily be verified by inspection of equations (2.4) and (2.5). Consequently, one can evaluate these parameters and then, if desired, recalculate the density distribution to the second order by retaining the second-order terms in the expression for  $V(s)$ . Such a refinement has not been carried out in the calculation of model planets below but could have easily been made.

Before taking up the solution of the equations

for  $\epsilon$  and  $\kappa$  it will be convenient to follow de Sitter and write the equations in dimensionless form by use of the following variables:

$$\rho_0 = \frac{M(R_1)}{\frac{4\pi}{3} R_1^3} \quad (2.10)$$

$$m = \frac{3\omega^2}{4\pi G \rho_0} \quad (2.11)$$

$$\beta = \frac{s}{R_1} \quad (2.12)$$

$$\rho_0 \delta(s) = \rho(s) \quad (2.13)$$

$$D = \frac{1}{\beta^3} \int_0^\beta \delta(t) dt^3 \quad (2.14)$$

$$S = \frac{1}{\beta^5} \int_0^\beta \delta(t) dt^5 \left[ \epsilon + \frac{1}{6} \epsilon^2 + \frac{4}{7} \kappa \right] \quad (2.15)$$

$$P = \frac{1}{\beta^7} \int_0^\beta \delta(t) dt^7 \left[ \epsilon^2 + \frac{8}{9} \kappa \right] \quad (2.16)$$

$$T = \int_\beta^1 \delta(t) dt \left[ \epsilon + \frac{9}{14} \epsilon^2 + \frac{4}{7} \kappa \right] \quad (2.17)$$

$$Q = \beta^2 \int_\beta^1 \delta(t) dt \left[ \frac{\kappa}{t^2} \right]. \quad (2.18)$$

(The notation is now de Sitter's except for the use in (2.11) of  $m$ , instead of his  $\rho_1$ , in order to avoid confusion with the various density symbols.) In terms of these new variables (2.4) and (2.5) become

$$D \left[ \epsilon + \frac{31}{42} \epsilon^2 + \frac{4}{7} \kappa \right] - \frac{3}{5} S \left( 1 + \frac{4}{7} \epsilon \right) - \frac{3}{5} T \left( 1 - \frac{8}{21} \epsilon \right) = \frac{1}{2} m \left( 1 + \frac{20}{21} \epsilon \right) \quad (2.19)$$

$$D[3\epsilon^2 - 8\kappa] - 6\epsilon S + 3P + \frac{8}{3} Q = 0. \quad (2.20)$$

These equations were given by de Sitter except for the coefficient of  $Q$  which he gave incorrectly as  $\frac{32}{3}$ . If the second-order terms in the first of these equations are neglected there results

$$D\epsilon - \frac{3}{5} (S + T) = \frac{1}{2} m, \quad (2.21)$$



an integral equation for  $\epsilon$ . By integrating by parts this can be brought to the form

$$\epsilon(\beta) = \frac{\frac{1}{2}m + \frac{3}{5}\delta_1\epsilon_1}{D(\beta)} + \int_0^1 \epsilon(t) dK(\beta, t), \quad (2.22)$$

where the subscript 1 signifies the evaluation of the relevant quantities at  $\beta = 1$  and

$$\begin{aligned} dK(\beta, t) &= -\frac{3}{5} \frac{t^5 d\delta(t)}{\beta^5 D(\beta)}, & t \leq \beta \\ &= -\frac{3}{5} \frac{d\delta(t)}{D(\beta)}, & t > \beta. \end{aligned}$$

When the density inside a planet has discontinuities, integration must conform to the definition of Stieltjes rather than Riemann.

If the density does not increase with  $\beta$ , a usual assumption,  $dK(\beta, t)$  is never negative; more precisely,

$$\frac{1}{b-a} \int_a^b dK(\beta, t) \geq 0.$$

Moreover,

$$\int_0^1 dK(\beta, t) = \frac{3}{\beta^5 D(\beta)} \int_0^\beta t^5 d\delta(t) - \frac{3}{5} \frac{\delta_1}{D(\beta)}.$$

Defining an average moment of inertia for the mass within the surface  $s = \text{const.}$  by  $I(s)$  this last integral can be expressed as

$$\int_0^1 dK(\beta, t) = \frac{3}{2} \frac{I(s)}{M(s) \cdot s^2} - \frac{3}{5} \frac{\delta_1}{D(\beta)}.$$

This quantity cannot exceed 0.6 at any value of  $\beta$  in a planet whose density never increases with radius. As a consequence of this fact, numerical solution of the integral equation is feasible by several methods. However, it is often more convenient to deal with the ellipticity by Radau's transformation using the variable

$$\eta = \frac{\beta}{\epsilon} \frac{d\epsilon}{d\beta},$$

which satisfies the differential equation

$$\begin{aligned} \beta \frac{d\eta}{d\beta} + \eta^2 + 5\eta - 6 \left[ 1 - \frac{\delta(\beta)}{D(\beta)} \right] \\ \times (1 + \eta) = 0 \quad (2.23) \end{aligned}$$

and can be transformed into a non-linear, but

easily solved, integral equation,

$$D(\beta) \beta^5 \sqrt{1 + \eta} = 5 \int_0^\beta t^4 D(t) F[\eta(t)] dt, \quad (2.24)$$

with

$$F(x) = \frac{1 + \frac{1}{2}x - \frac{1}{10}x^2}{\sqrt{1+x}}.$$

Once  $\epsilon(\beta)$  is known correct to the first order in  $\omega^2$ , the computation of  $\kappa(\beta)$  can also be reduced to the solution of an inhomogeneous integral equation

$$\kappa(\beta) = g(\beta) + \int_0^1 \kappa(t) d\Gamma(\beta, t), \quad (2.25)$$

with

$$\begin{aligned} g(\beta) &= -\frac{3}{8D(\beta)\beta^7} \int_0^\beta [\beta^2 \epsilon(\beta) - t^2 \epsilon(t)]^2 t^3 d\delta(t) \\ &\quad + \frac{1}{3} \frac{\beta^2 \delta_1 \kappa_1}{D(\beta)} \end{aligned}$$

and

$$\begin{aligned} d\Gamma(\beta, t) &= -\frac{1}{3} \frac{t^7 d\delta(t)}{\beta^7 D(\beta)} & t < \beta \\ &= -\frac{1}{3} \frac{\beta^2 d\delta(t)}{t^2 D(\beta)} & t > \beta. \end{aligned}$$

The contribution of the integral on the right to the surface value of  $\kappa$  is almost negligible for planets whose central condensation is as marked as those exhibited in Jupiter and Saturn. To demonstrate this we note that at the surface of the planet where  $\beta = 1$  we have

$$\int_0^1 d\Gamma(1, t) = -\frac{1}{3} \int_0^1 t^7 d\delta(t).$$

However, again on the assumption that the density does not increase with  $\beta$ , the integral on the right obviously satisfies the inequality

$$\begin{aligned} -\frac{1}{3} \int_0^1 t^7 d\delta(t) &\leq -\frac{1}{3} \int_0^1 t^5 d\delta(t) \\ &= \frac{1}{3} \left[ \frac{5}{2} \frac{I(R_1)}{M_1 R_1^2} - \delta_1 \right]. \end{aligned}$$

For Jupiter and Saturn the last term will be about 0.15. If the maximum value of  $\kappa$  is at the surface, and since  $\kappa(\beta)$  will be non-negative, it is a simple matter to derive the following double

inequality for  $\kappa_1$

$$\frac{-\frac{3}{8} \int_0^1 [\epsilon_1 - \ell^2 \epsilon(t)]^2 \ell^2 d\delta(t)}{1 - \frac{1}{3} \delta_1} \leq \kappa_1$$

$$\leq \frac{-\frac{3}{8} \int_0^1 [\epsilon_1 - \ell^2 \epsilon(t)]^2 \ell^2 d\delta(t)}{1 - \frac{5}{6} \frac{I}{MR^2}}. \quad (2.26)$$

The derivation is an elementary consequence of the remark that extreme values for the integral

$$\int_0^1 \kappa(t) d\Gamma(1, t)$$

are zero and

$$\kappa_1 \left[ \frac{5}{6} \frac{I}{MR_1^2} - \frac{1}{3} \delta_1 \right].$$

Consequently, by averaging the two extremes,  $\kappa$  can be computed to within  $\sim 8$  per cent without solving the integral equation. A more accurate evaluation can be made by assuming that  $\kappa$  is proportional to  $\epsilon^2$  in the outer layers of the planet, or by iteration. The surface value of  $\kappa$  has been derived for the model planets in this paper by each of the above three methods.

Once  $\epsilon$  is known to the first order and, consequently,  $\kappa$  to the second,  $\epsilon$  may also be computed to the second order by using  $\kappa$  and the first order  $\epsilon$  in the second-order terms in (2.19). The results were scarcely distinguishable from values calculated by de Sitter's modification of the Radau equation (see Spencer Jones 1954).

In the analysis of the motions of the satellites, the external potential of the planet is usually taken as

$$V_e = -\frac{GM_1}{r} \left[ 1 - \frac{2}{3} \frac{Ja^2}{r^2} P_2(\cos \phi) + \frac{4}{15} \frac{Ka^4}{r^4} P_4(\cos \phi) \right] \quad (2.27)$$

wherein  $a$  is the equatorial radius. In terms of the integrals  $S$  and  $P$ , we may write the external potential as

$$V_e = -\frac{GM_1}{r} \left[ 1 - \frac{2}{5} S_1 \frac{R_1^2}{r^2} P_2(\cos \phi) + \frac{12}{35} P_1 \frac{R_1^4}{r^4} P_4(\cos \phi) \right]. \quad (2.28)$$

Accordingly, to the second order,

$$J = \frac{3}{5} S_1 \left( 1 - \frac{2}{3} \epsilon_1 \right), \quad (2.29)$$

$$K = \frac{9}{7} P_1, \quad (2.30)$$

since the equatorial radius is given to the first order by

$$a = R_1 \left( 1 + \frac{1}{3} \epsilon_1 \right).$$

Equations (2.19) and (2.20) evaluated at the surface now become, in terms of  $J$  and  $K$ ,

$$\epsilon_1 - J = \frac{1}{2} m + \frac{1}{2} \epsilon_1^2 - \frac{1}{7} \epsilon_1 m - \frac{4}{7} \kappa_1 \quad (2.31)$$

$$K = \frac{24}{7} \kappa_1 + 3\epsilon_1^2 - \frac{15}{7} \epsilon_1 m \quad (2.32)$$

which were given by de Sitter. The relation (2.29) between  $J$  and  $S_1$  was given by de Sitter (1924) and also by Spencer Jones (1954) as

$$J = \frac{3}{5} S_1,$$

which is only correct to the first order. (It should be noted, however, that the mistakes in de Sitter's 1924 paper are not in the final equations generally used in connection with the figure of the Earth.)

In the following we shall assume that the density along any radius vector of the planet is a decreasing function of the distance from the center. Hydrostatic equilibrium will also be assumed unless otherwise indicated.

The equation of hydrostatic equilibrium (2.9) correct to the first order in the centrifugal effects is

$$\frac{dP}{ds} = -\frac{GM(s)}{s^2} \rho(s) + \frac{2}{3} \omega^2 s \rho(s),$$

in terms of the independent variable  $s$ . The pressure at the center is greater than for a planet of uniform density. Hence,

$$P_c \geq \frac{2\pi G}{3} \rho_0^2 R_1^2 \left[ 1 - \frac{\omega^2}{2\pi G \rho_0} \right].$$

Similarly, the pressure at the center is less than it would be if the density were equal to the central density,  $\rho_c$ , out to a value of  $s'$  given by

$$\rho_c s'^3 = \rho_0 R_1^3,$$

and zero outside this range, i.e.,

$$P_c \leq \frac{2\pi G}{3} \rho_c^2 s^{1/2} \left[ 1 - \frac{\omega^2}{2\pi G \rho_c} \right].$$

These two inequalities may be combined into the symmetrical form

$$\begin{aligned} \frac{1}{2} \left( \frac{4\pi}{3} \right)^{1/3} G M^{2/3} \rho_0^{4/3} \left[ 1 - \frac{\omega^2}{2\pi G \rho_0} \right] &\leq P_c \\ &\leq \frac{1}{2} \left( \frac{4\pi}{3} \right)^{1/3} G M^{2/3} \rho_c^{4/3} \left[ 1 - \frac{\omega^2}{2\pi G \rho_c} \right]. \end{aligned} \quad (2.33)$$

The lower limits to the central pressure for Jupiter and Saturn, respectively, are 11.4 and 2.1 in units of  $10^{12}$  dynes/cm<sup>2</sup>. The omission of the centrifugal correction would make these numbers 12.1 and 2.3.

On the assumption that the density does not increase with radius, Jeffreys (1924) has given an upper limit to the surface density of a planet whose average moment of inertia is known,

$$\rho_s \leq \frac{5}{2} \rho_0 (I/MR^2). \quad (2.34)$$

This inequality leads to surprisingly low numerical values for the surface density of Saturn but this result can be considerably sharpened. To see how this can be done consider the number  $J$  which appears in the potential of the planet. Combination of (2.15) and (2.29) gives

$$\begin{aligned} \left( 1 + \frac{2}{3} \epsilon_1 \right) J &= \frac{3}{5} \int_0^1 \delta(t) d \left[ t^5 \left( \epsilon + \frac{1}{6} \epsilon^2 + \frac{4}{7} \kappa \right) \right]. \end{aligned} \quad (2.35)$$

Integrating the right-hand side by parts this becomes

$$\begin{aligned} \left( 1 + \frac{2}{3} \epsilon_1 \right) J &= \frac{3}{5} \delta_1 \left( \epsilon_1 + \frac{1}{6} \epsilon_1^2 + \frac{4}{7} \kappa_1 \right) \\ &\quad - \frac{3}{5} \int_0^1 \left( \epsilon + \frac{1}{6} \epsilon^2 + \frac{4}{7} \kappa \right) t^5 d\delta(t). \end{aligned}$$

Using the assumption that the density decreases with  $\beta$  we can therefore write

$$\left( 1 + \frac{2}{3} \epsilon_1 \right) J \geq \frac{3}{5} \delta_1 \left( \epsilon_1 + \frac{1}{6} \epsilon_1^2 + \frac{4}{7} \kappa_1 \right),$$

which gives the desired inequality

$$\delta_1 \leq \frac{5}{3} \frac{\left( 1 + \frac{2}{3} \epsilon_1 \right)}{\left( \epsilon_1 + \frac{1}{6} \epsilon_1^2 + \frac{4}{7} \kappa_1 \right)} J. \quad (2.36)$$

It should be noted that this inequality depends upon  $J$  which is determined directly from the motions of the satellites while it is impossible to find the average moment of inertia without knowledge of the internal density distribution. To the first order, equation (2.35) is

$$J = \frac{3}{5} \int_0^1 \delta(\beta) d(\epsilon \beta^5) \leq \frac{3}{2} \epsilon_1 \frac{I}{MR_1^2}.$$

Hence,

$$\frac{I}{MR^2} \geq \frac{2}{3} \frac{J}{\epsilon_1},$$

equality obtaining only for a planet of constant density. But according to (2.36) we have, again to the first order,

$$\frac{2}{3} \frac{J}{\epsilon_1} \geq \frac{2}{5} \delta_1.$$

Again, equality only occurs if the planet is one of constant density. Hence, we can write

$$\delta_1 \leq \frac{5}{3} \frac{J}{\epsilon_1} \leq \frac{5}{2} \frac{I}{MR_1^2}. \quad (2.37)$$

From this we see immediately that the inequality (2.36) is sharper than that of Jeffreys. Exceptions are a homogeneous planet and the physically unrealistic planet with a massive point nucleus and weightless mantle for which the two inequalities agree. Even the inequality (2.36) can be sharpened, although at the expense of using observational data of lesser accuracy. This is done by combining (2.16) and (2.30) to obtain

$$\begin{aligned} K &= \frac{9}{7} \int_0^1 \delta d \left[ \left( \epsilon^2 + \frac{8}{9} \kappa \right) \beta^7 \right] \\ &= \frac{9}{7} \delta_1 \left( \epsilon_1^2 + \frac{8}{9} \kappa_1 \right) \\ &\quad - \frac{9}{7} \int_0^1 \left( \epsilon^2 + \frac{8}{9} \kappa \right) \beta^7 d\delta(\beta). \end{aligned} \quad (2.38)$$

Hence, as before,

$$K \geq \frac{9}{7} \delta_1 \left( \epsilon_1^2 + \frac{8}{9} \kappa_1 \right)$$

and

$$\delta_1 \leq \frac{7}{9} \frac{K}{\epsilon_1^2 + \frac{8}{9} \kappa_1}. \quad (2.39)$$

In Table (2.1) below are given the upper limits to the surface densities using Jeffreys' inequality (estimating  $I/MR^2$  from the Radau-Darwin approximation) and the last two inequalities. The empirical data employed are those of Brouwer

TABLE (2.1). ESTIMATES OF SURFACE DENSITY  
UPPER LIMIT

	Jupiter	Saturn
Jeffreys	.89 (g/cm <sup>3</sup> )	.42 (g/cm <sup>3</sup> )
From <i>J</i>	.77	.31
From <i>K</i>	.57	.19

and Clemence (1957). These figures, especially for Saturn, are surprisingly low. This is particularly true of the value for Saturn derived from the inequality (2.39). However, the empirical value of *K* is inherently much more uncertain than is that of *J*. Also, the meaning of a surface density is vague for planets, such as Jupiter and Saturn, whose outer layers are probably composed of fluids above their critical temperatures. Indeed, it might be argued that in such a case all the preceding inequalities are irrelevant since  $\delta_1$  vanishes. The following considerations show, however, that these numbers cannot be thus lightly dismissed.

Returning once again to the expression (2.38) for *K* we put  $\alpha < 1$  and write

$$\begin{aligned}
 K &= \frac{9}{7} \int_0^\alpha \delta(\beta) d\left[\beta^7 \left(\epsilon^2 + \frac{8}{9} \kappa\right)\right] \\
 &\quad + \frac{9}{7} \int_\alpha^1 \delta(\beta) d\left[\beta^7 \left(\epsilon^2 + \frac{8}{9} \kappa\right)\right] \\
 &= \frac{9}{7} \delta(\alpha) \left[\epsilon^2(\alpha) + \frac{8}{9} \kappa(\alpha)\right] \alpha^7 \\
 &\quad + \frac{9}{7} \int_\alpha^1 \delta(\beta) d\left[\beta^7 \left(\epsilon^2 + \frac{8}{9} \kappa\right)\right] \\
 &\quad - \frac{9}{7} \int_0^\alpha \beta^7 \left(\epsilon^2 + \frac{8}{9} \kappa\right) d\delta(\beta).
 \end{aligned}$$

The two last terms on the right are positive and hence

$$\delta(\alpha) \leq \frac{\frac{7}{9} K}{\left[\epsilon^2(\alpha) + \frac{8}{9} \kappa(\alpha)\right] \alpha^7}. \quad (2.40)$$

From the definition of  $\eta$  it follows that  $\epsilon_1^2 \beta^{2\eta_1}$  is a good approximation to  $\epsilon^2$  near  $\beta = 1$ . Also  $\kappa$  is small compared to  $\epsilon^2$  and varies in roughly the same fashion so that we may replace

$$\epsilon^2(\alpha) + \frac{8}{9} \kappa(\alpha)$$

by

$$\left(\epsilon_1^2 + \frac{8}{9} \kappa_1\right) \beta^{2\eta_1}$$

with good accuracy for  $1 - \beta \ll 1$  and hence

$$\delta(\alpha) \leq \frac{\frac{7}{9} K}{\left(\epsilon_1^2 + \frac{8}{9} \kappa_1\right) \alpha^{7+2\eta_1}} = \frac{.194}{\alpha^{10.34}} \quad (2.41)$$

for Saturn. Table (2.2) shows values of the right hand side of (2.41) for various depths below the surface.

TABLE (2.2). UPPER DENSITY LIMIT IN OUTER  
LAYERS OF SATURN

Depth (km)	Upper limit to density
0	.19 g/cm <sup>3</sup>
570	.22
1150	.24
1730	.27
2310	.30
2880	.33
5760	.58

Since the scale height for hydrogen gas at the surface of Saturn is about 42 km, it is easily seen that the outer layers of Saturn are peculiarly light.

### III. PHYSICAL PROPERTIES OF COLD HYDROGEN AND HELIUM

#### A. The Equations of State of Solid Hydrogen and Helium.

Experimental knowledge of the equations of state of solid hydrogen and helium is restricted to a single isotherm (4.2°K) and pressures less than 20,000 kg/cm<sup>2</sup>. Hydrogen and helium are both extremely compressible solids at low pressures, chiefly because of their large zero-point energies. With increasing pressure their compressibilities diminish by large amounts and are more or less normal at pressures above 10,000 kg/cm<sup>2</sup>. The densities of the solids measured by J. W. Stewart (1956) are given in Table (3.1) together

TABLE (3.1). EXPERIMENTAL DATA FOR SOLID  
HYDROGEN AND HELIUM

Pressure (kg/cm <sup>2</sup> )	Molar volume (cm <sup>3</sup> /mole)		$\frac{\alpha(\beta)}{(1.986 V_{H_2}/V_{He})}$
	Hydrogen	Helium	
0	22.65	(melted)	—
200	21.0	15.8	2.6
400	20.0	14.5	2.7
600	19.2	13.5	2.8
1,000	18.0	12.4	2.9
2,043	16.1	10.7	3.0
3,000	15.1	9.9	3.0
4,000	14.3	9.3	3.1
6,000	13.2	8.5	3.1
8,000	12.4	8.0	3.1
10,000	11.8	7.5	3.1
12,000	11.3	7.2	3.1
16,000	10.6	6.7	3.1
20,000	10.1	6.4	3.1



with their ratio, which we denote by  $\alpha(p)$  ( $= \rho_{He}(p)/\rho_{H_2}(p)$ ). In order to ascertain the chemical composition of the giant planets the equations of state of hydrogen and helium must be known at much higher pressures. For this purpose, the equations of state at pressures up to  $3 \times 10^6$  atmospheres have been obtained by an empirical extrapolation although quite legitimately one might describe the procedure as an interpolation. Fortunately, it will turn out that the derived compositions are quite insensitive to the equation of state of hydrogen at pressures less than  $3 \times 10^6$  atmospheres. In previous work the extended equations of state have been obtained "theoretically." Since theoretical calculations, no matter how insecure their bases, seem to carry with them a certain aura of respectability the writer feels compelled to explain at some length his reasons for abandoning such a course.

In many respects the theory of the equation of state of substances that solidify as molecular crystals is not nearly so advanced as the corresponding theory of the compressibility of metals, in particular the monovalent ones. The usual theory of molecular crystals assumes that the potential energy of the crystal as a whole can be written in the form

$$V = \frac{1}{2} \sum_{i \neq j} \phi(r_{ij}),$$

where  $r_{ij}$  is the distance between the mass centers of the pair of molecules ( $i, j$ ) and  $\phi$  is the interaction potential of an isolated pair of molecules. Such an assumption proves quite satisfactory for the calculation of the density and cohesive energy at zero pressures. Many objections can be raised against the use of this procedure for predicting the equation of state at high pressures. Against the computations actually carried out on these premises by several authors, only three objections will be mentioned specifically because their cogency is immediately evident. Firstly, the pressure-density relation derived by Kronig *et al.* and employed by Ramsey and the writer in his thesis crosses the Fermi gas curve. The possibility that the equation of state of any cold matter can cross its Fermi gas curve can be denied on general theoretical grounds. (The writer is indebted to Prof. E. P. Wigner for helpful correspondence on this point.) Secondly, the predicted equations of state are at variance with Stewart's experimental results. This is illus-

TABLE (3.2). COMPARISON OF STEWART'S MEASURED HYDROGEN DENSITIES WITH DE BOER'S THEORETICAL VALUES

Pressure (kg/cm <sup>2</sup> )	Density (Theory)	Density (Experiment)	Ratio
2,006	.111	.125	1.13
2,238	.119	.127	1.07
3,222	.128	.135	1.05
5,018	.139	.147	1.06
5,670	.142	.151	1.06
6,399	.145	.155	1.07
7,239	.148	.158	1.07
8,208	.151	.163	1.08
9,335	.155	.167	1.08
10,631	.159	.172	1.08
12,134	.163	.178	1.09
13,880	.167	.185	1.11
15,941	.171	.191	1.12
18,323	.175	.198	1.13

trated in Table (3.2) where the calculations of de Boer and Blaisse (1948) are compared with experiment. The peculiar behavior of de Boer's theoretical densities at low pressures is convincingly explained as due to the neglect of anharmonic terms in calculating the zero point energy. The subsequent gradual rise in the ratio of theoretical to experimental densities confirms that the Lennard-Jones potential is too hard. Thirdly, it can be shown (Appendix II) that even if the exact isolated pair potential could be found it would lead to a predicted equation of state which would be too "soft."

The extrapolation of the equation of state of helium was performed analytically. Raimes (1952) has applied the Wigner-Seitz-Fröhlich method to the divalent metals and to adapt his work to helium we have but to drop the term accounting for the presence of electrons in the ion-cores since there are no core electrons in helium. However, Raimes' results will only hold at sufficiently high densities when the wave functions are flat. To secure agreement with Stewart's measurements at low densities a second degree polynomial in  $r_s$  (the radius of the Wigner-Seitz sphere) has been added to Raimes' terms, with constants determined by Stewart's measures. This is the smoothest function which simultaneously yields Raimes' equation of state at high pressures and Stewart's results at low pressures. The resultant equation of state is shown in Fig. (3.1) and numerical values are listed in Table (3.5).

The equation of state of molecular hydrogen cannot be tied to a definite analytic form at high densities as was the case with helium. It is the considered opinion of the writer that a judicious extrapolation of Stewart's experimental data will

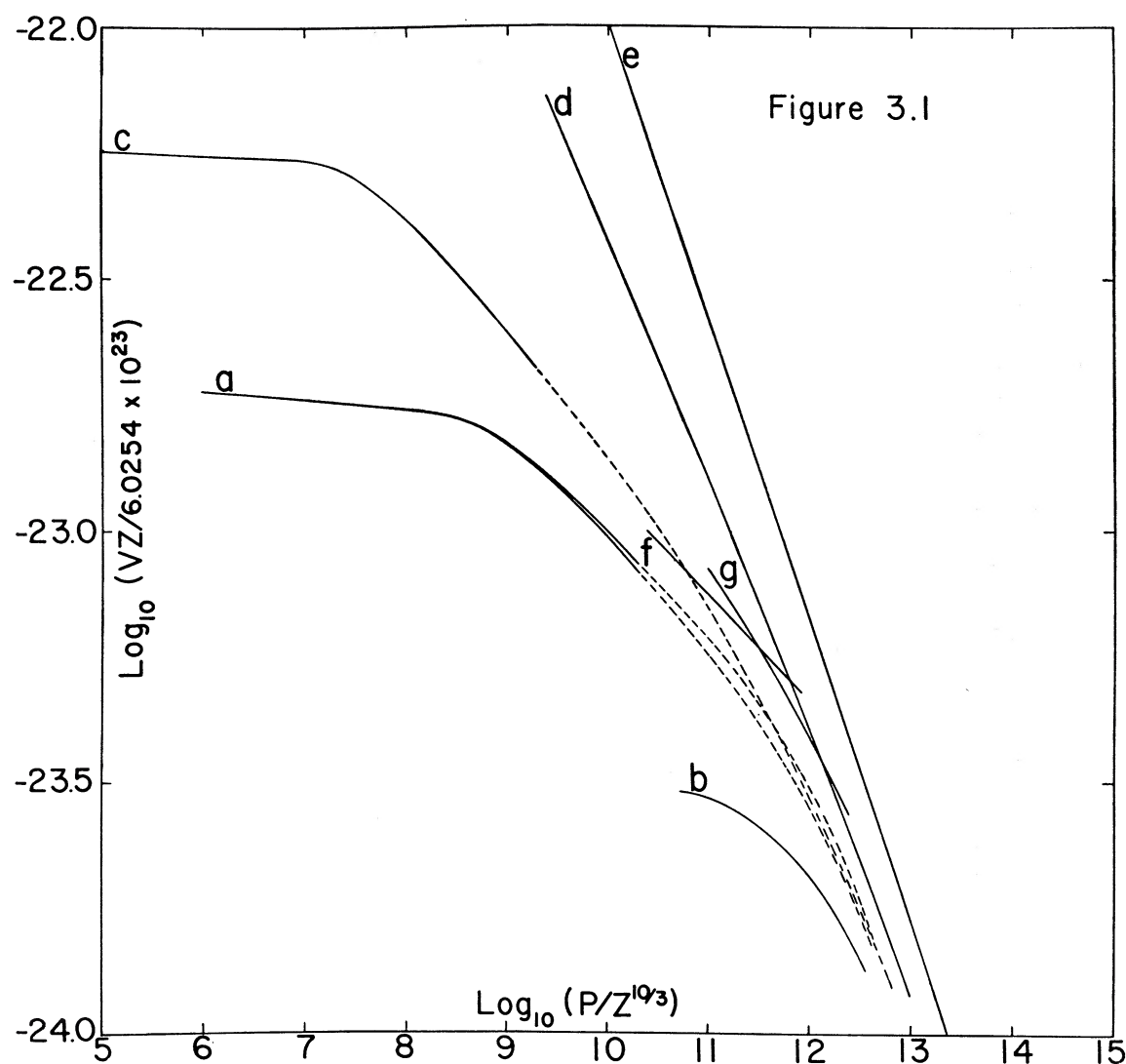


Figure 3.1. 0°K pressure-density relations. a. Equations of state of solid  $H_2$  (Tables 3.3 and 3.4). Solid portions cover pressure range of Stewart's experiments. b. Metallic hydrogen. c. Equation of state of helium (Table 3.5). d. Thomas-Fermi atom model. e. Fermi gas curve. f. Equation of state of  $H_2$  based on Lennard-Jones 6-12 potential. g. Equation of state of  $H_2$  given by Abrikosov (1956).

lead to a better equation of state than would result from inadequate theories. The confidence of the writer in his judgment of how the extrapolation should be performed is a result of protracted study of experiment and theory. No amount of explanation can disguise the fact that this is a high-handed procedure. Its ultimate justification rests on the realization (to be documented below) that the abundances of hydrogen in the planetary models of Jupiter and Saturn have proved to be rather insensitive to the extrapolation adopted.

The extrapolated equation of state will obviously be influenced mainly by Stewart's measures at the high end of the range of pressures covered in his experiments. Stewart estimates the error in his measurements as  $\sim 5$  per cent in the relative volume change from zero pressure so that the densities at the higher pressures are the most uncertain. Accordingly, two extrapolations of Stewart's data have been made, one assuming his data are correct as they stand (Table 3.3) and an alternative equation of state (Table 3.4) which is based on the assumption that Stewart's densi-

TABLE (3.3). EQUATION OF STATE OF  $H_2$ 

$p \times 10^{-12}$ (dynes/cm <sup>2</sup> )	$\rho$ (gm/cm <sup>3</sup> )	$p \times 10^{-12}$ (dynes/cm <sup>2</sup> )	$\rho$ (gm/cm <sup>3</sup> )
0	.089	.07	.270
.0002	.0961	.08	.279
.0004	.1010	.09	.287
.0006	.1052	.1	.295
.0008	.1090	.2	.356
.0010	.1123	.3	.401
.002	.1252	.4	.439
.003	.134	.5	.471
.005	.147	.6	.500
.007	.158	.7	.528
.009	.167	.8	.553
.011	.175	.9	.576
.013	.182	1.0	.601
.015	.188	1.5	.704
.02	.201	2.0	.797
.03	.220	2.5	.886
.04	.236	3.0	.970
.05	.248	3.5	1.047
.06	.261	4.0	1.114

ties are too high by 2 per cent at 20,000 kg/cm<sup>2</sup> and are progressively more accurate at the lower pressures. It must be admitted that the extrapolation of the alternative equation of state has been performed as an afterthought. Such an altered equation of state, which is consistent with Stewart's results within his experimental errors, has appeared as one possible resolution of a dilemma which arises when model planets of Saturn are calculated. This point will be discussed more fully when the model planets are described below.

*Metallic hydrogen.* The possible existence of a metallic phase of hydrogen was first analyzed in some detail by Wigner and Huntington (1935). They found the cohesive energy of the metallic phase to be 10 kilocalories per mole of hydrogen atoms, whereas the cohesive energy of the molec-

TABLE (3.5). EQUATION OF STATE OF HELIUM

$p \times 10^{-12}$ (dynes/cm <sup>2</sup> )	$\rho$ (gm/cm <sup>3</sup> )	$p \times 10^{-12}$ (dynes/cm <sup>2</sup> )	$\rho$ (gm/cm <sup>3</sup> )
.0002	.234	.80	1.77
.0010	.323	1.00	1.89
.0040	.431	2.00	2.45
.0100	.534	4.00	3.16
.0200	.626	8.00	4.15
.04	.742	10.00	4.56
.08	.884	20.00	6.21
.10	.936	30.00	7.58
.20	1.15	40.00	8.64
.40	1.40	80.00	11.41

ular phase is 50.5 kilocalories, this last figure being essentially one-half the heat of dissociation. In a later paper Kronig, de Boer and Korringa (1946) have given detailed formulae for computing the internal energy of metallic hydrogen as a function of density. Their calculations have been employed in almost all previous detailed work on homogeneous planets. However, there are some features of the calculations of Kronig *et al.* which render their results somewhat objectionable. As a consequence, the calculations in this paper are based on the paper of Wigner and Huntington. Wigner and Huntington employ the method of Wigner and Seitz (1933, 1934) which was developed for the computation of the internal energies of the alkali metals. The internal energy (at the absolute zero) is found to consist of the following quantities:

1. The energy of the bottom of the  $s$  band in the lattice. In the case of hydrogen where the ion core is a point charge this quantity is the lowest eigenvalue of the Hamiltonian

$$H = \left( -\nabla^2 + \frac{2}{r} \right)$$

subject to the condition that the normal derivative of the wave function vanishes on the surface of a sphere whose radius  $r_s$  is such that

$$\frac{4\pi}{3} N r_s^3 a_h^3 = V.$$

Here  $a_h$  is the Bohr radius,  $N$  the Avogadro Number and  $V$  the molar volume. The units are natural, i.e., energy in Rydbergs and length in Bohr radii. The writer has computed the eigenvalues by the variational method, using a three-parameter wave function. As a test, calculations were also made with one and two-parameter wave functions, in order to ascertain the rate of convergence of the eigenvalue with respect to the number of parameters. In every case, no need for more parameters was found.

TABLE (3.4). ALTERNATIVE EQUATION OF STATE OF  $H_2$ 

$p \times 10^{-12}$ (dynes/cm <sup>2</sup> )	$\rho$ (gm/cm <sup>3</sup> )	$p \times 10^{-12}$ (dynes/cm <sup>2</sup> )	$\rho$ (gm/cm <sup>3</sup> )
0	.089	.07	.254
.0002	.0961	.08	.262
.0004	.1010	.09	.268
.0006	.1051	.1	.274
.0008	.1089	.2	.324
.0010	.1122	.3	.365
.002	.1250	.4	.399
.003	.1336	.5	.428
.005	.146	.6	.455
.007	.157	.7	.480
.009	.166	.8	.503
.011	.173	.9	.526
.013	.180	1.0	.548
.015	.185	1.5	.643
.02	.197	2.0	.732
.03	.214	2.5	.819
.04	.227	3.0	.903
.05	.238	3.5	.984
.06	.247	4.0	1.07

2. The Fermi energy, which in first approximation is

$$\frac{2.21}{r_s^2}.$$

The correction for higher approximations is considered under Point 6.

3. The difference between the exchange energy and electron interactions. In first approximation,

$$\frac{.284}{r_s} - .0009.$$

Again, higher approximations are taken care of in Point 6.

4. The Correlation Energy. This term is included to correct for intrinsic errors in the Hartree-Fock equations. Wigner has estimated this quantity for  $r_s < 1$  and  $r_s \gg 1$  and has interpolated between these two limits to get an expression valid for the whole range of  $r_s$ . Wigner's original expression was

$$-\frac{.58}{r_s + 5.1}.$$

However, as Pines (1955) points out, Wigner later (1938) revised his earlier estimate of the low density limit and found, for large  $r_s$ , the value

$$-\frac{.88}{r_s}$$

which yields a revised interpolation formula

$$-\frac{.88}{r_s + 7.8}.$$

Using the collective coordinate technique of Bohm and Pines, this formula has been confirmed by Pines (1955) to within 6 per cent when  $r_s$  lies in the range ordinarily appropriate to the theory of metallic cohesion. However, Pines's expression is about 14 per cent higher than Wigner's at  $r_s = 1$  and we use as our final formula the last one given.

5. The Zero Point Energy. On the basis of the Debye Theory, this is

$$.0244 \left[ E''(r_s) - \frac{2}{r_s} E'(r_s) \right]^{1/2}.$$

6. A correction for the fact that the wave functions are not "flat." This consists of a group of terms whose structure is involved. The writer has taken them directly from Wigner and Huntington's graphs and has done so in two independent ways to guard against the usual errors

inherent in graphical readings. The combined effects are quite small. The only change that should have been made in Wigner and Huntington's curves is the correction to the correlation energy which ought to be made according to the new value. The difference between the old and new forms is small over the range of  $r_s$  of interest, so that the correction to the correction may be safely omitted.

The calculations with the changed correlation energy gives a cohesive energy of 13.3 kilocalories per mole. The volume at zero pressure is also "blown up" somewhat by the change and, whereas Wigner and Huntington found  $\rho$  ( $p = 0$ ) to be .59 g/cm<sup>3</sup>, we find  $\rho = .525$  g/cm<sup>3</sup>.

Table (3.6) gives the equation of state as calculated by the writer for 0°K. In addition the free enthalpy, which at  $T = 0$  implies

$$G = H = U + pV,$$

is also given, a quantity useful in predicting the phase transition.

The equation of state for pressures greater than 300,000 atmospheres is very close to the one obtained by Kronig, de Boer and Korringa if the zero point pressure is added to their results.

TABLE (3.6). METALLIC HYDROGEN AT 0°K

$r_s$	$\rho$ (gr/cm <sup>3</sup> )	$p \times 10^{-12}$ (cgs/cm <sup>2</sup> )	$U \times 10^{-12}$ (ergs/mole)	$G \times 10^{-12}$ (ergs/mole)
1.75	.5030	-.044	-.5568	-.6456
1.70	.5487	+.052	-.5567	-.4616
1.65	.6001	.182	-.5390	-.2336
1.60	.6582	.355	-.5001	+.0432
1.55	.7239	.586	-.4356	.3809
1.50	.7988	.893	-.3403	.7864
1.45	.8843	1.292	-.2069	1.266
1.40	.9824	1.834	-.0318	1.850
1.35	1.0957	2.547	+.1983	2.541
1.30	1.2270	3.495	.4928	3.364
1.25	1.3802	4.766	.8657	4.346
1.20	1.5601	6.476	1.3340	5.518
1.15	1.7725	8.795	1.9192	6.921
1.10	2.0254	11.963	2.6482	8.602
1.05	2.3287	16.336	3.5551	10.626
1.00	2.6958	22.437	4.6840	13.074
.95	2.1442	31.046	6.0923	16.045

#### B. The Equation of State of Fluid Hydrogen.

Since the critical temperature of molecular hydrogen is 33.2°K, only the hypercritical fluid state of hydrogen is of interest in relation to the giant planets. Excepting some measurements by Bridgman (1923), isotherm data are available only up to 3000 atmospheres. This range of pressures, at temperatures between 0 and 150°C, has been investigated with extreme accuracy by Michels and Goudekot (1941). For lower pres-

tures there are numerous sources of isotherm data which Woolley *et al.* (1948) combined into a single tabulation. Bridgman investigated the 65°C isotherm at pressures between 3000 and 14,000 atmospheres. This would be valuable information if it could be trusted. It appears that there is something seriously wrong with these results, as is readily seen by plotting the quantity

$$\sigma = 10^5 \frac{TV}{T_0 V_0} \log_{10} \frac{pV}{RT},$$

where  $T_0 = 273.16^\circ\text{K}$  and  $V_0 = 22,428.5 \text{ cm}^3$ . Cragoe (1941) and Woolley *et al.* (1948) have shown this quantity to be most useful in discussing the  $p, V, T$  data of hydrogen. Figure 3.2 shows the  $\sigma$  plots from the data of Michels and Goudekot along with the  $\sigma$  curve calculated from Bridgman's data. Bridgman actually measured volume changes from 3000 kg/cm<sup>2</sup> and took the volume at that pressure from some old data of Amagat. A small change ( $<0.5$  per cent) in this reference volume has been made in order to secure strict agreement with Michels and Goudekot. The change in the  $\sigma$  plot would scarcely be visible in the figure. The course of Bridgman's  $\sigma$  curve is markedly at variance with the Michels and Goudekot data. Bridgman's volume changes would be in better accord with Michels and Goudekot's curves if all volume changes were multiplied by  $4/3$ . The dashed curve of Figure 3.2 is an extrapolation of Michels and Goudekot's data attempted by the writer in a manner which will now be discussed.

The following method of discussing the equation of state of imperfect gases seems most suitable for the purpose at hand. Pressure and absolute temperature are taken as independent variables, and the equation of state written in the form

$$V = V^*(p, T) + \frac{RT}{p},$$

where  $V$  is the molar volume at fixed pressure  $p$  and temperature  $T$ ,  $R$  is the gas constant and  $V^*(p, T)$  will be termed the co-volume.  $V^*$  is a measure of the gas imperfection and is identically zero for a perfect gas. As is well known, from adequate knowledge of the molecular spectrum one can calculate the thermodynamic potentials of a fictitious ideal gas at, say, one atmosphere pressure (such quantities will be denoted by the superscript zero), and then the thermodynamic potentials of the real gas can be derived in a

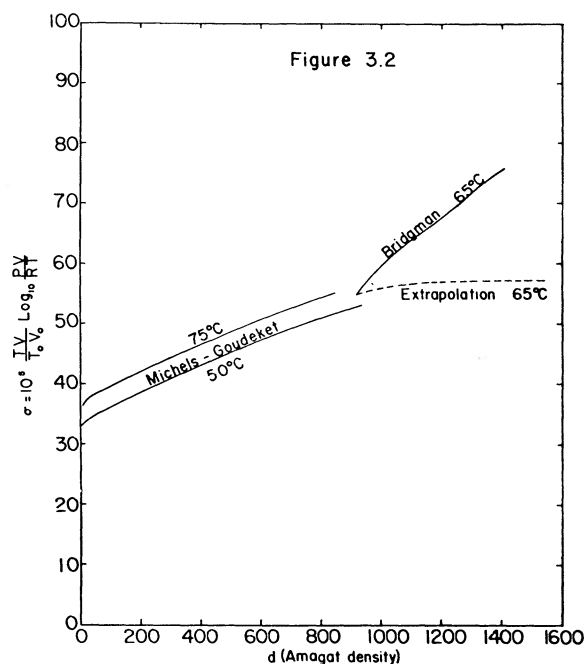


Figure 3.2.  $\sigma$ -plots of gaseous hydrogen data.

straightforward manner if  $V^*(p, T)$  is known. For example, the free enthalpy is given by

$$G = G^0(T) + RT \ln p + \int_0^p V^*(p, T) dp$$

and the entropy  $S$  by

$$S = S^0(T) - R \ln p - \int_0^p \frac{\partial V^*}{\partial T} dp,$$

etc. It is worth noting that a gas may be grossly imperfect, and yet its entropy in the  $p, T$  plane may follow the "ideal" relation

$$S = S^0(T) - R \ln p,$$

namely, if the co-volume depends only on the pressure. However, in the  $V, T$  plane the same gas would not obey the ideal relation. Figure 3.3 shows a series of isothermal plots of gaseous hydrogen formed from the data of Michels and Goudekot and the compilation of Woolley *et al.* Included is also a plot of the molar volume of the 4.2°K isothermal of the solid taken from Stewart's data. The striking features of this diagram are the confluence (for pressures in excess of 1000 atmospheres) of the several isothermals of  $V^*(p, T)$  into essentially a single curve and the suggestive approach of the values of  $V^*$  to the volume of the solid at still higher pressures.



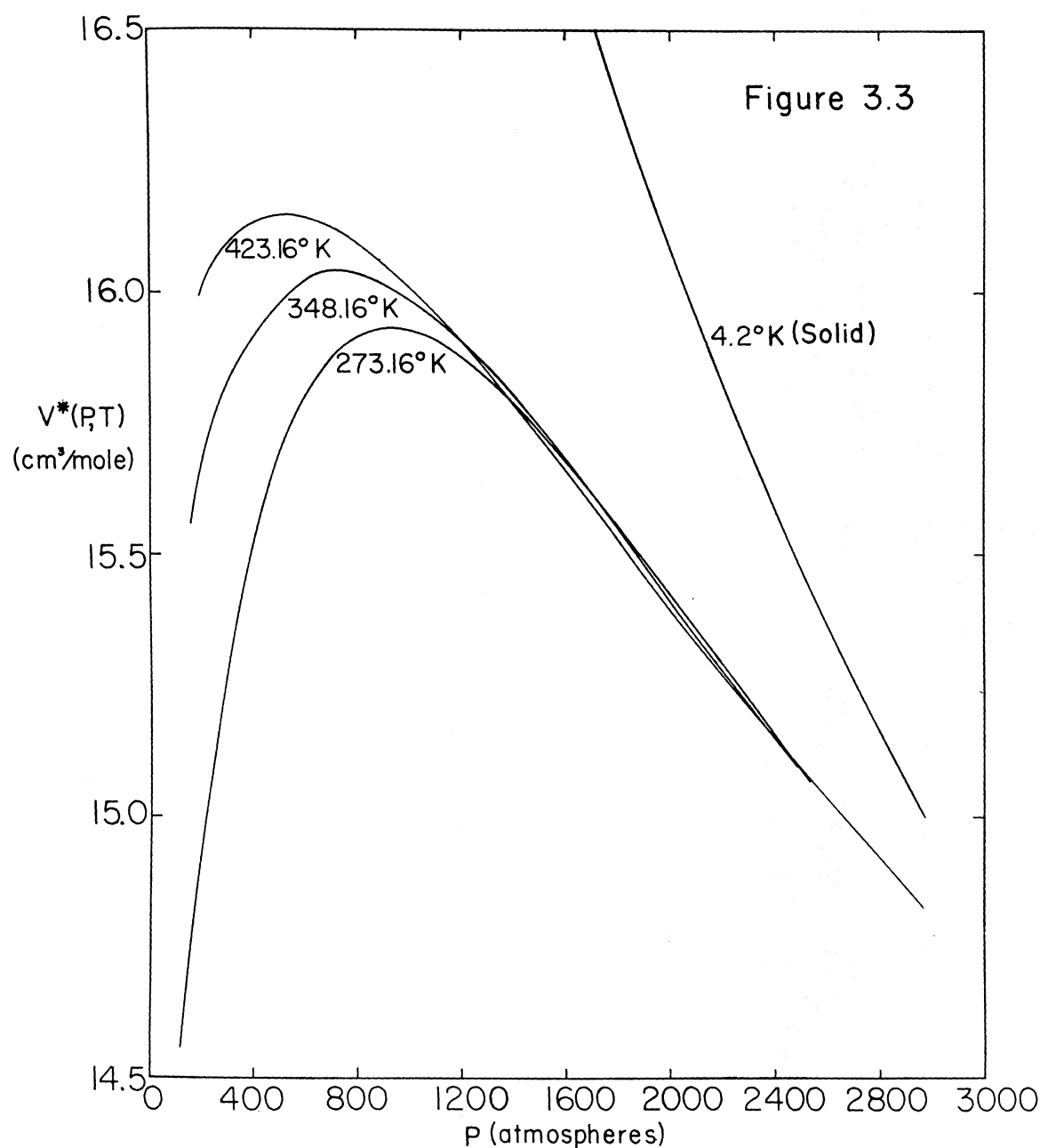


Figure 3.3. Co-volumes versus pressure for gaseous hydrogen.

From the thermodynamic equation

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p,$$

and by writing  $V = V^* + \frac{RT}{p}$ , the contribution

of gas imperfection to the entropy as a function of pressure is seen to tend at high pressures to a function of  $T$  alone, provided that

$$\frac{p}{R} \left(\frac{\partial V^*}{\partial T}\right)_p$$

tends to zero as rapidly as  $p^{-\nu}$  where  $\nu > 0$ .

Whether this actually happens cannot be decided merely from the experimental data, and the strict mathematical truth of such an assumption is not to be expected. What is needed here is assurance that no gross error is made by such an assumption. In Table (3.7) are given values of

$$\frac{p}{R} \left( \frac{\partial V}{\partial T} \right)_p$$

which support the contention that the adiabatic curves in the  $p, T$  plane depart from the ideal adiabatics only by amounts that are trivial in the present context. If this contention is accepted, it follows that the adiabatic compression

TABLE (3.7). AT  $T = 348.16^\circ\text{K}$

$p$ (atmospheres)	$\frac{p}{R} \left( \frac{\partial V^*}{\partial T} \right)_p$
400	.020
600	.013
800	.011
1000	.006
1200	.003
1400	.000
1600	-.002
1800	-.003
2000	-.003
2200	-.002
2400	-.001
2600	+.001

Calculated from data of Michels and Goudek (1941).

of hydrogen must result in solidification unless the melting line predicted from Simon's formula, discussed below, is much in error. This conclusion derives from the fact that the melting line according to Simon's formula behaves asymptotically as  $p \sim T^{1.83}$  whereas the "ideal" adiabatic of hydrogen behaves as  $p \sim T^{3.5+}$ . The approach of  $V^*(p, T)$  to the volume of solid hydrogen is also important because it implies that the density of hydrogen at pressures in excess of 100,000 atmospheres may be unexpectedly independent of temperature and state of aggregation.

The preceding discussion has centered on two rather bold extrapolations, namely, that  $V^*(p, T)$  ultimately merges with  $V_{\text{solid}}$ , and that Simon's formula actually represents the true course of the melting line. It is conceivable that either extrapolation might be justifiable without the other being so. That they are, in fact, complementary is a plausible inference from the following computations.

Melting occurs when the free enthalpies, the pressures and the temperatures of the solid and fluid phases are equal. The free enthalpy of the

fluid phase,  $G_f$ , is

$$G_f(p, T) = G^0(T) + RT \ln p + \int_0^p V^*(p, T) dp$$

while that of the solid,  $G_s(p, T)$ , is

$$G_s(p, T) = G_s(0, 0) + \int_0^p V_s(p) dp - \int_0^T S(p, T) dT,$$

wherein  $V_s(p)$  is the molar volume of the solid at the absolute zero. Hence, for melting at  $300^\circ\text{K}$  by Simon's relation we have  $p = 64,000$  atmospheres and at this pressure and temperature we must have

$$\begin{aligned} G^0(T) + RT \ln p + \int_0^{4000} V^*(p, T) dp \\ = G_s(0, 0) + \int_0^{4000} V_s(p) dp \\ - \int_0^{300^\circ\text{K}} S(64,000, T) dT, \end{aligned}$$

provided that  $V^*(p, T) = V_s(p)$  for  $p > 4000$  atmospheres. Now  $G^0(T) - G_s(0, 0)$  can be evaluated from the vapor pressure equation and all quantities except the last integral are known. On the assumption that the specific heat of the solid at constant pressure is a Debye function, the value of  $\theta_D$  necessary to secure equality of the two free enthalpies is  $\theta_D = 726^\circ\text{K}$ . This stands in good agreement with the value following from Lindeman's melting formula under the same conditions, namely,  $\theta_D = 656^\circ\text{K}$ .

### C. The Phase Diagram of Solid Hydrogen.

In order to find the phase boundaries of solid hydrogen one would have to know the free energies of all the phases involved. Unfortunately, one does not even know what phases are involved; for it is by no means certain that only the molecular and metallic phases can exist at low temperatures. Wigner and Huntington mention in particular the possibility of an intermediate layer-like lattice. This possibility has been completely ignored in all work on hydrogen planets up until now. It will appear presently that the existence of a layer-like modification of hydrogen, which is stable at pressures of the order of  $10^6$  atmospheres or even less, would not upset our conclusions regarding the composition of cold planets consisting mainly of pure hydrogen. For the time being, we shall ignore the possibility of

a layer lattice and deal only with the transition between the phase which is stable at low pressures and the metallic phase. The prediction of the phase transition, at absolute zero, can then be accomplished simply by plotting the free enthalpies of the two phases, per mole of H atoms, versus pressure. The point where they cross marks the pressure of the phase transition at absolute zero.

The free enthalpy of the molecular phase in ergs per mole of hydrogen atoms is

$$G = -2.164 \times 10^{12} + \int_0^p V(p) dp,$$

where the zero of energy is taken as the energy of isolated hydrogen atoms. Figure 3.4 shows the course of the free enthalpy of the molecular phase calculated from the alternative equation of state (3.4). The free enthalpy of the metallic phase is included in the same figure. The estimated pressure at which the transition between the molecular and metallic phases would occur can be read directly from such figures. The values of the transition pressure are  $3.5 \times 10^{12}$  dynes/cm<sup>2</sup> and  $1.93 \times 10^{12}$  dynes/cm<sup>2</sup> on the basis of the equations of state (3.3) and (3.4), respectively. Clearly, the transition pressure is quite sensitive to the density of the molecular phase. The free enthalpy of the molecular phase might always remain lower than that of the

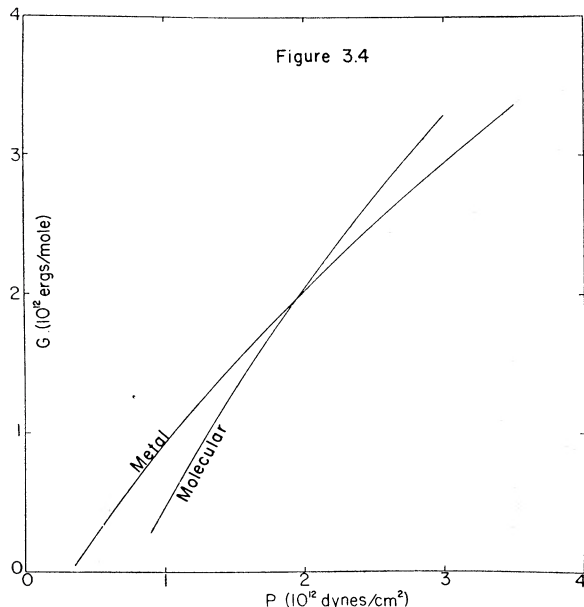


Figure 3.4. Graphical estimation of phase equilibrium pressure.

metal and approach it asymptotically from below while the metal, in turn, approaches asymptotically the Fermi gas curve.

Instead of attempting to evaluate from first principles the course of this phase boundary, it is more economical to use Simon's "universal" melting formula, and such a procedure probably yields the more accurate results. Simon's formula is

$$p = a[(T/T_0)^c - 1],$$

where  $a$  and  $c$  are empirical constants and  $T_0$  is the temperature of the triple point. Originally this relation was wholly empirical, although confirmed by a vast amount of experimental data, but now it enjoys a measure of theoretical support. This formula relating melting pressure to melting temperature has but two adjustable parameters (three in the original version by Simon *et al.* (1929)), which usually are determined in such a way as to accomplish a best fit to the experimental data. In estimating the Simon parameter, the writer gave more weight to the high-pressure end of the experimental range, because considerable extrapolation of the melting curve is desired. A tabulation of the adopted melting curve is given in Table (3.8).

TABLE (3.8). MELTING PRESSURES OF HYDROGEN

$$p = 242.328 \left[ \left( \frac{T}{13.96} \right)^{1.83} - 1 \right]$$

$T$	$p$ (kg/cm <sup>2</sup> )
20	226
40	1,420
60	3,250
80	5,670
100	8,660
150	18,400
200	31,400
300	66,200
500	169,000
1,000	601,000
2,000	2,140,000
3,000	4,490,000

For an isothermal hydrogen planet the melting pressure could be estimated from Table (3.8). For an adiabatic one the solidification point could be estimated by plotting Simon's curve and the adiabatic  $p, T$  curve on the same figure. It is assumed here that the adiabatic curve of hydrogen in the  $p, T$  plane is "ideal" (see the previous discussion of the properties of the fluid phase). Figure 3.5 exhibits such a plot when the adiabatics are specified by  $T_0 = 20^\circ\text{K}$ ,  $50^\circ\text{K}$ ,  $100^\circ\text{K}$ , and  $200^\circ\text{K}$ , when the pressure is one atmosphere. From this figure it would appear that solidification in such a case would occur at pressures of

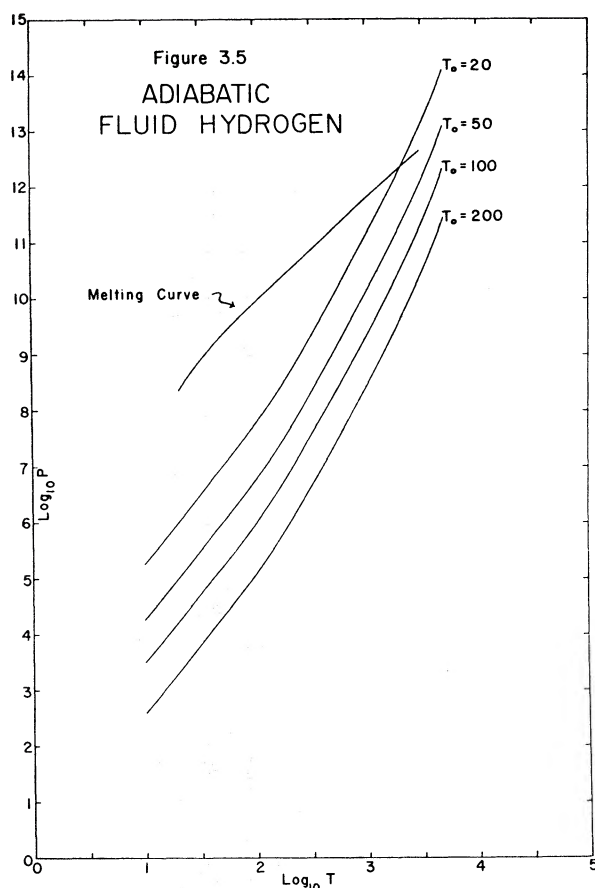


Figure 3.5. Adiabatic curves for gaseous hydrogen.

the order of  $10^{12}$  dynes/cm<sup>2</sup> and at temperatures of  $\sim 1000^\circ\text{K}$ . The elucidation of the physical state of Jupiter's or Saturn's outer shell is a delicate task, which is entirely analogous to the problem of delineating stellar convection zones and fitting radiative envelopes to them. This task is not undertaken in the present paper. However, it seems possible that substantial portions of the molecular phase might be fluid, perhaps even up to the point of stability of the metallic phase. Over a large portion of the pressure range from 0 to  $3.5 \times 10^{12}$  dynes/cm<sup>2</sup> the adiabatic fluid would still have an equation of state approximating that of the  $0^\circ\text{K}$  solid if the above conjectured behavior of the co-volume  $V^*(p, T)$  is correct.

It is believed that the qualitative conclusions drawn regarding the behavior of fluid hydrogen will also be applicable to helium. The experimental data are not extensive enough to justify a detailed discussion.

#### D. The Thermal Expansivity of Solid Hydrogen.

The conclusion of the preceding section was that heating of dense hydrogen in the fluid phase to super-critical temperatures would not render it markedly less dense than the corresponding solid at absolute zero. The implication is that the thermal expansivity of the solid can play only a minor role in Jupiter and Saturn. Detailed calculations on the basis of the Debye theory confirm these conclusions (Appendix III).

#### E. The Thermal Conductivity of Solid Hydrogen.

Since it will be concluded that the planets Jupiter and Saturn are predominantly composed of hydrogen if they are "cold," some estimates of the thermal conductivity of solid hydrogen are in order. The metallic phase has an enormous thermal conductivity so that the temperature drop across the metallic portion of a pure hydrogen planet can be only a few tens of degrees for any reasonably chosen radioactive heat flux. The deduction of the thermal conductivity of the metal is due to Critchfield (1942). Thermal conductivities of the metal calculated from Critchfield's formulae are listed in Table (3.9).

TABLE (3.9). THERMAL CONDUCTIVITY OF METALLIC HYDROGEN

Density (g/cm <sup>3</sup> )	$k$ (cal/cm sec $^\circ\text{K}$ )
.7	14
1.0	34
1.5	90
2.0	177
3.0	450

It is common knowledge that at room temperature and atmospheric pressure, metals are, as a rule, much better conductors of heat than are dielectric crystals. Therefore, one might expect the major temperature drop to occur across the molecular mantles of the planets. However, with most dielectric crystalline solids, measurement at room temperature is bound to show their powers of thermal conduction in a rather unfavorable light, because their thermal conductivities are, except at very low temperatures, rather strongly decreasing functions of their reduced temperatures,  $\tau = T/\theta_D$ . On the other hand, near room temperature the thermal conductivities of metals are roughly independent of temperature. These circumstances combine to give the impression that dielectric crystals are poor conductors of heat compared with metals.

Even at room temperature diamond is a better thermal conductor than any metal whose thermal conductivity is known. Table (3.10) illustrates this point.

TABLE (3.10). TEMPERATURE DEPENDENCE OF THERMAL CONDUCTIVITY

Substance	Thermal conductivity (cal/cm sec °K)	
	100°K	300°K
Cu	1.3	.94
Diamond	6.93	1.31

At lower temperatures the thermal conductivity of copper surpasses that of diamond, rising to a maximum value of 12 cal cm<sup>-1</sup> sec<sup>-1</sup> deg<sup>-1</sup> (Berman and MacDonald 1952); but artificial sapphire has a maximum thermal conductivity of 15.5 cal cm<sup>-1</sup> sec<sup>-1</sup> deg<sup>-1</sup> (Berman, Simon, and Wilks 1951).

The thermal conductivity of dielectrics is due mainly to the Debye waves, or phonons, as they are nowadays called. In metals, on the contrary, the major contribution is due to the free electrons, which in turn scatter the phonons and suppress their contribution to the thermal conductivity in an important way. By analogy to the thermal conductivity of gases it is convenient to write the thermal conductivity,  $k$ , as

$$k = \frac{1}{3} \lambda C v_s,$$

where  $C$  is the specific heat per unit volume,  $v_s$  is the average sound velocity in the crystal, and  $\lambda$  is of the nature of a mean free path for the phonons. The theoretical determination of  $\lambda$  is extremely complicated even for a perfect crystal. However, at low temperatures Peierls (1929) has found

$$\lambda \sim f(T) e^{+\theta_D/bT}.$$

$f(T)$  has algebraic rather than exponential behavior and according to Peierls  $b$  is about two. In reporting experimental results the function  $f(T)$  is replaced by a constant since its variation is in any case dominated by that of the exponential. Writing  $\lambda$  in the form

$$\lambda = A e^{+\theta_D/b'T},$$

it is found experimentally that  $b'$  generally falls in the range 2.1 to 2.7 while  $A$  ranges from  $6 \times 10^{-8}$  cm for solid helium to  $140 \times 10^{-8}$  cm for artificial sapphire. For helium  $b' = 2.3$ ,  $A = 6 \times 10^{-8}$  cm; and  $A$ , as well as  $b'$ , in this, the only case which has been investigated, does not

depend on the density in the range from .194 to .218 g cm<sup>-3</sup> covered by experiments (Webb, Wilkinson, and Wilks 1952). As a guess for solid  $H_2$  we take  $A = 10 \times 10^{-8}$  cm and  $b' = 2.3$ . Since, according to Berman (1953), this simple expression is not correct above a temperature of  $\theta_D/10$ , we modify the expression for  $\lambda$  to

$$\lambda = 10^{-7} (e^{\theta_D/2.3T} - 1).$$

This form gives the correct functional dependence on  $T$  at high temperatures (Debye 1914), though it lacks theoretical justification.

In the solid molecular phase of hydrogen, there is an extra possible mode of thermal conductivity which is absent in helium. This is the coupling between the librational degrees of freedom of the molecules which requires consideration once the rotational degrees of freedom have been lost. The ankylosis of rotational degrees of freedom has presumably been observed by Hill and Ricketson (1954). It is possible that the librations interact with the phonons and shorten their mean free path; but Gerritson and van der Star (1942), for the case of solid  $CH_4$ , give some evidence that the librational degrees of freedom contribute to the thermal conductivity without impairment of the other modes of thermal conduction. In the absence of further knowledge, we shall ignore this uncertain contribution to thermal conductivity and proceed as above. In Table (3.11), estimated

TABLE (3.11). THERMAL CONDUCTIVITY OF  $H_2$  AS FUNCTION OF ( $T$ )

$\rho = .15 \text{ g/cm}^3$		$\rho = .343 \text{ g/cm}^3$		$\rho = .729 \text{ g/cm}^3$	
$T$ (°K)	$k$	$T$	$k$	$T$	$k$
15	.49	75	4.1	172	16
19	.16	82	2.3	253	2.4
24	.059	164	.23	474	.66
34	.028	328	.11	948	.36
68	.014	468	.083	1260	.27
97	.010	528	.071	2530	.12

$k$  in units of cal cm<sup>-1</sup> sec<sup>-1</sup> °K<sup>-1</sup>.

values of the thermal conductivity of pure  $H_2$  as a function of temperature are given for several densities. The estimates of Debye temperature are based on Einstein's formula and the compressibility of the solid, which is experimentally known only for the .15 g/cm<sup>3</sup> case. The other compressibilities are estimates from the extrapolated equation of state as given in Table (3.3). These estimates indicate that a pure crystal of solid hydrogen is a much better conductor than "rocks and other compact non-metals," whose conductivity at room temperature and zero pressure



was used by Ramsey as an estimate of the conductivity of solid hydrogen. The thermal conductivity of the material forming the mantles of Jupiter and Saturn is quite a different problem, for the effects of impurities on the thermal conductivities of dielectrics are not too well understood. There is some work cited in the excellent review article of Berman (1953) which indicates that substitution of foreign atoms of the correct valency may not seriously impair the thermal conductivity of a crystal.

#### IV. THE MODEL PLANETS

Except for the masses, which are taken from Russell, Dugan and Stewart, the mechanical data used in this paper are taken from a consistent set recently derived by Brouwer and Clemence (1958). These are set forth in Table (4.1) below.

It has been noted in the introduction that it has proved impossible to construct model planets of pure hydrogen which reproduce simultaneously the observed masses and radii of Jupiter and Saturn. Originally this conclusion was based on the validity of the equation of state calculated from the Lennard-Jones potential function, but the equations of state used in the present paper have led to the same conclusion. Therefore, it is necessary to consider planetary models of heavier materials than pure hydrogen. An iterative procedure was used to arrive at a pair of models. In the initial attempts, the equation of state of hydrogen (Table 3.3) was used as a zeroth approximation to the relation between pressure and density for the outer layers of a satisfactory model of Saturn. The equation of hydrostatic equilibrium was integrated inward from the surface. When the integration had been carried down to small values of the radius it became apparent that the mass function  $M(s)$  would not vanish for  $s = 0$ . Instead of continuing the integration all the way to the center, the models were completed by placing a dense core in the center within which the density varied according

to the relation

$$\rho(s) = A - Bs^2 + Cs^4.$$

For a given value of the core radius, one relation between the constants is determined by the requirement that the mass be given correctly. The remaining two relations were obtained by specifying the density of the core at the core boundary and, within limits, the moment of inertia of the core. Having fixed the details of the core, the ellipticity was calculated. The calculated ellipticities were always larger than the ellipticity of Saturn regardless of the details of the core provided, of course, that the pressure-density relation implied by the details of the core was such that the density at a given pressure was always greater than the densities of Table (3.6). The failure to match the observed ellipticities was found to be due to the fact that the densities assumed applicable for pure hydrogen in the outer layers were so large that the moments of inertia of the model planets were too high. (It should be noted here that the Radau-Darwin approximation was not used to derive this conclusion; rather, account was taken of the variation of the function  $F(x)$ , Part II.) Evidently raising the densities of Table (3.3) as function of pressure to imply the presence of elements other than hydrogen in the outer layers would increase the discrepancy. Consequently, if the equation of state of cold hydrogen is correctly given by Table (3.3) and if the physical parameters of Saturn are correctly represented in Table (4.1), it would seem that Saturn cannot be a cold body in the sense that the densities throughout most of its bulk do not deviate significantly from the values they would assume at the same pressures when the temperature was 0°K. The possibility that Saturn is not a cold body in the above described sense should not be disregarded. However, there are many uncertainties in the mechanical data. The largest of these is the uncertainty associated with the rotational period. The amount

TABLE (4.1). MECHANICAL DATA OF THE PLANETS

Quantity	Unit	Jupiter	Saturn
Mass	Earth	318.35	95.3
Mass	gram	$1.902 \times 10^{30}$	$.5694 \times 10^{30}$
Mean Radius	Earth	10.965	9.046
Mean Radius	centimeter	$6.9861 \times 10^9$	$5.763 \times 10^9$
Mean Density	g/cm <sup>3</sup>	1.33	0.71
$\epsilon_1$		.0652	.0978
$m$		.0843	.1421
$J$		.02206	.02501
$K$		.00253	.00386
$k_1$		.00052	.00159
Rotational Period	hour	9.87	10.41

of the discrepancy between the ellipticities computed for the model planets and the ellipticity of Saturn derived by Brouwer and Clemence is only of the order of 10 per cent or less and could probably be removed by altering the rotational period and the equatorial semi-diameter by amounts which would be within the realm of possibility. Another resolution of the dilemma—which is perhaps the most obvious—is to alter the extrapolated equation of state of molecular hydrogen toward lower densities. In performing the extrapolation from which the numerical values of Table (3.3) resulted, care was taken to make the extrapolated values join smoothly with Stewart's measurements. However, Stewart estimates his experimental errors to be "better than 5 per cent in the relative volume changes." The possible existence of errors of this amount implies the possibility of correspondingly greater errors in the values of the compressibility calculated from his data. The alternative equation of state displayed in Table (3.4) has been extrapolated assuming Stewart's densities are correct for pressures up to 10,000 atmospheres but become progressively too high as the pressure increases being in error by 2 per cent at 20,000 atmospheres. This alteration is well within his estimated range of possible error. The extrapolated densities join smoothly with the altered Stewart data and deviate from the values of the original extrapolation by about 10 per cent throughout most of the range of stability of the molecular phase. The estimated pressure at which the transition to the metallic modification of hydrogen occurs, which was  $2.46 \times 10^{12}$  dynes/cm<sup>2</sup> on the basis of the original extrapolation, is lowered to  $1.93 \times 10^{12}$  dynes/cm<sup>2</sup>.

Before starting the model building process anew, it was decided to take some account of the fact that the planets are certainly not at the absolute zero of temperature. This fact is bound to appreciably affect the densities of at least a thin outer rind. Using the conclusions of Part III as regards the behavior of the co-volumes  $V^*(p, T)$  the isothermal equation of state of fluid hydrogen was calculated for pressures less than  $2 \times 10^{10}$  dynes/cm<sup>2</sup> which is the value of the melting pressure indicated by Simon's melting formula. For pressures greater than  $2 \times 10^{10}$  dynes/cm<sup>2</sup>, the alternative equation of state of molecular hydrogen was used. A core model of Saturn was constructed on exactly the same lines as indicated before except for the use of the "150°K" alternative equation of state. It turned

out to be possible to fit the observed ellipticity of Saturn by such a model. The details of the model of Saturn derived in this way are given in Table (4.2). Unfortunately, calculation of the

TABLE (4.2). SATURNIAN MODEL

$\beta$ ( $= r/R$ )	$\rho$ (g/cm <sup>3</sup> )	$M \times 10^{-27}$ (g)	$p \times 10^{-12}$ (dynes/cm <sup>2</sup> )	$W_{He}$
1.000	.00016	569.00	.000001	0
.995	.023	568.95	.00017	0
.990	.092	568.20	.00186	0
.985	.125	566.92	.00518	0
.980	.151	565.31	.00948	0
.975	.170	563.46	.0145	0
.970	.185	561.44	.0200	0
.970	.197	561.44	.0200	0
.95	.236	551.86	.0478	0
.925	.268	538.56	.0894	0
.90	.293	524.45	.137	0
.85	.347	495.18	.253	0
.80	.397	464.55	.396	0
.75	.446	434.33	.568	0
.70	.498	404.36	.775	0
.65	.552	375.80	1.02	0
.60	.611	348.36	1.32	0
.55	.678	322.96	1.69	0
.5227	.719	309.78	1.93	0
.5227	.999	309.78	1.93	0
.5	1.048	295.17	2.24	0
.45	1.163	265.26	3.02	0
.40	1.289	238.64	3.99	0
.35	2.166	211.69	5.49	.549
.30	4.155	173.00	8.74	.976
.25	6.730	124.33	14.8	1.0
.20	9.445	75.476	24.0	1.0
.15	11.93	36.157	34.8	1.0
.10	13.92	11.696	45.2	1.0
.05	15.18	1.539	52.7	1.0
0	15.62	0	55.5	1.0

quantity  $\kappa$  which measures the departure of the figure from true ellipticity again disclosed discrepancies, the computed  $\kappa$  being .00195, whereas Brouwer and Clemence give  $\kappa = .00159$ . The writer has in the past built other models of Saturn with "harder" equations of state for molecular hydrogen. These also gave too high values for  $\kappa$  when the ellipticity was correct. Moreover, Jeffreys (1954) has computed values of  $\kappa$  for a series of models of Saturn derived by Ramsey and a similar situation resulted, the model having the more nearly correct ellipticity yielding too high a value for  $\kappa$ .

The model of Jupiter was obtained in a similar fashion. The zero approximation calculated as "pure hydrogen" outside a central core gave in this case too small a moment of inertia. In the succeeding calculation the densities of Tables (3.4) and (3.6) were altered toward higher values. A function of pressure  $W_{He}(p)$  is defined by

$$\frac{1}{\rho(p)} = \frac{W_{He}}{\rho_{He}(p)} + \frac{1 - W_{He}}{\rho_H(p)}.$$

Here  $\rho(p)$  is the altered density of pressure  $p$ ,  $\rho_{He}(p)$  the estimated density of helium at the pressure  $p$  (Table 3.5) and  $\rho_H(p)$  is the density of hydrogen (Table 3.4 or 3.6) at pressure  $p$ . The altered densities were chosen in such a fashion that  $W_{He}(p)$  monotonically increased with pressure. The reason for altering the densities in this way will become clear in the following section concerning the elemental abundances of Jupiter and Saturn. It suffices here to say that the monotonic increase of  $W$  with  $p$  reflects the writer's feeling that elements heavier than hydrogen should be concentrated toward the planetary centers. Using a set of altered densities, the equation of hydrostatic equilibrium was integrated inward and a core added to force the model to have the correct total mass. On the first attempt it was found that a core could be chosen in such a manner that the density at the core boundary was continuous and, further, that the ellipticity calculated on the basis of the model agreed with the Brouwer and Clemence values for Jupiter. Moreover, the computed value of the quantity  $\kappa$  in this case also turned out to agree quite well with the one given by Brouwer and Clemence. The details of the Jovian model obtained are given in Table (4.3). The writer makes no claim

to the effect that these models necessarily reproduce the internal features of the two great planets. The derivation of a model which would faithfully reproduce the internal features of these two bodies does not seem to be possible on the basis of the present knowledge.

The extent of the agreement of the calculated external features of the model with observation can be judged from Table (4.4). Here the com-

TABLE (4.4). OBSERVABLE EXTERNAL POTENTIAL PARAMETERS

	Jupiter		Saturn	
	Calculated	Observed	Calculated	Observed
$\epsilon_1$	.0645	.0652	.0988	.0978
$\kappa_1$	.00049	.00052	.00195	.00159
$J$	.02130	.02206	.02594	.02501
$K$	.00250	.00253	.00588	.00386

puted values of the parameters characterizing the external potentials of the models are given along with the observed (Brouwer and Clemence) values.

#### V. ABUNDANCE ESTIMATES

Even if one possessed a formula or table giving exactly the density, pressure and temperature at every level surface within the two great planets, and in addition possessed exact equations of state for hydrogen and helium, i.e.,  $\rho = \rho(p, T)$ , it is correct to assert that one could not deduce the chemical composition of the planets. However, it is a conclusion of this paper that one could give a fairly precise lower limit to the percentage of the mass of the planets attributable to hydrogen. Even more important, however, is the conclusion that with only a rough knowledge of the internal densities and a fairly crude estimate of the equation of state of molecular hydrogen one can still give a fair estimate of a lower limit to the hydrogen content if certain plausible assumptions are made.

There is good reason to believe that the density of hydrogen,\* under any conditions of pressure and temperature which are at all likely to be applicable in Jupiter and Saturn, will be exceeded by the density of any other chemical substance. (This statement is subject to a trivial exception at low pressures due to the circumstance that hydrogen liquefies at higher temperatures than does helium.) One can also presume quite reasonably that the density of helium is less than or equal to the density of any substance excepting

\* In order to spare the reader a certain amount of excess verbiage, the term density will be used in many places where precision would require use of the phrase "quotient of the molecular weight by partial molar volume."

TABLE (4.3). JOVIAN MODEL

$\beta$ (= $r/R$ )	$\rho$ (g/cm <sup>3</sup> )	$M \times 10^{-27}$ (g)	$p \times 10^{-12}$ (dynes/cm <sup>2</sup> )	$W_{He}$
1.000	.00016	1902.0	0	0
.998	.032	1901.92	.00033	0
.996	.103	1901.34	.0027	0
.994	.138	1900.31	.0068	0
.992	.162	1899.04	.0121	0
.990	.181	1897.60	.0181	0
.98942	.185	1897.15	.0200	0
.98942	.197	1897.15	.0200	0
.98	.246	1888.49	.0568	.018
.94	.367	1840.11	.283	.038
.90	.479	1778.72	.614	.058
.86	.593	1707.76	1.06	.090
.82	.714	1628.92	1.62	.11
.802	.777	1591.13	1.93	.13
.802	1.08	1591.13	1.93	.13
.800	1.09	1585.16	1.98	.13
.75	1.31	1431.29	3.37	.14
.7	1.56	1270.27	5.07	.16
.65	1.83	1105.11	7.12	.20
.6	2.12	940.33	9.52	.23
.55	2.40	780.05	12.3	.24
.5	2.66	630.96	15.3	.25
.45	2.90	496.32	18.5	.26
.40	3.14	379.85	21.9	.26
.35	3.37	281.40	25.5	.26
.3	3.58	203.05	29.2	.26
.25	3.81	142.88	33.1	.26
.2	4.08	100.37	37.6	.27
.15	4.40	72.19	43.2	.27
.1	19.09	33.98	63.5	1.0
.05	27.90	5.19	96.3	1.0
0	30.84	0	110	1.0

hydrogen which is likely to be of importance in the interiors of Jupiter and Saturn. In this connection it is interesting to note that the density of solid helium as measured by Stewart already exceeds that of metallic lithium when both are subjected to pressures of 20,000 atmospheres. On general grounds lithium or lithium-rich compounds would be the most likely to violate the assumption about helium, but we may safely rule metallic lithium out of consideration in the outer regions of the giant planets; and whether or not we rule out the likelihood of the presence of lithium-rich compounds is only of minor importance for in general they are quite a bit denser than the pure metal.

In the estimation of hydrogen abundances it has been found convenient to make several restrictive assumptions and examine the consequences of their relaxation one at a time. The initial assumptions, aside from the two just stated, are:

1. The model planets specified in Tables (4.2) and (4.3) are correct in all details.
2. The equations of state of hydrogen and helium given in Tables (3.4), (3.5), and (3.6) are exact for pressures greater than 20,000 atmospheres and for whatever temperatures actually obtain in the two planets.
3. The partial molar volumes of hydrogen and helium when the two occur in mixtures are equal to the molar volumes of pure hydrogen and pure helium.
4. The empirical data given in Table (4.1) are correct.

With these assumptions the weight fraction of hydrogen,  $w_H(s)$ , at a given value of the parameter  $s$  inside one of the planets is not less than

$$w_0(s) = \frac{\rho_H}{\rho} \left\{ \frac{\rho_{He} - \rho}{\rho_{He} - \rho_H} \right\}$$

or zero, whichever is the greater numerically. In the expression  $\rho$ ,  $\rho_H$ , and  $\rho_{He}$  are, respectively, the density of the model planet and the densities implied by Tables (3.4), (3.5), and (3.6) at the pressures calculated by use of the model planet for the level surface of parameter  $s$ . "Helium" abundances *defined* by

$$w_{He}(s) = 1 - w_0(s)$$

are also given as function of  $s$  in Tables (4.2) and (4.3). Under the assumptions stated the fractional mass of the planet attributable to hy-

drogen is greater than

$$W_0 = 4\pi \int_0^{R_1} w_0(s) \rho(s) s^2 ds.$$

The values of  $W_0$  obtained in this way for Jupiter and Saturn are given in Table (5.1). The importance of assumption (1) can be judged as follows: Assume that only assumption (1) is in error. Ramsey and Miles (1952) have given a pair of models of Jupiter and Saturn which are in quite close agreement with the mechanical data of Brouwer and Clemence. These are the models  $S_2$  and  $J_3$ . If the densities of Ramsey's models  $S_2$  and  $J_3$  are used in conjunction with the hydrogen and helium densities from Tables (3.4), (3.5), and (3.6) a value of the hydrogen weight fraction,  $w_0^{(1)}(s)$ , can *formally* be calculated as

$$w_0^{(1)}(s) = \frac{\rho_H}{\rho_R} \left\{ \frac{\rho_{He} - \rho_R}{\rho_{He} - \rho_H} \right\}$$

wherein  $\rho_R$  denotes densities from Ramsey's models. The function,  $w_0^{(1)}(s)$ , obtained in this way assumes "impossible" values greater than unity. If, nevertheless, these values are used in the evaluation of the abundance integral,

$$W_0^{(1)} = 4\pi \int_0^{R_1} w_0^{(1)}(s) s^2 \rho_R(s) ds,$$

the values obtained do not differ very much from the values of  $W_0$  obtained using the model planets of the present paper. The amounts of the differences are given in Table (5.1). This result can be interpreted to mean that the computed hydrogen abundances are insensitive to the internal density distribution of the model planet employed so long as it is consistent with a fixed set of mechanical data. From the standpoint of the equations of state adopted in this paper, Ramsey's density distributions can be regarded as extreme.

The discussion of the effects of relaxation of assumption (2) is complicated by the fact that it can be invalid in several distinct ways. Firstly, the temperatures of the interiors of the planets would invalidate the equations of state of hydrogen and helium even if they were correct for the absolute zero. To give the reader a feeling for the magnitude of the effects to be expected the hydrogen weight fractions have been evaluated on the assumption that the densities of hydrogen and helium as tabulated are 3 per cent too high for pressures up to  $1.93 \times 10^{12}$  dynes/cm<sup>2</sup> and 1 per cent too high for larger pressures. The changes in the hydrogen abundances from the



basic value of  $W_0$  brought about by the changes in the equation of state are also given in Table (5.1). Secondly, the density of 0°K molecular hydrogen might be decidedly different from the alternative extrapolation of the equation of state of  $H_2$ . (As mentioned in Part III, there might be an intermediate hydrogen phase or phases. It is clear that the existence of such a phase could be dealt with for the purposes of the present discussion as an erroneous equation of state for the molecular phase.) To assess the magnitude of possible errors due to this source, Ramsey's models  $S_2$  and  $J_3$  were again pressed into service. Ramsey's equation of state was assumed to be correct for molecular hydrogen, Table (3.5) was adopted for helium, and Table (3.6) was assumed to give correctly the densities and free enthalpies of metallic hydrogen. The changes in the densities of  $H_2$  are  $\sim 30$  per cent. The phase transition pressure  $0.8 \times 10^{12}$  dynes/cm<sup>2</sup> found by Kronig *et al.* and by Ramsey was confirmed. Calculation of the hydrogen abundances of Ramsey's models using his equation of state of molecular hydrogen but the data of state for helium and metallic hydrogen of the present paper resulted in changes in  $W_0$  also detailed in Table (5.1). In the third instance, it might be that the values of the free enthalpy of metallic hydrogen obtained by the Wigner-Huntington method are incorrect but that the pressure-density values are correct. To be specific it was assumed that the free enthalpies should be raised by 5.3 kilocalories per mole of  $H$  atoms for pressures near  $2 \times 10^{12}$  dynes/cm<sup>2</sup>. In order to avoid unreasonable values of hydrogen weight fraction as function of  $s$ , the densities of the equation of state of molecular hydrogen were lowered 5 per cent. This procedure leads to the transition pressure of  $1.93 \times 10^{12}$  dynes/cm<sup>2</sup> used in the construction of the models (4.2) and (4.3). Again the changes incurred in the hydrogen abundances are given in Table (5.1). Fourthly, the densities of metallic hydrogen as given by Table (3.6) might be erroneous without the value of the free enthalpy at  $p = 1.93 \times 10^{12}$  dynes/cm<sup>2</sup> being so. The specific case investigated was a change in the densities of metallic hydrogen downward by 5 per cent. Finally, the equation of state of helium at 0°K could differ from the one obtained in this paper. The change in  $W_0$  arising when all the helium densities of Table (3.5) are raised by 5 per cent has also been evaluated.

The relaxation of assumption (3) with the retention of the other assumptions, particularly

assumption (2), is roughly equivalent to the relaxation of assumption (2) with the retention of assumption (3) and, hence, needs no special discussion. Non-ideal mixing would affect the equations of state of the mixture and the phase transition pressure.

The effects of relaxing the assumption (4) have been examined as follows. Ramsey has calculated four models for each of the planets with variable values for their external potentials but with fixed masses and radii. His derived abundance results may be interpreted as a demonstration that the computed abundances are insensitive to the moment of inertia of the mass distribution when the abundance estimates are based on definite equations of state for hydrogen and helium. Extreme values for the abundances and the extreme values of the quantity  $J$  calculated from Ramsey's models using the value of  $m$  from Table (4.1) are:

Jupiter		Saturn	
$J$	$W$	$J$	$W$
.015	.84	.018	.69
.026	.76	.046	.62

The effect of a change in the adopted numerical value of the mean radius has been investigated by applying a similarity transformation to the models (4.2) and (4.3). Specifically, it was assumed that the mean radii given in Table (4.1) were one per cent too large. This being the case the densities of Tables (4.2) and (4.3) must all be multiplied by 1.03 and the pressures by 1.04. The discontinuity in density of the transformed models corresponds to a pressure of  $2.01 \times 10^{12}$  dynes/cm<sup>2</sup> and is inconsistent with the alternative equation of state of molecular hydrogen. Multiplication of all densities of the alternative equation of state by 1.02 yields a new equation of state for which the calculated value of the phase transition pressure is  $2.01 \times 10^{12}$  dynes/cm<sup>2</sup>. The abundance estimates of the shrunken models were calculated using, for hydrogen densities  $\rho_H$ , values from Table (3.4) multiplied by 1.02. The changes made in the abundance estimates appear also in Table (5.1).

The changes in the bases of calculation of the hydrogen abundances should not be considered as reflecting the magnitude of the probable error. The writer does not feel probable errors can be realistically assessed at present. Nor do the signs chosen for the changes reflect any judgment on the part of the writer that future improvements of the bases are likely to be in the same direction



TABLE (5.1). HYDROGEN ABUNDANCES OF JUPITER AND SATURN  
Jupiter ( $W_0 = .782$ )      Saturn ( $W_0 = .632$ )

Alteration of Basis	Change in $W$	
	Jupiter	Saturn
Internal density distribution changed to conform to Ramsey's $J_3$ and $S_2$ models	+.023	+.100
Densities of hydrogen and helium lowered 3% if $p < 1.93 \times 10^{12}$ dynes/cm <sup>2</sup> and lowered 1% otherwise simulating temperature effects	-.023	-.017
Equation of state of $H_2$ altered from Table (3.4) to Ramsey's equation of state	+.023	+.069
Free enthalpy of metallic hydrogen raised 5.3 kilocalories per gram	-.011	-.031
Density values of metallic hydrogen lowered 5% from values in Table (3.6)	-.048	-.013
Helium densities raised 5% from values of Table (3.5)	+.004	+.001
Mean radii decreased 1% with respect to Brouwer and Clemence values	-.012	-.005

as the sample calculations. These sample calculations are rather intended to give the reader a feeling for the magnitudes of the "partial derivatives" of the minimal hydrogen abundances with respect to alterations in the bases of their computation. The principal result is the satisfactory stability of the abundance estimates with respect to the changes in basis.

#### VI. THE INTERNAL TEMPERATURES OF JUPITER AND SATURN

The fact that the radiometric temperatures of Jupiter and Saturn accord well with the values expected for bodies, with no internal heat sources, in equilibrium with the sun indicates that the outflow of heat from the planetary interiors is small compared with the flux of solar radiation. Numerically, this means that the emergent heat flux at the surface should be less than  $\sim 10^{-4}$  cal cm<sup>-2</sup> sec<sup>-1</sup>. The heavy central cores of the model planets are of the order of 10 Earth masses. For crude estimation we may regard the cores as identical chemically with the Earth and using the known radioactive heat flux for the Earth ( $\sim 10^{-6}$  cal cm<sup>-2</sup> sec<sup>-1</sup>) estimated fluxes of  $10^{-7}$  cal cm<sup>-2</sup> sec<sup>-1</sup> are obtained for the two great planets. The same order of magnitude results if the two planets are regarded as of cosmic composition and the radioactive heat generated in cosmic matter is considered (Ramsey 1951). The temperature drop across the molecular portion of the planets in case the heat is transported by conduction is, consequently,

$$\Delta T \sim 10^2 (\overline{1/k}) \text{ } ^\circ\text{K}$$

where  $(\overline{1/k})$  is a mean reciprocal thermal conductivity. The extreme variability of the thermal conductivity of dielectrics as evidenced in Table (3.10) makes it difficult to estimate a mean reciprocal thermal conductivity but, at the same time, it would appear from the table that solid mantles of hydrogen should have mean reciprocal

thermal conductivities less than  $\sim 10^2$  and accordingly for Jupiter and Saturn

$$\Delta T \lesssim 10^4 \text{ } ^\circ\text{K}.$$

The temperature  $10^4$  °K is incompatible with the assumption of a solid mantle. Unless the heat flux is large enough to maintain the convection process it is to be expected that the mantle would solidify starting from the bottom. In this case, Simon's melting formula indicates that the base of the molecular mantle should be at a temperature of the order of 2000°K or less, depending on the extent of the region of solidification.

The writer does not see at present how to refine the estimate nor how to delimit the region of solidity of the molecular phase. He also wishes to recall the reader's attention at this point to the difficulties experienced in building suitable model planets of hydrogen on the basis of the equation of state (3.3) and the possibility of resolving the difficulties by invoking appreciable temperature effects. The nature of the internal temperatures of Jupiter and Saturn cannot be regarded as very well understood. It would appear that temperature effects are likely to be more important for Saturn than for Jupiter because the molecular phase constitutes a larger portion of the total mass.

#### VII. SUMMARY CONCLUSIONS

On the basis of the evidence presented in the preceding portions of this paper it is concluded that over one-half of the masses of Jupiter and Saturn are to be attributed to hydrogen. If it is assumed that they are cold bodies in an appropriate sense, the weight fractions of hydrogen are estimated as .782 and .632 for Jupiter and Saturn. These numbers imply that the ratios of the total number of hydrogen atoms to the total number of helium atoms present in these bodies would exceed 14 and 7, respectively. No attempt has

been made to assign probable errors for the hydrogen weight fractions on the grounds that any such estimates given could not be logically defended. However, the insensitivity of the hydrogen abundances to relaxation of various assumptions made in their estimation gives strong indication that future improvements in the bases of calculation are not likely to lead to abundance estimates in which the predominance of hydrogen will no longer appear.

The most obvious means of improving our knowledge of these planets is via improvements in the equations of state of hydrogen and helium—in particular our knowledge of the equation of state of the molecular phase of hydrogen is vague, to say the least. Extension of the experimental pressure range beyond 20,000 atmospheres achieved by Stewart would yield most valuable information. A realistic calculation of the equation of state of solid molecular hydrogen seems at this stage to be practically unattainable but the situation could change drastically as progress is made in solid state physics. Also of interest would be the experimental determination of the pressure-density relation of mixtures of hydrogen with other substances, particularly helium.

An exploratory investigation of the internal temperatures of the planets would seem to be well worth while even if made on the basis of the physical properties of hydrogen inferred in Part II of this paper.

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#### APPENDIX I. PURE HYDROGEN PLANETS

This is not the place to give a detailed description of studies which have been made on pure cold hydrogen planets. However, the mass-radius diagram of planets composed solely of pure hydrogen is not without some interest in the present discussion. Figure A-I shows the mass-radius diagram of non-rotating planets of pure cold hydrogen taken from the writer's unpublished thesis (Yale 1951). To be sure, the equation of state of the molecular phase was calculated from the Lennard-Jones 6-12 interaction potential of de Boer and Michels and, as a consequence, the lower half of the curve needs revision. The diagram brings out very clearly a theorem due to

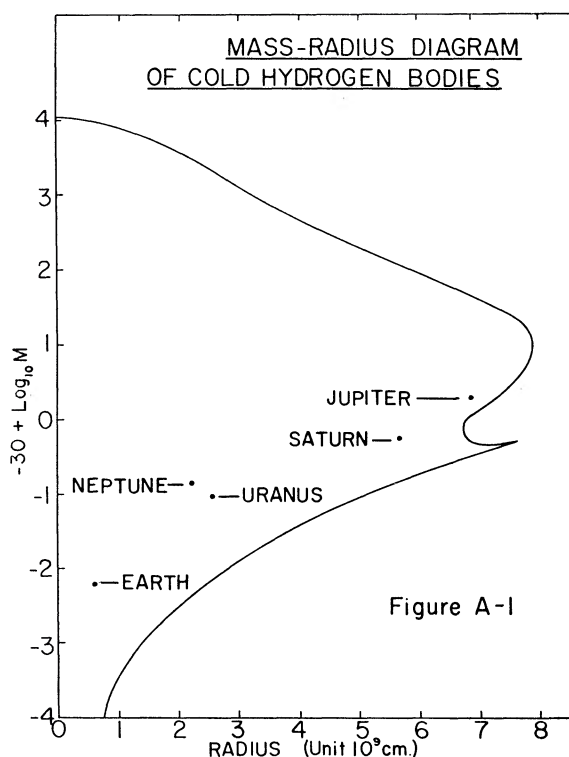


Figure A-I. Mass-radius diagram of cold hydrogen planets.

Ramsey and Lighthill (1950) that there will be cusps in the mass radius diagram if the density has a saltus of more than 50 per cent at some transition pressure. The existence of a maximum radius not much in excess of the radius of Jupiter is also shown, the maximum radius being  $7.93 \times 10^9$  cm, whereas the mean radius of Jupiter is  $6.99 \times 10^9$  cm. Ramsey's calculations on hydrogen planets give a maximum radius of  $8.5 \times 10^9$  cm. As the mass increases, the radius decreases along the high mass branch of the mass-radius relation until finally a smooth junction is made with the mass-radius relation of hydrogen white dwarfs as computed by Chandrasekhar (1939).

It must be pointed out that the interesting cusp in the mass-radius diagram would not be present if it were recomputed using either of the equations of state of this paper or any other equation of state exhibiting significantly higher compressibilities than those predicted on the basis of the Lennard-Jones 6-12 interaction potential.

#### APPENDIX II. SHORT RANGE INTERMOLECULAR INTERACTIONS

The argument presented in this appendix about the form of short range two-body forces is due to

Buckingham (1956). For simplicity, the discussion will deal with the case of helium.

The intermolecular energy of two helium atoms whose nuclei are at the fixed separation,  $R$ , is defined as the difference between the quantum mechanically calculated energy of the four-electron system and the two fixed nuclei and the energy of two helium atoms at infinite separation. The energy,  $E(R)$ , of the four electrons and two fixed nuclei can be written as the sum of two terms,

$$E(R) = \frac{4e^2}{R} + E_e(R),$$

thus accounting separately for the coulombic interaction of the two helium nuclei. The remaining term will be called the electronic energy and cannot be evaluated without precise knowledge of the wave functions of the four electrons in the field of the two helium nuclei. However, as  $R$  tends to zero the electronic energy approaches that of the ground state of the beryllium atom. Moreover, for sufficiently small  $R$  the electronic energy will depart from that of the beryllium atom quadratically in  $R$ , i.e.,

$$E_e(R) = E_{Be} + \alpha R^2 + \dots$$

where  $\alpha$  is constant and  $E_{Be}$  is the ground state energy of beryllium. The interaction energy is then

$$E(R) = E_{Be} - 2E_{He} + \frac{4e^2}{R} + \alpha R^2 + \dots,$$

$E_{He}$  being the ground state energy of helium. An obvious consequence of the form of the interaction potential is that the equation of state calculated on the assumption of additive two-body forces will behave at high densities as  $p \sim \rho^{4/3}$ , whereas the correct result has  $p \sim \rho^{5/3}$ .

#### APPENDIX III. THERMAL EXPANSION OF SOLID HYDROGEN AND HELIUM

The measurements of Stewart and of Stewart and Swenson give us knowledge of the 4.2°K isotherm of hydrogen and helium from zero pressure up to pressures of some 20,000 kg/cm<sup>2</sup>. Lacking other experimental data the caloric equation of state has to be estimated from theory. For this purpose, we use the Debye theory of dielectric solids. According to this theory, the complete caloric equation of state of monatomic solids is known if the  $pV$  isotherm at absolute zero is known and if the Debye characteristic temperature,  $\theta_D$ , is known as a function of its

argument  $V$ , the molar volume. For molecular crystals this information must be augmented to account for the internal degrees of freedom which we shall ignore with the expectation that the results will still be of the right order of magnitude. The pressure is then given at any temperature  $T$  and volume  $V$  by the expression

$$p(T, V) = p(0, V) + \frac{\gamma}{V} \cdot T \cdot F(T/\theta_D)$$

wherein  $p(0, V)$  is the pressure at 0°K corresponding to the volume  $V$ ,  $\gamma = -d \ln \theta_D / d \ln V$  is the so-called Grüneisen ratio and

$$F\left(\frac{T}{\theta_D}\right) = 9R \left(\frac{T}{\theta_D}\right)^3 \int_0^{\theta_D/T} \frac{\xi^3 d\xi}{e^\xi - 1},$$

$R$  being the gas constant per mole. The function  $F$  has been tabulated by Beattie (1926).

The characteristic temperature may be estimated by Einstein's formula,

$$\theta_D^2 = C \cdot V^{1/3} K,$$

where  $C$  is approximately a constant for a given solid and  $K$  is the incompressibility at the absolute zero of temperature. For  $H_2$  the constant  $C$  has been chosen so that  $\theta_D = 105^\circ\text{K}$  at  $V = 22.65$  cm<sup>3</sup>/mole (Bartholomé and Eucken 1936) and for helium  $\theta_D = 113^\circ\text{K}$  at  $V = 10.5$  cm<sup>3</sup>/mole (Dugdale and Simon 1953). The isochores are then easily determined and are terminated on the high pressure side when they cross the melting curve which is assumed to be known in the pressure-temperature plane. In this way it is found that up to melting pressures of 20,000 kg/cm<sup>2</sup> the density of the solid at the melting temperature differs from the density of the solid at the absolute zero by between 1 and 2 per cent, the difference decreasing at the higher pressures. Extrapolation of the melting curve and the equation of state confirms this trend at the higher pressures. As a consequence, for helium and for hydrogen, insofar as the molecular phase of  $H_2$  may be treated as a monatomic solid, temperature seems to play a very subordinate role in determining the equation of state.

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