

The Moist Pseudo-Adiabat for Multiple Condensing Species

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Theories of moist adiabats and pseudo-adiabats are commonplace in the Earth and planetary science literature. Generally, the atmosphere is assumed to consist of a background gas and a condensible species. The moist adiabat is derived assuming that the condensed-phase material remains in the parcel, which makes the condensation process reversible. The pseudo-adiabat, which is more commonly used, assumes that the condensed-phase material rains out of the parcel. This assumption makes the condensation process irreversible, although it is typically true that the energy exiting the parcel due to rain out is negligible.

It is likely true, however, that several co-occurring condensible species will be relevant over a range of conditions within the atmospheres of giant planets and brown dwarfs. To address this, the following note derives the moist pseudo-adiabatic lapse rate for multiple condensing species. The pseudo-adiabat is adopted as the common assumption in our thermo-chemical models is that of rain-out chemistry. We conclude the note with some comparisons to expressions from the literature for single-condensible cases.

Derivation

For a single condensible, conservation of energy relates the change in heat per mole of total gas in a parcel (dq) to changes in temperature (dT), volume (dv), and the volume mixing ratio of condensates (df_c) via,

$$dq = (f_b c_{v,b} + f_c c_{v,c} + f_s c_{\text{mol},s}) dT + p dv + L df_c , \quad (1)$$

where f indicates a volume mixing ratio (i.e., moles of a certain species per moles of *total* gas), a sub-script ‘b’ is for the background gas, a subscript ‘c’ is for the gas phase condensible, a subscript ‘s’ is for the solid (or liquid) phase of the condensible, c_v is a specific heat (per mole) at constant volume, $c_{\text{mol},s}$ is the solid phase molar heat capacity, p is pressure, v is volume *per mole* (i.e., $1/n$, where n is a molar number density), and L is a molar latent heat of condensation. For the pseudo-adiabat, the process is assumed adiabatic except for the loss of the solid (or liquid) phase, which would give,

$$dq = f_s c_{\text{mol},s} dT , \quad (2)$$

and the previous expression becomes,

$$0 = (f_b c_{v,b} + f_c c_{v,c}) dT + p dv + L df_c . \quad (3)$$

Generalizing to an arbitrary number of condensible species, our working equation would be,

$$0 = \left(f_b c_{v,b} + \sum_i f_i c_{v,i} \right) dT + p dv + \sum_i L_i df_i . \quad (4)$$

Taking the gas to be ideal, with

$$pv = RT , \quad (5)$$

where R is the universal gas constant, we have by the chain rule,

$$p dv + v dp = p dv + \frac{RT}{p} dp = R dT . \quad (6)$$

Inserting this into our working equation and re-arranging yields,

$$0 = \left(f_b c_{v,b} + \sum_i f_i c_{v,i} + R \right) dT - \frac{RT}{p} dp + \sum_i L_i df_i . \quad (7)$$

Then, with

$$c_v = c_p - R , \quad (8)$$

where c_p is the molar heat capacity at constant pressure, and noting that,

$$f_b + \sum_i f_i = 1 , \quad (9)$$

we have,

$$0 = \left(f_b c_{p,b} + \sum_i f_i c_{p,i} \right) dT - \frac{RT}{p} dp + \sum_i L_i df_i . \quad (10)$$

Next, we divide by $RT dp/p$ and re-arrange to yield,

$$1 = \left(f_b \frac{c_{p,b}}{R} + \sum_i f_i \frac{c_{p,i}}{R} \right) \frac{d \ln T}{d \ln p} + \frac{p}{RT} \sum_i L_i \frac{df_i}{dp} . \quad (11)$$

Noting that, in the absence of condensation, the dry adiabatic lapse rate would be,

$$\left(\frac{d \ln T}{d \ln p} \right)_{\text{dry}} = \nabla_d = \frac{R}{f_b c_{p,b} + \sum_i f_i c_{p,i}} , \quad (12)$$

we can write,

$$1 = \frac{1}{\nabla_d} \frac{d \ln T}{d \ln p} + \frac{p}{RT} \sum_i L_i \frac{df_i}{dp} . \quad (13)$$

We now focus on the final term in our working expression. Using the definition of the volume mixing ratio and the ideal gas law, we have,

$$\frac{df_i}{dp} = \frac{df_i}{dT} \frac{dT}{dp} = \frac{dp_i/p}{dT} \frac{dT}{dp} , \quad (14)$$

where p_i is the partial pressure of the condensible species. Differentiation by parts then yields,

$$\frac{df_i}{dp} = \frac{1}{p^2} \frac{dT}{dp} \left(p \frac{dp_i}{dT} - p_i \frac{dp}{dT} \right) . \quad (15)$$

Taking the parcel to be at saturation, so that p_i is the saturation vapor pressure, and noting that, by the Clausius-Clapeyron equation, we have,

$$\frac{dp_i}{dT} = \frac{p_i L_i}{RT^2} , \quad (16)$$

so that,

$$\frac{df_i}{dp} = \frac{f_i L_i}{RT^2} \frac{dT}{dp} - \frac{f_i}{p} . \quad (17)$$

Inserting this into our working equation and re-arranging yields the final result,

$$\frac{d \ln T}{d \ln p} = \frac{1 + \sum_i f_i \frac{L_i}{RT}}{\nabla_d^{-1} + \sum_i f_i \frac{L_i^2}{R^2 T^2}} . \quad (18)$$

Comparisons

Nearly every textbook and/or journal article has its own (slightly different) variation on the moist pseudo-adiabatic lapse rate. The final expression above (in the single condensible limit) agrees with results in Holton's *An Introduction to Dynamic Meteorology* (4th ed., p. 503), and with the expression in the soon-to-be published planetary atmospheres textbook from D. Catling and J. Kasting (D. Catling, personal communication). The equivalent expression in Pierrehumbert's *Principles of Planetary Climate* (1st ed., p. 108) is in rough agreement, but is expressed as an adiabat in the background gas pressure, which is less useful.

Sugiyama et al. (2005; their eqn. 5) present an expression (without derivation) that is nearly identical to the final expression above, except that their mixing ratios are everywhere relative to just the background component. [These authors cite two textbooks, Iribarne and Godson (1981) and Emanuel (1994), for their derivation.] Satoh's *Atmospheric Circulation*

Dynamics and General Circulation Models (1st ed., p. 252) fully derives an expression similar to Sugiyama et al. (2005), also containing mixing ratios that are relative to the background component. These solutions may be more correct, but would yield issues with division by zero when the atmosphere is composed entirely of a condensible.

Notable disagreements occur with Brunt (1933), Lasker (1963), and Fegley & Prinn (1986). The first (Brunt) begins with a conservation of energy that has dLf_c instead of Ldf_c . This leads to an errant $\frac{dL}{dT}$ term and where, without this term, we would be in agreement. Lasker (1963) and Fegley & Prinn (1986) adopt similar expressions to one another (noting that Fegley & Prinn cite Lasker). These each have an additional term in both their final numerator and denominator. We cannot reproduce these extra terms, and, worrisomely, these expressions, for $L \rightarrow 0$, give,

$$\frac{d \ln T}{d \ln p} \rightarrow \nabla_d \left(1 + \frac{p_c}{p_b} \right) , \quad (19)$$

which is not the correct limit.