Carnegie Mellon
THE ROBOTICS INSTITUTE

Kalman Filter

Robot Localization and Mapping
16-833

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September 20, 2023

Slides courtesy of Ryan Eustice and probabilistic-robotics.org

Discrete Kalman Filter

Estimates the (n x 1) state $\mathbf{x}_t$ of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$\mathbf{x}_t = A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t + \varepsilon_t$$

Observed through (k x 1) measurements $\mathbf{z}_t$

$$\mathbf{z}_t = C_t \mathbf{x}_t + \delta_t$$

Components of a Kalman Filter

$\boxed{A_t}$ Matrix (n x n) that describes how the state evolves from t-1 to t without controls or noise.

$\boxed{B_t}$ Matrix (n x m) that describes how the control u_t changes the state from t-1 to t .

$\boxed{C_t}$ Matrix (k x n) that describes a projection of state x_t to an observation z_t .

$\boxed{\varepsilon_t}$ $\boxed{\delta_t}$ Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t, respectively.

Reminder: Bayes Filters

$$\boxed{Bel(x_t)} = p(x_t \mid u_1, z_1, \dots, u_t, z_t)$$

z = observation
u = action
x = state

Bayes $$= \eta\, p(z_t \mid x_t, u_1, z_1, \dots, u_t)\, p(x_t \mid u_1, z_1, \dots, u_t)$$

Markov $$= \eta\, p(z_t \mid x_t)\, p(x_t \mid u_1, z_1, \dots, u_t)$$

Total prob. $$= \eta\, p(z_t \mid x_t) \int p(x_t \mid u_1, z_1, \dots, u_t, x_{t-1})\, p(x_{t-1} \mid u_1, z_1, \dots, u_t)\, dx_{t-1}$$

Markov $$= \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\, p(x_{t-1} \mid u_1, z_1, \dots, u_t)\, dx_{t-1}$$

Markov $$= \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\, p(x_{t-1} \mid u_1, z_1, \dots, z_{t-1})\, dx_{t-1}$$

$$\boxed{= \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\, Bel(x_{t-1})\, dx_{t-1}}$$

Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

$$bel(\mathbf{x}_0) = N\left(\mathbf{x}_0; \mu_0, \Sigma_0\right)$$

Linear Gaussian Systems: Dynamics

- Dynamics are linear function of state and control plus additive noise:

$$\mathbf{x}_t = A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t + \varepsilon_t$$

$$p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) = N\left(\mathbf{x}_t ; A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t, R_t\right)$$

$$\begin{array}{ll} \overline{bel}(\mathbf{x}_t) = \int p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) & bel(\mathbf{x}_{t-1}) d\mathbf{x}_{t-1} \\ \Downarrow & \Downarrow \\ \sim N\left(\mathbf{x}_t ; A_t \mathbf{x}_{t-1} + B_t \mathbf{u}_t, R_t\right) & \sim N\left(\mathbf{x}_{t-1} ; \mu_{t-1}, \Sigma_{t-1}\right) \end{array}$$

Linear Gaussian Systems: Dynamics

$$\overline{bel}(\mathbf{x}_t) = \int p(\mathbf{x}_t \mid \mathbf{u}_t, \mathbf{x}_{t-1}) \qquad bel(\mathbf{x}_{t-1}) d\mathbf{x}_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(\mathbf{x}_t; A_t\mathbf{x}_{t-1} + B_t\mathbf{u}_t, R_t\right) \sim N\left(\mathbf{x}_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

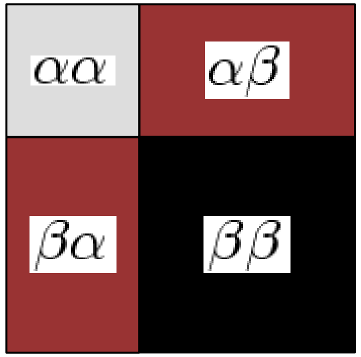
$$\Downarrow$$

$$\overline{bel}(\mathbf{x}_t) = \eta \int \exp\left\{-\frac{1}{2}(\mathbf{x}_t - A_t\mathbf{x}_{t-1} - B_t\mathbf{u}_t)^T R_t^{-1}(\mathbf{x}_t - A_t\mathbf{x}_{t-1} - B_t\mathbf{u}_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(\mathbf{x}_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1}(\mathbf{x}_{t-1} - \mu_{t-1})\right\} d\mathbf{x}_{t-1}$$

$$\overline{bel}(\mathbf{x}_t) = \begin{cases} \bar{\mu}_t = A_t\mu_{t-1} + B_t\mathbf{u}_t \\ \bar{\Sigma}_t = A_t\Sigma_{t-1}A_t^T + R_t \end{cases}$$

Reminder: Gaussian Parameterizations



	Covariance Form	Information Form
Marginalization $p(\boldsymbol{\alpha}) = \int p(\boldsymbol{\alpha}, \boldsymbol{\beta}) d\boldsymbol{\beta}$	$\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$ $\Sigma = \Sigma_{\alpha\alpha}$ (sub-block)	$\boldsymbol{\eta} = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\boldsymbol{\eta}_\beta$ $\Lambda = \Lambda_{\alpha\alpha} - \Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha}$ (Schur complement)
Conditioning $p(\boldsymbol{\alpha}\vert\boldsymbol{\beta}) = \frac{p(\boldsymbol{\alpha}, \boldsymbol{\beta})}{p(\boldsymbol{\beta})}$	$\boldsymbol{\mu}' = \boldsymbol{\mu}_\alpha + \Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}(\boldsymbol{\beta} - \boldsymbol{\mu}_\beta)$ $\Sigma' = \Sigma_{\alpha\alpha} - \Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\Sigma_{\beta\alpha}$ (Schur complement)	$\boldsymbol{\eta}' = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\boldsymbol{\beta}$ $\Lambda' = \Lambda_{\alpha\alpha}$ (sub-block)

Linear Gaussian Systems: Observations

- Observations are linear function of state plus additive noise:

$$\mathbf{z}_t = C_t \mathbf{x}_t + \delta_t$$

$$p(\mathbf{z}_t \mid \mathbf{x}_t) = N(\mathbf{z}_t; C_t \mathbf{x}_t, Q_t)$$

$$\begin{array}{ll} bel(\mathbf{x}_t) = \eta\, p(\mathbf{z}_t \mid \mathbf{x}_t) & \overline{bel}(\mathbf{x}_t) \\ \qquad\qquad\quad \Downarrow & \quad \Downarrow \\ \qquad \sim N(\mathbf{z}_t; C_t \mathbf{x}_t, Q_t) & \sim N(\mathbf{x}_t; \bar{\mu}_t, \bar{\Sigma}_t) \end{array}$$

Linear Gaussian Systems: Observations

$$bel(\mathbf{x}_t) = \eta \quad p(\mathbf{z}_t \mid \mathbf{x}_t) \qquad \overline{bel}(\mathbf{x}_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(\mathbf{z}_t; C_t \mathbf{x}_t, Q_t\right) \qquad \sim N\left(\mathbf{x}_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

$$\Downarrow$$

$$bel(\mathbf{x}_t) = \eta \exp\left\{-\frac{1}{2}(\mathbf{z}_t - C_t \mathbf{x}_t)^T Q_t^{-1} (\mathbf{z}_t - C_t \mathbf{x}_t)\right\} \exp\left\{-\frac{1}{2}(\mathbf{x}_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1} (\mathbf{x}_t - \bar{\mu}_t)\right\}$$

$$bel(\mathbf{x}_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(\mathbf{z}_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \qquad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Kalman Filter Algorithm

1: **Kalman_filter**$(\boldsymbol{\mu}_{t-1}, \Sigma_{t-1}, \mathbf{u}_t, \mathbf{z}_t)$**:**

2: $\bar{\boldsymbol{\mu}}_t = A_t \, \boldsymbol{\mu}_{t-1} + B_t \, \mathbf{u}_t$

3: $\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^\top + R_t$

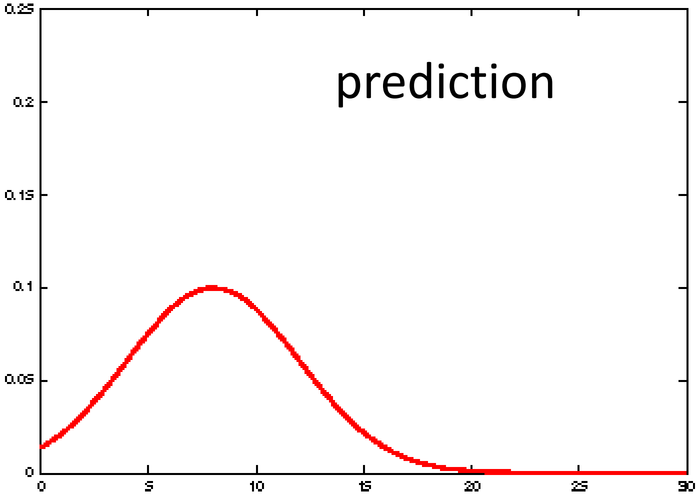
4: $K_t = \bar{\Sigma}_t \, C_t^\top (C_t \, \bar{\Sigma}_t \, C_t^\top + Q_t)^{-1}$

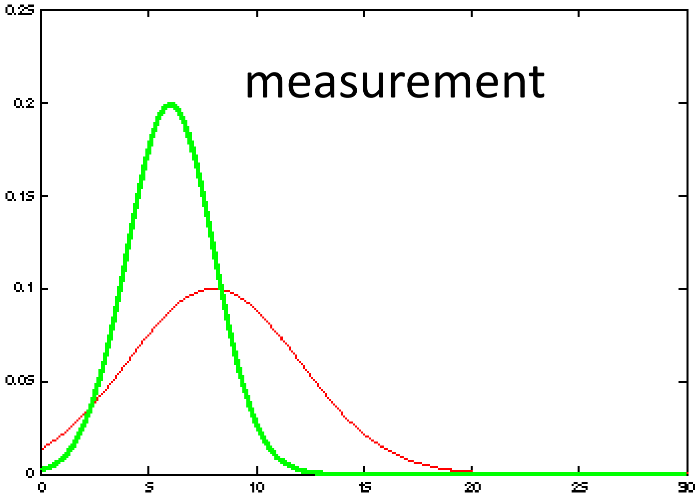
5: $\boldsymbol{\mu}_t = \bar{\boldsymbol{\mu}}_t + K_t(\mathbf{z}_t - C_t \, \bar{\boldsymbol{\mu}}_t)$

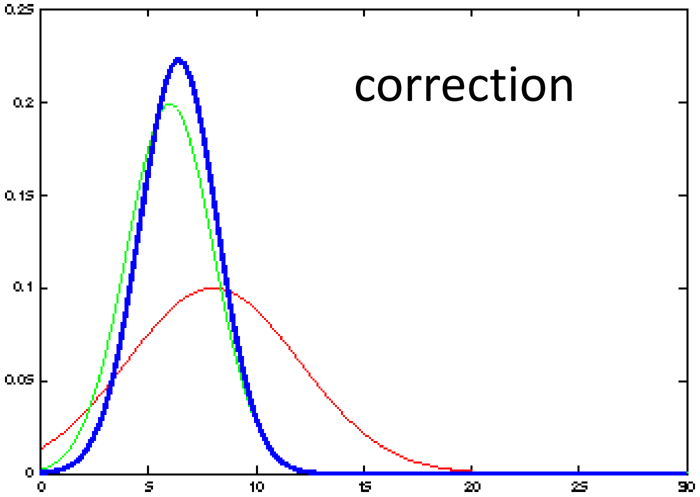
6: $\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$

7: *return* $\boldsymbol{\mu}_t, \Sigma_t$

1D Kalman Filter Example (1)

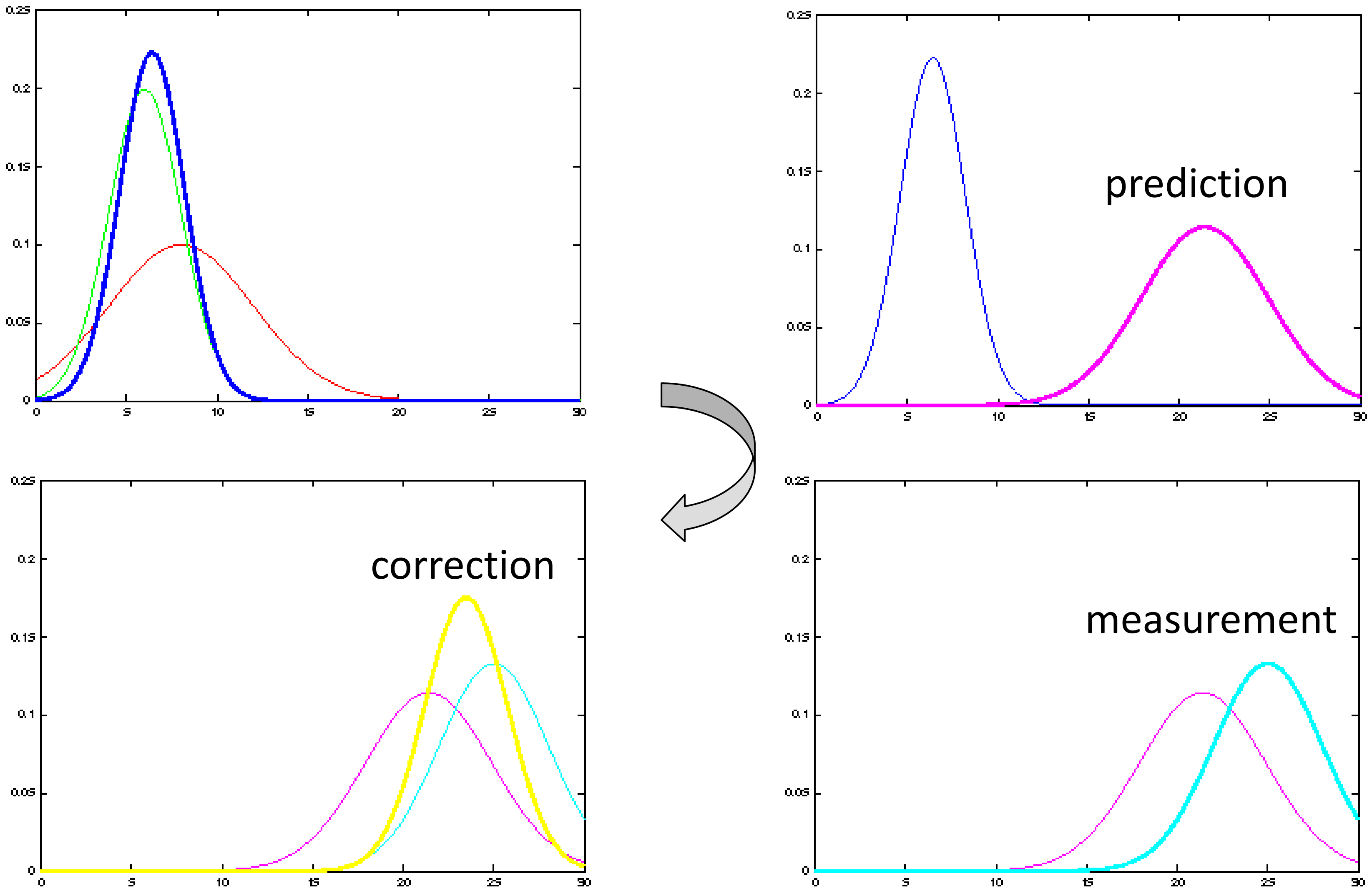




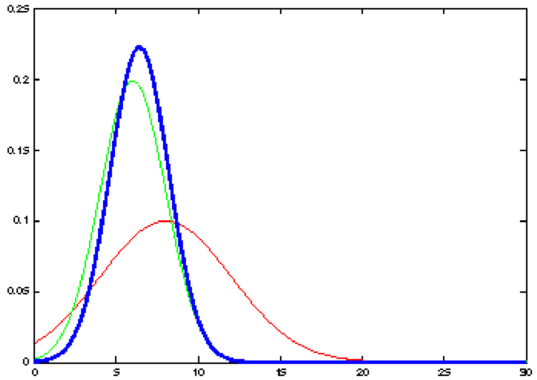


It's a weighted mean!

1D Kalman Filter Example (2)



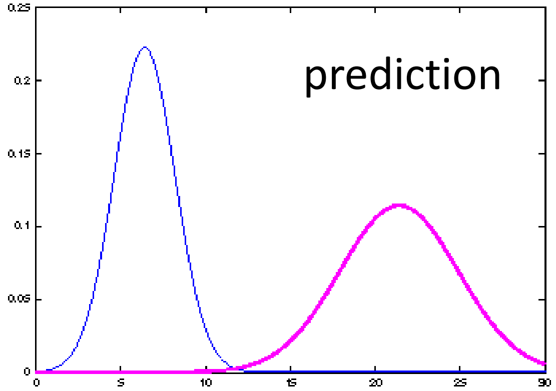
The Prediction-Correction-Cycle



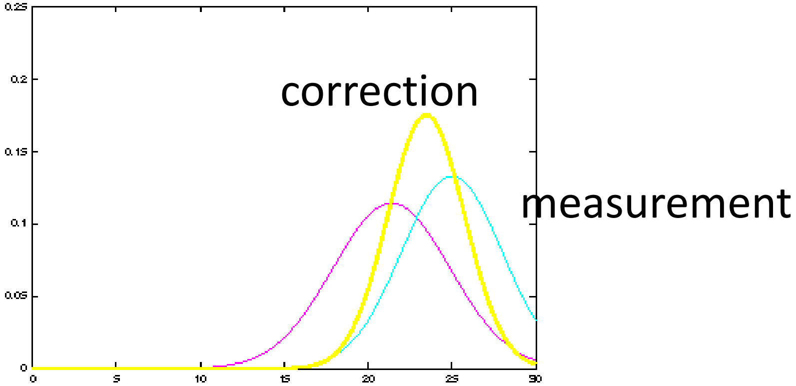
Prediction

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{\varepsilon_t}^2 \end{cases}$$

$$\overline{bel}(\mathbf{x}_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t \mathbf{u}_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

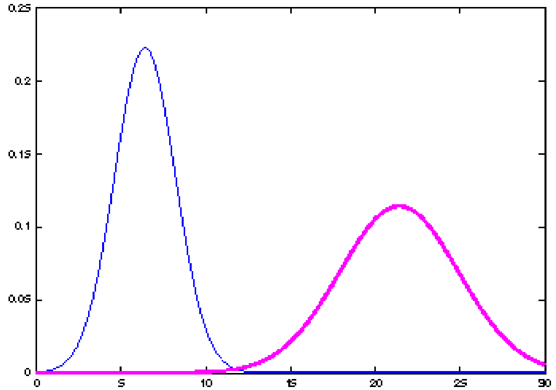


The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + k_t(z_t - c_t\bar{\mu}_t) \\ \sigma_t^2 = (1 - k_t c_t)\bar{\sigma}_t^2 \end{cases}, k_t = \frac{c_t \bar{\sigma}_t^2}{c_t^2 \bar{\sigma}_t^2 + \sigma_{\delta_t}^2}$$

$$bel(\mathbf{x}_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(\mathbf{z}_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$



Correction

The Prediction-Correction-Cycle

Prediction

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + k_t(z_t - c_t\bar{\mu}_t) \\ \sigma_t^2 = (1 - k_t c_t)\bar{\sigma}_t^2 \end{cases}, k_t = \frac{c_t \bar{\sigma}_t^2}{c_t^2 \bar{\sigma}_t^2 + \sigma_{\delta_t}^2}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{\varepsilon_t}^2 \end{cases}$$

$$bel(\mathbf{x}_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(\mathbf{z}_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$\overline{bel}(\mathbf{x}_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t \mathbf{u}_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

Correction

Kalman Filter Summary

- Highly efficient: Polynomial in measurement dimensionality k and state dimensionality n:

$$O(k^{2.376} + kn^2)$$

- Optimal **for linear Gaussian systems**!
  - No other estimator can do better

- Most robotics systems are nonlinear!

- Next: Extended KF, Unscented KF
  Probabilistic Robotics book 3.3, 3.4