Mathematical Tools for Elementary Psychics

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Abstract

Contents

1 Introduction

Probably I will discover nothing new here; that is fine. This will serve as a body of mathematical notes that will lay the foundations for the tools necessary to elementary psychics. I need to do this in order to assimilate all the data necessary in order to actually perform the calculations and synthesize everything. **This will take a long time; don't give up!**

1.1 Necessary branches

We will need to cover the foundations of probability theory, measure theory, and stochastic processes; category theory and the theory of (elementary) topoi; proof theory, metamathematics and elementary logic; lambda calculus, effective computability and (intuitionistic) type theory; synthetic differential geometry; Homotopy Type Theory and ∞-categories. These branches will be sufficient to construct the theory which I am after. The very basic idea is to interpret the sequents $\Gamma \vdash \Delta$ as affectively-mediated illative structures whose synthesis together forms a presentation of the universal psyche according to the symbolic unfolding of the $\lambda o \gamma o \varsigma$. Then one can assign (local) measures to the contents by assigning a functorial value to the terms inside of a primitive complete chain that functions as a model for affective valuation per Eliade in *Sacred and Profane*–this gives a presentation for an instantaneous expression of a psychic frame. Then extremized variations in time (I think) will correspond to evolutions of the distributions, which then detail the natural evolution of the psyche. This is the very basic idea. This requires notions of infinitesimals, variation, logical reasoning, truth-valuation, geometric morphisms, etc. Everything within these notes will serve to support these things. We will cite results where necessary. Lastly we mention that, given how far-reaching (surprisingly) this approach is, it is possible that there are other methods giving the description. The primary influences of the work are [11] and [32] so that functorial mappings into other forms of the theory are admissible as long as they preserve interpretation.

2 Category Theory

We will speak more philosophically about the nature of category theory and the manner in which it limits us in the actual work. For this part, we (mostly) restrict to actual calculations. The theory really needs to find its emphasis in *calculation* and especially *local* ones. The reason for this is that the individuation process is certainly *global* but the evolution of a *given* frame is *not* universal, since the distributions are not uniform (as far as we know; the other science will need to inform which bundle of distributions is legitimate and valid). We cover category theory and topoi first specifically because these are to lay the foundations for the remaining components, with the exception of type theory, which is the other foundational structure. Everything here follows primarily from [53, 2, 24, 8] (in order of relevance–with a close tie between the first two).

2.1 Basic Definitions

A *category* is an elementary notion exhibiting two data: that it consists of *objects*, and that it consists of *arrows* or *maps* (usually termed *morphisms*, but I don't like this terminology). The objects are to be regarded as static substances and the arrows as processes connecting these objects. If c, c' are objects and f is an arrow, these are synthesized together into a composite structure of the form

$$c \xrightarrow{f} c' \tag{1}$$

so that we can 'connect' two objects by means of an arrow. In general, these objects are *only* formable when $\text{dom}[f] = c$, the *domain* of f is the same as c, and $\text{cod}[f] = c'$, the *codomain* of f is the same as c'. Conversely, we can *define* these two properties as sending a diagram of the form (1) to c, c' respectively. Notice that if we adopt the constructive perspective, we *necessarily* cannot view these things as assignments because they are defined in terms of an elementary action. It is more reasonable in our view to look upon these as compatibility conditions which every arrow and object must satisfy.

Elementary category theory has three fundamental axioms which can be thought of in the following way: (1) indicates how objects may be connected; in general, one requires a mathematical entity of the sort 'arrow' to establish this connection. The following axiom, called *the axiom of composition*, details how arrows may be connected:

$$\begin{tikzcd} & c' \arrow[dr, "g"] & \\ c \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & c'' \end{tikzcd} \tag{2}$$

This just says that whenever $\text{cod}[f] = \text{dom}[g]$, there is a *unique* arrow, called the *composition of f with g*, that yields a connection or process from c to c''. This arrow is denoted $g \circ f$, but sometimes also $(f; g)$. The former is infinitely more common. The second axiom, *the axiom of identity*, states that there are specific arrows affiliated with objects

$$\begin{tikzcd} & c' \arrow[r, "id_{c'}"] & c' \\ c \arrow[ur, "f"] \arrow[urr, "f"] \arrow[r, "id_c"'] & c \arrow[ur, "f"'] & \end{tikzcd} \tag{3}$$

This asserts that for every object c there is an arrow $id_c : c \to c$ such that for arrows f, g with $\text{dom}[f] = c$ and $\text{cod}[g] = c$, $f \circ id_c = f$ and $id_c \circ g = g$[1]. In particular, we can think of this as asserting that there are (trivial) processes which correspond to the objects in such a way that the arrows themselves can't distinguish between the identities and the objects. Often you will see in texts on category theory that this means we can 'do away with the objects' and really just think about the arrows. The final axiom, *the axiom of associativity*, states that composition of arrows with composition of arrows is unique

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "h\circ g\circ f"'] \arrow[dr] & c' \arrow[d, "g"] \arrow[dl] \\ c''' & c'' \arrow[l, "h"] \end{tikzcd} \tag{4}$$

[1] Note that this predicate is bound in c and id_c and free in f, g.

The idea is that one can 'chase the diagram' along any of the possible presented paths. Specifically this shows that $h\circ(g\circ f) = (h\circ g)\circ f$. This implies that there is only one way to compose arrows with compositions of arrows[2]. If one knows a little bit of elementary algebra, he will observe that this implies that the collection of arrows in a category forms a *semigroup*. That is, a collection of objects with an associative binary operation. If we restrict to arrows f such that $\mathrm{dom}[f] = \mathrm{cod}[f] = c$, then this collection becomes a *monoid* (with identity id_c). These sorts of examples become useful in the considerations of what follows.

We can now define some basic properties of arrows. For example, it is often useful to know the degree to which some map is invertible. The strongest case is when some map f is *exactly* invertible. This means that if $f : c \to c'$ then there is a $g : c' \to c$ such that $g\circ f = id_c$ and $f\circ g = id_{c'}$. We can depict this diagramatically like so

$$\begin{tikzcd} c \arrow[loop left, "id_c"] \arrow[r, bend left, "f"] & c' \arrow[l, bend left, "g"] \arrow[loop right, "id_{c'}"] \end{tikzcd} \tag{5}$$

where one need only chase the diagram to see that the two equations hold. When this is the case, we say f is *iso*, g is the *(two-sided) inverse* of f, and that c and c' are *isomorphic* and we write $c \cong c'$. This induces an *equivalence relation* (reflexive, symmetric, transitive) on objects accordingly. It is easy to show that $c \cong c$, $c \cong c' \Leftrightarrow c' \cong c$, and $c \cong c'$, $c' \cong c''$ imply $c \cong c''$. Strictly speaking, these are elementary theorems which *induce* the relation '$\cong$' and can be proven immediately from the definitions and axioms listed above. For example, $c \cong c$ follows just by taking $f = g = id_c$ and then applying the identity axiom followed by the composition axiom. When a two-sided inverse exists, it is unique

Theorem 2.1. *Let f be iso with two-sided inverses g, g'. Then $g = g'$. We write f^{-1} for the (unique) two-sided inverse of f.*

Proof. We exhibit the diagrams directly.

$$\begin{tikzcd} c' \arrow[r, "g'"] \arrow[d] & c \arrow[d, "f"] \arrow[dl, "id_c = g\circ f"'] \\ c & c' \arrow[l, "g"] \end{tikzcd} \qquad \begin{tikzcd} c' \arrow[r, "g'"] \arrow[d] \arrow[dr, "f\circ g' = id_{c'}"'] & c \arrow[d, "f"] \\ c & c' \arrow[l, "g"] \end{tikzcd} \tag{6}$$

These diagrams exhibit the two necessary equations. By the associativity axiom, the arrow along the left edge in either diagram is unique. By employing the composition axiom and the definition of an iso arrow, these two diagrams exhibit the fact $g\circ(f\circ g') = g\circ id_{c'} = g = (g\circ f)\circ g' = id_c\circ g' = g'$, so that $g = g'$. □

In general, we will exhibit diagrams to indicate proofs. In principle, this means we can reduce everything to calculations on diagrams. This will leave us with immense freedom since we can then think: *for those diagrams such that...* In the long run, we'd like to argue that these essentially exhibit local differential calculations that encode local processes. Whenever these can be extended over the whole category, they become *global* and therefore indicate a fundamental property about the category itself. For example, this theorem demonstrates the global property that whenever f is iso in a category, its inverse is unique (everywhere in the category; i.e., whenever f appears in a diagram, it can be uniquely inverted regardless of the diagram it appears in).

We can relax the condition of invertibility. Since in general $f\circ g \neq g\circ f$, we can consider the two (distinct) diagrams

$$\begin{tikzcd} & c' \arrow[dr, shift left, "k"] \arrow[dr, shift right, "h"'] & \\ c \arrow[ur, "f"] \arrow[rr, "h\circ f = k\circ f"'] & & c'' \end{tikzcd} \qquad \begin{tikzcd} & c' \arrow[dr, "g"] & \\ c \arrow[ur, shift left, "h"] \arrow[ur, shift right, "k"'] \arrow[rr, "g\circ h = g\circ k"'] & & c'' \end{tikzcd} \tag{7}$$

Given these two conditions, we can ask when, for example, $h\circ f = k\circ f$ implies that $h = k$. *When* this is the case, we call f *epic* (or *right-cancellable*). In particular, this is a reasoning *on diagrams*, which means that *whenever* this circumstance appears, it is the case that f is right-cancellable. Logically, this says that f is epic if and only if for all h, k such that if $h\circ f = k\circ f$, then $h = k$. We claim the exact same assertion regarding the second diagram, but in the opposite order, i.e. if for all h, k, whenever $g\circ h = g\circ k$ this implies $h = k$, then we say g is *monic*. When this is the case, we often use the following notation on the heads/tails of arrows

[2] Because (2) implies $g\circ f$ is unique, we might actually assert that (4) is an elementary *theorem*, but actually it has some non-trivial content. This is because to 'prove' this axiom, one needs to leverage the fact that one is contracting along the two distinct diagonals. The fact that these *contractions* commute is really the novel content of the axiom.

$$
\begin{tikzcd}
 & c' \arrow[dr, shift left, "k"] \arrow[dr, shift right, "h"'] & \\
c \arrow[ur, two heads, "f"] \arrow[rr, "h\circ f=k\circ f"'] & & c''
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & c' \arrow[dr, tail, "g"] & \\
c \arrow[ur, shift left, "h"] \arrow[ur, shift right, "k"'] \arrow[rr, "g\circ h=g\circ k"'] & & c''
\end{tikzcd}
$$

If we think about this in the manner of a *diagrammatic sequence*, we can observe

$$
\begin{tikzcd}
 & c' \arrow[dr, shift left, "k"] \arrow[dr, shift right, "h"'] & \\
c \arrow[ur, two heads, "f"] \arrow[rr, "h\circ f=k\circ f"'] & & c''
\end{tikzcd}
\qquad \rightsquigarrow \qquad
c' \xrightarrow{h=k} c''
$$

where we are using '⇝' to denote the fact that we have applied a sequence of definitions, axioms, and theorems that results in a reduced diagram[3]. This case follows directly from the definition. It is a useful fact that all isos are monic-epic. The converse does not hold, however, and we will exhibit a counter-example.

Theorem 2.2. *All isos are monic-epic.*

Proof. The proof is exhibited again by contractions on the following two diagrams together with the hypotheses $f \circ h = f \circ k$ and $h' \circ f = k' \circ f$ (that is, *assume* these were to hold as in the diagrams below, then...)

$$
\begin{tikzcd}
c \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] \arrow[dr, bend right, "f\circ h=f\circ k"'] & c' \arrow[d, bend left, "f"] \\
 & c'' \arrow[u, bend left, "f^{-1}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
c \arrow[r, bend left, "f"] \arrow[dr, bend right, "h'\circ f=k'\circ f"'] & c' \arrow[l, bend left, "f^{-1}"] \arrow[d, shift right, "h'"'] \arrow[d, shift left, "k'"] \\
 & c''
\end{tikzcd}
$$

For the left-hand diagram, just compose $c \to c'' \to c'$ while observing the edge $c \to c''$ is unique, and the right-hand diagram compose $c' \to c \to c''$ while observing the same edge is unique, then contract via the axiom of composition. ◻

On the other hand, we can indicate monic-epics which are not iso. Consider the following

Example 2.3. Let **2** be the category

$$
\begin{tikzcd}
0 \arrow[loop above] \arrow[r] & 1 \arrow[loop above]
\end{tikzcd}
$$

Then the arrow $0 \to 1$ is monic-epic, since the only pair of arrows to consider on either side are the identities. On the other hand, this map has no inverse so it is not iso.

When monic-epics are iso, the category is called *balanced*. We can modify the definition slightly and recover the intuition that monic-epics should be isos, but we require one more property. An arrow f has a *right inverse* if there is an arrow g such that $f \circ g = id$. Any arrow with a right inverse is epic; it is easy to construct the diagrams. Syntactically this can be shown as $h \circ f = k \circ f \Rightarrow h \circ f \circ g = k \circ f \circ g \Rightarrow h = k$. Such an arrow will be called *split epic*. Thus every split epic is epic but not conversely. We get the

Theorem 2.4. *An arrow is iso if it is monic and split epic.*

Proof. Let f be monic and split epic. Then $f \circ h = f \circ k \Rightarrow h = k$ since f is monic. Thus if $f \circ g = id$ then g is unique. Since f is split epic, g exists. Then $f \circ g \circ f = f \circ id \Rightarrow g \circ f = id$, so that $g = f^{-1}$ is the two-sided inverse of f. ◻

so that as a corollary, we have

Corollary 2.5. *In any category, the arrow f is iso if and only if it is monic and split epic.*

Next we discuss some unique objects. If it is the case that for every object c, there is a unique arrow $c \dashrightarrow 1$, then 1 is called a *terminal* object. Conversely, if for every c there is a unique arrow $0 \dashrightarrow c$, then 0 is called *initial*. If an object z is both initial and terminal, it is called a *zero* object. Initial and terminal objects are unique up to isomorphism. If a category has a zero object, then all initial and terminal objects are zero objects. It is easy to prove all of these statements.

[3] We will see later that this is essentially what is called a 'meta-operator'–it is not explicit to the theory but is employed in the reasoning 'about' the theory from outside of it. Compare, for example, [39].

Lastly, we cover a couple of key results. First, monics, epics, and isos compose. It is easy to check the case of monics and epics. Composition of isos is given by chasing the diagram

$$
\begin{tikzcd}
c \arrow[r, "f"] \arrow[dr, "id_c"] & c' \arrow[r, "g"] \arrow[d, "f^{-1}"] \arrow[dr, "id_{c'}"] & c'' \arrow[d, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"] & \\
 & c \arrow[r, "f"] & c' \arrow[r, "f^{-1}"] & c
\end{tikzcd}
$$

When we discuss functors later, this while imply that the subcategory of a given category restricted to monics/epics/isos is well-defined because these compose. This is the categorial analogue of, say, the center of an arbitrary group. Additionally, the important elementary

Theorem 2.6. *If $f : 1 \to c$ exists for some object c, then f is monic*

Proof. Witness the diagram

$$
\begin{tikzcd}
 & 1 \arrow[dr, "f"] & \\
c' \arrow[ur, shift left, "h"] \arrow[ur, shift right, "k"'] \arrow[rr] & & c
\end{tikzcd}
$$

and observe that $h = k$ because 1 is terminal, so that if $f \circ h = f \circ k$, then $h = k$. □

This will be important when discussing topoi. There are also some important 'factoring' theorems

Theorem 2.7. *In any category **C**,*

1. *If f, g are monic, epic, or iso, then so is $g \circ f$ when it is defined.*
2. *If $g \circ f$ is monic, so is f. Dually, if $g \circ f$ is epic, then so is g.*
3. *If $g \circ f$ is iso, then f is monic and g is epic.*

Proof. We'll proceed in our proofs according to the above enumeration

1. Since the proof of each is the same (up to duality), we prove just the monic case. Let f, g be monic and suppose $(g \circ f) \circ h = (g \circ f) \circ k$. Then since g is monic, $f \circ h = f \circ k$, and since f is monic, $h = k$, but then $(g \circ f) \circ h = (g \circ f) \circ k \Rightarrow h = k$ and we're done.
2. We just need to show the first part of the statement; by duality the second part holds. Suppose $g \circ f$ is monic and let $f \circ h = f \circ k$. Compose with g on the left and apply the axiom of associativity. Then use the fact that $g \circ f$ is monic and conclude $h = k$. But then $f \circ h = f \circ k \Rightarrow h = k$ and we're done.
3. Let $g \circ f$ be iso. Suppose $f \circ h = f \circ k$. Then compose with g on the left and apply the iso condition. Conclude $h = k$. Suppose $h \circ g = k \circ g$. Compose with f on the right. Use the iso condition to invert the composition after applying the associativity axiom. Conclude $h = k$.

□

2.2 Products, Equalizers, and Their Duals

Now we consider some constructions on objects that play the role of operations in traditional mathematics. One of the simplest operations is multiplication, which has the categorial analogue in products. A *product* of two objects in a category is an object and two arrows satisfying the *span* $c \leftarrow c \times c' \rightarrow c'$ with the special property

$$
\begin{tikzcd}
 & c'' \arrow[dl, "f"'] \arrow[d, dashed, "(f,g)"] \arrow[dr, "g"] & \\
c & c \times c' \arrow[l, "\pi_c"] \arrow[r, "\pi_{c'}"'] & c'
\end{tikzcd}
\tag{8}
$$

Given any object c'' and span $c \leftarrow c'' \rightarrow c'$, the arrow $c'' \dashrightarrow c \times c'$ is unique. When $f : c'' \rightarrow c$ and $g : c'' \rightarrow c'$, we denote the arrow $(f, g) : c'' \dashrightarrow c \times c'$ and the arrows $\pi_c, \pi_{c'} : c \times c' \rightarrow c$ are called *projections*. Notice in particular that the projections are *part of the definition*. So far, we've only considered constructions either on arrows or on objects. We now relax this condition and consider constructions involving both. It is straight-forward to see that it is necessary to specify both the object and arrows, for let $i : c \times c' \xrightarrow{\sim} c \times c'$ be any automorphism of $c \times c'$ to itself and consider the diagram

$$
\begin{tikzcd}
& c'' \arrow[dl, "f"'] \arrow[d, dashed, "{(f,g)}"] \arrow[dr, "g"] & \\
c & c \times c' \arrow[l, "\pi_c"'] \arrow[r, "\pi_{c'}"] \arrow[d, "i"] & c' \\
& c \times c' \arrow[ul, "\pi'_c"] \arrow[ur, "\pi'_{c'}"'] &
\end{tikzcd}
$$

The we observe $\pi_c = \pi'_c \circ i$ and $\pi_{c'} = \pi'_{c'} \circ i$ for the same (f, g) so that the projections are not automatically selected upon selecting the product object. More generally, products are defined only up to isomorphism. Replace $\mathrm{cod}[i] = c \times c'$ above with any p' and $i : c \times c' \xrightarrow{\sim} c \times c'$ with an arbitrary iso $\phi : c \times c' \xrightarrow{\sim} p'$ and observe that since ϕ is fixed and (f, g) is unique that $\phi \circ (f, g)$ is unique[4]. The projections are defined then by $\pi_c \circ \phi^{-1}$ and $\pi_{c'} \circ \phi^{-1}$. This says

Theorem 2.8. *Let $c \times c'$ be a product object with projections $\pi_c, \pi_{c'}$. Then given any iso $\phi : c \times c' \xrightarrow{\sim} p'$, p' is also a product with projections $\pi_c \circ \phi^{-1} : p' \to c$ and $\pi_{c'} \circ \phi^{-1} : p' \to c'$.*

We can also show the converse

Theorem 2.9. *Let p, π_c, $\pi_{c'}$ and q, π'_c, $\pi'_{c'}$ be two products. Then $p \cong q$.*

Proof. This follows by witnessing the diagrams

$$
\begin{tikzcd}
& p \arrow[dl, "\pi_c"'] \arrow[d, dashed] \arrow[dr, "\pi_{c'}"] & \\
c & q \arrow[l, "\pi'_c"'] \arrow[r, "\pi'_{c'}"] \arrow[d, dashed] & c' \\
& p \arrow[ul, "\pi_c"] \arrow[ur, "\pi_{c'}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& q \arrow[dl, "\pi'_c"'] \arrow[d, dashed] \arrow[dr, "\pi'_{c'}"] & \\
c & p \arrow[l, "\pi_c"'] \arrow[r, "\pi_{c'}"] \arrow[d, dashed] & c' \\
& q \arrow[ul, "\pi'_c"] \arrow[ur, "\pi'_{c'}"'] &
\end{tikzcd}
$$

then contracting along the center lines. Since $p \to p$ and $q \to q$ are unique, these must be the identities. Consequently $p \cong q$. ◻

We can form (unique) maps between products given by

$$
\begin{tikzcd}
c \arrow[d, "h"] & c \times c' \arrow[l, "\pi_c"'] \arrow[r, "\pi_{c'}"] \arrow[d, dashed] & c' \arrow[d, "k"] \\
d & d \times d' \arrow[l, "\pi_d"'] \arrow[r, "\pi_{d'}"] & d'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& c \times c' \arrow[dl, "h \circ \pi_c"'] \arrow[d, dashed, "h \times k"] \arrow[dr, "k \circ \pi_{c'}"] & \\
d & d \times d' \arrow[l, "\pi_d"'] \arrow[r, "\pi_{d'}"] & d'
\end{tikzcd}
$$

where we define $h \times k = (h \circ \pi_c, k \circ \pi_{c'})$ to be the unique arrow along the center line. Clearly if $c \cong d$ and $c' \cong d'$, then $c \times c' \cong d \times d'$. We include some additional properties

Proposition 2.10. *Let $i : d'' \to c''$, $h : c'' \to c$, $k : c'' \to c'$. Then $(h, k) \circ i = (h \circ i, k \circ i)$.*

Proof. Witness the following diagram sequence

$$
\begin{tikzcd}
& d'' \arrow[d, dashed, "i"] & \\
& c'' \arrow[dl, "h"'] \arrow[d, dashed] \arrow[dr, "k"] & \\
c & c \times c' \arrow[l] \arrow[r] & c'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& d'' \arrow[d, dashed, "i"] \arrow[ddl, "h \circ i"'] \arrow[ddr, "k \circ i"] & \\
& c'' \arrow[dl, "h"] \arrow[d, dashed] \arrow[dr, "k"'] & \\
c & c \times c' \arrow[l] \arrow[r] & c'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& d'' \arrow[dl, "h \circ i"'] \arrow[d, dashed] \arrow[dr, "k \circ i"] & \\
c & c \times c' \arrow[l] \arrow[r] & c'
\end{tikzcd}
$$

[4] To see this, suppose $u, u' : c'' \to p'$ are distinct arrows in the relevant product diagram and $u \neq u'$. Then $u = \phi \circ (f, g) = u'$. Since (f, g) is unique, $\phi^{-1} \circ u = \phi^{-1} \circ u'$ so $u = u'$ since ϕ is iso.

This reduction sequence establishes the result. $\square$

Proposition 2.11. *Let $h : c'' \to c$, $k : c'' \to c'$, $f : c \to d$, $g : c' \to d'$. Then $(f \times g) \circ (h, k) = (f \circ h, g \circ k)$.*

Proof. Witness the reduction sequence

$$
\begin{tikzcd}
 & c'' \arrow[dl, "h"'] \arrow[d] \arrow[dr, "k"] & \\
c \arrow[d, dashed, "f"] & c \times c' \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "g"] \\
d & d \times d' \arrow[l] \arrow[r] & d'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & c'' \arrow[dl, "f \circ h"'] \arrow[d, dashed] \arrow[dr, "g \circ k"] & \\
d & d \times d' \arrow[l] \arrow[r] & d'
\end{tikzcd}
$$

The uniqueness of the center line guarantees the result. $\square$

Proposition 2.12. *Let $f : b \to c$, $h : c \to d$, $g : b' \to c'$, $k : c' \to d'$. Then $(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g)$.*

Proof. As always, witness the reduction sequence

$$
\begin{tikzcd}
b \arrow[d, dashed, "f"] & b \times b' \arrow[l] \arrow[r] \arrow[d, dashed] & b' \arrow[d, dashed, "g"] \\
c \arrow[d, dashed, "h"] & c \times c' \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "k"] \\
d & d \times d' \arrow[l] \arrow[r] & d'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
b \arrow[d, dashed, "h \circ f"] & b \times b' \arrow[l] \arrow[r] \arrow[d, dashed] & b' \arrow[d, dashed, "k \circ g"] \\
d & d \times d' \arrow[l] \arrow[r] & d'
\end{tikzcd}
$$

The result follows from the definition of $f \times g$. $\square$

Proposition 2.13. *Let $f : c \to d$ and $g : c' \to d'$. Then $(f \times id_{d'}) \circ (id_c \times g) = (id_{c'} \times g) \circ (f \times id_d)$.*

Proof. Witness the dual reduction sequence

$$
\begin{tikzcd}
c \arrow[d, dashed, "id_c"] & c \times c' \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "g"] \\
c \arrow[d, dashed, "f"] & c \times d' \arrow[l] \arrow[r] \arrow[d, dashed] & d' \arrow[d, dashed, "id_{d'}"] \\
d & d \times d' \arrow[l] \arrow[r] & d'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
c \arrow[d, dashed, "f"] & c \times c' \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "g"] \\
d & d \times d' \arrow[l] \arrow[r] & d'
\end{tikzcd}
\quad \leftsquigarrow \quad
\begin{tikzcd}
c \arrow[d, dashed, "f"] & c \times c' \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "id_{c'}"] \\
d \arrow[d, dashed, "id_d"] & d \times c' \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "g"] \\
d & d \times d' \arrow[l] \arrow[r] & d'
\end{tikzcd}
$$

$\square$

Proposition 2.14. $c \times (c' \times c'') \cong (c \times c') \times c''$

Proof. Witness the diagram (only containing the evident projections and identities)

$$
\begin{tikzcd}
 & c \times (c' \times c'') \arrow[dl] \arrow[dr] \arrow[rrrr, dash, dashed] & & & & (c \times c') \times c'' \arrow[dl] \arrow[dr] & \\
c \arrow[dr] & & c' \times c'' \arrow[d] \arrow[drrr] & & c \times c' \arrow[dlll] \arrow[dll] & & c'' \arrow[dl] \\
 & c & c' & & & c'' &
\end{tikzcd}
$$

Since both products reduce to the same collection of objects, it implies the existence of a pair of (unique) arrows $c \times (c' \times c'') \dashrightarrow (c \times c') \times c''$ and $(c \times c') \times c'' \dashrightarrow c \times (c' \times c'')$. Composing and contracting implies these are inverses, which establishes the isomorphism. $\square$

Proposition 2.15. $c \times c' \cong c' \times c$

Proof. Witness

$$
\begin{tikzcd}
c \arrow[d, dashed, "id_c"] & c \times c' \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "id_{c'}"] \\
c \arrow[d, dashed, "id_c"] & c' \times c \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "id_{c'}"] \\
c & c \times c' \arrow[l] \arrow[r] & c'
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[d, dashed, "id_c"] & c' \times c \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "id_{c'}"] \\
c \arrow[d, dashed, "id_c"] & c \times c' \arrow[l] \arrow[r] \arrow[d, dashed] & c' \arrow[d, dashed, "id_{c'}"] \\
c & c' \times c \arrow[l] \arrow[r] & c'
\end{tikzcd}
$$

These two together furnish the isomorphism[5]. This isomorphism will be called the *twist* map. □

This establishes a collection of results that allow for basic algebra of products. Notice that products (up to isomorphism) behave as they do in the usual algebraic fashion: products are associate and commutative. It is easy to show for a terminal object 1, $1 \times c \cong c$, so that the terminal object behaves like a unit. If we are in a category with all finite products and a terminal element, this turns the collection of objects into a monoid. It would be interesting to ask how we might turn this into a group (where '=' is replaced by '≅'). This is a tangential problem which we table for now.

In addition to the notion of products, there is also a formalized notion of restriction. This appears in the form of an *equalizer* for a parallel pair of arrows. Let $f,\ g : c \to c'$ be a pair of arrows from c to c'. Then an equalizer is a pair (e, i) where e is an object and $i : e \to c$ is an object such that $f \circ i = g \circ i$ and for any other object d and arrow $h : d \to c$ such that $f \circ h = g \circ h$, there is a unique arrow $u : d \dashrightarrow e$ satisfying the following diagram

$$\begin{tikzcd} e \arrow[r, "i"] & c \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & c' \\ d \arrow[u, dashed, "u"] \arrow[ur, "h"'] & & \end{tikzcd} \tag{9}$$

It is easy to show along the same lines of two products being isomorphic that two equalizers for a pair of arrows are also isomorphic. There is also the important

Theorem 2.16. *Every equalizer is monic.*

Proof. If this is true, then for an equalizer i for a parallel pair of arrows $f, g : c \to c'$, whenever $i \circ h = i \circ k$ then $h = k$. Let h, k be arrows such that $i \circ h = i \circ k$. Then witness the following reduction

$$\begin{tikzcd} & e \arrow[dr, "i"] & \\ d \arrow[ur, shift left, "h"] \arrow[ur, shift right, "k"'] \arrow[rr, "i\circ h = i\circ k"'] & & c \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & e \arrow[dr, "i"] & & \\ d \arrow[ur, shift left, "h"] \arrow[ur, shift right, "k"'] \arrow[rr, "i\circ h = i\circ k"'] & & c \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & c' \end{tikzcd}$$

By the definition of an equalizer, and since the arrow $d \to c$ is unique in the context of the diagram, the arrow $d \to e$ must also be unique, so that $h = k$. □

We also saw previously that not every monic-epic is iso. But it turns out that epic equalizers *are* iso

Theorem 2.17. *Let (e, i) be an equalizer for $f, g : c \to b$. Then the following are equivalent: a) i is epic; b) $f = g$; c) i is iso*

Proof. We proceed by $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (a)$. $\big((a) \Rightarrow (b)\big)$ Since i is an equalizer, $f \circ i = g \circ i$. But i is epic, so $f = g$. $\big((b) \Rightarrow (c)\big)$ Since $f = g$, the diagram

$$\begin{tikzcd} e \arrow[r, "i"] & c \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & c' \\ & c \arrow[u, "id_c"'] \arrow[ul, dashed, "u"] & \end{tikzcd}$$

holds. But then $i \circ u = id_c$, so i has a right inverse, and therefore is split-epic, so it is iso. $\big((c) \Rightarrow (a)\big)$ Every iso is epic. □

It is perhaps interesting to note that categories which only have *one* arrow between any given pair of objects immediately has products and equalizers. For example, it's easy to show in the category **2** considered above, $0 \times 0 = 0$, $0 \times 1 = 0$, and $1 \times 1 = 1$. Since the set of arrows for each pair of objects only has one element, every arrow that precomposes with a given arrow is an equalizer.

A notion in category theory is said to be *dual* to some other notion if the notion in question is obtained from the other notion merely by inverting the direction of the arrows. Any such notion is given the prefix *co-* to denote this fact. This means we obtain *coproducts* and *coequalizers* by inverting the directions of the arrows. Clearly epics are dual to monics and isos are self-dual. A *coproduct* is an object defined by a *cospan* $c \to c + c' \leftarrow c'$ such that for any other object c'' and cospan $c \to c'' \leftarrow c'$, there is a unique arrow $c + c' \dashrightarrow c''$ satisfying the diagram

[5] By our definition, the span for $c' \times c$ really should be $c' \leftarrow c' \times c \to c$. But then we have arrows to c', c from $c \times c'$, we just need to 'twist' them around (hence the name). By the definition of products, this means there *must be* a map to $c' \times c$ from $c \times c'$. This establishes the witness and the fact that it is an isomorphism follows.

$$
\begin{tikzcd}
c \arrow[r, "i_c"] \arrow[dr, "f"'] & c + c' \arrow[d, dashed, "\begin{pmatrix} f \\ g \end{pmatrix}"] & c' \arrow[l, "i_{c'}"'] \arrow[dl, "g"] \\
& c'' &
\end{tikzcd}
\tag{10}
$$

Similarly, a *coequalizer* for a parallel pair of arrows $f, g : c' \to c$ is a pair (z, q) such that $q \circ f = q \circ g$ and for any other arrow $h : c \to c''$ satisfying the same condition, there is a unique arrow $u : z \dashrightarrow c''$ satisfying

$$
\begin{tikzcd}
z \arrow[dr, dashed] & & c \arrow[ll, "q"'] \arrow[dl, "h"] & c' \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] \\
& c'' & &
\end{tikzcd}
\tag{11}
$$

Note that by theorem (2.16), we immediately have that coequalizers are epic. The duality principle applies to all the results above on product arrows too; we just need to swap domain and codomain. If a category has all finite products and is closed under duality, it also has all coproducts. Surprisingly, this isn't sufficient obtain the distributive property; in fact, you need exponentials to obtain this property, which means that a category with all finite products and closed under duality is a weaker condition than the ring condition in algebra. We'll see later that the idea of duality can be phrased in terms of a mathematical object called a *functor*.

2.3 Limits and Colimits

We can generalize the notion that leads to objects such as products, equalizers, and their duals. The common property shared in each case is the existence of a *unique* arrow as defining the way in which all other arrows factor through the given arrow, whenever such arrows exist. This is phrased as follows: define a *diagram* to be any collection of objects and arrows such that all arrows appearing in the diagram satisfy the axioms ((2)), ((3)), and ((4)). The diagram will be called *finite* whenever the number of arrows is finite and *compact* whenever the number of objects is finite. Obviously every finite diagram is compact, but not conversely. Let Δ be a diagram. A *cone* over Δ, $c \downarrow \Delta$ is an object c together with a collection of arrows $\{f\}$ such that for every object $c' \in \mathrm{Obs}(\Delta)$ there is an arrow $f_{c'} : c \to c'$ from c to c' and for every arrow $g : c' \to c''$, $f_{c''} = g \circ f_{c'}$ whenever $g \in \mathrm{Ars}(\Delta)$. A *limit* of a diagram Δ is a cone $c \downarrow \Delta$ together with the fact that for any other cone $u : c'' \downarrow \Delta$ there is a *unique* arrow $c'' \dashrightarrow c$ such that for every $c' \in \mathrm{Obs}(\Delta)$, the following diagram holds

$$
\begin{tikzcd}
c'' \arrow[rr, dashed, "u"] \arrow[dr] & & c \arrow[dl] \\
& c' &
\end{tikzcd}
\tag{12}
$$

We can now rephrase definitions of limits and consider generic limiting objects. For example, a cone over the empty diagram is just an object, so that the limit of the empty diagram is the *terminal object*. For a discrete number of objects with no arrows, a cone over them is an object with a single arrow to every object. The limit of this diagram is a *finite product* of these objects together. A cone for a parallel pair of arrows $c \rightrightarrows c'$ is an object $e \to c \rightrightarrows c'$ (following by the axiom of composition). The limit of this diagram is an *equalizer*. There is one other type of elementary diagram that needs to be considered, namely the cospan $c \to c' \leftarrow c''$. A cone for this diagram is

$$
\begin{tikzcd}
d \arrow[r] \arrow[d] & c'' \arrow[d] \\
c \arrow[r] & c'
\end{tikzcd}
$$

and a limit for this diagram is called a *pullback*, given by the condition

$$
\begin{tikzcd}
p \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& d \arrow[r] \arrow[d] & c'' \arrow[d] \\
& c \arrow[r] & c'
\end{tikzcd}
\tag{13}
$$

We can actually show that we can recover monics, equalizers, and products given pullbacks and terminal objects. To obtain monics, we observe that $f : c \to c'$ is monic if the following is a pullback

$$
\begin{tikzcd}
c \arrow[r, "id_c"] \arrow[d, "id_c"'] & c \arrow[d, "f"] \\
c \arrow[r, "f"] & c'
\end{tikzcd}
$$

which is easy to see. Let $h, k : t \rightrightarrows c$ be a test object and pair of arrows making the outer square commute. Then $f \circ h = f \circ k$. By the pullback condition, there is a unique $u : t \dashrightarrow c$ making $id_c \circ u = u = h = k$. But then $f \circ h = f \circ k$ implies $h = k$ and we're done. If we take the central object of the cospan to be the terminal element, then we have a diagram that appears as

$$
\begin{tikzcd}
d \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& c'' \arrow[r] \arrow[d] & c' \arrow[d, dashed] \\
& c \arrow[r, dashed] & 1
\end{tikzcd}
\cong
\begin{tikzcd}
d \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& c \times c' \arrow[r] \arrow[d] & c' \\
& c &
\end{tikzcd}
$$

where we can justify the isomorphism intuitively since every object in the diagram over the terminal element has exactly one arrow to 1. This means that there is a 1-1 correspondence between the two diagrams, implying that the pullback generates products. Equalizers can be obtained by the construction of a pullback as

$$
\begin{tikzcd}
d \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr, "h"'] & e \arrow[r] \arrow[d, tail, "i"] & c' \arrow[d, tail, "(id_{c'}, id_{c'})"] \\
& c \arrow[r, "(f,g)"] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & c' \times c' \arrow[d, "\pi_{c'}"] \\
& & c'
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
d \arrow[r, dashed] \arrow[dr, "h"'] & e \arrow[d, tail, "i"] & \\
& c \arrow[dr, shift left, "f"] \arrow[dr, shift right, "g"'] & \\
& & c'
\end{tikzcd}
$$

Conversely, equalizers imply pullbacks, so that a category with finite products and equalizers is a category with pullbacks and a terminal element. To see the initial assertion, observe the reduction

$$
\begin{tikzcd}
& c'' \arrow[d, "g"] \\
c' \arrow[r, "f"'] & c
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
e \arrow[dr, tail] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& c' \times c'' \arrow[r, "\pi_{c''}"] \arrow[d, "\pi_{c'}"'] \arrow[dr, shift left, "g \circ \pi_{c''}"] \arrow[dr, shift right, "f \circ \pi_{c'}"'] & c'' \arrow[d, "g"] \\
& c' \arrow[r, "f"'] & c
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
e \arrow[dr, tail] & & \\
& c' \times c'' \arrow[dr, shift left, "g \circ \pi_{c''}"] \arrow[dr, shift right, "f \circ \pi_{c'}"'] & \\
& & c
\end{tikzcd}
$$

where given the cospan $c' \to c \leftarrow c''$, we can form the product $c' \times c''$ which gives rise to the pair of arrows $c' \times c'' \rightrightarrows c$ as in the diagram. Then we can use the fact that the category has equalizers to construct the monic arrow $e \to c' \times c''$. This is a limit for the last diagram so that it is a limit for the first one (any arrow $t : T \to c' \times c''$ defines a pair of arrows to c by the relevant compositions, and the limit condition follows). The same sort of idea generalizes to arbitrary categories and leads to the important

Theorem 2.18. *A category has all finite limits if and only if it has pullbacks and a terminal element.*

We need to prove that a category with equalizers and products has all finite limits. Then since terminal objects and pullbacks are equivalent to products and equalizers, then the theorem follows.

Proof. Let the diagram Δ occur in a category $\mathbf{C}$ such that $\mathbf{C}$ has finite products and equalizers for every pair of parallel arrows. If $\prod_{d \in \mathrm{Obs}(\Delta)} d$ is the product of all objects appearing in the diagram, then we have a map from the product to every object in the diagram given by

$$\begin{tikzcd}
 & d' \\
\prod_{d\in\mathrm{Obs}(\Delta)} d \arrow[ur, "\pi_{d'}"] \arrow[dr, "\pi_{d''}"'] & \vdots \\
 & d''
\end{tikzcd}$$

which is *almost* a cone but for the fact that, given any arrow $f : d' \to d''$, $\pi_{d''} \neq f \circ \pi_{d'}$ in general. This latter observation implies the existence of a second arrow for any object that is the codomain of an arrow appearing in Δ. If $d' = \mathrm{dom}[f]$ and $d'' = \mathrm{cod}[f]$, then

$$\begin{tikzcd}
\prod_{d\in\mathrm{Obs}(\Delta)} d \arrow[d, "\pi_{d'}"] \arrow[dr, "f\circ\pi_{d'}"] & \\
d' \arrow[r, "f"'] & d''
\end{tikzcd}$$

Since a cone requires the commutativity condition, if we precompose with an equalizer for $f \circ \pi_{d'}$ and $\pi_{d''}$, we obtain a cone

$$\begin{tikzcd}
e \arrow[r, tail, "i"] & \prod_{d\in\mathrm{Obs}(\Delta)} d \arrow[r, shift left, "\pi_{d''}"] \arrow[r, shift right, "f\circ\pi_{d'}"'] & d''
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & d'' \\
e \arrow[ur, "\pi_{d''}\circ i = f\circ\pi_{d'}\circ i"] \arrow[dr, "\pi_{d'}\circ i"'] & \\
 & d' \arrow[uu, "f"']
\end{tikzcd}$$

for all arrows in the diagram, so that this is a cone. (Note in passing that e is a subobject of $\prod_{d\in\mathrm{Obs}(\Delta)} d$.) Given any other cone

$$\begin{tikzcd}
 & T \arrow[dl, "t_{d'}"'] \arrow[dr, "t_{d''}"] & \\
d' \arrow[rr, "f"] & & d''
\end{tikzcd}$$

This is universal if there is a unique $u : T \dashrightarrow e$. Since the category has products, this T has the universal property

$$\begin{tikzcd}
 & T \arrow[dl, "t_{d'}"'] \arrow[d, dashed, "t"] \arrow[dr, "t_{d''}"] & \\
d' & \prod_{d\in\mathrm{Obs}(\Delta)} d \arrow[l, "\pi_{d'}"] \arrow[r, "\pi_{d''}"'] & d''
\end{tikzcd}$$

But then this says that $t_{d''} = \pi_{d''} \circ t$. Furthermore, by the cone property, this has $t_{d''} = f \circ t_{d'}$. This yields the equation $f \circ \pi_{d'} \circ t = \pi_{d''} \circ t$, which is the same as the equalizing condition. This means there is a unique $u : T \dashrightarrow e$ in the map of cones. Then e is a limit for the diagram Δ. □

2.3.1 The Universal Mapping Property and Free Categories

The universal mapping property can be phrased like so: let c be an object in a category $\mathbf{C}$, and let $\mathcal{A} : \mathbf{C} \to \mathfrak{A}$ be a functor into a category with algebraic properties. Suppose there is a functor $U : \mathfrak{A} \to \mathbf{C}$. Then the *universal mapping property* is given by the pair of diagrams

$$\begin{tikzcd}
c \arrow[r, "i"] \arrow[dr, "f"'] & U(\mathfrak{A}(c)) \arrow[d, "|\phi_f|"] \\
 & U(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{A}(c) \arrow[d, dashed, "\phi_f"] \\
A
\end{tikzcd}
\tag{14}$$

so that ϕ_f is uniquely determined by f, i so that $f = |\phi_f| \circ i$. In particular, $|\phi_f| = U(\phi_f)$. (Obviously this section is not done yet.)

2.4 Functors and Natural Transformations

A *functor* is a map between categories which preserves categorial structure. That is, if $F : \mathbf{C} \to \mathbf{D}$ is defined by replacing the arrows and objects in (2), (3), and (4) by the image of each of the objects under F. In particular, F distributes over diagrams, so that

$$c \xrightarrow{f} c' \qquad \rightsquigarrow \qquad Fc \xrightarrow{Ff} Fc' \tag{15}$$

and is structure-preserving in the sense that if a diagram commutes, then the diagram of its image under F also commutes. Importantly, $F(id_c) = id_{Fc}$ for every object $c \in \mathrm{Obs}(\mathbf{C})$, which follows by the appropriate substitution above. In many regards, functors are generalizations of arrows, since they 'see' the structure within the category, although the arrows need not necessarily 'see' the structure within objects. Thus, we can generalize the notions of monic and epic along the following lines: a functor is *injective on objects* if whenever $Fc = Fc'$ then $c = c'$. The functor is *surjective on objects* if whenever $F : \mathbf{C} \to \mathbf{D}$ and $d \in \mathrm{Obs}(\mathbf{D})$, there is a $c \in \mathrm{Obs}(\mathbf{C})$ with $Fc = d$. Functors can be injective or surjective on objects without being so on arrows, exhibited by the following

$$\begin{tikzcd}
c \arrow[r, "F"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \overline{0} \arrow[d, "\leq"] \\
c' \arrow[r, "F"] & \overline{1}
\end{tikzcd}$$

where we've taken $c \rightrightarrows c'$ as a category of two objects and two arrows. The functor F is bijective on objects but not on arrows, since both f and g are sent to $\leq$. A functor is *injective (resp. surjective) on arrows* if it satisfies the relevant condition on the arrows part of the functor.

Proposition 2.19. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Then if F is injective on arrows it is injective on objects. If it is surjective on arrows, it is surjective on objects.*

Proof. Consider the first part. If F is injective on arrows, then $Ff = Fg$ implies $f = g$ for all arrows $f, g : c \to c'$ in $\mathbf{C}$. Then if $Fc = Fc'$ it follows that $F(id_c) = F(id_{c'})$. By the injective property, $id_c = id_{c'}$ implying $c = c'$. For the second part, let F be surjective on arrows and consider $d \in \mathrm{Obs}(\mathbf{D})$. Then by the surjective property there is an f in $\mathbf{C}$ with $Ff = id_d$. If f is the identity, then we're done. On the other hand, if not, then the following diagram must commute

$$\begin{tikzcd}
c \arrow[rr, "f"] \arrow[dr, "F"'] & & c' \arrow[dl, "F"] \\
& d \arrow[loop below, "id_d"] &
\end{tikzcd}$$

which takes c and c' to d so that there is some object in $\mathbf{C}$ that maps to d through F. □

These conditions can be generalized. A functor is called *faithful* if for every pair of objects in $\mathbf{C}$, if $Ff, Fg \in [Fc, Fc']$ and $Ff = Fg$, then $f = g$ in $[c, c']$. A functor is called *full* if for every arrow $g \in [Fc, Fc']$ there is an arrow $f \in [c, c']$ with $g = Ff$. To see that these notions are proper generalizations, consider the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[dr, "id_{\mathbf{C}}"'] & \mathbf{C} + \mathbf{C} \arrow[d, "\nabla"] & \mathbf{C} \arrow[l] \arrow[dl, "id_{\mathbf{C}}"] \\
& \mathbf{C} &
\end{tikzcd}$$

This sends

$$\begin{tikzcd}
(c, 1) \arrow[r, "\nabla"] \arrow[d, "(f,1)"'] & c \arrow[d, "f"'] & (c, 2) \arrow[l, "\nabla"'] \arrow[d, "(f,2)"] \\
(c', 1) \arrow[r, "\nabla"] & c' & (c', 2) \arrow[l, "\nabla"']
\end{tikzcd}$$

which is faithful but not injective on arrows (since there are two copies of every diagram in $\mathbf{C} + \mathbf{C}$).

A *natural transformation* is a map $\eta : F \to G$ between a pair of functors $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$ such that, given any $f : c \to c'$ in $\mathbf{C}$, the following holds

$$\begin{tikzcd} c \arrow[d, "f"] & Fc \arrow[r, "\eta_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ c' & Fc' \arrow[r, "\eta_{c'}"] & Gc' \end{tikzcd} \tag{16}$$

where η_c is called the *component* of η at c. As with arrows, functors can come equipped with various structural properties–e.g., we can define a functor F to be monic if whenever a pair of functors G, G' satisfy $FG = FG'$ then $G = G'$. If there are a pair of natural transformations $\eta : F \to G$, $\epsilon : G \to F$ such that

$$\begin{tikzcd} c \arrow[dd, "f"] & Fc \arrow[r, "Ff"] \arrow[d, "\eta_c"'] & Fc' \arrow[d, "\eta_{c'}"'] & Gc \arrow[r, "Gf"] \arrow[d, "\epsilon_c"'] & Gc' \arrow[d, "\epsilon_{c'}"'] \\ & Gc \arrow[r, "Gf"] \arrow[d, "\epsilon_c"'] & Gc' \arrow[d, "\epsilon_{c'}"'] & Fc \arrow[r, "Ff"] \arrow[d, "\eta_c"'] & Fc' \arrow[d, "\eta_{c'}"'] \\ c' & Fc \arrow[r, "Ff"] & Fc' & Gc \arrow[r, "Gf"] & Gc' \end{tikzcd} \tag{17}$$

for all objects and arrows in the categories, then F and G are called *naturally equivalent* and the natural transformation $\eta = \epsilon^{-1}$ is called a *natural isomorphism* and we write $F \simeq G$.

A pair of categories $\mathbf{C}$, $\mathbf{D}$ will be said to be *isomorphic* if there are a pair of functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ such that $GF = id_{\mathbf{C}}$ and $FG = id_{\mathbf{D}}$ the identity functors. If instead these are equal up to natural isomorphism, meaning that there is a pair of functors F, G such that $GF \simeq id_{\mathbf{C}}$ and $FG \simeq id_{\mathbf{D}}$, this will be called *an equivalence of categories* and we write $\mathbf{C} \simeq \mathbf{D}$. There is a natural way to generalize this further with the concept of the adjunction, which we will discuss below.

Functors can act in essentially two ways: the above definition ((15)) sends an arrow to an arrow *in the same direction*. Such a functor is called a *covariant* functor. On the other hand, there is one other logical possibility–viz.

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "F"'] & c' \arrow[d, "F"] \\ Fc & Fc' \arrow[l, "Ff"] \end{tikzcd}$$

Whenever this happens, that the arrow f is ‘flipped’ to its dual by the functor F, the functor is called *contravariant*.

2.4.1 Cayley Representation of a Small Category

A category is *small* whenever $\mathrm{Ars}(\mathbf{C}) \in \mathrm{Obs}(\mathbf{Set})$, so that the collection of arrows is a set. It is *locally small* whenever for any given pair of objects $c, c' \in \mathrm{Obs}(\mathbf{C})$, the collection of arrows $[c, c']_{\mathbf{C}} = \{f | f : c \to c'\}$ is a set. As soon as $\mathrm{Ars}(\mathbf{C}) \in \mathbf{Set}$, the axioms of a category require that $\mathrm{Obs}(\mathbf{C}) \cong \{id_c | c \in \mathrm{Obs}(\mathbf{C})\}$ so that $\mathrm{Obs}(\mathbf{C}) \subseteq \mathrm{Ars}(\mathbf{C})$ up to isomorphism. In the case that a category $\mathbf{C}$ is small, there is a special pair of functors from the category into $\mathbf{Set}$ which allows for the definition of a pair of representations of the category within $\mathbf{Set}$, meaning that the category is embedded in a Cartesian closed category. In the case that the limits and colimits are not in the original category, this suggests a way to construct the additional algebraic structure that the original category lacks. The construction is as follows:

Theorem 2.20. *Let $\mathbf{C}$ be a small category. Then there are a pair of functors*

$$dom : \mathbf{C} \to \mathbf{Set}$$
$$cod : \mathbf{C} \to \mathbf{Set}$$

where the former is contravariant and the latter covariant in its arguments. This representation is an isomorphism of categories.

Proof. The isomorphism will follow immediately from the construction. Define the functors dom and cod as follows

$$\begin{aligned}
\mathrm{dom}(c) &= \{f | \mathrm{dom}[f] = c\} \\
\mathrm{dom}(f) &= - \circ f \\
\mathrm{cod}(c) &= \{f | \mathrm{cod}[f] = c\} \\
\mathrm{cod}(f) &= f \circ -
\end{aligned}$$

where $- \circ f$ and $f \circ -$ are pre- and post-composition with the arrow f respectively. We just need to show that these are functorial. Consider $f : c \to c'$. Then

$$\begin{tikzcd}
c \arrow[r, "\mathrm{dom}"] \arrow[d, "f"'] & \mathrm{dom}(c) \\
c' \arrow[r, "\mathrm{dom}"] & \mathrm{dom}(c') \arrow[u, "-\circ f"']
\end{tikzcd}$$

is defined by $g \in \mathrm{dom}(c') \mapsto g \circ f \in \mathrm{dom}(c)$. This is just the same thing as the reduction sequence

$$c' \xrightarrow{g} c'' \quad \rightsquigarrow \quad \begin{tikzcd}
 & c' \arrow[dr, "g"] & \\
c \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & c''
\end{tikzcd}$$

for every arrow in $\mathrm{dom}(c')$. This assigns each such arrow to one in $\mathrm{dom}(c)$ so that the assignment is well-defined. We can check the functoriality of the the mapping for dom by considering

$$\begin{tikzcd}
c \arrow[r, "f"'] \arrow[rr, bend left, "g\circ f"] \arrow[d, "\mathrm{dom}"'] & c' \arrow[r, "g"'] \arrow[d, "\mathrm{dom}"'] & c'' \arrow[d, "\mathrm{dom}"'] \\
\mathrm{dom}(c) & \mathrm{dom}(c') \arrow[l, "-\circ f"'] & \mathrm{dom}(c'') \arrow[l, "-\circ g"'] \arrow[ll, bend left, "-\circ(g\circ f)"]
\end{tikzcd}$$

which is clearly functorial. The same line of reasoning remains valid for the functor cod. The isomorphism of categories is evident by the fact that

$$\begin{aligned}
c &\overset{\sim}{\longleftrightarrow} \mathrm{dom}(c) \\
f &\overset{\sim}{\longleftrightarrow} \mathrm{dom}(f)
\end{aligned}$$

□

We call the representation $\mathrm{cod}(\mathbf{C})$ the *Cayley representation* of the category $\mathbf{C}$. If instead we use $\mathrm{dom}(\mathbf{C})$, we call this the *dual Cayley representation*.

2.4.2 Representable Functors and Yoneda's Lemma

The reasoning in the previous section can be generalized through a relaxing of the condition that the category in question be small to the condition that it be merely locally small. If we relax to this case, we can define the analogues of the dom and cod functors. In this section we work within a specific category $\mathbf{C}$ so we don't need to include it as a subscript in our hom-sets. The natural analogy is as follows

$$\begin{aligned}
\mathrm{dom} &\leftrightarrow [-, c] \\
\mathrm{cod} &\leftrightarrow [c, -]
\end{aligned}$$

Whenever a functor has the form $[c, -]$ or $[-, c]$, we say it is *representable*. It is straight forward to show that these functors satisfy

$$
\begin{tikzcd}
c' \arrow[r, "{[-,c]}"] \arrow[d, "f"'] & {[c',c]} \\
c'' \arrow[r, "{[-,c]}"] & {[c'',c]} \arrow[u, "-\circ f"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
c' \arrow[r, "{[c,-]}"] \arrow[d, "f"'] & {[c,c']} \arrow[d, "f\circ -"] \\
c'' \arrow[r, "{[c,-]}"] & {[c,c'']}
\end{tikzcd}
$$

from which the analogy immediately follows. In fact, this can be made concrete. Recall that $\mathrm{dom}(c') = \{f|\mathrm{dom}[f] = c'\}$. We replace dom $\leftrightarrow [-, c]$. Then $[c', c] = \{f|\mathrm{dom}[f] = c'$ and $\mathrm{cod}[f] = c\}$. In this case, we are thinking of c in the functor as fixed, so that we can replace the notation $[-, c]$ with a generic functor F. Then we think of the logical predicate characterizing the intension of the hom-set $[c', c]$ as being variable in c' as we do with $\mathrm{dom}(c')$. Then we see we just restrict the set $\mathrm{dom}(c')$ to those functions $f : c' \to c$, which is the intersection of the set of functions in $\mathrm{dom}(c')$ and $\mathrm{cod}(c)$. Since the more general category is only locally small, this restriction only works when the whole category is small, otherwise this is a proper generalization. It's clear by the form of this argument that cod $\leftrightarrow [c, -]$ is also the correct replacement. It is further supported by the fact that the maps are sent to the same form under the functors, which we would expect when the representation is thought of as a set-theoretic restriction or inclusion; the form of the maps under the image doesn't change, just *which* maps we are referring to. In the case of a small category, $\mathrm{dom}(c') = \cup_c[c', c]$ and $\mathrm{cod}(c') = \cup_c[c, c']$, which solidifies the recovery of those objects.

When we work with the contravariant functors $[-, c]$, maps between objects $c \to \tilde{c}$ define natural transformations. We should note that this behaves like graph maps, in that if $F, G : \Delta \to \Gamma$ are graph maps, then the arrows in the domain define natural transformations. Given $g : c \to \tilde{c}$, this can be seen according to the commutativity of the following square

$$
\begin{tikzcd}
c' \arrow[d, "f"] \\
c''
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
{[c',c]} \arrow[r, "g\circ -"] \arrow[d, "-\circ f"'] & {[c',\tilde{c}]} \arrow[d, "-\circ f"] \\
{[c'',c]} \arrow[r, "g\circ -"] & {[c'',\tilde{c}]}
\end{tikzcd}
\tag{18}
$$

Indeed, this follows simply by the axiom of associativity. It is also interesting to note that the components are constant across the objects. Any functor $F : \mathbf{C} \to \mathbf{Set}$ which is contravariant is called a *presheaf*. We will see later that in a locally small category all presheaves can be expressed as the colimit of a diagram of representable prehseaves. In general, this means we just need to add and divide representable presheaves to create any other presheaf. These objects should be thought of as the analytic representation of the underlying object. In particular, one will obtain the symbolic decomposition of an arbitrary content by considering it as a presheaf. This suggests the definition of a functor, called the *Yoneda functor*, which sends the category into its category of presheaves. This is defined according to

$$\mathcal{Y} : \mathbf{C} \to [-, \mathbf{C}] = \mathbf{Set}^{\mathbf{C}^{op}} = \hat{\mathbf{C}} \tag{19}$$

$$\mathcal{Y} : c \mapsto [-, c] \tag{20}$$

Since the category is locally small, it defines a functor category $\mathbf{Set}^{\mathbf{C}^{op}}$, the category of presheaves over $\mathbf{C}$, which we'll denote by $\hat{\mathbf{C}}$ or $\mathbf{Psh}(\mathbf{C})$. This is called the *Yoneda embedding*, which means that it is injective on objects, full, and faithful. We'll prove this below. Before we state the important *Yoneda lemma*, notice that (18) implies that any map $g : c \to c'$ is a natural transformation whose components are all the same at each component. The converse can also be observed but with the arrows flipping under the natrual transformation.

Lemma 2.21. ***Yoneda.*** *Let* $\boldsymbol{C}$ *be locally small. For any* $c \in Obs(\boldsymbol{C})$ *and functor* $F : \boldsymbol{C}^{op} \to \boldsymbol{Sets}$*, there is a natural isomorphism*

$$\theta : Hom_{\hat{C}}([-, c], F) \cong Fc \tag{21}$$

which is natural in F *and* c*.*

Proof. We'll define the natural isomorphism $\theta : [[-, c], F] \cong Fc$ as follows. Let $\eta : [-, c] \to F$ be a natural transformation in the hom-set. Let $\theta(\eta) = \eta_c(id_c)$. To check that this makes sense, consider the following diagrams

$$
\begin{tikzcd}
c' \arrow[r, "{[-,c]}"] \arrow[d, "f"'] & {[c',c]} & & c' \arrow[r, "F"] \arrow[d, "f"'] & Fc' \\
c'' \arrow[r, "{[-,c]}"] & {[c'',c]} \arrow[u, "-\circ f"'] \arrow[rr, phantom, "\overset{\eta}{\Rightarrow}"] & & c'' \arrow[r, "F"] & Fc'' \arrow[u, "Ff"']
\end{tikzcd}
$$

Because η is a natural transformation, we have for *every* c', c''

$$
\begin{tikzcd}
c' \arrow[d, "f"'] & {[c',c]} \arrow[r, "\eta_{c'}"] & Fc' \\
c'' & {[c'',c]} \arrow[u, "-\circ f"] \arrow[r, "\eta_{c''}"'] & Fc'' \arrow[u, "Ff"']
\end{tikzcd}
$$

so in particular $c' = c'' = c$ is a valid substitution for c', c''. The components of η are set-functions. Then $\eta_c : [c, c] \to Fc$, and as $id_c \in [c, c]$ for any c, $\eta_c(id_c)$ is defined with $\eta_c(id_c) \in Fc$. This gives an assignment $\theta : \eta \mapsto \eta_c(id_c)$, or $\eta \in [[-, c], F] \mapsto \eta_c(id_c) \in Fc$. Now we need to consider a mapping in the opposite direction. Consider a map $\phi : Fc \to [[-, c], F]$, and we define this by $\phi : x \in Fc \mapsto \eta^{(x)} \in [[-, c], F]$. We define this natural transformation as follows: let $c' \in \mathrm{Obs}(\mathbf{C})$, then $\eta^{(x)}_{c'} : [c', c] \to Fc'$ by if $h \in [c', c]$, then $\eta^{(x)}_{c'}(h) = Fh(x)$, where Fh is the image of h under F. Now we need to check naturality. This just means that for all $c' \xrightarrow{f} c''$, we have

$$
\begin{tikzcd}
c' \arrow[d, "f"'] & {[c',c]} \arrow[r, "\eta^{(x)}_{c'}"] & Fc' \\
c'' & {[c'',c]} \arrow[u, "-\circ f"] \arrow[r, "\eta^{(x)}_{c''}"'] & Fc'' \arrow[u, "Ff"']
\end{tikzcd}
$$

follows by $h \in [c', c]$, then $h \mapsto h \circ f \mapsto \eta^{(x)}_{c''}(h \circ f) = F(h \circ f)(x) = Ff \circ Fh(x)$ (since F is contravariant). On the other hand, we have $h \mapsto \eta^{(x)}_{c'}(h) = Fh(x) \mapsto Ff \circ Fh(x)$, so that these commute for any c', c'', f so that $\eta^{(x)}$ is natural.

Next, we need to show that the composition of θ and ϕ yields the identity. To this end, we need to show

$$
\begin{tikzcd}
& Fc \arrow[dr, "\phi"] & \\
{[[-,c],F]} \arrow[ur, "\theta"] \arrow[rr, "\phi\circ\theta = id_{[[-,c],F]}"'] & & {[[-,c],F]}
\end{tikzcd}
\qquad
\begin{tikzcd}
& {[[-,c],F]} \arrow[dr, "\theta"] & \\
Fc \arrow[ur, "\phi"] \arrow[rr, "\theta\circ\phi = id_{Fc}"] & & Fc
\end{tikzcd}
$$

Since these are sets, we just need to show that their composition yields the identity function on the appropriate set. Checking the first, we have $\eta \mapsto \theta(\eta) = \eta_c(id_c) \mapsto \phi(\eta_c(id_c)) = \eta^{(\eta_c(id_c))}$. For any $h \in [c', c]$, $\eta^{(\eta_c(id_c))}_{c'}(h) = Fh(\eta_c(id_c))$. Now consider the naturality square for the given η

$$
\begin{tikzcd}
{[c,c]} \arrow[r, "\eta_c"] \arrow[d, "-\circ h"'] & Fc \arrow[d, "Fh"] \\
{[c',c]} \arrow[r, "\eta_{c'}"'] & Fc'
\end{tikzcd}
$$

which implies $Fh \circ \eta_c = \eta_{c'} \circ (-\circ h)$. Evaluating at id_c, $Fh \circ \eta_c(id_c) = \eta_{c'} \circ (-\circ h)(id_c) = \eta_{c'}(h)$. Then we have $\eta^{(\eta_c(id_c))}_{c'}(h) = \eta_{c'}(h)$, natural in c' so that $\eta^{(\eta_c(id_c))} = \eta$. This shows $\phi \circ \theta(\eta) = \eta$ for any η, so the composition is the (relevant) identity. Going in the other direction, given $x \in Fc$, consider $x \mapsto \phi(x) = \eta^{(x)} \mapsto \theta(\eta^{(x)}) = \eta^{(x)}_c(id_c)$. Then we have $\eta^{(x)}_c(id_c) = F(id_c)(x) = id_{Fc}(x) = x$, so that $x \mapsto x$ as required. This establishes the isomorphism.

To show that these are natural in F, c, we proceed as follows. Since for a natural isomorphism η, if η is natural in its components, so is its inverse. We therefore only need to show that θ is natural in its components. This amounts to checking the diagrams

$$
\begin{tikzcd}
c \arrow[d, "f"'] \\
c'
\end{tikzcd}
\qquad
\begin{tikzcd}
{[[-,c'],F]} \arrow[r, "\theta_{c',F}"] \arrow[d, "-\circ(f\circ -)"'] & Fc' \arrow[d, "Ff"] \\
{[[-,c],F]} \arrow[r, "\theta_{c,F}"'] & Fc
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[d, "\phi"'] \\
F'
\end{tikzcd}
\qquad
\begin{tikzcd}
{[[-,c],F]} \arrow[r, "\theta_{c,F}"] \arrow[d, "\phi\circ -"'] & Fc \arrow[d, "\phi_F"] \\
{[[-,c],F']} \arrow[r, "\theta_{c,F'}"] & F'c
\end{tikzcd}
$$

To do so, we consider the left square first, showing that f indeed defines the map $-\circ(f\circ -)$, the pre-composition with the post-composition functor induced by f, then we show the naturality in c. After this, we show the naturality in F. To that end, let $f : c \to c'$. In what follows, $\eta : [-,c] \to F$ with components $\eta_{c''} : [c'',c] \to Fc''$ and $\epsilon : [-,c'] \to F$ with components $\epsilon_{c''} : [c'',c'] \to Fc''$. Let $g \in [c'',c]$, then $f : c \to c'$ admits $f\circ g : c'' \to c \to c' \rightsquigarrow f\circ g : c'' \to c'$. Post composition then assigns $[c'',c] \xrightarrow{f\circ -} [c'',c']$. This defines a species of diagrams

$$
\begin{tikzcd}
{[c'',c]} \arrow[r, "f\circ -"] \arrow[dr, "\epsilon_{c''}\circ(f\circ -)"'] & {[c'',c']} \arrow[d, "\epsilon_{c''}"] \\
 & Fc''
\end{tikzcd}
$$

where the composition defines a natural transformation component-wise $\epsilon_{c''}\circ(f\circ -) : [c'',c] \to Fc''$ which is a member of the hom-object $[[-,c],F]$ so that it defines an $\eta : [-,c] \to F$ with components $\eta_{c''} = \epsilon_{c''}\circ(f\circ -)$. This therefore defines an assignment for every $-\circ(f\circ -) : \epsilon \mapsto \epsilon\circ(f\circ -) = \eta$, so that $-\circ(f\circ -) : [[-,c'],F] \to [[-,c],F]$. This is defined locally $g \mapsto \epsilon_{c''}\circ(f\circ -)(g) = \epsilon_{c''}(f\circ g) \in Fc''$. This implies the *possibility* of a naturality square

$$
\begin{tikzcd}
{[[-,c'],F]} \arrow[r, "\theta_{c',F}"] \arrow[d, "-\circ(f\circ -)"'] \arrow[dr] \arrow[dr, shift right] & Fc' \arrow[d, "Ff"] \\
{[[-,c],F]} \arrow[r, "\theta_{c,F}"] & Fc
\end{tikzcd}
$$

which we can check. Take $\epsilon : [-,c'] \to F$. Then the bottom left triangle is $\theta_{c,F}\circ[-\circ(f\circ -)](\epsilon) = \theta_{c,F}(\epsilon\circ(f\circ -)) = \epsilon_c\circ(f\circ -)(id_c) = \epsilon_c(f\circ id_c) = \epsilon_c(f) \in Fc$. Then top right. This reads $Ff\circ\theta_{c',F}(\epsilon) = Ff(\epsilon_{c'}(id_{c'})) = Ff\circ\epsilon_{c'}(id_{c'})$. Then by using the naturality of ϵ

$$
\begin{tikzcd}
c''' \arrow[d, "h"'] \\
c''
\end{tikzcd}
\qquad
\begin{tikzcd}
{[c'',c']} \arrow[r, "\epsilon_{c''}"] \arrow[d, "-\circ h"'] & Fc'' \arrow[d, "Fh"] \\
{[c''',c']} \arrow[r, "\epsilon_{c'''}"] & Fc'''
\end{tikzcd}
$$

we have $Ff\circ\epsilon_{c'} = \epsilon_c\circ(-\circ f)$ implying $Ff\circ\epsilon_{c'}(id_{c'}) = \epsilon_c(id_{c'}\circ f) = \epsilon_c(f)$. Therefore the square commutes and $\theta_{c,F}$ is natural in c. Naturality in F is easier. Take $\phi : F \to F'$ is a natural transformation. Then the hypothetical naturality square is

$$
\begin{tikzcd}
F \arrow[d, "\phi"'] \\
F'
\end{tikzcd}
\qquad
\begin{tikzcd}
{[[-,c],F]} \arrow[r, "\theta_{c,F}"] \arrow[d, "\phi\circ -"'] & Fc \arrow[d, "\phi_c"] \\
{[[-,c],F']} \arrow[r, "\theta_{c,F'}"] & F'c
\end{tikzcd}
$$

Given $\eta : [-,c] \to F$, since η and ϕ are natural, $(\phi\circ\eta)_{c''} = \phi_{c''}\circ\eta_{c''}$. Then we chase either leg of the diagram. $\theta_{c,F'}\circ(\phi\circ -)(\eta) = \theta_{c,F'}(\phi\circ\eta) = (\phi\circ\eta)_c(id_c) = \phi_c\circ\eta_c(id_c)$. The other leg is $\phi_c\circ\theta_{c,F}(\eta) = \phi_c\circ\eta_c(id_c)$, demonstrating the naturality in F. □

This gives rise to the important

Theorem 2.22. *The Yoneda functor $\mathcal{Y} : \boldsymbol{C} \to \hat{\boldsymbol{C}}$ is a fully faithful functor that is injective on objects. Therefore, it is an embedding of $\boldsymbol{C}$ into the category of presheaves $\hat{\boldsymbol{C}}$.*

Proof. Recall that a functor is *fully faithful* if for each pair of objects c, c', then $[c,c'] \cong [Fc,Fc']$. Notice $[c,c'] \cong [[-,c],[-,c']] = [\mathcal{Y}(c),\mathcal{Y}(c')]$ by the lemma. For each $h \in [c,c']$, it sends $h \mapsto \mathcal{Y}(h) = h\circ -$, the natural transformation $h \circ - : [-,c] \to [-,c']$. To see that it is injective on objects, let $\mathcal{Y}(c) = \mathcal{Y}(c')$, then $[-,c] = [-,c']$, which means for all $c'' \in \mathrm{Obs}(\mathbf{C})$, we have $[c'',c] = [c'',c']$ with $id_c \in [c,c']$ implying $c = c'$. ◻

Further, we can show that for any objects in a locally small category, it's sufficient to show that the hom-set into the objects being isomorphic implies the objects themselves are.

Theorem 2.23. *Let $\mathbf{C}$ be a locally small category, then $[-,c] \cong [-,c']$ implies $c \cong c'$.*

Proof. Suppose F is any fully faithful functor, and suppose

$$\begin{tikzcd} & Fc' \arrow[dr, "g"] & \\ Fc \arrow[ur, "f"] \arrow[rr, "id_{Fc}=F(id_c)"'] & & Fc \end{tikzcd}$$

Because F is fully faithful, then there are f', g' with $f = Ff'$, $g = Fg'$ so that $g \circ f = Fg' \circ Ff' = F(g' \circ f') = F(id_c)$. Since F faithful, $g' \circ f' = id_c$. Furthermore, it is clear by the same sort of reasoning, $f' \circ g' = id_{c'}$. Taking $F = \mathcal{Y}$ proves the theorem. ◻

To continue down this line of reasoning, we'll need the

Proposition 2.24. *Let $\mathbf{C}$ be a locally small category. Then the covariant representable functors preserve limits and the contravariant representable functors send limits to colimits.*

Proof. It suffices to show that this is the case for products and equalizers, by the theorem. For the empty product, $[c,1] \cong \{!_c : c \dashrightarrow 1\}$. For (binary) products, any $h \in [c, a \times b]$ maps to $(\pi_a \circ h, \pi_b \circ h) \in [c,a] \times [c,b]$. Clearly the inverse mapping is just $\langle h_a, h_b \rangle \in [c,a] \times [c,b] \mapsto h = (h_a, h_b) \in [c, a \times b]$. As for equalizers, let the diagram be given as in (9). If $[c,-]$ is the relevant functor, then $f : c' \to c'' \mapsto f \circ - : [c,c'] \to [c,c'']$. This means $i \mapsto i \circ -$, $f, g \mapsto (f \circ -), (g \circ -)$ with the relevant compositions, so $(f \circ i) \circ - = (g \circ i) \circ -$. If $u, u' : [c,d] \mapsto [c,e]$ given $(f \circ h) \circ - = (g \circ h) \circ -$, then $u = u'$ for if not, then $(i \circ -) \circ (u \circ -) = (i \circ u) \circ - = h \circ - = (i \circ -) \circ (u' \circ -) = (i \circ u') \circ -$. But this would requzire that $i \circ u = h = i \circ u'$ and $u \neq u'$, but as i is part of an equalizer, then $u = u'$ (or also because i is monic). By duality, this holds for the functor $[-,c]$ for every c. ◻

Importantly, this is *not* true for colimits in general. It's easy to see why if we consider just diagrams. Consider the Yoneda functor $\mathcal{Y}(\Delta + \Gamma)$. This preserves colimits if $\mathcal{Y}(\Delta + \Gamma) \cong \mathcal{Y}(\Delta) + \mathcal{Y}(\Gamma)$. This is the same claim as $[-, \Delta + \Gamma] \cong [-,\Delta] + [-,\Gamma]$. But if we consider any test diagram Ξ, and evaluate at this test diagram, we have the important functors $F : \Xi \to \Delta + \Gamma$ that are characterized by the following type of diagram

$$\begin{tikzcd} \Xi_\Delta \arrow[r, "F|_\Delta"] \arrow[d, tail] & \Delta \arrow[d, tail] \\ \Xi \cong \Xi_\Delta + \Xi_\Gamma \arrow[r, "F = F|_\Delta + F|_\Gamma"] & \Delta + \Gamma \\ \Xi_\Gamma \arrow[r, "F|_\Gamma"] \arrow[u, tail] & \Gamma \arrow[u, tail] \end{tikzcd}$$

This functor F cannot be given by the coproduct of a pair of functors from Ξ, since functors are required to be total in their assignments. This shows in general that $\mathcal{Y}$ *does not* preserve colimits (although by duality and the above proposition, its dual does).

Corollary 2.25. *The assignment $c \mapsto [-,c]$ preserves limits and therefore the embedding into $\hat{\mathbf{C}}$ is complete. The evaluation functor (at c') $[-,c] \mapsto [c',c]$ therefore preserves limits. The dual statements are also the case.*

If we substitute for **Sets** any other category **D**, these statements hold whenever the codomain of the functor category in question is (co)complete. If we further relax the locally-small condition, these (co)limits *only* make sense 'point-wise' as the proof shows. These considerations lead us to an essential result, which gives us the essential structure of $\hat{\mathbf{C}}$.

Theorem 2.26. *Let $\mathbf{C}$ be a small category, then any presheaf $P \in \hat{\mathbf{C}}$ is a colimit of representable functors*

$$\varinjlim_{c \in \Delta} [-,c] \cong P \tag{22}$$

where Δ is a diagram in $\mathbf{C}$.

Proof. We will use the Grothendieck construction (with P valued in **Sets** rather than **Cat**). In this case, we have $\int_{\mathbf{C}} P$ where our objects are (x, c) where $c \in \mathrm{Obs}(\mathbf{C})$ and $x \in Pc$. Given $f : c \to c'$, we define $(x, c) \to (x', c')$ by $(Pf^*, f) : (x, c) \to (x', c')$ with $Pf(x') = x$. In this case, $\int_{\mathbf{C}} P$ is small, which is the same claim as $\mathrm{Ars}(\int_{\mathbf{C}} P) \in \mathbf{Sets}$. Since $\mathbf{C}$ is small, $\mathrm{Ars}(\mathbf{C}) \in \mathbf{Sets}$. The arrows $(Pf^*, f) \leftrightarrow f$ are in bijection. Then $\int_{\mathbf{C}} P$ is small. Furthermore, the projection functor: $\pi : \int_{\mathbf{C}} P \to \mathbf{C}$ by $(x, c) \mapsto c$, with the relevant projection onto functions. Now $\int_{\mathbf{C}} P \cong \mathcal{Y} \downarrow P$, which we can see by

$$\begin{tikzcd} (x,c) \arrow[d, "(Pf^*,f)"] \\ (x',c') \end{tikzcd} \qquad \begin{tikzcd} {[-,c]} \arrow[rr, "f\circ -"] \arrow[dr, "x"'] & & {[-,c']} \arrow[dl, "x'"] \\ & P & \end{tikzcd}$$

In this sense, $\mathcal{Y} \downarrow P$ is the category of all diagrams over P with the objects valued in the representables, which gives a construction to evaluate the relevant colimit. Hence a cocone in over P is just a diagram in $\int_{\mathbf{C}} P$. To see that this is limiting, consider $\int_{\mathbf{C}} Q$, with $Q \in \hat{\mathbf{C}}$, we have the diagrams

$$\begin{tikzcd} {[-,c]} \arrow[rr, "f\circ -"] \arrow[dr, "x"'] & & {[-,c']} \arrow[dl, "x'"] \\ & P & \end{tikzcd} \qquad \begin{tikzcd} {[-,c]} \arrow[rr, "f\circ -"] \arrow[dr, "y"'] & & {[-,c']} \arrow[dl, "y'"] \\ & Q & \end{tikzcd}$$

which defines a diagram

$$\begin{tikzcd} {[-,c]} \arrow[rr, "f\circ -"] \arrow[dr, "x"] \arrow[ddr, "y"'] & & {[-,c']} \arrow[dl, "x'"'] \arrow[ddl, "y'"] \\ & P \arrow[d, "\eta"] & \\ & Q & \end{tikzcd}$$

and $\eta : P \to Q$ the natural transformation defined by $y = \eta(x)$ for every object c and natural transformation $[-, c] \xrightarrow{x} P$. By the Yoneda lemma, these define the components of η. It is unique since for every $c \in Pc$, $\eta(x) = y = \eta'(x)$, and this will be true for every object in $\mathbf{C}$. □

2.4.3 Sites and Sheaves

The previous section considers functors of the form $[-, c]$ and their duals, as well as the limits/colimits of these diagrams. We also mentioned that in any locally small category, $[-, c] : \mathbf{C} \to \mathbf{Set}$ is called a *representable presheaf* and that *any* contravariant functor $F : \mathbf{C} \to \mathbf{Set}$ is a presheaf, and that this is the colimit of some diagram of representables. These objects are of special interest in many circumstances. They contain a special subclass of objects called *sheaves* which we will discuss now.

2.4.4 Grothendieck's Construction

There is a special construction that is worth mentioning. It called a *Grothendieck construction* after the famous mathematician Alexander Grothendieck. Given any category $\mathbf{C}$ and a functor $F : \mathbf{C} \to \mathbf{Cat}$, a functor to the category of (small) categories, there is a construction $\int_{\mathbf{C}} F$ which essentially constructs a fibration over $\mathbf{C}$. The objects of this category are pairs (c, x) where $c \in \mathrm{Obs}(\mathbf{C})$ and $x \in \mathrm{Obs}(F(c))$. The maps are pairs (f, g) with $f : c \to c'$ and $g : F(f)(x) \to x'$. The key idea is that at each c there is a fiber whose structure is that of a category. Furthermore, there is an evident projection back down into $\mathbf{C}$. We can think of this schematically according to the diagram

$$\begin{tikzcd}
x_1 \arrow[r] \arrow[rr, bend left, "F(f)"] \arrow[dr, dotted, no head] & x_2 \arrow[d, dotted, no head] & x_1' \arrow[r, "g"] \arrow[d, dotted, no head] & x_2' \arrow[dl, dotted, no head] \\
& c \arrow[r, "f"] & c' &
\end{tikzcd} \tag{23}$$

See [2, 30] for more details. Notice in particular, since at each c we have a category $F(c)$, if we take $c' = c$ and $f = id_c$, then we can work 'within' the category at c. The image of $F(id_c)(x)$ is just x since $F(id_c)$ is the identity functor for the category $F(c)$, or $F(id_c) = id_{F(c)}$.

2.5 Adjunctions

Adjunctions generalize invertibility of functors. In the case of equivalence of categories, we observed that $F \simeq G$ whenever there are a pair of natural isomorphisms $\eta : id_{\mathbf{C}} \xrightarrow{\sim} GF$ and $\epsilon : FG \xrightarrow{\sim} id_{\mathbf{D}}$. If we relax these conditions and instead demand merely that such natural transformations exist, we obtain a pair of *adjoint functors* and we write $F \dashv G$, where F is called the *left adjoint* and G the *right adjoint*. We have two essential claims that we would like to ultimately demonstrate: 1) all adjunctions can be characterized either by considering a triple $F \dashv G \dashv H$ where G is either covariant or contravariant (that is, the characterization comes fundamentally by whether the functor flips arrows or not); 2) Every adjunction decomposes into a 'pair' where one is covariant, the other is contravariant, and we obtain the whole adjunction by application of a type of coequalizer onto something like the free diagram that enters into the adjunction. Neither of these claims is currently well-posed.

More generally, the importance of adjunctions are expressed in the fact that they are, in a sense, the most generalized notion of inversion we've discovered so far. We will argue that a function F has a left adjoint G in the case that the codomain of F needs to be fully synthetically extended into the domain of F in order to obtain an inversion according to the triangle diagrams (which we will define). In this sense, a functor F has a left adjoint whenever one requires a synthetic representation of the domain of F to 'undo' its action. Dually, it has a right adjoint when the inversion process requires an analytic representation to 'undo' its action. This is the fundamental heuristic for understanding the significance and meaning of adjunctions. From this perspective, it is easy to see that left adjoints must be right exact (i.e., preserve colimits) and right adjoints are left exact (i.e., preserve limits).

2.5.1 Cohesion

2.6 Exponentiation and Cartesian Closed Categories

Some categories have objects which internalize the arrows–that is, there is an object which represents the 'space' of arrows between a pair of objects in the category. When this happens, we say the category has *exponentials* and whenever a category is finitely bicomplete and has exponentials, it is called *Cartesian closed*. We define it as follows: In a category with (binary) products, the *exponential* of c by c' is an object $c^{c'}$ and an arrow $\mathrm{ev}_{(c',c)} : c^{c'} \times c' \to c$ such that for any other object c'' and arrow f there is a unique $\overline{f} : c'' \to c^{c'}$ making the following commute

$$\begin{tikzcd}
c^{c'} & c^{c'} \times c' \arrow[r, "\mathrm{ev}_{(c',c)}"] & c \\
c'' \arrow[u, dashed, "\overline{f}"] & c'' \times c' \arrow[u, "\overline{f} \times id_{c'}"] \arrow[ur, "f"'] &
\end{tikzcd} \tag{24}$$

$\overline{f}$ is called the *transpose* of f.

There is an important relationship between exponentiation and products.

2.7 Topoi

Topoi are cartesian closed categories with a subobject classifier. We saw that the previous category is a place where formal compuations can be carried out. When we include the subobject classifier, we obtain the ability to endow the formal calculations with an internal logic.

2.7.1 Subobjects, Classifiers, and the Ω-axiom

Logical data begins with the notion of *subobjects*, which are monics. If **C** is (locally) small, then $\mathrm{Sub}_{\mathbf{C}}(c) = \{i | i : c'' \rightarrowtail c\}$. Furthermore, if **C** is finitely complete, $\mathrm{Sub}_{\mathbf{C}} : \mathbf{C} \to \mathbf{Sets}$ is a contravariant functor. Given $f : c \to c'$, we can assign $\mathrm{Sub}_{\mathbf{C}}(c') \xrightarrow{\mathrm{Sub}(f)} \mathrm{Sub}_{\mathbf{C}}(c)$ via the pullback

$$\begin{tikzcd}
c''' \arrow[r, tail, "f^{-1}(i)"] \arrow[d] & c \arrow[d, "f"] \\
c'' \arrow[r, tail, "i"] & c'
\end{tikzcd}$$

so that given $f : c \to c'$, we can assign a monic $f^{-1}(i)$ to each $i : c'' \rightarrowtail c'$, which establishes $\mathrm{Sub}_{\mathbf{C}}$ as a contravariant functor. Note that even if **C** is not (locally) small, the subobjects at an object c is defined, but the notion of the subobject category need not a priori be defined as being 'in' some other category. A *subobject classifier* is an object Ω together with a monic $\top : 1 \rightarrowtail \Omega$ such that it satisfies the

Axiom 2.27. (**Ω-axiom**) Let **C** be a finitely complete category. For any subobject $e \rightarrowtail c$, there is a unique arrow $\chi_f : c \to \Omega$ such that the following diagram is a pullback square

$$\begin{tikzcd}
e \arrow[r, tail, "i"] \arrow[d, "!"'] & c \arrow[d, "\chi_i"] \\
1 \arrow[r, tail, "\top"] & \Omega
\end{tikzcd} \tag{25}$$

The axiom (2.27) implies that $\mathrm{Sub}_{\mathbf{C}}(-) \cong [-, \Omega]_{\mathbf{C}}$ is representable. Clearly this just means that we need to exhibit a natural isomorphism $\theta : \mathrm{Sub}_{\mathbf{C}}(-) \cong [-, \Omega]_{\mathbf{C}}$. This can be done by observing the diagram

$$\begin{tikzcd}
c \arrow[d, "f"'] & \mathrm{Sub}_{\mathbf{C}}(c') \arrow[r, "\{u'\}"] \arrow[d, "f^*"] & {[c', \Omega]_{\mathbf{C}}} \arrow[d, "- \circ f"] \\
c' & \mathrm{Sub}_{\mathbf{C}}(c) \arrow[r, "\{u\}"] & {[c, \Omega]_{\mathbf{C}}}
\end{tikzcd}$$

where the elements of $\theta_c = \{u\}$ take an equivalence class of subobjects $[i]$ to the arrow χ_i. We can see this by realizing that if $i \simeq i'$ (given by the existence of an iso $i' = i \circ \phi$), then $\chi_i = \chi_{i'}$. This follows by the uniqueness of the classifying arrow here

$$\begin{tikzcd}
 & e' \arrow[dl, bend right, "\phi"'] \arrow[dr, tail, "i'"] \arrow[dd, "!_{e'}"] & \\
e \arrow[ur, bend right, "\phi^{-1}"] \arrow[rr, tail, "i"] \arrow[dr, "!_e"'] & & c \arrow[d, "\chi_i = \chi_{i'}"] \\
 & 1 \arrow[r, tail, "\top"] & \Omega
\end{tikzcd}$$

Then recalling $i' = i \circ \phi$, we have from the diagram $\chi_i \circ i = \top_e = \top \circ !_e$ and the *unique* arrow which does so. Then $\chi_{i'} \circ i' = \top_{e'}$ implies $\chi_{i'} \circ i' = \chi_{i'} \circ i \circ \phi = \top_{e'}$ which means $\chi_{i'} \circ i = \top_{e'} \circ \phi^{-1} = \top_e$. This last equality follows from the fact that $!_{e'} \circ \phi^{-1} = !_e$ which follows from the property of terminal objects. But by the Ω-axiom, we have χ_i is the unique arrow making this statement true, hence $\chi_{i'} = \chi_i$[6]. From here, we'd have the uniqueness of u for each equivalence class of monics establishes the bijection. A *topos* $\mathcal{E}$ is a cateogory that is finitely bicomplete, has exponentiation, and has a subobject classifier.

2.7.2 Logical Connectives as Arrows

The requirement of the existence of a subobject classifier guarantees the existence of the arrow $\top : 1 \rightarrowtail \Omega$; the Ω-axiom implies that every (equivalence class of) subobject(s) can be uniquely classified. Consequently, we can begin to build up

[6]This 'proof' is *not* elementary–the preferable method follows by exhibiting a sequence of contractions on diagrams. In principle, we'd need to *construct* the sequence of diagrams exhibiting the 1-1 correspondence for any subobject/characteristic arrow pair. This can be done by 'contradiction' since the local reasoning will be among finitely differentiated data. Just include the arrow $!_{e'}$ explicitly above. Then as soon as $!_{e'} = !_e \circ \phi$ we have $\chi_{i'} = \chi_i$

the local logical data within a given topos. First, because we are in a category that is finitely bicomplete, we know that $0 \in \mathrm{Obs}(\mathcal{E})$. By the definition of the initial object, we know there exists an arrow $\leq$: $0 \to 1$ and furthermore that this is unique. Consequently this arrow is monic and we can consider the diagram

$$\begin{tikzcd}
0 \arrow[r, tail, "\leq"] \arrow[d, "!_0"'] & 1 \arrow[d, "\chi_{\leq}"] \\
1 \arrow[r, tail, "\top"] & \Omega
\end{tikzcd} \tag{26}$$

where we can ask: since $0 \to 1$ is *unique*, one way to ensure this is a pullback is to suppose that $\chi_{\leq} = \top$. If we do so, then we have the diagram

$$\begin{tikzcd}
1 \arrow[dr, dashed] \arrow[drr, bend left, "id_1"] \arrow[ddr, bend right, "id_1"'] & & \\
& 0 \arrow[r, tail, "\leq"] \arrow[d, "!_0"'] & 1 \arrow[d, "\chi_{\leq} \stackrel{?}{=} \top"] \\
& 1 \arrow[r, tail, "\top"] & \Omega
\end{tikzcd}$$

which would have to exist because the outside square commutes. But then this says $0 \cong 1$ which implies that the logic this topos encodes is trivial.[7] Consequently, we conclude $\chi_{\leq} \neq \top$ and the outer square above does not exist since the only arrow $1 \to 1$ is id_1. But then what is this arrow? Consider the diagram

$$\begin{tikzcd}
T \arrow[dr, dashed] \arrow[drr, bend left, "!_T"] \arrow[ddr, bend right, "!_T"'] & & \\
& 0 \arrow[r, tail, "\leq"] \arrow[d, "!_0"'] & 1 \arrow[d, "\chi_{\leq}"] \\
& 1 \arrow[r, tail, "\top"] & \Omega
\end{tikzcd}$$

which then says, given the fact that $\top_T = \top \circ !_T = \chi_{\leq} \circ !_T$, there is a unique map $u : T \dashrightarrow 0$ making $!_T = \leq \circ u = !_0 \circ u$. Since $\leq = !_0$ by properties of initial and terminal objects, it is sufficient to consider $\leq \circ u$. Now we want to argue that this is $\chi_{\leq}$ should be interpreted as $\perp$, i.e., *false*. False is the canonical element which is contradictory with $\top$–i.e., $\top \wedge \perp = \perp$. But at this moment we don't have a concept of $\wedge$, so we need to introduce this and leave $\chi_{\leq}$ alone for now. We turn to this definition now.

The canonical defining property of $\wedge$ is $\top \wedge \top = \top$, and $\wedge : X \times X \to X$ when it is defined for an object X. Consequently, we need to define $\wedge : \Omega \times \Omega \to \Omega$. It naturally follows that we would want to define $\top \wedge \top = \wedge \circ \langle \top, \top \rangle$. The enforcement of the property then implies

$$\begin{tikzcd}
1 \arrow[r, tail, "\langle \top, \top \rangle"] \arrow[d, "id_1"'] & \Omega \times \Omega \arrow[d, "\chi_{\langle \top, \top \rangle} = \wedge"] \\
1 \arrow[r, tail, "\top"] & \Omega
\end{tikzcd}$$

is a pullback. This means $\wedge$ is the characteristic arrow of $\langle \top, \top \rangle : 1 \to \Omega \times \Omega$.

2.7.3 Calculating in Topoi

We discussed at the beginning of this chapter that one calculates in categories by exhibiting sequences of diagrams and application of the associated reduction induced by axioms or derived rules. But with topoi, as a consequence of the Ω-axiom, we get another rule. For a simple example, we would intuitively expect $\top \wedge \perp = \perp$. How can we show this?

[7] We will need to 'prove' this.

2.8 The Category (Topos) of Diagrams

The category of diagrams is our prototypical example[8]. It really is a meta-category, since we are going to use diagrams to indicate the properties of diagrams. See for example [40, 39] for a discussion on this. Since it is a meta-category, the claims made must be understood as purely intuitive and not formal. We'll have more to say on this later. Up to this point, we've been reasoning tacitly in terms of diagrams, but we can express this is a semi-formal way if we want to take, say, **Set** as our foundational system (cf. [44] and compare the notion of Cantor's 'Kardinalen'). Ultimately, we will want to think of the vertices as valued in some type that is really to be taken as primitive. Its substantiveness will be *postulated* (we refrain from asserting it as an *axiom*) which will follow from something like a postulate of consciousness (?). In any case, a *diagram* consists of a pair of collections: 1) objects assigned to vertices; 2) arrows assigned to (directed) edges. In this sense, a diagram is a directed graph subjected to the axioms (2), (3), and (4). Given any directed graph, we can form the *categorial closure* by appending to its collection of edges those additional edges implied by the axioms[9]. For example

$$\begin{tikzcd} & c' \arrow[dr, "g"] & \\ c \arrow[ur, "f"] & & c'' \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & c' \arrow[dr, "g"] \arrow[loop above] & \\ c \arrow[ur, "f"] \arrow[rr, "g\circ f"'] \arrow[loop left] & & c'' \arrow[loop right] \end{tikzcd}$$

If we work locally, then diagrams will present themselves as *finite* and the appending process will be well-defined (for example, as an algorithmic process). We can interpret every diagram as a category and define a *diagram map* between a pair of diagrams Δ, Γ as a map $F : \Delta \to \Gamma$ which is functorial in the sense above. These compose as we showed. Furthermore, it implies that the algebraic operations of product, equalizer, pullback, and their duals are well-defined. This is also the case for exponentiation. The *product diagram* for a pair of diagrams Δ, Γ has as its objects the pairs (d, g) and has as its arrows $\langle h, k\rangle : (d, g) \to (d', g')$. Given a parallel pair of diagram maps $F, G : \Delta \rightrightarrows \Gamma$, an *equalizer diagram* is a diagram $E \xrightarrow{I} \Delta \rightrightarrows \Gamma$ satisfying $F \circ I = G \circ I$, so that Δ 'restricted' to $\delta \leq \Delta$ equalizes the functors F, G. We can define subdiagrams of a given diagram. A *subdiagram* $\delta \leq \Delta$ of a diagram Δ is a diagram with a functor $I : \delta \to \Delta$ which is monic. It is an equalizer for every pair of functors restricted to it such that they are equal there. Given any diagram Δ, this defines a lattice over the diagram, where diagram meets and joins can be defined as intersections and unions in the expected way. The terminal diagram is just the categorial closure of a single object or vertex. Pullbacks follow. Exponentiation is then the diagram consisting of all functors between the two diagrams in question, so that the category of finite diagrams is Cartesian closed. In fact, it is a topos, with the subobject classifier determining essentially one of five cases: 1) a diagram which is contained entirely within the diagram in question; 2) a diagram with arrows that exit the diagram in question; 3) a diagram with arrows that enter into the diagram in question; 4) a diagram with arrows which leave then re-enter the diagram in question; 5) a diagram with no arrows within the diagram in question. It seems like a natural conjecture to suppose that every topos will have a subobject classifier that is 'built out of' one like this. That's something for a different time.

For *any* category (whether it is presented as a finite species of finite diagrams, or in any other manner), it will always be isomorphic to some species of diagrams whose vertices and edges are valued in some type[10]. The diagrams together with these types *are* the presentation of the unique "equivalence class of categories". The diagrams by themselves present a *frame* of the category. The types which populate the arrows and objects are the substance of that category. Indeed, we should even refrain from the statement 'isomorphic' and declare them as literally equal, which follows on the basis of the nature of the axioms[11]. This means that calculations in categories really should be thought of as calculations on diagrams, by stitching them together, or by applying the axioms[12]. In this sense, there is no 'category of diagrams'; a category *is* its diagrams[13], which constitute

[8] Technically, this is the category of *small* or *finite* diagrams, but the circumstances where this particular property is invoked are usually obvious. On the other hand, the claims made within the meta-category always rely on 'small' diagrams but the 'size' of the diagrams at each vertex really determine the size of the presented diagram. This is why we need to work at the first-order level and this aspect of the theory is really just a linguistic aid in the actual context of reasoning–cf. [11, 40].

[9] This is also the *free category* on a directed graph. See the notes above for the definition of a free category and its related universal mapping property.

[10] Just take the unity of all types that the possible objects and arrows exist in.

[11] Obviously if one has a different disposition towards mathematical entities (e.g., as an analytical perspective), then the 'choice' of a given species of diagrams renders the category into an *analytic presentation*, as is more so the disposition in the theory of sets. But this perspective requires that the content being so analyzed be supposed to be 'compact' in a very specific and peculiar way–viz. that its data can be encapsulated entirely within the process of analysis. In our view, this perspective more accurately captures, say, Lawvere's point of view (e.g. [46]) as well as other mathematicians (cf esp. [47]). It also entails, I think, a more 'globalist' perspective, whereas the local one always requires synthetic extension to build up the structure. Then rules of invariance allow for *defining* global data. These rules are not diagramatic, but again are meta-rules about, say, all diagrams of a particular structure.

[12] There is a *very* strong suggestion here: this sort of calculation is the same as the one which characterizes the proof that every presheaf is the colimit of a diagram of representables (in a small category), or that calculation proceeds either by application, λ-abstraction, or β-reduction. It also has the flavor of taking (free) coproducts, then applying quotients to obtain a relevant diagram.

[13] This is the only perspective of the elementary theory of categories that is consistent with the metamathematical disposition due to Hilbert and others.

together in their unity the (synthetic) extension of the category[14]. Each 'atomic' diagram (which features in the process of this extending as a base in any extension in which it appears–therefore what we call elsewhere a *symbol*) is a *geometric realization* or *geometric presentation*[15] of a fundamental intension which cannot be captured more clearly, since every such presentation requires a *geometric projection* (i.e., the realization of the unity in intuition in terms of its (unified) geometric realization) and necessarily severs connection among the unity by declaring an object which satisfies the principle of non-contradiction–for as soon as any other presentation is made manifest, the possibility of comparison by way of extension asserts itself. As soon as we have a process of comparison, we have the ability to form judgments of equality, and as soon as this is the case, whether the two objects are equal is rendered. In that case, the *mere possibility* of inequality implies the formation of the absurdity of data which are the same as the initial emergent content, and the principle of non-contradiction presents itself–cf [11], for example. Therefore reasoning on diagrams captures the very heart of category theory and should be considered as the first order logical structure of category theory as a whole and for mathematics insofar as category theory is foundational[16]. It is for this reason that we analyze this specific 'category' since it really is the universal one (as opposed to **Cat**, cf [46]).

The core of the idea is to think of diagrams as analogues of points; in the case of $C^\infty_X(K) = \{f | f : X \to K\}$, the set of smooth functions from the manifold X into the field K, one has that this set *inherits* the algebraic properties *point-wise*–viz., $f + g := f(p) + g(p)$, any $p \in X$ (obviously similar for multiplication, distributivity). In our case, we are thinking of the *diagrams* now as the points (together with how they 'hang' together–as determined by $\mathrm{Ars}(\Delta)$)[17]. As a result, whenever something holds *on diagrams*, it should hold in the category in general. We should therefore be able to: 1) given a general notion of a subobject classifier; 2) show how this yields colimits in general in topoi assuming limits, exponentials, and a subobject classifier; 3) construct adjunctions 'diagram-wise' (especially the possible classes of them). As a side remark, we reasoned *exactly* in this way when we proved that a category with products and equalizers has all finite limits.

2.8.1 Algebraic Semantics of the Category of Diagrams

Traditionally, a *graph* is determined by a pair of sets, the *vertex* set, X, and the *edge* set, $E \subseteq X \times X/tw$, where $tw : (x, y) \xrightarrow{\sim} (y, x)$ is the twist isomorphism. *Directed graphs* relax the quotient condition, so that $(x, y) \nsim (y, x)$ and an edge can be thought of as an arrow $x \to y$. There appears to be an analogy between $\emptyset \le \theta \le 1$, the adjoint relationships between an arbitrary topology and the discrete, 1, and codiscrete, 0, topologies, and the edge sets $\emptyset \subseteq E \subseteq X \times X$. In the future, we'd like to replace sets with species, so vertices are more like $\xi : X$ rather than $x \in X$. Edges are more complex–we will touch on this later. Since the replacement occurs at the local level, for specific graphs, then if we have a predicate $\alpha(\Gamma_1, ..., \Gamma_n)$ depending on several (traditional) graphs, we can just do $\alpha(\gamma_j/\Gamma_j)$, where we substitute at each graph. We define the category of diagrams to be **Dia**. We first demonstrate that this is in fact a category, then we show that it has all finite limits. After this, we show it has exponentials. Finally, we show the existence of a subobject classifier. The finite co-completeness condition follows immediately. (Actually, we should be able to prove it.)

A *diagram* is a directed graph satisfying the categorial closure axioms (2), (3), (4). Formally, this implies that whenever $e \in E$, its canonical form is $\langle x, x' \rangle$ with $x, x' \in X$ and: 1) if $e, e' \in E$ such that $e = \langle x, x' \rangle$ and $e' = \langle x', x'' \rangle$, then there is an edge $e \sim e' = e'' = \langle x, x'' \rangle \in E$ (we use $e \sim e'$ to denote *contraction*); 2) for ever $x \in X$ there is a special edge $id_x \in E$ given by $\langle x, x \rangle$ with the contraction rules $id_x \sim e = e$ and $e \sim id_x = e$ whenever these make sense; 3) whenever there is a triple e, e', e'' such that $(e \sim e') \sim e''$ and $e \sim (e' \sim e'')$ make sense, then these are equal. We can pass from finite diagrams to compact ones by introducing an *ambient set, H*, which labels our edges, so that we can have $\langle x, x', \eta \rangle$ and $\langle x, x', \eta' \rangle$ as *distinct* edges between the same pair of vertices. Morally, we will operate under the assumption that our edges in the diagrams given do not need this ambient set, but adding it in is effectively trivial and doesn't alter our reasoning. A *map* between diagrams will be called a *(local) functor* and it is given by a pair of *rules* (F_0, F_1). *It should be stressed that these rules are **not** edges.* If we regard them as such, the interpretation of a functor as a *diagram* is problematic. In particular, if $F : \Delta \to \Gamma$ is a 'functor,' do we define it to be the categorial closure of the graph $\langle \mathrm{Obs}(\Delta) + \mathrm{Obs}(\Gamma), \mathrm{Ars}(\Delta) + \mathrm{Ars}(\Gamma) + \sum_{x \in \mathrm{Obs}(\Delta)} \langle x, Fx \rangle \rangle$ or $\langle \mathrm{Obs}(\Delta) + F_0(\mathrm{Obs}(\Delta)), \mathrm{Ars}(\Delta) + F_1(\mathrm{Ars}(\Delta)) + \sum_{x \in \mathrm{Obs}(\Delta)} \langle x, Fx \rangle \rangle$ or $\langle F_0(\mathrm{Obs}(\Delta)), F_1(\mathrm{Ars}(\Delta)) \rangle$? It can't be the first, since this

See below for the section on logic and metamathematics for a discussion of this.

[14] We cannot seem to free ourselves from the need for an extended content that enables (algebraic) manipulation.

[15] There is no 'formal mathematical' definition of this concept–we have to take it as primitive–because doing so introduces an object which itself is a geometric presentation (which therefore induces a vicious circle, cf [70]). Loosely, the reader can think of this as denoting the experience in which a content emerges before him in a way that induces its immediate perception. That emergence manifests itself through the content in a way in which the content is experienced as wholly other to him, as if he were watching it and participating in it rather than producing it or creating it (as though it were a shared musical experience between the observer and observed). The object emerging in the awareness in this process *is* the geometric presentation (of the content). It follows immediately that any attempt at a formal definition of the term is absurd. Indeed, it is on the basis of this observation (together with Kant's *Transcendental Aesthetic*) that we've been led to speculate that this is something like the application of the intuition of space to the inner realm that gives rise to the whole process. It also suggests–if this speculation is told–that this application *cannot possible be sufficient to span the whole of the data in question*. For this reason we place *very strong* emphasis on the constructive point of view, especially in light of the discovery of the unconscious.

[16] Although in our view, this suggests that really type theory should be taken as the fundamental tool.

[17] From this point of view, a category is a special case of something more general–which probably are the ∞-categories. Can come back to this later.

introduces edges like $e' \sim F_1(e)$, with $e' \in \mathrm{Ars}(\Gamma)$. The second introduces problems when we try to define exponentiation. For example, if we take the following

$$\begin{tikzcd} 0 \arrow[r, "\perp"] \arrow[d, "\leq"'] & 0 \\ 1 \arrow[ur, "\perp"] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[d, "\leq"'] \arrow[dr, "\top"] & \\ 1 \arrow[r, "\top"] & 1 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, "Id"] \arrow[d, "\leq"'] & 0 \arrow[d, "\leq"] \\ 1 \arrow[r, "Id"] & 1 \end{tikzcd}$$

How should we define exponentiation? Is it the disjoint sum of these? If so, then it would be no better than $\mathrm{Hom}(\Delta, \Gamma)$ (as a *set* and we lose the fact that $\mathbf{C}^{\mathbf{D}}$ has the intuition of a *category*). Furthermore, how can we define the evaluation functor, and what about the intuition that $\Gamma^\Delta \cong \mathbf{Func}[\Delta, \Gamma]$ (so that it should also include the natural transformations)? On the other hand, if we try to glue these together, we are left with the previous problem (for example, we'd have to add $\leq \circ \perp : 0 \mapsto 1$). Defining it as a rule allows us to take subdiagrams of the codomain diagram as defining a functor, as long as the following rules hold: 1) for each $x \in X$, $F_0(x) \in X'$; 2) for each $e \in E$, $F_1(e) \in E'$; 3) $F_1(e) = F_1(\langle x, y\rangle) = \langle F_0(x), F_0(y)\rangle$. If there is an ambient set, F requires three components, with the third a mapping $F_\smile : H \to H'$ (i.e., short for $\eta \in H \Rightarrow F_\smile(\eta) \in H'$) and with 3′) now reading $F_1(e) = F_1(\langle x, y, \eta\rangle) = \langle F_0(x), F_0(y), F_\smile(\eta)\rangle$. Under this view, the functor itself is not a diagrammatic object, but its image is. On the other hand, natural transformations *can* be defined graphically in an unambiguous way[18]–interestingly, this seems to parallel the natural isomorphism $V \cong V^{**}$ independent of basis, which is one of the original motivating examples of Eilenberg and Mac Lane's original 1945 paper that started the theory of categories. A *natural transformation* between a pair of functors $\eta : F \to G$, with $F, G : \Delta \rightrightarrows \Gamma$ is a subset of the edge set in the codomain diagram specifying a subgraph determined by (the categorial closure of) $\langle F_0(X) + G_0(X), F_1(E) + G_1(E) + \sum_{x\in X}\langle F_0(x), G_0(x)\rangle\rangle$–indeed, η can be defined by $\sum_{x\in X}\langle F_0(x), G_0(x)\rangle$ and the *components* of the natural transformation are $\eta_x = \langle F_0(x), G_0(x)\rangle$. Now the problem that functors posed just becomes the *naturality condition* $\langle F_0(x), F_0(y)\rangle \sim \langle F_0(y), G_0(y)\rangle = \langle F_0(x), G_0(x)\rangle \sim \langle G_0(x), G_0(y)\rangle$, so that the categorial closure is indeed a diagram and introduces no additional non-trivial arrows. From this point of view, a functor can be thought of as selecting a subdiagram from the codomain diagram (subjected to structural conditions of the domain) and natural transformation is a method of passing between (admissible) subdiagrams selected by a given domain diagram. The presheaf object $[-, \Gamma]$ then can be viewed as the possible functors into Γ and is synonymous with the possible subdiagrams which can be selected. Substitution (or really evaluation) at the contravariant component of this bifunctor selects *which* species of diagrams one wishes to consider, and (I think) parallels a (constructive) method for selecting a topology on a space. Compare our comments above regarding edge sets and the adjoint relationship to topology. Conversely, $[\Delta, -]$ becomes the species of functors which Δ *can select*–this object is a bit more difficult to envision. The presheaf object can be thought of as a sort of bundle over the codomain diagram, but this one seems to require us to think of the presheaf object *first* and then conceptualize this by considering the dual to that notion.

To demonstrate finite limits, we just show finite products and equalizers; we sketched this above. The empty product is just the terminal diagram, $\mathbf{1} = \langle *, id_*\rangle$, and binary products are defined by $\Gamma \times \Delta = \langle X \times Y, E \times E'\rangle$, with the rules $x \in X$, $y \in Y \Rightarrow (x, y) \in X \times Y$ (this is just a sloppy way of writing a natural introduction rule), and $e \in E$, $e' \in E' \Rightarrow \langle e, e'\rangle \in E \times E'$. Then $e = \langle x, x'\rangle$, $e' = \langle y, y'\rangle$ implies $\langle e, e'\rangle = \langle\langle x, x'\rangle, \langle y, y'\rangle\rangle = \langle (x, y), (x', y')\rangle$. (This is an example of a *computation rule*–see the theory of types below.) We just need to check product diagrams are in fact diagrams and that this is a limit. Ultimately, the fact that this is a diagram just means that composition is well-defined when it is in the diagrams forming the product. We check $\langle e_1, e_1'\rangle \sim \langle e_2, e_2'\rangle := \langle e_1 \sim e_2, e_1' \sim e_2'\rangle$, which defines an edge $(x, y) \to (x'', y'')$ given the edges $\langle e_1, e_1'\rangle : (x, y) \to (x', y')$ and $\langle e_2, e_2'\rangle : (x', y') \to (x'', y'')$. Clearly this extends to the categorial closure. Induction furnishes the case for $n > 2$ diagrams. Uniqueness follows after defining the projection functors $\pi_{\Delta,\Gamma}(\Delta\times\Gamma) = \Delta, \Gamma$ respectively. Given a test diagram Ξ and a pair of functors F, G with $F : \Xi \to \Delta$ and $G : \Xi \to \Gamma$, then the functor $u : \Xi \to \Delta\times\Gamma$ must be given by $F = \pi_\Delta \circ u$ and $G = \pi_\Gamma \circ u$. This requires $u = \langle F, G\rangle$, particularly because the functors are defined on the *whole* diagram Ξ. *Equalizers* for a pair of functors $F, G : \Delta \rightrightarrows \Gamma$ will be defined as a pair consisting of a subdiagram δ and a functor $\iota : \delta \leq \Delta$. This is abuse of notation. In reality δ is 'built' by the condition: *those* $x \in X$ *for which* $F_0(x) = G_0(x)$ *and those edges* $e \in E$ *for which* $F_1(e) = G_1(e)$. Now this doesn't imply the existence of a mode of constructing the $\delta \leq \Delta$ such that $F(\delta) = G(\delta)$ (but it does for finite diagrams; see [11, 39]), but supposing we *had* one that satisfied these criteria, we first show that δ is a diagram–which again amounts to showing that it is closed under contraction. If $e, e' \in \delta$, then $F(e) = G(e)$ and $F(e') = G(e')$. By the properties of functors, $F(e) \sim F(e') = F(e \sim e') = G(e \sim e')$, so that $e \sim e' \in \delta$. For this δ (and its inclusion) to be an equalizer, suppose we had any other $h : \Xi \to \Delta$ such that $Fh(\Xi) = Gh(\Xi)$, then $F(h(\Xi)) = G(h(\Xi))$, so that $h(\Xi) \leq \delta$, since δ defined to be the *maximal* subdiagram in Δ such that $F(\delta) = G(\delta)$. Then put $u = h|_{\mathrm{cod}(h)=\delta}$–which is well-defined by $h(\Xi) \leq \delta$–this is uniquely defined

[18] Although if we wanted to think that a natural transformation should give a G-diagram given an F-diagram, then we would need to regard it as a rule as well. On the other hand, we don't *need* to, but to recover this 'usual' perspective we *would* need another functor, viz. something like cod. In either case, we would need to rely on a process or rule which cannot be represented internally. This is why I've thought that exponential objects are in a sense absurd.

because its only variable aspect depends on h, but then given h this map is totally determined. All finite limits follow[19,20].

Exponentiation can be conceived of as a diagram whose vertices are the functors and the edges are the natural transformations. This is short-hand for the subdiagrams affiliated with the functors and the subsets of the edge set in the codomain diagram that are the natural transformations between any given pair. That is,

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & G \arrow[d, "\tau"] \\ & K \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(\Delta) \arrow[r, "\{\eta_\Delta\}"] \arrow[dr] & G(\Delta) \arrow[d, "\{\tau_\Delta\}"] \\ & K(\Delta) \end{tikzcd}$$

where we use the abuse of notation $\{\eta_\Delta\}$ to denote the components of the natural transformations as we outlined above. In order to obtain the structure of exponentiation, we also need to define the evaluation map–which is determined by $ev_{\Delta,\Gamma} : \Gamma^\Delta \times \Delta \to \Gamma$, and can be thought of as $\langle F, x\rangle \mapsto F_0(x)$ (i.e., just selects the object determined by x in the (sub)diagram indicated by (the rule) F) and $\langle \eta, e\rangle = \langle \eta, \langle x, y\rangle\rangle \mapsto \langle F_0(x), G_0(x)\rangle \sim \langle G_0(x), G_0(y)\rangle = \langle F_0(x), F_0(y)\rangle \sim \langle F_0(y), G_0(y)\rangle$, which is well-defined by the naturality condition and gives us an arrow in Γ as an output. Importantly, this makes good sense when we consider the following pair of diagrams

$$\begin{tikzcd} \Gamma^\Delta \\ \\ \mathbf{1} \arrow[uu, "\ulcorner F \urcorner"] \end{tikzcd} \qquad \begin{tikzcd} \Gamma^\Delta \times \Delta \arrow[rr, "ev_{\Delta,\Gamma}"] & & \Gamma \\ \\ \mathbf{1} \times \Delta \arrow[uu, "\ulcorner F \urcorner \times Id_\Delta"] \arrow[uurr] \arrow[rr, "\pi_\Delta"'] & & \Delta \arrow[uu, "F"'] \arrow[ul, phantom, "\Leftarrow \sim"] \end{tikzcd} \tag{27}$$

where we should obtain the image of F in Γ. It's easy to check that this is the case. If we replace the short-hand functor category by the elementary version, how do our definitions change? We will need to check the calculations, but *I think* it should be done as follows: put $F \to \{Fx\}$ and $\eta \to \{Ff\} + \{\eta_x\}$, all F, f, η, x respectively, for the objects and arrows of Γ^Δ. Then the evaluation functor can be given by the rules: $ev_{\Delta,\Gamma}[(F_0(x), y)] = F_0[y/x]$ for the object evaluation and $ev_{\Delta,\Gamma}[(\langle F_0(x), F_0(y)\rangle, \langle u, v\rangle)] = \langle F_0[u/x], F_0[v/y]\rangle$ and $ev_{\Delta,\Gamma}[(\langle F_0(x), G_0(x)\rangle, \langle u, v\rangle)] = \langle F_0[u/x], G_0[v/x]\rangle$ for the arrows. Then if we check the same diagram as above in this definition, we should be able to tell whether this definition is correct. Actually, it's easy to see that this should work at least in the case of the assignment of the natural transformation; for there is assigns at each component the arrow $\langle F_0(x), G_0(y)\rangle$, which is the naturality condition. In a sense, the natural transformation is overspecified, since the substitution condition as evaluation implies that *any* $\langle F_0(x), G_0(x)\rangle$ will do. Indeed, if one carries out the same calculation with the diagram $\mathbf{1} = \langle *, \langle *, *\rangle\rangle$, then one selects in Γ^Δ the assignments $F_0(x)$ for the object and $F_1(id_x)$ for the arrow; the substitution rules will then generate all edges.[21] Another way to look at this, in accordance with the outlined method of describing functors as rules, we can take $\Gamma^\Delta \times \Delta$ as

$$\begin{tikzcd} F(\Delta) \times \Delta \arrow[r, "\{\eta_\Delta\} \times \mathrm{Ars}(\Delta)"] \arrow[dr] & G(\Delta) \times \Delta \arrow[d, "\{\tau_\Delta\} \times \mathrm{Ars}(\Delta)"] \\ & K(\Delta) \times \Delta \end{tikzcd}$$

Then evaluation ought to be defined as analogous to the above.[22] It's important to note that *if* we admit the formation of function types, so that spaces of functions becomes a sensible notion (i.e., processes can be viewed as reduced to contents), then phrasing the diagram with vertices valued in the function type becomes well-defined and we can work with the functors

[19] Again, this *requires* the *assumption* that we have a rule $(F, G) \to \delta$ satisfying the appropriate properties; clearly if we assume our diagrams are finite then there are brute force methods for determining this. On the other hand, if the diagram is merely compact, the conditions for determining δ are insufficient to totally determine it. Upon reflection, it appears that finding pullbacks instead would suffer from the same problem.

[20] Actually, it seems that colimits will be more easily defined in terms of the manner of thought here; clearly if we are working classically and with sets, there are no issues, but equalizers pose a problem. On the other hand, coproducts are easily conceived (just by laying diagrams down together) and quotients can be specified as a rule that is given by a type of contraction on the codomain diagram. Incidentally, this actually really shows Grothendieck's genius–because the maps *out of* the codomain are always *co*variant and therefore to enforce changes, one need only propagate the necessary changes *forward* rather than having to invoke pullbacks.

[21] Is this an example of Yoneda?

[22] This method seems to suggest $\Gamma^\Delta \cong \Gamma$. I think the difference here is that the objects of Γ become equivalence classes of arrows in Δ that allow for exponential diagrams as we've outlined previously.

'directly'.

At this point, we've been working with F as though it comes from an *ambient* species which must accompany the structure; we might imagine circumventing this issue by employing a pairing $\langle x, F_0(x)\rangle$ and $\langle e, F_1(e)\rangle$. At this time, I am unsure whether this is the appropriate solution–since it suggests redefining F in this way; *certainly* F as a rule *implies* this construction, but that presupposes a name for the process prior to the construction, which is problematic. On the other hand, we might say that these pairs *represent* F, and that $F(\Delta)$ for any diagram Δ is just the image of the rule $\langle \Delta, \gamma\rangle \xrightarrow{\pi_\gamma} \gamma = F(\Delta)$ subjected to the compatibility conditions.[23] What I think we need to avoid is the intuition that what constitutes a functor admits the construction of a functor *space*. In the case of exponentiation, this is possible insofar as the contents which form the substance of the diagram are differentiated. I *think* the correct intuition for the notion of function is along the lines of Curry's idea of operation in [16] and [17]. He juxtaposes this with the algebraic perspective, which he asserts is conceptualized "as a function which assigns a value to each finite sequence of specified number of obs. This terminology suggests that the obs are given beforehand and the operators are correspondences established among them." In [16], they "are regarded as steps of construction, which may be iterated indefinitely, for forming new obs...what is given beforehand if a class of primitive obs and a class of operations...the obs are generated from the primitive obs by the operations, and eventually an ob is identified...with a construction from the primitive obs by the operations...an operation is here regarded as forming a new object rather than as assigning a value." Jung views the function as a substantive complex of affective data which together form a unity and whose action on the psyche is given by the result of assimilation of experiential data. Actually, in this sense, Jung would not view the $\lambda o\gamma o\varsigma$ as an archetype but as a mere function–although probably he would be open to the argument that these find their genesis in archetypal data. Regardless of this, it is certainly the case that a function names a process far more similar to the notion Curry presents. Indeed, the essential property of Jung's approach is its *synthetic* character. In this sense, an analytic representation of a functor really ought to be thought of a sequence of diagrams–as in the sense of Brouwer. For example, in our various proofs above, we often employ $\Delta \rightsquigarrow \Gamma$. This reduction really is a 'function' (or a functor).

In this sense, we can 'get around' the problem of a function space by adopting the following disposition: begin with the objects of sets–i.e. fully differentiated contents. Then we *define* one of the sets to be the vertices and the other to be edges. An arrow between a pair of vertices exists as soon as the set which we've identified as the representation of the collection of edges between the vertices in question can be proven to be substantive, and we've given the rule(s) for associating an edge to a pair of vertices. In this sense, we can obtain the following construction, as an example

$$
\begin{tikzcd}
c & \rightsquigarrow & c & & c' & \rightsquigarrow & c \arrow[r, "f"] & c' \\
 & \rightsquigarrow & c \arrow[rr, "f"] & & c' & & c'' & \\
 & \rightsquigarrow & c \arrow[rr, "f"] & & c' \arrow[rr, "g"] & & c'' & \\
 & \rightsquigarrow & & & c' \arrow[drr, "g"] & & & \\
 & & c \arrow[urr, "f"] \arrow[rrrr, "g\circ f"] & & & & c'' & \\
 & \rightsquigarrow & & c \arrow[r, "g\circ f"] & c'' & & &
\end{tikzcd}
$$

which is accompanied by the following reasoning: we observe the existence of c; we observe the existence of c' along side c; we observe the existence of a substantive connection $f : c \to c'$; we observe the existence of c'' along side this process; we observe the existence of a substantive connection $g : c' \to c''$; we employ the axiom that facilitates the construction $(f; g) = g \circ f$. This perspective does not a priori require the notion of a function space, but rather the determination of a

[23] We need to continually invoke things like this. They are problematic as well, since they implicitly contain the predicate: *those diagrams satisfying the conditions...*–but this *sort* of phrasing always implies a comprehension principle, which is inherently nonconstructive, since it really ought to be phrased: *there **exist** diagrams such that...*

species as exhibiting contents whose *interpretation* are these primitive connectors. This is primitive (in fact, it's essentially a form of an introduction rule); we might think $f := \langle c, c' \rangle$, but this is only the case when no ambient set H is needed to label edges. The need to invoke this actually demonstrates the perspective we are proposing here. A diagram then can be viewed as a species together with a (disjoint) union of species indexed by the substance of the object species. In this sense, we have a species X, then the edges are $\sum_{x,x' \in X} E(x, x')$ so that whenever $e \in E(x, x')$, then $e = \langle x, x' \rangle$ or $e : x \to x'$. Thus, a *canonical* element of this species is an object *indexed* by the substance of X, which conforms to Curry's perspective. (Technically, if $y \in \sum_{x,x' \in X} E(x, x')$, then $y = (x, x', e(x, x'))$ as a canonical element. Compare [50].)

In regards to the subobject classifier, we recall that this essentially entails an assignment $\delta \xrightarrow{\top_\delta} \Omega$ of the subdiagram $\delta \leq \Delta$ to *true* $\in \Omega$. If we notice that the arrows are functors in the metacategory, it is easy to see how the subobject classifier Ω is

$$\begin{tikzcd} \perp \arrow[loop left, "id_\perp"] \arrow[r, bend right, "u"'] & \top \arrow[l, bend right, "d"'] \arrow[loop above, "l"] \arrow[loop below, "id_\top"] \end{tikzcd} \tag{28}$$

2.8.2 Reasoning in Categories as Diagrams

The essential idea is that we want to quantify over the diagrams given a canonical diagram. In the phrasing we've outlined in the algebraic sematics, how would this express itself? Letting $\Delta = \langle \mathrm{Obs}(\Delta), \mathrm{Ars}(\Delta) \rangle$. If Δ exhibits a prototypical property, then *I think* the idea is that whenever a substitution in the diagram makes sense, the property it exhibits continues to hold–for example, (2) holds *for all objects c, c', c'' and arrows f, g whenever these make sense*.

3 Synthetic Differential Geometry

The essential idea here is that we'd like to analyze an object that is supposed to be conceptualized as a synthetic unity, whose essence cannot be captured it its entirety, and which requires the introduction of a local analytical representation in terms of infinitesimals. These infinitesimals are supposed to arise out of an exhaustive analysis of the continuum, and they are geometrically to be thought of as infinitesimal straight line segments. To proceed with the synthetic method, one selects two distinguished points; first, the point of fundamental symmetry, which we denote by 0, and second, the point of unit measure, which we denote by 1. Clearly sums of segments are just given by concatenation of them (assuming translations are native to the geometry). We want to define products. [41] suggests that we should use Euclid's method in the plane to find a fourth proportional. But this requires the synthetic space of at least two 'dimensions'. For now, we might assume there exists a method of multiplication $R \times R \xrightarrow{\mu} R$ satisfying the usual axioms. In fact, algebra on the line appears to be highly nontrivial if we wish to preserve the purely synthetic structure. The very notion of segmentation of the space implies the ability to impose a type of distinguishability. For if the object is a synthetic whole, the idea that it can be fragmented is absurd; on the other hand, the ability to impose algebraic operations requires the notion of, for example, a segment length s. Indeed, the rule above would produce $(s_1, s_2) \mapsto s_1 \cdot s_2 \subset R$. Here it seems clear that we need to invoke the analytic faculty that can facilitate this segmentation. In our case, we see this as the hypothesis due to Brouwer that neither the discrete nor the continual can be 'done away with', and that each is a fundamental intuition. In that case, the 'synthetic' method consists in an approach where the synthetic *disposition* dominates, but not to the extent where analytic *contents* are totally removed. In fact, in the approach that follows, we still speak about 'membership' and use expressions like $s \in R$. Even supposedly 'purely' synthetic methods in, say, computability theory are not really that–they require the initial hypothesis of analytically precipitated data which one can then manipulate according to the 'geometry' of the space in which they live. This is the tie back to the geometric presentation and adds *body* to the calculus so considered. As for this section, we will denote this geometric object by R and for the moment we will take for granted the usual algebraic axioms in R.

In fact, we will impose abstract *processes* which then will need to find their home in a *geometric presentation* as with multiplication and Euclid's proportional fourth. This will be generalized. Furthermore, an algebra of infinitesimals will characterize *completely* the local data which then will be subjected to an invariance principle which will allow the extraction of *global* structure. This will play the role of the event space, and the measure will be the 'mapping' assigning affective data to the (relevant) processes within the synthetic unity of a given archetype.

3.1 Infinitesimals

The synthetic method is wholly contingent upon the introduction of a particular species–viz.

$$D := \{d | d^2 = 0\} \tag{29}$$

where these $d \in R$ so that $D \subset R$. We further want to lay the foundation of what Bell calls *microlinearity* [6], which is the intuition that for points sufficiently close to a given point, the geometry of the space is Euclidean so that there is a *unique* line segment connecting the point to its neighboring points. That is

Axiom 3.1. For all $g : D \to R$ there is a unique $b \in R$ such that for all $d \in D$

$$g(d) = g(0) + d \cdot b \tag{30}$$

which produces as an immediate consequence

Theorem 3.2.

$$D \neq \{0\} \tag{31}$$

Proof. Take the following functions: $g_a(x) = a \cdot x$, $g_b(x) = b \cdot x$. Then $g_a(0) = g_b(0) = 0$. Then we get $g_a(d) = a \cdot d$, $g_b(d) = b \cdot d$, all $d \in D$. By the axiom, a and b are unique (and therefore distinct) if $g_a \neq g_b$. On the other hand, if for all $d \in D$, $a \cdot d = b \cdot d$, then the axiom implies $a = b$. But if $D = \{0\}$, then for every $(a, b) \in R^2$, $a \cdot 0 = b \cdot 0$ so one cannot evaluate whether $a = b$ or $a \neq b$, immediately contradicting the axiom. Then $D \neq \{0\}$. □

Strangely, despite this fact, whenever $d \in D$, we cannot evaluate the proposition $(d = 0) \vee \neg(d = 0)$. Consequently, we cannot deduce $\neg(D \neq \{0\})$. We can see this due to the following construction: let $g : D \to R$ be defined by

$$g(d) = \begin{cases} 1 & d \neq 0 \\ 0 & d = 0 \end{cases}$$

We can effect such a construction under the hypothesis of the law of excluded middle, for in the case that it holds in R, we have that for every $x \in R$, $(x = 0) \vee (x \neq 0)$. Consequently, we can define a function

$$g(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

By our Axiom 3.1, and using $g(0) = 0$, we observe that the previous theorem requires $D \neq \{0\}$ so that there must be $d' \in D$ with $d' \neq 0$. This gives rise to the above function. The axiom then requires $1 = g(d) = b \cdot d' \Rightarrow 1 = 0$. Now this approach isn't 'fool-proof' (for what is b here?)–furthermore, I think it suggests strongly that the *very notion* of a point in R cannot be considered; in fact, if $S \subset R$, then $S + D \sim S$. In particular, $\{x\} + D \sim \{x\}$ accordingly. But then there is no *point*, but really something more like a smoothly translated species of infinitesimals. Here '$\sim$' denotes the fact that the two species in question are *congruent*, meaning that we cannot distinguish between them (for if we could, then $D = \{0\}$ or not $D = \{0\}$, which is absurd).

4 Mathematical Logic and Computation

Logic and computation have become effectively synonymous as a consequence of Post's observation in his PhD thesis that one can render logical structures in a computable manner, and that every such logical assertion is representable as such. This was extended by Kleene, for example, in [39]. This section in particular includes a discussion of the metamathematical perspective, the development of the λ-calculus, elementary logical systems, Gödel's theorems, and the notion of a Turing machine. We then will touch on how this connects back to Brouwer's original vision and discuss the modern theory of Intuitionistic Type Theory.

4.1 Intuitionistic Type Theory

This section follows [50].

5 Probability and Measure Theory

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