Notes on Path Integration in Micro-Ring Resonators

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Contents

1 Overview

The purpose of these notes is to summarize the basic findings of path integration in the context of micro-ring resonators (MRRs). The principle reason for doing this is to give a physically motivated foundation for mathematical methods in calculating what is known as the S matrix. This matrix is responsible for relating input states to output states and therefore encodes information about the transition amplitudes of the linear optical devices we will consider. Generally, our goal is to lay forth this foundation for a natural project that could expand the methods here to allow for generalized analysis and characterization of arbitrary but finite planar arrays of MRRs.

2 Theoretical Introduction

For the purposes of these notes, the elementary quantum circuit element is depicted in Fig. 1. Importantly, the local couplings are represented by linear transformations from the group $SU(2)$ (which gives some consistency conditions on the coupling parameters). Often we label the coupling parameters at the top η, γ and at the bottom τ, κ. The S matrix can then be extracted by various means. The existing standard methods are the mode swap algebra, the boundary condition method, the path integral formulation, and the Mason gain method. The existing literature on the first two methods is well-explored [4, 3, 1]. In these notes, we are interested in the relationship between the latter two, which lays the foundation for the application of the latter more generally among the ring resonator circuits one would wish to consider.

2.1 Path Integration in MRRs

Here we develop the formal theory of the path integral treatment. It is based on the intuition that one can think of the quantum systems in a 'classical' way (we don't mean this technically). Loosely, one can think of a photon as a localized quantum of electromagnetic energy with a well-defined position and momentum, whose dynamics are determined by the confinement and scattering processes occurring within the circuit element. Each coupling then becomes a location of a possible scattering event, and the amplitude associated to the event outcome is determined by the parameters at the coupler itself. For example, the probability amplitude associated for a photon to pass from the input port $\hat{a}_{in}$ (resp. $\hat{b}_{in}$) to the output port $\hat{a}_{out}$ (resp. $\hat{b}_{out}$) without entering the ring itself is just η (resp. τ) and can be represented by a diagram. This diagram is used to denote this process. The path integral formulation then just says that the total amplitude for a given transition is just the sum over all possible (time-ordered) paths that connect a given input and output port. To illustrate this method, we can work out the basic details for the single ring case of Fig. 1. By symmetry, there are only two distinct calculations: $\hat{a}_{in} \to \hat{a}_{out}$ and $\hat{a}_{in} \to \hat{b}_{out}$. The other two will be obtained by an appropriate set of substitutions which we will detail shortly. Considering first $\hat{a}_{in} \to \hat{a}_{out}$, if

$$\begin{pmatrix} \hat{a}_{out} \\ \hat{b}_{out} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_{in} \\ \hat{b}_{in} \end{pmatrix} \tag{1}$$

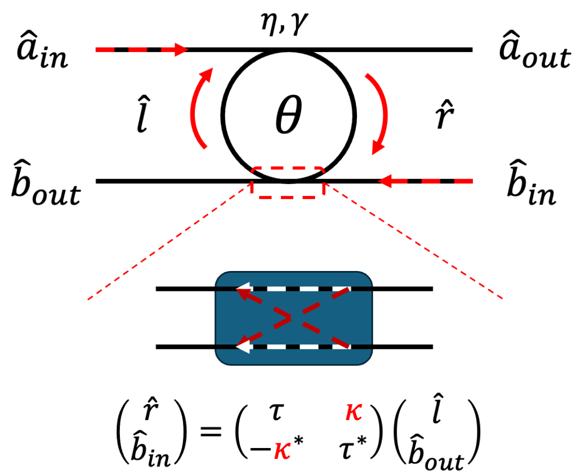


Figure 1: Elementary circuit element.

then clearly $S_{11} : \hat{a}_{in} \mapsto \hat{a}_{out}$, so that we are calculating the S_{11} transition element. If Feynman's observation is correct, then

$$S_{ij} = \sum_{\text{path } n} S_{ij}^{(n)} \tag{2}$$

where $S_{ij}^{(n)}$ is the amplitude associated with *some* path $i \to j$ and the sum is taken over all such (ordered and indexed by n). Geometrically, for S_{11}, this means we have the trivial path, then the path with a single loop around the ring, then two loops, etc.

Now let us introduce the formal structure we will be using to reason about these processes. Let us first give an example, then lay down the general rules and discuss them. For this case we are considering, consider the case where we pass around the ring n times. Evidently this means in the time ordered process, we need to first scatter off the coupler into the ring, pass around the first half of the ring to the bottom coupler, then scatter through the bottom coupler, up around the ring, through the top coupler, down around the ring, and repeat this set of steps $n-1$ additional times, then scatter up through the left half of the ring and out at the top coupler through the out port. In particular, notice we need to distinguish between whether we are entering a half ring from the rail or the ring itself. This means we need a pair of distinct symbols for either case. For this case, we will present this process as

$$1_{in} \to \bigcirc^n \to 1_{out} = ⤵, ⤹, ⤵, ..., ⤹, ⤵, ⤹, \uparrow \tag{3}$$

Now we should always keep in mind that this is an *amplitude*, so we just need to work out what that would be. Using our intuition of the process as a scattering process, the amplitude obtained at the first coupler is γ, the amplitude associated with scattering into the ring, this means the output state after this first scattering event is just

$$\psi_{out}^{(1)} = \gamma \psi_{in} \tag{4}$$

This output state becomes the input state for the next event, which is the accumulation of the phase ϕ, say, along the right half of the ring. The output is then

$$\psi_{out}^{(2)} = e^{-i\phi} \psi_{out}^{(1)} = e^{-i\phi} \gamma \psi_{in} \tag{5}$$

We continue in this way; evidently this means that the total output is just the product of all the amplitudes which contribute to the outcome in question–and in this case is

$$S_{11}^{(n+1)} = \gamma e^{-i\phi} (\tau^* e^{-i(\theta-\phi)} \eta^* e^{-i\phi})^n \tau^* e^{-i(\theta-\phi)} (-\gamma^*) \tag{6}$$

which is the basis for the rule formation which we outline now. To do this, we define a function which assigns to each arc an associated amplitude, this assignment essentially serving as an explanation for the interpretation of the arc (or whatever other symbol we need). Clearly if $\Sigma = \{\sigma_1, ..., \sigma_N\}$ denotes the collection of elementary scattering processes $\{\sigma_j\}$, then a process $i \to j = \langle \sigma_{k_1}, ..., \sigma_{k_n} \rangle$ is a sequence of these processes. Here, i is the i^{th} input port, and j is the j^{th} output port. The amplitude $S_{ij}^{(n)}$ associated to this process is defined by a map $\chi^n : \Sigma^n \to \mathbb{C}$ given by the rule

$$\chi^n(\langle \sigma_{k_1}, ..., \sigma_{k_n} \rangle) = \prod_{l=1}^{n} \chi(\sigma_{k_l}) \tag{7}$$

where $\chi = \chi^1 : \Sigma \to \mathbb{C}$ assigns an elementary amplitude from the circuit element in question into the complex numbers. We can therefore see that the map is totally determined by specifying χ. Since Σ is a finite set, we just need to give the rules for each of its elements. We can do this as follows

$$\chi\left(\overset{\circ}{)}\right) = \gamma e^{-i\phi} \tag{8}$$

$$\chi\left(\overset{\bullet}{)}\right) = \eta^* e^{-i\phi} \tag{9}$$

$$\chi\left(\underset{\circ}{(}\right) = \kappa e^{-i(\theta-\phi)} \tag{10}$$

$$\chi\left(\underset{\bullet}{(}\right) = \tau^* e^{-i(\theta-\phi)} \tag{11}$$

$$\chi\left(\uparrow\right) = -\gamma^* \tag{12}$$

$$\chi\left(\downarrow\right) = -\kappa^* \tag{13}$$

$$\chi\left(\longrightarrow\right) = \eta \tag{14}$$

$$\chi\left(\longleftarrow\right) = \tau \tag{15}$$

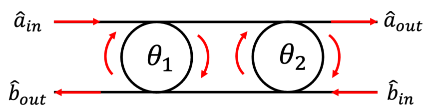
Here we assume directionality and only consider a single ring. Obviously there are other arcs which are logical possibilities generally but which do not apply in this case (e.g., $\overset{\circ}{(}$) due to time-ordering and the directionality of the circuit element in consideration. One can generalize this mapping; this specifies the amplitude associated to a specific process totally. Importantly, because these are $\mathbb{C}$-valued, there is a natural commutativity in the ordering which allows the reduction of a process down to an *effective* representation. For example, in this case the whole process is

$$S_{11}^{(n+1)} \doteq \overset{\circ}{)}, \bigcirc^n, \underset{\bullet}{(}, \uparrow \tag{16}$$

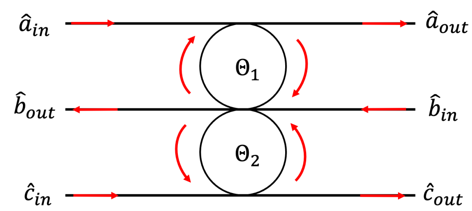
This suggests the definition of rules more generally which we might call 'Feynman rules' (although probably only suggestively). These rules are as follows

1. There are *open arcs* and *closed arcs*. Closed arcs are always composed of concatenations of open arcs and will be called *loops*.
2. Arcs have an *orientation* (right or left respectively). Whenever open arcs appear in succession, they must alternate in orientation. If any process other than an arc occurs after an arc, and if any scattering event represented by an arc occurs again, it must also be in the opposite orientation to the one nearest to it in reverse time-order. When two arcs pair, we call it a *proper pairing*.[1] An *even process* is one in which every open arc is properly paired with another; an *odd process* is one in which there is exactly one unpaired open arc remaining after all proper pairings. All other processes where two or more open arcs occur will be called *nonphysical*. Non-arc processes are always properly paired with themselves.
3. Open arcs are always input-inclusive and output-exclusive. An open circle denotes the transition from a rail into a ring, and a closed circle (dot) denotes a transition from within the ring into the ring. This will be called the *arc filling*, with the open circle called an *open arc filling* and dot called a *closed arc filling*.
4. The trivial scattering processes are denoted by left and right arrows respectively. Every sequence ends with an output–the transition out of the ring into the rail is denoted by a vertical arrow. The vertical arrow points in the direction of the rail from the ring.
5. Closed loops *always* start at the bottom of the bottom most ring participating in the process. Closed loops always have to end where they start.
6. Paired open arcs of the same arc filling concatenate into loops. For the simplest case, only open arcs of closed arc filling concatenate, but larger arrays will allow for more complex loops.

[1] This is old terminology due to Kleene regarding pairings of parenthases in well-formed logical formulas. See, for example, his *Introduction to Metamathematics*.



(a) MRRs in parallel



(b) MRRs in series

Then using our mapping rules, we have

$$S_{11}^{(n)} = \chi^n \left(\circlearrowright, \bigcirc^n, \circlearrowleft, \uparrow\right) = \chi\left(\circlearrowright\right)\chi\left(\bigcirc\right)^n\chi\left(\circlearrowleft\right)\chi\left(\uparrow\right) = -|\gamma|^2\tau^* e^{-i\theta}(\eta^*\tau^* e^{-i\theta})^n \tag{17}$$

Then using this reasoning for the whole amplitude

$$S_{11} \doteq \longrightarrow + \circlearrowright, \circlearrowleft, \uparrow + \circlearrowright, \bigcirc, \circlearrowleft, \uparrow + \circlearrowright, \bigcirc, \bigcirc, \circlearrowleft, \uparrow + ... = \eta - |\gamma|^2\tau^* e^{-i\theta}\sum_{n=0}^{\infty}(\eta^*\tau^* e^{-i\theta})^n = \frac{\eta - \tau^* e^{-i\theta}}{1 - \eta^*\tau^* e^{-i\theta}} \tag{18}$$

Carrying this calculation out explicitly shows, for example, that there is an effective procedure for a reduction of a given process to an effective one. It also shows that, while we use the diagrams to keep track of time-ordering, each diagram corresponds to an amplitude (i.e., a complex number), so that the diagrams can be permuted around *within a given process* without changing the value of the amplitude itself. Ultimately this suggests the possibility of the construction of a closed form for general S-matrix elements. It becomes even clearer when we consider more complex circuit elements like those shown in Figs. 2a, 2b. For the parallel case, we do not need to modify the set of elementary open arcs, but we do need to introduce an indexing to denote whether we are considering the first or second ring. We *will* observe an additional loop, however, obtained by concatenating $\circlearrowleft_1$ with $\circlearrowright_2$, producing a loop which passes down into the rightmost arc of ring 2, exits out into the bottom rail, passes into the leftmost arc of ring one, exits the ring, and completes the loop. We can denote this using $\bigcirc_{1+2}$.

MRRs in series are significantly more complicated. The reason for this is that the central junction in Fig. 2b is actually a 3×3 unitary matrix. This is reflected in Fig. 3. This is due to the so-called *supermodes* at the central coupler which allow the photons to transition directly to the upper/lower ring from the lower/upper ring respectively. Interestingly, the open arcs remain the same, but we need to introduce an additional symbol for the arc fillings. We can do this just by using a different filling color, for example. Doing this, we still see that our rules basically hold, but how we form sequences of processes requires additional care in our thought.

2.2 Mason's Gain Formula

See [2]. The basic idea is to map a ring resonator circuit into a directed, reflexive graph and then calculate something called a graph determinant. One can analyze general circuits by considering the natural isomorphism suggested by replacing the basic circuit element (Fig. 1) by the graph

$$\begin{tikzcd}
\circ_{1,in} \arrow[r, "\eta"] \arrow[dr, "\gamma"] & \circ_{1,out} \\
\circ \arrow[r, "\eta^*"] \arrow[ur] & \circ \arrow[d, "e^{-i\phi_1}"] \\
\circ \arrow[u, "e^{-i\phi_2}"] & \circ \arrow[l, "\tau^*"] \arrow[dl] \\
\circ_{2,out} & \circ_{2,in} \arrow[l, "\tau"] \arrow[ul, "\kappa"]
\end{tikzcd} \tag{19}$$

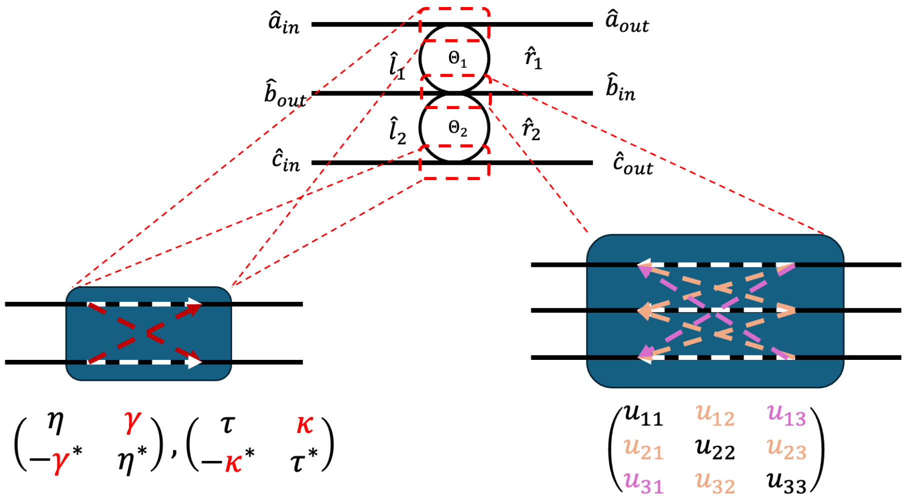


Figure 3: MRRs in series with expanded couplers

where the arrows are labeled by the transfer amplitude accrued along the elementary path. We take the convention of γ, κ indicating paths into the ring. To avoid messy notation, we suppress the amplitudes out of the ring, which are given by $-\gamma^*, -\kappa^*$ respectively. We encourage the interested reader to read Mason's papers. To calculate the amplitude for a given $in \rightarrow out$ amplitude, one sums over open paths connecting the input and output nodes,

References

[1] E. E. Hach III. Afrl report. 2012.

[2] Samuel J Mason. Feedback theory: Further properties of signal flow graphs. Technical report, Research Laboratory of Electronics, Massachusetts Institute of Technology, 1956.

[3] D. G. Rabus. *Integrated Ring Resonators.* Springer-Verlag, Berlin, 2007.

[4] R. E. Scott, P. M. Alsing, A. M. Smith, M. L. Fanto, C.C. Tison, and E. E. Hach III. Scalable controlled-not gate for linear optical quantum computing using microring resonators. *Phys. Rev. A*, 100:022322, 2019.