Kalman Filter Algorithm in Track Reconstruction

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Why is Track Fitting Important?

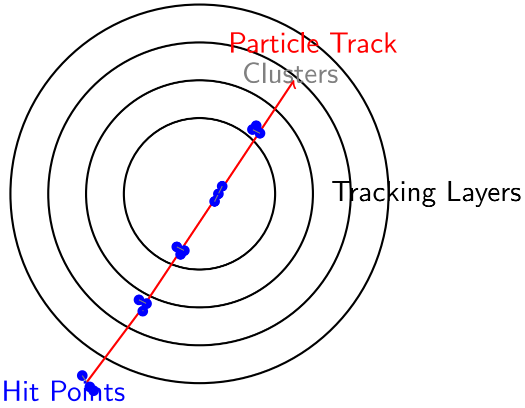
Tracking in High-Energy And Nuclear Physics

- Precise measurement of particle momenta.
- Identification of different particle species.
- Understanding interactions in detectors.

Applications:

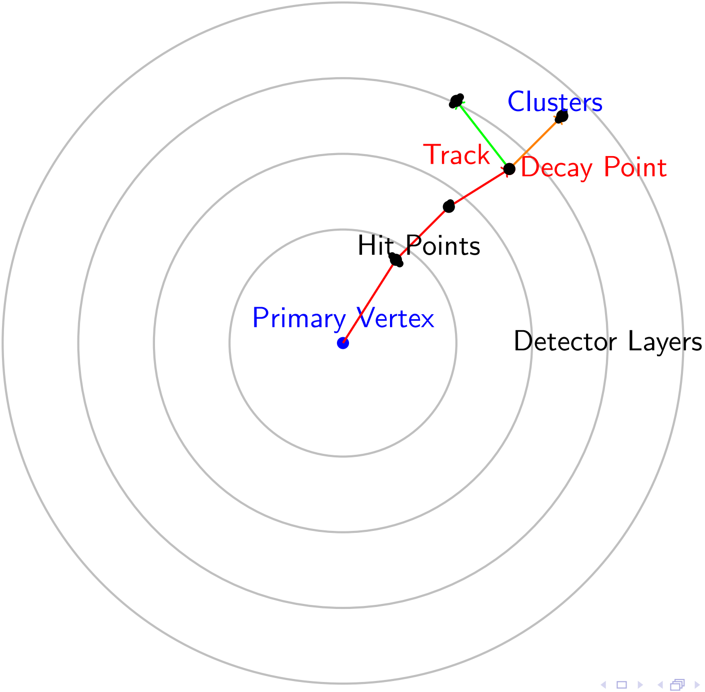
- Large-scale experiments: LHC, Belle II, FOOT.
- Space science
- medical imaging (PET scanners).
- Robotics and autonomous navigation.

Track in a Multi-layer detector



Track

A track is the reconstructed trajectory of a particle as it traverses multiple detector layers. It is formed by connecting interaction points (hits) recorded in the detector, providing insight into the otherwise invisible particle's motion, origin, and interactions.


Clusters
Track
Decay Point
Hit Points
Primary Vertex
Detector Layers

Track Reconstruction Concepts

- **Vertex:** The point where a particle originates, typically from a collision or interaction.
- **Hit Points:** The recorded positions where a particle crosses a detector layer, used to reconstruct the track.
- **Clustering:** Groups of nearby hit points formed due to detector resolution or secondary interactions.
- **Track:** The reconstructed trajectory of a charged particle as it moves through the detector.
- **Decay Points:** The locations where a particle decays into secondary particles.
- **Secondary Tracks:** New particle trajectories originating from a decay point, forming part of a particle's decay chain.

Challenges in Track Reconstruction

Factors Affecting Tracking:

- **Detector Resolution:** Finite precision in position measurements.
- **Multiple Scattering:** Deflections due to interactions with detector material.
- **Energy Loss:** Bremsstrahlung, ionization, and other effects.
- **Magnetic Field Effects:** Curvature of charged particle trajectories.
- **Background Noise & Fake Hits:** Distinguishing true hits from noise.

Why Kalman Filter?

- Handles **noisy** and **incomplete** data efficiently.
- Provides **dynamical updates** of position and momentum.
- **Computationally efficient** for real-time applications.

What is the Kalman Filter?

Definition

A recursive Bayesian estimation method for tracking objects in noisy environments.

Basic Idea:

- Predict the next state (extrapolation).
- Compare prediction with new measurement.
- Correct the estimate using **Kalman Gain**.

Equation (Conceptual Form):

$$\text{New Estimate} = \text{Prediction} + \text{Correction (Weighted by Kalman Gain)}$$

Kalman Filter

- The Kalman Filter (KF) is a recursive algorithm for estimating the state of a system in the presence of noise.
- In Nuclear Physics, KF is used to reconstruct particle trajectories from detector hits.
- It accounts for measurement uncertainties and multiple scattering effects.

State Vector Representation

The 5D state vector:

$$X = \begin{bmatrix} x \\ y \\ dx/dz \\ dy/dz \\ q/p \end{bmatrix} \tag{1}$$

- x, y - Position in the detector plane.
- $dx/dz, dy/dz$ - Slopes describing the trajectory.
- q/p - Charge-to-momentum ratio.

Kalman Filter Steps

The KF consists of two main steps:

1. **Prediction (Extrapolation)**
2. **Update (Correction)**

Prediction Step: State Estimation

State Prediction Equation:

$$\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_{k-1} \tag{2}$$

where:

- $\mathbf{x}_k$ is the predicted state vector at step k.
- $\mathbf{F}$ is the state transition matrix modeling system dynamics.
- $\mathbf{B}$ is the control matrix accounting for external influences.
- $\mathbf{u}_{k-1}$ is the control input vector (if applicable).

Prediction Step: Error Covariance Propagation

Covariance Prediction Equation:

$$\mathbf{P}_k = \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^T + \mathbf{Q} \tag{3}$$

where:

- $\mathbf{P}_k$ is the predicted error covariance matrix.
- $\mathbf{Q}$ is the process noise covariance matrix, modeling system uncertainty.

This step ensures that uncertainties are appropriately propagated with the predicted state.

Prediction Step: Process Noise Considerations

- The process noise covariance $\mathbf{Q}$ accounts for unknown variations in system dynamics.
- In tracking applications, $\mathbf{Q}$ is derived based on expected physical uncertainties such as measurement inaccuracies.
- A well-tuned $\mathbf{Q}$ matrix improves prediction accuracy and stability.

Update Step: Measurement Innovation

Innovation Calculation:

$$\mathbf{y}_k = \mathbf{z}_k - \mathbf{H}\mathbf{x}_k \tag{4}$$

where:

- $\mathbf{y}_k$ is the innovation (difference between measurement and prediction).
- $\mathbf{z}_k$ is the actual measurement at step k.
- $\mathbf{H}$ is the measurement matrix relating state to observed measurement.

Update Step: Innovation Covariance

Innovation Covariance Matrix:

$$\mathbf{S}_k = \mathbf{H}\mathbf{P}_k\mathbf{H}^T + \mathbf{R} \tag{5}$$

where:

- $\mathbf{S}_k$ quantifies uncertainty in the innovation.
- $\mathbf{R}$ is the measurement noise covariance matrix, representing sensor inaccuracies.

Update Step: Kalman Gain Computation

Kalman Gain Calculation

$$\mathbf{K}_k = \mathbf{P}_k \mathbf{H}^T \mathbf{S}_k^{-1} \tag{6}$$

where:

- $\mathbf{K}_k$ determines how much the prediction should be adjusted based on new measurement.
- A higher gain gives more weight to measurement, while a lower gain prioritizes prediction.

Update Step: Kalman Gain Computation

- The Kalman gain is computed to balance prediction and measurement uncertainties.
- If the measurement noise is low ($\mathbf{R}$ is small), the Kalman gain increases, giving more importance to measurement.
- If the process model is accurate, the prediction is more reliable, reducing the impact of measurement updates.

Update Step: Kalman Gain Computation

- The Kalman gain acts as an adaptive weight, ensuring optimal state estimation.
- Proper tuning of **R** and **Q** affects how much the filter relies on new measurements.
- A high Kalman gain can lead to overfitting to noisy measurements, while a low gain may ignore useful updates.

State and Covariance Update: State Correction

State Update Equation:

$$\mathbf{x}_k = \mathbf{x}_k + \mathbf{K}_k \mathbf{y}_k \tag{7}$$

- The new state estimate combines the predicted state and measurement innovation, weighted by the Kalman gain.
- This correction step ensures better accuracy based on available measurements.

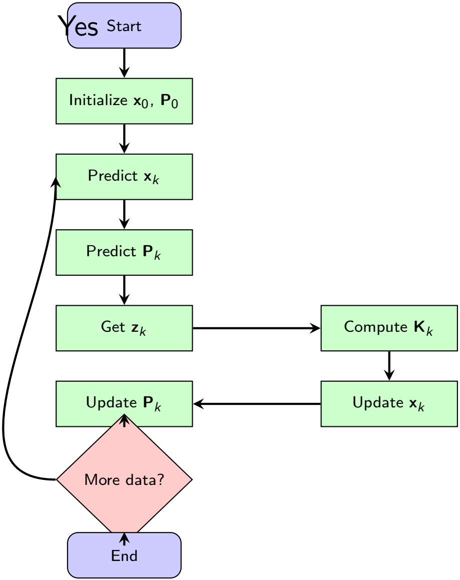
State and Covariance Update: Covariance Adjustment

Covariance Update Equation:

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k \mathbf{H}) \mathbf{P}_k \tag{8}$$

- The updated covariance reflects reduced uncertainty after measurement incorporation.
- The filtering process continuously refines state estimation over time.

Kalman Filter Flowchart



State Transition Matrix F

State Vector

$$X_k = \begin{bmatrix} x \\ y \\ dx/dz \\ dy/dz \\ q/p \end{bmatrix} \tag{9}$$

State Transition Matrix

$$F = \begin{bmatrix} 1 & 0 & \Delta z & 0 & 0 \\ 0 & 1 & 0 & \Delta z & 0 \\ 0 & 0 & 1 & 0 & f_1 \\ 0 & 0 & 0 & 1 & f_2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

where f_1, f_2 depend on the magnetic field and charge-to-momentum ratio.

Linear Equations of Motion

State Transition Equation

$$X_k = FX_{k-1} \tag{11}$$

State Propagation

$$x_k = x_{k-1} + (dx/dz)_{k-1} \cdot \Delta z \tag{12}$$

$$y_k = y_{k-1} + (dy/dz)_{k-1} \cdot \Delta z \tag{13}$$

$$\left(\frac{dx}{dz}\right)_k = \left(\frac{dx}{dz}\right)_{k-1} + f_1 \cdot \left(\frac{q}{p}\right)_{k-1} \tag{14}$$

$$\left(\frac{dy}{dz}\right)_k = \left(\frac{dy}{dz}\right)_{k-1} + f_2 \cdot \left(\frac{q}{p}\right)_{k-1} \tag{15}$$

$$\left(\frac{q}{p}\right)_k = \left(\frac{q}{p}\right)_{k-1} \tag{16}$$

These equations describe how each state variable evolves in a magnetic

field.

Covariance Matrix P_k and Its Evolution

State Covariance Matrix

$$P_k = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{x,dx/dz} & \sigma_{x,dy/dz} & \sigma_{x,q/p} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{y,dx/dz} & \sigma_{y,dy/dz} & \sigma_{y,q/p} \\ \sigma_{x,dx/dz} & \sigma_{y,dx/dz} & \sigma_{dx/dz}^2 & \sigma_{dx/dz,dy/dz} & \sigma_{dx/dz,q/p} \\ \sigma_{x,dy/dz} & \sigma_{y,dy/dz} & \sigma_{dx/dz,dy/dz} & \sigma_{dy/dz}^2 & \sigma_{dy/dz,q/p} \\ \sigma_{x,q/p} & \sigma_{y,q/p} & \sigma_{dx/dz,q/p} & \sigma_{dy/dz,q/p} & \sigma_{q/p}^2 \end{bmatrix} \tag{17}$$

Evolution of P_k

$$P_k = FP_{k-1}F^T + Q \tag{18}$$

- Diagonal elements are the variances of each state variable.
- Off-diagonal elements are the covariances different state variables

Process Noise Covariance Matrix (Q)

The process noise covariance matrix Q accounts for uncertainties in system dynamics due to multiple scattering and energy loss

$$Q = \begin{bmatrix} \sigma_x^2 & 0 & \sigma_{x,dx/dz} & 0 & 0 \\ 0 & \sigma_y^2 & 0 & \sigma_{y,dy/dz} & 0 \\ \sigma_{x,dx/dz} & 0 & \sigma_{dx/dz}^2 & 0 & 0 \\ 0 & \sigma_{y,dy/dz} & 0 & \sigma_{dy/dz}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{q/p}^2 \end{bmatrix} \tag{19}$$

Components

- σ_x^2, σ_y^2 - Position uncertainties due to multiple scattering.
- $\sigma_{dx/dz}^2, \sigma_{dy/dz}^2$ - Angular uncertainties from scattering.
- $\sigma_{q/p}^2$ - Uncertainty in momentum due to energy loss.
- Off-diagonal terms represent correlations between position and angle.

Sources of Uncertainty in Q

1. **Multiple Scattering:**
   - A charged particle undergoes small deflections due to Coulomb interactions with nuclei.
   - Causes deviations in track parameters, introducing non-Gaussian noise.
   - Follows Molière theory.
   - Affects angular components $(dx/dz, dy/dz)$.
2. **Energy Loss Fluctuations:**
   - Described by Landau distribution.
   - Leads to variations in q/p.
3. **Material Effects:**
   - Each detector layer contributes to uncertainties.
   - Depends on material thickness and radiation length.

Measurement Noise Covariance Matrix R

- The measurement noise matrix R represents the uncertainties in position measurements.
- It accounts for detector resolution and intrinsic noise.
- In our case, measurements are taken at 15 detector planes.
- Each plane records (x, y) positions with independent uncertainties.

Definition of R for a Single Detector Plane

The measurement noise covariance for a single plane is given by

$$R = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \tag{20}$$

where:

- σ_x^2: variance in x-position measurement.
- σ_y^2: variance in y-position measurement.

Example: If the detector resolution is 1 mm (0.1 cm), then:

$$R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \tag{21}$$

Measurement Noise Matrix for 15 Detector Planes

For 15 detector planes, R is a block diagonal matrix

$$R = \begin{bmatrix} \sigma_x^2 & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma_y^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \sigma_x^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \sigma_y^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \sigma_y^2 \end{bmatrix} \tag{22}$$

- Each plane contributes two measurements, (x, y).
- Each measurements may have independent uncertainty.

Measurement Innovation (Residue) in Kalman Filtering

- Measurement innovation (residue) represents the difference between the actual measurement and the predicted measurement.
- It quantifies how much the prediction deviates from the observation.

Mathematical Formulation

$$y_k = z_k - HX_k \tag{23}$$

where,

- y_k is the measurement innovation (residual) at step k.
- z_k is the actual measurement vector at step k.
- H is the measurement matrix mapping state space to measurement space.
- X_k is the predicted state vector before measurement update.

Interpretation in Our Case

Muon Tracking

- In our setup, the state vector is $X_k = \begin{bmatrix} x & y & dx/dz & dy/dz & q/p \end{bmatrix}^T$.
- The measurements z_k come from detector hits, providing observed x and y positions.
- The measurement matrix H selects position components from the state vector

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

- The innovation (residual) tells us how much the Kalman filter's prediction deviates from actual detector hits.

Kalman Gain

The Kalman Gain, K_k, is computed as

$$K_k = P_k H^T (H P_k H^T + R)^{-1} \tag{25}$$

where:

- P_k is the predicted state covariance matrix.
- H is the measurement matrix.
- R is the measurement noise covariance matrix.

The Kalman Gain determines how much the new measurement updates the state estimate.

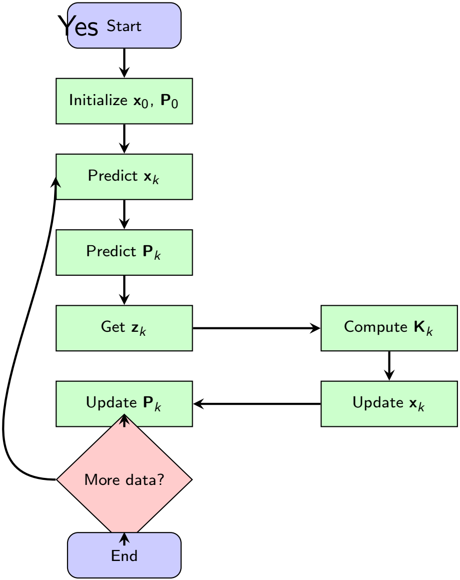
State and Covariance Update: Role of Kalman Gain

- If measurement uncertainty is low, the Kalman gain gives more weight to measurements.
- If measurement noise is high, the Kalman gain favors the predicted state.
- This dynamic balance enables optimal estimation in varying conditions.

State and Covariance Update: Practical Interpretation

- The iterative update process improves the state accuracy with each measurement.
- Optimal filtering ensures smooth and reliable tracking of particle trajectories.
- The Kalman filter is widely used in high-energy physics, robotics, and navigation systems.

Kalman Filter Flowchart



Iteration Through Detector Layers

- The filter processes hits layer by layer.
- At each step, it refines the trajectory estimate.
- The final track is the optimal estimate of the trajectory of the particles.