Equivalent circuit models (ECM): A dynamic programming approach to battery charging

The purpose of this document is to elucidate the governing electrical equations used in the equivalent circuit model (ECM) for both continuous and dynamic programming (DP) frameworks. Namely, we investigate the OCV-R and OCVR-RC ECM models. In both cases, the current I is the controllable input. OCV-R has a single state: state-of-charge (SOC), and the OCVR-RC adds the capacitor voltage as a second state. Parameter values are taken from a A123 2300 mAh Li-ion battery using an equilibrium SOC of 0.5, and temperature of 25 °C. Our optimization function minimizes the least squares error of the SOC to the target SOC.

Dynamic programming approach

The dynamic programming (DP) framework converts the optimization problem into a discrete form. We evoke Bellman's Principal of Optimality, i.e.,:

Let V_k represent the value function for the minimum least squares error (LSQ) problem at each time step $k \in \{0, 1, .., N-1\}$:

$$\boxed{V_k = \min_{I_k} \left\{ \left(z_k - z_{target}\right)^2 + V_{k+1} \right\} + V_N} \tag{1}$$

$$V_N = \left(z_N - z_{target}\right)^2$$

We solve this objective by identifying optimal control I_k^* for all k. We utilize a forward finite difference method for discretization of the continuous state dynamics, with $\Delta t = 1$. We now break down the dynamic programming formulation of each model.

Model 1: OCV-R

Model description

This model is the simplest circuit model using a voltage source V_{oc} and a resistor R_0. We have a single state: the state-of-charge of the battery, hereafter denoted as z. V_{oc} depends on z, and for the purpose of this report we use the data from an A123 2300 mAh battery.

Dynamics

First we understand the state equation for SOC evolution as a function of the current:

$$\frac{dz}{dt} = \frac{I}{C_{\text{batt}}}$$

We now convert the single continuous state equation to the discrete form.

$$\frac{z_{k+1} - z_k}{\Delta t} = \frac{I_k}{dt}$$

$$\boxed{z_{k+1} = \frac{I_k}{C_{\text{batt}}} \cdot \Delta t + z_k} \tag{2}$$

Constraints

We present the constraints for the state and control separately:

$$z_{min} \leq z_k \leq z_{max}$$

$$V_{t,min} \leq V_{t,k} \leq V_{t,max}$$

$$I_{min} \leq I_k \leq I_{max}$$

We rewrite the state constraint given the time-stepping dynamics:

$$z_{min} \leq z_{k+1} - \frac{I_k}{C_{\text{batt}}} \cdot \Delta t \leq z_{max}$$

We convert this form into a bound on the control I_k:

$$C_{\text{batt}} \cdot \frac{z_{k+1} - z_{\text{max}}}{\Delta t} \leq I_k \leq C_{\text{batt}} \cdot \frac{z_{k+1} - z_{\text{min}}}{\Delta t}$$

We now rewrite the terminal voltage constraint also as a bound on I_k:

$$V_{t,min} \leq V_{oc} + I_k R_0 \leq V_{t,max}$$

$$\frac{V_{t,min} - V_{oc}}{R_0} \leq I_k \leq \frac{V_{t,max} - V_{oc}}{R_0}$$

Combining the constraints on I_k, we can tighten the bounds with the following expression:

$$\max\left\{C_{\text{batt}} \cdot \tfrac{z_{k+1} - z_{\text{max}}}{\Delta t}, \tfrac{V_{t,\text{min}} - V_{oc}}{R_0}, I_{\text{min}}\right\} \leq I_k \leq \min\left\{C_{\text{batt}} \cdot \tfrac{z_{k+1} - z_{\text{min}}}{\Delta t}, \tfrac{V_{t,\text{max}} - V_{oc}}{R_0}, I_{\text{max}}\right\} \quad (3)$$

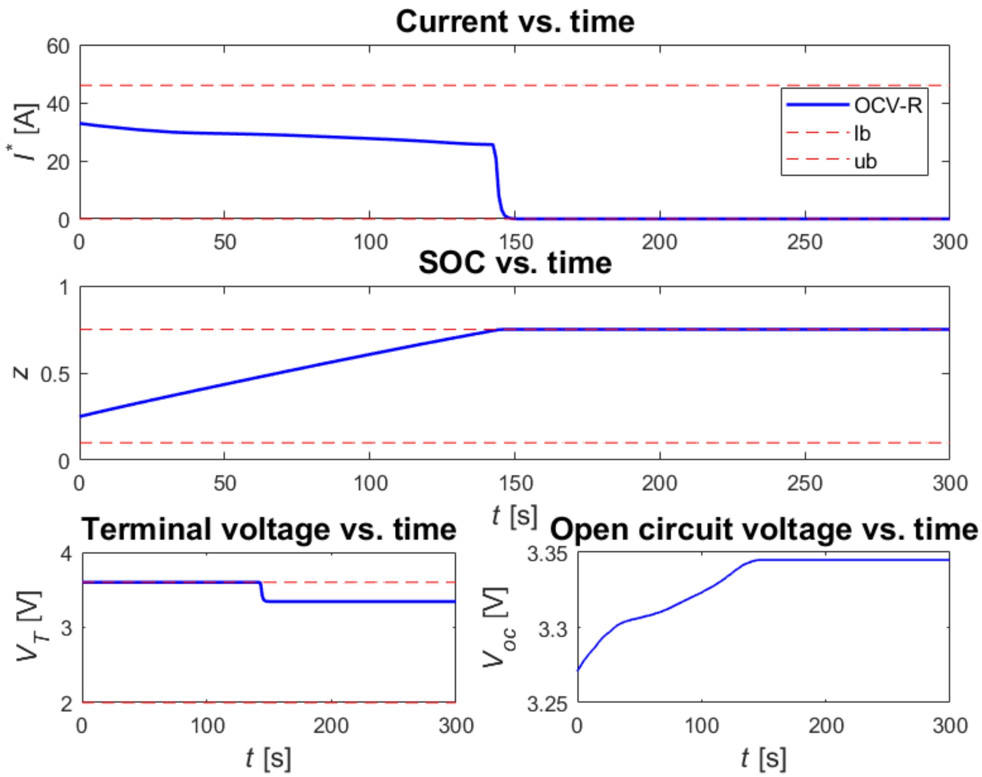
For notation purposes, hereafter we will note the state of charge lower/upper bounds as z_{lb}, z_{ub} and the terminal voltage lower/upper bounds as $V_{t,lb}, V_{t,ub}$ respectively. We can rewrite Equation 3 with an easier framework:

$$\boxed{\max\left\{z_{lb}, V_{t,lb}, I_{\text{min}}\right\} \leq I_k \leq \min\left\{z_{ub}, V_{t,ub}, I_{\text{max}}\right\}} \quad (4)$$

Results

We present the OCV-R solution with a target SOC of 0.75 and initial SOC of 0.25[1]. We note that the optimal policy is constant voltage (CV). The maximum permissible current is drawn so as to charge the battery as quickly as possible while operating at the upper terminal voltage limit. Once the target SOC constraint is reached, the circuit no longer draws current and the terminal voltage decreases slightly. The target SOC is reached around 148s with a 300s charging window.

[1] For a full list of the other parameters, please refer to the Appendix



Model 2: OCV-R-RC

Model description

This model adds an RC pair to the OCV-R model. The capacitor contributes a time-varying voltage V_1. We now have two states and one control.

Dynamics

The SOC dynamics remain the same. We now describe the state dynamics for the capacitor voltage with the continuous equation:

$$C_1 \frac{dV_1}{dt} = -\frac{V_1}{R_1} + I$$

We use the finite difference method to convert the continuous form to a dynamic one.

$$C_1 \frac{V_{1,k+1} - V_{1,k}}{\Delta t} = -\frac{V_{1,k}}{R_1} + I_k$$

Rewriting,

$$\boxed{V_{1,k+1} = V_{1,k}\left(1 - \tfrac{\Delta t}{R_1 \cdot C_1}\right) + \frac{I_k \cdot \Delta t}{C_1}} \quad (5)$$

State space

We described the state space for z in the OCV-R model. We consider the minimum voltage for the capacitor. We assume we charge from rest, and as we are only interested in the charging problem, we take $\boxed{V_{1,\min} = 0}$.

The maximum capacitor voltage comes from the steady-state solution to: $C_1 \cdot 0 = -\frac{V_1}{R_1} + I$. Solving, we get $\boxed{V_{1,\max} = I_{\max} \cdot R_0}$

Constraints

We retain the SOC constraints z_{lb}, z_{ub} from before, in addition to the current constraints $I_{\min}, I_{\max}$. The terminal voltage constraint only changes slightly:

$$V_{t,min} \leq V_{oc} + V_{1,k} + I_k R_0 \leq V_{t,max}$$

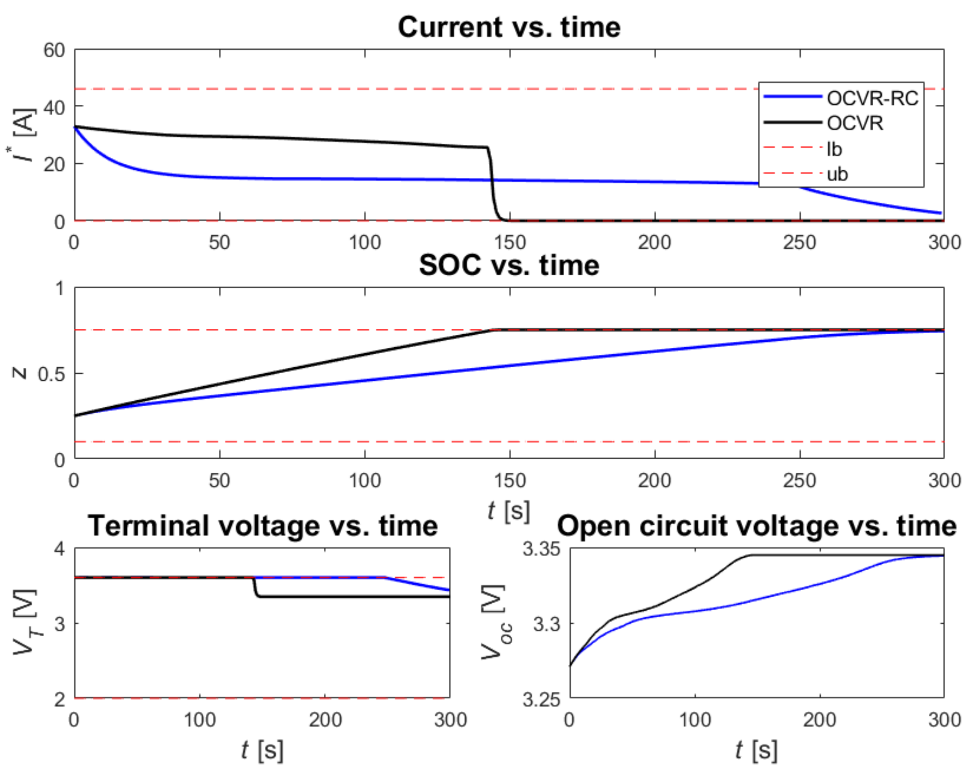
$$\frac{V_{t,min} - V_{oc} - V_{1,k}}{R_0} \leq I_k \leq \frac{V_{t,max} - V_{oc} - V_{1,k}}{R_0}$$

We use this updated framework for $V_{1,lb}, V_{1,ub}$. Our control constraints remain the same as described in Equation 4. For completeness, the full version is included below:

$$\max\left\{C_{\text{batt}} \cdot \frac{z_{k+1} - z_{\max}}{\Delta t}, \frac{V_{t,\min} - V_{1,k} - V_{oc}}{R_0}, I_{\min}\right\} \leq I_k \leq \min\left\{C_{\text{batt}} \cdot \frac{z_{k+1} - z_{\min}}{\Delta t}, \frac{V_{t,\max} - V_{1,k} - V_{oc}}{R_0}, I_{\max}\right\} \quad (6)$$

Results

We present the OCVR-RC solution with a target SOC of 0.75 and initial SOC of 0.25. We note that the optimal policy is constant voltage (CV). The maximum permissible current is drawn so as to charge the battery as quickly as possible while operating at the upper terminal voltage limit. Once the target SOC constraint is reached, the circuit no longer draws current and the terminal voltage decreases slightly. The target SOC is reached around 148s with a 300s charging window.



Appendix

Parameter	Value	Units
z_{min}	0.1	[-]
z_{max}	0.95	[-]
z_{target}	0.75	[-]
$V_{t,\text{min}}$	2	[V]
$V_{t,\text{max}}$	3.6	[V]
I_{min}	0	[A]
I_{max}	46	[A]
z_{eq}	0.5	[-]
T_{∞}	25	[$\circ$ C]
Δt	1	[s]
R_0	0.01	[Ω]
C_{batt}	8280	[coulombs]
C_1	2500	[F]
R_1	0.01	[Ω]
$V_{1,\text{min}}$	0	[V]
$V_{1,\text{max}}$	0.46	[V]