The Pure Spinor correspondence as Koszul Duality

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1 What's the plan?

We goal is to make precise and generalize the statement of the derived pure spinor formalism in [5]. The derived pure spinor equivalence says that for $\mathfrak{p}$ a super-poincaré algebra of the form

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n} \tag{1.1}$$

with $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ a super-translation algebra and $\mathfrak{g}_0 = \mathfrak{so}(d) \times \mathfrak{r}$ such that $\mathfrak{p}$ is a graded Lie algebra.

The statement of the derived pure spinor correspondence is

derivedPS

derivedPS **Theorem 1.1** ([5]). *Given a super Poincaré algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}$, the pure spinor functor defines an equivalence of dg categories*

$$\mathrm{CE}^\bullet(\mathfrak{t})\text{-}\mathsf{Mod}^{\mathfrak{p}_0} \cong \mathsf{Mult}_{\mathfrak{g}}^{\textit{strict-ob}} \tag{1.2}$$

between the category of $\mathfrak{g}_0$-equivariant $\mathrm{CE}^\bullet(\mathfrak{n})$-modules and the full dg subcategory of the category of multiplets whose objects are strict multiplets.

The upshot of the theorem is twofold: on the one hand, from a physical point of view, it says that (up to quasi-isomorphism) all multiplets can be constructed via a suitable derived "enhancement" of the pure spinor formalism. On the other hand, mathematically, it relates a geometric category, to a representation-theoretic one, in a Koszul duality-like fashion. The first goal is to make the relation to Koszul duality precise and concret.

2 Algebraic Beilinson-Bernstein localization

In this section we follow the notes [10].

Let $\mathfrak{g}$ be a reductive coplex Lie algebra. Let Fl be the flag variety of all Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. There's the universal Cartan quotient $\mathfrak{b} \to \mathfrak{t}$, giving a sheaf of commutative Lie

algebras over Fl, which is constant, so there is a canonical commutative Lie algebra $\mathfrak{h}$ over $\mathbb{C}$ ("the universal Cartan") with an isomorphism $\mathfrak{h}|_{Fl} \cong \mathfrak{t}$.
Also note that $\mathfrak{g}$ integrates to a smooth formal group $\hat{G}$ (in the level of functions, dual to $U(\mathfrak{g})$ and $D_{qc}(*/\hat{g})$ gives the derived category of $\mathfrak{g}$-modules.

3 Pure Spinors

We review and generalize the derived pure spinor formalism for a general class of super Lie algebras. This might involve super flag varieties and a super version of Springer theory. A starting point for the desired categorical equivalences can be the work of Arkivpov, Bezrukavnikov and Ginzburg [1].
A version of Koszul duality which takes advantage of derived algebraic geometry and makes a connection with factorization algebras is the chiral Koszul duality of [6].

4 The relation the Koszul duality and the geometry of loop spaces

There is a close relation between the geometry of derived loop spaces and Koszul duality. We investigate this starting from the work of Ben-Zvi, Nadler, Preygel [9] and Chen [4].
As explained for example in [3], the derived loop space is related the the algebraic de Rham complex. This indicates that the chiral de Rham complex as discussed in [2] and [8] are related to the chiral Koszul duality in [6]. An explicit instance of a relation between loop spaces and supersymmetry can be found in [7].

5 Further directions

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