## CHIRAL DE RHAM COMPLEX

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#### Introduction

**0.1.** The aim of this note is to define certain sheaves of vertex algebras on smooth manifolds. In this note, "vertex algebra" will have the same meaning as in Kac's book [K]. Recall that these algebras are by definition  $\mathbb{Z}/(2)$ -graded. "Smooth manifold" will mean a smooth scheme of finite type over  $\mathbb{C}$ .

For each smooth manifold X, we construct a sheaf  $\Omega_X^{ch}$ , called the **chiral de Rham complex** of X. It is a sheaf of vertex algebras in the Zarisky topology, i.e. for each open  $U \subset X$ ,  $\Gamma(U; \Omega_X^{ch})$  is a vertex algebra, and the restriction maps are morphisms of vertex algebras. It comes equipped with a  $\mathbb{Z}$ -grading by fermionic charge, and the chiral de Rham differential  $d_{DR}^{ch}$ , which is an endomorphism of degree 1 such that  $(d_{DR}^{ch})^2 = 0$ . One has a canonical embedding of the usual de Rham complex

$$(\Omega_X, d_{DR}) \hookrightarrow (\Omega_X^{ch}, d_{DR}^{ch}) \tag{0.1}$$

The sheaf  $\Omega_X^{ch}$  has also another  $\mathbb{Z}_{\geq 0}$ -grading, by conformal weight, compatible with fermionic charge one. The differential  $d_{DR}^{ch}$  respects conformal weight, and the subcomplex  $\Omega_X$  coincides with the conformal weight zero component of  $\Omega_X^{ch}$ . The wedge multiplication on  $\Omega_X$  may be restored from the operator product in  $\Omega_X^{ch}$ , see 1.3. The map (0.1) is a quasiisomorphism, cf. Theorem 4.4. Each component of  $\Omega_X^{ch}$  of fixed conformal weight admits a canonical finite filtration whose graded factors are symmetric and exterior powers of the tangent bundle  $\mathcal{T}_X$  and of the bundle of 1-forms  $\Omega_X^1$ .

Similar sheaves exist in complex-analytic and  $C^{\infty}$  settings.

If X is Calabi-Yau, then the sheaf  $\Omega_X^{ch}$  has a structure of a topological vertex algebra (i.e. it admits N=2 supersymmetry, cf. 2.1), cf. 4.5. (For an arbitrary X, the obstruction to the existence of this structure is expressible in terms of the first Chern class of  $\mathcal{T}_X$ , cf. Theorem 4.2.)

One may hope that the vertex algebra  $R\Gamma(X;\Omega_X^{ch})$  defines the conformal field theory which is Witten's "A-model" associated with X.

The intuitive geometric picture behind our construction is as follows. Let LX be the space of "formal loops" on X, i.e. of the maps of the punctured formal disk to X. Let  $L^+X \subset LX$  be the subspace of loops regular at 0. Note that we have a natural projection  $L^+X \longrightarrow X$  (value at 0). We have a functor

$$p: (Sheaves \ on \ LX) \longrightarrow (Sheaves \ on \ X),$$

namely, if  $\mathcal{F}$  is a sheaf on LX, then  $\Gamma(U; p(\mathcal{F})) = \Gamma(LU; F)$  for an open  $U \subset X$ . Now, the sheaf  $\Omega_X^{ch}$  is the image under p of the semiinfinite de Rham complex of the  $\mathcal{D}$ -module of  $\delta$ -functions along  $L^+X$ . This sheaf is a particular case of a more general construction which associates with every  $\mathcal{D}$ -module  $\mathcal{M}$  over X its "chiral de Rham complex"  $\Omega_X^{ch}(\mathcal{M})$  which is a sheaf of vertex modules over the vertex algebra  $\Omega_X^{ch}$ . Its construction is sketched in §6, cf. 6.11.

**0.2.** One can also try to define a purely even sheaf  $\mathcal{O}_X^{ch}$  of vertex algebras, which could be called **chiral structure sheaf**. Here the situation is more subtle than in the case of  $\Omega_X^{ch}$ , where "fermions cancel the anomaly". On can define this sheaf for curves, cf. 5.5. If  $\dim(X) > 1$ , then there exists a non-trivial obstruction of cohomological nature to the construction of  $\mathcal{O}_X^{ch}$ . This obstruction can be expressed in terms of a certain homotopy Lie algebra, cf. §5 A.

However, one can define  $\mathcal{O}_X^{ch}$  for the flag manifolds X = G/B (G being a simple algebraic group and B a Borel subgroup), cf. §5 B, C. This sheaf admits a structure of a  $\widehat{\mathfrak{g}}$ -module at the critical level (here  $\mathfrak{g} = Lie(G)$  and  $\widehat{\mathfrak{g}}$  is the corresponding affine Lie algebra). The space of global sections  $\Gamma(X; \mathcal{O}_X^{ch})$  is the irreducible vacuum  $\widehat{\mathfrak{g}}$ -module for  $\mathfrak{g} = sl(2)$ ; conjecturally, it is also true for any  $\mathfrak{g}$ , cf. 5.13. The sheaf  $\mathcal{O}_X^{ch}$  may be regarded as localization of Feigin-Frenkel bosonization.

More generally, if we start from an arbitrary  $\mathcal{D}$ -module  $\mathcal{M}$  on X = G/B correponding to some  $\mathfrak{g}$ -module M, then we can define its "chiralization"  $\mathcal{M}^{ch}$  which is an  $\mathcal{O}^{ch}$ -module. It seems plausible that the space of global sections  $\Gamma(X; \mathcal{M}^{ch})$  coincides with the Weyl module over  $\widehat{\mathfrak{g}}$  corresponding to M (on the critical level).

- **0.3.** A few words about the plan of the note. In §1 we recall the basic definitions, to fix terminology and notations. In §2, some "free field representation" results are described. No doubt, they are all well known (although for some of them we do not know the precise reference). They are a particular case of the construction of  $\Omega_X^{ch}$ , given in §3. In §4, we discuss the topological structure, and in §5 the chiral structure sheaf. In §6 we outline another construction of our vertex algebras, and some generalizations.
- **0.4.** The idea of this note arose from reading the papers [LVW] and [LZ]. We have learned the first example of "localization along the target space" from B. Feigin, to whom goes our deep gratitude. We thank the referee for the useful remarks which helped to improve the exposition. We are thankful to V. Gorbounov for catching many misprints.

After the submission of this note to the alg-geom server, an interesting preprint [B] has appeared, where a possible application of the chiral de Rham complex to Mirror Symmetry was suggested.

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# $\S 1.$ Recollections on vertex algebras

For more details on what follows, see [K]. For a coordinate-free exposition, see [BD1].

**1.0.** For a vector space V,  $V[[z, z^{-1}]]$  will denote the space of all formal sums  $\sum_{i \in \mathbb{Z}} a_i z^i$ ,  $a_i \in V$ . V((z)) will denote the subspace of Laurent power series, i.e. the sums as above, with  $a_i = 0$  for  $i << \infty$ . We denote by  $\partial_z$  the operator of differentiation by z acting on  $V[[z, z^{-1}]]$ . We set  $\partial_z^{(k)} := (\partial_z)^k/k!$ , for an integer  $k \geq 0$ . For  $a(z) \in V[[z, z^{-1}]]$ , we will also use the notation f(z)' for  $\partial_z f(z)$ , and  $\int a(z)$  for the coefficient at  $z^{-1}$  (i.e. the residue).

 $V^*$  will denote the dual space. In the sequel, we will omit the prefix "super" in the words like "superalgebra", etc. For example, "a Lie algebra" will mean "a Lie superalgebra".

**1.1.** Let us define a **vertex algebra** following [K], 4.1. Thus, a vertex algebra is the data  $(V, Y, L_{-1}, \mathbf{1})$ . Here V is a  $\mathbb{Z}/(2)$ -graded vector space  $V = V^{ev} \oplus V^{odd}$ . The  $\mathbb{Z}/(2)$ -grading is called *parity*; for an element  $a \in V$ , its parity will be denoted by  $\tilde{a}$ . The space of endomorphisms  $\mathrm{End}(V)$  inherits the  $\mathbb{Z}/(2)$ -grading from V.

1 is an even element of V, called vacuum. Y is an even linear mapping

$$Y: V \longrightarrow \operatorname{End}(V)[[z, z^{-1}]]$$
 (1.1)

For  $a \in V$ , the power series Y(a) will be denoted a(z), and called the *field* corresponding to a. The coefficients of the series a(z) are called *Fourier modes*.

 $L_{-1}$  is an even endomorphism of V.

The following axioms must be satisfied.

**Vacuum Axiom.**  $L_{-1}(1) = 0$ ;  $\mathbf{1}(z) = \mathrm{id}$  (the constant power series). For all  $a \in V$ ,  $a(z)(1)|_{z=0} = a$  (in particular, the components of a(z) at the negative powers of z act as zero on the vacuum).

**Translation Invariance Axiom.** For all  $a \in V$ ,  $[L_{-1}, a(z)] = \partial_z a(z)$ 

**Locality Axiom.** For all 
$$a, b \in V$$
,  $(z - w)^N[a(z), b(w)] = 0$  for  $N >> 0$ .

The meaning of this equality is explained in [K].

Given two fields a(z), b(z), their operator product expansion is defined as in [K], (2.3.7a)

$$a(z)b(w) = \sum_{j=1}^{N} \frac{c^{j}(w)}{(z-w)^{j}} + : a(z)b(w) :$$

where

$$: a(z)b(w) := a(z)_{+}b(w) + (-1)^{\tilde{a}\tilde{b}}b(w)a(z)_{-}$$
(1.2)

Here

$$a(z)_{-} = \sum_{n \ge 0} a_{(n)} z^{-n-1}; \quad a(z)_{+} = \sum_{n < 0} a_{(n)} z^{-n-1}$$

for  $a(z) = \sum a_{(n)} z^{-n-1}$ , cf. [K], (2.3.5), (2.3.3).

A morphism of vertex algebras  $(V, \mathbf{1}, L_{-1}, Y) \longrightarrow (V', \mathbf{1}', L'_{-1}, Y')$  is an even linear map  $f: V \longrightarrow V'$  taking  $\mathbf{1}$  to  $\mathbf{1}'$ , such that  $f \circ L_{-1} = L'_{-1} \circ f$ , and for each  $a \in V$ , if  $a_n$  is a Fourier mode of the field a(z), we have  $f(a)_n \circ f = f \circ a_n$ .

1.2. A conformal vertex algebra (cf. [K], 4.10) is a vertex algebra  $(V, Y, L_{-1}, \mathbf{1})$ , together with an even element  $L \in V$  such that if we write the field L(z) as

$$L(z) = \sum_{n} L_n z^{-n-2},$$

the endomorphisms  $L_n$  satisfy the Virasoro commutation relations

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{n^3 - n}{12} \cdot c \cdot \delta_{n, -m}$$
(1.3)

Here c is a complex number, to be called the *Virasoro central charge* of V (in [K] it is called the rank).

The component  $L_{-1}$  of the series (1.2) must coincide with the endomorphism  $L_{-1}$  from the definition of a vertex algebra.

The endomorphism  $L_0$  must be diagonalizable, with integer eigenvalues. Set  $V^{(n)} = \{a \in V | L_0(a) = na\}$ . We must have  $V = \bigoplus_{n \in \mathbb{Z}} V^{(n)}$ . For  $a \in V^{(n)}$ , the number n is called the **conformal weight** of a and denoted |a|. This  $\mathbb{Z}$ -grading induces a  $\mathbb{Z}$ -grading on the space  $\operatorname{End}(V)$ , also to be called the conformal weight. By definition, an endomorphism has conformal weight n if it maps  $V^{(i)}$  to  $V^{(i+n)}$  for all i.

The conformal weight grading on V should be compatible with the parity grading, i.e. both gradings should come from a  $\mathbb{Z} \times \mathbb{Z}/(2)$ -bigrading.

We require that if a has conformal weight n, then the field a(z) has the form

$$a(z) = \sum_{i} a_i z^{-i-n},$$
 (1.4)

with  $a_i$  having the conformal weight -i.

We have the important **Borcherds formula**. If a, b are elements in a conformal vertex algebra, with |a| = n then

$$a(z)b(w) = \sum_{i} \frac{a_i(b)(w)}{(z-w)^{i+n}} + : a(z)b(w) :$$
 (1.5)

cf. [K], Theorem 4.6.

The Virasoro commutation relations (1.3) are equivalent to the operator product

$$L(z)L(w) = \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L(w)'}{z-w}$$
(1.6)

(we omit from now on the regular part).

**1.3.** Let V be a conformal vertex algebra, such that  $V^{(n)} = 0$  for n < 0. For  $a \in V^{(n)}, b \in V^{(m)}$ , consider the operator product

$$a(z)b(w) = \sum_{i} \frac{a_{i}(b)(w)}{(z-w)^{i+n}}$$
(1.7)

We have  $|a_i(b)| = m - i$ . Therefore,  $a_m(b)z^{-n-m}$  is the most singular term in (1.5).

If n = m = 0 then a(z)b(w) is non-singular at z = w. Define the multiplication on the space  $V^{(0)}$  by the rule

$$a \cdot b = \left( a(z)b(w) \big|_{z=w} \mathbf{1} \right) \bigg|_{z=0}$$
 (1.8)

This endows  $V^{(0)}$  with a structure of an associative and commutative algebra with unity 1.

**1.4. Heisenberg algebra.** In this Subsection, we will define a conformal vertex algebra, to be called **Heisenberg vertex algebra**.

Fix an integer N > 0. Let  $H_N$  be the Lie algebra which as a vector space has the base  $a_n^i$ ,  $b_n^i$ , i = 1, ..., N;  $n \in \mathbb{Z}$ , and C, all these elements being even, with the brackets

$$[a_m^j, b_n^i] = \delta_{ij} \delta_{m,-n} C, \tag{1.9}$$

all other brackets being zero.

Our vertex algebra, to be denoted  $V_N$ , as a vector space is the **vacuum representation** of  $H_N$ . As an  $H_N$ -module, it is generated by one vector  $\mathbf{1}$ , subject to the relations

$$b_n^i \mathbf{1} = 0 \text{ if } n > 0; \ a_m^i \mathbf{1} = 0 \text{ if } m \ge 0; \ C\mathbf{1} = \mathbf{1}$$
 (1.10)

The mapping

$$P(b_n^i,a_m^j)\mapsto P(b_n^i,a_m^j)\cdot \mathbf{1}$$

identifies  $V_N$  with the ring of commuting polynomials on the variables  $b_n^i$ ,  $a_m^j$ ,  $n \le 0$ , m < 0, i, j = 1, ..., N. We will use this identification below.

Let us define the structure of a vertex algebra on  $V_N$ . The  $\mathbb{Z}/(2)$  grading is trivial: everything is even. The vacuum vector is 1.

The fields corresponding to the elements of  $V_N$  are defined by induction on the degree of a monomial. First we define the fields for the degree one monomials by setting

$$b_0^i(z) = \sum_{n \in \mathbb{Z}} b_n^i z^{-n}; \ a_{-1}^j(z) = \sum_{n \in \mathbb{Z}} a_n^j z^{-n-1}$$
 (1.11)

Here  $b_n^i$ ,  $a_n^j$  in the right hand side are regarded as operating on  $V_N$  by multiplication.

We set

$$b_{-n}^i(z) = \partial_z^{(n)} b_0^i(z), \quad a_{-n-1}^j(z) = \partial_z^{(n)} a_{-1}^i(z) \tag{1.12} \label{eq:1.12}$$

for n > 0.

The fields corresponding to the monomials of degree > 1 are defined by induction, using the *normal ordering*. Let us call the operators  $b_n^i$ , n > 0, and  $a_n^j$ ,  $n \ge 0$ , acting on  $V_N$ , annihilation operators.

For  $x = b_n^i$  or  $a_n^j$ ,  $n \in \mathbb{Z}$ , and  $b \in \text{End}(V_N)$ , the **normal ordered product** : xb : is given by

$$: xb := \begin{cases} bx & \text{if } x \text{ is an annihilation operator} \\ xb & \text{otherwise} \end{cases}$$
 (1.13)

Define by induction

$$: x_1 \cdot \ldots \cdot x_k :=: x_1 \cdot (: x_2 \cdot \ldots \cdot x_k :) :, \tag{1.14}$$

for  $x_p = b_n^i$  or  $a_n^j$ , p = 1, ..., k.

Given two series  $x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n+p}$  and  $y(z) = \sum_{n \in \mathbb{Z}} y_n z^{-n+q}$ , with  $x_n$  as above, we set

$$: x(z)y(w) := \sum_{n,m \in \mathbb{Z}} : x_n y_m : z^{-n+p} w^{-m+q}$$
 (1.15)

For any finite sequence  $x_1, \ldots, x_p$ , where each  $x_j$  is equal to one of  $a_n^i$  or  $b_n^i$ , we define the series :  $x_1(z)x_2(z) \cdot \ldots \cdot x_p(z) :\in \operatorname{End}(V_N)[[z,z^{-1}]]$  by induction, as in (1.14). This expression does not depend on the order of  $x_i$ 's.

Given a monomial  $x_1 \cdot ... \cdot x_p \mathbf{1} \in V_N$ , with  $x_i$  as above, we define the corresponding field by

$$x_1 \cdot \ldots \cdot x_p(z) =: x_1(z) \cdot \ldots \cdot x_p(z) : \tag{1.16}$$

Since every element of  $V_N$  is a finite linear combination of monomials as above, this completes the definition of the mapping (1.1).

We will use the shorthand notations

$$b^{i}(z) = b_{0}^{i}(z); \ a^{j}(z) = a_{-1}^{j}(z)$$
 (1.17)

The operator products of these basic fields are

$$a^{j}(z)b^{i}(w) = \frac{\delta_{ij}}{z-w} + (\text{regular})$$
 (1.18a)

$$b^{i}(z)b^{j}(w) = (\text{regular}); \ a^{i}(z)a^{j}(w) = (\text{regular})$$
(1.18b)

where "(regular)" means the part regular at z = w. These operator products are equivalent to the commutation relations (1.9).

Other operator products are computed by differentiation of (1.18), and using Wick theorem, cf. [K], 3.3.

One can say that the vertex algebra  $V_N$  is generated by the even fields  $b^i(z)$ ,  $a^j(z)$ , of conformal weights 0 and 1 respectively, subject to the relations (1.18).

The Virasoro field is given by

$$L(z) = \sum_{i=1}^{N} : b^{i}(z)' \cdot a^{i}(z) :$$
 (1.19)

The central charge is equal to 2N. Let us check this. Assume for simplicity that N = 1, and let us omit the index 1 at the fields a, b. Thus, we have

$$L(z) =: b(z)'a(z):$$

Let us compute the operator product L(z)L(w) using the Wick theorem. We have

$$: b(z)'a(z) :: b(w)'a(w) := [b(z)'a(w)][a(z)b(w)'] +$$

$$+[b(z)'a(w)]: a(z)b(w)': +[a(z)b(w)']: b(z)'a(w):=$$

(we have  $b(z)'a(w) = a(z)b(w)' = 1/(z-w)^2$ )

$$=\frac{1}{(z-w)^4}+\frac{2:b(w)'a(w):}{(z-w)^2}+\frac{:b(w)''a(w):+:b(w)'a(w)':}{z-w}$$

Hence,

$$L(z)L(w) = \frac{1}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{L(w)'}{z-w}$$

which is (1.6) with c=2.

**1.5. Clifford algebra.** Let  $Cl_N$  be the Lie algebra which as a vector space has the base  $\phi_n^i$ ,  $\psi_n^i$ ,  $i=1,\ldots,N,\ n\in\mathbb{Z}$ , and C, all these elements being odd, with the brackets

$$[\phi_m^j, \psi_n^i] = \delta_{ij}\delta_{m,-n} \cdot C \tag{1.20}$$

Clifford vertex algebra  $\Lambda_N$  is defined as in the previous subsection, starting with the odd fields

$$\psi^{i}(z) = \sum_{n \in \mathbb{Z}} \psi_{n}^{i} z^{-n-1}; \ \phi^{j}(z) = \sum_{n \in \mathbb{Z}} \phi_{n}^{j} z^{-n}$$
 (1.21)

and repeating the definitions of *loc. cit.*, with a (resp. b) replaced by  $\psi$  (resp.  $\phi$ ). One must put the obvious signs in the definition of the normal ordering. Thus,  $\Lambda_N$  is generated by the odd fields  $\phi^i(z)$ ,  $\psi^i(z)$ , subject to the relations

$$\phi^{i}(z)\psi^{j}(w) = \frac{\delta_{ij}}{z-w} + \text{regular};$$
 (1.22a)

$$\phi^{i}(z)\phi^{j}(w) = \text{regular}; \ \psi^{i}(z)\psi^{j}(w) = \text{regular}$$
 (1.22b)

The Virasoro field is given by

$$L(z) = \sum_{i=1}^{N} : \phi^{i}(z)' \cdot \psi^{i}(z) :$$
 (1.23)

The central charge is equal to -2N.

**1.6.** If  $\mathcal{A}$  and  $\mathcal{B}$  are vertex algebras then their tensor product  $\mathcal{A} \otimes \mathcal{B}$  admits a canonical structure of a vertex algebra, cf. [K], 4.3. The Virasoro element is given by

$$L_{\mathcal{A}\otimes\mathcal{B}} = L_{\mathcal{A}}\otimes 1 + 1\otimes L_{\mathcal{B}} \tag{1.24}$$

We will use in the sequel the tensor product of the Heisenberg and Clifford vertex algebras  $\Omega_N = V_N \otimes \Lambda_N$ . Its Virasoro central charge is equal to 0.

**1.7.** Let  $\mathcal{A}$  be a vertex algebra. A linear map  $f: \mathcal{A} \longrightarrow \mathcal{A}$  is called *derivation* of  $\mathcal{A}$  if for any  $a \in \mathcal{A}$ ,

$$f(a)(z) = [f, a(z)]$$
 (1.25)

Note that an invertible map  $g: \mathcal{A} \longrightarrow \mathcal{A}$  is an automorphism of  $\mathcal{A}$  iff

$$g(a)(z) = ga(z)g^{-1} (1.26)$$

and (1.25) is the infinitesimal version of (1.26).

It follows from Borcherds formula that for every  $a \in \mathcal{A}$ , the Fourier mode  $\int a(z)$  is a derivation, cf. Lemma 1.3 from [LZ]. Consequently, if the endomorphism  $\exp(\int a(z))$  is well defined, it is an automorphism of  $\mathcal{A}$ .

#### §2. De Rham chiral algebra of an affine space

**2.1.** A topological vertex algebra of rank d is a conformal vertex algebra  $\mathcal{A}$  of Virasoro central charge 0, equipped with an even element J of conformal weight 1, and two odd elements, Q, of conformal weight 1, and G, of conformal weight 2.

The following operator products must hold

$$L(z)L(w) = \frac{2L(w)}{(z-w)^2} + \frac{L(w)'}{z-w}$$
(2.1a)

$$J(z)J(w) = \frac{d}{(z-w)^2}; \ L(z)J(w) = -\frac{d}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{J(w)'}{z-w}$$
(2.1b)

$$G(z)G(w) = 0; \ L(z)G(w) = \frac{2G(w)}{(z-w)^2} + \frac{G(w)'}{z-w}; \ J(z)G(w) = -\frac{G(w)}{z-w} \qquad (2.1c)$$

$$Q(z)Q(w) = 0; \ L(z)Q(w) = \frac{Q(w)}{(z-w)^2} + \frac{Q(w)'}{z-w}; \ J(z)Q(w) = \frac{Q(w)}{z-w}$$
 (2.1d)

$$Q(z)G(w) = \frac{d}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{L(w)}{z-w}$$
 (2.1e)

Note the following consequence of (2.1e),

$$[Q_0, G(z)] = L(z)$$
 (2.2)

**2.2.** In this subsection, we will intorduce a structure of a topological vertex algebra of rank N on the vertex algebra  $\Omega_N$  from 1.6. This topological vertex algebra will be called **de Rham chiral algebra of the affine space**  $\mathbb{A}^N$ .

Recall that the Virasoro element is given by

$$L = \sum_{i=1}^{N} \left( b_{-1}^{i} a_{-1}^{i} + \phi_{-1}^{i} \psi_{-1}^{i} \right)$$
 (2.3a)

Define the elements J, Q, G by

$$J = \sum_{i=1}^{N} \phi_0^i \psi_{-1}^i; \ Q = \sum_{i=1}^{N} a_{-1}^i \phi_0^i; \ G = \sum_{i=1}^{N} \psi_{-1}^i b_{-1}^i$$
 (2.3b)

The corresponding fields are

$$L(z) = \sum \left( :b^{i}(z)'a^{i}(z) : + :\phi^{i}(z)'\psi^{i}(z) : \right)$$
 (2.4a)

and

$$J(z) = \sum : \phi^{i}(z)\psi^{i}(z) :, \ Q(z) = \sum : a^{i}(z)\phi^{i}(z) :, \ G(z) = \sum : \psi^{i}(z)b^{i}(z)' :$$
(2.4b)

The relations (2.1) are readily checked using the Wick theorem.

**2.3.** Let us define the **fermionic charge** operator acting on  $\Omega_N$ , by

$$F := J_0 = \sum_{i} \sum_{n} : \phi_n^i \psi_{-n}^i :$$
 (2.5)

We have

$$F\mathbf{1} = 0 \tag{2.6}$$

and

$$[F, \phi_n^i] = \phi_n^i; \ [F, \psi_n^i] = -\psi_n^i; \ [F, a_n^i] = [F, b_n^i] = 0 \tag{2.7}$$

We set

$$\Omega_N^p = \{ \omega \in \Omega_N | F\omega = p\omega \}$$
 (2.8)

Obviously,

$$\Omega_N = \bigoplus_{p \in \mathbb{Z}} \Omega_N^p \tag{2.9}$$

We define an endomorphism d of the space  $\Omega_N$  by

$$d := -Q_0 = -\sum_{i,n} : a_n^i \phi_{-n}^i :$$
 (2.10)

(we could omit the normal ordering in the last formula since the letters a and  $\phi$  commute anyway). We have  $d^2 = 0$ . Indeed, by Wick theorem Q(z)Q(w) = regular, hence all Fourier modes of Q(z) (anti)commute. The map d is called **chiral de Rham differential**.

The map d increases the fermionic charge by 1, by (2.7). Thus, the space  $\Omega_N$  equipped with the fermionic charge grading and the differential d, becomes a complex (infinite in both directions), called **chiral de Rham complex** of  $\mathbb{A}^N$ .

Consider the usual algebraic de Rham complex  $\Omega(\mathbb{A}^N) = \bigoplus_{p=0}^N \Omega^p(\mathbb{A}^N)$  of the affine space  $\mathbb{A}^N$ . We identify the coordinate functions with the letters  $b_0^1, \ldots, b_0^N$ , and their differentials with the fermionic variables  $\phi_0^1, \ldots, \phi_0^N$ .

Thus, we identify the commutative dg algebras

$$\Omega(\mathbb{A}^N) = \mathbb{C}[b_0^1, \dots, b_0^N] \otimes \Lambda(\phi_0^1, \dots, \phi_0^N), \tag{2.12}$$

the second factor being the exterior algebra. The grading is defined by assigning to the letters  $b_0^i$  (resp.  $\phi_0^j$ ) the degree 0 (resp. 1).

The usual de Rham differential is given by

$$d_{DR} = \sum_{i} a_0^i \phi_0^i, \tag{2.13}$$

as follows from the relations (1.9).

**2.4.** Theorem. The obvious embedding of complexes

$$i: (\Omega(\mathbb{A}^N), d_{DR}) \longrightarrow (\Omega_N, d)$$
 (2.14)

is compatible with the differentials, and is a quasiisomorphism.

We identify the space  $\Omega_N$  with the space of polynomials in the letters  $b_n^i, \phi_n^i$   $(n \leq 0)$  and  $a_n^i, \psi_n^i$  (n < 0). One sees that on the subspace  $\mathbb{C}[b_0^i, \phi_0^i]$ , all the summands  $a_n^i \phi_{-n}^i$  with  $n \neq 0$  act trivially. It follows that the map i is compatible with the differentials.

To prove that it is a quasiisomorphism, let us split d in two commuting summands  $d = d_+ + d_-$  where

$$d_{+} = \sum_{i} \sum_{n \geq 0} a_{n}^{i} \phi_{-n}^{i}, \ d_{-} = \sum_{i} \sum_{n \leq 0} a_{n}^{i} \phi_{-n}^{i}$$
 (2.15)

We think of the space  $\Omega_N$  as of the tensor product  $\mathbb{C}[a_n^i, \psi_n^i] \otimes \mathbb{C}[b_m^j, \phi_m^j]$ . The differential  $d_-$  acts trivially on the second factor, and on the first one it is the Koszul differential. So, the cohomology of  $d_-$  is  $\mathbb{C}[b_m^j, \phi_m^j]_{m \leq 0}$ .

Now, we have to compute the cohomology of this space with respect to  $d_+$ . For this purpose, split  $d_+$  once again as  $d_+ = d_{DR} + d_+^{>0}$ , and our space as  $\mathbb{C}[b_0^j, \phi_0^j] \otimes \mathbb{C}[b_m^j, \phi_m^j]_{m<0}$ . The differential  $d_+^{>0}$  acts trivially on the first factor, and on the second one, it is the de Rham differential. Hence (by Poincaré lemma), taking the cohomology of  $d_+^{>0}$  kills all non-zero modes, and we are left precisely with the usual de Rham complex.

Alternatively, it follows from (2.2) that

$$[G_0, d] = L_0 (2.16)$$

The operator  $G_0$  commutes with  $L_0$ , and it follows from (2.16) that it gives a homotopy to 0 for the operator d on all the subcomplexes of non-zero conformal weight. Therefore, all cohomology lives in the conformal weight zero subspace.  $\triangle$ 

**2.5.** The vertex algebra  $\Omega_N$  satisfies the assumptions of 1.3. The subspace  $\Omega(\mathbb{A}^N)$  coincides with the conformal weight zero component of it. If we apply the definition of 1.3, we get the structure of a commutative algebra on  $\Omega(\mathbb{A}^N)$  which is given by the usual wedge product of differential forms.

# §3. Localization

**3.1.** Consider the Heisenberg vertex algebra  $V_N$  defined in 1.4. As in *loc. cit.*, we will identify the space  $V_N$  with the space of polymonials  $\mathbb{C}[b_{-n}^i, a_{-m}^j]_{n\geq 0, m>0}$ . To simplify the notations below, let us denote the zero mode variables  $b_0^i$  by  $b^i$ .

Let  $A_N$  denote the algebra of polynomials  $\mathbb{C}[b^1,\ldots,b^N]$ . The space  $A_N$  is identified with the subspace  $V_N^{(0)} \subset V_N$  of conformal weight zero. The space  $V_N$  has an obvious structure of an  $A_N$ -module. Let  $\widehat{A}_N$  denote the algebra of formal power series  $\mathbb{C}[[b^1,\ldots,b_N]]$ . Set

$$\widehat{V}_N = \widehat{A}_N \otimes_{A_N} V_N \tag{3.1}$$

We are going to introduce a structure of a conformal vertex algebra on the space  $\hat{V}_N$ . Let us define the map (1.1).

Let  $f(b^1, \ldots, b^N)$  be a power series from  $\widehat{A}_N$ . We claim that the expression  $f(b^1(z), \ldots, b^N(z))$  makes sense as an element of  $\operatorname{End}(\widehat{V}_N)[[z, z^{-1}]]$ . (We are grateful to Boris Feigin who has shown us a particular case of the following construction.) Let us express the power series  $b^i(z)$  as

$$b^{i}(z) = b^{i} + \Delta b^{i}(z) \tag{3.2}$$

Thus,

$$\Delta b^{i}(z) = \sum_{n>0} \left( b_{n}^{i} z^{-n} + b_{-n}^{i} z^{n} \right)$$
 (3.3)

Let us define  $f(b^1(z),\ldots,b^N(z))$  by the Taylor formula

$$f(b^{1}(z), \dots, b^{N}(z)) = \sum \Delta b^{1}(z)^{i_{1}} \cdot \dots \cdot \Delta b^{N}(z)^{i_{N}} \partial^{(i_{1}, \dots, i_{N})} f(b^{1}, \dots, b^{N}) \quad (3.4)$$

where

$$\partial^{(i_1,\dots,i_N)} = \frac{\partial_{b^1}^{i_1}}{i_1!} \cdot \dots \cdot \frac{\partial_{b^N}^{i_N}}{i_N!}$$
 (3.5)

We will show that the series (3.4) gives a well-defined element of  $\operatorname{End}(\widehat{V}_N)[[z,z^{-1}]]$ . Let us write

$$\Delta b^{1}(z)^{i_{1}} \cdot \ldots \cdot \Delta b^{N}(z)^{i_{N}} = \sum_{k} c_{k}^{i_{1}, \ldots, i_{N}} z^{-k}$$

The coefficient  $c_k^{i_1,\ldots,i_N}$  is an infinite sum of the monomials

$$b_{k_1}^{j_1} \cdot \ldots \cdot b_{k_I}^{j_I}, \tag{3.6}$$

with  $I=i_1+\ldots+i_N,\,k_1+\ldots+k_I=k$ . Pick an element  $v\in\widehat{V}^N$ . There exists M such that  $b_{l_1}^{j_1}\cdot\ldots\cdot b_{l_N}^{j_N}v=0$  if  $\sum l_i>M$ .

We have

$$k = \sum k_i = + \sum |k_i| - - \sum |k_i|$$

where  $^+\sum$  (resp.  $^-\sum$ ) denotes the sum of all positive (resp. negative) summands. If  $b_{k_1}^{j_1}\cdot\ldots\cdot b_{k_l}^{j_l}v\neq 0$ , then

$$+\sum |k_i| \le M \tag{3.7a}$$

On the other hand,

$$-\sum |k_i| = +\sum |k_i| - k \le M - k \tag{3.7b}$$

There exists only a finite number of tuples  $(k_1, \ldots, k_I)$  satisfying (3.7a) and (3.7b). Therefore,  $c_k^{i_1, \ldots, i_N}$  are well-defined endomorphisms of  $\widehat{V}_N$ .

We have

$$f(b^{1}(z), \dots, b^{N}(z)) = \sum_{k} \left( \sum_{i_{1}, \dots, i_{N}} c_{k}^{i_{1}, \dots, i_{N}} \partial^{(i_{1}, \dots, i_{N})} f(b^{1}, \dots, b^{N}) \right) z^{-k}$$
 (3.8)

All numbers  $k_i$  are non-zero. Let  $I_+$  (resp.,  $I_-$ ) be the number of positive (resp. negative  $k_i$ 's). We have  $I_- \leq -\sum k_i \leq M - k$ , hence  $M \geq +\sum k_i \geq I_+ = I - I_- \geq I - M + k$ . Therefore,

$$I \le 2M - k \tag{3.9}$$

Therefore, when we apply the series (3.8) to the element v, only a finite number of terms in the sum over  $(i_1, \ldots, i_N)$  survives. Therefore, the series (3.8) is a well-defined element of  $\operatorname{End}(\widehat{V}_N)[[z, z^{-1}]]$ .

Every element of  $\widehat{V}_N$  is a finite sum of products g(a)f(b) where g(a) is a polynomial in the letters a and f(b) is a power series as above. We have already defined f(b)(z). The definition of g(a)(z) is the same as in the case of  $V_N$ . We define (g(a)f(b))(z) by

$$(g(a)f(b))(z) =: g(a)(z)f(b)(z):$$
 (3.10)

where the normal ordering is defined in (1.2). This completes the definition of the mapping

$$Y: \widehat{V}_N \longrightarrow \operatorname{End}(\widehat{V}_N)[[z, z^{-1}]]$$
 (3.11)

The following version of the definition of the map (3.11) is helpful in practice. Every element  $c \in \widehat{V}_N$  is a limit of the elements of  $c_i \in V_N$  (in the obvious topology). We can regard the fields  $c_i(z)$  as the elements of  $\operatorname{End}(\widehat{V}_N)[[z,z^{-1}]]$ . The field c(z) is the limit of the fields  $c_i(z)$ .

We define the vacuum and Virasoro element  $\mathbf{1}, L \in \widehat{V}_N$  as the image of the corresponding element of  $V_N$  under the natural map  $V_N \longrightarrow \widehat{V}_N$ .

**3.2. Theorem.** The construction of 3.1 defines a structure of a conformal vertex algebra on the space  $\widehat{V}_N$ .

This follows from [K], Theorem 4.5.

**3.3.** Let  $A_N^{an} \subset \widehat{A}_N$  denote the subalgebra of power series, convergent in a neighbourhood of the origin. Set

$$V_N^{an} = A_N^{an} \otimes_{A_N} V_N \subset \widehat{V}_N \tag{3.12}$$

It is clear from the inspection of the Taylor formula (3.4) that for  $f(a,b) \in V_N^{an}$ , the Fourier modes of f(a,b)(z) which belong to  $\operatorname{End}(\widehat{V}_N)$ , respect the subspace  $V_N^{an}$ . Therefore, the conformal vertex algebra structure on  $\widehat{V}_N$  defined in 3.1 induces the structure of a conformal vertex algebra on  $V_N^{an}$ .

In this argument,  $A_N^{an}$  can be replaced by any algebra of functions containing  $A_N$  and closed under derivations. More precisely, one has the following general statement. Let A' be an arbitrary commutative  $A_N$ -algebra, given together with an action of the Lie algebra  $\mathcal{T} = Der(A_N)$  by derivations, extending the natural action of  $\mathcal{T}$  on  $A_N$ . Then the space  $V_{A'} := A' \otimes_{A_N} V_N$  admits a natural structure of a vertex algebra. For the details, see 6.9.

For example, let  $A_N^{sm}$  denote the algebra of germs of smooth  $(C^\infty)$  functions. Then we get a vertex algebra

$$V_N^{sm} = A_N^{sm} \otimes_{A_N} V_N \tag{3.13}$$

Another natural example is that of localization of A. It is treated in the next subsection.

**3.4. Zariski localization.** Let  $f \in A_N$  be a nonzero polynomial. Let  $A_{N,f}$  denote the localization  $A_N[f^{-1}]$ . Set

$$V_{N:f} = A_{N:f} \otimes_{A_N} V_N \tag{3.14}$$

Consider the Taylor formula (3.4) applied to the function  $f^{-1}$ . We have evidently

$$\left[\partial^{(i_1,\dots,i_N)}f^{-1}\right](b^1,\dots,b^N)\in A_{N;f}$$

In more concrete terms, let  $f(z) = \sum f_n z^{-n}$  is the field corresponding to f, then we want to define the field corresponding to  $f^{-1}$  as

$$f(z)^{-1} = (f_0 + f_{-1}z + f_1z^{-1} + \dots)^{-1} = f_0^{-1} (1 + f_0^{-1}(f_{-1}z + f_1z + \dots))^{-1} =$$

(we use the geometric series)

$$= f_0^{-1}(1 + f_0^{-2}(2f_{-1}f_1 + 2f_{-2}f_2 + \dots) + \dots)$$

(we started to write down the coefficient at  $z^0$ ). Now, in the right hand side, the coefficient at each power of z is an infinite sum, but as an operator acting on  $A_{N,f}$  it is well defined since only finite number of terms act nontrivially. We need only to invert  $f_0 = f$ .

Therefore, the construction 3.1 provides a conformal vertex algebra structure on the space  $V_{N;f}$ .

Let X denote the affine space  $\operatorname{Spec}(A_N)$ . Let  $\mathcal{O}_X^{ch}$  be the  $\mathcal{O}_X$ -quasicoherent sheaf corresponding to the  $A_N$ -module  $V_N$ . We have just defined the structure of a conformal vertex algebra on the spaces  $V_{N;f} = \Gamma(U_f; \mathcal{O}_X^{ch})$  where  $U_f = \operatorname{Spec}(A_{N;f})$ . If  $U_f \subset U_g$  then the restriction map  $V_{N;g} \longrightarrow V_{N;f}$  is a morphism of conformal vertex algebras.

If  $U \subset X$  is an arbitrary open, we have  $U = \bigcup U_f$ , and

$$\Gamma(U; \mathcal{O}_X^{ch}) = \operatorname{Ker} \left( \prod V_{N;f} \xrightarrow{\longrightarrow} \prod V_{N;fg} \right)$$

Using this formula, we get a structure of a conformal vertex algebra on the space  $\Gamma(U; \mathcal{O}_X^{ch})$ . Therefore,  $\mathcal{O}_X^{ch}$  gets a structure of a sheaf of conformal vertex algebras.

**3.5.** We can add fermions to the picture. Consider the spaces  $\widehat{\Omega}_N = \widehat{A}_N \otimes_{A_N} \Omega_N$ ,  $\Omega_N^{an} = A_N^{an} \otimes_{A_N} \Omega_N \subset \widehat{\Omega}_N$ ,  $\Omega_N^{sm} = A_N^{sm} \otimes_{A_N} \Omega_N$ . The construction 3.1 provides a structure of a topological vertex algebras on these spaces.

Let X be as in 3.4; let  $\Omega_X^{ch}$  denote the  $\mathcal{O}_X$ -quasicoherent sheaf associated with the  $A_N$ -module  $\Omega_N$ . The construction 3.4 provides a structure of a sheaf of topological vertex algebras of rank N on  $\Omega_X^{ch}$ .

**3.6.** Now we want to study coordinate changes in our vertex algebras. Let X be the formal scheme  $\mathrm{Spf}(\mathbb{C}[[b^1,\ldots,b^N]])$ . Consider the formal N|N-dimensional

superscheme  $\tilde{X} = \Pi TX$  (here TX is the total space of the tangent bundle,  $\Pi$  is the parity change functor). Thus,  $\tilde{X}$  has the same underlying space as X, and the structure sheaf of  $\tilde{X}$  coincides with the de Rham algebra of differential forms X. On  $\tilde{X}$ , we have N even coordinates  $b^1, \ldots, b^N$  and odd ones  $\phi^1 = db^1, \ldots, \phi^N = db^N$ .

To this superscheme  $\tilde{X}$ , with the above coordinates, we have assigned a (super)vertex algebra  $\hat{\Omega}_N$ , generated by the fields  $b^i(z)$ ,  $a^i(z)$  (even ones) and  $\phi^i(z)$ ,  $\psi^i(z)$  (odd ones). The fields  $a^i(z)$  (resp.  $\psi^i(z)$ ) correspond to the vector fields  $\partial_{b^i}$  (resp.  $\partial_{\phi^i}$ ) on  $\tilde{X}$ .

These fields satisfy the relations (cf. (1.18), (1.22))

$$a^{i}(z)b^{j}(w) = \frac{\delta_{ij}}{z - w} \tag{3.15a}$$

$$b^{i}(z)b^{j}(w) = (\text{regular}); \ a^{i}(z)a^{j}(w) = (\text{regular})$$
(3.15b)

$$\phi^{i}(z)\psi^{j}(w) = \frac{\delta_{ij}}{z - w} \tag{3.15c}$$

$$\phi^{i}(z)\phi^{j}(w) = (\text{regular}); \ \psi^{i}(z)\psi^{j}(w) = (\text{regular})$$
 (3.15d)

$$b^{i}(z)\phi^{j}(w) = (\text{regular}); \ b^{i}(z)\psi^{j}(w) = (\text{regular})$$
 (3.15e)

$$a^{i}(z)\phi^{j}(w) = (\text{regular}); \ a^{i}(z)\psi^{j}(w) = (\text{regular})$$
 (3.15f)

Consider an invertible coordinate transformation on X,

$$\tilde{b}^i = g^i(b^1, \dots, b^N); \ b^i = f^i(\tilde{b}^1, \dots, \tilde{b}^N)$$
 (3.16a)

where  $g^i \in \mathbb{C}[[b^j]]$ ;  $f^i \in \mathbb{C}[[\tilde{b}^j]]$ . It induces the transformation of the odd coordinates  $\phi^i = db^i$ ,

$$\tilde{\phi}^i = \frac{\partial g^i}{\partial b^j} \phi^j; \ \phi^i = \frac{\partial f^i}{\partial \tilde{b}^j} \tilde{\phi}^j \tag{3.16b}$$

(the summation over the repeating indices is tacitly assumed). The vector fields transform as follows,

$$\partial_{\tilde{b}^i} = \frac{\partial f^j}{\partial \tilde{b}^i}(g(b))\partial_{b^j} + \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b)) \cdot \frac{\partial g^l}{\partial b^r} \cdot \phi^r \partial_{\phi^k}$$
(3.16c)

and

$$\partial_{\tilde{\phi}^i} = \frac{\partial f^j}{\partial \tilde{h}^i}(g(b))\partial_{\phi^j} \tag{3.16d}$$

We want to lift the transformation (3.16a) to the algebra  $\Omega_N$ . Define the tilded fields by

$$\tilde{b}^i(z) = g^i(b)(z) \tag{3.17a}$$

$$\tilde{\phi}^{i}(z) = \left(\frac{\partial g^{i}}{\partial b^{j}}\phi^{j}\right)(z) \tag{3.17b}$$

$$\tilde{a}^{i}(z) = \left(a^{j} \frac{\partial f^{j}}{\partial \tilde{h}^{i}}(g(b))\right)(z) + \left(\frac{\partial^{2} f^{k}}{\partial \tilde{h}^{i} \partial \tilde{h}^{l}}(g(b)) \frac{\partial g^{l}}{\partial b^{r}} \phi^{r} \psi^{k}\right)(z)$$
(3.17c)

$$\tilde{\psi}^{i}(z) = \left(\frac{\partial f^{j}}{\partial \tilde{b}^{i}}(g(b))\psi^{j}\right)(z) \tag{3.17d}$$

**3.7.** Theorem. The fields  $\tilde{b}^i(z)$ ,  $\tilde{a}^i(z)$ ,  $\tilde{\phi}^i(z)$  and  $\tilde{\psi}^i(z)$  satisfy the relations (3.15).

We will use the relations

$$h(b)(z)a^{i}(w) = -\frac{\partial h/\partial b^{i}(w)}{z - w}; \ a^{i}(z)h(b)(w) = \frac{\partial h/\partial b^{i}(w)}{z - w}$$
(3.18)

for each  $h \in \widehat{A}_N$ , which follow from (3.15a) and the Wick theorem. Let us check (3.15a) for the tilded fields. We have

$$\tilde{b}^{i}(z)\tilde{a}^{j}(w) = g^{i}(b)(z)a^{k}\frac{\partial f^{k}}{\partial \tilde{b}^{j}}(g(b))(w) -$$

$$-g^{i}(b)(z)\frac{\partial^{2} f^{k}}{\partial \tilde{b}^{i} \partial \tilde{b}^{l}}(g(b))\frac{\partial g^{l}}{\partial b^{r}}\psi^{k}\phi^{r}(w)$$

By the Wick theorem, the first summand is equal to

$$-\frac{\partial g^i}{\partial b^k}\frac{\partial f^k}{\partial \tilde{b}^j}(g(b))(w)\cdot\frac{1}{z-w} = \frac{\delta_{ij}}{z-w},$$

by (3.18). The second summand is zero. Let us check (3.15b). The first identity is clear. We have

$$\begin{split} \tilde{a}^i(z)\tilde{a}^j(w) &= a^k \frac{\partial f^k}{\partial \tilde{b}^i}(g(b))(z)a^n \frac{\partial f^n}{\partial \tilde{b}^j}(g(b))(w) + a^k \frac{\partial f^k}{\partial \tilde{b}^i}(g(b))(z) \frac{\partial^2 f^n}{\partial \tilde{b}^j \partial \tilde{b}^l}(g(b)) \frac{\partial g^l}{\partial b^r} \psi^n \phi^r(w) - \\ &\qquad \qquad - \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b)) \frac{\partial g^l}{\partial b^r} \psi^k \phi^r(z)a^n \frac{\partial f^n}{\partial \tilde{b}^j}(g(b))(w) + \\ &\qquad \qquad + \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b)) \frac{\partial g^l}{\partial b^r} \psi^k \phi^r(z) \frac{\partial^2 f^n}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b)) \frac{\partial g^l}{\partial b^r} \psi^n \phi^r(w) \end{split}$$

When we compute each term using the Wick theorem, there appear single and double pairings. The part corresponding to the single pairings coincides with the expression of the bracket  $[\partial_{\tilde{b}^i}, \partial_{\tilde{b}^j}]$  in old coordinates  $b^p$ , so it vanishes. The "anomalous" part comes from the double pairings.

One double pairing appears in the first term and is equal to

$$-\frac{\partial}{\partial b^n} \bigg[ \frac{\partial f^k}{\partial \tilde{b}^i} (g(b)) \bigg] (z) \frac{\partial}{\partial b^k} \bigg[ \frac{\partial f^n}{\partial \tilde{b}^j} (g(b)) \bigg] (w) \cdot \frac{1}{(z-w)^2}$$

another one appears in the fourth term and equals

$$\delta_{km}\delta_{rn}\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^r}(z)\frac{\partial^2 f^n}{\partial \tilde{b}^j \partial \tilde{b}^p}(g(b))\frac{\partial g^p}{\partial b^m}(w)\cdot\frac{1}{(z-w)^2}=$$

$$=\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l}(g(b))\frac{\partial g^l}{\partial b^n}(z)\frac{\partial^2 f^n}{\partial \tilde{b}^j \partial \tilde{b}^p}(g(b))\frac{\partial g^p}{\partial b^k}(w)\cdot \frac{1}{(z-w)^2}.$$

We see that these terms cancel out.

The remaining relations, (3.15c - f), contain only single pairings, and are easily checked.  $\triangle$ 

Thus, for each automorphism  $g = (g^1, \dots, g^N)$  of  $\mathbb{C}[[b^1, \dots, b^N]]$ , (3.16a), the formulas (3.17) determine a morphism of vertex algebras

$$\tilde{g}: \widehat{\Omega}_N \longrightarrow \widehat{\Omega}_N$$
 (3.19)

More precisely, each element c of  $\widehat{V}_N$  is an (infinite) sum of finite products of  $a_{-n}^i, b_{-m}^j, \psi_{-n}^i, \psi_{-m}^j$ . We have  $c = (c(z)\mathbf{1})(0)$ . By definition,

$$\tilde{g}(c) = \left(\tilde{g}(c(z))\mathbf{1}\right)(0) \tag{3.20}$$

Thus, we have to define the field  $\tilde{g}(c(z))$  for each  $c \in \hat{V}_N$ . If c is one of the generators  $a_{-1}^i, b_0^i, \phi_0^i, \psi_{-1}^i$ , we define  $\tilde{g}(c(z))$  by formulas (3.17). We set

$$\tilde{g}(a_{-1-n}^{i}(z)) = \partial_{z}^{(n)} \tilde{g}(a_{-1}^{i}(z)), \tag{3.21}$$

and the same with  $b, \phi, \psi$ , cf. (1.12). Finally, if  $c = c^1 c^2 \cdot \ldots \cdot c^p$  where each  $c^i$  is one of the letters  $a, b, \phi$  or  $\psi$ , we set

$$\tilde{g}(c(z)) =: \tilde{g}(c^1(z))\tilde{g}(c^2(z)) \cdot \ldots \cdot \tilde{g}(c^p(z)) : \tag{3.22}$$

where the normal ordered product of two factors is defined by (1.2), and if p > 2, we use the inductive formula (1.14).

Equivalently, if  $c^j = x^j_{k_j}$  where  $x^j = a^i, b^i, \phi^i$  or  $\psi^i$ , we have

$$\tilde{g}(c^1 \cdot \ldots \cdot c^p \mathbf{1}) = \left[ \tilde{g}(x^1(z)) \right]_{k_1} \cdot \ldots \cdot \left[ \tilde{g}(x^p(z)) \right]_{k_p} \mathbf{1}$$
(3.23)

(Here we use the following notation. If  $a(z) = \sum_i a_i z^{-i-n}$  is a field corresponding to an element of conformal weight n, we denoted the Fourier mode  $a_i$  by  $a(z)_i$ .)

Let  $G_N$  denote the group of automorphisms (3.16a).

**3.8. Theorem.** The assignment  $g \mapsto \tilde{g}$  defines the group homomorphism  $G_N \longrightarrow Aut(\widehat{\Omega}_N)$ .

Let us consider two coordinate transformations,  $b^i = g_1^i(b)$ , and  $b^i = g_2^i(b)$ . Let  $f_j$  denote the transformation inverse to  $g_j$ . We have to check that

$$\widetilde{g_2g_1} = \widetilde{g}_2\widetilde{g}_1 \tag{3.24}$$

By Theorem 4.5 from [K], it suffices to check this equality on the generators. Let us begin with  $a^i$ . The element " $a_{-1}^i \mathbf{1}$  is expressed in the coordinates 'a, etc., as follows

$$''a_{-1}^{i}\mathbf{1} = \left( \ 'a_{-1}^{j} \frac{\partial f_{2}^{j}}{\partial''b^{i}} (g_{2}('b_{0})) - \frac{\partial^{2}f_{2}^{k}}{\partial''b^{i}\partial''b^{l}} (g_{2}('b_{0})) \frac{\partial g_{2}^{l}}{\partial'b^{r}} ('b_{0}) \ '\psi_{-1}^{k} \ '\phi_{0}^{r} \right) \mathbf{1}$$

Expressing it in the coordinates a, etc., we get the element

$$\tilde{g}_{2}\tilde{g}_{1}(a^{i}) = \\
= \left[ a^{p} \frac{\partial f_{1}^{p}}{\partial' b^{j}}(g_{1}(b))(z) + \frac{\partial^{2} f_{1}^{p}}{\partial' b^{j} \partial' b^{q}}(g_{1}(b)) \frac{\partial g_{1}^{q}}{\partial b^{s}} \psi^{p} \phi^{s}(z) \right]_{-1} \frac{\partial f_{2}^{j}}{\partial'' b^{i}}(g_{2}g_{1}(b))(z)_{0} \mathbf{1} - \\
- \frac{\partial^{2} f_{2}^{k}}{\partial'' b^{i} \partial'' b^{l}}(g_{2}g_{1}(b)) \frac{\partial g_{2}^{l}}{\partial' b^{r}}(g_{1}(b))(z)_{0} \left[ \frac{\partial f_{1}^{p}}{\partial' b^{k}}(g_{1}(b)) \psi^{p}(z) \right]_{-1} \left[ \frac{\partial g_{1}^{r}}{\partial b^{q}} \phi^{q}(z) \right]_{0} \mathbf{1} \quad (3.25)$$

(we have used (3.23)). Now, the action of our group  $G_N$  on the classical de Rham complex is associative. It follows that the expression (3.25) is equal to  $\widetilde{g_2g_1}(a^i)$  plus two anomalous terms:

$$a_0^p \left[ \frac{\partial f_1^p}{\partial' b^j}(g_1(b))(z) \right]_{-1} \frac{\partial f_2^j}{\partial'' b^i}(g_2 g_1(b_0)) \mathbf{1} = \left[ \frac{\partial f_1^p}{\partial' b^j}(g_1(b))(z) \right]_{-1} \frac{\partial}{\partial b_0^p} \frac{\partial f_2^j}{\partial'' b^i}(g_2 g_1(b_0)) \mathbf{1}$$

coming from the first summand, and

$$-\frac{\partial^2 f_2^k}{\partial''b^i\partial''b^l}(g_2g_1(b_0))\frac{\partial g_2^l}{\partial'b^r}(g_1(b_0))\left[\frac{\partial f_1^p}{\partial'b^k}(g_1(b))(z)\right]_{-1}\frac{\partial g_1^r}{\partial b^q}(b_0)\psi_0^p\phi_0^q\mathbf{1} =$$

$$=-\frac{\partial^2 f_2^k}{\partial''b^i\partial''b^l}(g_2g_1(b_0))\frac{\partial g_2^l}{\partial'b^r}(g_1(b_0))\left[\frac{\partial f_1^p}{\partial'b^k}(g_1(b))(z)\right]_{-1}\frac{\partial g_1^r}{\partial b^p}(b_0)\mathbf{1}$$

coming from the second one. One sees that these two terms cancel out, which proves (3.24) for  $a^i$ . For the generators  $b, \psi$ , and  $\phi$ , the anomaly does not appear at all.  $\triangle$ 

**3.9.** Theorems 3.7 and 3.8 allow one to define the sheaf of conformal vertex algebras  $\Omega_X^{ch}$  for each smooth manifold X in an invariant way, by gluing the sheaves defined in 3.5. This can be done in each of the three settings: in algebraic, complex analytic or smooth one.

In the complex analytic situation, we have our sheaves of vertex algebras over the coordinate charts, and the formulas (3.17) allow to glue these sheaves in a sheaf over X.

In the algebraic situation, Theorem 3.8 ensures the existence of our sheaves by the standard arguments of "formal geometry" of Gelfand and Kazhdan, cf. [GK]. Consider the formal situation. By 3.8, the vertex algebra  $\widehat{\Omega}_N$  is a  $G_N$ -module. Therefore, the Lie algebra  $W_N' = Lie(\widehat{\Omega}_N)$  of formal vector fields vanishing at 0 acts on  $\widehat{\Omega}_N$  by derivations. In fact, since the proof of Theorem 3.8 never uses the fact that the automorphisms in question preserve the origin, the infinitesimal version of formulas (3.17) shows that the entire algebra  $W_N$  of formal vector fields operates on  $\widehat{\Omega}_N$ . (Alternatively, this can be shown by the computation similar to the one from 5.1 (in the case of  $\widehat{\Omega}_N$  the anomaly vanishes!)). In other words,  $\widehat{\Omega}_N$  is a  $(W_N, G_N)$ -module (cf. [BS], [BFM]). Now, the standard result, [GK], says that such a module defines naturally a sheaf  $\Omega_X^{ch}$  on each smooth algebraic variety X. They are sheaves of vertex algebras since  $G_N$  (resp.,  $W_N$ ) acts by vertex algebra automorphisms (resp., derivations).

A more direct construction of these sheaves is outlined in §6, see 6.10.

**3.10.** Consider the formal situation. Let us show that the algebra  $\widehat{\Omega}_N$  admits a canonical filtration whose graded factors are standard tensor fields.

Placing the formulae (3.16) and (3.17), (3.23) on the desk next to each other, one realizes that the "symbols of fields" transform in the same way as the corresponding geometric quantities: functions, 1-forms, and vector fields. To be more precise, introduce a filtration on  $\widehat{\Omega}_N$  as follows.

The space  $\widehat{\Omega}_N$  is a free  $\widehat{A}_N$ -module with a base consisting of monomials in letters  $a_n^i, \psi_n^i \ (n < 0); \ b_m^i, \phi_m^i \ (m \le 0)$ . Define a partial ordering on this base by

- (a)  $a > \phi$ ,  $a > \psi$ , a > b,  $\psi > \phi$ ,  $\psi > b$ ,  $\phi > b$ ;  $x_n^i > x_m^j$  if n < m, x being  $a, b, \phi$  or  $\psi$ ;
  - (b) extending this order to the whole set of monomials lexicographically.

This partial order on the base naturally determines an increasing exhausting filtration on the spaces of fixed conformal weight,

$$F_0\widehat{\Omega}_N^{(i)} \subset F_1\widehat{\Omega}_N^{(i)} \subset \dots \tag{3.27}$$

For example,  $F_0\widehat{\Omega}_N^{(0)}=\widehat{A}_N$ ,  $F_0\widehat{\Omega}_N^{(-1)}=\oplus_{i=1}^N\widehat{A}_Nb_{-1}^i$ , etc. A glance at (3.17), (3.23) shows that the corresponding graded object  $\operatorname{Gr}_{\bullet}^F\Omega_N^{(i)}$  is a direct sum of symmetric powers of the tangent bundle, symmetric powers of the bundle of 1-forms, and tensor products thereof. For example, the image of  $b_0^i$  in  $\operatorname{Gr}_{\bullet}^F\widehat{\Omega}_N^{(0)}$  is a function, that of  $b_{-n}^i$  (n>0) is a 1-form, that of  $a_{-n}^i$  is a vector field, etc.

This filtration is stable under coordinate changes. Therefore all the sheaves  $\Omega_X^{ch}$  acquire the natural filtration with graded factors being the bundles of tensor fields.

# §4. Conformal and topological structure

**4.1.** Let us return to the formal setting 3.6. Recall that we have in our vertex algebra  $\widehat{\Omega}_N$  the fields L(z), J(z), Q(z) and G(z), defined by the formulas (2.4), which make it a topological vertex algebra. Let us study the effect of the coordinate changes (3.16a) on these fields.

Let us denote by  $\tilde{L}(z)$ , etc., the field L(z), etc., written down using formulas (2.4), in terms of the tilded fields  $\tilde{a}^i(z)$ , etc., and then expressed in terms of the old fields  $a^i(z)$ , etc.

# **4.2.** Theorem. We have

$$\tilde{L}(z) = L(z) \tag{4.1a}$$

$$\tilde{J}(z) = J(z) + \left( Tr \log(\partial g^i / \partial b^j(z)) \right)' \tag{4.1b}$$

$$\tilde{Q}(z) = Q(z) + \left(\frac{\partial}{\partial \tilde{b}^r} \left[ Tr \log(\partial f^i / \partial \tilde{b}^j) \right] \tilde{\phi}^r(z) \right)'$$
(4.1c)

$$\tilde{G}(z) = G(z) \tag{4.1d}$$

We have  $J = \phi_0^i \psi_{-1}^i \mathbf{1}$ . Therefore, (cf. (3.23)).

$$\tilde{J} = \left[ \frac{\partial g^i}{\partial b^j} \phi^j(z) \right]_0 \left[ \frac{\partial f^k}{\partial \tilde{b}^i} \psi^k(z) \right]_{-1} \mathbf{1} = J + \delta_{jk} \left[ \frac{\partial g^i}{\partial b^j}(z) \right]_{-1} \left[ \frac{\partial f^k}{\partial \tilde{b}^i}(z) \right]_0 \mathbf{1} = J + \left( \frac{\partial g^i}{\partial b^j}(z) \right)_0' \left( \frac{\partial f^j}{\partial \tilde{b}^i}(z) \right)_0 \mathbf{1},$$

which implies (4.1b). We have  $G = \psi_{-1}^i b_{-1}^i \mathbf{1}$ . Therefore,

$$\tilde{G} = \left[\frac{\partial f^{j}}{\partial \tilde{b}^{i}} \psi^{j}(z)\right]_{-1} \left[g^{i}(b)(z)\right]_{-1} \mathbf{1} = \left[\frac{\partial f^{j}}{\partial \tilde{b}^{i}} \psi^{j}(z)\right]_{-1} \left[\frac{\partial g^{i}}{\partial b^{k}} b^{k}(z)'\right]_{0} \mathbf{1} =$$

$$= \delta_{jk} \psi^{j}_{-1} b^{k}_{-1} \mathbf{1} = G$$

This proves (4.1d). We have  $Q = a_{-1}^i \phi_0^i \mathbf{1}$ . Therefore,

$$\tilde{Q} = \left[ a^j \frac{\partial f^j}{\partial \tilde{b}^i}(z) - \frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l} \frac{\partial g^l}{\partial b^r} \psi^k \phi^r(z) \right]_{-1} \left[ \frac{\partial g^i}{\partial b^q} \phi^q(z) \right]_0 \mathbf{1}$$

The classical terms:

$$\left[a_{-1}^{j}\left(\frac{\partial f^{j}}{\partial \tilde{b}^{i}}\right)_{0}-\left(\frac{\partial^{2} f^{k}}{\partial \tilde{b}^{i} \partial \tilde{b}^{l}}\frac{\partial g^{l}}{\partial b^{r}}\right)_{0} \psi_{-1}^{k} \phi_{0}^{r}\right]\left(\frac{\partial g^{i}}{\partial b^{q}} \phi^{q}\right)_{0} \mathbf{1}=Q,$$

since the second summand is zero, due to the anticommutation of  $\psi_{-1}^k$  and  $\phi_0^q$ . Quantum corrections (anomalous terms):

$$\left[ \left( \frac{\partial f^{j}}{\partial \tilde{b}^{i}} \right)_{-1} \left( \frac{\partial^{2} g^{i}}{\partial b^{j} \partial b^{q}} \right)_{0} \phi_{0}^{r} + \left( \frac{\partial^{2} f^{k}}{\partial \tilde{b}^{i} \tilde{b}^{l}} \frac{\partial g^{l}}{\partial b^{r}} \phi^{r} \right)_{-1} \left( \frac{\partial g^{i}}{\partial b^{k}} \right)_{0} \right] \mathbf{1} =$$

$$= \left( \left[ \left( \frac{\partial f^{j}}{\partial \tilde{b}^{i}} \right)_{-1} \left( \frac{\partial^{2} g^{i}}{\partial b^{j} \partial b^{r}} \right)_{0} + \left( \frac{\partial^{2} f^{k}}{\partial \tilde{b}^{i} \partial \tilde{b}^{l}} \frac{\partial g^{l}}{\partial b^{r}} \right)_{-1} \left( \frac{\partial g^{i}}{\partial b^{k}} \right)_{0} \right] \phi_{0}^{r} +$$

$$+ \left( \frac{\partial^{2} f^{k}}{\partial \tilde{b}^{i} \partial \tilde{b}^{l}} \frac{\partial g^{l}}{\partial b^{r}} \frac{\partial g^{i}}{\partial b^{k}} \right)_{0} \phi_{-1}^{r} \right) \mathbf{1} = \left( \frac{\partial f^{k}}{\partial \tilde{b}^{i} \partial \tilde{b}^{l}} \frac{\partial g^{l}}{\partial b^{r}} \frac{\partial g^{i}}{\partial b^{k}} \phi^{r} \right)_{-1} \mathbf{1}, \tag{4.2}$$

since

$$\begin{split} &\left(\frac{\partial^{2} f^{k}}{\partial \tilde{b}^{i} \partial \tilde{b}^{l}} \frac{\partial g^{l}}{\partial b^{r}}\right)_{0} \left(\frac{\partial g^{i}}{\partial b^{k}}\right)_{-1} \mathbf{1} = -\left(\frac{\partial f^{k}}{\partial \tilde{b}^{l}} \frac{\partial^{2} g^{l}}{\partial b^{t} \partial b^{r}} \frac{\partial f^{t}}{\partial \tilde{b}^{i}}\right)_{0} \left(\frac{\partial g^{i}}{\partial b^{k}}\right)_{-1} \mathbf{1} = \\ &= \left(\frac{\partial f^{k}}{\partial \tilde{b}^{l}} \frac{\partial^{2} g^{l}}{\partial b^{t} \partial b^{r}}\right)_{0} \left(\frac{\partial f^{t}}{\partial \tilde{b}^{i}}\right)_{-1} \left(\frac{\partial g^{i}}{\partial b^{k}}\right)_{0} \mathbf{1} = \left(\frac{\partial^{2} g^{i}}{\partial b^{t} \partial b^{r}}\right)_{0} \left(\frac{\partial f^{t}}{\partial \tilde{b}^{i}}\right)_{-1} \mathbf{1} \end{split}$$

Returning to (4.2), we have

$$\frac{\partial^2 f^k}{\partial \tilde{b}^i \partial \tilde{b}^l} \frac{\partial g^l}{\partial b^r} \frac{\partial g^i}{\partial b^k} \phi^r = \frac{\partial^2 f^k}{\partial \tilde{b}^l \partial \tilde{b}^i} \frac{\partial g^i}{\partial g^k} \tilde{\phi}^l = \frac{\partial}{\partial \tilde{b}^l} \bigg( \mathrm{Tr} \log(\partial f^i / \partial \tilde{b}^j) \bigg) \tilde{\phi}^l,$$

which proves (4.1c). It follows from (4.1c) that the operator  $Q_0$  is invariant. Hence, (4.1a) follows from (4.1d) and (2.2).  $\triangle$ 

**4.3.** It follows from (4.1a) that for an arbitrary smooth manifold X, the field L(z) is a well-defined global section of the sheaf  $\operatorname{End}(\Omega_X^{ch})[[z,z^{-1}]]$ , i.e.  $\Omega_X^{ch}$  is canonically a sheaf of conformal vertex algebras.

It follows from (4.1b) and (4.1c) that the Fourier modes  $F = J_0$  ("fermionic charge") and  $d_{DR}^{ch} = Q_0$  ("BRST charge") are well-defined endomorphisms of the sheaf  $\Omega_X^{ch}$ . Thus,  $(\Omega_X^{ch}, d_{DR}^{ch})$  becomes a complex of sheaves, graded by F. This is a localization of definition 2.3.

**4.4. Theorem.** For any smooth manifold X, the obvious embedding of complexes of sheaves

$$i: (\Omega_X, d_{DR}) \longrightarrow (\Omega_X^{ch}, d_{DR}^{ch})$$
 (4.3)

is a quasiisomorphism. This is true in algebraic, analytic and  $C^{\infty}$  settings.

Indeed, the problem is local along X, and we are done by Theorem 2.4.

**4.5.** If X is Calabi-Yau, i.e.  $c_1(\mathcal{T}_X) = 0$ , then the fields J(z) and Q(z) are globally well defined, by (4.1b) and (4.1c). Here  $\mathcal{T}_X$  denotes the tangent bundle. Therefore, in this case the sheaf  $\Omega_X^{ch}$  is canonically a sheaf of topological vertex algebras.

# §5. Chiral structure sheaf

#### A. OBSTRUCTION

**5.1.** Consider the formal setting 3.1, 3.2, 3.6. We have the vertex algebra  $\widehat{V}_N$  of "chiral functions" over the formal disk  $D_N = \operatorname{Spf}(\widehat{A}_N)$  where  $\widehat{A}_N = \mathbb{C}[[b^1, \dots, b^N]]$ . Let  $W_N$  denote the Lie algebra of vector fields  $f^i(b)\partial_{b^i}$  on  $D_N$ . Let  $\Omega^1(\widehat{A}_N)$  denote the module of one-forms  $f^i(b)db^i$ . The spaces  $\widehat{A}_N$  and  $\Omega^1(\widehat{A}_N)$  are naturally  $W_N$ -modules. Recall that the action of  $W_N$  on  $\Omega^1(\widehat{A}_N)$  is given by

$$f^{i}\partial_{b^{i}} \cdot g^{j}db^{j} = f^{i}\partial_{b^{i}}g^{j}db^{j} + g^{j}df^{j}$$

$$(5.1)$$

The de Rham differential  $d: \widehat{A}_N \longrightarrow \Omega^1(\widehat{A}_N)$  is compatible with the  $W_N$ -action.

Let us define a map

$$\pi: W_N \longrightarrow \operatorname{End}(\widehat{V}_N)$$
 (5.2)

For a vector field  $\tau = f^i(b)\partial_{b^i}$ , let  $\tau(z)$  denote the field  $f^i(b)a^i(z)$  (of conformal weight 1) of our vertex algebra  $\widehat{V}_N$ . Let  $\pi(\tau) \in \operatorname{End}(\widehat{V}_N)$  denote the Fourier mode

$$\pi(\tau) := \int \tau(z) = \tau(z)_0 \tag{5.3}$$

Note that by 1.7, the maps  $\pi(\tau)$  are derivations of  $\widehat{V}_N$ .

The mapping  $\pi$  does not respect the Lie bracket. Let us compute the discrepancy. Let  $\tau_1 = f^i(b)\partial_{b^i}$ ,  $\tau_2 = g^i(b)\partial_{b^i}$  be two vector fields. We have the operator product

$$\tau_1(z)\tau_2(w) = -\frac{\partial_{b^j} f^i(b(z))\partial_{b^i} g^j(b(w))}{(z-w)^2} +$$

$$+\frac{f^{i}(b(w))\partial_{b^{i}}g^{j}(b(w))a^{j}(w)-g^{j}(b(w))\partial_{b^{j}}f^{i}(b(w))a^{i}(w)}{z-w}=$$

$$= -\frac{\partial_{b^{j}} f^{i}(b(w)) \partial_{b^{i}} g^{j}(b(w))}{(z-w)^{2}} + \frac{[\tau_{1}, \tau_{2}](w)}{z-w} - \frac{\left[\partial_{b^{j}} f^{i}(b(w))\right]' \partial_{b^{i}} g^{j}(b(w))}{z-w}$$
(5.4)

It follows that

$$[\pi(\tau_1), \tau_2(w)] = [\tau_1, \tau_2](w) - [\partial_{b^j} f^i(b(w))]' \partial_{b^i} g^j(b(w))$$
(5.5)

In particular,

$$[\pi(\tau_1), \pi(\tau_2)] = \pi([\tau_1, \tau_2]) - \int (\partial_{b^j} f^i(b(w)))' \partial_{b^i} g^j(b(w))$$
 (5.6)

**5.2.** For  $\omega = f^i(b)db^i \in \Omega^1(\widehat{A}_N)$ , let us denote by  $\omega(z)$  the field  $f^i(b)b^i(z)'$  of our vertex algebra. Denote by  $\pi(\omega)$  the Fourier mode  $\omega(z)_0 = \int \omega(z)$ .

For  $f = f(b) \in \widehat{A}^N$ , let f(z) denote the corresponding field of  $\widehat{V}_N$ . Its conformal weight is 0, and  $f(z)_0 = f$ . We have

$$df(z) = f(z)' (5.7)$$

Given  $\tau = f^i \partial_{b^i} \in W_N$ ,  $\omega = g^j db^j \in \Omega^1(\widehat{A}_N)$ , we have the operator product

$$\tau(z)\omega(w) = \frac{f^{i}(b(z))g^{i}(b(w))}{(z-w)^{2}} + \frac{f^{i}(b(w))\partial_{b^{i}}g^{j}b^{j}(w)'}{z-w} =$$

$$= \frac{f^{i}(b(w))g^{i}(b(w))}{(z-w)^{2}} + \frac{f^{i}\partial_{b^{i}}g^{j}(w)b^{j}(w)' + f^{i}(b(w))'g^{i}(b(w))}{z-w}$$
(5.8)

It follows that

$$[\pi(\tau), \omega(z)] = (\tau\omega)(z) \tag{5.9}$$

and

$$[\pi(\tau), \pi(\omega)] = \pi(\tau\omega) \tag{5.10}$$

Let  $\tilde{W}_N$  denote the linear subspace of  $\operatorname{End}(\hat{V}_N)$  generated by the Fourier modes  $\pi(\tau)$  ( $\tau \in W_N$ ) and  $\pi(\omega)$  ( $\omega \in \Omega^1(\hat{A}_N)$ ). Let  $I_N \in \tilde{W}_N$  be the linear subspace generated by the Fourier modes  $\pi(\omega)$ .

It follows from (5.7) that if  $\omega$  is exact then  $\pi(\omega) = 0$ . Thus,  $\pi$  induces an epimorphic map

$$\Omega^1(\widehat{A}_N)/d\widehat{A}_N \longrightarrow I_N$$
 (5.11)

#### **5.3.** Lemma. The map (5.11) is an isomorphism.

This can be proved by writing down the Fourier mode as an infinite sum of monomials in  $b_n^i$  and comparing the coefficients of like terms. In fact, a more general statement, namely, that  $\int Q(z) = 0$  if and only if  $Q = \text{const} \cdot \mathbf{1}$  or Q(z) = P(z)' for some P, seems to be valid for a broad class of vertex algebras, cf. a similar statement in [FF3].

¿From our point of view, this phenomenon has topological nature. It is amusing to exhibit an example of a vertex algebra, for which the lemma above is false. Namely, take  $b^{-1}b_{-1} \in A_1[b^{-1}]$ , see 3.4; then  $(b^{-1}b_{-1})(z) = b(z)^{-1}b(z)'$  and  $\int b(z)^{-1}b(z)' = 0$ , but  $b(z)^{-1}b(z)'$  is not a total derivative.  $\triangle$ 

**5.4.** Obviously,  $\omega_1(z)\omega_2(w) = 0$  for all  $\omega_1, \omega_2 \in \Omega^1(\widehat{A}_N)$ . It follows from (5.6) and (5.10) that  $\widetilde{W}_N$  is a Lie algebra,  $I_N$  is its abelian ideal, and we have the canonical extension

$$0 \longrightarrow I_N \longrightarrow \tilde{W}_N \longrightarrow W_N \longrightarrow 0 \tag{5.12}$$

The action of  $W_N$  on  $I_N$  arising from this extension coincides with the canonical action of  $W_N$  on  $\Omega^1(\widehat{A}_N)/d\widehat{A}_N$ , by (5.10). Note that we have defined this extension together with its splitting (5.2). It is given by the two-cocycle  $c \in Z^2(W_N; \Omega^1(\widehat{A}_N)/d\widehat{A}_N)$  of  $W_N$  with values in  $\Omega^1(\widehat{A}_N)/d\widehat{A}_N$ , read from (5.5),

$$c(f^{i}\partial_{b^{i}}, g^{j}\partial_{b^{j}}) = -\partial_{b^{i}}g^{j}d(\partial_{b^{j}}f^{i}) \pmod{d\widehat{A}_{N}}$$
(5.13)

**5.5.** Let us consider the truncated and shifted de Rham complex

$$\Omega^{\bullet}: 0 \longrightarrow \widehat{A}_N \longrightarrow \Omega^1(\widehat{A}_N) \longrightarrow 0$$
(5.14)

where we place  $\Omega^1(\widehat{A}_N)$  in degree zero. It is a complex of  $W_N$ -modules. We have an obvious map of complexes of  $W_N$ -modules

$$\Omega^{\bullet} \longrightarrow \Omega^{1}(\widehat{A}_{N})/d\widehat{A}_{N}$$
(5.15)

where the target is regarded as a complex sitting in degree zero.

Let us write down a two-cocycle  $\tilde{c} \in Z^2(W_N; \Omega^{\bullet})$  which is mapped to c, (5.13), under the map (5.14). Such a cocycle is by definition a pair  $(c^2, c^3)$  where  $c^2 \in C^2(W_N; \Omega^1(\widehat{A}_N))$  is a two-cochain, and  $c^3 \in C^3(W_N; \widehat{A}_N)$  is a three-cochain, such that

$$d_{Lie}(c^2) = d_{DR}(c^3) (5.16a)$$

$$d_{Lie}(c^3) = 0 (5.16b)$$

Let us define

$$c^{2}(f^{i}\partial_{i}, g^{j}\partial_{j}) = \partial_{i}g^{j}d(\partial_{j}f^{i}) - \partial_{j}f^{i}d(\partial_{i}g^{j})$$

$$(5.17)$$

and

$$c^{3}(f^{i}\partial_{i}, g^{j}\partial_{j}, h^{k}\partial_{k}) = \partial_{j}f^{i}\partial_{k}g^{j}\partial_{i}h^{k} - \partial_{k}f^{i}\partial_{i}g^{j}\partial_{j}h^{k}$$

$$(5.18)$$

We write for brevity  $\partial_i$  instead of  $\partial_{b^i}$ . One checks the compatibilities (5.16) directly. Thus, we have defined  $\tilde{c}$ . One sees that  $\tilde{c}$  is mapped to -2c under (5.15).

For N=1 the space  $\Omega^1(\widehat{A}_N)/d\widehat{A}_N$  is trivial. This allows one to define the sheaf  $\mathcal{O}_X^{ch}$  for curves, acting as in 3.9, and starting from  $\widehat{V}_N$  instead of  $\widehat{\Omega}_N$ .

Assume that N > 1. Using the computations of Gelfand-Fuchs, cf. [F], Theorems 2.2.7 and 2.2.4, one can show that the map in cohomology

$$H^2(W_N; \Omega^{\bullet}) \longrightarrow H^2(W_N; \Omega^1(\widehat{A}_N)/d\widehat{A}_N)$$
 (5.19)

induced by (5.15) is an isomorphism. We have the canonical short exact sequence

$$0 \longrightarrow H^2(W_N; \Omega^1(\widehat{A}_N)) \longrightarrow H^2(W_N; \Omega^{\bullet}) \longrightarrow H^3(W_N; \widehat{A}_N) \longrightarrow 0$$
 (5.20)

the left- and right-most terms being one-dimensional. Under the second map of this sequence, our cocycle  $\tilde{c}$  is mapped to its second component  $c^3$  which is a canonical representative of a generator of the space  $H^3(W_N; \widehat{A}_N)$ , cf. [F], Theorem 2.2.7′ and Chapter 2, §1, no. 4. In particular, our cocycle  $\tilde{c}$  is non-trivial. It follows that the cocycle c defining the extension (5.12) is also non-trivial.

What kind of an object does the cocycle  $\tilde{c}$  define? Recall that a homotopy Lie algebra  $L^{\bullet}$  is a complex of vector spaces equipped with a collection of brackets

$$[,\ldots,]_i:\Lambda^iL^{\bullet}\longrightarrow L^{\bullet}[-i+2],\ i\geq 2,$$
 (5.21)

satisfying certain compatibility conditions, cf. for example [HS],  $\S 4$ . In particular  $[\ ,]_2$  is a skew symmetric map, satisfying the Jacobi identity up to the homotopy (given by the third bracket).

Let us define a complex  $L^{\bullet}$  as follows. Set  $L^0 = W_N \oplus \Omega^1(\widehat{A}_N)$ ,  $L^{-1} = \widehat{A}_N$ , the other components being trivial. The differential  $L^{-1} \longrightarrow L^0$  is the composition of the de Rham differential and the obvious embedding  $\Omega^1(\widehat{A}_N) \hookrightarrow L^0$ .

Let the second bracket  $[\ ,\ ]_2$  be given by the usual bracket of vector fields, and the action of vector fields on  $\widehat{A}_N$  and  $\Omega^1(\widehat{A}_N)$ . Define the third bracket with the only nontrivial component being the three-cocycle  $c^3$ , (5.18). We set the other brackets equal to zero. This way we get a structure of a homotopy Lie algebra on  $L^{\bullet}$ .

We have a canonical extension of homotopy Lie algebras

$$0 \longrightarrow \Omega^{\bullet} \longrightarrow L^{\bullet} \longrightarrow W_N \longrightarrow 0 \tag{5.22}$$

Here  $\Omega^{\bullet}$  is an abelian ideal in  $L^{\bullet}$  (all brackets are zero). This is a refinement of extension (5.12).

### B. PROJECTIVE LINE

**5.6.** Let X be the projective line  $\mathbb{P}^1$ . Let us fix a coordinate b on  $\mathbb{P}^1$ , and consider the open covering  $X = U_0 \cup U_1$  where  $U_0 = \operatorname{Spec}(\mathbb{C}[b]), \ U_1 = \operatorname{Spec}(\mathbb{C}[b^{-1}]).$ 

Consider the sheaves  $\mathcal{O}_{U_0}^{ch}$  on  $U_0$  with coordinate b, and  $\mathcal{O}_{U_1}^{ch}$  on  $U_1$  with coordinate  $b^{-1}$ , which were defined in 3.4. Let us glue them on the intersection  $U_{01} = U_0 \cap U_1$  using the transition function

$$\tilde{b}(z) = b(z)^{-1}$$
 (5.23a)

$$\tilde{a}(z) = b^2 a(z) + 2b(z)' \tag{5.23b}$$

In this way, we get the sheaf on the X, to be denoted  $\mathcal{O}_X^{ch}$ .

**5.7. Theorem.** The space of global sections  $\Gamma(X; \mathcal{O}_X^{ch})$  admits a natural structure of an irreducible vacuum  $\widehat{sl}_2$ -module on the critical level.

We have to compute  $\mathcal{O}_{U_0}^{ch} \cap \mathcal{O}_{U_1}^{ch}$  where both  $\mathcal{O}_{U_i}^{ch}$  are regarded as subspaces of  $\mathcal{O}_{U_{01}}^{ch}$ . It is the essence of the Wakimoto construction, [W], that the fields  $a(z)b(z)^2 + 2b(z)'$ , a(z) (resp.,  $\tilde{a}(z)\tilde{b}(z)^2 + 2\tilde{b}(z)'$ ,  $\tilde{a}(z)$ ) generate an  $\widehat{\text{sl}}_2$ -action on  $\mathcal{O}_{U_0}^{ch}$  (resp., on  $\mathcal{O}_{U_1}^{ch}$ ), and under this action,  $\mathcal{O}_{U_i}^{ch}$ , i=1,2, become the restricted Wakimoto module with zero highest weight. (Restricted here means that the level is critical, and the Sugawara operators act by zero.) It follows from (5.23) that the  $\widehat{\text{sl}}_2$ -action comes from  $\Gamma(X;\mathcal{O}_X^{ch})$  and, therefore,  $\mathcal{O}_{U_0}^{ch} \cap \mathcal{O}_{U_1}^{ch}$  is also an  $\widehat{\text{sl}}_2$ -module. It follows from [FF1] or [M] that each  $\mathcal{O}_{U_i}^{ch}$  contains a unique proper submodule which is isomorphic to the irreducible vacuum representation. To complete the proof, it remains to show that  $\mathcal{O}_{U_0}^{ch} \neq \mathcal{O}_{U_0}^{ch} \cap \mathcal{O}_{U_1}^{ch}$ , and this is obvious.  $\triangle$ 

**5.8.** In fact, the first cohomology space  $H^1(X; \mathcal{O}_X^{ch})$  is also isomorphic to the same irreducible  $\widehat{\mathrm{sl}}_2$ -module.

To prove this, let us compute the Euler character

$$ch(X;\mathcal{O}_X^{ch}) = \sum_{N=0}^{\infty} \ \chi(X;\mathcal{O}_X^{ch(N)}) \cdot q^N$$

in two different ways. First, by definition

$$ch(X; \mathcal{O}_X^{ch}) = ch(\Gamma(X; \mathcal{O}_X^{ch})) - ch(H^1(X; \mathcal{O}_X^{ch}))$$

By Theorem 5.7 and [M],

$$ch(\Gamma(X; \mathcal{O}_X^{ch})) = (1-q)^{-1} \prod_{N=1}^{\infty} (1-q^N)^{-2}$$

On the other hand, formulas (5.23) imply that  $\mathcal{O}_X^{ch}$  carries a filtration F such that the image of  $a_{-n}$  (resp.,  $b_{-n}$ ) ( $n \geq 1$ ) in  $\operatorname{Gr}^F$  is a vector field (resp., a 1-form). It follows that each monomial  $a_{-n_1} \cdot \ldots \cdot a_{-n_r} b_{-m_1} \cdot \ldots \cdot b_{-m_s}$  contributes 2s - 2r + 1 in  $\chi(X; \mathcal{O}_X^{ch(N)})$ , where  $N = \sum n_i + \sum m_j$ . Therefore,

$$ch(X; \mathcal{O}_X^{ch}) = \prod_{N=1}^{\infty} (1 - q^N)^{-2},$$

hence

$$ch(H^1(X; \mathcal{O}_X^{ch})) = q \cdot ch(\Gamma(X; \mathcal{O}_X^{ch}))$$

In other words,  $H^1(X; \mathcal{O}_X^{ch})$  has the same (up to the shift by q) character as  $\Gamma(X; \mathcal{O}_X^{ch})$ . Again by [M], these two spaces are isomorphic as  $\widehat{\mathfrak{g}}$ -modules.

## C. FLAG MANIFOLDS

**5.9.** Let G be a simple algebraic group,  $B \subset G$  a Borel subgroup,  $N \subset B$  the maximal nilpotent subgroup. The manifold N is isomorphic to the affine space, and

is a  $(\mathfrak{g}, B)$ -scheme, where B acts by conjugation. Consider the Heisenberg vertex algebra V associated with the affine space N. According to [FF2], V admits a structure of a  $\widehat{\mathfrak{g}}$ -module (Wakimoto module); in particular, V is a  $(\mathfrak{g}, B)$ -module. Note that  $x \in \mathfrak{g}$  acts on V as  $\int X(z)$  for some  $X \in V$ .

Consequently, considered as an affine space, V admits a structure of a  $(\mathfrak{g}, B)$ -scheme. Let M be the algebra of functions on V. Proceeding as in 3.9, with K = B,  $\hat{X} = G$ , and X = G/B, we get the sheaf of ind-schemes

$$U \mapsto \operatorname{Spec}(H_{\nabla}(\Delta(M)))$$

on X. The sheaf of its  $\mathbb{C}$ -points is called the *chiral structure sheaf* of X and denoted by  $\mathcal{O}_X^{ch}$ .

**5.10.** If G = SL(n) then the sheaf  $\mathcal{O}_X^{ch}$  admits a more explicit construction, using charts and gluing functions. In this case  $X = GL(n)/(B \times \mathbb{C}^*)$ . The Weyl group is identified with the symmetric group  $S_n$  and can be realized as the subgroup of GL(n) consisting of permutation matrices. One checks that in terms of the Lie algebra gl(n), the simple permutation  $r_i$  (interchanging i and i+1) can be written as follows

$$r_i = \exp(\pi \sqrt{-1}E_{ii}) \exp(E_{i+1,i}) \exp(-E_{i,i+1}) \exp(E_{i+1,i})$$
(5.24)

where  $E_{ij}$   $(1 \le i, j \le n)$  form the standard base of gl(n).

The manifold X is covered by |S| = n! charts, the chart associated with an element  $w \in S_n$  being  $U_w = wNw_0B$ , where  $N \subset B$  is the unipotent subgroup consisting of all upper-triangular matrices and  $w_0 \in S_n$  is the element of maximal length. Let us identify  $U_w$  with N using the bijection  $n \mapsto wnw_0B$ . Under this identification, if  $x \in U_{w_1} \cap U_{w_2}$ , then the change from the coordinates determined by  $U_{w_1}$  to the ones determined by  $U_{w_2}$ , is given by  $x \mapsto w_2^{-1}w_1x$ .

Each  $U_w$  may be identified with the affine space  $\mathbb{C}^{n(n-1)/2}$ . To define  $\mathcal{O}_X^{ch}$ , we declare that  $\mathcal{O}_X^{ch}|_{U_w} = \mathcal{O}_{U_w}^{ch}$  where the last sheaf is defined in 3.4. Now we have to glue these sheaves over the pairwise intersections in a consistent manner.

Let V denote the vertex algebra  $V_{n(n-1)/2} = \Gamma(U_w; \mathcal{O}_{U_w}^{ch})$ . First, we extend the  $\widehat{sl}(n)$ -module structure on V to a  $\widehat{gl}(n)$ -module structure. For that, in addition to the formulae in [FF1], p. 279, define

$$E_{ii}(z) = -\sum_{j>i} b^{ij} a^{ij}(z) + \sum_{j (5.25)$$

It is easily checked that in this way we indeed get an action of  $\widehat{\mathrm{gl}}(n)$  on V.

For any  $A \in \text{End}(V)$ , introduce the formal exponent  $\exp(tA): V \longrightarrow V \otimes \mathbb{C}[[t]]$ ,

$$\exp(tA)(v) = \sum_{i=0}^{\infty} \frac{A^i(v)}{i!} t^i$$
(5.26)

Working over  $\mathbb{C}[[t]]$ , we can easily compose such maps. Motivated by (5.24), set

$$r_i(t) = \exp(t\pi\sqrt{-1}\int E_{ii}(z))$$

$$\cdot \exp(t \int E_{i+1,i}(z)) \exp(-t \int E_{i,i+1}(z)) \exp(t \int E_{i+1,i}(z))$$
 (5.27)

Thus,  $r_i(t)$  is a map  $V \longrightarrow V[[z, z^{-1}]][[t]]$ . Note that V[[t]] is naturally a vertex algebra. It contains the vertex subalgebra "generated by functions, rational in t", that is,

$$V(t) = R(N \times \mathbb{C}) \otimes_A V \subset V[[t]]$$
(5.28)

Here  $A = \Gamma(N; \mathcal{O}_N)$ , and  $R(N \times \mathbb{C})$  denotes the ring of rational functions on  $N \times \mathbb{C}$ , regular at  $N \times \{0\}$ .

**5.11. Lemma.** (a)  $r_i(t)V \subset V(t)$ . Further,  $r_i(t)V$  is generated by functions well-defined for any value of t.

(b)  $r_i(1)$  is well defined (by (a)), and determines a map

$$r_i(1): \Gamma(U_w; \mathcal{O}_{U_w}^{ch}) \longrightarrow \Gamma(U_w \cap U_{r_iw}, \mathcal{O}_{U_w}^{ch})$$
 (5.29)

As  $\int X(z)$ ,  $X \in V$ , is a derivation, the map  $r_i(t)$  is an embedding of vertex algebras. Therefore, it is enough to compute  $r_i(t)$  on generators.

There are two types of generators in V: (i) those coming from the subspace  $W \subset V$ , canonically isomorphic to  $\mathfrak{g} = sl(n)$ , such that the fields X(z) ( $X \in W$ ) generate the action of  $\widehat{\mathfrak{g}}$ ; (ii) those coming from  $\mathbb{C}[b_n^{ij}]\mathbf{1} \subset V$ . Obviously, the endomorphisms  $\int E_{ij}(z)$  ( $i \neq j$ ), act on W as  $E_{ij}$  on sl(n); in particluar, this action is nilpotent, and  $r_i(t)$  is polynomial in t. As for  $\mathbb{C}[b_n^{ij}]\mathbf{1}$ , this subspace is identified with the symmetric algebra  $S^{\bullet}(\Omega^1(U_w))$ . For any  $X \in gl(n) \subset \Gamma(X; \mathcal{T}_X)$ , the element  $\int X(z)$  acts on this space as the Lie derivative along X. Consequently,  $\exp(\int X(z))$  maps  $S^{\bullet}(\Omega^1(U_w))$  into  $S^{\bullet}(\Omega^1(\exp(X) \cdot U_w))$ .  $\triangle$ 

Repeated application of this lemma gives for any  $v = r_{i_1} \cdot \ldots r_{i_k} \in S_n$  the map

$$v(1) = r_{i_1} \cdot \ldots \cdot r_{i_k} : \Gamma(U_w; \mathcal{O}_{U_w}^{ch}) \longrightarrow \Gamma(U_w \cap U_{v \cdot w}; \mathcal{O}_{U_w}^{ch})$$
 (5.30)

Finally, to complete our construction of  $\mathcal{O}_X^{ch}$ , glue the sheaves  $\mathcal{O}_{U_{w_1}}$  and  $\mathcal{O}_{U_{w_2}}$  using the maps

$$\Gamma(U_{w_1}; \mathcal{O}_{U_{w_1}}^{ch}) \longrightarrow \Gamma(U_{w_1} \cap U_{w_2}; \mathcal{O}_{U_{w_2}}^{ch}) \longleftarrow \Gamma(U_{w_2}; \mathcal{O}_{U_{w_2}}^{ch})$$
 (5.31)

where the first (resp. second) arrow is  $w_2w_1^{-1}(1)$  (resp.,  $w_1w_2^{-1}(1)$ ). Since the gluing maps are induced by the action of  $S_n$ , they are transitive, and the sheaf  $\mathcal{O}_X^{ch}$  is well defined.

**5.12. Example.** Let  $\mathfrak{g} = sl(2)$ . We have  $N = \mathbb{A}^1 \subset X = \mathbb{P}^1$ ; b is the coordinate on N such that generators of gl(2) act as the following vector fields:  $E_{21} \mapsto -\partial_b$ ,  $E_{12} \mapsto b^2 \partial_b$ ,  $E_{11} \mapsto b \partial_b$ .

One easily calculates that

$$\exp(E_{21}): b \mapsto b-1; \exp(-E_{12}): b \mapsto b/(b+1); \exp(\pi\sqrt{-1}E_{11}): b \mapsto -b$$

The simple reflection is

$$r = \exp(\pi\sqrt{-1}E_{11})\exp(E_{21})\exp(-E_{12})\exp(E_{21}): b \mapsto b^{-1}$$
(5.32)

To do the chiral analogue of this computation, recall (cf. [W]) that  $E_{21}(z) = -a(z)$ ,  $E_{12}(z) = b^2 a(z) + 2b(z)'$ ,  $E_{11}(z) = ba(z)$ . A direct computation using Wick theorem yields

$$r(1)a_{-1}\mathbf{1} = (b^2a_{-1} + 2b_{-1})\mathbf{1}$$
(5.33)

which coincides with (5.23b). Computation of  $r(1)b\mathbf{1}$  does not differ from the classical one, see (5.32).

**5.13. Theorem.** The space  $\Gamma(X; \mathcal{O}_X^{ch})$  admits a natural structure of a  $\widehat{\mathfrak{g}}$ -module at the critical level, such that it is a submodule of the restricted Wakimoto module, and its conformal weight zero component is 1-dimensional.

Proof is not much different from that of Theorem 5.7. The space of sections  $W = \Gamma(N; \mathcal{O}_X^{ch})$  over the big cell is the restricted Wakimoto module with the zero highest weight. Restricted here again means that the center of  $U(\widehat{\mathfrak{g}})_{loc}$  acts trivially. As the action of B on V preserves  $\widehat{\mathfrak{g}} \subset \operatorname{End}(V)$ ,  $\widehat{\mathfrak{g}}$  is spanned by the Fourier modes of fields associated with certain elements of  $\Gamma(X; \mathcal{O}_X^{ch})$ . Therefore,  $\Gamma(X; \mathcal{O}_X^{ch})$  is a  $\widehat{\mathfrak{g}}$ -module at the critical level, and there arises a  $\widehat{\mathfrak{g}}$ -morphism

$$\Gamma(X; \mathcal{O}_X^{ch}) \longrightarrow W$$
 (5.34)

By construction, the sheaf  $\mathcal{O}_X$  admits a canonical filtration whose associated quotients are free  $\mathcal{O}_X$ -modules of finite rank. Therefore, the map (5.34) is an injection. By construction, the conformal weight zero component of  $\Gamma(X; \mathcal{O}_X^{ch})$  is equal to  $\Gamma(X; \mathcal{O}_X) = \mathbb{C}$ .  $\triangle$ 

This theorem is less precise than 5.7, the reason being that the representation-theoretic result of [FF1], [M] mentioned in 5.7 is only available in the sl(2)-case. However, Theorem 5.1 of [FF1] makes it plausible that this representation-theoretic statement carries over to any  $\mathfrak{g}$ , and hence that  $\Gamma(X; \mathcal{O}_X^{ch})$  is in fact the irreducible vacuum  $\widehat{\mathfrak{g}}$ -module.

- **5.14.** Note that the whole sheaf  $\mathcal{O}_X^{ch}$  admits a structure of a  $\widehat{\mathfrak{g}}$ -module at the critical level.
- **5.15.** Localization for non-zero highest weight. For any integral weight of  $\mathfrak{g}$ , there exists a twisted analogue of  $\mathcal{O}_X^{ch}$ , to be denoted by  $\mathcal{L}_{\lambda}^{ch}$ . Its construction repeats word for word 5.9, except that the action of  $(\mathfrak{g}, B)$  on V is to be twisted by  $\lambda$ , see [W, FF1]. If  $\lambda$  is a regular dominant weight, then Theorems 5.7 and 5.13, along with their proofs, generalize in the obvious way. For example, the word "vacuum" in the formulation of Theorem 5.7 is to be replaced with "highest weight  $\lambda$ ". Likewise, the claim that "conformal weight zero component is 1-dimensional" should be replaced with the following: "conformal weight zero component is isomorphic to  $\Gamma(X; \mathcal{L}_{\lambda})$ ", where  $\mathcal{L}_{\lambda}$  is the standard line bundle on X.

One may want to regard  $\mathcal{L}_{\lambda}^{ch}$  as a sheaf of modules over a sheaf of vertex algebras, just as  $\mathcal{O}_{X}^{ch}$  is a sheaf of modules over itself. From this point of view,  $\mathcal{O}_{X}^{ch}$  is a chiral analogue of the sheaf  $\mathcal{D}_{X}$  of differential operators on X. Thus, one cannot expect  $\mathcal{L}_{\lambda}^{ch}$  to be a module over  $\mathcal{O}_{X}^{ch}$ . We believe (and have checked this for  $\mathfrak{g}=sl(2)$ ) that one can define a sheaf of vertex algebras  $\mathcal{O}_{\lambda}^{ch}$  which acts on  $\mathcal{L}_{\lambda}^{ch}$  and is locally isomorphic to  $\mathcal{O}_{X}^{ch}$ . Therefore, this sheaf can be regarded as a chiral partner of the sheaf  $\mathcal{D}_{\lambda}$  of twisted differential operators on X.

## §6. Alternative construction

In this section we will outline another construction of our vertex algebras, and some generalizations. The details will appear in a separate paper, see [MS].

**6.1.** Recall (cf. [K], 4.9) that a *graded* vertex algebra is a pair (V, H) where V is a vertex algebra and  $H: V \longrightarrow V$  is a diagonalizable linear operator (*Hamiltonian*) such that

$$[H, a(z)] = z\partial_z a(z) + (Ha)(z)$$
(6.1.1)

for each  $a \in V$ . For example, a conformal vertex algebra is graded by the Hamiltonian  $L_0$ . The eigenvalues of H are called conformal weights. By  $V^{(\Delta)}$  we will denote the eigenspace of conformal weight  $\Delta$ . We have

$$a_{(n)}b \in V^{(\Delta + \Delta' - n - 1)}$$
 for  $a \in V^{(\Delta)}, b \in V^{(\Delta')}$  (6.1.2)

Let us call a graded vertex algebra *restricted* if it has no negative integer conformal weights.

Let us fix a restricted vertex algebra V. To simplify the notations, we assume that V is purely even. We would like to describe the structure which is induced on the space  $V^{\leq 1} := V^{(0)} \oplus V^{(1)}$  by the vertex algebra structure on V. We omit all computations; all the claims below are deduced directly from the Borcherds identity [K], Proposition 4.8 (b) and from *op. cit.* (4.2.3).

- **6.2.** The operation  $a_{(-1)}b$  in V will be denoted simply by ab.
- (a) The space  $V^{(0)}$  is a commutative associative unital  $\mathbb{C}$ -algebra with respect to the operation ab. This algebra will be denoted A. The unity is the vacuum, to be denoted by  $\mathbf{1}$ .

The space A acts by the left multiplication on  $V^{(1)}$ . However, this does not make  $V^{(1)}$  an A-module: the multiplication by A is not associative in general.

We have the map  $L_{-1}: A \longrightarrow V^{(1)}$ . Let  $\Omega \subset V^{(1)}$  denote the subspace spanned by the elements  $a\partial b$ ,  $a, b \in A$ . Thus,  $L_{-1}$  induces the map

$$d: A \longrightarrow \Omega$$
 (6.2.1)

(b) The left multiplication by A makes  $\Omega$  a left A-module. We have

$$a(db) = (db)a \quad (a, b \in A) \tag{6.2.2}$$

(c) The map d is a derivation, i.e.

$$d(ab) = adb + bda (6.2.3)$$

Let us denote by  $\mathcal{T}$  the quotient space  $V^{(1)}/\Omega$ .

(d) The left multiplication by A makes  $\mathcal{T}$  a left A-module.

Consider the operation

$$(0): V^{(1)} \otimes V^{(1)} \longrightarrow V^{(1)}$$
 (6.2.4)

(e) The operation (6.6.4) induces a Lie bracket on  $\mathcal{T}$ , to be denoted  $[\ ,\ ]$ .

Consider the operation

$$(0): V^{(1)} \otimes A \longrightarrow A$$
 (6.2.5)

(f) The operation (6.2.5) vanishes on the subspace  $\Omega \otimes A$ , and induces on A a structure of a module over the Lie algebra  $\mathcal{T}$ .

This action will be denoted by  $\tau(a)$   $(a \in A, \tau \in T)$ .

(g) The Lie algebra  $\mathcal{T}$  acts on A by derivations,

$$\tau(ab) = \tau(a)b + a\tau(b) \tag{6.2.6}$$

(h) We have

$$(a\tau)(b) = a\tau(b) \tag{6.2.7}$$

The properties (d) — (h) mean that  $\mathcal{T}$  is a *Lie algebroid* over A.

(i) The operation (6.2.4) induces a structure of a module over the Lie algebra  $\mathcal{T}$  on the space  $\Omega$ .

This action will be denoted  $\tau(\omega)$  or  $\tau\omega$  ( $\tau\in\mathcal{T},\ \omega\in\Omega$ ).

(j) We have

$$\tau(a\omega) = a\tau(\omega) + \tau(a)\omega \ (a \in A, \ \tau \in \mathcal{T}, \ \omega \in \Omega)$$
 (6.2.8)

- (k) The differential  $d: A \longrightarrow \Omega$  is compatible with the  $\mathcal{T}$ -module structure.
- It follows from (j) and (k) that
- (l) we have

$$\tau(adb) = \tau(a)db + ad(\tau(b)) \quad (\tau \in \mathcal{T}, \ a, b \in A)$$
(6.2.9)

Consider the operation

$$(1): V^{(1)} \otimes V^{(1)} \longrightarrow A$$
 (6.2.10)

(m) The map (6.2.10) vanishes on the subspace  $\Omega \otimes \Omega$ . Therefore, it induces the pairing

$$\langle , \rangle : \Omega \otimes \mathcal{T} \oplus \mathcal{T} \otimes \Omega \longrightarrow A$$
 (6.2.11)

This pairing is A-bilinear and symmetric. We have

$$\langle \tau, adb \rangle = a\tau(b) \quad (\tau \in \mathcal{T}, \ a, b \in A)$$
 (6.2.12)

(n) We have

$$(a\tau)(\omega) = a\tau(\omega) + \langle \tau, \omega \rangle da \quad (a \in A, \ \tau \in \mathcal{T}, \ \omega \in \Omega)$$
 (6.2.13)

**6.3.** Let us denote by  $\widehat{T}$  the space  $V^{(1)}/dA$ . The operation (6.6.4) induces a Lie bracket on the space  $\widehat{T}$ . The subspace  $\Omega/dA \subset \widehat{T}$  is an abelian Lie ideal. The adjoint action of  $\mathcal{T} = \widehat{\mathcal{T}}/(\Omega/dA)$  coincides with the action defined by (i) and (l).

Thus, we have an extension of Lie algebras

$$0 \longrightarrow \Omega/dA \longrightarrow \widehat{T} \longrightarrow T \longrightarrow 0 \tag{6.3.1}$$

Note that this extension is not central in general.

**6.4.** Let us denote the space  $V^{(1)}$  by  $\tilde{\mathcal{T}}$ . Thus, we have an exact sequence of vector spaces

$$0 \longrightarrow \Omega \longrightarrow \tilde{\mathcal{T}} \longrightarrow \mathcal{T} \longrightarrow 0 \tag{6.4.1}$$

Both arrows are compatible with the left multiplication by A. Let  $\pi$  denote the projection  $\pi: \tilde{\mathcal{T}} \longrightarrow \mathcal{T}$ .

Let us define the "bracket"  $[ , ] : \Lambda^2 \tilde{\mathcal{T}} \longrightarrow \tilde{\mathcal{T}}$  by

$$[x,y] = \frac{1}{2}(x_{(0)}y - y_{(0)}x) \quad (x,y \in \tilde{\mathcal{T}})$$
(6.4.2)

This bracket does not make  $\tilde{\mathcal{T}}$  a Lie algebra: the Jacobi identity is in general violated. Set

$$J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y] \quad (x, y, z \in \tilde{\mathcal{T}})$$

$$(6.4.3)$$

Consider the operation (6.2.10).

(a) The operation (6.2.10) is symmetric. It will be denoted by  $\langle x, y \rangle$ .

Let us define the map  $I: \Lambda^3 \tilde{\mathcal{T}} \longrightarrow A$  by

$$I(x, y, z) = \langle x, [y, z] \rangle + \langle y, [z, x] \rangle + \langle z, [x, y] \rangle \tag{6.4.4}$$

(b) We have

$$J(x, y, z) = \frac{1}{6}dI(x, y, z)$$
(6.4.5)

**6.5.** Let us choose a splitting

$$s: \mathcal{T} \longrightarrow \tilde{\mathcal{T}}$$
 (6.5.1)

of the projection  $\pi$ . Let us define the map

$$\langle , \rangle = \langle , \rangle_s : S^2 \mathcal{T} \longrightarrow A$$
 (6.5.2)

by

$$\langle \tau_1, \tau_2 \rangle = \langle s(\tau_1), s(\tau_2) \rangle \tag{6.5.3}$$

(we put the lower index s in the notation if we want to stress the dependence on the splitting s). Let us define the map

$$c^2 = c_s^2: \Lambda^2 \mathcal{T} \longrightarrow \Omega$$
 (6.5.4)

by

$$c^{2}(\tau_{1}, \tau_{2}) = s([\tau_{1}, \tau_{2}]) - [s(\tau_{1}), s(\tau_{2})]$$
(6.5.5)

Let us define the map  $K: \Lambda^3 \mathcal{T} \longrightarrow A$  by

$$K(\tau_1, \tau_2, \tau_3) = \langle s(\tau_1), s([\tau_2, \tau_3]) \rangle + \langle s(\tau_2), s([\tau_3, \tau_1]) \rangle + \langle s(\tau_3), s([\tau_1, \tau_2]) \rangle \quad (6.5.6)$$

Let us define the map

$$c^3 = c_s^3 : \Lambda^3 \mathcal{T} \longrightarrow A$$

by

$$c^{3}(\tau_{1}, \tau_{2}, \tau_{3}) = -\frac{1}{2}K(\tau_{1}, \tau_{2}, \tau_{3}) + \frac{1}{3}I(s(\tau_{1}), s(\tau_{2}), s(\tau_{3}))$$
(6.5.7)

cf. (6.4.4). Let us regard  $c^2$  (resp.  $c^3$ ) as Lie algebra cochains living in  $C^2(\mathcal{T};\Omega)$  (resp., in  $C^3(\mathcal{T};A)$ ).

(a) We have

$$d_{Lie}(c^2) = dc^3 (6.5.8)$$

(b) We have

$$d_{Lie}(c^3) = 0 (6.5.9)$$

The identities (a) and (b) mean that the pair  $c = (c^2, c^3)$  form a 2-cocycle of the Lie algebra  $\mathcal{T}$  with coefficients in the complex  $A \longrightarrow \Omega$ .

(c) We have

$$\langle [\tau_1, \tau_2], \tau_3 \rangle + \langle \tau_2, [\tau_1, \tau_3] \rangle = \tau_1(\langle \tau_2, \tau_3 \rangle) - \frac{1}{2} \tau_2(\langle \tau_1, \tau_3 \rangle) - \frac{1}{2} \tau_3(\langle \tau_1, \tau_2 \rangle) - \langle \tau_2, c^2(\tau_1, \tau_3) \rangle - \langle \tau_3, c^2(\tau_1, \tau_2) \rangle$$

$$(6.5.10)$$

Let us investigate the effect of the change of a splitting. Let  $s': \mathcal{T} \longrightarrow \tilde{\mathcal{T}}$  be another splitting of  $\pi$ . The difference s'-s lands in  $\Omega$ ; let us denote it

$$\omega = \omega_{s,s'}: \ \mathcal{T} \longrightarrow \Omega$$
 (6.5.11)

We regard  $\omega$  as a 1-cochain of  $\mathcal{T}$  with coefficients in  $\Omega$ . Let us define a 2-cochain  $\alpha = \alpha_{s,s'} \in C^2(\mathcal{T};A)$  by

$$\alpha(\tau_1, \tau_2) = \frac{1}{2} (\langle \omega(\tau_1), \tau_2 \rangle - \langle \tau_1, \omega(\tau_2) \rangle)$$
(6.5.12)

$$c_s^2 - c_{s'}^2 = d_{Lie}(\omega) - d\alpha$$
 (6.5.13)

$$-d_{Lie}(\alpha) = c_s^3 - c_{s'}^3 \tag{6.5.14}$$

The properties (d) and (e) mean that

$$c_s - c_{s'} = d_{Lie}(\beta)$$
 (6.5.15)

where  $\beta = \beta_{s,s'} := (\omega, \alpha) \in C^1(\mathcal{T}; A \longrightarrow \Omega).$ 

Therefore, we have assigned to our vertex algebra a canonically defined "characteristic class"

$$c(V) = c(V^{\leq 1}) \in H^2(\mathcal{T}; A \longrightarrow \Omega)$$
 (6.5.16)

as the cohomology class of the cocycle  $c_s$ .

# **6.6.** Let us introduce the mapping

$$\gamma = \gamma_s: \ A \otimes \mathcal{T} \longrightarrow \Omega \tag{6.6.1}$$

by

$$\gamma(a,\tau) = s(a\tau) - as(\tau) \tag{6.6.2}$$

(a) We have

$$\gamma(ab,\tau) = \gamma(a,b\tau) + a\gamma(b,\tau) + \tau(a)db + \tau(b)da \tag{6.6.3}$$

(b) We have

$$\langle a\tau_1, \tau_2 \rangle = a\langle \tau_1, \tau_2 \rangle + \langle \gamma(a, \tau_1), \tau_2 \rangle - \tau_1 \tau_2(a)$$
(6.6.7)

(c) We have

$$c^{2}(a\tau_{1}, \tau_{2}) = ac^{2}(\tau_{1}, \tau_{2}) + \gamma(a, [\tau_{1}, \tau_{2}]) - \gamma(\tau_{2}(a), \tau_{1}) + \tau_{2}\gamma(a, \tau_{1}) - \frac{1}{2}\langle \tau_{1}, \tau_{2}\rangle da + \frac{1}{2}d(\tau_{1}\tau_{2}(a)) - \frac{1}{2}d(\langle \tau_{2}, \gamma(a, \tau_{1})\rangle)$$

$$(6.6.8)$$

 $(a \in A, \ \tau_i \in \mathcal{T}).$ 

(d) We have

$$c^{3}(a\tau_{1}, \tau_{2}, \tau_{3}) = ac^{3}(\tau_{1}, \tau_{2}, \tau_{3}) + \frac{1}{2}\tau_{1}[\tau_{2}, \tau_{3}](a) -$$

$$-\frac{1}{2} \{ \langle \tau_{2}, \gamma(a, [\tau_{3}, \tau_{1}]) \rangle - \langle \tau_{3}, \gamma(a, [\tau_{2}, \tau_{1}]) \rangle + \langle \tau_{2}, \gamma(\tau_{3}(a), \tau_{1}) \rangle - \langle \tau_{3}, \gamma(\tau_{2}(a), \tau_{1}) \rangle \} +$$

$$+ \frac{1}{2} \{ \langle \tau_{2}, \tau_{3}\gamma(a, \tau_{1}) \rangle - \langle \tau_{3}, \tau_{2}\gamma(a, \tau_{1}) \rangle \} - \frac{1}{2} \langle [\tau_{2}, \tau_{3}], \gamma(a, \tau_{1}) \rangle$$
(6.6.9)

- 6.7. Let us call a **prevertex algebra** the data (a) (f) below.
- (a) A commutative algebra A.

- (b) An A-module  $\Omega$ , together with an A-derivation  $d: A \longrightarrow \Omega$ . We assume that  $\Omega$  is generated as a vector space by the elements adb  $(a, b \in A)$ , i.e. the canonical map  $\Omega^1(A) := \Omega^1(A) \longrightarrow \Omega$  is surjective.
  - (c) An A-Lie algebroid  $\mathcal{T}$ . Define the action of  $\mathcal{T}$  on  $\Omega$  by

$$\tau(adb) = \tau(a)db + ad(\tau(b)), \tag{6.7.1}$$

cf. (6.2.9). We assume that this action is well defined. It follows that d is compatible with the action of  $\mathcal{T}$ .

We assume that the formula

$$\langle \tau, adb \rangle = a\tau(b) \tag{6.7.2}$$

gives a well defined A-bilinear pairing  $\mathcal{T} \times \Omega \longrightarrow A$ .

- (d) A  $\mathbb{C}$ -bilinear mapping  $\gamma: A \times \mathcal{T} \longrightarrow \Omega$  satisfying 6.6 (a).
- (e) A  $\mathbb{C}$ -bilinear symmetric mapping  $\langle \ , \ \rangle : \ \mathcal{T} \times \mathcal{T} \longrightarrow A$  satisfying 6.6 (b).
- (f) A  $\mathbb{C}$ -bilinear skew symmetric mapping  $c^2: \mathcal{T} \times \mathcal{T} \longrightarrow \Omega$ . This map should satisfy 6.5 (c), 6.6 (c), and the property (6.7.7) below. Let us define the map

$$[\ ,\ ]' := [\ ,\ ] - c^2 :\ \Lambda^2 \mathcal{T} \longrightarrow \mathcal{T} \oplus \Omega$$
 (6.7.3)

Let us define the map

$$c^{3} := -\frac{1}{2}\tilde{K} + \frac{1}{3}\tilde{I}: \Lambda^{3}\mathcal{T} \longrightarrow A$$
 (6.7.4)

where

$$\tilde{K}(\tau_1, \tau_2, \tau_3) = \langle \tau_1, [\tau_2, \tau_3] \rangle + \langle \tau_2, [\tau_3, \tau_1] \rangle + \langle \tau_3, [\tau_1, \tau_2] \rangle \tag{6.7.5}$$

and

$$\tilde{I}(\tau_1, \tau_2, \tau_3) = \langle \tau_1, [\tau_2, \tau_3]' \rangle + \langle \tau_2, [\tau_3, \tau_1]' \rangle + \langle \tau_3, [\tau_1, \tau_2]' \rangle$$
(6.7.6)

In the last formula, we imply the symmetric pairing  $\langle , \rangle$  to be extended to the whole space  $\mathcal{T} \oplus \Omega$  using (6.7.2), (e), and setting it equal to zero on  $\Omega \times \Omega$ .

Now, with  $c^3$  defined above, we require that

$$d_{Lie}(c^2) = dc^3; \quad d_{Lie}(c^3) = 0$$
 (6.7.7)

Let us call a restricted vertex algebra V split if it is given together with a splitting (6.5.1). We have constructed in 6.2 - 6.6 a functor

$$\mathcal{P}: (Split\ restricted\ vertex\ algebras) \longrightarrow (Prevertex\ algebras)$$
 (6.7.8)

# **6.7.1.** Claim. Functor $\mathcal{P}$ admits a left adjoint, to be denoted $\mathcal{V}$ .

In other words, given a prevertex algebra  $P = (A, \Omega, \mathcal{T}, \dots)$ , the corresponding vertex algebra  $\mathcal{V}(P)$  is defined by its universal property: to give a morphism of vertex algebras from  $\mathcal{V}(P)$  to an arbitrary split restricted vertex algebra V' is the

same as to give a morphism of prevertex algebras  $P \longrightarrow \mathcal{P}(V')$ . (Morphisms of prevertex algebras are defined in the obvious manner.)

The construction of  $V = \mathcal{V}(P)$  goes in two steps. First, the components  $V_0, V_1$  and the operations (i), i = -2, -1, 0, 1 acting on them, are recovered by inverting the discussion 6.2 - 6.6. For example,  $V^{(0)} = A$ ;  $V^{(1)} = \mathcal{T} \oplus \Omega$ ;

$$\tau_{1(0)}\tau_2 = [\tau_1, \tau_2] - c^2(\tau_1, \tau_2) + d\langle \tau_1, \tau_2 \rangle \tag{6.7.9}$$

$$\tau_{1(1)}\tau_2 = \langle \tau_1, \tau_2 \rangle \tag{6.7.10}$$

 $(\tau_i \in \mathcal{T})$ , etc. Now, the components of weights  $\geq 2$  are recovered by "bootstrap" from the universal property.

Note that the set of conformal weights of  $\mathcal{V}(P)$  is equal to  $\mathbb{Z}_{>0}$  if  $\mathcal{V}(P) \neq \mathbb{C}$ .

**6.8. Example.** Let  $\mathcal{T}$  be a Lie algebra over  $\mathbb{C}$  equipped with an invariant bilinear form  $\langle \ , \ \rangle$ . Set  $A=\mathbb{C},\ \Omega=0,\ c^2=0,\gamma=0$ . This defines a prevertex algebra P. Note that the component defined by the rule 6.7 (f) is not equal to zero, but is given by

$$c^{3}(\tau_{1}, \tau_{2}, \tau_{3}) = -\frac{1}{2} \langle \tau_{1}, [\tau_{2}, \tau_{3}] \rangle$$
(6.8.1)

The vertex algebra  $\mathcal{V}(P)$  coincides with the vacuum (level 1) representation of the affine Kac-Moody Lie algebra corresponding to  $(\mathcal{T}, \langle , \rangle)$ .

**6.9. Example.** Let A be a  $\mathbb{C}$ -algebra; set  $\Omega = \Omega^1_{\mathbb{C}}(A)$ . Let  $\mathcal{T}_0$  be an abelian Lie algebra over  $\mathbb{C}$  acting by derivations on A.

Set  $\mathcal{T} = A \otimes_{\mathbb{C}} \mathcal{T}_0$ . There is a unique Lie bracket on  $\mathcal{T}$  making it a Lie algebroid over A,

$$[a\tau_1, b\tau_2] = a\tau_1(b)\tau_2 - b\tau_2(a)\tau_1 \quad (\tau_i \in \mathcal{T}_0, a, b \in A)$$
 (6.9.1)

We set  $\langle \tau_1, \tau_2 \rangle = 0$ ;  $\gamma(a, \tau) = 0$ ;  $c^2(\tau_1, \tau_2) = 0$ ;  $c^3(\tau_1, \tau_2, \tau_3) = 0$  for  $a \in A$ ,  $\tau_i \in \mathcal{T}_0$ . Then the formulas 6.6 (a) — (d) define the unique extension of the operations  $\gamma, \langle \cdot, \cdot \rangle, c^2$  and  $c^2$  to the whole space  $\mathcal{T}$ .

Namely,

$$\gamma(a, b\tau) = -\tau(a)db - \tau(b)da \tag{6.9.2}$$

$$\langle a\tau_1, b\tau_2 \rangle = -a\tau_2\tau_1(b) - b\tau_1\tau_2(a) - \tau_1(b)\tau_2(a)$$
 (6.9.3)

It is convenient to write down  $c = (c^2, c^3)$  as a sum of a simpler cocycle and a coboundary,

$$c^{2}(a\tau_{1}, b\tau_{2}) = 'c^{2}(a\tau_{1}, b\tau_{2}) + d\beta(a\tau_{1}, b\tau_{2})$$
(6.9.4)

where

$$'c^{2}(a\tau_{1},b\tau_{2}) = \frac{1}{2} \{ \tau_{1}(b)d\tau_{2}(a) - \tau_{2}(a)d\tau_{1}(b) \}$$
(6.9.5)

$$'c^{2}(a\tau_{1},b\tau_{2}) = \frac{1}{2} \{ \tau_{1}(b)d\tau_{2}(a) - \tau_{2}(a)d\tau_{1}(b) \}$$
 (6.9.5)

$$\beta(a\tau_1, b\tau_2) = \frac{1}{2} \{ b\tau_1 \tau_2(a) - a\tau_2 \tau_1(b) \}$$
 (6.9.6)

and

$$c^{3}(a\tau_{1}, b\tau_{2}, c\tau_{3}) = 'c^{3}(a\tau_{1}, b\tau_{2}, c\tau_{3}) + d_{Lie}\beta(a\tau_{1}, b\tau_{2}, c\tau_{3})$$
(6.9.7)

where

$$'c^{3}(a\tau_{1},b\tau_{2},c\tau_{3}) = \frac{1}{2} \{ \tau_{1}(b)\tau_{2}(c)\tau_{3}(a) - \tau_{1}(c)\tau_{2}(a)\tau_{3}(b) \}$$
 (6.9.8)

This gives the a prevertex algebra P .

Note that the cocycle ( ${}'c^2$ ,  ${}'c^3$ ) coincides with (minus one half of) the cocycle (5.17-18) if A is the polynomial ring.

For example, let A be smooth, and assume that there exists a base  $\{\tau_i\}$  of the left A-module  $\mathcal{T} := Der_{\mathbb{C}}(A)$  consisting of commuting vector fields. Let  $\mathcal{T}_0 \subset \mathcal{T}$  be the  $\mathbb{C}$ -vector space spanned by  $\{\tau_i\}$ . This gives a prevertex algebra P. The vertex algebra  $A^{ch} := \mathcal{V}(P)$  may be called a "chiralization of A". This definition depends on the choice of  $\{\tau_i\}$ , and this is essential; when we change the basis, we may get a non-isomorphic vertex algebra: here the "anomaly" appears.

Specifying even more, let A be a polynomial algebra  $A_N$ , cf. 3.1. Let  $\tau_i = \partial_{b^i}$ . Then the vertex algebra  $\mathcal{V}(P)$  coincides with the Heisenberg vertex algebra  $V_N$ .

If A' is an arbitrary commutative A-algebra given together with an action of  $\mathcal{T}_0$  extending its action on A, then the base change  $P_{A'} = (A', \mathcal{T}_{A'} := A' \otimes_A \mathcal{T}, \dots)$  has an obvious structure of a prevertex algebra. We have  $\mathcal{V}(P_{A'}) = A' \otimes_A \mathcal{V}(P)$ . This explains the remark about the base change in 3.3.

There exists a common generalization of the above two examples.

**6.10.** All the above considerations have an obvious "super" ( $\mathbb{Z}/(2)$ -graded) version. Let us consider the super version of the Example 2.9. Let us start from the de Rham superalgebra  $\Omega_A$  of differential forms over an arbitrary smooth algebra A

Let us assume that there exists an étale map  $\operatorname{Spec}(A) \longrightarrow \mathbb{A}^N$  given by coordinate functions  $\{b^i\}$  (this is true maybe after some Zariski localization). Lifting the coordinate vector fields  $a^i = \partial_{b^i}$  to A, we get an abelian base in the Lie algebra  $\operatorname{Der}(A)$  which gives rise to an abelian base in Lie superalgebra  $\operatorname{Der}(\Omega_A)$ . Now, we proceed as in 6.9 (in its super version), and get a vertex superalgebra  $\Omega^{ch} = \mathcal{V}(P)$ . The calculation in the proof of Theorem 3.7 shows that this vertex superalgebra does not depend on the choice of local étale coordinates.

This is nothing but  $\Gamma(X; \Omega_X^{ch})$  for X = Spec(A). This may be viewed as an alternative (or a version of) construction of the chiral de Rham complex.

**6.11. Chiral Weyl modules.** Let us return to the even situation again. Let V be a restricted vertex algebra, let  $\mathcal{P}(V) = (A, \Omega, \mathcal{T}, \dots)$  be the corresponding prevertex algebra. Assume that the Lie algebra  $\mathcal{T}$  coincides with Der(A).

Let  $\mathcal{M}$  be a graded vertex module over V (the definition of such an object is an obvious modification of the definition of a graded vertex algebra). Assume that  $\mathcal{M}$  is restricted, i.e. has no negative integer conformal weights. Consider the weight zero component  $M = \mathcal{M}^{(0)}$ . Then the operations  $am = a_{(-1)}m$  ( $a \in A, m \in M$ ) and  $\tau m = x_{(0)}m$  ( $\tau \in \mathcal{T}, x \in \tilde{\mathcal{T}}$  is any representative of  $\tau, m \in M$ ) makes M a left  $\mathcal{D}$ -module over A.

This way we get a functor

$$(Restricted\ V-modules) \longrightarrow (D_A-modules)$$
 (6.11.1)

This functor admits a left adjoint

$$W: (D_A - modules) \longrightarrow (Restricted \ V - modules)$$
 (6.11.2)

For a  $\mathcal{D}$ -module M, the vertex module  $\mathcal{W}(M)$  is called the *chiral Weyl module* corresponding to M.

It is natural to hope this construction applied to flag spaces G/B gives a functor from  $\mathcal{D}$ -modules over G/B to modules over  $\mathcal{O}_{G/B}^{ch}$  which corresponds to the Weyl module construction in the language of representations.

A similar construction gives for an arbitrary smooth manifold X a functor

$$\Omega^{ch}: (\mathcal{D}_X - mod) \longrightarrow (\Omega_X^{ch} - mod)$$
(6.11.3)

called the chiral de Rham complex of a D-module.

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