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Calibrations and mathematical aspects of BPS Branes

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Calibrations and mathematical aspects of BPS Branes:

The aim of this thesis is to present the theory of calibrations and show their relevance in physics. We first define calibrations and calibrated submanifolds and show that these submanifolds have minimal volume in their homology class. Next, we discuss the most important examples of calibrations on $\mathbb{R}^n$. After getting familiar with these calibrations on affine space we make a transition to the setting of general manifolds and discuss the connection between manifolds with special holonomy and calibrations. We will study the geometry of spinors in order to understand how special holonomy, parallel spinors and calibrations are related. The main result here will be the construction of calibrations from parallel spinors. We will conclude with an application to physics and discuss some aspects of BPS membranes in D=11 supergravity wrapping supersymmetric cycles. We show that in the case of a static background geometry and vanishing 3-form potential the BPS condition for the membrane is equivalent to a calibration condition for the membrane.

Kalibrierungen und mathematische Aspekte von BPS Branes:

Das Ziel dieser Arbeit ist es, die Theorie der Kalibrierungen einzuführen und zu zeigen, an welchen Stellen sie in der Physik relevant sind. Zunächst definieren wir Kalibrierungen und kalibrierte Untermannigfaltigkeiten und zeigen, dass diese in ihrer Homologieklasse volumenminimierend sind. Anschließend führen wir die wichtigsten Beispiele für Kalibrierungen auf dem $\mathbb{R}^n$ ein. Nach der Behandlung des affinen Falls gehen wir dazu über, Kalibrierungen auf allgemeinen Mannigfaltigkeiten zu betrachten. Hierbei legen wir besonderen Fokus auf die Verbindung von Mannigfaltigkeiten mit eingeschränkter Holonomie und Kalibrierungen. Um den Zusammenhang zwischen eingeschränkter Holonomie, parallelen Spinoren und Kalibrierungen aufzuzeigen, werden wir die Geometrie von Spinoren diskutieren. Das Hauptresultat dieses Abschnitts wird die Konstruktion von Kalibrierungen aus parallelen Spinoren sein. Den Abschluss der Arbeit bildet die Anwendung der Theorie in der Physik. Wir diskutieren hier BPS-Membranen in D=11 Supergravitation, die supersymmetrische Zykel umschließen. Wir zeigen, dass in einer statischen Raum-Zeit für verschwindendes 3-Form Potential BPS-Membranen kalibrierte Untermannigfaltigkeiten sind.

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Introduction

In the contexts of string theory and supergravity there arise nonperturbative corrections to couplings from branes wrapping supersymmetric cycles. Under certain restrictions their energy is given by their volume times their tension. To minimize the energy, we have to minimize the volume of the brane. Thus the task is to find minimal submanifolds inside the target space of the theory. Furthermore, we want the theory containing such a brane to still be supersymmetric. One can use calibrations in order to find such minimal submanifolds. The purpose of this thesis is to understand the mathematical aspects of calibrations and show the connection to physics.
Calibrations were first defined by Harvey and Lawson in 1982 [1]. These are closed p-forms φ on a smooth manifold X which satisfies the condition

$$\varphi_x \leq \mathrm{vol}|_\xi \tag{0.1}$$

for all p-dimensional subspaces $\xi \subseteq \mathrm{T}_x X$ at every point $x \in X$. The tangent planes of a p-dimensional submanifold N of X are certainly such p-dimensional subspaces. If we have equality in (0.1) for all of these tangent planes (i.e. $\varphi_x = \mathrm{vol}|_{\mathrm{T}_x N}$ for all $x \in N$) we call N a calibrated submanifold. These submanifolds are minimal surfaces and volume minimizing in their homology classes. This was the original motivation for the introduction of calibrations.
Although the concept of calibrations is not interesting for the general case (since for generic calibrations the set of p-planes which saturate (0.1) is too small to give rise to interesting calibrated submanifolds), however there are specific examples that give rise to very rich geometries.
In section 1 we will introduce the general theory of calibrations and thereafter discuss the most important examples which are complex geometry, special Lagrangian geometry and the exceptional geometries. We will find that all the arising calibrations are invariant under some Lie group action and interestingly all of the arising Lie groups appear in Berger's classification of holonomy groups [2]. This suggests that there is a connection between Riemannian holonomy and calibrations. To further explore this connection we review certain aspects of spin geometry in section 2.
It turns out that many manifolds with special holonomy (i.e. manifolds for which the holonomy group is a proper subgroup of SO_n) admit a parallel spinor. As demonstrated in section 3 this kind of spinors can be used to construct calibrations on the manifold by "squaring" it, more precisely, we construct a bilinear map, that sends a pair of spinors to a p-form.
Finally, in section 4 we discuss the application of calibrated geometries to D=11 supergravity. The goal is to find stable bosonic solutions that preserve some of the super-

symmetries. Under suitable assumptions the background geometry will take the form $\mathbb{R}^{d-p-1,1} \times M_p$ where M_p is a compact manifold which admits a parallel spinor ε and hence has special holonomy. Considering a membrane propagating in this background we see that if we demand the solution to preserves some of the supersymmetries, we get the BPS condition

$$(1-\Gamma)\varepsilon = 0. \tag{0.2}$$

This condition implies that the spatial part of the membrane has to be a submanifold, calibrated with respect to the calibration that can be constructed from the parallel spinors on the background.

1 Calibrated Geometries

In this section we lay the foundations for the rest of the thesis and introduce the concept of a calibration. A large portion will concern linear algebra and we will consider only examples of calibrations on $\mathbb{R}^n$. We will then see that the structure of this examples for calibrations allows us to transport these calibrations to manifolds other than flat $\mathbb{R}^n$ with very little effort. The main reference in this part is the original paper on calibrated geometries by Harvey and Lawson [1].

1.1 Grassmann Geometries

Consider an oriented manifold X (for simplicity take it to be C^∞) of dimension n. Let $G(p, T_xX)$ denote the collection of oriented p-planes in T_xX (this is precisely the oriented p-Grassmannian of the tangent space of X at x). This gives rise to a bundle $G(p, TX)$ which we will call the *Grassmann bundle* over X.
Let $\mathscr{G} \subseteq \mathrm{G(p, T\,X)}$ be an arbitrary subset (note that $\mathscr{G}_x := \mathscr{G} \cap G(p, T_xX)$ may be empty). Let S be a p-dimensional oriented (C^∞-) submanifold (possibly with boundary in X). We call S a $\mathscr{G}$-submanifold of X if $T_xS \in \mathscr{G}$ for all $x \in S$. The family of all $\mathscr{G}$-submanifolds of X defines the $\mathscr{G}$-geometry of X. We call any geometry defined in that way a *Grassmann geometry*.
Remark: There is a natural embedding $G(p, T_xX) \hookrightarrow \bigwedge^p \mathrm{T_xX}$, which will be used extensively later in the text. The image of the embedding is given by

$$\mathrm{G(p, T_xX)} = \{\xi \in \bigwedge\nolimits^p \mathrm{T_xX} | \xi \,\text{unit simple}\}. \tag{1.1}$$

A p-vector is called simple if it is of the form $v_1 \wedge ... \wedge v_p$. Other authors use the term decomposable instead of simple. The embedding sends an oriented orthonormal basis $f_1, ..., f_p$ (where we choose an inner product $\langle \cdot, \cdot \rangle$ on the tangent space) of a p-plane in

T_xX to $f_1 \wedge ... \wedge f_p$. Given the inner product on T_xX, the scalar product on $\bigwedge^\bullet T_xX$ is defined by

$$\langle v_1 \wedge ... \wedge v_p, w_1 \wedge ... \wedge w_{p'} \rangle := \begin{cases} \det(\langle v_i, w_j \rangle), & \text{if } p = p' \\ 0, & \text{else} \end{cases} \tag{1.2}$$

We should check that the map is well defined. Suppose $(f_1, ..., f_p)$ and $(f_1', ..., f_p')$ are two oriented orthonormal bases for $\xi \in \mathrm{G(p, T_xX)}$ with the same orientation. They are related by some special orthogonal transformation A. We can consider $f_1 \wedge ... \wedge f_p$ and $f_1' \wedge ... \wedge f_p'$ to be in $\bigwedge^p \xi$. Then both are top-forms and hence they are proportional. The proportionality constant is just $\det(A) = 1$ and thus the definition is independent of the choice of basis.
We want to look at some examples to see why Grassmann geometries might be interesting:

Example I. *(Complex Geometries)* Let X be complex manifold, $\mathscr{G} \subseteq \mathrm{G}(2p, \mathrm{T}X)$ the subset of canonically oriented complex p-planes. In this case the $\mathscr{G}$-submanifolds are just the complex submanifolds of (complex) dimension p. Thus this $\mathscr{G}$-geometry is just the geometry of complex submanifolds in $\mathbb{C}^n$.

Example II. *(Lagrangian Geometries)* Suppose (X, ω) is a $2n$-dimensional symplectic manifold. Let $\mathscr{G} \subseteq \mathrm{G(n, TX)}$ be the subset of oriented Lagrangian planes L ($\dim(L)$ = n and $\omega|_L$ = 0). A $\mathscr{G}$-submanifold in this case is called a Lagrangian submanifold of X.

1.1.1 Grassmann Geometries determined by a Differential Form

We now assume X to be a Riemannian manifold. Consider $\varphi \in \Omega^p(X)$ an exterior p-form on X. We define the comass at each point $x \in X$ as

$$\|\varphi\|_x^* := \sup\{\langle \varphi_x, \xi_x \rangle \,|\, \xi_x \text{ is a unit simple } p\text{-vector}\} \tag{1.3}$$

i.e. $\|\varphi\|_x^*$ is the supremum of φ restricted to $\mathrm{G}(p, \mathrm{T}_xX) \subseteq \bigwedge^p \mathrm{T}_xX$. The comass for a subset $A \subseteq X$ is now defined in an obvious way as

$$\|\varphi\|_A^* = \sup_{x \in A} \|\varphi\|_x^* \tag{1.4}$$

(1.5) Definition: Suppose φ is a smooth p-form of comass 1 on X. We define $\mathscr{G}(\varphi)$ to be the union of the sets $\mathscr{G}_x := \{\xi_x \in \mathrm{G}(p, \mathrm{T}_xX) \,|\, \langle \varphi_x, \xi_x \rangle = 1\}$. The Grassmann geometrie defined by $\mathscr{G}(\varphi)$ will be called φ-geometry.

Example I. *(Complex geometries).* Let us continue the discussion from above. Choose a Riemannian metric $g(\cdot,\cdot)$ on X such that the map $J : TX \to TX$ which acts by multiplication with $\sqrt{-1}$, is orthogonal on every tangent space. We define the Kähler 2-form $\omega(V,W) = g(JV,W)$. Obviously $\omega(V,W) = -\omega(W,V)$ so that ω is an exterior 2-form. We set $\Omega_p = \frac{\omega^p}{p!}$. It follows from Wirtinger's inequality $\omega^p(\xi) \leq p!$ for all $\xi \in G(p,TX)$ that Ω_p has comass 1 and thus Ω_p defines a Grassmann geometry. The set $\mathscr{G}_x(\Omega_p)$ consists of all complex p-planes with positive orientation. Consider the case $p = 1$. Then $\omega(\xi) = 1$ precisely when ξ is of the form $\xi = v \wedge Jv$ for some unit vector v. Note that $-\xi = Jv \wedge v$ also defines a complex plane (in fact it is the same plane just with the opposite orientation), but since $\omega(-\xi) = -\omega(\xi)$, the plane $-\xi$ is not in $\mathscr{G}_x(\Omega_p)$. If we chose $\bar{J} = -J$ as the complex structure on X, we would get $\mathscr{G}_x(\overline{\omega}) = -\mathscr{G}_x(\omega)$ as the subset that defines our geometry. That is $\overline{\omega}$-submanifolds are just ω-submanifolds with the opposite orientation. The reason for this is that choosing a complex structure from J and $\bar{J}$ corresponds to choosing an orientation of X. Then the orientation of the ω-submanifold is just the induced orientation from X.

Example II. For *Lagrangian geometry* we have $\mathscr{G}_x = -\mathscr{G}_x$. Hence it is not possible to realize it as a φ-geometry. We will soon consider the subset that consists of special Lagrangian planes in $T_xX = \mathbb{C}^n$. In particular, as in the case of complex geometries the definition involves choosing an orientation, such that $\mathscr{G}'_x \cap -\mathscr{G}'_x = \emptyset$.

1.2 Calibrated Geometries

Consider a φ-geometry on an oriented Riemannian manifold X. Let S be a φ-submanifold of X. We know that $\varphi(\xi) = 1 = \mathrm{vol}_S(\xi)$ for any $\xi \in G(p,TX)$. From the fact that both are multilinear functionals we can deduce that $\varphi|_S = \mathrm{vol}_S$ and hence we get

$$\int_S \varphi = \mathrm{vol}(S). \tag{1.6}$$

We need an extra property to really use this to our advantage, namely we require φ to be a closed form such that we can apply Stoke's theorem.

(1.7) Definition: A smooth p-form on a Riemannian manifold X is said to be a *calibration* if φ is of comass one on X and $d\varphi = 0$. A Riemannian manifold together with a calibration is called a *calibrated manifold*.

With this definition and (1.6) we can now prove the following.

(1.8) Lemma: Suppose X is a calibrated manifold with calibration φ and suppose S is a compact φ-submanifold. Let S' be any compact submanifold that is homologous to S

(i.e $\partial S = \partial S'$ and $S \sqcup_\partial -S'$ is a boundary). Then $\mathrm{vol(S)} \leq \mathrm{vol(S')}$ with equality if and only if $\mathrm{vol(S')}$ is a φ-submanifold.

Proof. We want to show that the following is true:

$$\mathrm{vol(S)} \overset{(1.6)}{=} \int_S \varphi \overset{(2)}{=} \int_{S'} \varphi \overset{(3)}{\leq} \mathrm{vol(S')}. \tag{1.9}$$

(2) This follows immediately from the fact that S and S′ are homologous and φ is closed by applying Stoke's Theorem.
(3) We know that equality holds in the case that S′ is calibrated from (1.6). For S′ not calibrated the strict inequality easily follows from the comass condition. □

The lemma says that any calibrated submanifold is a minimal submanifold (since being calibrated is a local property, this is still true for non-compact submanifolds).

(1.10) Corollary: Any closed, compact calibrated submanifold of X is volume minimizing in its deRham homology class. Furthermore, every other closed submanifold in this homology class with the same volume must also be calibrated.

(1.11) Corollary: Let T be a compact calibrated submanifold with boundary B = ∂T. If $\mathrm{H_p}(X;\mathbb{R}) = 0$, then T is a solution of the plateau problem for B. This means that T is a submanifold of X that has minimal volume among all submanifolds in X with boundary B.

Example I. If $\mathrm{d}\omega = 0$, then X is said to be a Kähler manifold. It follows that complex subvarieties of Kähler manifolds are homologically volume minimizing.
We want to show in the case of $\mathbb{C}^2$ that the condition for a 2-dimensional submanifold of $M \subseteq \mathbb{C}^2$ to be a calibrated submanifold is equivalent to M being locally the graph of a holomorphic function, i.e. one that satisfies the Cauchy-Riemann equations. We know that every point $p \in M$ has an open neighborhood $U \subseteq M$ of the form M = $\{(x, y, u(x, y), v(x, y)) | x, y \in \mathbb{R}\}$ for some smooth functions $u, v : V \to \mathbb{R}$ where $V \subseteq \mathbb{R}^2$ is an open subset. We introduce the notation $u_x := \frac{\partial u}{\partial x}|_p$, $v_x := \frac{\partial v}{\partial x}|_p$, $u_y := \frac{\partial u}{\partial y}|_p$ and $v_y := \frac{\partial u}{\partial y}|_p$. Then we find that $T_pM = \mathrm{span}\{(1, 0, u_x, v_x), (0, 1, u_y, v_y)\}$. A simple calculation shows that the metric pulled back to T_pM in a basis is given by

$$g = (1 + u_x^2 + v_x^2)\,\mathrm{d}x^2 + 2(u_x u_y + v_x v_y)\,\mathrm{d}x\mathrm{d}y + (1 + u_y^2 + v_y^2)\,\mathrm{d}y^2$$

For the Kähler form on T_pM we have

$$\begin{aligned}\omega &= \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2 = \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}u \wedge \mathrm{d}v \\ &= \mathrm{d}x \wedge \mathrm{d}y + (u_x\mathrm{d}x + u_y\mathrm{d}y) \wedge (v_x\mathrm{d}x + v_y\mathrm{d}y) = (1 + u_x v_y - u_y v_x)\,\mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

Recall now the condition for M to be calibrated which is $\omega = \mathrm{vol}|_M = \sqrt{\mathrm{g}}\, \mathrm{d}x \wedge \mathrm{d}y$. Plugging in our results from above and squaring both sides leads to the equation

$$(1 + u_x v_y - u_y v_x)^2 = (1 + u_x^2 + v_x^2)(1 + u_y^2 + v_y^2) - (u_x u_y + v_x v_y)^2$$

We can reduce this to the equation

$$(\mathrm{u}_x - \mathrm{v}_y)^2 + (u_y + v_x)^2 = 0 \tag{1.12}$$

which is equivalent to the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$.

1.3 Special Lagrangian Geometry

1.3.1 The Special Lagrangian Calibration

Consider $\mathbb{C}^n$ with coordinates $z = (z_1, ..., z_n)$ where $z_i = x_i + iy_i$. We can identify $\mathrm{R}^n \subseteq \mathbb{C}^n$ as the subset where $y = 0$ with the standard orientation. We now introduce the form on $\mathbb{C}^n$ we want to study:

$$\alpha = Re(\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge ... \wedge \mathrm{d}z_n) \in \bigwedge\nolimits^n \mathbb{C}^n. \tag{1.13}$$

Our goal is to show that this is a calibration on $\mathbb{C}^n$, but first, we want to describe the subspaces of $\mathbb{C}^n$ contained in $\mathscr{G}(\alpha)$.
First we need to define totally real n-planes in $\mathbb{C}^n$. A n-plane ξ is called *totally real* (or just *real*) if $u \in \xi$ implies $\mathrm{J}u \notin \xi$. Furthermore a real n-plane ξ is called *Lagrangian* if $u \perp \mathrm{J}u$ (this is equivalent to the vanishing of the Kähler form on ξ, i.e. $\omega|_\xi = 0$).
We now consider $G(n, 2n)$ the Grassmannian of oriented, real n-planes in $\mathbb{C}^n$ and let Lag denote the subset of Lagrangian n-planes. The group U_n acts transitively on Lag, since it acts transitively on the set of orthonormal frames of $\mathbb{C}^n$. We want to find the stabilizer of $\xi_0 \in Lag$, but this is just SO_n acting diagonal on $\mathbb{R} \oplus \mathbb{R}$. Hence we get

$$Lag \cong U_n / SO_n. \tag{1.14}$$

(1.15) Definition: An oriented n-plane ξ is called *special Lagrangian* if

1. ξ is Lagrangian.
2. $\xi = A\xi_0$, where $A \in SU_n$.

$SLag$ denotes the set of special Lagrangian n-planes.

Remark: There is a fibration

$$SLag \longrightarrow Lag \stackrel{\det_{\mathbb{C}}}{\longrightarrow} S^1 \tag{1.16}$$

and $SLag$ is precisely the fiber above $1 \in S^1$. We could choose the fiber over any other $0 \le \theta < 2\pi$ and would get the calibrated geometry defined by $\alpha_\theta = Re(e^{i\theta} \mathrm{d}z_1 \wedge dz_2 \wedge ... \wedge \mathrm{d}z_n)$. All these geometries are isomorphic and thus we will only consider the case $\theta = 1$ from now on.
We set $\beta := \mathrm{Im}(\mathrm{d}z)$ so that $\mathrm{d}z = \alpha + i\beta$. For $\zeta \in \bigwedge^{2n}\mathbb{R}^{2n}$ we set $|\zeta| := |\mathrm{vol}_{2n}(\zeta)|$ where the volume form is given by $\mathrm{vol}_{2n} = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge ... \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$ and for $\xi \in G(n, 2n)$ we set $|\xi| := |\mathrm{vol}_{2n}|_\xi(\xi)|$. By $|_\xi$ we mean restriction to the plane defined by ξ. One can show by using the basic identity $\mathrm{d}z(A\varepsilon) = \det_{\mathbb{C}}(A)$ for $\varepsilon = \varepsilon_1 \wedge ... \wedge \varepsilon_n$ the standard $\mathbb{C}$-basis of $\mathbb{C}^n$ that for real n-planes $|\mathrm{d}z(\xi)|^2 = \alpha^2(\xi) + \beta^2(\xi) = |\xi \wedge \mathrm{J}\xi|$. Furthermore $|\xi \wedge \mathrm{J}\xi| \le |\xi|^2$ with equality if and only if ξ is Lagrangian because for equality $(\varepsilon_1, ..., \varepsilon_n, J\varepsilon_1, ..., J\varepsilon_n)$ must be an orthonormal basis. Also, $\beta(\xi) = 0$ holds if and only if $\det_{\mathbb{C}}(A) = \pm 1$. Altogether we get that $\alpha(\xi) \le |\xi|$ with equality if and only if $\xi \in G(n, 2n)$ is special Lagrangian, hence $\alpha = Re(\mathrm{d}z)$ is a calibration on $\mathbb{C}^n$ called the *special Lagrangian calibration*. It is also true that if ξ is Lagrangian and $\beta(\xi) = 0$ then either ξ or -ξ is special Lagrangian.

1.3.2 Special Lagrangian Submanifolds of $\mathbb{C}^n$

(1.17) Definition: An n-dimensional oriented submanifold M of $\mathbb{C}^n$ is called a *(special) Lagrangian submanifold* of $\mathbb{C}^n$ if the tangent plane to M at each point is (special) Lagrangian.
Note that the special Lagrangian submanifolds are exactly the calibrated submanifolds for the special Lagrangian calibration and hence they are volume minimizing.

We know that every real submanifold of $\mathbb{C}^n \cong \mathbb{R}^n \oplus \mathbb{R}^n$ locally looks like the graph of a smooth function $f : \Omega \to \mathbb{R}$. We want to find differential equations, such that for any solution f the graph of f (denoted with $\mathrm{Graph}(f)$) is special Lagrangian.
Since - as mentioned before - being (special) Lagrangian is a local property one only has to consider linear maps (by passing to the differential of f). We then have (maybe after translating)

$$\mathrm{Graph}(f) = \{x + i f_* x | x \in \mathbb{R}^n\} \tag{1.18}$$

Then being Lagrangian means that $\langle x + i f_* x, -f_* x' + i x' \rangle = 0$ and thus $\langle f_* x, x' \rangle = \langle x, f_* x' \rangle$ for all x, x' in $\mathbb{C}^n$. Hence $\mathrm{Graph}(f)$ is Lagrangian if and only if the Jacobian matrix $\mathrm{D}f_x$ is symmetric at each point $x \in \Omega$. In that case, if Ω is simply connected we can find a potential F, such that $\nabla F = f$.
We need another condition to obtain a criterion for $\mathrm{Graph}(f)$ to be special Lagrangian.

Consider $F : \Omega \to \mathbb{R}$. Define the Hessian matrix as usual

$$\mathrm{Hess}(F) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right) \tag{1.19}$$

Then the linear map $A = I + if_*$ rotates ξ_0 into the Graph(f) (w.l.o.g f is linear). So we find that Graph(f) is special Lagrangian if and only if $\mathrm{Im}\{\det_{\mathbb{C}}(I + i\mathrm{Hess}(f))\} = 0$.
We can also define a submanifold implicitly for smooth real valued functions $f_1, ... f_n$ with $\mathrm{d}f_1, ..., \mathrm{d}f_n$ linearly independent at every point of $M = \{\, z \in \Omega | f_1(z) = ... = f_n(z)\}$. Using that the Kähler form locally can be written $\omega = \sum_{k=1}^n dx_k \wedge dy_k$, we find, that M is Lagrangian if and only if all Poisson brackets $\{f_j, f_k\}$ vanish. Furthermore we can show that M is special Lagrangian if and only if $\mathrm{Im}(\det_{\mathbb{C}}(2i\partial f_j / \partial \bar{z}_j)) = 0$.

Example: Let M_c denote the locus of the equations: $|z_1|^2 - |z_j|^2 = c_j, j = 2, ..., n$ and $\mathrm{Im}((i)^n z_1 ... z_n) = c_1$. Then M_c (with the correct orientation) is a special Lagrangian submanifold of $\mathbb{C}^n$. One can check that the defining functions for M satisfy the conditions for implicit defined special Lagrangian submanifolds of $\mathbb{C}^n$.
It's interesting to consider the case $c = 0$. The resulting manifold will have a singularity. Hence there are special Lagrangian varieties that are not smooth.

1.3.3 Minimality of Special Lagrangian Planes

To conclude this section we want to show that special Lagrangian submanifolds are actually minimal submanifolds of $\mathbb{C}^n$.
Let us start with some basics about minimal submanifolds. Since this topic is not very central to us, we will only introduce the ideas and definitions needed for our purpose very briefly. For a detailed discussion see for example [3]. A discussion of the minimality of special Lagrangian submanifolds can be found in [4] and [5].
We consider a submanifold N of a Riemannian manifold (M, g). A variation of N is a smooth map $F : N \times (-\epsilon, \epsilon) \to M$ with $F(p, 0) = p$ such that $F(p, t)$ is an immersion for every t. We assume that our variations have compact support meaning that for some compact set $K \subseteq N$ it is $F|_{K^c} = \mathrm{Id}|_{K^c}$. For simplicity we will assume that N is compact, such that this condition is always true. We define the variational vector field as $X = F_*(\frac{\partial}{\partial t})|_{t=0}$. We denote by $A(t)$ the volume of the immersed submanifold $N_t = F(N, t)$, i.e. let $\mathrm{d}V_t$ be the induced volume element and set $A(t) = \int_N \mathrm{d}V_t$.
We wish to calculate $\frac{d}{\mathrm{d}t} A(t)|_{t=0}$. In local coordinates we have

$$A(t) = \int_N \sqrt{\det \left(g \left(\frac{\partial F(t)}{\partial x_i}, \frac{\partial F(t)}{\partial x_j} \right) \right)} \mathrm{d}V_0 \tag{1.20}$$

We fix $p \in N$ and choose our local coordinates x_i to be normal coordinates at p. This means that $e_i(t) := \frac{\partial F}{\partial x_i}(p,t)$ satisfy $g(e_i(0), e_j(0)) = \delta_{ij}$. We know that the derivative of the determinant is the trace and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\det(g(e_i(t), e_j(t)))}|_{t=0} = \frac{1}{2}\sum_i g'(e_i(0), e_i(0)) = \frac{1}{2}\sum_i \frac{\mathrm{d}}{\mathrm{d}t} g\left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_i}\right)|_{t=0} \tag{1.21}$$

From the definition of the Levi-Civita connection it follows that

$$\frac{1}{2}\sum_i \frac{\mathrm{d}}{\mathrm{d}t} g\left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_i}\right) = \sum_i g(\nabla_X e_i, e_i) = \sum_i g(\nabla_{e_i} X, e_i) = \mathrm{div}_N(X) \tag{1.22}$$

The equality between the second and the third term holds since $[X, e_i] = 0$ (i.e. the t and x_i derivatives commute).
The mean curvature vector is defined as the trace of the second fundamental form α given by

$$\alpha(X,Y) := \nabla_X^\perp Y \qquad \forall X, Y \in \Gamma(TN) \tag{1.23}$$

In local coordinates the second fundamental form then reads as

$$H = \sum_i (\nabla_{e_i} e_i)^\perp \tag{1.24}$$

where $(\cdot)^\perp$ is the projection onto the normal bundle to N ($\mathrm{T}M = \mathrm{T}N \oplus \nu$ with ν the normal bundle to N in $\mathrm{T}M$).
The following is the central calculation in this discussion. First note that $\nabla_{e_i} g(X^\perp, e_i) = 0$. Using this, a simple calculation shows that

$$\mathrm{div}_N(X) = \mathrm{div}(X^{\mathrm{T}}) - \sum_i g(X^\perp, \nabla_{e_i} e_i) = \mathrm{div}_N X^{\mathrm{T}} - g(X, H) \tag{1.25}$$

Combine all of this and noting that integration over a normal vector field is zero we have established

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t)|_{t=0} = \int_N \mathrm{div}_N(X) dV = -\int_N g(X,H) dV \tag{1.26}$$

Therefore, we have two equivalent definitions for a minimal submanifold. First we can define it to be a critical point of the volume functional. The second definition, which by (1.26) is equivalent to the first one, defines a minimal submanifold as a submanifold for which the mean curvature vector vanishes everywhere, i.e. $H = 0$.
Clearly a variation as defined above doesn't change the homology class of the manifold since the variation is a homotopy of the immersions of N into M. So since calibrated submanifolds are volume minimizing in their homology class they certainly will be critical points of the volume functional. That in turn means that the mean curvature

vector must vanish. We use this to show the following:

(1.27) Proposition: *Let $M \subseteq \mathbb{R}^{2n} \cong \mathbb{C}^n$ be a connected Lagrangian submanifold. Then M is a minimal submanifold if and only if M is special Lagrangian with respect to one of the calibrations $\alpha_\theta = \mathrm{Re}(e^{i\theta}\mathrm{d}z)$.*

Proof. Let M be a Lagrangian submanifold. We know from above that the condition for T_pM to be a Lagrangian subspace is equivalent to the condition that $|\mathrm{d}z(T_pM)| = 1$ which is equivalent to $\mathrm{d}z(T_pM) = e^{i\theta} =: f$. This defines a map $\theta : M \to \mathbb{R}/2\pi\mathbb{Z}$.
In the following we use the definition of the second fundamental form in (1.23) and the identity $g(\alpha(X,Y), JZ) = g(\alpha(X,Z), JY)$ where J is the standard complex structure on $\mathbb{C}^n$. Recall that in the Euclidean case, the Levi-Civita connection is just the ordinary directional derivative. We choose a local frame $(e_1, ..., e_n)$ at p with dual frame $(e_1^*, ..., e_n^*)$ and find that $\mathrm{d}z = e^{i\theta(p)}(e_1^* + i\mathrm{J}e_1^*) \wedge ... \wedge (e_m^* + i\mathrm{J}e_m^*)$. Let V be an arbitrary tangent vector. Without loss of generality we assume that $(\nabla^{\|}e_i)_p = 0$ and hence $(\nabla_V e_i)_p = \alpha(e_i, V)$ Furthermore note that $\mathrm{d}z$ is closed and hence covariantly constant, i.e. $\nabla_X \mathrm{d}z = 0$. Hence we find

$$\begin{aligned}
V(f) &= V(\mathrm{d}z(e_1, ..., e_m)) = \sum_i \mathrm{d}z(e_1, ..., \alpha(V, e_i), ..., e_m) \\
&= \sum_i f(p)\,((e_1^* + i\mathrm{J}e_1^*) \wedge ... \wedge (e_m^* + i\mathrm{J}e_m^*))\,(e_1, ..., \alpha(V, e_i), ..., e_m) \\
&= if(p)\sum_i g(\mathrm{J}e_i, \alpha(V, e_i)) = if(p)\sum_i g(\mathrm{J}V, \alpha(e_i, e_i)) \\
&= if(p)\, g(\mathrm{J}V, H)
\end{aligned}$$

so

$$V(\theta) = V(-i\log(f)) = -i\frac{V(f)}{f} = \mathrm{g}(\mathrm{J}V, H) \tag{1.28}$$

Thus theta is constant if and only if $H = 0$, i.e. M is a minimal surface. $\square$

We have in particular shown that for special Lagrangian submanifolds of $\mathbb{C}^n$ the mean curvature always vanishes. The proof given above still holds under less restrictive assumptions. The crucial point is that there must exist a covariantly constant holomorphic n-form which is exactly the case for Calabi-Yau manifolds.

1.4 The Exceptional Geometries

In the classification of Riemannian holonomy groups two exceptional cases for possible holonomy groups appear which are G_2 and Spin_7. These are closely related to the structure of the octonions. We want to explore how these groups appear in the context of geometry over the octonions.

1.4.1 The Octonions

The octonions are constructed from $\mathbb{R}$ via repeated application of the Cayley-Dickson process. The construction can be found in [1] (Ch.4, Appendix A). As an $\mathbb{R}$-vector space $\mathbb{O}$ has the basis $\{1, i, j, ij, e, ei, ej, ek\}$. The octonions are the largest normed division algebra over $\mathbb{R}$. The norm is induced by an inner product given by $\langle x, y \rangle = \mathrm{Re}(x\bar{y})$. Now we want to construct automorphisms of $\mathbb{O}$ by using the construction given by the Cayley-Dickson process.

(1.29) Lemma: Suppose e_1, e_2, e_3 is an orthonormal triple in $\mathrm{Im}\,\mathbb{O}$ with $e_3 \perp e_1e_2$. Then there exists a (unique) automorphism g of $\mathbb{O}$ sending $i \to e_1$, $j \to e_2$ and $e \to e_3$.

We need to define several cross products on $\mathrm{Im}\,\mathbb{O}$.

(1.30) Definition: The two-fold cross product of $x, y \in \mathbb{O}$ is given by $x \times y := -\frac{1}{2}(\bar{x}y - \bar{y}x) = \mathrm{Im}(\bar{y}x)$.

Clearly the two-fold cross product is alternating. The three-fold cross product is just the alternation of the two-fold cross product, i.e. $x \times y \times z = \frac{1}{3}(\bar{x}(y \times z) + \bar{y}(z \times x) + \bar{z}(x \times y))$ and hence also is alternating. It can be simplified to the expression $x \times y \times z := \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x))$. The four-fold cross product is again the alternation of the three-fold cross product. The definition of the cross product allows us to give a nice geometric interpretation in the sense that we find $|x \times y| = |x \wedge y|$ and the analogous expressions for higher cross products.

(1.31) Definition: (i) The *associator* is given by $[x, y, z] = (xy)z - x(yz)$ for all $x, y, z \in \mathbb{O}$.
(ii) The *coassociator* is given by $[x, y, z, w] = -(\langle y', z'w' \rangle x' + \langle z', x'w' \rangle y' + \langle x', y'w' \rangle z' + \langle y', x'z' \rangle w')$ for all $x, y, z \in \mathbb{O}$ where $x' = \mathrm{Im}(x)$.
The associator and the coassociator are both alternating.

We will state more properties of these objects when needed. The proofs can be found in the literature, e.g. [1].

1.4.2 The Associative Calibration

Consider the trilinear form φ on $\mathrm{Im}\,\mathbb{O}$ given by

$$\varphi(x,y,z) = \langle x, yz \rangle. \tag{1.32}$$

The form φ is alternating since for $x \in \mathrm{Im}\,\mathbb{O}$ it holds that $\bar{x} = -x$. We take $\{e_1, ...e_8\}$ to be the standard basis of $\mathbb{O}$ and $\{\omega_1, ..., \omega_8\}$ the dual basis for $\mathbb{O}^*$. We define $\omega_{pqr} = \omega_p \wedge \omega_q \wedge \omega_r$. From that we find

$$\varphi = \omega_{234} - \omega_{278} - \omega_{638} - \omega_{674} - \omega_{265} - \omega_{375} - \omega_{485} \tag{1.33}$$

If we consider $\mathrm{Im}\,\mathbb{H} = \mathrm{i} \wedge \mathrm{j} \wedge \mathrm{k} = \mathrm{e}_{234}$ as the imaginary quaternions embedded into $\mathrm{Im}\,\mathbb{O}$, we find that $\varphi(\mathrm{Im}\,\mathbb{H}) = 1$. This observation leads us to the following definition.

(1.34) Definition: If $\xi \in \mathrm{G}(3,7) \subseteq \bigwedge^3 \mathrm{Im}\,\mathbb{O}$ is the canonically oriented imaginary part of any quarternion subalgebra of $\mathbb{O}$ ($\xi = x_1 \wedge x_2 \wedge x_3$ such that $x_1 x_2 = x_3$), then ξ is said to be *associative*. We call $\mathrm{G}(\varphi)$ the associative Grassmannian. In particular, the associator vanishes on all associative planes.

It is an immediate consequence of the following theorem that φ is actually a calibration on $\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O}$.

(1.35) Theorem: *The form φ has comass one. In fact, $\phi(\xi) \leq 1$ for all $\xi \in \mathrm{G}(3,7) \subseteq \bigwedge^3 \mathrm{Im}\,\mathbb{O}$ with equality if and only if ξ is associative.*

One can easily prove this fact by noting that $|x \times y \times z| = |x \wedge y \wedge z|$ (where the first pair of vertical bars denotes the norm on $\mathbb{O}$ and the second one the induced norm on the outer product) and $x \times y \times z = \mathrm{Re}(x \times y \times z) + \mathrm{Im}(x \times y \times z)$ with $\mathrm{Re}(x \times y \times z) = \langle x, yz \rangle$ and $\mathrm{Im}(x \times y \times z) = \frac{1}{2}[x,y,z]$. From this, one furthermore deduces that if for $\xi = x \wedge y \wedge z \in \mathrm{G}(3,7)$ the equation $[x,y,z] = 0$ is satisfied then either ξ or $-\xi$ is associative. We call φ the *associative calibration* on $\mathrm{Im}\,\mathbb{O}$.
The next step is to study the connection between the associative calibration φ and the automorphisms of the octonions. This automorphism group is by definition the exceptional Lie group G_2. Explicitly this means

$$\mathrm{G}_2 = \{g \in \mathrm{GL}_8(\mathbb{R}) | \, \forall x, y \in \mathbb{O} : g(xy) = g(x)g(y)\} \tag{1.36}$$

In particular the action of G_2 is $\mathbb{R}$-linear and sends 1 to 1 (and hence fixes $\mathbb{R} \subseteq \mathbb{O}$). The square of an octonion is real if and only if it is real or purely imaginary. If we now take a purely imaginary octonion we have

$$x^2 = g(x^2) = g(x)^2 \tag{1.37}$$

for all $g \in \mathrm{G}_2$. From that we see that g(x) is purely imaginary (the square of $g(x)$ is real and since g is an automorphism on $\mathbb{O}$, $g(x)$ can't be purely real. Furthermore x^2 is a negative number so there is no real root). $\mathbb{R}$-linearity now implies that $g(\overline{x}) = \overline{g(x)}$. We see that the elements of G_2 act as isometries on $\mathbb{O}$. So G_2 fixes the inner product and thus also the associative calibration. The converse is also true, so we have the following alternative characterization of G_2

$$G_2 = \{ g \in \mathrm{O}_7 | \, g^* \varphi = \varphi \} \qquad \textbf{(1.38)}$$

Since we just have seen that G_2 is precisely the group preserving φ the following proposition should not come as a surprise (for a proof we refer to [1]).

(1.39) Proposition: *The group* G_2 *acts transitively on* $\mathrm{G}(\varphi)$ *with isotropy subgroup* SO_4. *Thus* $\mathrm{G}(\varphi) \cong \mathrm{G}_2 / \mathrm{SO}_4$. *The action of* $\mathrm{SO}_4 = \mathrm{Sp}_1 \times \mathrm{Sp}_1 / \mathbb{Z}_2$ *on* $\mathrm{Im}\, \mathbb{O}$ *is given by* $\mathrm{g(a,b)} = (\mathrm{q_1 a \bar{q}_1, q_2 b \bar{q}_1})$ *for* $\mathrm{q_1, q_2} \in \mathrm{Sp}_1 \cong \mathbb{H}$.

The Lie group G_2 also appears is Berger's classification of Riemannian holonomy groups. There is indeed an intimate relationship between calibrations and reduced holonomy. In particular in all considered examples we can read of the required holonomy for a manifold on which the calibration extends to a closed differential form from the homogeneous space $\mathrm{G}(\varphi)$, e.g. we always get an associative calibration on G_2 manifolds and a special Lagrangian calibration on Calabi-Yau manifolds (one characterization of the Calabi-Yau condition is that the holonomy group of the manifold is reduced to SU_n).

1.4.3 The Coassociative Calibration

There is a second calibration on $\mathrm{Im}\, \mathbb{O}$ that is closely related to the associative calibration.

(1.40) Definition: The 4-form $\psi \in \bigwedge^4 \mathrm{Im}\, \mathbb{O}^*$ is defined by

$$\psi(x,y,z,w) := \frac{1}{2}\langle x, [y,z,w] \rangle.$$

in coordinates this can be expressed as

$$\psi = \omega_{5678} - \omega_{5634} - \omega_{5274} - \omega_{5238} + \omega_{3478} + \omega_{2468} + \omega_{2367} \qquad \textbf{(1.41)}$$

By recalling the vector space isomorphism $* : \bigwedge^3 \mathbb{R}^7 \longrightarrow \bigwedge^4 \mathbb{R}^7$ given by the hodge-$*$ operator and comparing the expressions in coordinates one concludes that $\psi = *\varphi$.

From that it is clear how to proceed. The discussion is dual to the one above. For example it is obvious that ψ is a calibration on $\mathbb{R}^7$ called the *coassociative calibration*.

(1.42) Definition: An oriented 4-plane $\xi \in G(4,7) \subseteq \bigwedge^4 \mathrm{Im}\,\mathbb{O}$ is said to be *coassociative* if the canonically oriented normal 3-plane $*\xi$ is associative. These are the planes calibrated with respect to the coassociative calibration.

1.4.4 The Cayley Calibration

The previously discussed forms provide calibrations on the imaginary octonions $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. We now want to define a calibration on $\mathbb{O} \cong \mathbb{R}^8$.

(1.43) Definition: The 4-form $\Phi \in \bigwedge^4 \mathbb{O}^*$ defined by

$$\Phi(x,y,z,w) := \frac{1}{2}\langle x, y \times z \times w\rangle \tag{1.44}$$

is called the *Cayley calibration* on $\mathbb{O}$.

The coordinate expansion shows that we can write this form as

$$\Phi = 1^* \wedge \varphi + \psi \tag{1.45}$$

where 1^* denotes the basis vector of $\mathbb{O}^*$ dual to $1 \in \mathbb{O}$, hence the form is actually alternating. A plane $\xi \in G(4,8) \subseteq \bigwedge^4 \mathbb{O}$ which satisfies $\Phi(\xi) = 1$ is called a Cayley 4-plane. The terminology already suggests that Φ actually is a calibration on $\mathbb{R}^8$. We justify this claim with the following theorem.

(1.46) Theorem: *The form Φ has comass one and hence is a calibration. The calibrated planes are precisely the Cayley 4-planes.*

Proof. That the form has comass one is an immediate consequence of Schwarz's inequality: $\langle x, y \times z \times w\rangle \leq |x||y \times z \times w| = 1$, which shows that Φ indeed has comass one. The characterization of the calibrated planes is true by definition. □

We want to give a more concrete characterization of the calibrated planes. First note that the fact that Φ is alternating shows that $y \times z \times w$ always is perpendicular to y, z and w for all y, z, $w \in \mathbb{O}$. On the other hand, we find that a 4-plane $\xi = x \wedge y \wedge z \wedge w \in G(4,8)$ is Cayley if and only if $x = y \times z \times w$. Together this implies that either ξ or $-\xi$ are Cayley if and only if for each $y, z, w \in \xi$ we also know that $y \times z \times w \in \xi$.
It is a non-trivial fact that $\Psi(x \wedge y \wedge z \wedge w) = \mathrm{Re}(x \times y \times z \times w)$ such that we get the following equation:

$$|\Psi(x \wedge y \wedge z \wedge w)| + |\mathrm{Im}(x \times y \times z \times w)| = |x \wedge y \wedge z \wedge w| \tag{1.47}$$

and we conclude that either ξ or $-\xi$ is Cayley if and only if $\mathrm{Im}(x \times y \times z \times w) = 0$.
Another way to characterize these planes is by considering complex structures on the octonions. Consider the 6-Sphere of imaginary unit quaternions

$$S^6 = \{u \in \mathrm{Im}\,\mathbb{O} |\, \langle u, u \rangle = 1\} \qquad \textbf{(1.48)}$$

Given any $u \in S^6$ let us denote by $\mathrm{J}_u : \mathbb{O} \longrightarrow \mathbb{O}$ the multiplication from the right by u, i.e. $\mathrm{J}_u v = vu$. Since $u^2 = -1$ and $(vu)u = v(u^2) = -v$ we have $\mathrm{J}_u^2 = -\mathbb{I}$ and hence, every imaginary quarternion defines a complex structure on $\mathbb{O}$. From the definition of the two-fold cross product it follows that for any two octonions x and y the product $x \times y$ will be purely imaginary. Further, for x and y chosen to be an orthonormal pair, $x \times y$ will be of unit norm. Last, note that for an other orthonormal x', y' spanning same plane as x, y we have $x \times y = x' \times y'$ (given that the orientation is chosen consistently) such that we get a well defined map

$$\begin{aligned} \mathrm{G}(2,8) &\longrightarrow S^6 \\ x \wedge y &\mapsto x \times y \end{aligned}$$

One can check that $\mathrm{J}_{x \times y}$ is just the canonical complex structure on the surface $y \wedge x \subseteq \mathbb{O}$. With this on hand we can give yet another criterion for a 4-plane to be Cayley.

(1.49) Proposition: *$\xi \in \mathrm{G}(4,8) \subseteq \bigwedge^4 \mathbb{O}$ is Cayley if and only if $-\xi$ is a complex 2-plane with respect to one of the complex structures determined by the 2-planes contained in ξ.*

Proof. We choose z, w to be orthonormal in ξ and complete it to an orthonormal basis of ξ. We have the canonical complex structure $\mathrm{J}_{w \times z}$ on $z \wedge w$. This is a complex structure on ξ if and only if $\mathrm{J}_{w \times z} y = x$. For orthonormal y, z, w this is just

$$x = \mathrm{J}_{w \times z} y = y(\bar{z}w) = y \times z \times w \qquad \textbf{(1.50)}$$

which is exactly the condition for ξ to be Cayley found above. □

Let us now recall that the Cayley calibration basically is the sum of the associative and the coassociative calibration (1.45). So we see that the coassociative and associative geometries are actually restrictions of Cayley geometry in the sense that every coassociative plane is Cayley and if we have an associative plane ξ in $\mathrm{Im}\,\mathbb{O}$ that we complete to a quaternionic subalgebra $1 \wedge \xi$ of $\mathbb{O}$ it is also an Cayley 4-plane.
It would be nice to have a connection between special Lagrangian geometry, complex geometry and Cayley geometry. This can actually be achieved. We consider the complex structure associated to $\mathrm{e} = (0, 1) \in \mathbb{H} \oplus \mathbb{H} = \mathbb{O}$. We can associate $\mathbb{O} \cong \mathbb{C}^4$ via the complex structure J_e which induces a Kählerform on $\mathbb{O}$. Furthermore we can identify

$\mathbb{H} \subseteq \mathbb{O}$ with $\mathbb{R}^4 \subseteq \mathbb{O}^4$. Then we find

$$\Phi = -\frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\mathrm{dz}) \tag{1.51}$$

This shows that 4-dimensional complex geometry (with reversed orientation) and special Lagrangian geometry on $\mathbb{R}^8$ are both special cases of Cayley geometry. There are however "mixed"-planes that are neither (reversed) complex nor Lagrangian but still Cayley so that Cayley geometry is more general.
We now want to find the group that fixes the Cayley calibration. Recall that for the associative calibration (and by duality also for the coassociative calibration) the group that preserves the form is the exceptional Lie group G_2. It will turn out that for the Cayley calibration this will be Spin_7. In the following this claim will be made precise.
For each $u \in \mathbb{O}$ we denote the right-multiplication by $R_u : \mathbb{O} \longrightarrow \mathbb{O}$ given by $R_u(x) = xu$. Note that for $u \neq 0$ this lies in $\mathrm{GL}^+(\mathbb{O})$. The inner product on the octonions satisfies $\langle xu, yu \rangle = \langle x, y \rangle |u|^2$ and hence for non-zero u the determinant does not vanish. Since $u' = u/|u|$ has unit norm, $R_{u'}$ defines a complex structure on $\mathbb{O}$. It preserves the orientation and hence has positive determinant.

(1.52) Definition: Spin_7 is the subgroup of SO_8 generated by $S_6 := \{\, R_u |\, u \in \mathrm{Im}\,\mathbb{O}$ and $\langle u, u \rangle = 1\}$. Note that we regard the R_u's as automorphisms of $\mathbb{R}^8$ and therefore have an associative composition by definition, however in general $R_u \circ R_v \neq R_{vu}$.

Using the identity $((zx)y)x = z(xyx)$ we see that $R_u \circ R_v \circ R_u^{-1} = R_{-uvu}$. We can use this to define an action by Spin_7 on the vector space $W := \{\, R_v |\, v \in \mathrm{Im}\,\mathbb{O}\} \ \cong \mathbb{R}^7$ by

$$\chi_{\mathrm{g}}(R_v) := g \circ R_v \circ \mathrm{g}^{-1} \tag{1.53}$$

To see that this is well defined, consider $u, v \in \mathrm{Im}\,\mathbb{O}$. Then we have $\overline{uvu} = \bar{u}(\overline{uv}) = \bar{u}\bar{v}\bar{u} = (-u)(-v)(-u) = -uvu$. In fact, Spin_7 is precisely the subgroup of SO_8 that conjugates W to itself. That is, given $\mathrm{g} \in \mathrm{SO}_8$, g belongs to Spin_7 if and only if for every $v \in \mathrm{Im}\,\mathbb{O}$, there exists $w \in \mathrm{Im}\,\mathbb{O}$ such that $\mathrm{g} \circ R_v \circ \mathrm{g}^{-1} = R_w$. From this we see, that $\chi_{\mathrm{g}}(R_v)(1) = \mathrm{g}(\mathrm{g}^{-1}(1)v) = w$. This gives an action of Spin_7 on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$, i.e. a morphism $\bar{\chi} : \mathrm{Spin}_7 \longrightarrow \mathrm{SO}_7$ given by $\bar{\chi}_{\mathrm{g}}(v) = \mathrm{g}(\mathrm{g}^{-1}(1)v)$ which is the standard double cover. Putting all together we find that for $\mathrm{g} \in \mathrm{SO}_8$ to lie in Spin_7 it must satisfy $\mathrm{g}(\mathrm{g}^{-1}(y)v) = y\bar{\chi}_{\mathrm{g}}(v)$ for all $y \in \mathbb{O}$. If we replace y by $\mathrm{g}(u)$ we find

$$\mathrm{Spin}_7 = \{\mathrm{g} \in \mathrm{SO}_8 |\, \mathrm{g}(uv) = \mathrm{g}(u)\bar{\chi}_{\mathrm{g}}(v)\, \forall u, v \in \mathbb{O}\}. \tag{1.54}$$

The Cayley calibration is fixed by the action of Spin_7, i.e. for all $\mathrm{g} \in \mathrm{Spin}_7$ it is true that $\mathrm{g}^*\Phi = \Phi$. In analogy to the discussion of the associative calibration, the space of Cayley 4-planes can be described as the homogeneous space $G(\Phi) = \mathrm{Spin}_7/K$ where $K = \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2/\mathbb{Z}_2$.

2 Spin Geometry

In section 1 most of the discussion was on linear algebra. To make contact with differential geometry, we now review some basic notions of spin geometry. A comprehensive discussion of this topic can be found in [6]. The schedule is as follows: First, we review the theory of Clifford algebras and Clifford modules. A good reference for this is [7]. We then want to define spin manifolds and spinor bundles on them. The goal is to provide the tools for the construction of calibrations from covariant constant spinors carried out in section 3.

2.1 Clifford Algebras

Let V be a finite dimensional vector space over a field k (for us only the cases $\mathbb{R}$ and $\mathbb{C}$ will be of interest, so we always will assume $\operatorname{char} k = 0$) with a quadratic form Q (which we assume to be non-degenerate) on it. The *Clifford algebra* $Cl(V, Q)$ associated to (V, Q) is the tensor algebra $T(V) = \bigoplus_r (\otimes^r V)$ modulo the ideal $I_Q(V)$ generated by the elements $v \otimes v + Q(v)$ for $v \in V$:

$$Cl(V, Q) = T(V)/I_Q(V) \tag{2.1}$$

The Clifford algebra contains the field k and the vector space V. The elements of V (together with $1 \in k$) generate the Clifford algebra and the defining relation is just $v \cdot v = -Q(v)1$. As a vector space the Clifford algebra is isomorphic to the exterior algebra and hence it has dimension 2^n, i.e.

$$Cl(V, Q) \cong \bigwedge^{\bullet} V \tag{2.2}$$

The Clifford algebra has the following universal property: Let $f : V \longrightarrow A$ be a linear map into an associative k-algebra with unit, such that $f(v) \cdot f(v) = -Q(V)1$ for all $v \in V$. Then f uniquely extends to a k-algebra homomorphism $\tilde{f} : Cl(V, Q) \longrightarrow A$. This characterizes the Clifford algebra uniquely up to unique isomorphism.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ Cl \arrow[ur, "\exists! \tilde{f}"'] & \end{tikzcd} \tag{2.3}$$

As a consequence we see that the construction of the Clifford algebra is functorial (a morphism of vector spaces induces a morphism of the Clifford algebra and the uniqueness guarantees that the composition commutes with the functor). Thus the group $\mathrm{O}(V,Q) := \{\mathrm{f} \in \mathrm{GL}(V) | \, f^*Q = Q\}$ extends to a group of automorphisms of $Cl(V,q)$. An especially interesting automorphism is

$$\alpha : Cl(V,Q) \longrightarrow Cl(V,Q) \tag{2.4}$$

which extends the map $\alpha(v) = -v$. This map has eigenvalues 1 and -1, and we denote the eigenspaces corresponding to the eigenvalue $(-1)^i$ for $i = 0, 1$ by $Cl^i(V,Q)$. We get a decomposition

$$Cl(V,Q) = Cl^0(V,Q) \oplus Cl^1(V,Q) \tag{2.5}$$

This is the same grading as the one inherited from the $\mathbb{Z}$-grading on the tensor algebra. Since $I_Q(V)$ is homogeneous mod 2, there will remain a $\mathbb{Z}_2$ grading on the quotient. For elements of definite grading let $|\cdot|$ denote this grading. For homogeneous elements $v, w \in Cl(V,Q)$ we then have the relation $|v \cdot w| = |v| + |w|$. Hence the Clifford algebra is a $\mathbb{Z}_2$-graded algebra.
Besides (2.5) there is another important $T(V)$ given on simple elements by reversal of the order, i.e. $v_1 \otimes ... \otimes v_r \to v_r \otimes ... \otimes v_1$. The induced map

$$(\cdot)^t : Cl(V,Q) \longrightarrow Cl(V,Q) \tag{2.6}$$

is called transposition. This map is an antiautomorphism which means that $(v \cdot w)^t = \mathrm{w}^t \cdot \mathrm{v}^t$.

2.1.1 The Groups Pin and Spin

The unit group $Cl^\times(V,Q)$ of the Clifford algebra admits a representation onto $Cl(V,Q)$ called the adjoint representation given by

$$\begin{aligned} \mathrm{Ad} : Cl^\times(V,Q) &\longrightarrow \mathrm{Aut}(Cl(V,Q)) \\ \varphi &\mapsto \mathrm{Ad}_\varphi(x) := \varphi x \varphi^{-1} \end{aligned} \tag{2.7}$$

If we define $2Q(v,w) = Q(v+w) - Q(v) - Q(w)$ we see that $\{v,w\} = v \cdot w + w \cdot v = -2Q(v,w)$. Now consider $v \in V$ with $Q(v) \neq 0$. Then clearly $v^{-1} = -v/Q(v)$. So we find

$$-Q(v)\mathrm{Ad}_v(w) = -Q(v)vwv^{-1} = vwv = -v^2w - 2Q(v,w)v = Q(v)w - 2Q(v,w)v. \tag{2.8}$$

Dividing both sides by $Q(v)$ yields

$$-\mathrm{Ad}_v(w) = w - 2\frac{Q(v,w)}{Q(v)}v = -R_v(w) \tag{2.9}$$

Here R_v denotes the reflection of a vector on the hyper plane perpendicular to v. Since the reflections lie in $O(V,Q)$ each of these maps extends to a map on the Clifford algebra given by $\mathrm{Ad}_v(x) = vxv^{-1}$ for $v, x \in Cl(V,Q)$ Note that one might want to use the twisted adjoint representation given by $\widetilde{\mathrm{Ad}}_v(w) = (-v)wv^{-1} = R_v(w)$ to get rid of the sign. The induced map on the Clifford algebra then reads $\widetilde{\mathrm{Ad}}_v(x) = \alpha(v)xv^{-1}$.
We are now ready to define the groups $\mathrm{Pin}(V,Q)$ and $\mathrm{Spin}(V,Q)$:

(2.10) Definition: The *Pin group* of (V,Q) is the subgroup $\mathrm{Pin}(V,Q)$ of $Cl^{\times}(V,Q)$ generated by the elements $v \in V$ with $Q(v) = 1$. The corresponding *Spin group* of (V,Q) is defined by

$$\mathrm{Spin}(V,Q) = \mathrm{Pin}(V,Q) \cap Cl^0(V,Q). \tag{2.11}$$

The twisted adjoint representation maps the Pin group to the orthogonal group $\mathrm{O}(V,Q)$ and thus gives rise to a group homomorphism. Certainly all reflections lie in the image. By a theorem of Cartan-Dieudonné (see [6] p.17) which says that for a non-degenerate quadratic form on V the set of reflections generates the group $\mathrm{O}(V,Q)$ we find that the homomorphism

$$\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V,Q) \longrightarrow \mathrm{O}(V,Q) \tag{2.12}$$

is surjective. Furthermore the theorem tells us that it restricts to a surjective homomorphism

$$\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V,Q) \longrightarrow \mathrm{SO}(V,Q). \tag{2.13}$$

From now on we will focus on real vector spaces. A quadratic form on a real vector space is determined by its signature. Let's write $V = \mathbb{R}^{r+s}$ for an $(r+s) = n$-dimensional real vector space V equipped with non-degenerate quadratic form with r positive and s negative eigenvalues which we denote by $Q_{r,s} := Q$ and the corresponding Clifford algebra by $Cl_{r,s} := Cl(V)$. For $s = 0$ we further simplify the notation and just write Cl_n. We use similar notations for O, SO, Pin and Spin. There is one more interesting result. Let's consider the kernels of the homomorphisms in (2.12) and (2.13). It turns out that in the real case we get the following exact sequences

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}_{r,s} \xrightarrow{\tilde{Ad}} \mathrm{SO}_{r,s} \longrightarrow 1 \tag{2.14}$$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_{r,s} \xrightarrow{\tilde{Ad}} \mathrm{O}_{r,s} \longrightarrow 1 \tag{2.15}$$

Especially interesting is the case when $r = n$, $s = 0$. The second sequence is then the universal covering map of SO_n by Spin_n.

2.1.2 The Classification

The classification of the Clifford algebras $Cl_{r,s}$ is a very important task in the theory of Clifford algebras. However we don't want to discuss it in detail since it is treated in every standard textbook on the subject (e.g. [6]). Nevertheless we will state some results to make the text self-contained. The examples in low dimensions can be calculated directly from the definition, after some simple calculations one finds

$$
\begin{aligned}
&Cl_{1,0} \cong \mathbb{C} && Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R} \\
&Cl_{2,0} \cong \mathbb{H} && Cl_{0,2} \cong \mathbb{R}(2) = \mathrm{Mat}(2, \mathbb{R}) \\
&Cl_{1,1} \cong \mathbb{R}(2) &&
\end{aligned}
$$

The following identities are then the crucial tool in order to determine the Clifford algebras in higher dimensions:

$$
\begin{aligned}
Cl_{n,0} \otimes Cl_{0,2} &\cong Cl_{0,n+2} \\
Cl_{0,n} \otimes Cl_{2,0} &\cong Cl_{n+2,0} \\
Cl_{r,s} \otimes Cl_{1,1} &\cong Cl_{r+1,s+1}
\end{aligned} \tag{2.16}
$$

One also should recall some simple rules for the tensor product of $\mathbb{R}$-algebras. The ones useful for our purpose are (with all tensor products in this list over $\mathbb{R}$)

$$
\begin{aligned}
&\mathbb{R}(n) \otimes \mathbb{R}(m) \cong \mathbb{R}(nm) \\
&\mathbb{R}(n) \otimes k \cong k(n) \quad \text{for } k = \mathbb{C}, \mathbb{H} \\
&\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \\
&\mathbb{C} \otimes \mathbb{H} \cong \mathbb{C}(2) \\
&\mathbb{H} \otimes \mathbb{H} \cong \mathbb{R}(4)
\end{aligned}
$$

The universal property implies that for there is following relation between real and complex Clifford algebras:

$$
Cl_{r,s} \otimes \mathbb{C} \cong Cl(\mathbb{C}^{r+s}, Q \otimes \mathbb{C}) \tag{2.17}
$$

Since over $\mathbb{C}$ all quadratic forms Q are equivalent this must be isomorphic to the unique complex Clifford algebra that we will denote $\mathbb{C}l_n$. One can then prove the following isomorphisms

$$
Cl_{n+8,0} \cong Cl_{n,0} \otimes Cl_{8,0}
$$

$$Cl_{0,n+8} \cong Cl_{0,n} \otimes Cl_{0,8}$$
$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2$$

From this isomorphisms we see, that there is a periodicity modulo 8 in the real and modulo 2 in the complex case. From the following table one can deduce all possible Clifford algebras (using (2.16) and the fact that $Cl_{1,1} \cong \mathbb{R}(2)$).

n		1	2	3	4	5	6	7	8
$Cl_{n,0}$		$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$Cl_{0,n}$		$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$\mathbb{C}l_n$		$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

Tab. 1: Clifford algebras with pure signature

2.2 Clifford Modules

The reason why we are interested in Clifford algebras is that their representations yield representations of the contained Pin and Spin groups by restriction. The representation theory of Clifford algebras was first systematically studied in [7]. We start our brief overview of the topic with the following definition:

(2.18) Definition: Let A be a real associative algebra and let $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. A K-representation of the A is an $\mathbb{R}$-linear homomorphism

$$\rho : A \longrightarrow \mathrm{End}_K(W) \qquad \textbf{(2.19)}$$

for some K-vector space W. In the case $A = Cl(V,Q)$ we call W a $Cl(V,Q)$-module or just *Clifford module* and write $\phi \cdot w := \rho(\phi)(w)$. This product will be called Clifford multiplication.
Two K-representations $\rho : A \to \mathrm{End}_K(E)$ and $\rho' : A \longrightarrow \mathrm{End}_K(E')$ are equivalent if there is a K-linear isomorphism $F : E \to E'$ such that $F \circ \rho(a) \circ F^{-1} = \rho'(a)$ for all $a \in A$.

As for group representations there is an equivalent notion of irreducible K-representations, i.e. one that cannot be decomposed into a non-trivial direct sum of other K-representations. Of course we are interested in the case $A = Cl(V,Q)$ and define what we mean by spinor and pinor representations.

(2.20) Definition: A *pinor representation* of $\mathrm{Pin}(V,Q)$ is the restriction of an irreducible representation of $Cl(V,Q)$.
A *spinor representation* of $\mathrm{Pin}(V,Q)$ is the restriction of an irreducible representation of $Cl^0(V,Q)$.

The pinor and spinor representations are irreducible group representations. From the classification of Clifford algebras we know that they are all of the form $K(2^m)$ or $K(2^m) \oplus K(2^m)$ for $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Thus we can apply the following theorem:

(2.21) Theorem: *The natural representation ρ of $K(n)$ on the vector space K^n is the only irreducible real representation of $K(n)$ up to equivalence.*
The algebra $K(n) \oplus K(n)$ has exactly two equivalence classes of irreducible real representations. They are given by $\rho_1(v_1, v_2) := \rho(v_1)$ and $\rho_2(v_1, v_2) := \rho(v_2)$ acting on K^n.

In the following table we present the Clifford representations for Euclidean signature. By ν_n we denote the number of inequivalent irreducible representations of Cl_n, by d_n the real dimension of these representations and by K_n the field that underlies the representation space. For $\mathbb{C}l_n$ we denote the number of inequivalent irreducible representations by $\nu_n^{\mathbb{C}}$ and by d_n their complex dimension. Note that $K_n^{\mathbb{C}}$ is $\mathbb{C}$ for all n, so we omit it in the table

n		1	2	3	4	5	6	7	8
Cl_n		$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
ν_n		1	1	2	1	1	1	2	1
d_n		2	4	4	8	8	8	8	16
K_n		$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{C}l_n$		$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$
$\nu_n^{\mathbb{C}}$		2	1	2	1	2	1	2	1
$d_n^{\mathbb{C}}$		1	2	2	4	4	8	8	16

Tab. 2: Clifford Modules over Cl_n and $\mathbb{C}l_n$

We state another important fact about Clifford algebras:

$$Cl_{r,s} \cong Cl^0_{r+1,s} \qquad \textbf{(2.22)}$$

hence, since $\mathrm{Spin}_{r,s} \subseteq Cl^0_{r,s} \subseteq \mathbb{C}l^0_{r+s}$, the relevant representations for constructing Spin representations are the irreducible representations of $Cl_{r-1,s}$ and $\mathbb{C}l_{r+s-1}$.
The interesting question now is, when an irreducible representation of the Clifford algebra restricts to an irreducible representation of the corresponding Spin group. We consider the two cases of real and complex spinor representation for pure signature. The other cases are similar using (2.16). Consider the homomorphism

$$\Delta_n : \mathrm{Spin}_n \longrightarrow \mathrm{GL}(S) \qquad \textbf{(2.23)}$$

given by restriction of an irreducible real $\mathbb{R}$-representation of Cl_n.

(2.24) Theorem: *When $n \equiv 3 \bmod 4$ then Δ_n is independent of which irreducible representation of Cl_n is used. For $n \equiv 1{,}2 \bmod 4$, Δ_n is the direct sum of two equivalent spin representations. For the remaining case $n \equiv 0 \bmod 4$, we find that*

$$\Delta_{4m} = \Delta_{4m}^{+} \oplus \Delta_{4m}^{-} \tag{2.25}$$

where Δ_{4m}^{+} and Δ_{4m}^{-} are inequivalent irreducible representation of Spin_{4m}.

Next consider

$$\Delta_n^{\mathbb{C}} : \mathrm{Spin}_n \longrightarrow \mathrm{GL}_{\mathbb{C}}(S) \tag{2.26}$$

given by restriction of an irreducible real $\mathbb{C}$-representation of $\mathbb{C}l_n$.

(2.27) Theorem: *When n is odd, the definition of $\Delta_n^{\mathbb{C}}$ is independent of which irreducible representation of $\mathbb{C}l_n$ is used and is itself irreducible. When n is even, there is a decomposition*

$$\Delta_n^{\mathbb{C}} = \Delta_{2m}^{\mathbb{C}+} \oplus \Delta_{2m}^{\mathbb{C}-} \tag{2.28}$$

into a direct sum of two inequivalent irreducible complex representations of Spin_7.

To finish the discussion about the representation theory of the Clifford algebra we want to introduce an explicit description of $\mathbb{C}$-representations of Cl_n. Therefore we need to introduce the contraction. For $v \in \mathbb{R}^n$, this is a linear map $(v \llcorner) : \bigwedge^p \mathbb{R}^n \to \bigwedge^{p-1} \mathbb{R}^n$ given on simple vectors by

$$v \llcorner (v_1 \wedge ... \wedge v_p) := \sum_{i=1}^{p} (-1)^{i+1} \langle v_i, v \rangle v_1 \wedge ... \wedge \hat{v}_i \wedge ... \wedge v_p \tag{2.29}$$

where $(\hat{\cdot})$ means that we leave out this entry. The contraction has the following properties

$$v \llcorner (\varphi \wedge \psi) = (v \llcorner \varphi) \wedge \psi + \varphi \wedge (v \llcorner \psi) \tag{2.30}$$

$$v \llcorner (v \llcorner \psi) = 0 \tag{2.31}$$

for all $\varphi \in \bigwedge^p \mathbb{R}^n$, $\psi \in \bigwedge^\bullet \mathbb{R}^n$. From this one deduces that the contraction extends to a bilinear map $\bigwedge^\bullet \mathbb{R}^n \times \bigwedge^\bullet \mathbb{R}^n \to \bigwedge^\bullet \mathbb{R}^n$. With this, the Clifford multiplication can be described on $\bigwedge^\bullet \mathbb{R}^n \cong Cl_n$. For $v \in \mathbb{R}^2$ and $\varphi \in Cl_n \cong \bigwedge^\bullet \mathbb{R}^n$ it is

$$v \cdot \phi \cong v \wedge \varphi - v \llcorner \varphi \tag{2.32}$$

We now return to the representation theory of Clifford algebras. It is possible to introduce an inner product on a real Clifford representation W with the property

$$\langle e \cdot w, e \cdot w' \rangle = \langle w, w' \rangle \tag{2.33}$$

for all $w, w' \in W$ and $e \in \mathbb{R}^n$ with $\|e\| = 1$ which means that the inner product is invariant under the action of Pin_n on Cl_n and hence also under Spin_n.
Consider now the standard hermitian metric $(\cdot, \cdot)$ on $\mathbb{C}^n$. Just as in the case of a real vector space with inner product discussed above, one can define the contraction

$$(v \llcorner\,) : \bigwedge\nolimits_{\mathbb{C}}^{p} \mathbb{C}^n \longrightarrow \bigwedge\nolimits_{\mathbb{C}}^{p-1} \mathbb{C}^n$$

and gets a map $f_v : \wedge_{\mathbb{C}}^{\bullet} \mathbb{C}^n \to \wedge_{\mathbb{C}}^{\bullet} \mathbb{C}^n$ by setting

$$f_v(\varphi) = v \wedge \varphi - v \llcorner \varphi \tag{2.34}$$

From (2.30) and (2.31) and $v \llcorner v = \|v\|^2$ it follows that

$$f_v \circ f_v(\varphi) = -\|v\|^2 \varphi \tag{2.35}$$

Since $(\cdot, \cdot)$ is only sesquilinear, the map $v \mapsto f_v$ is only $\mathbb{R}$-linear but not $\mathbb{C}$-linear. We can however identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The universal property (2.3) tells us that this map extends uniquely to a representation

$$f : Cl_{2n} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\wedge_{\mathbb{C}}^{\bullet} \mathbb{C}^n, \wedge_{\mathbb{C}}^{\bullet} \mathbb{C}^n) \tag{2.36}$$

Since the dimension is 2^n this is the irreducible representation of the Clifford algebra.

2.3 Spinor Bundles

We now arrived at the point where differential geometry gets involved. What we want to do is to construct vector bundles with an action of a Spin group, such that this action restricts to a representation of the Spin group on each fiber. This is not possible on all manifolds but only on manifolds that carry a Spin structure. To define Spin structures we use the bundle of orthonormal frames, i.e. let $\pi : E \longrightarrow X$ be an oriented vector bundle and X be an oriented manifold. $P_{\mathrm{SO}}(E)$ denotes the principal SO_n-bundle with fiber $P_{\mathrm{SO}}(E)_x = V_n(E_x) \cong \mathrm{SO}_n$ where $V_n(E_x)$ denotes the Stiefel manifold of orthonormal frames (the reduction of the structure group to SO_n is equivalent to the choice of a metric on each fiber).
The morphism $\widetilde{Ad}$ in (2.14) gives a 2-fold covering map which is the universal cover

homomorphism if $n \geq 3$:

$$\xi_0 : \mathrm{Spin}_n \longrightarrow \mathrm{SO}_n \tag{2.37}$$

This then leads us to the following definition:

(2.38) Definition: A *Spin structure* on E is a principal Spin_n-bundle $P_{\mathrm{Spin}}(E)$ together with a bundle morphism $\xi : P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$ which restricts fiberwise to the covering homomorphism ξ_0. In particular $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{\mathrm{Spin}}(E)$ and $g \in \mathrm{Spin}_n$.

The fibration $SO_n \to P_{\mathrm{SO}}(E) \to X$ gives a long exact sequence of homology groups

$$0 \to H^1(X;\mathbb{Z}_2) \to H^1(P_{SO}(E);\mathbb{Z}_2) \to H^1(\mathrm{SO}_n;\mathbb{Z}_2) \overset{\mathrm{w}_E}{\to} H^2(X;\mathbb{Z}_2) \to \ldots \tag{2.39}$$

The group $H^2(\mathrm{SO}_n;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$ and hence has a unique generator g_2. The image $\mathrm{w}_2(E) := \mathrm{w}_E(g_2) \in H^2(X;\mathbb{Z}_2)$ is the second Stiefel-Whitney class of the oriented bundle E. The existence of a Spin structure on E is equivalent to the vanishing of the second Stiefel-Whitney class of E.
The next step is to define spinor bundles. They will be constructed as vector bundles associated to a principal Spin_n-bundle. We will briefly sketch the construction.
Let $\pi : P \to X$ be a principal G-bundle. Further take a vector space V and a representation $\rho : G \to \mathrm{GL}(V)$. Consider now the free left action of G on $P \times V$ given by $\phi_g(p,v) = (pv^{-1}, \rho(g)v)$ for $g \in G$, $p \in P$ and $v \in V$. With that we define

$$P \times_\rho V := P \times V / G \tag{2.40}$$

Since the projection $P \times V \to P \to X$ is G-invariant this again is a fiber bundle over X. The fiber is V and hence the construction yields a vector bundle. This construction is dual to the frame bundle construction. Thus whether one studies vector bundles or principal GL_n-bundles is just a different point of view of essentially the same underlying theory.

(2.41) Definition: The *Clifford bundle* of the oriented Riemannian vector bundle E is the bundle

$$Cl(E) = P_{SO}(E) \times_{\rho_n} Cl(\mathbb{R}^n) \tag{2.42}$$

associated to the representation

$$\rho_n : \mathrm{SO}_n \longrightarrow \mathrm{Aut}(Cl(\mathbb{R}^n)) \tag{2.43}$$

This is the extension of the standard representation of SO_n on $\mathbb{R}^n$ to the Clifford algebra.

The Clifford bundle is a bundle of algebras over X. As usual for vector bundles all properties carry over from the linear algebra setting to the bundle setting, e.g. we get a decomposition

$$Cl(E) = Cl^0(E) \oplus Cl^1(E) \tag{2.44}$$

We finally reached the point that we are primarily interested in, i.e. we can define bundles of modules over the bundle of Clifford algebras $Cl(E)$.

(2.45) Definition: Let E be an oriented Riemannian vector bundle with a spin structure $\xi : P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$. A real *spinor bundle* of E is a bundle of the form

$$S(E) = P_{\mathrm{Spin}}(E) \times_\mu M, \tag{2.46}$$

where M is a left module for $Cl(\mathbb{R}^n)$ and $\mu : \mathrm{Spin}_n \to \mathrm{SO}(M)$ is the representation given by left multiplication of $\mathrm{Spin}_n \subseteq Cl^0(\mathbb{R}^n)$.
For a complex spinor bundle one replaces the representation M of $Cl(\mathbb{R}^n)$ by a complex representation $M_{\mathbb{C}}$ of $Cl(\mathbb{R}^n) \otimes \mathbb{C}$.

Since the fiber over every point is M, we get a representation of the Clifford algebra on the corresponding spinor bundle. This extends to an action of the Clifford bundle on the corresponding spinor bundle:

$$\mu : Cl(E) \oplus S(E) \longrightarrow S(E) \tag{2.47}$$

2.3.1 The Spin Connection

In the next section we want to construct calibrations from parallel spinors on a spin manifold. However it is not yet clear what we mean by parallel. To understand this we need to study the concept of a connection on a principal bundle.
Let again $\pi : P \to X$ be a smooth principal bundle with structure group G where G is a Lie group with Lie algebra $\mathfrak{g}$. By definition $\mathfrak{g} \cong V_p$, where V_p denotes the tangent space to the orbit through p, since each orbit is isomorphic to G. The isomorphism $\sigma : \mathfrak{g} \to V_p$ is given for $V \in \mathfrak{g}$ by

$$\sigma(V) = \mathrm{d}/\mathrm{d}t(p \cdot exp(tV))|_{t=0} \tag{2.48}$$

Since the V_p are tangent to the fibers one should think of them as being vertical to $\mathrm{T}_pX \subseteq \mathrm{T}_pP$. It is a prior not clear however what the horizontal directions are. This is what a connection provides

(2.49) Definition: A *connection* on P is a G-invariant field of tangent n-planes, i.e. a global section τ of the bundle $\mathrm{G}(n, \mathrm{T}P)$ such that $(R_g)_*\tau_p = \tau_{pg}$, where R_g acts by multiplication with g from the right, and such that the linear map $\pi_* : \tau_p \longrightarrow \mathrm{T}_{\pi(p)}X$ is an isomorphism for all $p \in P$. In particular, we have

$$\mathrm{T}_pP = V_p \oplus \tau_p \tag{2.50}$$

This decomposition comes together with linear projections onto V_p and τ_p. Further we have the isomorphism $\sigma : V_p \to \mathfrak{g}$. Composing these maps gives

$$\omega_p : \mathrm{T}_p P \longrightarrow \mathfrak{g} \tag{2.51}$$

This is a $\mathfrak{g}$-valued 1-form on P, called the *connection 1-form*. The data of this 1-form is equivalent to the connection, since $\tau_p = \ker(\omega_p)$. The *curvature* of the connection is the the $\mathfrak{g}$-valued 2-form ω given by the equation

$$\omega = \mathrm{d}\omega + [\omega, \omega] \tag{2.52}$$

with $[\omega, \omega](v, w) = [\omega(v), \omega(w)]$. On vector bundles we can also define what we mean by a connection.

(2.53) Definition: A *connection* or *covariant derivative* on a smooth vector bundle $E \to X$ is linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E) \tag{2.54}$$

such that

$$\nabla(fe) = df \otimes e + f\,\nabla e \tag{2.55}$$

for all $f \in C^\infty(X)$ and $e \in \Gamma(E)$. $\Gamma(E')$ denotes the smooth sections of any smooth vector bundle E'.

Every connection 1-form ω on $P_{\mathrm{SO}}(E)$ determines an unique covariant derivative on E by

$$\nabla e_i = \sum_j \tilde{\omega}_{ij} \otimes e_j \tag{2.56}$$

where $\tilde{\omega} = \mathscr{E}^*\omega$ with $\mathscr{E}$ being a local section of $P_{\mathrm{SO}}(E)$. The covariant derivative then satisfies

$$V\langle e, e'\rangle = \langle \nabla_V e, e'\rangle + \langle e, \nabla_V e'\rangle \tag{2.57}$$

for all $V \in \mathrm{T}X$ and $e, e' \in \Gamma(E)$. The formula (2.56) also shows that a Riemannian connection on E determines a unique connection 1-form on $P_{\mathrm{SO}}(E)$. We see once more that a vector bundle and its frame bundle are equivalent.
A connection on a principal G-bundle P can always be extended to any vector bundle associated to P by a representation $\rho : G \longrightarrow \mathrm{SO}_n$ by extending the connection trivially on $P \times \mathrm{SO}_n$ and then push it forward to the quotient $P \times_\rho \mathrm{SO}_n = P_{\mathrm{SO}}(P \times_\rho \mathbb{R}^n)$. This then induces a connection on $P \times_\rho \mathbb{R}^n$.
Let us now assume that E carries a Spin structure. Then a connection 1-form on $P_{\mathrm{SO}}(E)$ pulls back via the bundle homomorphism $\xi : P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$ to a connection 1-form on $P_{\mathrm{Spin}}(E)$. In particular we get a connection on every spinor bundle since the

connection on $P_{\text{Spin}}(E)$ induces a connection on $S(M) = P_{\text{Spin}}(E) \times_\mu M$, where M is a Clifford module and $\mu : \text{Spin}_n \longrightarrow \text{SO}(M)$ the corresponding spin representation. We also get an induced connection on the Clifford bundle which satisfies $\nabla(\phi \cdot \psi) = (\nabla\phi) \cdot \psi + \phi \cdot (\nabla\psi)$ for any two sections ϕ, ψ of $Cl(E)$. The connection on $S(E)$ then acts like

$$\nabla(\phi \cdot \sigma) = (\nabla\phi) \cdot \sigma + \phi \cdot (\nabla\sigma) \tag{2.58}$$

for all $\phi \in \Gamma(Cl(E))$ and $\sigma \in \Gamma(S(E))$.
In particular the Levi-Civita connection on a Riemannian manifold with spin structure induces a connection on the spinor bundle $S(TM) := S(X)$ on that manifold. We will refer to this as the Spin connection of the spinor bundle.
We call a section $\sigma \in \Gamma(S(X))$ *parallel* or *covariantly constant* if $\nabla\sigma = 0$. One can show that manifolds that admit a parallel spinor σ are Ricci-flat by applying the spin connection twice to the covariantly constant spinor.

3 Manifolds with special Holonomy and Calibrations

3.1 Riemannian Holonomy

The setting where the previously discussed concepts come together are manifolds with special holonomy. First we introduce the concept of holonomy. Let X be an connected complete n-dimensional manifold. We can consider a piecewise smooth loop $\gamma : [0,1] \to X$ in X with endpoint $x \in X$. Parallel transport (with respect to the Levi-Civita connection) along this loop defines a map $h_\gamma : T_xX \longrightarrow T_xX$. These transformations form a group $\mathscr{H}_x$ since composing with the reveresed path, i.e. $\gamma^{-1}(t) = \gamma(1-t)$, provides the inverse. Since parallel transport acts as an isometry on T_xX the holonomy group $\mathscr{H}_x$ must lie inside of O_n. The holonomy groups at different points are related by conjugation and hence are isomorphic. Thus the definition as an abstract subgroup is independent of the base point. We call $\mathscr{H}(X)$ the *holonomy group* of X. The inclusion $\mathscr{H}(X) \hookrightarrow \text{O}_n$ is the so called *holonomy representation*. One should note that the holonomy is really a property of the connection, however we will only consider the Levi-Civita connection on a manifold.
An alternative way to think about holonomy is by considering the frame bundle $P_O(X)$ with the induced connection τ_p. It is true that a path γ ,which does not need to be closed, lifts to a unique path $\widetilde{\gamma}$ in $P_{\text{O}}(X)$ with $\dot{\widetilde{\gamma}} \in \tau_b riefly\gamma(t)$ for all t, once we specify $\widetilde{\gamma}(0) = p$. Such a path is called horizontal with respect to the connection on $P_O(X)$. If we now take the path to be closed in X the endpoint of the path lies in the same fiber, however in general it does not equal p but rather $g \cdot p$ for some $g \in O_n$. This defines an

equivalence relation and we define

$$\mathscr{H}'_p(X) = \{\, g \in G \,|\, p \sim p \cdot g\} \tag{3.1}$$

This group is isomorphic to the holonomy group of the corresponding covariant derivative. By abuse of notation we also denote it as $\mathscr{H}_p(X)$. Since for oriented manifold we can reduce the structure group of the frame bundle to SO_n, the holonomy group should actually lie in SO_n. Now, one can consider a special subbundle of $P_{\mathrm{SO}}(X)$, the so called *holonomy bundle*. We fix a point $p \in P$. Let $H(p)$ be the set of points in $P_{\mathrm{O}}(X)$ that can be joined to p by a horizontal path in $P_{\mathrm{O}}(X)$. This is a principal bundle with structure group $\mathscr{H}_p(X)$.
Since the holonomy group of the product of manifolds $X = Y \times Z$ decomposes into $\mathscr{H}(X) = \mathscr{H}(Y) \times \mathscr{H}(Z)$ we only consider manifolds that are not a Riemannian product. Such manifolds are called irreducible. There is a complete classification of the possible holonomy groups for irreducible oriented Riemannian manifolds due to Berger [2]. We summarize the results in the following table:

$\mathscr{H}(X)$	$\dim(X)$	Geometry
SO_n	n	Generic
U_m	2m	Kähler
SU_m	2m	Calabi-Yau
$\mathrm{Sp}_1 \cdot \mathrm{Sp}_m$	4m	Quarternionic-Kähler
Sp_m	4m	Hyperkähler
G_2	7	Exceptional
Spin_7	8	Exceptional

Tab. 3: List of possible Riemannian holonomy groups

3.2 Calibrations on Manifolds with special Holonomy

It might be interesting to have a closer look at which groups appear in this classification. Then one should note that the groups are in some way related to isometries on division algebras over $\mathbb{R}$. Particularly interesting is that also that the groups G_2 and Spin_7 appear which, as we have seen before, are connected to the octonions. There, G_2 appeared as the group that fixes the associative and coassociative calibrations on $\mathbb{R}_7$. So one can suspect that G_2 holonomy means that the tangent spaces carry in some sense the structure of $\mathrm{Im}\,\mathbb{O}$. Let X be an oriented Riemannian manifold with $\mathscr{H}(X) = G_2$. Then we can consider the associative form φ by choosing a local frame at $p \in X$ and defining it locally by (1.33). We can pull back the form at the point p by an element of the holonomy group. Since the form is fixed by G_2 acting on its entries,

it is parallel and hence we can extend it to the whole manifold by parallel transport. Hence we get a globally defined parallel form. This then implies that the form is closed since $(\nabla_k \omega)(\partial_i) - (\nabla_i \omega)(\partial_k) = \mathrm{d}\omega(\partial_k, \partial_i)$ for any one-form ω. Hence φ extends to a calibration on X which we also denote by φ and call it a *calibration of associative type*. Note that the construction is not unique since we could choose a different frame and possibly get another form. Similarly, one can consider the form Φ on an 8-dimensional manifold with Spin_7-holonomy to obtain *calibrations of Cayley type*.
For manifolds with SU_n-holonomy consider $A \in \mathrm{SU}_n \subseteq \mathrm{SO}_{2n}$. Then we find that $\mathrm{d}z(A\xi) = \det_{\mathbb{C}}(A)\mathrm{d}z(\xi) = \mathrm{d}z(\xi)$ for $\xi \in \mathrm{G}(n, 2n)$ and $\mathrm{d}z$ the complex volume form on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. This shows that $\mathrm{d}z$ is invariant under SU_n and hence defines a closed form Ω_n on manifolds with holonomy group SU_n. The existence of this form is a key feature of Calabi-Yau manifolds. The Kähler-form ω is invariant under U_n and defines a closed form on manifolds with holonomy U_n and hence U_n-manifolds are actually Kähler. Conversely, given a closed Kähler form on X we know that the holonomy group acts on it by pullback. Since the form is closed the holonomy group must preserve it and we find that $\mathscr{H}(X) \subseteq \mathrm{U}_n$.

3.3 Parallel Spinors

Since we are actually interested in spin manifolds equipped with a spin connection, the classification of Riemannian holonomy groups in table 3 is not exactly the tool that we need. The following proposition from [8] will solve this problem.

(3.2) Proposition: *Let X be a Riemannian manifold and suppose that there exists an embedding* $\Phi : \mathscr{H}(X) \to \mathrm{Spin}_m$ *such that the following diagram commutes:*

$$\begin{tikzcd} & \mathrm{Spin}_n \arrow[d] \\ \mathscr{H}(X) \arrow[ur, "\Phi"] \arrow[r, "i"] & \mathrm{SO}_n \end{tikzcd} \qquad \textbf{(3.3)}$$

The X carries a spin structure with holonomy group $\Phi(\mathscr{H}(X)) \cong \mathscr{H}(X)$.

Proof. With $\xi_0 : \mathrm{Spin}_n \to \mathrm{SO}_n$, the covering map from (2.37), we have that $\xi_0 \circ \Phi = i$. Fix $u \in P_{\mathrm{SO}}(X)$. Above we defined the holonomy bundle $H(p)$. We get a canonical bundle isomorphism $H(p) \times_i \mathrm{SO}_n \cong P_{\mathrm{SO}}(X)$ and a canonical projection from $H(p) \times_\Phi \mathrm{Spin}_n$ onto $H(p) \times_i SO_n$ since $(pg^{-1}, \xi_0(\Phi(g))\xi_0(S)) = (pg^{-1}, i(g)\xi_0(s))$ for all $(p, s) \in H(p) \times \mathrm{Spin}_n$ and $g \in \mathscr{H}(X)$. It is clear that this defines a spin structure on X. The spin connection comes from the restriction of the Levi-Civita connection of X. This implies that the holonomy of the spin connection is given by $\Phi(\mathscr{H}(X))$. $\square$

Hence the classification in table 3 is essentially still true for the case of spin connections. The groups from table 3 for which such an embedding into Spin_n exist for some $n = 2m$ are SU_m, Sp_m, $\mathrm{G}_2 \subseteq \mathrm{Spin}_7$ and $\mathrm{Spin}_7 \subseteq \mathrm{Spin}_8$. This is a result by Wang in [9]. We now to want study parallel spinors on manifolds with special holonomy. Recall that a spinor $\sigma \in \Gamma(S(X))$ is called parallel if $\nabla\sigma$ vanishes. We first prove the following theorem

(3.4) Theorem: *Let X be an n-dimensional spin manifold on which there exists a globally parallel spinor σ. Then at any point x the holonomy group satisfies*

$$\mathscr{H}_x \subseteq \mathrm{G}_{\sigma(x)} \tag{3.5}$$

Here $G_{\sigma(x)} := \{\, g \in \mathrm{Spin}(T_x X) : g\sigma_x = \sigma_x \}$ *is the stabilizer subgroup of* $\mathrm{Spin}(T_x X)$ *with respect to* σ_x*. Conversely, if* (3.5) *is satisfied for some spinor* σ_x *at some point x, then* σ_x *extends to a globally parallel field* σ*.*

Proof. The argument is the same as above when we discussed how one can extend calibrations to manifolds. If one has a parallel spinor σ then the holonomy group must act trivially on it at every point, hence $\mathscr{H}_x \subseteq \mathrm{G}_{\sigma_x}$. Conversely just as in the case of forms one can extend an invariant spinor to a parallel spinor on the manifold via parallel transport. □

In the case of G_2-manifolds we can see the existence of a parallel spinor explicitly. Recall from table 2 that the irreducible representation of Spin_7 has dimension 8. As we have seen in section 1.4.4, Spin_7 is a subgroup of SO_8 and hence acts on S^7. One can check that this action is transitive. Note that $\mathrm{G}_2 \subseteq \mathrm{SO}_7 \subseteq \mathrm{SO}_8$ and from (1.54) one can then see that $\mathrm{G}_2 \subseteq \mathrm{Spin}_7$. It is also clear that G_2 preserves $1 \in S^7$. One can show that this already is the full stabilizer subgroup and hence $S^7 \cong \mathrm{Spin}_7/\mathrm{G}_2$ and hence all non-zero spinors are essentially equivalent. Let X be a spin manifold with holonomy G_2. Now take a spinor in the irreducible representation of Spin_7 on $S(\mathrm{T}_x X)$. Then any spinor $\sigma_x \in S(\mathrm{T}_x X)$ is fixed by a G_2 subgroup of Spin_7 and hence $\mathscr{H}_x \cong G_{\sigma_x}$. By (3.4) one can extend this spinor to a global parallel spinor on X.
The following proposition provide the structure of the space of parallel spinors on manifolds with special holonomy:

(3.6) Proposition: *Let (M,g) be a complete, simply connected, irreducible Riemannian spin manifold of dimension n. Let N denote the dimension of the space of parallel spinors. Then*

(a) If $n = 2m$, $m \geq 2$ *and* $\mathscr{H}(X) = \mathrm{SU}_m$*, then* $N = 2$

(b) If $n = 4m$, $m \geq 2$ *and* $\mathscr{H}(X) = \mathrm{Sp}_m$*, then* $N = m + 1$

(c) If $n = 8$ *and* $\mathscr{H}(X) = \mathrm{Spin}_7$*, then* $N = 1$

(d) If $n = 7$ *and* $\mathscr{H}(X) = \mathrm{G}_2$*, then* $N = 1$

This result was proved by Wang in [9]. His proof also gives an explicit expression for the parallel spinors in terms of the representation (2.36).
We give the idea of the proof for the case (a). The holonomy representation of SU_m is just the vector representation μ_m on $\mathbb{C}^m$. We can lift that by an embedding $\mathrm{SU}_m \longrightarrow \mathrm{Spin}_{2m}$ as in (3.2). Consider then the spin representation $\Delta_{2m} : \mathrm{Spin}_{2m} \to \mathrm{Hom}_{\mathbb{C}}(\wedge^{\bullet}_{\mathbb{C}}\mathbb{C}^m, \wedge^{\bullet}_{\mathbb{C}}\mathbb{C}^m)$ induced from (2.36). The restriction of this representation to SU_m is

$$\Delta_{2m}|_{\mathrm{SU}_m} = \wedge^m \mu_m \oplus \wedge^{m-1}\mu_m \oplus ... \oplus \wedge^0 \mu_m \tag{3.7}$$

Parallel spinors correspond to the trivial representation in this composition. But these are just $\wedge^m \mu_m$ and $\wedge^0 \mu_m$. Hence the space of parallel spinors has dimension 2 and $1 \in \bigwedge^0 \mathbb{C}^m$ and $e_1 \wedge ... \wedge e_m \in \bigwedge^m \mathbb{C}^m$ form a basis.

3.4 Squaring parallel Spinors

We are now in the position to describe the construction of calibrations from parallel spinors. We will follow [9] here. In the context of calibrations this was first discussed in [10] and [11]. For the first construction of calibrations we heavily rely on the irreducible Clifford representation (2.36). Recall that $\mathbb{C}$ carries a hermitian metric $\langle \cdot, \cdot \rangle$ which we assume to be $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second argument. We can extend it to $\wedge^{\bullet}_{\mathbb{C}}\mathbb{C}^n$. The contraction $(v \llcorner)$ is then just the hermitian adjoint of $(v \wedge)$.
Let $(e_1, ..., e_n)$ be the standard basis of $\mathbb{C}^n$ and define $e_{m+i} := J(e_{m+i})$. Then $(e_1, ..., e_{2m})$ is a basis of the real vector space $\mathbb{R}^{2m} \cong \mathbb{C}^m$. We define a map $\lambda : \wedge^{\bullet}_{\mathbb{C}}\mathbb{C}^m \to \wedge^{\bullet}_{\mathbb{C}}\mathbb{C}^m$ by $\lambda = \mathrm{f}_{e_1} \circ ... \circ \mathrm{f}_{e_m}$. Given two spinors σ_1, σ_2 in $\wedge^{\bullet}_{\mathbb{C}}\mathbb{C}^m$ we can consider forms defined by:

$$\eta_p(X_1, ..., X_p) := \langle \lambda(\overline{\sigma_1}), (X_1 \wedge ... \wedge X_p) \cdot \sigma_2 \rangle \tag{3.8}$$

where $X_i \in \mathbb{R}^{2m}$ and we use the isomorphism (2.2) where the algebra structure translates like in (2.32). The action is then given by (2.36). Note that we can do this construction with parallel spinors on a spin manifold evaluated at a point. The form then is invariant under the holonomy action since it is constructed from invariant objects and hence extends to a global parallel form on the manifold. Since it is parallel it is in particular closed.
Now we want to discuss an example for this construction. We will indeed recover a calibration that we introduced in section 1.
Let X be a $2m$-dimensional manifold with SU_m holonomy. We have seen above that the two parallel spinors are given by $\sigma_1 = e_1 \wedge ... \wedge e_n$ and $\sigma_2 = 1$.

Take $1 \leq i_1 < ... < i_p \leq 2m$. Then we consider

$$\begin{aligned}\eta_p(e_{i_1}, ..., e_{i_p}) &= \langle \lambda(\overline{\sigma_1}), (e_{i_1} \wedge ... \wedge e_{i_p}) \cdot \sigma_2 \rangle = \langle (e_1 \wedge ... \wedge e_m)^2, (e_{i_1} \wedge ... \wedge e_{i_p}) \cdot 1 \rangle \\ &= (-1)^{m(m+1)/2} \langle 1, (e_{i_1} \wedge ... \wedge e_{i_p}) \cdot 1 \rangle\end{aligned}$$

This is always 0 for odd p. For $p = 2$ we get

$$\begin{aligned}\eta_2(e_i, e_j) &= (-1)^{m(m+1)/2} \langle 1, (e_i \wedge e_j) \cdot 1 \rangle \\ &= (-1)^{m(m+1)/2} (\langle 1, e_i \cdot e_j \rangle + \langle 1, \mathrm{e}_i \llcorner e_j \rangle \\ &= (-1)^{m(m+1)/2} (-\delta_{ij} + \langle e_i, e_j \rangle)\end{aligned}$$

This is only non-zero if $e_j = \sqrt{-1}e_i = e_{m+i}$. If we set $\omega = (-1)^{m(m+1)/2+1}\sqrt{-1}\eta_2$ and define J by $\omega(X, Y) = \mathrm{g}(JX, Y)$ we thus find that ω is the Kähler form and J the complex structure. Both give rise to parallel forms by the argument given above. For $p > 2$ we will get higher powers of the Kähler form. Next consider

$$\langle \alpha(\overline{\sigma_2}), (X_1 \wedge ... \wedge X_p) \cdot \sigma_2 \rangle = \langle e_1 \wedge ... \wedge e_m, (X_1 \wedge ... \wedge X_p) \cdot 1 \rangle. \tag{3.9}$$

For $p = m$ this is exactly the volume form Ω_m on the Calabi-Yau manifold. From this we recover the special Lagrangian calibration. The formular (3.8) will not reconstruct calibrations in the general case and we need to do some modifications, however this example is useful since the idea of the general construction is similar and one can explicitly calculate how the forms act on vectors.

Now, let S be an irreducible real Cl_n-module where $n \equiv 6$ or 8 (mod 8) and equip S with a Pin_n invariant inner product $\langle \cdot, \cdot \rangle$. Recall the isomorphism $Cl_n \to \mathrm{Hom}(S, S) \cong S \otimes S^*$. We can identify S with S^* via $\langle \cdot, \cdot \rangle$. We then have the inverse isomorphism

$$S \otimes S \longrightarrow Cl_n \tag{3.10}$$

This sends $\sigma_1 \otimes \sigma_2 \in S \otimes S$ on the unique element $\varphi \in Cl_n$ such that $\varphi \cdot \tau = \langle \tau, \sigma_2 \rangle \sigma_1$ for all $\tau \in S$. From $Cl_n \cong \bigwedge^\bullet \mathbb{R}^n$ we get an isomorphism $S \otimes S \to \bigwedge^\bullet \mathbb{R}^n$. The statement is also true for $n \equiv 7$ (mod 8) if one considers one of the two inequivalent representations S^+ or S^-. In the complex case we get for $n = 2m$ an analogous isomorphism

$$S_{\mathbb{C}} \otimes \bar{S}_{\mathbb{C}} \longrightarrow \mathbb{C}l_{2m}. \tag{3.11}$$

for $S_{\mathbb{C}}$ the irreducible representation of $\mathbb{C}l_{2m}$. Take now an irreducible representation S of Cl_8. The so called *volume element* $\omega = e_1...e_8$ acts on spinors by ± 1 since $\omega^2 = 1$. Take a spinor $\sigma \in S$ with $\omega\sigma = \sigma$. Then one finds that

$$\sigma \otimes \sigma = 1 + \Phi + \omega \tag{3.12}$$

where $\Phi \in \bigwedge^4 \mathbb{R}^8$ is the Cayley 4-form. By identifying $\mathbb{R}^8$ with $(\mathbb{R}^8)^*$ and restricting the form to $\bigwedge^4 \mathbb{R}^8$ one recovers the Cayley calibration. For G_2 manifolds one can consider an irreducible representation S^+ (i.e. the irreducible representation on which $\omega = e_1 ... e_7$ acts by $+1$) of Cl_7 , then for any $\sigma \in S$ we get a decomposition

$$1 + \varphi + \psi + \omega \tag{3.13}$$

from the isomorphism (3.10), where $\varphi \in \bigwedge^3 \mathbb{R}^7$ is the associative calibration and $\psi \in \bigwedge^4 \mathbb{R}^7$ is the coassociative calibration which satisfies $\psi = *\varphi$.
In some cases the calibration condition $\varphi(\xi) \leq \mathrm{vol}(\xi)$ for all $\xi \in G(p, n)$ can be checked explicitly. In order to do that we need the explicit form of the isomorphism (3.10). For a fixed non-zero spinor $\sigma \in S$ one can consider the image of $\sigma \otimes \sigma$ under the map

$$S \otimes S \xrightarrow{\cong} \bigwedge^{\bullet} \mathbb{R}^n \xrightarrow{\pi} \bigwedge^{p} \mathbb{R}^n \xrightarrow{\cong} \bigwedge^{p} (\mathbb{R}^n)^* \tag{3.14}$$

where the last isomorphism to $\bigwedge^{\mathrm{p}} (\mathbb{R}^n)^*$ is induced by the bilinear form on the vector space V that underlies the Clifford algebra. This image is the map

$$\bigwedge^{\mathrm{p}} \mathbb{R} \ni e_{i_1} \wedge ... \wedge e_{i_p} \mapsto \langle \sigma, e_{i_1} ... e_{i_k} \cdot \sigma \rangle \in \mathbb{R} \tag{3.15}$$

where $e_1 ... e_n$ are orthonormal in $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ denotes the Pin-invariant inner product on the spinors space S and the dot denotes Clifford multiplication. Choosing $e_1, ..., e_n$ to be orthonormal allows us to identify the wedge product with the Clifford product if all $e_{i_1}, ... e_{i_n}$ are pairwise distinct. In this case we find that $e_{i_1} \wedge ... \wedge e_{i_p} \cong e_{i_1} ... e_{i_p}$. If we take $u = \mathrm{e}_{i_1} ... \mathrm{e}_{i_p} \in Cl_n$ we find that $u^2 = (-1)^{\frac{p(p+1)}{2}}$. We show that for p satisfying $u^2 = 1$ the p-form defined by (3.15) is a calibration. In euclidean signature this is the case exactly for $p \equiv 3, 4 \bmod 4$. Let's consider $\langle (1-u) \cdot \sigma, (1-u) \cdot \sigma \rangle$. By semi-positivity of $\langle \cdot, \cdot \rangle$ we see that this quantity must be greater or equal than 0. Recall now that the inner product is Pin invariant. Since $\|u\| = 1$ and hence $u \in \mathrm{Pin}_n$, we see that $\langle u \cdot \tau, u \cdot \tau' \rangle = \langle \tau, \tau' \rangle$ for all $\tau, \tau' \in S$. This means that $\langle u \cdot \tau, \tau' \rangle = \langle \tau, u \cdot \tau' \rangle$ and hence

$$0 \leq \langle (1-u) \cdot \sigma, (1-u) \cdot \sigma \rangle = \langle \sigma, (1-u)^2 \cdot \sigma \rangle = \langle \sigma, (1-u) \cdot \sigma \rangle \tag{3.16}$$

so that

$$\langle \sigma, \sigma \rangle \geq \langle \sigma, u \cdot \sigma \rangle \tag{3.17}$$

By normalization we can assume that $\langle \sigma, \sigma \rangle = 1$. For an arbitrary simple p-vector u with non-zero norm we can apply the inequality to $u / \|u\|$ which has norm 1. Altogether we get

$$\phi(\tilde{u}) := \langle \sigma, u \cdot \sigma \rangle \leq \|u\| \tag{3.18}$$

where we denote by $\tilde{u} \in \bigwedge^p \mathbb{R}^n$ the element that corresponds to $u \in Cl_n$ under the isomorphism (2.2). The inequality (3.18) is the linear algebra condition for $\phi \in \bigwedge^p (\mathbb{R}^n)^*$ to be a calibration. If we apply this to a parallel spinor on a manifold we get a closed form and hence a calibration on the manifold.
To conclude this section we want to discuss some additional aspects of the isomorphism (3.11) between pairs of spinors and exterior vectors. Recall that $\mathbb{C}l_{2m} \cong \bigwedge^\bullet \mathbb{C}^{2m}$. The exterior product decomposes into a direct sum of homogeneous polyvectors and hence comes with canonical projection maps onto $\bigwedge^p \mathbb{C}^{2n}$.
First consider $p = 0$, we get a map $(\cdot, \cdot) : S \otimes \bar{S} \to \mathbb{C}$ which defines the spin invariant hermitian metric on the space of spinors. In physics literature the matrix C corresponding to this form is usually called charge conjugation matrix.
For $p = 1$ we get a bilinear $\mathrm{Spin}_2 m$-invariant homomorphism $\Gamma : S \otimes \bar{S} \to \mathbb{C}^{2m}$. In a super Poincaré Lie algebra extension of the Poincaré Lie algebra by S this constitutes the odd-odd Lie bracket.

4 Supersymmetric Cycles

The goal of this section to make contact with physics. The standard tool to calculate couplings in string theory is perturbation theory. There are however contributions to the couplings that are nonperturbative. For example, massless states that come from branes wrapping supersymmetric cycles give such contribution to the low-energy effective action. A supersymmetric cycle is a non-trivial homology cycle of the internal manifold such that, wrapping a brane on it, the resulting theory on the non-compact directions is supersymmetric. Thus we want to find BPS branes in the theory, that wrap non-trivial cycles.
We want to study how in D=11 supergravity calibrated submanifolds arise as BPS brane solutions. The steps are the following: First we want to give a brief overview of D=11 supergravity and discuss simple bosonic solutions that preserve some of the supersymmetries. One solution is given by compactification of the 11-dimensional theory on $\mathbb{R}^{1,D-1}$ to a theory on a manifold of the form $\mathbb{R}^{1,10-d} \times M_d$ for a d-dimensional compact manifold M_d with special holonomy. Then we go over to the discussion of branes inside the compactified manifold and show that BPS branes (i.e. branes that preserves some supersymmetry) are calibrated submanifolds with respect to the calibration that corresponds to a parallel spinor. In this section the main references are [12] and [13]. For the original treatment of the connection between BPS branes and calibrations we refer to [14] and [15]

4.1 D=11 Supergravity

We will not do a complete presentation of supergravity but rather merely introduce the basic notions such that the reader gets familiar with the terminology used. We will switch to the physics notation from now on to align with the literature, however we will try to make the correspondence to the previous discussion clear. Supergravity in 11 space-time dimensions was first discussed by Cremmer, Julia and Scherk [16]. For a detailed discussion on supergravity we refer to [17]. We follow the discussion in [13] which will be sufficient for our purpose.

Since supergravity contains gravity it has to contain a graviton that is represented by a vielbein field E_M^A, where $M, N, ..$ are the curved indices on the base space and $A, B, ..$ the flat indices on the tangent space. Since in a suitable coordinate system massless particles can be represented as $(p^\mu, 0...0, \mathrm{p}^\mu)$ we see that the little group (or stabilizer subgroup) is $SO(D-2)$. Massless particles are symmetric, traceless tensors of the little group and hence have $\frac{1}{2}(D-1)(D-2)-1 = 44$ physical degrees of freedom in $D = 11$.

Remark: Since we are about to introduce fermions in the thoery we briefly discuss the physicists' conventions concerning the Clifford algebra and spinors. Physicists are usually interested in quadratic forms for which the corresponding bilinear form has signature $(-1, +1, .., +1)$. The complex irreducible representation of the complex Clifford algebra $\mathbb{C}l_d$ induces a representation of any spin group $\mathrm{Spin}(p, q)$ with $p + q = d$. The induced representation of $\mathrm{Spin}(d-1, 1)$ is often referred to as the Dirac representation. Choosing an orthonormal basis $(e_0, ..., e_{d-1})$ for the vector space $\mathbb{R}^{d-1,1}$ we have elements Γ^μ, $\mu = 0, ..., d-1$ which are the images of the basis vectors under the representation map. We denote anti-symmetric products of gamma matrices as $\Gamma^{\mu\nu} := \Gamma^{[\mu}\Gamma^{\nu]}$. These are the so called gamma matrices. Recall now the decomposition in (2.24). In even dimensions (or more precisely for $(d-1)-1$ even) the Dirac representation of the Spin group splits into two irreducible representations $\Delta^{\mathbb{C}} = \Delta^{\mathbb{C}+} \oplus \Delta^{\mathbb{C}-}$. The representations $\Delta^{\mathbb{C}+}$ and $\Delta^{\mathbb{C}-}$ are called positive and negative chirality Weyl representations. The next question one could ask is, in what dimensions there are "real spinors", i.e. in what dimension there exists an irreducible representation of $Cl_{1,d-1}$ on a real vector space. At this point one has to be careful. Usually the definition for the Gamma anticommutation relation for the Dirac matrices is

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\mathrm{g}_{\mu\nu}. \tag{4.1}$$

This has the opposite sign compared to the mathematicians' definition for the Clifford algebra, hence one has to consider $Cl_{1,d-1}$ and not $Cl_{d-1,1}$ as one would expect. From now on we will follow the physics convention, i.e. when we write $\mathrm{Spin}(p, q)$, we mean

Spin(q, p) in the mathematicians' definition. From table 1 one can see that there are real representations in $d-2=0,1,2 \bmod 8$. These representations are Majorana representations of Spin$(1, d-1)$. In terms of the dirac matrices, this means that one can find a basis in which the Dirac matrices have only real entries. Then it is consistent to restrict the representation space to the real part of the complex representation. For $d=10$ we further see that the real Majorana representation decomposes into two real irreducible representations using (2.24). These are just the restriction of the Weyl representations to the real part of the complex representation. These spinors are called Majorana-Weyl spinors. They exist precisely in $d-2 \equiv 0 \bmod 8$ dimensions. Since they exist in 10 dimensions they are very useful in string theory, since they provide spinors with a quite simple structure. In 11 dimensions there are no Majorana-Weyl spinors, but we still get Majorana spinors.

To get a supersymmetric theory one needs to include a spinor field Ψ_M which is the gauge field for local supersymmetry. For each $M=0,\ldots,11$, Ψ_M is a 32-component Majorana spinor. The little group for a massless d-dimensional spinor is the double cover of SO$(D-2)$. For $D=11$ that is Spin(9). This has a real 16-dimensional representation. Thus we want to consider the vector-spinor representation $\mathbf{9} \otimes \mathbf{16} = \mathbf{128} + \mathbf{16}$. There is a local gauge invariance $\delta\Psi_M = \partial_M \varepsilon$. Using this transformation we can gauge away the degrees of freedom coming from the spinor representation **16** and we are left with 128 physical degrees of freedom. To get an supersymmetric theory we need an equal number of fermions and bosons hence we are missing $128-44=84$ bosonic degrees of freedom. The missing degrees of freedom come from a rank-3 antisymmetric tensor A_{MNP}, which is equivalent to a 3-form A_3. We want this potential to be invariant under the gauge transformation

$$A_3 \mapsto A_3 + \mathrm{d}\Lambda_2 \tag{4.2}$$

These fields transform in the antisymmetric tensor representation of SO$(d-2)$ and hence the dimension is given by $\binom{9}{3} = 84$, such that we get the correct number of propagating bosonic degrees of freedom to match the fermionic degrees of freedom as required for supersymmetry.
The action of D=11 supergravity is strongly constrained by the symmetries it should respect. In fact, the restrictions are so strong that there is a unique supergravity theory in D=11. The bosonic part of the action is given by

$$2\kappa_{11}^2 S = \int \mathrm{d}^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4. \tag{4.3}$$

Here R is the scalar curvature. The first term in the action can be identified as the Einstein-Hilbert term. This is necessary to make the theory a generalization of gen-

eral relativity. $F_4 = \mathrm{d}A_3$ is the 4-form field strength associated with the potential A_3. The constant κ_{11} denotes the 11-dimensional gravitational coupling. The term $\int A_3 \wedge F_4 \wedge F_4$ is independent of the metric (the vielbein). The first term in the action depends on the metric $G_{MN} = \eta_{AB} E^A_M E^B_N$. The quantity $|F_4|^2$ is defined via $|F_n|^2 = \frac{1}{n!} G^{M_1 N_1} G^{M_2 N_2} \dots . G^{M_n N_n} F_{M_1 M_2 \dots M_n} F_{N_1 N_2 \dots N_n}$.
The full action including terms which contain fermions is invariant under local supersymmetry transformations. These transformations are

$$
\begin{aligned}
\delta E^A_M &= \bar{\varepsilon} \Gamma^A \Psi_M \\
\delta A_{MNP} &= -3 \bar{\varepsilon} \Gamma_{[MN} \Psi_{P]} \\
\delta \Psi_M &= \nabla_M \varepsilon + \frac{1}{12} \left(\Gamma_M \mathbf{F}^{(4)} - 3 \mathbf{F}^{(4)}_M \right) \varepsilon
\end{aligned}
\tag{4.4}
$$

where $\mathbf{F}^{(4)} = \frac{1}{4!} F_{MNPQ} \Gamma^{MNPQ}$ and $\mathbf{F}^{(4)}_M = \frac{1}{4!} F_{MNPQ} \Gamma^{NPQ}$. We use here the spin connection ∇ introduced in 2.3.1. In a local frame it is given by

$$
\nabla_M \varepsilon = \partial_M \varepsilon + \frac{1}{4} \omega_{MAB} \Gamma^{AB} \varepsilon. \tag{4.5}
$$

If there are fermions in the theory there are additional terms such the connection is no longer torsion free. However, for the case we are interested in, i.e. $\Psi = 0$, the expression for the connection given above holds. In terms of the vielbein ω can be expressed by $\omega_{MAB} = (-\eta_{BC} \partial_{[A} E^C_{M]} + \eta_{MC} \partial_{[B} E^C_{A]} - \eta_{AC} \partial_{[M} E^C_{B]})$.
We now want to justify that only the bosonic part of the action is given. The reason is that we are mainly interested in classical solutions of the theory and a classical solution has vanishing fermion fields. In this context one is interested in solutions that preserve some of the supersymmetries of the full theory. Setting the fermionic fields to zero it follows that the fermionic supersymmetry variation in (4.4) must vanish in order for some supersymmetry to be preserved, i.e. $\delta \Psi_M = 0$. This gives rise to a Killing equation

$$
\nabla_M \varepsilon = -\frac{1}{12} \left(\Gamma_M \mathbf{F}^{(4)} - 3 \mathbf{F}^{(4)}_M \right) \varepsilon \tag{4.6}
$$

We want to consider a theory with vanishing field strength 4-form F_4. We then obtain an equation of motion and a Killing spinor equation of the form

$$
R_{\mu\nu} = 0 \tag{4.7}
$$

$$
\nabla_M \varepsilon = 0 \tag{4.8}
$$

This means that solutions are Ricci-flat manifolds with a parallel spinor which are just manifolds with special holonomy.

4.2 Membranes

The original idea of branes arose in string theory where they are simply defined as objects in the target space on which open strings can end. However they are much more than that. In string theory they are extended objects in the directions in which Neumann boundary conditions are imposed and degenerate in directions with Dirichlet boundary conditions. A Dp-brane denotes a brane with p spatial dimensions. One can generalize this and regard a brane itself as a dynamical object that propagates in the target space. A generic Dp-brane in a theory will break all supersymmetry, however these are not stable. Stable branes have to couple to a generalized Maxwell type charge. An n-form gauge field couples electrically to a p-brane with $p = n - 1$ and magnetically to a p-brane with $p = 7 - n$. The conservation of this charge implies stability. Stable D-branes are exactly the branes preserving some part of the supersymmetry in the theory. They are generally called BPS branes. Sometimes the amount of supersymmetry preserved is included in the nomenclature (e.g. half-BPS D-brane for a brane that preserves one half of the supersymmetries).

The natural guess for D=11 supergravity is that we should look for membranes (i.e. D2-branes) since they can couple electrically to the 3-form gauge field present in the theory. The action is a generalization of the Green-Schwarz action for superstrings introduced in [18]. The bosonic part of membrane action is given by

$$S = T_2 \int_W d^3\sigma [-\det(\partial_i X^\mu \partial_j X^\nu g_{\mu\nu})]^{1/2} + \frac{1}{3!}\epsilon^{ijk}\partial_i X^{\mu_1}\partial_j X^{\mu_2}\partial_k X^{\mu_3} C_{\mu_1\mu_2\mu_3} \tag{4.9}$$

The first term is just the Nambu-Goto action for the world volume. The second term corresponds to the coupling of the membrane to an electric 4-form charge. This has the form of a Chern-Simons term and gives rise to a local fermionic symmetry. The explicit form of this so called κ-symmetry is given below. It is of great importance for the derivation of the calibration condition which is our ultimate goal. As before we are only interested in bosonic solutions that preserve some supersymmetry.

We further assume a static bosonic D=11 background with $A_3 = 0$. In local coordinates the metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{MN}\,\mathrm{d}X^M \mathrm{d}X^N \tag{4.10}$$

where $M, N = 1, ...10$ are spatial indices. If we insert this into (4.9) and fix $\sigma^0 = t$ the time dependent term decouples and from the spatial part of the world-volume W' we get the energy functional

$$E = T_2 \int_{W'} \mathrm{d}^2\sigma [\det(m_{ab})]^{1/2} \tag{4.11}$$

where $m_{ab} = \partial_a X^M \partial_b X^N g_{MN}$ and $a, b = 1, 2$. Hence the integral is just the spatial area of the embedded membrane with the induced metric. Static solutions to the equations of motion minimize the energy functional and hence the area of the membrane. This implies that a stable membrane is a minimal surface.
We now restrict ourselves to backgrounds of the form $\mathbb{R}^{1,10-d} \times M_d$ with $A_3 = 0$. As discussed above in order for this geometry to preserve some supersymmetry we must take M_d to have special holonomy such that the manifold accepts a parallel spinor. Static membrane configurations that preserve supersymmetry wrap homology cycles in the internal manifold M_d which are called *supersymmetric cycles*.
We finally can discuss what role calibrations play. For that lets consider the already mentioned κ-symmetry transformations:

$$\delta_\kappa \Theta = 2P_+ \kappa(\sigma) \tag{4.12}$$
$$\delta_\kappa X^M = 2i\bar{\Theta}\Gamma^M P_+ \kappa(\sigma) \tag{4.13}$$

The projection operators $P_\pm$ are given by

$$P_\pm = \frac{1}{2}(1 \pm \Gamma) \tag{4.14}$$

Where

$$\Gamma = \frac{1}{\sqrt{\det(m)}} \Gamma^0 \gamma \tag{4.15}$$
$$\gamma = \frac{i}{2!} \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \Gamma_{MN} \tag{4.16}$$

The gamma matrices satisfy $\{\Gamma_M, \Gamma_N\} = 2g_{MN}$. Since the kappa symmetry is a local symmetry it is a gauge symmetry which can be used to gauge away half of the components of Θ. This is crucial since otherwise the theory would have the wrong number of propagating fermionic degrees of freedom. For details we refer to [19]. The normalization of Γ is chosen such that $\Gamma^2 = 1$ and hence

$$P_\pm^2 = P_\pm \tag{4.17}$$
$$P_+ P_- = 0 \tag{4.18}$$
$$P_+ + P_- = 1 \tag{4.19}$$

The global supersymmetry can be represented as

$$\delta_\varepsilon \Theta = \varepsilon \tag{4.20}$$
$$\delta_\varepsilon X^M = i\bar{\varepsilon}\Gamma^M \Theta \tag{4.21}$$
$$\tag{4.22}$$

Since we are only interested in bosonic solutions we set $\Theta = 0$ and the global supersymmetry variation of X^M always vanishes. For a global supersymmetry generator ε to be preserved there must be a κ-transformation that undoes the effect of ε. This means that the effect of this transformation can be gauged away and hence it remains a symmetry of the theory, i.e.

$$\delta_\varepsilon \Theta + \delta_\kappa \Theta = \varepsilon + 2P_+ \kappa(\sigma) = 0. \tag{4.23}$$

Applying P_- to the equation gives rise to the BPS condition

$$P_- \varepsilon = 0. \tag{4.24}$$

We have $\Gamma^2 = 1$ and moreover $\Gamma^\dagger = \Gamma$. In this situation we can follow the same steps which lead to (3.18) and we find that

$$1 \geq \langle \varepsilon, \Gamma \varepsilon \rangle \tag{4.25}$$

where now $\langle \cdot, \cdot \rangle$ is a hermitian inner product on a spinor space. In the explicit representation with gamma matrices this reads

$$\sqrt{\det(m)} \geq \varepsilon^\dagger \Gamma^0 \gamma \varepsilon = -\bar{\varepsilon}\gamma\varepsilon. \tag{4.26}$$

Thus

$$\varphi = -\frac{1}{2!}\,\bar{\varepsilon}\,\Gamma_{MN}\,\varepsilon\,\mathrm{d}X^M \wedge \mathrm{d}X^N \tag{4.27}$$

is a calibration on the background manifold. Recall now the condition (4.24). We see that the configuration is supersymmetric if and only if we have equality in (4.25), i.e.

$$1 = \langle \varepsilon, \Gamma \varepsilon \rangle. \tag{4.28}$$

Since we took ε to be parallel, φ is closed. We arrive at the desired result: We showed that the spatial part of a BPS membrane is calibrated with respect to φ.
In the previous discussion the metric of the world sheet was taken to have Minkowski signature. Now we want to consider the Euclideanized membrane, i.e. we take the metric on the world-volume to be positive definite. This approach is due to [14] and is also discussed in [13]. We fix the internal manifold M_d to be a Calabi-Yau three fold. In

this case the projection operators for the kappa symmetry are of the form

$$P_{\pm} = \frac{1}{2}(1 \pm \frac{i}{6\sqrt{G}}\varepsilon^{\alpha\beta\gamma}\partial_{\alpha}X^{M}\partial_{\beta}X^{N}\partial_{\gamma}X^{P}\Gamma_{MNP}) \tag{4.29}$$

Where G denotes the determinant of the induced metric. The BPS condition takes the same form as in the previous example:

$$P_{-}\varepsilon = 0. \tag{4.30}$$

We want to analyze this further. We discussed the covariantly constant spinors on Calabi-Yau manifolds around (3.7). On the Calabi-Yau three-fold there exist parallel Weyl spinors ε_{+} and ε_{-}. The spinor ε_{-} is the one squaring to a holomorphic volume form under the map (3.14) which was explicitly performed in (3.9), if one introduces complex coordiantes X^{a} and $X^{\bar{a}}$ we can write

$$\Omega_{abc} = e^{-\mathcal{K}}\eta_{-}^{\mathrm{T}}\gamma_{abc}\eta_{-} \tag{4.31}$$

We will not discuss $\mathcal{K}$ further and refer to [14] for an explanation. The Kähler form is $J_{a\bar{b}} = ig_{a\bar{b}}$. One can choose the normalization such that the following holds.

$$\Gamma_{\bar{a}\bar{b}\bar{c}}\varepsilon_{+} = e^{-\mathcal{K}}\overline{\Omega}_{\bar{a}\bar{b}\bar{c}}\varepsilon_{-} \tag{4.32}$$

$$\Gamma_{a\bar{b}\bar{c}}\varepsilon_{+} = 2iJ_{a[\bar{b}}\gamma_{\bar{c}]}\varepsilon_{+} \tag{4.33}$$

The BPS conditon in terms of the Weyl spinors is given by

$$\frac{1}{2}(1 - \frac{i}{6\sqrt{G}}\varepsilon^{\alpha\beta\gamma}\partial_{\alpha}X^{M}\partial_{\beta}X^{N}\partial_{\gamma}X^{P}\Gamma_{MNP})(\mathrm{e}^{-i\Theta}\eta_{+} - \mathrm{e}^{i\Theta}\eta_{-}) = 0 \tag{4.34}$$

Rewriting these conditions we eventually obtain

$$\partial_{[\alpha}X^{a}\partial_{\beta]}X^{\bar{b}}J_{a\bar{b}} = 0 \tag{4.35}$$

$$\partial_{[\alpha}X^{a}\partial_{\beta}X^{b}\partial_{\gamma]}X^{c}\Omega_{abc} = \sqrt{G}\,e^{-i\varphi}e^{\mathcal{K}}\varepsilon_{\alpha\beta\gamma} \tag{4.36}$$

The phase φ is constant and hence this implies that the membrane must be a special Lagrangian submanifold of the Calabi-Yau three-fold. It is located in time and preserves supersymmetry (since it satisfies a BPS condition). This kind of solution is called a supersymmetric instanton. They give non-perturbative corrections to the theory. We want to mention that if we choose the internal manifold M_d to be a manifold with G_2 holonomy the Euclineanized membrane has to be calibrated with respect to the associative calibration in order to be BPS. The derivation of this fact is similar as in the Calabi-Yau case. The G_2 case is especially interesting for phenomenology since compactifications on G_2 manifolds give rise to $\mathcal{N} = 1$ theories in four dimensions.

5 Summary and Outlook

Finally let us sum up the results and mention some perspectives for further research. The first section mainly dealt with linear algebra and the introduction of the most important examples. In the context of complex and special Lagrangian geometry we saw that we can derive differential equations for calibrated submanifolds of $\mathbb{R}^{2m}$. One can also derive such differential equations for the exceptional geometries. A possible next step would be to find solutions for the differential equations. Some examples have been presented by Harvey and Lawson in [1].

In section 3 we saw that calibrations are closely linked to manifolds with special holonomy and in many cases to the parallel spinors on these manifolds. Besides the cases we discussed in the text, the list in (3.6) includes the case of hyperkähler manifolds with Sp_m-holonomy. This gives rise to a quarternionic structure, meaning that there are three distinct complex structures which satisfy

$$J^a \cdot J^b = -\delta^{ab} + \varepsilon^{abc} J^c \tag{5.1}$$

These manifolds admit $m+1$ parallel spinors. In the case $m = 2$ (i.e. $d = 8$), where $\mathrm{Sp}_m \subseteq \mathrm{Spin}_7 \subseteq \mathrm{SO}_8$, we get a particularly rich geometry which includes complex, special Lagrangian and Cayley geometry. For a detailed treatment we refer to [20]. Next one can investigate the implications to physics where hyperkähler manifolds appear in theories with $\mathcal{N} = 2$ supersymmetry.

A truly remarkable result is that the calibration condition translates to a BPS condition in the context of supergravity. In this thesis we considered D=11 supergravity in the very special case of vanishing four-form field-strength F_4. That lead us to the conclusion that our solution must admit a parallel spinor which gives rise to a calibration. Stable membrane solutions have to be calibrated by this calibration. Moreover, one could ask if there is still a differential form that controls the geometry of BPS solutions if one allows for non-vanishing four-form field-strength F_4. This leads to the concept of generalized calibrations discussed in [21].

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Declaration

I assure that this thesis is my own work and I only used the sources listed in the reference section.

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, March 13, 2019 . . . .