Études in Arithmetic Physics

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1 Introduction

This notes are supposed to be a rough overview of some connections between number theory and physics. At this point, the presentation is rather sketchy and by now means complete.
Most of the connections originate from either modular invariance or arithmetic geometry and often from both. Modularity play an important role in both physics and number theory. More precisely, it's often interesting to consider objects that are invariant under the action of $SL_2(\mathbb{Z})$ by Möbius transformations

$$\gamma : z \mapsto \gamma \cdot z = \frac{az+b}{cz+d} \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{1}$$

In physics this kind of symmetry for instance shows up in conformal field theory in 2 dimensions. Another example is $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions. Here the $SL_2(\mathbb{Z})$ symmetry gives rise to an S-duality between gauge theories with gauge groups that are dual to each other.
In mathematics, the most prominent examples of an object that is invariant under the $SL_2(\mathbb{Z})$-action are Modular forms. Which are holomorphic functions on the upper half-plane that transform like

$$f(\gamma z) = (cz+d)^k f(z) \tag{2}$$

under $\gamma \in SL_2(\mathbb{Z})$, satisfying some analytical properties.
Conformal invariance of a theory implies, that it's amplitudes one calculates will have some modular properties as well. Modular forms contain a lot of arithmetic information and hence the appearance of modular invariance hints at a deeper relationship between number theory and physics.
There are different kinds of connections between physics and number theory. The plan is to discuss the following example:

1. The moonshine conjecture originally relates finite groups and modular forms. Recently discovered moonshine indicate that the objects involved may carry interesting arithmetic information that goes beyond what one would usually expect. The very first moonshine conjecture relates the Monster group and the J-invariant and the proof uses a certain vertex operator algebra constructed from a conformal field theory that. To find the physical theories underlying other instances of moonshine is an active area of research.

2. From the physical point of view most interesting example of arithmetics arising from physics was pioneered by G. Moore in his work on attractors and arithmetic [Moo98]. From a mathematical point of view it is well known that Calabi-Yau varieties defined over number fields contain a lot of arithmetic information. It is however very surprising, that some physical mechanism singles out arithmetic Calabi-Yau varieties (i.e. Calabi-Yau varieties that can be defined over number fields). This raises the question if this is just a coincidence or if there is really a deeper connection between string theory on Calabi-Yau manifolds and number theory.
   There is some evidence, that the connection is not accidental. Taking this connection seriously, one can ask what the arithmetic information contained in a Calabi-Yau 3-fold is mapped to under Mirror symmetry. In this context one should also study the arithemtic properties of the periods of the given Calabi-Yau varieties, connection to L-function, etc.

3. Another place where the arithmetic properties of Calabi-Yau varieties shows up in physics is in mirror symmetry. Similar to the attractor mechanism some physical mechanism singles out point in the moduli space with enhanced arithmetics. In the case of mirror symmetry the points of interest are MUM points. The prepotential in topological string theory at a MUM points is a so called s-function. An s-function over some number field K is a function $W \in zK[[z]]$ such that for each unramified prime p and $\mathfrak{p}|p$

$$\frac{1}{p^s} \operatorname{Frob}_p W_{\mathfrak{p}} - W_{\mathfrak{p}} \in z\mathcal{O}_{K_{\mathfrak{p}}}[[z]] \tag{3}$$

   One can ask if MUM points that are not defined over $\mathbb{Q}$ show up and indeed there are examples of MUM points that are not rational, but defined over some finite extension of $K/\mathbb{Q}$. s-function where introduced in [SVW15] and further studied in [SVW17].

4. The last appearance connection between number theory and physics we discuss, is philosophically different from the previous ones, that it goes into the opposite direction compared to the previous ones. The Langlands correspondence originally is a conjecture in number theory connecting automorphic forms and Galois representations. Later there was a geometric version of the Langlands conjecture. This was interpreted by Kapustin and Witten in terms of electric-magnetic duality of some super Yang-Mills theory. In recent time there have been attempts to relate the physical approach to number theory. For instance Minhyong Kim proposed the idea of arithmetic gauge theory. More recently David Ben-Zvi and collaborator are working on an interpretation of the arithmetic Langlands correspondence in terms of topological quantum field theory.
   The origin of this topological quantum field theory is not clear yet, but on possible approach to understand this arithmetic field theory is via p-adic geometry. We discuss some recent speculation of physics on arithmetic spaces presented in [Hec21].

2 Modularity in Mathemtics

We first introduce the mathematical objects involved. The most prominent modular objects are Elliptic curves and Modular forms which are closely related by the Shimura-Taniyama-Weil Conjecture.

2.1 Elliptic Curves

Over the the field of complex number $\mathbb{C}$ an elliptic curve is easy to describe. Topologically an elliptic curve is just the torus T^2 with a fixed choice of complex structure. The moduli space of complex structures is given by $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$ (note that the choice of the subgroup $SO_2(\mathbb{R}) \subset SL_2(\mathbb{R})$ is by no means random, but it is the maximal compact subgroup of $SL_2(\mathbb{R})$). Equivalently, this is the quotient of $\mathfrak{h} := \{z \in \mathbb{C} \,|\, Im(z) > 0\}$ by the action of $SL_2(\mathbb{Z})$ by Möbius transformations, i.e. for $\tau \in \mathbb{H}$ and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma\tau := \frac{a\tau + b}{c\tau + d} \tag{4}$$

For arithmetic questions, one has to consider elliptic curves over number fields (sometimes also over fields of positive characteristic) K instead of the complex numbers. Ever elliptic curve is isomorphic to the vanishing set of a polynomial of the form

$$y^2 = x^3 + ax + b \tag{5}$$

For $a, b \in K$. We define the discriminant $\Delta = -16(4a^3 + 27b^2)$. If $\Delta \neq 0$ the equation defines a smooth curve. We mention that the points of a ellitpic curve form an abelian group.

2.2 Modular forms

The theory of elliptic curves is closely related to the theory of modular forms. The idea of modular forms is to consider functions on the upper-half plane (often extended by the cusp points $\mathbb{P}^1(\mathbb{Q}) = \{i\infty\} \cup \mathbb{Q}$) with a certain transformation property under a discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$.

Definition 2.1 *A modular form of weight k and level N on*

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}) \,|\, c \equiv 0 \ mod \ N \right\} \tag{6}$$

is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ satisfying

1. $$f(\gamma\tau) = f\left(\frac{a\tau + b}{c\tau + d} = (c\tau + d)^k f(\tau) \right) \tag{7}$$

   for all $\gamma \in \Gamma_0(N)$.

2. *f is holomorphic at all cusps.*

The modular group $PSL_2(\mathbb{Z})$ is generated by the two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For transformation induced by $T \in \Gamma_0(N)$ implies that $f(\tau+1) = f(\tau)$ and hence, every modular form admits a Fourier expansion

$$f(q) = \sum_n a_n q^n \tag{8}$$

with $q = e^{2\pi i \tau}$. A modular form which vanishes at all cusps is called a cusp-form.
The simplest non-trivial example of a modular form are the so-called Eisenstein series. For $k \in \mathbb{N}_{\geq 2}$ the k-th Eisenstein series is defined as:

$$G_{2k}(\tau) = \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \frac{1}{\omega^{2k}} \tag{9}$$

The ring of modular forms is generated by G_4 and G_6 and is a graded commutative ring graded by the weight k.
One easily sees that the smallest k for which M_k has dimension bigger then 1 is $k = 12$. So by taking quotients of two linearly independent modular forms of weight 12 we can find meromorphic modular functions (i.e. modular forms of weight zero). Klein's J-invariant is defined as:

$$J(\tau) = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} \tag{10}$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$. This is an important modular function since the field of modular functions is $\mathbb{C}(J)$, i.e. every modular function is a rational function of J.
The J-invariant has exceptional arithmetic properties. Consider τ satisfying the quadratic equation $a\tau^2 + b\tau + c = 0$ with $g.c.d.(a,b,c) = 1$. This is condition is equivalent to the condition that the elliptic curve associated to τ admits complex multiplication. We call such a τ a complex multiplication point in $SL_2(\mathbb{Z}) \setminus \mathfrak{h}$.

Proposition 2.2 *Let τ be a complex multiplication point. Let D be the discriminant of the associated primitive quadratic form. Then,*

1. *$j(\tau)$ is an algebraic integer of degree $h(D)$.*

2. *If $[\tau_i]$ correspond to the distinct ideal classes in the order $\mathcal{O}(D)$ and the minimal polynomial of $j(\tau_i)$ is*

$$P(X) = \prod_{k=1^{h(D)}} (X - j(\tau_k)) \in \mathbb{Z}[X] \tag{11}$$

3. *$\hat{K}_D := K_D(j(\tau_i)$ is Galois over K_D and is independent of i.*

A more geometric and conceptual view on the subject of modular forms is the following. Let $\Gamma \subset SL_2(\mathbb{Z})$ be subgroup. A modular form of level Γ and weight k is an element of

$$f \in H^0(X_\gamma, K^k) \tag{12}$$

where K is the canonical bundle on the modular curve

$$X_\Gamma = \Gamma \backslash (\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})) \tag{13}$$

In particular the classical modular forms of $\Gamma = SL_2(\mathbb{Z})$ are sections of a line bundle on the moduli stack of elliptic curves.
We can now make first contact with number theory. The proof of Fermat's last theorem by Andrew Wiles is largely based on the fact that certain objects, namely L-functions associated elliptic curves and modular forms agree and give rice to a 1-1 correspondence between modular forms and isogeny classes of elliptic curves.
The L-functions associated to elliptic curve contain information about the number of points of a elliptic curve in a finite fields.

2.3 Calabi-Yau Varieties

We have already introduced Elliptic curves. They are special among all curves in that there canonical bundle is trivial. A first generalization of elliptic curves are abelian varieties. These are just projective abelian group-schemes of higher dimensions and as for elliptic curves, abelian varieties over $\mathbb{C}$ are given by complex tori. They share the property, that there canonical bundle is trivial and this for instance makes their derived category of coherent sheaves particularly interesting. Calabi-Yau varieties are all varieties with trivial canonical bundle. Let's be more precise:

Definition 2.3 *A smooth projective variety X over a field k of dimension d is called Calabi-Yau if*

1. $H^i(X, \mathcal{O}_X)$ *for every* i, $0 < i < d$

2. *The canonical bundle* K_X *is trivial, i.e.* $K_X \simeq \mathcal{O}_X$

Working over $k = \mathbb{C}$ we can define the Hodge numbers

$$h^{i,j}(X) = \dim_{\mathbb{C}} H^j(X, \Omega^i_X) \tag{14}$$

for all $0 \leq i, j \leq d$. The Hodge diamond of a Calabi-Yau 3-fold is given by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

This mean, a Calabi-Yau 3-fold is topologically governed by only 2 degrees of freedom

1. The Kähler parameters, controlled by $H^{1,1}(M)$

2. The complex structure parameter, controlled by $H^{2,1}(M)$.

Consider a Calabi-Yau Manifold 3-fold X, with holomorphic $(3,0)$-form Ω. Depending on the given choice of Kähler class on X, the homology group, we can choose a symplectic basis (A^a, B_b). The periods in the given basis are

$$Z^a = \int_{A^a} \Omega$$

$$F_a = \int^{B_a} \Omega$$

These periods encode arithmetic properties of the underlying variety.
Considert the example of the family of quintic 3-folds M_ϕ, defined by the vanishing of the polynomials

$$P(X,\phi) = \sum_{i=1}^{5} X_i^5 - 5\phi X_1 X_2 X_3 X_4 X_5 \tag{15}$$

It is a consequence of mirror symmetry that the Yukawa coupling may be expended in the form

$$y_{ttt} = 5 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1-q^k}, \quad q = e^{2\pi i t} \tag{16}$$

where the coordinate t and the Yukawa coupling can both be expressed in terms of certain periods ϖ_0 and ϖ_1

$$y_{ttt} = \left(\frac{5}{2\pi i}\right)^3 \frac{5\phi^2}{\varpi_0(\phi)^2(1-\phi^5} \left(\frac{d\phi}{dt}\right)^3, \quad t = \frac{1}{2\pi i}\frac{\varpi_1(\phi)}{\varpi_0(\phi)}. \tag{17}$$

The n_k are integers that count the numbers of rational curves of M_ϕ. With the relation above we see that we can calculate this numbers from certain periods of the Calabi-Yau.
The periods satisfy a system of differential equations, the Picard-Fuchs equations, with respect to the parameter, with respect to the parameters. For the family (15) there are 204 periods in total.
In general we have the period map $H_{d/2}(X,\mathbb{Z}) \to \mathbb{C}, \Gamma \mapsto \int_\Gamma \Omega$, where d is the real dimension of X and Ω the holomorphic top-form. The kernel of this mapping inside $H_{d/2}(X,\mathbb{Z})$ is called the Neron-Severi lattice $NS(X)$. The transcendental lattice is the orthogonal complement: $T(X) := (NS(X))^\perp$.

2.4 L-functions

The prototypical example of an L-function is Riemann ζ-function

$$\zeta(s) = \sum_{n\geq 1} n^{-s} \quad (\mathrm{Re}(s) > 1) \tag{18}$$

Writing the ζ-function as an Euler-product, it is obvious, that it is really an object of number theory since it contains important information about prime numbers in $\mathbb{Z}$

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \quad (\mathrm{Re}(s) > 1) \tag{19}$$

The ζ-function depends on all finite places of $\mathbb{Q}$. At the infinite place $|\cdot|_\infty$ we can attach the factor $\pi^{-s/2}\Gamma(s/2)$. Then we define the completed ζ-function

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{20}$$

which is the product over the factors for all places of $\mathbb{Q}$. $Z(s)$ can be continued to a meromorphic function on the whole complex plane. It satiesfies the functional equation

$$Z(s) = Z(1-s) \tag{21}$$

In general one thinks of an L-function as attached to some object X. The one thing that all L-functions have in common is a Dirichlet series expansion

$$L(X,s) = \sum_{n\geq 1} a_n n^{-s} \tag{22}$$

where s is a complex variable and the a_n are complex numbers determined by X and growing at most polynomially as $n \to \infty$. This growth condition implies that for some $c > 0$, that the series converges on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > c\}$ and defines there a holomorphic function. Some properties we would like to have for L-functions are the following

- If $a_{nm} = a_m a_n$ for m and n coprime natural numbers, and moreover the a_{p^r} for p prime and $r \geq 1$ satisfy suitable recurrence relations, then we can write the L-function as an Euler product:

  $$L(X,s) = \prod_{p \text{ prime}} \frac{1}{F_p(p^{-s}} \tag{23}$$

  where $F_p \in \mathbb{C}[t]$ is a polynomial of the form $1 - a_p t + \dots$.

- We want he L-function has an analytic continuation to the complex plane.

- In many cases, one can degine a dual object $X^\vee$, a completed L-function $\Lambda(X,s)$, an integer k and a complex number $\epsilon(X)$ such that

  $$\Lambda(X,s) = \epsilon(X)\Lambda(X^*, k-s). \tag{24}$$

  In the case where $L(X,s)$ admits such a functional equation, the vertical line mit real part $k/2$ is called the critical line.

- The Riemann hypothesis for L-functions says that all non-trivial zeros, lie on the critical line.

There are essentially two ways of constructing L-functions:

1. from number theory and arithmetic geometry
2. from automorphic forms and automorphic representations

We give some concrete construction of L-functions from different objects. The first obvious generalization of Riemann's ζ-function are so called Dirichlet L-series. They can be seen as a twist of the ζ-function by a character. Let for some positive integer m

$$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times \tag{25}$$

be a group homomorphism. One extends this to a function $\chi : \mathbb{Z} \to \mathbb{C}$ by putting $\chi(a) = 0$ if a is not coprime to m. One then defines the L-function associated to this character

$$\begin{aligned} L(\chi, s) &= \sum_{n=1}^{\infty} \chi(n) n^{-s} \\ &= \prod_{p|m \text{ prime}} (1 - \chi(p) p^{-s})^{-1}. \end{aligned} \tag{26}$$

Next we define the ζ-function for a generalisation of the Riemann ζ-function to an arbitrary number field K. It is defined by

$$\begin{aligned} \zeta_K(s) &= \prod_{\mathfrak{p} \in Specm(\mathcal{O}_K)} (1 - N\mathfrak{p}^{-s})^{-1} \\ &= \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} (N\mathfrak{a})^{-s}. \end{aligned} \tag{27}$$

The norm $N\mathfrak{a}$ is defined as the cardinality of $\mathcal{O}_K/\mathfrak{a}$. The sum $\mathfrak{a}$ runs over all non-zero ideals of $\mathcal{O}_K$.
Let $\Delta_K \in \mathbb{Z}$ be the discriminant of K, and let r_1 and r_2 denote the number of real and complex places of K, respectively. Then the completed ζ-function

$$Z_K(s) = |\Delta_K|^{s/2} (\pi^{-s/2} \Gamma(s/2))^{r_1} ((2\pi)^{1-s} \Gamma(s))^{r2} \zeta_K(s) \tag{28}$$

this again satisfies

$$Z_K(s) = Z_K(1-s). \tag{29}$$

We know return to modular form. Let f be a modular form of weight k and level n. To define the L-function consider the Fourier expansion of the modular form

$$f(\tau) = \sum_{m\geq 0} a_m(f)q^m \quad q = e^{2\pi i\tau} \tag{30}$$

for certain complex number $a_m(f)$. The L-function of a modular form f as above is defined by the Dirichlet series

$$L(f,s) = \sum_{m\geq 1} a_m(f)m^{-s} \tag{31}$$

As before, one can define a completed L-function $\Lambda(f,s)$ by multiplying $L(f,s)$ by certain elementary factors. An important property of L-functions of modular forms of modular forms is that one can express $\Lambda(f,s)$ as the Mellin transform of f:

$$\Lambda(f,s) = \int_0^\infty (f(iy) - a_0(f))y^s \frac{dy}{y} \tag{32}$$

For certain f (so-called primitive cusp forms, which in particular satisfy $a_0(f) = 0$ and $a_1(f) = 1$), the L-function $\Lambda(f,s)$ can be analytically continued to an entire function satisfying a functionla equation linking f with a dual form $\overline{f}$ satisfying $\overline{f}(z) = \sum_{m\geq 1} \overline{a_m(f)}q^m$.
We can also attach L-functions to Galois representations. Consider a finite Galois extension E/F of number fields, with Galois group $G = Gal(E/F)$. Consider a representation

$$\rho : G \to Aut_{\mathbb{C}}(V) \tag{33}$$

of G on an n-dimensional complex vector space V. For every finite place $\mathfrak{p}$ of F, we choose a place $\mathfrak{P}$ of E lying over F, giving us a decomposition group $D_{\mathfrak{P}}$, an inertia group $I_{\mathfrak{P}}$ and a Frobenius element $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}/I_{\mathfrak{P}}$. (We recall that $I_{\mathfrak{P}} = 1$ if $\mathfrak{P}$ is unramified, which is the case for all but finitely many $\mathfrak{P}$). We put

$$\chi_{\mathfrak{p}}(t) = \det(\mathrm{id} - t\rho(\sigma_{\mathfrak{P}}) \mid V^{I_{\mathfrak{P}}}) \in \mathbb{C}[t]. \tag{34}$$

This polynomial whose coefficients lie in some cyclotomic extension of $\mathbb{Q}$, and which is independent of the choice of prime $\mathfrak{P}$ over $\mathfrak{p}$.
We then define

$$L(\rho, s) = \prod_{\mathfrak{p}} \chi_{\mathfrak{p}}(N\mathfrak{p})^{-1} \tag{35}$$

where the product runs over all finite places of F.
The power of L-functions lies in the fact that we can attach L-functions to geometric objects and connect them to L-functions of other kinds.
Let E be an elliptic curve over $\mathbb{Q}$ with $\Delta_E \neq 0$. For any prime number $p \nmid \Delta$ we study the points of E over $\mathbb{F}_p$. More precisely, we define

$$a_p = p - |E(\mathbb{F}_p)| \tag{36}$$

Next, we define $L(E, s)$ by the Euler product

$$L(E, s) = \prod_{p \in Specm(\mathbb{Z})} (1 - a_p p^{-s} + p \cdot p^{-2s})^{-1} \quad (\mathrm{Re}(s) > 3/2). \tag{37}$$

One can extend this to a completed L-function $\Lambda(E, s)$ using specific factors for primes dividing Δ and for the infinite place of $\mathbb{Q}$. The modularity theorem conjectured by Taniyama-Shimura and proven by Taylor and Wiles connects L-function of Elliptic curves coming from number theory and L-function of modular curves.

Theorem 2.4 *Let E be an elliptic curve over $\mathbb{Q}$ of conductor n. Then $L(E, s) = L(f, s)$ for some primitive cusp form f of weight 2 and level n.*

The conductor can be derived from the discriminant and contains information about the bad primes.
The Hasse-Weil L-function of an elliptic curve defined over a number field K contains information information about the points of E mod p. The BSD conjecture (after Birch and Swinnerton-Dyer) states that the rank of the abelian group $E(K)$ of points of E is the order of the zero of $L(E, s)$ at $s = 1$.
We would like to attach L-functions to geometric objects of higher dimensions. This can be done, by studying étale cohomology groups of arithmetic varieties. These étale cohomology groups carry Galois representations and we can define the L-function of a Calabi-Yau variety defined over a number field via the L-function of these Galois representations.

A very important tool for the study of L-functions is there integral representation. For example the completed Riemann ζ-function can be written as

$$\int_0^\infty y^{s/2} \sum_{n=0}^{\infty} e^{-n^2 \pi y} dy \tag{38}$$

It's useful to note that we integrate some expression containing the ϑ-function

$$\vartheta(z;\tau) = \sum_{-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z} \tag{39}$$

at $z = 0$.

3 Moonshine

The Moonshine conjecture originally predicted a connection between a certain finite group called the Monster group and the J-function introduced above. The connection between these seemingly very different objects was understood by constructing a certain conformal field theory, that had the Monster group as a symmetry. Since then many new instances of moonshine have been found. Very recently, Duncan, Mertens and Ono established a Moonshine conjecture for the O'Nan group, which connects weight 3/2 modular forms with representations of the O'Nan group [DMO17]. The modular forms appearing can be associated to class numbers, L-functions and other data attached to certain elliptic curves. This is the first example of moonshine containing arithmetic information of this kind.
For a nice introduction to moonshine including more recent developments see [AC18]. For a more physical presentation we refer to [Kac16].

3.1 Classification of finite groups

The ultimate goal is to find a Moonshine connection for *all* finite groups. For this is it natural to focus on simple groups. Recall that a group is called simple if it has no proper normal subgroups. It's one of the big achievements in group theory in this century that we now have a full classification of all finite simple groups. They belong to one of the following four categories:

- Cyclic groups $\mathbb{Z}/p$ for a prime p.

- Alternating groups $\mathcal{A}_n$ for $n > 5$.
- 16 families of Lie type.
- 26 sporadic groups.

The monster group is the largest of the sporadic groups. The original motivation for studying groups is to encode symmetries. This means, that we want the group to act on some object. In the following this will usually be a complex vector space. This means we want to study representations of finite groups.

3.2 Monsterous Moonshine

The original Moonshine conjecture is about a connection between the J-function and representations of the Monster group. For an elementary introduction to the subject we refer to [Tat19]. To sketch the connection, we consider the Fourier expansion of the J-function, given by

$$J(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 8642999970q^3 + \dots \tag{40}$$

The Fourier coefficients of the normalized J-invariant $\tilde{J} = J - 744$ are linear combinations of the dimensions of irreducible representations of the Monster group.
The conjecture first formulated by Thompson can be stated as follows:

Conjecture 3.1 *There exists a natural infinite-dimensional graded representation* $\left(\rho_\natural, V^\natural = \oplus_{i\geq -1} V_i^\natural\right)$ *of the monster group such that each graded part is finite dimensional, and such that the generating series of the dimension of these graded parts is the q-expansion of the normalized J-invariant:*

$$\tilde{J}(\tau) = \sum i \geq -1 \dim \left(V_i^\natural q^i.\right) \tag{41}$$

Given such a representation $V^\natural$ one can attach to it more than just this series. Given a representation ρ we can define the character

$$\begin{aligned} \xi_\rho : & G \to \mathbb{C} \\ & g \mapsto \mathrm{Tr}(\rho(g)) \end{aligned}$$

Characters are class function on G, i.e. they are constant on conjugacy classes of G. The dimension of a representation is the trace of $\rho(e)$ and hence it might be useful to study the family of series

$$T_{[g]} = \sum_{i\geq -1} \mathrm{Tr}\left(\rho_\natural(g)|_{V_i^\natural}\right) q^i = \frac{1}{q} + \sum_{n=0}^{\infty} H_n([g]) q^n \tag{42}$$

which are known as *McKay-Thompson series*. There is one such series for each conjugacy class $[g]$ of the monster group.

Conjecture 3.2 *There exists a natural infinite-dimensional graded representation* $\left(\rho_\natural, V^\natural = \oplus_{i\geq -1} V_i^\natural\right)$ *of the monster group such that each graded part is finite dimensional, and such that for each conjugacy class* $[g]$ *the McKay-Thompson series* $T_{[g]}$ *is the* q*-expansion of the normalized Hauptmodul of a subgroup* $\Gamma_{[g]}$ *of* $PSL_2(\mathbb{R})$ *commensurable with* $PSL_2(\mathbb{Z})$*.*

To understand this connection one has to look at certain vertex operator algebras. These are realized as certain $2d$ CFTs that have the relevant finite group as symmetry groups.

The most elementary setting where modular forms show up is the torus partition function of a $2d$ CFT with left/right Virasoro generators L_n, $\overline{L}_n$ and central charge c is

$$Z(\tau, \bar{\tau}) = Tr(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}) \tag{43}$$

In general, if one has a quantum system with a Hilbert space $\mathcal{H}$ of physical states, and partition function

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) \tag{44}$$

then (for integer energy levels) one would have

$$Z = \sum_n c_n e^{-\beta n} \tag{45}$$

The coefficient $c_n \in \mathbb{Z}$ counts the numbers of states at energy level n (this is a general phenomenon that modular forms are in some way connected with

some kind of counting problem).
The question we ask for the monster is: Is there a $2d$ CFT such that

$$J(\tau) = \mathrm{Tr}(q^{L_0 - \frac{c}{24}})? \tag{46}$$

In that case we decompose the Hilbert space as

$$\mathcal{H} = \oplus_n \mathcal{H}_n \tag{47}$$

with

$$J = \sum \dim(\mathcal{H}_n) q^n. \tag{48}$$

If the $2d$ CFT has Monster symmetry, then it would be natural for the states at a given energy to transform in interesting Monster representations. So we would expect the j_n to be simple sums of dimensions of irreducible representations of the Monster.
If we have such a CFT the McKay-Thompson series should agree with

$$Z_g := \mathrm{Tr}\left(g q^{L_0 - \frac{c}{24}}\right) \tag{49}$$

On a physical level of rigor the problem was settled by Frenkel, Lepowsky and Meurman in [FLM89].

Consider an even, self-dual, Euclidean lattice Λ of dimension r. This gives rise to a consistent $c = r$ holomorphic CFT. Let's first introduce r fields $X^i(z)$, viewed as coordinates on the Torus

$$T^r = \mathbb{R}^r / \Lambda \tag{50}$$

Todo/Question: Why are the field parametrized by complex coordinates?
The case we are interested in is the so called *Leech lattice*. It is given by the densest way to pack spheres in 24 dimensions.
One can show that

$$Z_{Leech}(q) = \frac{\Theta_{Leech}}{\eta^{24}} = J(q) + 24 \tag{51}$$

The Θ function gives the sum over the momenta/windings in the lettice, while the η functions come from the bosonic oscillator modes.
Frenkel, Lepowsky and Meurman proved that an appropriate $\mathbb{Z}_2$ quotient of this theory removes the 24 in the partition function. Moreover it gives a

theory where the monster group M acts as a symmetry, commuting with the Hamiltonian and leaving the OPEs unchanged. The existence of this CFT gives a heuristic explanation for some aspects of Monstrous moonshine. A mathematically rigorous treatment was provided by Borcherds a few years later in 1992.

3.3 Generalized Moonshine

Monsterous moonshine provides a connection between finite groups and modular forms. It's remarkable that we can find the monster group in a rather simple way realized as a physical symmetry. Monsterous moonshine is the first discovery of that kind. For further investigation we hope to extend the connection in different ways

- The most obvious generalization is to establish moonshine for all finite groups. The case of the pariah groups (i.e. the finite sporadic groups that are not related to the monster group) and find the field theory that realizes the corresponding moonshine module as its Hilbert space.
- Add algebraic geometry and number theory to the picture. One of the deepest connections in mathematics arise from the connection between modular forms, which are analytic objects, and number theory.

The geometric objects that lie in the intersection of string theory and arithmetics are Calabi-Yau varieties. Algebraically a Calabi-Yau variety is is characterized by triviality of its canonical bundle. The Calabi conjecture, proved by Yau, that a (projective smooth) Calabi-Yau variety viewed as a complex manifoldadmits a Ricci flat metric for each choice of Kähler class. The simplest Calabi-Yau space is the *K3* surface. A concrete example for a K3 surface is given by the hypersurface

$$\sum_{i=1}^{4} z_i^4 = 0 \tag{52}$$

in $\mathbb{P}^3$. The moduli space of vacua for Type IIA string theory on K3 is 80-dimensional:

$$\mathcal{M}_{K3} = O(4,20;\mathbb{Z})\backslash O(4,20;\mathbb{R})/(O(4)\times O(20)) \tag{53}$$

Since it is in practice impossible to compute the Ricci flat metric on a Calabi-Yau manifold, we can't compute the partition function but we can use the supersymmetry present in superstring theory to compute so called supersymmetry indices.
Consider a $(2,2)$ SCFT with Calabi-Yau target. The $(2,2)$ algebra comes with a $U(1)$ current of each chirality, in addition to the stress tensor and supercharges. For such a theory the elliptic genus is defined as

$$\phi(\tau,z) = \mathrm{Tr}_{R,R}\left(q^{L_0-\frac{c}{24}}y^{J_L}(-1)^{F_R}\overline{q}^{\overline{L}_0-\frac{c}{24}}\right) \tag{54}$$

Here

$$q = e^{2\pi i\tau},\ y = e^{2\pi iz} \tag{55}$$

The trace is computed in the Ramond sector for both left and right movers. This object is a *weak Jacobi form*. They have the property

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = (c\tau+d)^w e^{2\pi im\frac{cz^2}{c\tau+d}}\phi(\tau,z)$$
$$\phi(\tau,z+l\tau+l') = e^{-2\pi im(l^2\tau+2lz)}\phi(\tau,z)$$

The elliptic genus of a Calabi-Yau n-fold is a weak Jacobi form of weight 0 and index $n/2$. The structure of the vector space of jacobi forms is similar to that of modular forms. It's the polynomial ring generated by E_4, E_6 and two new generators

$$\phi_{-2,1}(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta^6}$$
$$\phi_{0,1}(\tau,z) = 4\left(\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}+\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}+\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right)$$

where θ_i are the standard Jacobi functions. The subscripts denote the weight and index. Note that E_4 and E_6 clearly have index 0. For Calabi-Yau 2-folds, the story is very simple. The K3 surface must yield a Jacobi form of weight 0 and index 1. There is one possibility, up to scale, i.e. $\phi_{0,1}$. In fact, it turns out that with these standard definitions,

$$\phi_{K3}(\tau,z) = 2\phi_{0,1} \tag{56}$$

The K3 CFT has $N = (4, 4)$ supersymmetry. In addition to the Virasoro generators and 4 supercharges, each copy of the $2D$ $N = 4$ superconformal algebra comes with an $SU(2)$ R-symmetry. The representations of this algebra are labeled by a weight and a $U(1)$ charge J_3. The resulting heighest weight representation then haven known superconformal characters. There are two types of characters: BPS characters and non-BPS characters. One can expend

$$\phi_{K3} = (\text{BPS characters}) \; + \sum_{\text{non-BPS}} A_n ch_n(\tau, y) \tag{57}$$

where ch_n is the appropriate character for a $\Delta \sim n$ super-Verma module with fixed $U(1)$ charge allowed by the $c = 6$, $N = 4$ symmetry. The resulting numbers are

$$A_1 = 90,\; A_2 = 462,\; A_3 = 1540,\; \ldots \tag{58}$$

In 2010 Eguchi, Ooguri and Tachikawa noticed that these numbers constitute (2 times the) dimensions of the sporadic simple group M_{24}, which is the largest of the so called Mathieu groups.
It is not clear yet, why this version of Moonshine appears, but we want to make some remark about what is known:

Remark: *It is known that the symplectic automorphisms of any K3 surface lie in subgroups of M_{23}. M_{23} is a subgroup of M_{24}, however there is no simple analogy to the Monstrous moonshine construction, that explains Mathieu moonshine in the elliptic genus since it is known that there is no CFT with K3 target that admits M_{24} symmetry.*
One of the latest attempts to connect Moonshine with physics by Miranda Cheng et. al. uses the attractor mechanism discussed below [Che+18]. There insight was, that the Jacobi forms and mock Jacobi forms involved in moonshine also play important roles as counting functions governing black hole entropy in string theory. They claim, that BPS counting functions appearing in the theory of strings and wrapped fivebranes at rational and attractor points provide a rich source of objects and suggest further new possibilities for connections between moonshine, black holes, and BPS state counting. Given the many connection to arithmetic all the involved objects have, it seems likely, that moonshine itself contains a lot of arithmetic informations. The connection between attractors, mock modular forms and class numbers was studied in [KT17].

3.4 String Modular K3 Surface

In [Sch06] Schimmrigk studies string compactification on arithmetic Calabi-Yau hypersurfaces.
We consider very special K3 surfaces. Constructed as hypersurfaces in projective space

$$\begin{aligned} S^4 &= \{(z_0 : \cdots : z_3) \in \mathbb{P}_3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \\ S^{6A} &= \{(z_0 : \cdots : z_3) \in \mathbb{P}_{1,1,1,3} \mid z_0^6 + z_1^6 + z_2^6 + z_3^2 = 0\} \\ S^{6B} &= \{(z_0 : \cdots : z_3) \in \mathbb{P}_{1,1,2,2} \mid z_0^6 + z_1^6 + z_2^6 + z_3^2 = 0\} \end{aligned}$$

All these surfaces have maximal Picard number, i.e. $\operatorname{rank}(NS(S)) = 20$. We have already encountered this condition in the discussion of the attractor mechanism above. The following theorem proves a string theoretic interpretation of the motivic L-function associated to the holomorphic Ω-form of K3 surfaces in terms of the affine Lie algebra $A_1^{(1)}$ on the worldsheet:

Theorem 3.3 *Let $M_\Omega \subset H^2(S^d)$ be the irreducible representation of* $\mathrm{Gal}(\mathbb{Q}(\mu_d)/\mathbb{Q})$ *associated to the holomorphic 2-form $\Omega \in H^{2,0}(S^d)$ be the irreducible representation of $Gal(\mathbb{Q}(\mu_d/\mathbb{Q}))$ associated to the holomorphic 2-form $\Omega \in H^{2,0}(S^d)$ of the K3 surface S^d, where $d = 4, 6A, 6B$. Then the Mellin transforms $f_\Omega(S^d, q)$ of the L-functions $L_\Omega(S^d, s)$ associated to M_Ω are given by*

$$\begin{aligned} f_\Omega(S^4, q) &= \eta^6(q^4) \\ f_\Omega(S^{6A}, q) &= \theta(q^3)\eta^2(q^3)\eta 62(q^9) \\ f_\Omega(S^{6B}, q) &= \eta^3(q^2)\eta^3(q^6) \otimes \chi_3. \end{aligned}$$

All these functions are cusp forms of weight three with respect to $\Gamma_0(N)$ with levels $16, 27$ and 48, respectively. For S^4 and S^{6A} the L-functions can be written as $L_\Omega(S^d, s) = L(\wedge^2 f_d, s)$, where $f_d(q)$ are cusp forms of weight two and level 64 and 27 respectively. $\wedge^2 f(q) = \sum_n b_n q^n$ with $b_n = a_p^2 - 2p$.

It is an interesting question to ask how common modularity is for K3 surfaces. An argument that points to modularity as a common property can be made by combining mirror symmetry with the elliptic modularity theorem. The idea of the SYZ conjecture is based on a toroidal fibration structure of general Calabi-Yau varieties that is suggested by D-branes.

The modular forms that appear in these constructions admit complex multiplication, and allow an interpretation as generalized McKay-Thompson series associated to the Mathieu and Conway groups and thus hints at some connection to a string motivated notion of arithmetic moonshine.

4 Arithmetic of U-duality

4.1 U-duality

We give a brief and by no means complete of some aspects of U-duality following [Cec15]. For discussion of U-duality in M-theory we refer to [BBS06]. In $\mathcal{N} \geq$, $D = 4$ ungauged SUGRA the complete equations of motion are invariant under the non-compact 'hidden symmetry' G. G acts on the field strength F through a real symplectic representation V of generalized duality, and all couplings are uniquely determined once we know the G-representation V. U-duality can also be extended to gauged SUGRA with $\mathcal{N} = 1, 2$.
In the superstring/M-theory, the G symmetry is broken down to a discrete subgroup $G_{\mathbb{Z}}$. This symmetry is a good quantum invariance in the stringy framework. The discrete subgroups should be of arithemtic type. Let's consider for simplicity $G = SL(2, \mathbb{R})$. The subgroups that may be realized as the U-symmetry of a physical theory are the congruence subgroups, namely

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ mod N} \right\} \tag{59}$$

If the SUGRA is the low-energy limit of some fundamental theory, there are vector fields on which G is acting, which come form some unified non-Abelian gauge theory. In particular, the vectors carry electric and magnetic charges m^x, e_x. The dirac charge quantization condition implies that

$$[Q_1, Q_2] = m_1^x e_{2,x} - e_{1,x} m_2^x \in \mathbb{Z} \tag{60}$$

hence Q takes value in some lattice $L \subset \mathbb{R}^{2n}$, equipped with a symplectic form given by 60. The U-duality group should preserve this structure and hence the symmetry group should be some arithmetic group $G_S \subset G_{\mathbb{Z}} := G \cap Sp(2n, \mathbb{Z})$. The Vafa conjecture states that the non-compact space $G_S \backslash G/H$ has finite volume. These locally symmetric space also play an important role in the discussion of the Langlands conjecture below. The volume of $G_{\mathbb{Z}} \backslash G/H$

is

$$\mathrm{Vol}(G_{\mathbb{Z}}\backslash G/H) = \#Z(H)\frac{\mathrm{Vol}(G_{\mathbb{Z}}\backslash G)}{\mathrm{Vol}(H)} \tag{61}$$

The hard factor is $\mathrm{Vol}(G_{\mathbb{Z}}\backslash G)$. This is given by the following very elegant formula

$$\mathrm{Vol}(G_{\mathbb{Z}}\backslash G) = \#(\pi_1(G))\prod_{k=1}^{l}\zeta(a_i) \tag{62}$$

containing the Riemann ζ-function. Here $l = \mathrm{rank}(G)$ and $(a_1 = 2, a_2, \ldots, a_l)$ are the degrees of the basic Killing invariants of G.
In **??** discusses connections of U-duality to the Swampland conjectures formulated in [Ark+07]. In particular the role of Shimura varietues for $\mathcal{N} = 2$ supergravity theories is discussed.
Both tt^* geometry and the varation of the hodge structure both have deep connections. The problem one is interested in, is the classification of consistent $4d$ $\mathcal{N} = 2$ quantum gravity theories. On source of such theories is the compactification of Type II theory on Calabi-Yau varities. This indicates, that in order to understand the theory on the compactified space, one has to study the algebraic geometry of Calabi-Yau 3-folds.
The differential geometric description associates $4d$ $\mathcal{N} = 2$ ungauged SUGRA and special Kähler geometry to the real variation of the real variation of Hodge structure of weight 3 with $h^{3,0} = 1$. The Swampland describes the low energy effective field theories that look consistent but cannot be completed to a consistent theory of quantum gravity. In [Cec20] Cecotti argues, that a necessary condition for a effective theory to come from some UV theory should be of arithmetic nature.

4.2 The attractor mechanism

The attractor mechanism was first discovered by Ferrara, Kallosh and Strominger [FK96] and is related to dyonic black holes in supergravity. Moore discovered that fixed points of the attractor equations have interesting arithmetic properties [Moo98] [Moo04].
Type IIB supergravity is a theory of gravity together with extra $U(1)$ gauge fields, this means that there is a lattice of electric and magnetic charges. Since we are dealing with an abelian gauge group, this must be a real torus of the for $\mathbb{R}^r/2\pi L$ where L is a rank r lattice. The charge charge lattice is

then just

$$\Lambda = L \oplus L^* \tag{63}$$

In general the charge lattice for the compactification of IIB theory on a Calabi-Yau 3-fold X is

$$\lambda = H^3(X;\mathbb{Z}) \tag{64}$$

We are interested in black hole solutions in $d = 4, \mathcal{N} = 2$ compactification of type IIB string theory on a CY 3-fold X with the following properties:

1. Static, spherically, asymptotically Minkowski spacetimes M_4.
2. Dyonic of charge $\hat{\gamma} \in H^3(X,\mathbb{Z})$.
3. BPS, i.e. satisfy $\delta\lambda = \delta\phi = 0$ for a 4 real dimensional space of spinors.

From condition one we ontain

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}(d\vec{x})^2 \tag{65}$$

where $e^{2U} \to 1 + \mathcal{O}(1/r)$ for $r \to \infty$. Condition two and three lead to the so called attractor equations

$$\begin{aligned} \frac{d}{dr}(e^{-U}) &= \frac{|Z(\Omega;\gamma)|}{r^2} \\ \pi^{2,1}\left(e^{K/2}\frac{d\Omega}{dr}\right) &= i\frac{e^U}{r^2}\frac{Z}{|Z|}\gamma^{2,1} \end{aligned} \tag{66}$$

where the central charge is given by

$$Z(\Omega,\gamma) = e^{K/2}\int_\gamma \Omega. \tag{67}$$

These equations define a dynamical system on the complex structure moduli space of the Calabi-Yau 3-fold X and under mild conditions, the fixed points of this flow are the ones satisfying

$$\gamma = \gamma^{3,0} + \gamma^{0,3} \tag{68}$$

In the following we want to investigate some of the arithmetic properties of these so called attractor points. Suppose $\gamma \in H_3(X;\mathbb{Z})$ defines an attractor point $z_*(\gamma) \in \tilde{\mathcal{M}}$. Then we define the discriminant $D(\gamma) \leq 0$ via

$$|Z(z_*(\gamma);\gamma)|^2 = \sqrt{-D(\gamma)} \tag{69}$$

Let's focus on X of the form $K3 \times T^2$. The charge lattice is given as

$$H^3(K3 \times T^2; \mathbb{Z}) \cong II^{19,3} \oplus II^{19,3} \tag{70}$$

where $II^{19,3}$ is the even unimodular lattice of signature $((-1)^{19}, (+1)^3)$. So the charges are of the for (p, q) for $p, q \in II^{19,3}$. We can organize these data in a matrix

$$Q_{p,q} := \frac{1}{2} \begin{pmatrix} p^2 & -p \cdot q \\ -p \cdot q & q^2 \end{pmatrix} \tag{71}$$

we find that

$$D(\gamma) = -\det(Q_{p,q}) = (p \cdot q)^2 - p^2 q^2 \tag{72}$$

So the discriminant of a BPS state coincides with the discriminant of this quadratic form.
U-duality of type IIB acts on the BPS-states

$$\mathcal{H}_{BPS} = \oplus_\gamma \mathcal{H}^\gamma_{BPS} \tag{73}$$

the discriminant $D(\gamma)$ is invariant under the group $U(\mathbb{Z})$, but there might be different γ with the same discriminant. For $X = K3 \times T^2$, the U-duality group is a product:

$$U(\mathbb{Z}) = SL_2(\mathbb{Z}) \times O(6, 22; \mathbb{Z}) \tag{74}$$

In fact, two $\gamma = (p, q)$ and $\gamma' = (p', q')$ are equivalent if and only if the induced quadratic form are $SL_2(\mathbb{Z})$ equivalent, i.e. there exists $s \in SL_2(\mathbb{Z})$ such that

$$s Q_{p,q} s^t = Q_{p',q'} \tag{75}$$

A primitive discriminant is primitive if and only if

- $D \equiv 1 \pmod 4$ and is square-free
- $D = 4m$, where $m \equiv 2, 3 \pmod 4$ and m is square-free.

For a primitive discriminant D we define the class-number $h(D)$ to be the number of inequivalent quadratic forms with discriminant D. The set of classes $C(D)$ forms an abelian group of order $h(D)$ which is naturally identified with the ideal class group in the quadratic field $K_D \equiv \mathbb{Q}[i\sqrt{|D|}]$. *Example.* A first example for a non-trivial class number is provided by $D = -20$. There are two inequivalent quadratic forms

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \; x^2 + 5y^2 \tag{76}$$

$$Q_2 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \; 2x^2 + 2xy + 3y^2 \tag{77}$$

Assume $D < 0$. We can associate to a quadratic form a modulus $\tau \in \mathfrak{h}/PSL_2(\mathbb{Z})$. We write

$$ax^2 + bxy + cy^2 = a|x - \tau_Q y|^2 \tag{78}$$

where

$$\tau_Q := \frac{-b + \sqrt{D}}{2a} \tag{79}$$

The action of $SL_2(\mathbb{Z})$ on Q then becomes the standard action by Möbius transformation. In our example, we have

$$\begin{aligned} \tau_1 &= i\sqrt{5} \\ \tau_2 &= \frac{-1 + i\sqrt{5}}{2} \end{aligned} \tag{80}$$

A quadratic form is called reduced, if τ is in the fundamental domain of $\mathfrak{h}/PSL_2(\mathbb{Z})$, so Q_1 and Q_2 are both reduced.
Obviously $C(-20) = \mathbb{Z}/2\mathbb{Z}$. The class given by $[\tau_1]$ is the unit and $[\tau_2]^2 = [\tau_1]$.
We return to black holes in $K3 \times T^2$. For primitive discriminant, the number of U-inequivalent $\mathcal{N}(D) = h(D)$. For general discriminants, the number of inequivalent black holes is

$$\mathcal{N}(D) = \sum_m h(D/m^2) \tag{81}$$

where the sum is over m such that $D/m^2 \equiv 0, 1 \bmod 4$. The modulus for a quadratic form coming from a attractor black hole is given by

$$\tau(p,q) = \frac{p \cdot q + \sqrt{D_{p,q}}}{p^2} \tag{82}$$

where $D_{p,q} = (p \cdot q)^2 - p^2 q^2$. One can show that the attractor equations imply that this coincides with the complex structure moduli of the T^2 factor.
Let's denote the $K3$ factor of X as S. And consider the Neron-Severi lattice. The attractor equations imply that $NS(S) = \langle p, q \rangle_{\mathbb{Z}}^{\perp}$. The lattice has rank 20 and signature $(+1, (-1)^{19})$. Equivalently, we have:

$$H^{2,0}(S) \oplus H^{0,2}(S) = T_S \otimes \mathbb{C}, \quad \mathrm{rank}(T_S) = 2 \tag{83}$$

$K3$ surfaces satisfying these conditions are called *attractive K3 surfaces*. The key result about attractor K3 surfaces is the Shioda-Inose theorem:
Attractor points of this kind will be called rank two. On the other hand, attractor point which satisfy must be linearly dependent for some real constant λ. If an attractor complex structure does satisfy $\langle\hat{\gamma}_1, \hat{\gamma}_2\rangle \neq 0$ for some pair of charges γ_1, γ_2 is called an attractor point of rank one.
Moore formulated a row of conjectures concerning attractor varieties:

Attractor Conjecture 1: Suppose $\gamma \in H^3(X;\mathbb{Z})$ for a polarized CY 3-fold X defines an attractor point $z_*(\gamma) \in \tilde{\mathcal{M}}$ in the Teichmüller space of complex structures. Then the period vector is valued in a number field $\mathbb{P}^{h^{2,1}}(K(\gamma))$.

Attractor Conjecture 2: Suppose $\gamma \in H^3(X;\mathbb{Z})$ for a polarized CY 3-fold X defines an attractor point $z_*(\gamma) \in \tilde{\mathcal{M}}$ in the Teichmüller space of complex structures. Then the corresponding variety X_γ is arithmetic, and defined over a number field $\hat{K}(\gamma)$.

Attractor Conjecture 3: $\hat{K}(\gamma)$ can be choosen so that $\hat{K}(\gamma)K(\gamma)/K(\gamma)$ is Galois.

The attractor conjectures can be stated in a strong and a weak form. The strong form of the attractor conjectures state that the conjectures hold for all attractor points. The weak form claims them only for rank-two attractor points.
In [LT20a] Joshua Lam and Arnav Tripathy proof that the strong attractor conjectures do not hold in general in dimensions other than $1, 3, 5, 9$. The emphasised the role of Shimura varieties in the theory of attractor points. It is conjectured that Calabi-Yau varities who's moduli space is a Shimura variety has very special properties.
In [Yan21] Yang studies the arithmetics of rank two attractor points and the connection to special values of L-functions. This work builds on the work of Candelas, de la Ossa, Elmi and van Straten [Can+20] they study one parameter families of Calabi-Yau manifolds X_φ determined by the equation

$$1 - \varphi(X_1 + X_2 + X_3 + X_4 + X_5)\left(\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4} + \frac{1}{X_5}\right) \tag{84}$$

This family has at least three attractor points of rank two, occurring at a rational value

$$\varphi = -1/7 \tag{85}$$

and a pair of values corresponding to the roots of the quadratic equation $\varphi^2 - 66\varphi + 1 = 0$

$$\varphi_\pm = 33 \pm 8\sqrt{17} \tag{86}$$

As we have seen a rank two attractor point, the two dimensional vector space $V = H^{3,0} \oplus H^{0,3}$ is the complexification of a rank two lattice in $H^3(X, \mathbb{Z})$. The space $V^\perp = H^{2,1} \oplus H^{1,2}$ is orthogonal to V under the natural symplectic product on three forms and is also the complexification of a rank two sublattice of $H^3(X, \mathbb{Z})$. This splitting must be visible in the arithmetic structure of X. It follows from arithmetic considerations that, the splitting at a rank two attractor point gives rise to modular forms of weight two and four that are determined by the way that the two factors of $R(T)$ way with p. The modular groups that arise in that way have far reaching consequences. For example, the periods of the attractor variety and further quantities like the central charge can be expressed in terms of critical L-values of these modular forms.

Any projective variety defined over $\mathbb{Q}$ can be defined by polynomial equations with integral coefficients. Tor any prime we may then ask how many solutions these equations have over $\mathbb{F}_q$. Let N^r for be this number for $q = p^r$. These numbers are collected into the generating function

$$\zeta_p(T) = e^{\sum\limits_{r=1}^{\infty} N_r \frac{T^r}{r}} \tag{87}$$

known as the Artin-Weil Zeta Function. The form of $\zeta(T)$ is governed by the Weil Conjectures. The important point for our discussion is that $\zeta_p(T)$ is a rational function of T. Let H^k be any Weil cohomology (for example l-adic étale cohomology) and $\mathrm{Frob}_{p,k} : H^k((X) \to H^k(X)$ be the induced p-Frobenius homomorphism. The we define

$$R_{p,k}(T) = \det(1 - T\,\mathrm{Frob}_{p,k}^{-1}) \in \mathbb{Z}[T] \tag{88}$$

In particular, the degree of R_k is equal to the k-th Betti-number b^k of the complex variety defined over X.

The function ζ_p is then given by

$$\zeta(T) = \frac{R_1 R_3 \ldots R_{2n-1}}{R_0 R_2 \ldots R_{2n}} \tag{89}$$

where we dropped the p dependence in our notation.
Consider a Calabi-Yau 3-fold with $h^{21} = 1$. Then

$$\zeta(T) = \frac{R(T)}{(1-T)(1-pT)^{h_{11}}(1-p^2T)^{h_{11}}(1-p^3T)} \tag{90}$$

The polynomial $R(T)$ Is of degee 4 if the reduction mod p is smooth and we refer to it as the Frobenius polynomial. It is of the form

$$R(T) = 1 + aT + bpT^2 + ap^3T^2 + p^6T^4 \tag{91}$$

and so is determined by two integers $a, b \in \mathbb{Z}$.
We return now to our rank two attractor variety X_{φ_*}. The third cohomology group splits as

$$H^3(X, \mathbb{Q}) = \Lambda_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}}^{\perp} \tag{92}$$

By the Hodge conjecture this splitting is supposed to have a geometric origin. This means that the splitting should induce a decomposition of $R(T)$ over $\mathbb{Z}$ into two quadratic factors as

$$R(T) = (1 - \alpha pT + p^3T^2)(1 - \beta T + p^3T^2). \tag{93}$$

The first factor comes from $H^{2,1} \oplus H^{1,2}$ and looks like the factor associated to an elliptic curve:

$$r1 - \alpha T + pT^2 \tag{94}$$

The second factor has the form of the numberator of the ζ-function of a rigid Calabi-Yau manifold.
The arithmetic information of the Frobenius transformations for various p can conveniently be backed into a Galois representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_4(\mathbb{Q}_l) \tag{95}$$

that maps a Frobenius element at p to the matrix Frob_p. In the case of a splitting, we end up with 2-dimensional representations

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_l) \tag{96}$$

By Serre's conjecture (which is by now proven), such representations are atteched to modular forms of specific weight and conductor. Gouvea and Yui have shwn the modularity of rigid Calabi-Yau threefolds over $\mathbb{Q}$ can be derived from it. Furthermore, for 3-folds that split in the above way the

modularity has been proven, which means that the coefficients a_p and b_p are Fourier coefficients of cusp forms of weight 2 and 4 for some congruence group $\Gamma_0(N)$ of the modular group.
If the variety X is not defined over $\mathbb{Q}$ but over some number field K, the situation is more complicated, as we have to then study representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. In the case of a totally real field one in general expects Hilbert modular forms.
Let's briefly discuss the physical significance of this considerations. The well known conjecture Ooguri, Strominger and Vafa states that one may compute the entropy of $\mathcal{N}=2$ black holes, that arise in Type II compactifications, by topological string free energies. This means one calculates Gopakumar–Vafa invariants of the mirror Calabi-Yau. At genus zero and one, when evaluated at a rank two attractor point, this may be expressed in terms of L-function values and the modulus of an elliptic curve.
On the other hand, the area of a black hole is expected to have an expression in terms of the periods of a rigid Calabi-Yau manifold. The periods of a rigid Calabi-Yau manifold are expected to be given by critical L-values.

Let X be an attractor variety for which the attractor conjectures hold. In that case X is defines over (some finite abelian extension of) the class field of K_D. The Galois group $Gal(\hat{K}_D/K_D)$ is just the ideal class group and hence, the Galois group of this extension acts on permutes the attractor moduli. Thus the Galois group extends the U-duality group and "unifies" the different attractor points at discriminant D.
The Galois symmetry is not a symmetry of string theory since it for instance the BPS spectrum at different attractor points does not agree, so the physical meaning of the this symmetry is unclear.
More relations to arithmetics could be found when studying F-theory compactifications. For instance Moore conjectured that the 7-brane of F-theory will be located at arithmetic points, and their position will be permuted by $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

5 Mirror Symmetry

Maybe the greatest achievements of String Theory are the implications for pure mathematics. The most prominent example is mirror symmetry. The mirror conjecture predict the existence of mirror pairs (M, W) of Calabi-Yau

threefolds such that the complexified Kähler moduli space $\mathcal{M}_K(M)$ of M is isomorphic to an open subset of the complex moduli space $\mathcal{M}_C(W)$ of W. The isomorphism between these spaces is called the mirror map. We already have seen some evidence, that Calabi-Yau 3-folds and string theory compactified on them contain interesting arithmetic information.
In algebraic geometry, given a family of varieties, usually the singular fibers encode a lot of essential information about the properties of the family of varieties, usually the singular fibers encode a lot of essential information about the properties of the family. The most important types of singular fibers that occur in the mirror family are

1. the large complex structure limit.

2. the conifold singularity of one parameter mirror family.

In the section on L-functions we briefly introduced the BSD-conjecture. There is a far-reaching generalization known as Beilinson's conjecture. The conjecture says the following

Conjecture 5.1 *Suppose X is a smooth projective variety defined over $\mathbb{Q}$ and n is an integer such that $0 < 2n - 1 < 2\dim(X)$ then*

1. *The height pairing*

$$h : (CH^n(X)_0 \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} (CH^{\mathrm{X}+1-n}(X)_0 \otimes_{\mathbb{Z}} \mathbb{Q}) \to \mathbb{R} \tag{97}$$

   is non-degenerate.

2. *The dimension of $CH^n(X)_0 \otimes +\mathbb{Z}\mathbb{Q}$ is finite which equals the order of vanishing of $L(h^{2n-1}(X), s)$ at $s = n$.*

3. *The leading coefficient $L^*(h^{2n-1}(X), n)$ is in $c^+(h^{2n-1}(X)(n)) \cdot \det(h)\mathbb{Q}^*$.*

In [Yan18] provides an example that supports Beilion's conjecture, by studying the arithmetic geometry of the conifold transition in the mirror family of the quintic in $\mathbb{P}^4$.
This gives a first example, where mirror symmetry exhibits arithmetic properties. In mirror symmetry one often studies families of varieties parametrized by varieties. As mentioned above the singular fibers are of special interest. One way the study the discriminant locus, i.e. the locus in the base space at which the fibers are singular, is to study the monodromy group, i.e. the

action of the fundamental group of the complement of the singular locus on the cohomology ring of a fixed non-singular fiber [LTY00]. In number theory the action of the Galois group on étale cohomology plays the role of the monodromy representation for a family of varieties.
Consider a one parameter family of n dimensional Kähler manifolds

$$\pi : \mathcal{X} \to D \tag{98}$$

over a disk D. Assume that X is a smooth algebraic manifold and that for each $t \neq 0$, $\pi^{-1}(t) = X_t$ is a non-singular n dimensional Kähler manifold. In this case the singular locus is just the origin $t = 0$ and the we get a representation of $\pi_1(S^1) = \mathbb{Z}$ in the group of diffeomorphisms of X_t. We get a generator $\phi \, in \, \mathrm{Diff}(X_t)$ of the monodromy group as the image of $1 \in \pi_1(S^1)$. We call ϕ the geometric monodromy. The induced action of ϕ on $H_n(X_t, \mathbb{Z})$ will be called the monodromy operator and will be denoted by T. The main result about T is that we have always:

$$(T^N - \mathrm{id})^{n+1} = 0 \tag{99}$$

for some positive integer N. A point of maximal unipotent monodromy satiesfies:

$$(T^n - \mathrm{id})^n \neq 0 \tag{100}$$

Classical mirror symmetry takes place at points of Maximal Unipotent Monodromy (MUM). It relates enumerative geometry (A-model) and variation of Hodge structure (B-model). Underlying this correspondence is a physical duality which mathematically is described as an equivalence of symplectic categories and complex categories. In the B-model we find some interesting arithmetic properties, as shown by Kontsevich, Schwarz and Vologodsky [KSV06]. They showed the integrality of Gopakumar-Vafa invariant, which we already briefly came across in the discussion of attractor points of rank two above and gave an expression for the instanton number in terms of p-adic cohomology. It is a natural question to ask how this arithmetic properties are mapped to the A-model.
In [SVW15] Schwarz, Vologodsky and Walcher introduced s-functions motivated by the expansion of the $\mathcal{N} = 2$ prepotential in the large complex structure limit (that is at a point of maximal unipotent monodromy) which take the form

$$F^(0) = \cdots + \sum_d M_d q^d = \sum_d N_d Li_3(q^d) \tag{101}$$

where $M_d \in \mathbb{Q}$ are Gromov-Witten invariants and $N_d \in \mathbb{Z}$ are BPS invariants.

Definition 5.2 *Let $s \in \mathbb{N}$. An s-function with rational coefficients is $W \in z\mathbb{Q}[[z]]$ such that*

$$W = \sum_{n=1}^{\infty} n_d Li_s(z^d) \tag{102}$$

where $n_d \in \mathbb{Z}$ and

$$Li_s(z) = \sum_{d=1}^{\infty} \frac{z^k}{k^s} \in z\mathbb{Q}[[z]] \tag{103}$$

is a s-logarithm. For instance $Li_1(z) = -ln(1-z)$

Let's consider the dilogarithm Li_2. We define Rogers' dilogarithm

$$L_2(z) = Li_2(z) + \frac{1}{2}log(z)log(1-z) - \frac{\pi^2}{6}. \tag{104}$$

For $1 > x > y > 0$ this function satiesfies the functional equation

$$L_2(x) - L_2(y) + L_2(\frac{y}{x}) - L_2(\frac{1-x^{-1}}{1-y^{-y}} + L^2(\frac{1-x}{1-y} = 0 \tag{105}$$

This is just one of many functional equation that is satiesfied by the dilogarithm. The dilogarithm has a single-valued cousin: the Bloch-Wigner function

$$\mathcal{L}_2(z) := \mathrm{Im}(Li_2(z)) + \arg(1-z)log(|z|) \tag{106}$$

Let (z_1, z_2, z_3, z_4) be four distinct points in $\mathbb{C}P^1$. We define the cross-ration

$$r(z1, z_2, z_3, z_4) := \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \tag{107}$$

Then the Bloch-Wigner function satisfies

$$\sum_{i=0}^{4} \mathcal{L}_2(r(z_0, \dot{;}\hat{z}_i, \ldots, z_4)) = 0. \tag{108}$$

This equation plays a similar role for the dilogarithm as $log(|xy|) = log(|x|) + log(|y|)$. Any measurable function satisfying 108 is proportional to $\mathcal{L}_2(z)$. As mentioned before definition of s-functions is motivated by the problem of calculating the instanton corresctions in the A-model. Let's give yet another example where an s-function shows up. The instanton corrections induce a superpotential which we may write as

$$W(q) = \sum_{\beta \in H_2(X,L)} n_\beta Li_2(q^\beta) \tag{109}$$

where X is a Calabi-Yau 3-fold and $L \subset X$ is a Lagragian submanifold. So we see that a 2-function shows up.
We define the logarithmic derivative

$$\delta := z\frac{d}{dz} \tag{110}$$

If W is an s-function, δW is an $(s-1)$-function and hence δ^{s-1} is a 1-function, i.e. an expansion in powers over logarithms. This motivates the following definition

Definition 5.3 *An s-function W is called*

- ***rational*** *if $e^{\delta^{s-1}W}$ has an expansion of rational functions*
- ***locally analytic*** *if it has finite radius convergence in the complex topology*
- ***analytic*** *if it can be continued to a dense subset of $\mathbb{C}$*

One observes, that W is an s-function if and only if for all prime numbers p

$$\frac{1}{p^s}\operatorname{Frob}_p W_p - W_p \in z\mathbb{Z}_p[[z]] \tag{111}$$

where W_p is just W viewed as a power series over $\mathbb{Q}_p$ and Frob_p sends z to z^p.
The dilogarithm is special among the polylogarithms since it is given as a period on some curve:

$$Li_2(z) = \int log(x) d\, log(z) \tag{112}$$

on $C = \{x + z - 1 = 0\} \subset \mathbb{C}^\times \times \mathbb{C}^\times$. This means $Y = 1 - z$. If we choose some $Y \in 1 + z\mathbb{Z}[[z]]$, this defines an automorphism of $z\mathbb{Z}[[z]]$ via

$$z \mapsto \tilde{z} = -zY \tag{113}$$

and we define $\tilde{Y} = \frac{z}{\tilde{z}}$. Then we have the following theorem

Theorem 5.4 *If $W = \delta^{-1}log(Y) \in z\mathbb{Q}[[z]]$ is a 2 function, then so is $\tilde{W} = \tilde{\delta}^{-1}log(\tilde{Y}) \in \tilde{z}\mathbb{Q}[[\tilde{z}]]$.*

This is what is called framing. For $s > 2$ the theorem is not two in general. In the geometric picture, such a framing corresponds to a different choice of coordinates for C.
We can generalize the notion of a s-function to algebraic number fields K. With ring of integers $\mathcal{O}_K$. We get a Frobenius lift on $\mathcal{O}_{K_\mathfrak{p}} \subset K_\mathfrak{p}$ which we can extend to K. We then define an s-function to be an element is $W \in zK[[z]]$ such that for all unramified primes

$$\frac{1}{p^s}\,\mathrm{Frob}_p W_p - W_p \in z\mathcal{O}_{K_p}[[z]] \tag{114}$$

For example, for ζ some root of unity, $Li_s(\zeta z)$ is an s-function over $\mathbb{Q}(\zeta)$. The structure of rational 2-functions over a number field is rather simple:

Theorem 5.5 *If $W \in zK[[z]]$ is a rational 2-function, then K is an abelian extension of $\mathbb{Q}$.*

We now go back to points with maximal unipotent monodromy. Recall that we consider a smooth family of Calabi-Yau manifolds

$$\pi : Y \to B \tag{115}$$

where B is smooth quasi-projective complex curve, and the generic member $Y_b = \pi^{-1}, b \in B$ is simply connected and $b_3 = 4$.

5.1 Quick recaps of variation of Hodge structure

A variation of Hodge structure of weight w on a complex manifold X is a couple $(V_\mathbb{Z}, \mathcal{F}^\bullet)$ where $V_\mathbb{Z}$ is a locally constant sheaf of finitely- generated Abelian groups on X, and $\mathcal{F}^\bullet$ is a finite descreasing filtration of $V = V_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_X$, by holomorphic subbundles, subject to the following conditions:

1. The flat connection ∇ on V defined by

$$\nabla(v \otimes f) = v \otimes df \tag{116}$$

   for $v \in H^0(V_\mathbb{Z}, X)$ and $f \in H^0(\mathcal{O}_X, X)$ satisfies

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_X \tag{117}$$

2. For each $x \in X$, the pair $(V_{\mathbb{C},s}, \mathcal{F}^\bullet(s))$ is a Hodge structure of weight w.

The variation of Hodge structure is polarized if there exists a flat bilinear pairing

$$\langle \cdot, \cdot \rangle : V_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \to V_{\mathbb{Z}} \tag{118}$$

which is a polarization on every fiber.
Going back to our family of Calabi-Yau manifolds, we take $V_{\mathbb{Z}} = (R^3\pi_*\mathbb{Z})_0$ with fibers $H^3(Y_b, \mathbb{Z})$ of rank 4. $H^3(Y_b, \mathbb{Z})$ naturally carries the hodge filtration and we have a polarization

$$\{\cdot, \cdot\} : V_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \to \mathbb{Z}(-3) \tag{119}$$

Consider now a MUM point a and restrict to a small neighbourhood U around that point. We get a monodromy weight filtration $W_0 \subset W_2 \subset W_4 \subset W_6$ of $V_{\mathbb{Q}}$ on U.
$W_\bullet$ and $F^\bullet$ pair up to define a variation of mixed Hodge structure over U that is Hodge-Tate, i.e., it admits a composition series

$$\mathcal{L}_0 \to \mathcal{L}_2 \to shL_4 \to shL_6 \tag{120}$$

with

$$\mathcal{L}_{2s}/\mathcal{L}_{2(s-1)} = Gr^W_{2s} \cong \mathbb{Z}(-s) \tag{121}$$

From this one can derive the data of a "canonical coordinate" q and the prepotential $\mathcal{F}$ mentioned before. Defining the logarithmic derivative $\delta = q\frac{d}{dq}$ we can calculate the Yukawa coupling

$$\mathcal{C} = \delta^3 \mathcal{F} \tag{122}$$

In [KSV06] showed that for rational MUM points q and $\mathcal{C}$ can be equivalently be calculated over $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}_p$. One is also interested in irrational MUM points. For these one finds that the prepotential gives 3-functions over some number fields [**Walcher2020AGQFT**].
If one extends Y by a family of 2-cycles over B, one can construct example such that the solutions of the Picard-Fuchs equation give analytic 2-functions over *non-Abelian* number fields.
The number fields that appear in the extended theory seem to arbitrary. Recall that we found the s-functions by studying the large complex structure limit. It is not clear, what the interpretation of the arithmetic properties in enumerative geometry could be.

6 The Langlands Correspondence

The Langlands correspondence is one of the most far reaching set of conjectures in all in mathematics and involves almost all areas of pure mathematics, in particular number theory, geometry, representation theory and analysis. Thanks to the work of Kapustin and Witten [KW06] it is now known that on the physical side supersymmetric gauge theories is also part of Langlands. It is hence a natural place to search for connection between number theory and physics.

6.1 Electric-Magnetic Duality

Many theories in physics are described by a Lagrangian, which is invariant under some local or global symmetry transformation.
The usual example for a theory with local gauge symmetry is Yang-Mills theory. Lets for now focus on 4 dimensions. Let X be a 4-dimensional (pseudo-)Riemannian manifold X. Let G be a Lie group and (E, ω) a principal G-bundle E on X with connection $\omega \in \Omega^1(E, \mathfrak{g})$. The action of this field is given by

$$S(E, \omega) = \frac{1}{e^2} \int_X tr(F_\omega) \wedge \star F_\omega) + i\theta \int_X tr(F_\omega \wedge F_\omega) \tag{123}$$

where $F_\omega \in \Omega(X, \mathfrak{g}_E)$ is the curvature of ω given by

$$F_\omega = d\omega + \frac{1}{2} [\omega \wedge \omega] \tag{124}$$

The great inside by Montonen and Olive was that if we consider Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry and introduce the complex coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \tag{125}$$

with is an element of the upper half-plane. Then the quantum theory is invariant under the action of $SL_2(\mathbb{Z})$ on the upper half-plane. To be more precise. The path integral in invariant under $\tau \mapsto \tau + 1$. Further more, the original theory is equivalent to the theory with gauge group ${}^L G$ and coupling constant

$$\frac{-1}{n_{\mathfrak{g}} \tau} \tag{126}$$

Where $n_{\mathfrak{g}}$ is the lacing number of the root lattice of $\mathfrak{g}$. Looking at $\theta = 0$ the duality relates e^2 with $(\frac{1}{e'})^2$. That means, that under this S-duality transformation a strongly coupled theory is equivalent to a weakly coupled theory.

6.2 Arithmetic Topology

It is not obvious how one can relate theoretical physics which is mainly concerned with manifolds to the theory of number field who's geometric properties are very different. We already gave seen a few examples how such connections arise. The arithmetic topology analogy states that the number fields in some sense build a category of generalized 3-manifolds and places of the number field are in some sense knots embedded in the manifold. It has itself so far not manifested in a concrete theorem, but it is still conceptual very interesting and the analogy seems to be to deep to be pure coincidence. Recently arithmetic topology has been the starting point to understand the role of physics in number theory. This viewpoint is among others promoted by Minhyong Kim [Kim18b] [Kim18a] [Chu+20]. We want to give a brief overview of the analogy between knots and primes following [Mor09].
First, let us recall the objects we are dealing with in algebraic number theory. Usually we want to study number fields, i.e. finite extensions of $\mathbb{Q}$. Inside of $\mathbb{Q}$ we have the ring of integers $\mathbb{Z}$. It's integral closure in a finite extension field $K|\mathbb{Q}$ is denoted by $\mathcal{O}_K$, which are the integers inside K. We can think of the ring of integers in geometric terms as $\mathrm{Spec}(\mathcal{O}_K)$. The points of this space are the primes of $\mathcal{O}_K$. Furthermore, we can associate infinite primes to a number fields which correspond to embedding $K \hookrightarrow \mathbb{C}$ determine the Archimedean absolute values on K. Each maximal prime $\mathfrak{p}$ gives rise to an embedding $\mathrm{Spec}(\mathcal{O}_K/\mathfrak{p}) \to \mathrm{Spec}(\mathcal{O}_K)$.
We want to attach some geometrical interpretation to this data. For this we consider at the étale cohomology of $\mathrm{Spec}(\mathcal{O}_K)$. Artin and Verdier proved that $Spec(\mathcal{O}_K)$ has cohomological dimension 3 (up to 2-torsion) and enjoys some version of Poincaré duality and hence we might view $\mathrm{Spec}(\mathcal{O}_K)$ as some kind of 3-manifold.
Continuing to $\mathrm{Spec}(\mathcal{O}_K/\mathfrak{p}) \simeq \mathrm{Spec}(\mathbb{F}_q)$ one naively might think this is just a point. However it posses non-trivial coverings (i.e. étale maps $X \to \mathrm{Spec}(\mathbb{F}_q)$, which means that X is the spectrum of a finite product of seperable extensions of $\mathbb{F}_q$). When we study fundamental groups in topology, they are just given by the group of deck transformations. In our setting, this is precisely

the galois group of the seperable extension and the fundamental group is the group group of deck transformations of the universal cover. Hence we see that $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{F}_q)) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$. In topology and physics we don't think about pro-finite completions and we think of $\hat{\mathbb{Z}}$ just as the natural arithmetic analog of $\mathbb{Z}$. We can thus think of $\mathrm{Spec}(\mathbb{F}_q)$ as an analog to the Eilenberg-Maclane space $K(\mathbb{Z}, 1) = S^1$. This S^1 is embedded in the "3-manifold" $\mathcal{O}_K$ as a knot. This is the fundamental observation by Mazur, which leads to an interesting analogy between knots and primes.
Given a prime $\mathfrak{p}$ we can complete K with respect to $\mathfrak{p}$ and denote the completion $K_\mathfrak{p}$. Geometrically completing with respect to a prime ideal means that we consider an infinitesimal neighborhood of the prime p. Thus we should think of the ring of integers $\mathcal{O}_{K_\mathfrak{p}}$ as a tubular neighborhood of the knot given by $\mathfrak{p}$ inside $\mathcal{O}_K$. As the field of fractions $K_\mathfrak{p} = \mathrm{Frac}(\mathcal{O}_{K_\mathfrak{p}})$ is the the complement of $\mathbb{F}_q$ (looking at the spectrum $\mathrm{Spec}(\mathcal{O}_{K_\mathfrak{p}})$, then $\mathbb{F}_q$ corresponds to the closed point and $K_\mathfrak{p}$ corresponds to the open point). So in this analogy $\mathrm{Spec}(K_\mathfrak{p})$ should be thought of as the boundary torus of a tubular neighborhood.

6.3 Arithmetic Field Theory

To make this analogy precise and put the ideas of Kim on solid foundations is ongoing research by C. Barwick and P. Haine [BH21]. So far their have been no publication discussing the physical result of their work, but we present here the content of a talk given by Barwick at AGQFT [Bar]. We might review this work at a later point.

6.4 Topological Quantum Field Theory

Topological field theories are the simplest field theories we know since their Hamiltonian is trivial. This makes things a lot easier than in ordinary QFTs. The basic idea is that we describe the a TFT by some functor $\mathcal{F}$. This functor should attach a number to a closed n-manifold P which we may think of as the partition function. To a $(n-1)$-manifold, we attach a vector space, which we think of as a space of functionals on the space of field on M. Now we would like to attach some ($\mathbb{C}$-linear) category to any $(n-2)$-manifold, representing boudnary conditions. The natural choice is the category of sheaves on $\mathcal{F}(N)$. Mathematically one has to consider the symmetric monoidal

2-category $(Bord_{n-2,n}, \sqcup)$ with objects being closed $(n-2)$-manifolds, 1-morphisms are $(n-2)$-manifolds with boundary $N_1 \sqcup N_2$, and 2-morphisms are n-manifolds with corners. A 2-extended TFT is a symmetric monoidal functor:

$$Z : (Bord_{n-2,n}, \sqcup) \to (\mathcal{C}, \otimes) \tag{127}$$

where $(\mathcal{C}, \otimes)$ is some symmetric monoidal 2-category. Moreover we want it to satisfy the following conditions:

$$\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}}) = \mathrm{Vect} \tag{128}$$

and

$$Z(\emptyset_{n-1}) = \mathbb{C} \tag{129}$$

6.5 Gauge-Theoretic Langlands correspondence

Recall what Kapustin-Witten taught us: The starting point is the electric magnetic duality of $\mathcal{N} = 4$ $d = 4$ SYM discussed above.
This theory can be twisted in a few different ways and one of these twists (called "geometric Langlands" or "GL_n-twist) leads to a ($\mathbb{P}^1(\mathbb{C})$ family of topological field theories. They are parametrized by complex parameters ϕ and $\psi^\vee$ for G and ${}^LG$ respectively. Then equivalence under $\psi^\vee = -1/n_{\mathfrak{g}}$ (or more generally under $SL_2(\mathbb{Z})$), becomes the duality of "quantum geometric Langlands".
The basic case the recovers the ordinary geometric Langlands correspondence is the duality between $\psi = 0$ for G ("the A-model") and $\psi^\vee = \infty$ for $G^\vee = {}^LG$ ("the B-model").
For the geometric langlands correspondence we study compactify the theory on a fixed oriented 2-manifold C i.e. we only consider 4-manifolds of the form $\Sigma \times C$ where Σ is some other 2-manifold (possibly with corners). We study the effective 2-dimensional TFT on Σ.
Assume Σ has a boundary. The we have to assign a boundary condition $\mathcal{B}$ to this boundary. Boundary conditions make a category and the morphisms between boundary conditions have the structure of a vector space over $\mathbb{C}$. In particular $\mathrm{Hom}(\mathcal{B}, \mathcal{B})$ is a $\mathbb{C}$-algebra for every $\mathcal{B}$ and $\mathrm{Hom}(\mathcal{B}, \mathcal{B})$ is a left module for $\mathrm{Hom}(\mathcal{B}, \mathcal{B})$.
The key observation is that electric-magnetic duality will give an isomorphism between the category $\mathcal{C}$ associated to C in the G theory at $\psi = 0$ and the category $\mathcal{C}^\vee$ associated to C in the $G^\vee$ theory at $\psi^\vee = \infty$. Moreover there

is a natural mapping between certain functors on $\mathcal{C}$ and $\mathcal{C}^\vee$. These functors are so called line operators inserted along 1-dimensional submanifolds of Σ. In the A-model at $\psi = 0$ the operators are so called 't Hooft operators on the B-side at $\psi = \infty$ we find the so called Wilson operators. In this context we want to think of line operators as functors from the category of boundary conditions to itself. Given a line operator T that runs along a boundary with some boundary condition $\mathcal{B}$ makes a new boundary condition $T\mathcal{B}$. We think of T being inserted arbitrarily close to the boundary and $T\mathcal{B}$ be the effective composite boundary condition. Remembering that our theory is really 4-dimensional rather that 2-dimensional, we can use the compact dimensions of C to commute two line operators, so we see, that the Line operators are commuting functors on the category of boundary conditions.
The 't Hooft operators $T(p)$ labelled by $p \in C$ correspond to the Hecke functors of the geometric Langlands correspondence whereas the Wilson operators $W(p)$ correspond to its dual. These functors also depend on a representation R of $G^\vee$.

6.6 Arithmetic quantum Langlands correspondence

We want to connect this story to number theory. This work is very recent due to Ben-Zvi, Sakellaridis and Venkatesh. There's no paper out yet, but Ben-Zvi gave a course on the fundations of the topic in Spring 2021 [Ben21a] and also gave some talks [Ben21b]. The first idea is to view the theory of automorphic forms as quantum mechanics on arithmetic locally symmetric spaces. To be more precise, we study harmonic analysis on spaces of the form

$$\mathcal{M}_G = \Gamma \backslash G / K \tag{130}$$

where Γ is an arithmetic lattice and K is a maximal compact subgroup. The standard example is

$$G = SL_2(\mathbb{R}) \qquad \Gamma = SL_2(\mathbb{Z}) \qquad K = SO_2(\mathbb{R})$$

So we can consider the theory of modular forms as quantum mechanics on this arithmetic locally symmetric space. This in principal works for any reductive group G. I.e. study spectral theory of some Laplacian acting on $L^2([G])$, where

$$[G] = G(\mathbb{Z}) \backslash G(\mathbb{R}) / K \tag{131}$$

or some variant of this. This quantum mechanics problem is very special. For instance there are Hecke operators T_p for each prime p.
For general G the Hecke operator at p is additionally labelled by a representation of the Langlands dual group of $G^\vee_{\mathbb{C}} \circlearrowright V$.
To make contact with number theory we look at number fields: Let $F/\mathbb{Q}$ finite

$$[SL_2]_F := SL_2(\mathcal{O}_F)\backslash SL_2(F \otimes_{\mathbb{Q}} \mathbb{R})/K \tag{132}$$

For instance when we take a some imaginary quadratic field $F = \mathbb{Q}(\sqrt{-d})$, we get an arithmetic quotient of $\mathbb{H}^3$, i.e. a hyperbolic 3-manifold.
The Langlands correspondence relates automorphic forms with Galois representations. In the quantum mechanics interpretation of the Langlands correspondence, the Galois side or B-side is given by the spectral theory that solves the quantum system. This means the Galois side tells us how to simultaneously diagonalize all Hecke operators at p. Recall that the Hecke operators are labelled by some representation of $G^\vee$. Then we can associate to a Hecke operator a function on the space of Galois representations on $G^\vee$, by taking the trace of the image of the Frobenius element under $Gal(\overline{F}/F) \to G^\vee \to GL(V)$. This corresponds to the Wilson operators we saw in the gauge theoretic geometric Langlands correspondence.
We would like to construct an arithmetic extension of the $4d$ $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group G which Kapustin-Witten related to the geometric langlands correspondence.
The idea is to use the arithmetic topology dictionary discussed above to interpret number fields (or rather there ring of integers) as 3-manifolds.
To do construct a full arithmetic quantum field theory is difficult, however one can consider a topological quantum field theory. For us that means, we consider our space of fields to be $L^2([G]_F)$-value differential forms and take cohmology.
In the Kapustin-Witten interpretation of geometric Langlands, the 't Hooft operators are inserted along a knot K and labelled by some representation of the Langlands dual group $G^\vee_{\mathbb{C}}$ using the arithmetic topology, a 't Hooft operator should correspond to some prime and a representation of the dual group. Indeed this is exactly the data labelling the Hecke operators on the automorphic side of the geometric Langlands correspondence.
Electric magnetic duality indicates that there should be a duality between the for defects of higher codimension as well. We summarize the duality between the A- and B-side in the following table 1.

A-side: automorphic / magnetic	B-side: spectral / electric
topology of arithmetic loc. sym. spaces	algebraic geometry of Galois representations
1d defects Hecke / 't Hooft	Wilson / trace
2d defects congruence subgroups	singularities of flat connections
3d defects (boundary conditions) periods	L-functions

Table 1: Langlands/Electric-Magnetic Duality

In the correspondence between L-functions and periods we are most interested in the so called *Hecke period* of a cusp form φ:

$$\mathcal{P}_T(\varphi) = \int_0^\infty \varphi(iy) y^s \frac{dy}{y} \tag{133}$$

This is the integral over the vertical line in $\mathfrak{h}$. This Hecke period is related to the L-function of the cusp form by

$$\mathcal{P}_T(\varphi) = \frac{\Gamma(s)}{(2\pi)^s} L(\varphi, s) \tag{134}$$

The L-function of the cusp form is equal to the L-function of some $2d$ Galois representation ρ

$$L(\varphi, s) = L(\rho, s) := \prod \frac{1}{\det(1 - p^{-s}\rho(F_p))} \tag{135}$$

To calculate more general periods one to integrates some automorphic form, i.e. some function in $[G]_F$, over some subspace associated to a subgroup $[H]_G$. This integral might be ill defined. In order to describe a class of spaces on which the integral is well defined, it is actually better to think about a G-space X over which we integrate. For some subgroup $H \subset G$ we would for instance consider the space $X = G/H$ on which G is acting.
For the integral to be well defined, we need some finiteness condition. It turns out, that a sufficient and still quite general condition is for X to be *spherical*:

Definition 6.1 *A variety X with an action of some reductive algebraic G on X is called spherical if for a Borel subgroup B of G, there are only finitely many orbits of the B-action, equivalently, X contains an open, dense B-orbit.*

Some examples of spherical varieties are

- Toric varieties
- Flag varieties
- Symmetric spaces
- ...

We want to match the theory of Periods and the theory of L-functions. The L-function of $\rho : Gal \to G^\vee$ is labelled by representations V of $G^\vee$.
This data is completely unrelated to data of a spherical varieties $G^\vee_{\mathbb{C}} \circlearrowright X$.
To match this data we have to enrich the theory of periods by studying version of Langlands relative to X. Sakellaridis and Venkatesh proposed to study the harmonic analysis of $G \circlearrowright L^2(X)$ [SVV17].
We want to interpret Periods and L-functions in terms of the much richer structure of boundary theories for A_G and $B_{G^\vee}$.
A Quantum field theory on $M \times [0,1]$ with local boundary conditions produces a state on any M. A_G boundary theory uniformly encodes relative Langlands. The global theory includes the period $\mathcal{P}_X$, the ϑ-series, the relative Trace Formula, while the local theory structure contains information about representations of $L^2(X)$ and the Plancherel measure.
In the physical picture Gaiotto-Witten showed that SUSY boundary theories in $\mathcal{N} = 4$ SYM for G are labelled by holomorphic Hamiltonian G-spaces [GW09]. Therefore there is a duality between Hamiltonian actions of G and $G^\vee$.
In order to obtain a Hamiltonian action it is natural to pass to the cotangent bundle $M = T^*X$. The cotangent bundle has the Fourier transform as a natural symmetry. One can consider more general Hamiltonian G-varieties M satisfying some finiteness conditions. These Hamiltonian G-spaces is to get σ-models on M. In that way, we have arrived at boundary theories for A_G.
On the B-side we are talking about L-function and we want once again interpret this in terms of a boundary theory. On the B-side we started with

a representation of the Langlands dual group $G^\vee$ and attached to the representation the inverted characteristic polynomial

$$\frac{1}{(1 - t\rho(F))} = \mathrm{Tr}_{gr}(F, Sym^\bullet(V) = \mathcal{O}(V^\vee)) \tag{136}$$

We thus can replace the representation V by $G^\vee$-varieties $X^\vee$ with carry a representation of $G^\vee$ on the space of global sections. Like before we the go over to Hamiltonian $G^\vee$-space $M^\vee$ and thus have found the link between L-functions and boundary theories for $B_{G^\vee}$.
Physics tells us that we expect a duality between the A- and B-side boundary theory. We first consider line defects on the A-side. The category that describes this line defects on the boundary is

$$Sh(LX/LG_+) \tag{137}$$

where LX is denotes the loop space of X and LG_+ describes the positive part of the loop group LG_+. This forms a tensor category. The bulk line operator are just the 't Hooft line and the category of 't Hooft lines is equivalent to

$$Sh(LG_+ \backslash LX/LG_+). \tag{138}$$

The category of line defects on the boundary is linear over the category of 't Hooft operators.
Under electric magnetic duality. The 't Hooft line operators correspond to Wilson line operators which are given by

$$QC(\mathfrak{g}^{\vee *}/G^\vee) \tag{139}$$

mathematically this correspondence is what is known as geometric Satake equivalence. The line defects on the B-side form a tensor category over the category of line operators. The module structure is induce by the moment map

$$\mu : M^\vee/G^\vee \to g^{\vee *}/G^\vee \tag{140}$$

that means by the Hamiltonian action on the Hamiltonian $G^\vee$-space $M^\vee$.
The conjectured duality takes the following form

- On the A-side we study the M-relative automorphic theory (periods, harmonic analysis, ...)
- On the B-side we study $M^\vee$-relative spectral theory (L-functions, alg. geometry of $M^\vee$).

These data should match in some way.

7 Physics and Arithemtic Geometry

In the previous section we have seen that geometry plays an important role in most connection between number theory and physics. Previously, arithmetic geometry showed up when varieties over $\mathbb{C}$ where already defined over number fields as for attractor varieties or in U-duality where some symmetry group was broken to a arithmetic subgroup. In the section we want to study to what extend we can apply physical principles directly to objects from arithmetic geometry.

7.1 Physical Discretization and Arithmetic Geometry

Consider a field theory with field given by maps $\phi : \mathbb{R}_s^d \to \mathbb{R}_t^{d'}$. When defining quantum field theroy, we have to discretize the source, i.e. we consider fields $\phi : \mathbb{Z}^d \to \mathbb{Z}_t^{d'}$. In some contexts like condensed matter physics and quantum gravity it is also natural to discretize the target, i.e. we consider fields $\phi : \mathbb{Z}_s^d \to \mathbb{Z}_t^{d'}$. The action is given by

$$S[\phi] := \frac{2\pi}{N} \sum_{j \in \mathbb{Z}} \left(\alpha(\phi(j+1) - \phi(j))^2 - \beta\phi(j)^2 \right) + \dots \tag{141}$$

This discretization has some serious flaws. For instance some symmetries are explicitly broken and it is very hard to incorporate chiral fermions.
Instead of discretize to $\mathbb{Z}$ we can instead fix $N = p$ to be some odd prime and consider fields $\mathbb{F}_p[x]$. The action then becomes

$$S[\phi] = \sum_{t \in \mathbb{F}_p} \mathrm{ev}_{x=t} \left(\alpha \partial_x \phi \partial_x \phi - V(\phi) \right) \in \mathbb{F}_p \tag{142}$$

The part integral phase is given by the character

$$\chi(\phi) = e^{\left(\frac{2\pi i}{p} S[\phi] \right)} \in \mathbb{C} \tag{143}$$

Operators are given by

$$s\hat{h}O(t) = e^{\left(\frac{2\pi i}{p} a\phi(t) \right)} \tag{144}$$

And we can compute the correlators via

$$\langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_m \rangle = \frac{1}{Z} \sum_{\phi \in \mathbb{F}_p[X]} \chi(\phi) \mathcal{O}_1 \dots \mathcal{O}_m \tag{145}$$

More geometrically we want to consider σ-models over $\mathbb{F}_p$. Let X, Y be varieties over $\mathbb{F}_p$. Fields are rational functions $\phi : X \dashrightarrow Y$ and the correlators are

$$\langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_m \rangle = \frac{1}{Z} \sum_{fields} \chi(\phi) \mathcal{O}_1 \dots \mathcal{O}_m \tag{146}$$

We allow X and Y to be defined to be over $\mathbb{F}_q$. However $S[\phi]$ still has to lie in $\mathbb{F}_p$. This can be always be achived by summing over Frobenius conjugates. We can introduce Fermions in this formalism. The Grassmann algebra of an $\mathbb{F}_p$ vector space is just the exterior algebra. For odd elements χ, ψ we have $\chi\psi = -\psi\chi$. We extend the Frobenius to the Grassmann algebras by

$$F(\chi) = \chi,\ F(\psi) = \psi; F(\chi\psi) = \psi\chi \tag{147}$$

The fermionic action will look like

$$S[\chi, \psi] = \sum_{t \in \mathbb{F}_p} \mathrm{ev}_{x=t}\, \hat{i} \chi \partial_x \psi + \dots \tag{148}$$

We need an Element $\hat{i} \in \mathbb{F}_q$ such that $F(\hat{i}) = -\hat{i}$. Hence we can't write the fermionic action over $\mathbb{F}_p$, but we need to consider some extension field. Next we want to study supersymmetry for this quantum field theory. Consider the Lagragian

$$L = \frac{1}{2}(\partial_x \phi)^2 + \hat{i}\chi\partial_x\phi - \frac{\hat{i}^2}{2} f^2 + W' + \hat{i} W'' \chi\psi \tag{149}$$

where $W(\phi)$ is a superpotential, i.e. some polynomial in ϕ. The SUSY variation tells us

$$\delta_{SUSY} L = \text{ exact differential} \tag{150}$$

This differential satiesfies $Q^2 = 0$ and hence we can define the Q-cohomology. In characteristic 0 the resulting cohomology theory is de Rham cohomology. The analogue in characteristic p is crystalline cohomology $H^{\bullet}_{cris}$. More generally for non-smooth varieties one should study rigid cohomology instead.
An important invariant in supersymmetric field theory is the Witten index $(\text{-}1)^{\mathrm{F}} = \mathrm{Tr}_{Fr}(\dim(ker(Q)) - \dim(coker(Q)))$. This is very analogous to the Hasse-Weil Zeta Function

$$\log\ Z_{V,\mathbb{F}_q}(z) = \sum_n \# V(\mathbb{F}_{q^n}) \frac{z^n}{n} = \sum_n \mathrm{Tr}_n (-1)^F \frac{z^n}{n} \tag{151}$$

So the theory of ζ- and L-functions get a physical interpretation.
Recall that in our discretization we had identified $\hbar = \frac{N}{2\pi}$. What happens if we choose $N = p^a$ and consider the limit $a \to \infty$? This procedure mean, that we are taking an inverse limit over the $\mathbb{Z}_p$ with transition maps given by projections, hence our fields are valued in $\mathbb{Z}_p$. Allowing for rational functions means that the values are actually in $\mathbb{Q}_p$. It is natural to consider fields with values in $\mathbb{C}_p$, the p-adic analog of $\mathbb{C}$. If S converges in $\mathbb{C}_p$, $\lim_{a\to\infty} e^{\frac{2\pi i}{p^a} S}$ converges in $\mathbb{C}$.
Let's consider X, Y now defined over $\mathbb{Z}_p$. Note that integral models contain the most information, since $\mathbb{Z}_p$ has $\mathbb{Q}_p$ at it's generic point and $\mathbb{F}_p$ at it's unique closed point. We have the following picture:

$$\begin{array}{ccccc} X_{\mathbb{Q}_p} & \longrightarrow & X & \longleftarrow & X_{\mathbb{F}_p} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Q}_p) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}_p) & \longleftarrow & \mathrm{Spec}(\mathbb{F}_p) \end{array}$$

Consider the Lagragian $L = \alpha \partial_x \phi \partial_x \phi - V(\phi)$. With the theory of p-adic integration we obtain

$$S[\phi] = \int_{\mathbb{Z}_p} L[\phi(t)]dt \in \mathbb{Z}_p \tag{152}$$

The classical equation of motion for this Lagragian is just

$$\partial_x^2 \phi = -V'(\phi) \tag{153}$$

This is a p-adic differential equation. The theory of p-adic differential equations was developed by Dwork, Kedlaya and others (see [TODO: Reference]).

Example: *We consider*

$$\partial_x^2 \phi = -\Omega^2 \phi \tag{154}$$

for some $\Omega \in \mathbb{Q}_p$. This differential equation is solved by

$$exp(\sqrt{-1}\Omega x) := \sum_{n\geq 0} \frac{(\sqrt{-1}\Omega t)^n}{n!} \tag{155}$$

with radius of convergence $|\Omega t|_p < p^{-1/(p-1)}$. If we want to consider general maps $X \to Y$ we run into the problem, that a priori $X(\mathbb{C}_p)$ and $Y(\mathbb{C}_p)$ carry

no topology. The minimal solution for this problem would be to consider the Rigid Analytic Varieties introduced by Tate. However Berkovic Spaces maybe suit our problem better, since they are path connected.
There have been many recent development in p-adic geometry due to Scholze. One can consider more general spaces like perfectoid space defined over large fields that have $\mathbb{C}_p$ as it's quotient field. He uses the formalism of adic spaces. It might be interesting to generalize quantum field theory to these more complex objects of p-adic geometry.
Lets return to the discretization. A general N has a prime factorization of the form $N = p_1^{a_1} \dots p_m^{a_m}$ and hence $\hbar = \frac{p_1^{a_1} \dots p_m^{a_m}}{2\pi}$. We consider X defined over $\mathbb{Z}$. In that way we arrive at adelic physics. As shown by Freund and Witten in there study of the Veneziano amplitude for the p-adic string, this can be related to archimedean physics.
A powerful in mathematical physics is so called geometric engineering. This can also be done in arithmetic physics. Consider Type $II/\mathbb{R}^{3,1} \times T^2 \times \mathbb{C}^2/\Gamma_{ADE}$. This engineers $\mathcal{N} = 4$ SYM on $\mathbb{R}^{3,1}$.
Working over $\mathbb{Q}$ we consider $S \times \mathbb{E} \times M_{ADE}$ where S is some arithmetic surface over $\mathbb{Z}$. It's however not clear, what the physics on this arithmetic surface looks like.

7.2 HMS for the Fargues-Fontaine curve

There has been recent work on mirror symmetry for the Fargues-Fontaine curve by Lekili and Treumann [LT20b].
Let E be local field be a local field and C a perfectoid field of characteristic p. We denote the Fargues-Fontaine curve by $FF_E(C)$. The Fargues-Fontaine curve is a E-scheme. It is no curve over E or even a variety, since it is not of finite tyoe over E or any other field, but in some way the Fargues-Fontaine curve resembles a closed Riemann surface:

- It is noetherian of Krull dimension one. Moreover it is regular, so that the local ring at each closed point of $FF_E(C)$ has a discrete valuation.
- A nonzero rational function f (tat is, a section of $\mathcal{O}_{FF}$ over the generic point) has $v(f) \neq 0$ for at most finitely many of these valuations v adn $\sum_v v(f) = 0$.

In fact FF_E resembles the Riemann sphere: one has

$$\mathrm{Pic}(FF_E(C)) = \mathbb{Z} \tag{156}$$

and

$$H^1(\mathcal{O}_{FF}) \tag{157}$$

when $E = \mathbb{Q}_p$, $FF_E(C)$ is an important object in p-adic cohomology. The category of vector bundles on $FF_E(C)$ is equivalent to the category of (φ, Γ)-Modules and thus is of great importance for p-adic Hodge theory. In [LT20b] mirror symmetry for $E = \mathbb{F}_p((z))$ is discussed.
In [LP12] a refinement of homological symmetry which relates the Fukaya category of the 2-torus, relative to a basepoint, with the category of perfect complexes of coherent sheaves on the Tate curve over the formal disc $\mathrm{Spec}(\mathbb{Z}[[q]])$ was established. To get from the Tate curve to the Fargues-Fontaine curve, one has to introduce two changes:

- One couples Lagragian Floer theory to a locally constant sheaf of rings on the torus. The fiber of the sheaf of rings has characteristic p, and going around one of the circles is the p-th power map, while going around the other is the identity.
- We set the Novikov parameter (this is the element $\mathbb{Z}((t))$ in the ground ring of the Tate curve) to $t = 1$. Symplectically this is sort of like studying the limit as the symplectic form goes to 0.

One can speculate that there is a similar story for $E = \mathbb{Q}_p$. In this case, this would establish a surprising between symplectic geometry and p-adic hodge theory. Further there might even be connections to the work of Fargues-Scholze on the local langlands correspondence [FS21].

8 Further Directions and Speculations

We have seen how one might construct quantum field theory in characteristic p and how to lift that to the p-adic world. There have been attempts to extend the framework of string theory and quantum field theory to the p-adic number for a long time. In recent years, most noteably there has been developed a p-adic version of the AdS/CFT correspondence. With the recent advances in p-adic geometry. It might be interesting to study sting theory on other object of p-adic geometry.

We discussed a recent attempt to understand the arithmetic Langlands program from a physical point of view. The origin of topological quantum field theory that plays a central role in the discussion is not yet clear. We have seen, that the framework of p-adic physics allows us to geometrically engineer physical theories. If one can reproduce the Kapustin-Witten construction in the case of the arithmetic spaces, this may shed some light on the origin of the arithmetic TQFT discussed above. From the p-adic view point one can ask if one can understand Fargue-Scholzes work on geometrization of the local Langlands conjecture in terms of a p-adic quantum field theory.

The definition of s-functions over some arbitrary number fields K only depends on the unramified primes. It might be an interesting question to ask what happens at the ramified primes. It might be beneficial to study the local s-functions over p-adic number fields since the investigation of s-functions with ramification might be easier.
Using ramified Witt vectors one can define a Frobenius lift on ramified extensions of $\mathbb{Q}_p$ and thus one can define s-functions. One might however have to replace p by some uniformizer $\pi \in \mathcal{O}_K$.

There are a lot of open questions concerning the connections of number theory and physics. We have seen implications in either directions, but it is likely, that so far we only stretched the surface.

9 More Interesting Paper

- Antoine Bourget and Jan Troost -The Arithmetic of Supersymmetric Vacua (arxiv.org/abs/1606.01022)
- Steven S. Gubset - A p-adic version of AdS/CFT (arxiv.org/abs/1705.00373)
- Diego Delmastro and Jaume Gomisa - Symmetries of Abelian Chern-Simons Theories and Arithmetic (arxiv.org/abs/1904.12884)
- Branki Dragovich - From p-Adic to Zeta Strings (arxiv.org/abs/2007.13628)

- David A. McGady - L-functions for Meromorphic Modular Forms and Sum Rules in Conformal Field Theory (arxiv.org/abs/1806.09874)
- Yang-Hui Hea, Vishnu Jejjalab, Djordje Minic - From Veneziano to Riemann: A String Theory Statement of the Riemann Hypothesis (arxiv.org/abs/1501.01975)
- An Huang, Bogdan Stoica, and Shing-Tung Yau - General relativity from p-adic strings (arxiv.org/abs/1901.02013)

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