Notes on Rigid Cohomology and Crystalline Cohomology

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1 Introduction

Both crystalline and rigid cohomology are p-adic cohomology theory. In particular both of them are Weil cohomology theories. The original motivation to search for Weil cohomology theories was in order to prove the Weil conjectures. The first discovered example of a cohomology theory for varieties in characteristic $p > 0$ was étale cohomology with l-adic coefficients where $l \neq p$ is a prime. This would have sufficed to prove the Weil conjectures, but it would have been unsatisfying not also to have a p-adic cohomology theory in characteristic p. Furthermore, p-adic cohomology provides many useful tools for computations of, for instance, zeta functions. In 1966 Grothendieck found for smooth proper varieties over a field of characteristic $p > 0$ crystalline cohomology to be the correct p-adic Weil cohomology theory, however it wasn't clear how to deal with non-smooth or non-proper varieties. In 1968 Monsky–Washnitzer found the description of a p-adic cohomology theory for non-singular affine variety of positive characteristic p. Soon after, in 1986 Berthelot introduced in [1] rigid cohomology which generalizes MW cohomology and crystalline cohomology. Rigid cohomology is a straightforward generalization of MW cohomology, but the relation between rigid cohomology and crystalline cohomology is not quite as obvious. The main goal of this paper is to explain the following comparison isomorphism between rigid and crystalline cohomology:

$$H^i_{rig}(X/K) = H_{crys}(X/(W(k)) \otimes_{W(k)} K \tag{1}$$

where k is a perfect field of characteristic $p > 0$, X a smooth proper variety over k, $W(k)$ the ring of Witt-vectors over k and $K = W(k)[\frac{1}{p}]$. Given this isomorphism, one could wonder why one should care about crystalline cohomology at all. Recall that étale cohomology is defined with coefficients in $\mathbb{Z}_l$ and we get the l-adic groups by tensoring, i.e. $H^i_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_l) = H^i_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. So in l-adic cohomology we know how to define a integral model of our cohomology groups which can explain l-torsion phenomena. It is however not at all clear in rigid cohomology how to define an integral model of rigid cohomology. Recall now that for $k = \mathbb{Z}/p\mathbb{Z}$. Hence crystalline cohomology provides us with an integral model of p-adic cohomology, at least in the case when X is a proper smooth variety. The second reason why crystalline cohomology is important is a technical one. The prove of finite-dimensionality of rigid cohomology reduces to the case of a proper smooth variety. Since one already knows that the crystalline cohomology groups are finite dimensional, we can apply the comparison isomorphism in order to prove that the rigid cohomology groups have to be of finite dimension as well.

In the section 2 we will revisit the definition and properties of the ring of Witt-vectors over a perfect field k of positive characteristics. Next we will briefly introduce crystalline cohomology without going into much detail. In order to motivate the isomorphism between crystalline and de Rham cohomology, we will discuss the de Rham-Witt complex. In section 5 we finally can prove the isomorphism between rigid cohomology and de Rham cohomology. This leads us to the insight that for proper smooth varieties we can compute rigid cohomology via the cohomology of the de Rham-Witt complex. In section 6 we want to generalize this idea and introduce the overconvergent de Rham-Witt complex, which allows us to compute rigid cohomology of quasi-projective varieties.

2 Witt vectors

Theorem: *For every perfect field k of characteristic p, there exists a unique (up to unique isomorphism) complete discrete valuation ring which is absolutely unramified (i.e. the maximal ideal is (p)) and has k as its residue field.*

We define this as the ring of Witt vectors $W(k)$. More precisely, we know that $W(k)$ satisfies the following:

1. $W(k)$ is a complete discrete valuation ring of characteristic zero.
2. The unique maximal ideal $\mathfrak{m}$ of $W(k)$ is generated by p, and the residue field $W(k)/\mathfrak{m}$ is isomorphic to k.
3. Every $\mathfrak{m}$-adically complete discrete valuation ring of char 0 with residue field k contains $W(k)$ as a subring.
4. The Witt ring $W(k)$ is functorial in k, i.e. every $\phi : k \to k'$ induces a unique $f : W(k) \to W(k')$.

For computations it is useful to have an explicit construction of $W(k)$. Let's define the Witt polynomials:

$$\begin{aligned} W_0(x_0) &:= x_0 \\ W_1(x_0, x_1) &:= x_0^p + px_1 \\ &\vdots \\ W_n(x_0, ..., x_n) &:= \sum_{i=0}^{n} p^i x_i^{p^{n-i}} \end{aligned}$$

There are unique polynomials S_n and P_n in $2n+2$ variables with coefficients in $\mathbb{Z}$ such that

$$W_n(x_0, ..., x_n) + W_n(y_0, ..., y_n) = W_n(S_0(x_0, y_0), ..., S_n(x_0, ..., x_n, y_0, ..., y_n))$$
$$W_n(x_0, ..., x_n) \cdot W_n(y_0, ..., y_n) = W_n(P_0(x_0, y_0), ..., P_n(x_0, ..., x_n, y_0, ..., y_n))$$

We now construct $W(R)$ for an arbitrary ring R (not necessarily of characteristic p). First, we define the truncated Witt ring $W_{n+1}(R)$ to be the set R^{n+1}, together with the operations:

$$(x_0, ..., x_n) \oplus (y_0, ..., y_n) := (S_0(x_0, y_0), ..., S_n(x_0, ..., x_n, y_0, ..., y_n))$$
$$(x_0, ..., x_n) \odot (y_0, ..., y_n) := (P_0(x_0, y_0), ..., P_n(x_0, ..., x_n, y_0, ..., y_n))$$

With this operations, $W_n(R)$ is a ring with zero $0 = (0, ..., 0)$ and unit $1 = (1, .., 0)$. For instance, $W_1(R)$ is just R with the usual operations.
Next assume R to be of characteristic p. We introduce two operations:

$$\begin{array}{cccc} \tilde{V}: & W_n(R) & \to & W_{n+1} \\ & (x_0, ..., x_{n-1}) & \mapsto & (0, x_0, ..., x_{n-1}) \end{array}$$

$$\begin{array}{cccc} \sigma: & W_n(R) & \to & W_{n-1} \\ & (x_0, ..., x_{n-1}) & \mapsto & (x_0^p, ..., x_{n-2}^p) \end{array}$$

We call $\tilde{V}$ *Verschiebung*. This map is additive. The map σ is called the *Frobenius* and it is a homomorphism of rings. One can check that the following identity holds:

$$\sigma \circ \tilde{V} = \tilde{V} \circ \sigma = p \cdot id_{W_n(R)} \tag{2}$$

Notice that multiplication by p is given by repeated addition p times and not by coordinate-wise multiplication (since that would be the same as multiplication by zero).
We have one more homomorphism, namely the projection $W_n(R) \to W_{n-1}(R)$ forgetting the last component. This homomorphism is clearly surjective and we can take the projective limit if this projective system to get $W(R)$.
For example if $R = \mathbb{F}_p$, the Frobenius is trivial, so the Verschiebung V is just multiplication by p. Hence, $W_n(\mathbb{F}_p) \subset \mathbb{Z}/p^n\mathbb{Z}$. Moreover, by noting that there is an element of order p^n and comparing sizes, we conclude equality. Therefore,

$$W(\mathbb{F}_p) = \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p. \tag{3}$$

As for most constructions in algebraic geometry, one can globalize this construction to schemes over a field of char p, by gluing together the Witt vectors on the affine parts.

3 Crystalline Cohomology

Let k be an algebraically closed field of characteristic $p > 0$. Before discussing the construction of crystalline cohomology, we state the main properties:

Proposition: If X is a proper and smooth scheme over k, then we can define a cohomology satisfying the following properties:

1. $H^n_{crys}(X/W)$ is a contravariant functor in X. These groups are finitely generated W-modules and zero if $n \notin [0, 2dim(X)]$.

2. There is a cup-product $\cup_{i,j}$ structure modulo torsion that induces a perfect pairing at $\cup_{n,2dim(X)-n}$ when $n \in [0, 2dim(X)]$

3. $H^n_{crys}(X/W)$ defines an integral structure on $H^n_{crys}(X/W) \otimes_W K$.

4. If l is a prime different from p, then

$$\dim_{\mathbb{Q}_l} H^n_{\acute{e}t}(X, \mathbb{Q}_l) = \mathrm{rank}_W H^n_{crys}(X/W) \tag{4}$$

   so crystalline cohomology computes l-adic Betti numbers.

5. Crystalline cohomology computes the de Rham cohomology:

$$0 \to H^n_{crys}(X/W) \otimes_W k \to H^n_{dR}(X/k) \to Tor_1^W(H^{n+1}_{crys}(X/W), k) \to 0 \tag{5}$$

6. There is a Lefschetz fixed point formula, there exist base change formulas, cycles classes in $H^{2q}_{crys}(X/W)$ of codimension q subvarieties,...

Definition: *Let A be a ring. Let I be an ideal of A. A collection of maps $\gamma_n : I \to I$ is called a divided power structure on I if for all $n \geq 0$, $m > 0$, $x, y \in I$, and $a \in A$ we have*

(1) $\gamma_1(x) = x$ *and we see* $\gamma_0(x) = 1$.

(2) $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$

(3) $\gamma_n(ax) = a^n \gamma_n(x)$

(4) $\gamma_n(x + y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$

(5) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$.

Remark: Let A be a ring. And I an ideal of A. If γ is a divided power structure on I, then $n!\gamma_n(x) = x^n$.

Example:

(1) Let K be a field of characteristic zero (e.g. $K = \mathbb{Q}$). Consider the polynomial ring $K[T]$. Let $I = (T)$. Then

$$\gamma_n(T) = \frac{T^n}{n!} \tag{6}$$

defines a divided power structure on I.

(2) Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e. A is a $\mathbb{Z}_{(p)}$-algebra. The $I = pA$ has a canonical divided power structure. Namely, given $x = pa \in I$ we set

$$\gamma_n(x) = \frac{p^n}{n!}a^n. \tag{7}$$

We have that $p^n/n! \in p\mathbb{Z}_{(p)}$, so that the definition makes sense and gives us a sequence of maps $\gamma_n : I \to I$. This defines a divided power structure on I.

Definition: *A divided power ring is a triple (A, I, γ) where A is a ring, $I \subset A$ is an ideal and $\gamma = (\gamma_n)_{n\geq 1}$ is a divided power structure on I. A morphism of divided power rings $\phi : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $\phi : A \to B$ such that $\phi(I) \subset J$ and such that $\delta_n(\phi(x)) = \phi(\gamma_n(x))$ for all $x \in I$ and $n \geq 1$.*

We now turn to the discussion of the crystalline site. Our setting is as follows. Fix a prime number p. Let (A, I, γ) be a divided power ring such that A is a $\mathbb{Z}_{(p)}$-algebra and $A \to C$ is a ring map such that p is nilpotent in C. (Usually the prime number p will be contained in the divided power ideal I.

Definition: *A divided power thickening of C over (A, I, γ) is a homomorphism of divided power algebras $(A, I, \gamma) \to (B, J, \delta)$ such that p is nilpotent in B and a ring map $C \to B/J$ such that the following diagram commutes*

$$
\begin{tikzcd}
B \arrow[r] & B/J \\
 & C \arrow[u] \\
A \arrow[uu] \arrow[r] & A/I \arrow[u]
\end{tikzcd}
$$

A homomorphism of divided power thickenings

$$(B, J, \delta, C \to B/J) \to (B', J', \delta', C \to B'/J') \tag{8}$$

is a homomorphism $\phi : B \to B'$ of divided power A-algebras such that the following diagram commutes:

$$
\begin{tikzcd}
 & B' \arrow[rrr] & & & B'/J' \\
B \arrow[ur, "\phi"] \arrow[rr] & & B/J \arrow[urr] & & \\
 & & & C \arrow[ul] \arrow[uur] & \\
 & A \arrow[uul] \arrow[uuu] \arrow[rr] & & A/I \arrow[u] &
\end{tikzcd}
$$

We denote CRYS$(C/A, I, \gamma)$ or simply CRYS(C/A) the category of divided power thickenings of C over (A, I, γ). We denote Crys(C/A) the full subcategory of objects such that $C \to B/J$ is an isomorphism. We often denote such an object $(B \to C, \delta)$ with $J = Ker(B \to C)$ being understood.

Let's focus on the following special case: Let k be a perfect field of characteristic p, and we denote $W := W(k)$ and $W_n := W_n(k) = W/p^n$. Let X be a scheme over k. Then $Crys(X/W_n)$ can be described as follows: The objects are commutative diagrams

$$
\begin{tikzcd}
U \arrow[r, "i"] \arrow[d] & V \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & Spec(W_n)
\end{tikzcd}
$$

where $U \subset X$ is a Zariski open subscheme, and $i : U \to V$ is a divided power thickening. In our setting that means, that i is a closed immersion of W_n-schemes such that the ideal $\mathrm{Ker}(\mathcal{O}_V \to \mathcal{O}_U)$ is endowed with a divided power structure δ compatible with the canonical divided powers structure on $pW_n \subset W_n$, i.e. $\delta(pa) = \gamma_n(p)a^n$ for any $pa \in \mathrm{Ker}(\mathcal{O}_V \to \mathcal{O}_U)$.
The morphisms from (U, V, δ) to (U', V', δ') are commutative diagrams formed by an open immersion $U \hookrightarrow U'$ and a morphism $V \to V'$ compatible with the divided power structure.

To obtain the crystalline site we need to put a Grothendieck topology on $Crys(X/W_n)$. A cover is a collection of morphisms $(U_i, V_i, \delta_i) \to (U, V, \delta)$, where $V_i \to V$ is an open immersion and $V = \cup_i U_i$.
Recall that a sheaf on a site C is a contravariant functor, such that compatible sections glue with respect to a give grothendieck topology. Let F be a sheaf (with values in SETS) on $Crys(X/W_n)$ and (U, V) an object of $Crys(X/W_n)$. For a Zariski open $W \subset V$, we define $\tilde{F}_{(U,V)} := F(U \times_V W, W)$, and this way we get a sheaf over the scheme V. Moreover for $g : (U, V) \to (U', V')$ in $Crys(X/W_n)$ we have a morphism $g_F^* : g^{-1}\tilde{F}_{(U',V')} \to \tilde{F}_{(U,V)}$. The construction is functorial and give $V \to V'$ is an open immersion such that $U = U' \times_{V'} V$, then g_F^* is an isomorphism. Conversely,, if fore every object (U, V) of $Crys(X/W_n)$ we have a sheaf $\tilde{F}_{(U,V)}$ for the Zariski topology of V s.t. for every morphism $g : (U, V) \to (U', V,)$ in $Crys(X/W_n)$ we have a morphism $g_F^* : g^{-1}\tilde{F}_{(U',V')} \to \tilde{F}_{(U,V)}$ satisfying the conditions above, we get a sheaf F on the crystalline site.
With this insight, we can define a structure sheaf in $Crys(X/W_n)$, $\mathcal{O}_{X/W_n}$, that associates to every object (U, V, δ) the sheaf $\mathcal{O}_V$.

4 De Rham-Witt complex I

From now on we assume X to be smooth and projective over k a field of characteristic p. In particular X is proper. The sheaves $W_n\mathcal{O}_X$ and $W\mathcal{O}_X$ are just the zero part of the complexes $W_n\Omega_X^\bullet$ and $W\Omega_X^\bullet$. We can define two operators $F : W_n\Omega_X^\bullet \to W_{n+1}\Omega\bullet_X$ and $F : W_n\Omega\bullet_X \to W_{n-1}\Omega_X^\bullet$ between the complexes satisfying the following properties:

1. Both operators are additive.

2. Over the degree zero piece F restricts to σ and V restricts to $\tilde{V}$.

3. $FV = VF = p$

4. $FdV = d$. Hence we know from the previous condition that $dF = pFd$.

5. $Fdx = x^{p-1}dx$. Here by abuse a notation we denote by x the Teichmüller representative of $x \in \mathcal{O}_X$. This is given by $(x, 0, 0, ...)$.

6. $FxFy = F(xy)$, $xVy = xVy$ and $V(xdy) = V(x)dV(y)$.

Example: Assume that $k = \mathbb{F}_p$. Then, we have the absolute Frobenius $F_X : X \to X$, which is the identity on the topological spaces and taking about the p-th power on the ring structure. This absolute Frobenius induces a morphism F_X^* on the complex $W_n\Omega_X^\bullet$. Let's look at the degree 1 piece, i.e. $W_n\Omega_X^1 \to W_n\Omega_X^1$. It is a fact that we can write every element of $W_n\Omega_X^1$ as a sum of $V^i(a)dV^j(b)$, with $a, b \in \mathcal{O}_X$. Then, the induced Frobenius acts as follows

$$\begin{gathered} F_X^* : W_n\Omega_X^1 \to W_n\Omega_X^1 \\ V^i(a)dV^j(b) \mapsto V^i(a^p)dV^j(b^p) \end{gathered}$$

we get:

$$F_X^*(V^i(a)dV^j(b)) = F(V^i(a))dFV^j(b) = pF(V^i(a)V^j(b)) \tag{9}$$

so we have that:

$$F_X^*\alpha = pF\alpha \tag{10}$$

The technical part of divided powers in the construction of crystalline cohomology is motivated by this factor p, because we want to make sense of the statement "$F = F_X^*/p$ in characteristic p.

Let's now consider the following filtration of the complex $W\Omega^\bullet$.

$$\begin{array}{lccccccc}
W\Omega^\bullet : & W\mathcal{O} & \longrightarrow & W\Omega^1 & \longrightarrow & W\Omega^2 & \longrightarrow & \dots \\
\cup & \downarrow & & \downarrow & & \downarrow & & \\
W\Omega^{\geq 1} : & 0 & \longrightarrow & W\Omega^1 & \longrightarrow & W\Omega^2 & \longrightarrow & \dots \\
\cup & \downarrow & & \downarrow & & \downarrow & & \\
W\Omega^{\geq 2} : & 0 & \longrightarrow & 0 & \longrightarrow & W\Omega^2 & \longrightarrow & \dots
\end{array}$$

By abuse of notation we denote by $W\Omega^i$ the complex that is zero everywhere except in degree zero, where is is precisely $W\Omega^i$. The we get the equality of complexes:

$$\frac{W\Omega^{\geq i}}{W\Omega^{\geq i+1}} = (W\Omega^i)[i] \tag{11}$$

This double complex induces a spectral sequence, and we have

$$E_1^{i,n-i} = H^n(W\Omega^{\geq i}/W\Omega^{\geq i+1}) = H^{n-i}(W\Omega^i_X) \tag{12}$$

Assuming that X is smooth and proper over k, and that $k$is perfect and after tensoring with $K := Frac(W(k))$, we have that the spectral sequence degenerates at $E_!$, i.e.

$$H^{n-i}(W\Omega^i_X)_K \Rightarrow H^n(X_{Zar}, W\Omega^\bullet)_K. \tag{13}$$

The right hand side means the hypercohomology of the total complex obtained double complex obtained by considering an injective resolution of each Ω^i_X.
One nice property of crystalline cohomology is that we get the equality

$$H^n(X_{Zar}, W\Omega^\bullet)_K = H^n_{crys}(X/W)_K =: H^n_{crys}(X/K) \tag{14}$$

5 The Comparison Isomorphism

Now it's time to state the main theorem of this paper. It was first proven by Berthelot in [1]. A detailed discussion of rigid cohomology that includes all results we use in this section can be found in [7].

Theorem: *Suppose k is a perfect field. $K := Frac(W(k))$. Let X be a smooth proper variety over k such that there exist a lift to a formal smooth $\mathcal{O}_K$-scheme $\mathfrak{X}$ then there is a natural isomorphism*

$$H^i_{rig}(X/K) \cong H^i_{crys}(X/W(k)) \otimes_{W(k)} K \tag{15}$$

Our strategy to prove this statement is as follows: We first recall how (14) induces an isomorphism:

$$H^i_{crys}(X/W(k)) \otimes_{W(K)} K \longrightarrow H^i_{dR}(\mathfrak{X}_K) \tag{16}$$

Next we will prove that for a proper smooth variety X we will have an isomorphism

$$H^i_{dR}(\mathfrak{X}_K) \cong H^i_{rig}(X/K). \tag{17}$$

Putting these together we get the desired result. Let's start with the isomorphism between crystalline and de Rham cohomology. We don't give proves of the statements here. They can be found in [2].

Theorem: *Suppose $X \hookrightarrow Y$ is a closed embedding of S-schemes with Y/S smooth. Let E be a coherent a $\mathcal{O}_Y$-module with integrable connection. Then there is a canonical isomorphism*

$$H^i_{crys}(X/S, E) \longrightarrow H^i_{dR}(X, E) \tag{18}$$

Furthermore if we denote by $U_{X/S}$ the canonical projection from the crystalline site to the Zariski site we get a natural isomorphism in the derived category of abelian groups on X_{Zar}:

$$Ru_{X/S*} \to E \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S} \tag{19}$$

Together with the fact that $R\Gamma_{crys} = R\Gamma_{Zar} \circ Ru_{X/S*}$ this implies (18).

As a consequence of this fact we have to following corollary:

Corollary: Suppose $\mathfrak{X}/S$ is smooth, $S_0 \hookrightarrow S$ is defined by a sub divided power ideal of I and $X = S_0 \times_S \mathfrak{X}$. Then there is a natural quasi-isomorphism:

$$H^i_{crys}(X/W(k)) \longrightarrow H^i_{dR}(\mathfrak{X}/W(k)) \tag{20}$$

Take now $\mathfrak{X}$ to be the lift of X in the main theorem above. Then the corollary implies that

$$H^i_{crys}(X/W(k)) \longrightarrow H^i_{dR}(\mathfrak{X}/W(k)) \tag{21}$$

by a calculation involving a suitable Čech complex one can show that

$$H^i_{dR}(\mathfrak{X}/W(k)) \otimes_{W(k)} K \cong H^i_{dR}(\mathfrak{X}_K). \tag{22}$$

We know come to the crucial point in the proof of (5). That we so far proved that in the proper smooth case the crystalline cohomology is isomorphic to de Rham cohomology if one takes care of the coefficients. Now we want to prove that rigid cohomology is isomorphic to this de Rham cohomology as well.

Recall the definition of the rigid cohomology of a variety X/k. Given a proper smooth frame $X \hookrightarrow Y \hookrightarrow \mathcal{P}$ the rigid cohomology of X is defined as

$$H^i_{rig}(X/K) = H^i(]Y[_{\mathcal{P}}, j^\dagger_X \Omega^\bullet_{]Y[_{\mathcal{P}}/K}) \tag{23}$$

To get the desired result we first need to find a frame such that the tube of Y in $\mathcal{P}$ is just $\mathcal{P}_K$. This is an easy exercise. Indeed, the trivial frame:

$$X \hookrightarrow X \hookrightarrow \mathfrak{X} \tag{24}$$

has the desired property. That is $\mathfrak{X}_k = X$ and $]X[_{\mathfrak{X}} = \mathfrak{X}_K$. The second assertion follows from the following proposition:

Proposition: Any formal embedding $\iota : X \hookrightarrow \mathcal{P}$ factors through a closed formal embedding $\iota' : X \hookrightarrow \mathcal{P}'$ where $\mathcal{P}'$ is an open formal subscheme of $\mathcal{P}$ and we have $]X[_P =]X[_{\mathcal{P}'}$. When $\iota(X)$ is open in $\mathcal{P}_k$, we may choose $\mathcal{P}'$ with $\mathcal{P}'_k = X$, and then $]X[_{\mathcal{P}} = \mathcal{P}'_K$.

Proof. Since $\mathcal{P}$ and $\mathcal{P}_k$ have the same underlying space, if X' is an open subset of $\mathcal{P}_k$, we can write $X' \simeq \mathcal{P}'_k$ with $\mathcal{P}'$ an open subset of $\mathcal{P}$. Furthermore, we know that $\mathcal{P}'_K = sp^{-1}(\mathcal{P}')$. Putting everything together we get a factorization

$$X \hookrightarrow X' \simeq \mathcal{P}'_k \hookrightarrow \mathcal{P}' \hookrightarrow \mathcal{P} \tag{25}$$

with X is closed in X' and $\mathcal{P}'$ open in $\mathcal{P}$. Taking the trivial frame we have $\iota(X) = \mathfrak{X}_k$ and hence it is open. Hence it follows that $]X[_{\mathfrak{X}} = \mathfrak{X}'_K$. □

With this result we can rewrite the definition of the rigid cohomology of a proper smooth variety X with lift $\mathfrak{X}$ as follows:

$$H^i_{rig}(X/K) = H^i(\mathfrak{X}_K, j^\dagger_X \Omega^\bullet_{\mathfrak{X}/K}) \tag{26}$$

It remains to show that this is isomorphic to the de Rham cohomology of the K-scheme $\mathfrak{X}$. We want to show the isomorphism on the level of complexes, i.e. $j^\dagger_X \Omega^\bullet_{\mathfrak{X}/K} \cong \Omega^\bullet_{\mathfrak{X}/K}$. This is most easily done by showing that $j^\dagger_X$ is the identity functor.

First we briefly revisit the definition of the functor $j^\dagger_X$ in a general setting. Let X be an variety over k and $X \hookrightarrow Y \hookrightarrow \mathcal{P}$ proper smooth frame. Let $\mathcal{F} \in Sh_{Ab}(]Y[_{\mathcal{P}})$. For a strict neighborhood $]X[_{\mathcal{P}} \subset V \subset]Y[_{\mathcal{P}}$ let $j_V : V \hookrightarrow]Y[_{\mathcal{P}}$ be the canonical inclusion. Then $j_{V*} j_V^{-1} \mathcal{F}$ are the sections of $\mathcal{F}$ convergent on V. We define the sections of $\mathcal{F}$ overconvergent on $]X[_{\mathcal{P}}$ to be

$$j^\dagger_X \mathcal{F} := \varinjlim j_{V*} j_V^{-1} \mathcal{F} \tag{27}$$

where the colimit runs through all j_V. Consider now the trivial frame

$$X \hookrightarrow X \hookrightarrow \mathfrak{X} \tag{28}$$

as above. In this case $]X[_{\mathcal{P}}=]Y[_{\mathcal{P}}= \mathcal{P}_K$ and hence the colimit runs only over the identity $j_{]Y[_{\mathcal{P}}} :]Y[_{\mathcal{P}} \hookrightarrow]Y[_{\mathcal{P}}$. More general, for any proper scheme we get that $j_X^\dagger \mathcal{F} = \mathcal{F}$. Hence we find that

$$H^i_{rig}(X/K) = H^i(\mathfrak{X}_K, j_X^\dagger \Omega^\bullet_{\mathfrak{X}/K}) = H^i(\mathfrak{X}_K, \Omega^\bullet_{\mathfrak{X}/K}) = H^i_{dR}(\mathfrak{X}_K) \tag{29}$$

Together with (18) this proves the main theorem, i.e.

$$H^i_{crys}(X/W(k)) \otimes_{W(k)} K \cong H^i_{dR}(\mathfrak{X}_K) = H^i_{rig}(X/K) \tag{30}$$

6 Overconvergent de Rham-Witt complex

In section 5 we saw that we can compute crystalline cohomology via the de Rham-Witt complex. Next we saw that for proper smooth schemes crystalline cohomology computes rigid cohomology. Hence in this special case we can compute rigid cohomology via a de Rham-Witt complex. Our goal in this section is to construct a suitable generalization of the de Rham Witt complex that can compute the rigid cohomology for quasi-projective varieties. This complex we will call the overconvergent de Rham-Witt complex. We have following result:

Theorem: *Let X be a smooth quasiprojective scheme over k. Then we have a natural quasi-isomorphism*

$$R\Gamma_{rig}(X) \longrightarrow R\Gamma(X, W^\dagger \Omega_{X/k}) \otimes \mathbb{Q}. \tag{31}$$

This section is a brief summary of the work of Davis, Langer and Zink [6], [4] and [3].

6.1 De Rham-Witt complex II

We first review some properties of the de Rham-Witt complex: Let A be a unitary commutative ring and B be a unitary commutative A-algebra. Assume that $\mathfrak{b} \subset B$ is an ideal, which is equipped with divided powers $\mathfrak{b} \to \mathfrak{b}$ for $n \geq 1$. We set $\gamma_0(b) = 1$ for $b \in \mathfrak{b}$

Definition: *Let M be a B-module. A pd-derivation $\nu : B \to M$ over A is a A-linear derivation ν which is satisfies:*

$$\nu(\gamma_n(b)) = \gamma_{n-1}(b)\nu(b) \tag{32}$$

for $n \geq 1$ and each $b \in \mathfrak{b}$. The pd-derivations form a B-module which we denote by

$$\check{Der}_{B/A}(B, M). \tag{33}$$

There is a universal pd-derivation $d : B \to \check{\Omega}^1_{B/A}$. The $\check{\Omega}^1_{B/A}$ is obtained as the factor module of $\Omega^1_{B/A}$ by the submodule generated by the elements $d(\gamma_n(b)) - \gamma_{n-1}(b)d(b)$. A derivation is a pd-derivation if and only if for all $b \in \mathfrak{b}$ it satisfies:

$$\nu(\alpha_p(b)) = b^{p-1}\nu(b) \tag{34}$$

A differential graded B/A-algebra will be a unitary graded B-algebra $P = \oplus_{i \in \mathbb{Z}_{i \geq 0}} P^i$ equipped with an A-linear differential of degree one $d : p \to P$ such that:

$$\begin{aligned} \omega\eta &= (-1)^{ij}\eta\omega \quad \omega \in P^i, \eta \in P^j \\ d(\omega\eta) &= (d\omega)\eta + (-1)^i \omega d\eta \\ d^2 &= 0 \end{aligned}$$

A pd-differential graded algebra is a differential graded algebra such that the composition of maps $B \to P^0 \to P^1$ is a pd-derivation. We set $\check{\Omega}^i_{B/A} = \bigwedge^i \check{\Omega}^i_{B/A}$ and form the pd-complex. This is a unique object in the category of differential graded pd-algebras. Let now R be a $\mathbb{Z}_{(p)}$-algebra and S a R-algebra. Remember that for each m we have a Verschiebungsmap between rings of truncated Witt vectors

$$V_m : W_{m-1}(S) \longrightarrow W_m(S) \tag{35}$$

Then $I_S = im(V_m)$ is a divided power ideal of $W_m(S)$ with divided power structure given by

$$\gamma_n(V(\xi)) = \frac{p^{n-1}}{n!} V(\xi^n) \tag{36}$$

for and $\xi \in W_{m-1}(S)$.

Recall that $W(S)$ is the colimit over all $W_m(S)$ and we get a canonical map $W(S) \to W_m(S)$ by truncation. We call a $W(S)$-module discrete M discrete,

if it is obtained by restriction of scalars from a $W_m(S)$ module for some natural number m. A map $W(S) \to M$ is called continuous, if it factors through $W_l(S)$ for some number l.
Let us consider any continuous pd-derivation $\nu : W(S) \to M$ to a discrete $W(S)$-module M. We want define a Frobenius on this pd-derivation ${}^F\nu : W(S) \to M$ as follows:
An arbitrary $\xi \in W(S)$ has a unique representation $\xi = [x] + V(\rho)$ for $[x] \in S$ and $\rho \in W(S)$. We set

$$ {}^F\nu(\xi) = [x^{p-1}]\nu([x]) + \nu(\rho). \tag{37} $$

Let us denote by $M_{[F]}$ the $W(S)$-module obtained via restriction of scalars by $\sigma : W(S) \to W(S)$. If $\nu : W(S) \to M$ is a continuous $W(R)$-linear pd-derivation, then ${}^F\nu : W(S) \to M_{[F]}$ is a continuous $W(R)$-linear pd-derivation. We have the relation $\nu(\sigma(\xi)) = p^F\nu(\xi)$. If we start with the universal pd-derivative we obtain a homomorphism of $W_{m+1}(S)$-modules:

$$ F : \check{\Omega}_{W_{m+1}(S)/W_{m+1}(R)} \to \left(\check{\Omega}_{W_m(S)/W_m(R)}\right)_{[F]} \tag{38} $$

Let $\xi \in W_{m+1}$, $\eta \in W_m(S)$, $x \in S$. The map satisfies the following properties:

1. $F(d\xi) = {}^F d(\xi)$
2. ${}^F d(V(\eta)) = d\eta$
3. ${}^F d([x]) = [x]^{p-1} d[x]$
4. $d(F(\xi)) = p\, {}^F d\xi$

Let us now construct the de Rham Witt complex in a more careful fashion as before:
We set

$$ W_1\Omega^\bullet_{S/R} = \Omega^\bullet_{S/R} = \check{\Omega}^\bullet_{W_1(S)/W_1(R)}. \tag{39} $$

We assume that we have already constructed a system $\left\{W_m\Omega^\bullet_{S/R}\right\}_{m\le n}$ of pd-differential graded $W_m(S)/W_m(R)$-algebras:

$$ W_n\Omega^\bullet_{S/R} \to W_{n-1}\Omega^\bullet_{S/R} \to \cdots \to \Omega^\bullet_{S/R} \tag{40} $$

and for all $m \le n$ surjective homomorphisms of differential graded algebras:

$$ \check{\Omega}^\bullet_{W_m(S)/W_m(R)} \to W_m\Omega^\bullet_{S/R} \tag{41} $$

which are compatible with the restriction maps and with F. Moreover we assume that for $1 \leq m < n$ there are additive maps

$$V : W_m\Omega^{\bullet}_{S/R} \to W_{m+1}\Omega^{\bullet}_{S/R}. \tag{42}$$

We require that $W_m\Omega^0_{S/R} = W_m(S)$, and that for $\omega \in W_m\Omega^{\bullet}_{S/R}$, $\eta \in W_{m+1}\Omega^{\bullet}_{S/R}$ the following relations holds:

$$\begin{aligned} FV(\omega) &= p\omega \\ {}^F dV(\omega) &= \omega \\ V(\omega F(\eta)) &= V(\omega) \cdot \eta \end{aligned}$$

We define an ideal $I \subset \check{\Omega}^{\bullet}_{W_{n+1}(S)/W_{n+1}(R)}$ as follows. We start with an arbitrary relation in $W_n\Omega^i_{S/R)}$:

$$\sum_{l=1}^{M} \xi^{(l)} d\eta_1^{(l)} \cdot \dots \cdot d\eta_i^{(l)} = 0. \tag{43}$$

Here i and M are positive integers and $\xi^{(l)}, \eta_k^{(l)} \in W_n(S)$ for $l = 1, \dots, M$ and $k = 1, \dots, i$. Then we consider the following elements of $\check{\Omega}^{\bullet}_{W_{n+1}(S)/W_{n+1}(R)}$:

$$\begin{gathered} \sum_{l=1}^{M} V(\xi^{(l)}) dV(\eta_1^{(l)}) \cdot \dots \cdot dV(\eta_i^{(l)}) \\ \sum_{l=1}^{M} dV(\xi^{(l)}) dV(\eta_1^{(l)}) \cdot \dots \cdot dV(\eta_i^{(l)}) \end{gathered}$$

These elements for all relations (43) generate a homogeneous ideal $I \subset \check{\Omega}^{\bullet}_{W_{n+1}(S)/W_{n+1}(R)}$. We see that $dI \subset I$. Moreover I is mapped to 0 by the map

$$F : \check{\Omega}^{\bullet}_{W_{n+1}(S)/W_{n+1}(R)} \to \check{\Omega}^{\bullet}_{W_n(S)/W_n(R)} \to W_n\Omega^{\bullet}_{S/R}. \tag{44}$$

We set

$$\bar{\Omega}^{\bullet}_{n+1} = \check{\Omega}^{\bullet}_{W_{n+1}(S)/W_{n+1}(R)}/I \tag{45}$$

This is a differential graded algebra. The map F factors through a map of algebras

$$F : \bar{\Omega}^{\bullet}_{n+1} \to W_n\Omega^{\bullet}_{S/R}. \tag{46}$$

By definition of I we also have a map

$$V : W_n\Omega^\bullet_{S/R} \to \bar{\Omega}^\bullet_{n+1} \tag{47}$$

which satisfies ${}^F dV(\omega) = \omega$ for all $\omega \in W_n\Omega^\bullet_{S/R}$. Consider now the ideal $\mathcal{I} \subset \bar{\Omega}^\bullet_{n+1}$, which is generated by the following elements $V(\omega F(\eta)) - V(\omega)\eta$ and $d(V(\omega F(\eta)) - V(\omega)\eta)$. Then we set

$$W_{n+1}\Omega^\bullet_{S/R} = \bar{\Omega}^\bullet_{n+1}/\mathcal{I} \tag{48}$$

Obviously, F maps $\mathcal{I}$ to zero so we get maps

$$F : W_{n+1}\Omega^\bullet_{S/R} \to W_n\Omega^\bullet_{S/R}$$
$$V : W_n\Omega^\bullet_{S/R} \to W_{n+1}\Omega^\bullet_{S/R}$$

6.2 Overconvergent Witt vectors

Let A be a discrete valuation ring of characteristic p with valuation ν. For any Witt vector

$$\alpha = (a_0, a_1, a_2, \dots) \in W(A) \tag{49}$$

we say that α has radius of convergence $\epsilon > 0$, if there is a constant $c \in \mathbb{R}$, such that

$$i \geq -\epsilon^{-i}\nu(A_i) + c. \tag{50}$$

We denote the ring of these Witt vectors by $W^\epsilon(A)$. We define the Gauss norm

$$\gamma_\epsilon(\alpha) = \inf\left\{i + \epsilon p^{-i}\nu(a_i)\right\} \tag{51}$$

Definition: *The union of the rings $W^\epsilon(A)$ for $\epsilon > 0$ is called the ring of overconvergent Witt vectors $W^\dagger(A)$. Let $\alpha \in W^\dagger(A)$ and let $\delta > 0$ a real number. Then there is an $\epsilon > 0$ such that $\gamma_\epsilon(\alpha) > -\delta$.*

Let R be an integral domain and endow it with the trivial valuation. Consider on the the polynomial ring $A = R[T_1, \dots, T_d]$ a degree valuation ν, such that $\nu(T_i) = -\delta_i < 0$. Let γ_ϵ be the associated Gauss norms on $W(A)$. Let us denote by $[1, d]$ the set of natural numbers between 1 and d. A weight k is a function $k : [1, d] \to \mathbb{Z}_{\geq 0}[1/p]$. Its values are denoted by k_i. The denominator of k is the smallest number u such that $p^u k$ takes values in $\mathbb{Z}$. We set $\delta(k) = k_1\delta_1 + \dots + k_d\delta_d$. We write $X_i = T_i$ for the Teichmüller

representative and we set $X^k = X_1^{k_1} \cdot \dots \cdot X_d^{k_d}$.
Any element $\alpha \in W(A)$ has a unique expansion:

$$\alpha = \sum_k \xi_k X^k, \ \xi_k \in V^u W(R). \tag{52}$$

This series is convergent in the $V(W(R))$-adic topology.
For $\xi \in W(R)$ we define:

$$\mathrm{ord}_V \xi = \min \{m | \xi \in V^m W(R)\}. \tag{53}$$

6.3 Witt differentials

For each weight k we fix once and for all a total order on the arguments where k doesn't vanish. We denote it by $\mathrm{supp}(k)$. We choose the total orders in such a way that for each integer a and for each weight k the orders on $\mathrm{supp}(k) = \mathrm{supp}(p^a(k))$ agree. We will call a weight k primitive if it is integral and not all k_i are divisible by p. The orders for primitive weights are chosen arbitrarily. We set $t(k_{i_l}) = -\mathrm{ord}_p k_{i_l}$ and $u(k_{i_l}) = max(0, t(k_{i_l}))$. We will denote by I an interval of $supp(k)$ in the given order.

$$I = \{i_l, i_{l+1}, \dots, i_{l+m}\}. \tag{54}$$

The restriction of k to I will be denoted by k_I and the extension by zero to $[1, d]$ will be denoted by the same letter k_I. Then we set

$$t(k_I) = \max\{t(k_i) | i \in I\}$$
$$u(k_I) = \max(0, t(k_I))$$

In degree zero we have $W\Omega^0_{S/R} = W(S)$ with expansion given in (52). We have two types of basic Witt differentials of degree one. If the weight k_I is not integral we consider for $I \neq$:

$$dV^{u(I)}(\eta X^{p^{u(I)} \cdot k_I}). \tag{55}$$

If the weight k_I is integral we have the basic differential

$$F^{-t(I)}(dX^{t(I)k_I}) = X^{(k_I - p^{t(I)}k_I)} dX^{p^{t(I)}k_I}. \tag{56}$$

In general a basic Witt differential is obtained by talking products of these elememts in a certain way:

Let $\mathcal{P} = I_0, \dots, I_l$ be a partition of $supp(k)$ in disjoint intervals. The interval I_0 may be empty but the interval $I_1, \dots, I_l$ assumed to be non-empty. For $\xi \in V^{u(I)}W(R)$ we define a basic Witt differential

$$e = e(\xi, k, \mathcal{P}) \in W\Omega^l_{R[T_1,\dots,T_d]/R} \tag{57}$$

of degree l as follows:
We set $\xi = V^{u(I)}\eta$. Let us denote by $r \in [0, \dots, l-1]$ the first index such that $K_{I_{r+1}}$ is integral. We set $r = l$ if k_{I_l} is not integral.
We distinguish 3 cases in the definition of e:
First case: $I_0 \neq$.

$$\begin{aligned} e =& V^{u(I_0)}(\eta X^{p^{u(I_0)}k_{I_0}})(dV^{u(I_1)}X^{p^{u(I_1)})k_{I_1}}) \dots (dV^{u(I_r)}X^{p^{u(I_r)}}) \\ & ({}^{F^{-t(I_{r+1})}}dX^{p^{t(I_{r+1})}k_{I_{r+1}}}) \dots ({}^{F^{-t(I_l)}}dX^{p^{t(I_l)}k_{I_l}}) \end{aligned}$$

Second case: $I_0 =$ and k not integral, i.e. $r > 0$.

$$\begin{aligned} e =& (dV^{u(I_1)}(\eta X^{p^{u(I_1)}k_{I_1}}))(dV^{u(I_2)}X^{p^{u(I_2)})k_{I_2}}) \dots (dV^{u(I_r)}X^{p^{u(I_r)}}) \\ & ({}^{F^{-t(I_{r+1})}}dX^{p^{t(I_{r+1})}k_{I_{r+1}}}) \dots ({}^{F^{-t(I_l)}}dX^{p^{t(I_l)}k_{I_l}}) \end{aligned}$$

Third case: $I_0 =$ and k integral.

$$e = \eta({}^{F^{-t(I_1)}}dX^{p^{t(I_1)}k_{I_1}}) \dots ({}^{F^{-t(I_l)}}dX^{p^{t(I_l)}k_{I_l}})$$

With these differentials each element $\omega \in W\Omega^r_{A/R}$ has a unique decomposition as a sum of basic differentials

$$\omega = \sum_{k,\mathcal{P}} e(\xi_{k,\mathcal{P}}, k, \mathcal{P}). \tag{58}$$

where $\xi_{k,\mathcal{P}} \in V^m W(R)$ for all but finitely many weights k. We can extend the Gauss norms γ_ϵ to the de Rham=Witt complex. Given the expansion (58) we set:

$$\gamma_\epsilon(\omega) = \inf_{k,\mathcal{P}}\{\mathrm{ord}_V \xi_{k,\mathcal{P}} - \epsilon|k|\}. \tag{59}$$

We will call ω overconvergent, if there is an $\epsilon > 0$ such that ω has radius of convergence ϵ. The overconvergent Witt differentials form a subgroup of $W\Omega_{A/R}$ which is denoted by $W^\dagger\Omega_{A/R}$. Let now be A a smooth finitely generated k-algebra, $S = k[T_1, \dots, T_r]$ the polynomial algebra. Then the epimorphism $S \to A$ induces a canonical epimorphism

$$\lambda : W\Omega^\bullet_{S/k} \longrightarrow W\Omega^\bullet_{A/k} \tag{60}$$

of de Rham-Witt complexes.

Definition: *We set* $W^\dagger\Omega^\bullet_{A/k} = \lambda(W^\dagger\Omega^\bullet_{S/k})$.

Next we want to turn the overconvergent de Rham-Witt complex into a sheaf over Spec(A). Let us denote by $f \in A$ an arbitrary element. The presheaf

$$W^\dagger\Omega^d_{\text{Spec } A/k}(\text{Spec } A_f) := W^\dagger\Omega^d_{A_f/k} \tag{61}$$

is a sheaf for the Zariski topology on Spec A. The Zariski cohomology of these sheaves vanishes in degrees $j > 0$, i.e.

$$H^j_{Zar}(\text{Spec } A, W^\dagger\Omega^d_{\text{Spec } A/k}) = 0. \tag{62}$$

The complex $W^\dagger\Omega_{\text{Spec } A/k}$ extends to a complex of Zariski sheaves $W^\dagger\Omega_{X/k}$ on any variety X/k. Let X be a smooth variety. Then $W^\dagger\Omega^\bullet_{X/k}$ defines a complex of étale sheaves on X.

6.4 Comparison with Monsky-Washnitzer cohomology

Let B/k be a finitely generated, smooth algebra over a perfect field k of char $p > 0$. Let $\tilde{B}^\dagger$ be the weak completion of a smppth finitely generated $W(k)$-algebra $\tilde{B}$ lifting B. There exist a map

$$t_F : \tilde{B}^\dagger \longrightarrow W(B) \tag{63}$$

described in [5]. This map depends on the choice of a lift F of the Frobenius on B. Fix a lift. Then the image of this map is $W^\dagger(B)$. Let $B = A[X]/(f(X))$ where A is smooth over k, B étale over A, $f(X)$ monic of degree $m = [B : A]$ such that $f'(X)$ is invertible in B. Let $[x]$ be the Teichmüller of the element X mod $f(X)$ in $W(B)$. Then we have for each $d \geq 0$ a direct sum decomposition of $W^\dagger(A)$-modules

$$W^\dagger\Omega^d_{B/k} = W^\dagger\Omega^d_{A/k} \oplus W^\dagger\Omega^d_{A/k}[x] \oplus \cdots \oplus W^\dagger\Omega^d_{A/k}[x]^{m-1}. \tag{64}$$

Let $A = k[T_1, \dots, T_n]$, $f \in A$. Let now be $\tilde{f} \in \tilde{A} := W(k)[T_1, ..., T_n]$ be a lifting of f and $\tilde{A}_{\tilde{f}} := W(k)[T_1, \dots, T_n]_{\tilde{f}}$. B lifts to a finite étale extension $\tilde{B}$ over $\tilde{A}_{\tilde{f}}$. If $B = A_f[x]$, then $\tilde{B} = \tilde{A}_{\tilde{f}}[x]$. We write $u = [x]$ for the Teichmüller representative of $x \in W(B)$. Consider the canonical map

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\sigma} & W^\dagger(B) \\ \uparrow & & \uparrow \\ \tilde{A}_{\tilde{f}} & \longrightarrow & W^\dagger(A_f) \end{array}$$

Hence B has a canonical overconvergent Witt lift. Let $\tilde{B}^\dagger$, $\tilde{A}^\dagger_{\tilde{F}}$ be the weak completion of $\tilde{B}$, $\tilde{A}_{\tilde{f}}$. Then $\tilde{B}^\dagger = \tilde{A}^\dagger_{\tilde{f}}[x]$ is finite étale over $\tilde{A}^\dagger_{\tilde{f}}$. Using the canonical composition of $W^\dagger\Omega^\bullet_{B/k}$ described above we get a canonical map

$$\sigma : \Omega^\bullet_{\tilde{B}^\dagger/W(k)} = \tilde{B}^\dagger \bigotimes_{\tilde{A}^\dagger_{\tilde{f}}} \Omega^\bullet_{\tilde{A}^\dagger_{\tilde{f}}/W(k)} \to W^\dagger\Omega^\bullet_{B/k} = \bigoplus_{i=0}^{m-1} W^\dagger\Omega^\bullet_{A_f/k} x^i \tag{65}$$

This map is a quasi-isomorphism.
For an arbitrary smooth algebra A, consider an overconvergent Witt lift

$$\phi : \tilde{A} \to W^\dagger(A) \tag{66}$$

which is uniquely determined by a lifting of the Frobenius to $\tilde{A}^\dagger$. It induces a map of complexes

$$\psi : \Omega_{\tilde{A}^\dagger/W(k)} \to W^\dagger\Omega_{A/k} \tag{67}$$

If dim $A < p$. Then ϕ_* is an isomorphism. In general, there is n isomorphism

$$H^*_{MW}(A/K) \cong H^*(W^\dagger\Omega_{A/k} \otimes_{W(k)} K) \tag{68}$$

between Monsky-Washnitzer cohomology and overconvergent de Rham-Witt cohomology, where again $K = W(k)[\frac{1}{p}]$. In general there a even stronger result holds:

Theorem: *Let X be a smooth quasi-projective scheme over k. Then we have a natural quasiisomorphism*

$$R\Gamma_{rig}(X) \to R\Gamma(X, W^\dagger\Omega_{X/k}) \otimes \mathbb{Q} \tag{69}$$

Heuristically we can interpret this theorem as follows: If X/k can be embedded in an projective space, we can compute the p-adic cohomology of a X by a certain kind of de Rham-complex. This extends the philosophy that motivates all kinds of p-adic cohomology theories:

$\{$p-adic cohomology in char p$\} \leftrightarrow \{$de Rham cohomology of a lift to char 0$\}$.

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