Sommerakademie zur p-adischen Hodge Theorie

Dieses Bericht entstand im Rahmen der CIMPA Sommerakademie[1], vom 2.-10. August 2021 am CIMAT in Guanajuato, Mexico, an der ich online teilgenommen habe. Die Vorlesungen fanden über 7 Tage jeweils von 9 Uhr bis 18 Uhr (lokaler Zeit) statt. Dies entspricht insgesamt 63 Stunden. Für den Erwerb von ECTS war außerdem ein Vortrag und dieser Bericht notwendig. Der Vortrag fand in Präsenz in Heidelberg unter der Aufsicht von Rustam Steingart statt. Die Vorbereitung des Vortrags und die Ausarbeitung des Berichts benötigten nochmal ungefähr 40 Stunden. Der gesamte Zeitaufwand betrug somit circa 100h. In diesem Bericht geben wir eine Zusammenfassung der Themenblöcke ”p-adische Galois Darstellungen” und ”Kristalline Darstellungen”. Die weiteren Themenblöcke waren: ”Klassiche Hodge Theorie”, ”Étale Kohomologie”, ”p-adische Geometrie” und ”1-Motive als Hodgestrukturen auf Level 1”.

Ich möchte mich herzlich bei Rustam für die Betreuung wärend der Sommerakademie und sein Feedback zu diesem Bericht bedanken.

Summer School on p-adic Hodge Theory

This report is part of the requirements to earn credits for the 2021 CIMPA Research School on Hodge Theory and p-adic Hodge Theory[1], which took place online from August, 2nd-10th, 2021. The school included lectures between 9am-18pm for 7 days (i.e approx. 63h in total). Additionally, it was required to give a presentation about a selected topic from the courses. The presentation took place in Heidelberg under the supervision of Rustam Steingart. Together with additional readings and reviewing, this took about an extra 40h. All in all, the time requirement for this school was about 100h.

In this report we give a summary of the courses ”p-adic Galois Representations” and ”Crystalline Representations”. The other courses at the school were ”Classical Hodge Theory”, ”p-adic Geometry”, ”Étale Cohomology” and ”1-motives as Hodge structures of level 1”.

I’d like to thank Rustam for his mentoring during the summer school and his helpful feedback to this report.

1 p-adic Galois Representations from Geometry

One motivation for the development of p-adic Hodge theory historically was the study of p-adic Galois representations. Geometry provides a lot of interesting Galois representations. Let’s start with a concrete example:

Let K be a field of char $K \neq 2,3$ and E an elliptic curve over K. We denote the separable closure of K by K^s. Then there exists $a_i \in K$, $i = 1,2,3$ such that with $P(X) = \lambda(X - a_1)(X - a_2)(X - a_3)$ the K^s points of E are given by

$$E(K^s) = \left\{(x,y) \in (K^s)^2 \,|\, y^2 = P(x)\right\} \cup \{\infty\}$$

[1] https://cimparesearchschool.eventos.cimat.mx/

Since $P \in K[X]$, the absolute Galois group G_K is acting on $E(K^s)$. For any field F amd some prime ℓ we can define the subgroup of l^n torsion points

$$E_{\ell^n}(F) := \{p \in E(F) \mid \ell_n p = 0\}$$

one can show that

$$E_{\ell^n}(K^s) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} \times \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } \ell \neq \operatorname{char} K \\ 0, \mathbb{Z}/\ell^n\mathbb{Z} & \text{if } \ell = \operatorname{char} K \end{cases}$$

In particular $E_{\ell^n}(K^s)$ is a $\mathbb{Z}/\ell^n\mathbb{Z}$-module. There is an obvious map $E_{\ell^{n+1}}(K^s) \to E_{\ell^n}(K^s)$ given by multiplication with l.

Definition 1.1. The Tate module of E is the $\mathbb{Z}_\ell$

$$T_\ell(E) = \varprojlim_n E_{\ell^n}(K^s) \tag{1.1}$$

The G_K action on $E(K^s)$ induces a G_K action on $T_\ell(E)$. Let's assume $\ell \neq \operatorname{char} K$, then $T_\ell(K) \cong \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell$. This is the H^1 of a Torus with coefficients in $\mathbb{Z}_\ell$ and indeed we have

$$T_\ell(E)^\vee \cong H^1_{\acute{e}t}(E_{K^s}, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell. \tag{1.2}$$

This is one example, of how ℓ-adic Galois representations arise from ℓ-adic étale cohomology. In the case that $\ell = \operatorname{char} K$ one has to be more careful and we need to consider crystalline cohomology. In general if X is proper smooth over K and $\ell \neq \operatorname{char} K$, then $H^m_{\acute{e}t}(X_{K^s}, \mathbb{Q}_\ell)$ is an ℓ-adic Galois representation.

Before discussing we discuss the role of p-adic Hodge Theory in this story, we give a brief review of ordinary Hodge Theory.

Let $X \to S$ be a smooth scheme of relative dimension n, we can define $\Omega^1_{X/S}$ its sheaf of Kähler differentials. We have the de Rham complex

$$\Omega^\bullet_{X/S} := 0 \longrightarrow \mathcal{O}_{X/S} \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{X/S} \longrightarrow 0.$$

In particular we have $d^2 = 0$ and hence we can define relative de Rham cohomology as the cohomology of this complex

$$H^*_{dR}(X/S) := \mathbb{H}^*(X, \Omega^\bullet_{X/S}). \tag{1.3}$$

Let's focus on the case where $S = \mathbb{C}$ first.

Theorem 1.2 (Hodge Theorem)**.** *Let X be a smooth scheme over $\mathbb{C}$. There is a canonical isomorphism from algebraic to analytic de Rham cohomology*

$$H^*_{dR}(X/\mathbb{C}) \simeq H^*_{dR}(X(\mathbb{C})) \otimes_\mathbb{R} \mathbb{C} \simeq H^*_{sing}(X(\mathbb{C}), \mathbb{C}). \tag{1.4}$$

Furthermore, the first isomorphism is $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ *equivariant.*

In the analytic setting we can decompose the complexified de Rham cohomology into Dolbeault cohomology groups $H^{p,q}(X,\mathbb{C})$. From this we can define the Hodge filtration:

$$\mathrm{Fil}^i H^r_{dR}(X)\otimes_{\mathbb{R}}\mathbb{C} := \bigoplus_{\substack{a+b=r\\ a\leq i}} H^{a,b}(X(\mathbb{C}))$$

On the other hand we can define the filtration on $H^*_{dR}(X/\mathbb{C})$. We first define a filtration on $\Omega^\bullet_{X/\mathbb{C}}$ by

$$\mathrm{Fil}^i\Omega^\bullet_X := 0\longrightarrow 0\longrightarrow \ldots \longrightarrow \Omega^i_{X/S}\xrightarrow{d}\Omega^{i+1}_{X/S}\xrightarrow{d}\ldots\xrightarrow{d}\Omega^n_{X/S}\longrightarrow 0.$$

We have an obvious map $\mathrm{Fil}^i\Omega^\bullet_X\to\Omega^\bullet_X$. This induces a map on the cohomology. With that we get the algebraic Hodge filtration

$$\mathrm{Fil}^i H^r_{dR}(X/\mathbb{C}) := \mathrm{im}(H^r(X,\mathrm{Fil}^i\Omega^\bullet_X)\to H^r(X,\Omega^\bullet_{X/S}))$$

There is an analogous statement in the p-adic world. Let $K/\mathbb{Q}_p$ be a finite extension and $C := \hat{\overline{K}}$. The action of $G_K := \mathrm{Gal}(\overline{K}/K)$ on $\overline{K}$ extends to an action on C by continuity. Let X be a smooth and proper scheme over K.

Theorem 1.3 (Faltings)**.** *For $n\geq 0$ there exists a natural G_K-equivariant isomorphism*

$$H^n_{\acute{e}t}(X_{\overline{K}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} C\simeq\bigoplus_{i+j=n} H^i(X,\Omega^j_{X/K})\otimes_K C(-j) \tag{1.5}$$

Here $C(-j)$ is the $(-j)$-th Tate twist of M is defined by

$$M(j) = M\otimes^{\otimes j}_{\mathbb{Z}_p},\ j\in\mathbb{Z}$$

with the diagonal G_K-action, where

$$\mathbb{Z}_p(1) := \varprojlim_k \mu_{p^k}(\overline{K})$$

is the Tate module of the p^∞-roots of unity in $\overline{K}$ (with its canonical Galois action). As $\mathbb{Z}_p$-modules, $\mathbb{Z}_p(1)\cong\mathbb{Z}_p$. From the theorem we get

Corollary 1.4. *For $n\geq 0$, $j\geq 0$*

$$H^{n-j}(X,\Omega^j_{X/K})\cong(H^n_{\acute{e}t}(X_{\overline{K}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} C(j))^{G_K}. \tag{1.6}$$

As a slogan: "p-adic étale cohomology knows Hodge cohomology".

The de Rham cohomology $H^n_{dR}(X)$ of a proper, smooth scheme over K together with its filtration is a slightly finer invariant than the Hodge cohomology.
This leads to the following question: Does the G_K-representation $H^n_{\acute{e}t}(X_{\overline{K}},\mathbb{Q}_p)$ determine $H^n_{dR}(X)$ together with its filtration? The answer is yes.

Theorem 1.5. *For $n\geq 0$, there exists a natural G_K-equivalent, filtered isomorphism*

$$H^n_{\acute{e}t}(X_{\overline{K}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} B_{dR}\cong H^n_{dR}\otimes B_{dR}. \tag{1.7}$$

Here B_{dR} is Fontaine's field of p-adic periods, which is the fraction field of a complete discrete valuation ring B_{dR}^+ with residue field C. As such B_{dR} is naturally filtered by

$$\mathrm{Fil}^j B_{dR} := \xi^j B_{dR}^+,\ j \in \mathbb{Z},$$

where $\xi \in B_{dR}^+$ is a uniformizer. The G_K-action is diagonally on LHS, via B_{dR} on RHS. The filtration is via B_{dR} on the LHS, and diagonally on the RHS.
We give a brief reminder of the construction of B_{dR}.
Consider the non-archimedean field $\mathbb{C}_K$ with ring of integers $\mathcal{O}_{\mathbb{C}_K}$. The tilt of an $\mathcal{O}_{\mathbb{C}_K}$ algebra A is defined by

$$A^\flat = \left\{ (x^{(n)})_{n\geq 0} \mid x^{(n)} \in A,\ \left(x^{(n+1)}\right)^q = x^{(n)} \right\}.$$

This defines a functor $(-)^\flat : \mathrm{Alg}_{\mathcal{O}} \to \mathrm{Menge}$. If we restrict this functor to $\hat{\mathrm{Alg}}_{\mathcal{O}}$, the categaory of π-adically complete and seperated (i.e. $\cup_n \pi^n \mathcal{O} = 0$), this factor canonically through Perf_k which where $A^\flat$ is isomorphic to $\varprojlim A/\pi A$ where the transition functions are given by the Frobenius $\varphi_q : A/\pi \to A/\pi$. This functor is right adjoint to the functor

$$W(-)_L : \mathrm{Perf}_k \to \hat{\mathrm{Alg}}_{\mathcal{O}}.$$

We are now in a position to construct B_{dR}. The ring $\mathcal{O}^\flat_{\mathbb{C}_K}$ is ϖ-adically complete and separated for any pseudo-uniformizer ϖ. The tilt $\mathcal{O}^\flat_K$ containts an element $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ and $\varpi_\flat := \varepsilon - 1$. We define $\tilde{\mathbb{A}}^+ := W(\mathcal{O}^\flat_K)$ and $\tilde{\mathbb{B}}^+ := \tilde{\mathbb{A}}\left[\frac{1}{p}\right]$. The counit of the adjunction above gives a map

$$\theta : \tilde{\mathbb{A}}^+ \twoheadrightarrow \mathcal{O}_{\mathcal{O}_K}.$$

One can show that the kernel of this map is $\ker\theta = ([\tilde{p}] - p) = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}$. The kernel is G_K-stable, but not φ-stable. Inverting p gives a map

$$\theta_{\mathbb{Q}} : \tilde{\mathbb{B}}^+ \twoheadrightarrow \mathbb{C}_K.$$

Then we define B_{dR}^+ as

$$\varprojlim_j \tilde{\mathbb{B}}^+ / (\ker\theta_{\mathbb{Q}})^j.$$

This is a complete discrete valuation ring with maximal ideal $\ker\theta_{\mathbb{Q}}$ and residue field $\mathbb{C}_K$. Finally, $B_d R = \mathrm{Quot}(B_{dR}^+)$. One can show that $B_{dR}^{G_K} = K$. Let

$$t := \log([\varepsilon]) \in B_{dR}.$$

This element transforms as $g(t) = \chi(g)t$ for all $g \in G_K$. Thus we see that B_{dR} contains a canonical G_K-stable line $\mathbb{Q}_p t \subseteq B_{dR}$ on which G_K acts via the cyclotomic character

$$\chi_{cycl} : G_K \to \mathbb{Z}_p,$$

i.e. $\mathbb{Q}_p t \cong \mathbb{Q}_p(1)$. The element t is the analogue of the period $2\pi i \in \mathbb{C}$ in complex geometry.

Assuming now that X/K has good reduction, i.e. it exists a proper, smooth scheme $\mathcal{X}/\mathrm{Spec}(\mathcal{O}_K)$ such that $X \cong \mathcal{X}_K$ is isomorphic to the generic fiber of $\mathcal{X}$. Let $X_0 := \mathcal{X} \times_{\mathcal{O}} k$ be the special fiber living over $k = \mathcal{O}/\mathfrak{m}$. In this case we can compute the crystalline cohomology of X_0 with coefficients in the Witt vectors. Roughly the crystalline cohomology of X_0 is isomorphic to the de Rham cohomology of a smooth lift to $\mathcal{O}_{K_0}$ (note that $\mathcal{X}$ is just a smooth lift to $\mathcal{O}_K$ but not to $\mathcal{O}_{K_0}$). There is a crystalline comparison theorem similar to the de Rham comparison theorem:

Theorem 1.6. *For $n \geq 0$ there exists a natural G_K-equivariant, filtered φ-equivariant isomorphism*

$$H^n_{ét}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} \cong H^n_{cris}(X_0/\mathcal{O}_{K_0}) \otimes_{\mathcal{O}_{K_0}} B_{cris}. \tag{1.8}$$

Here $K_0 \subseteq K$ is the maximal unramified subextension. We have $K_0 \cong W(k)$ and hence the Frobenius φ lifts to $\mathcal{O}_{K_0}$. On the other by functoriality there exists a natural Frobenius endomorphism φ on $H^n_{cris}(X_0/\mathcal{O}_{K_0})$.
The data $H^n_{cris}(X_0/\mathcal{O}_{K_0})$ with its Frobenius and the Hodge filtration over K (induced by the filtration on H^n_{dR}) is an example of a filtered φ-module $(D, \varphi_D, Fil^i(D_K))$ over K, i.e. a K_0-vector space D together with an isomorphism $\varphi_D : \varphi^* D \cong D$ and a decreasing separated and exhaustive filtration on $D_K := D \otimes_{K_0} K$. The period Ring B_{cris} can be defined in terms of crystalline cohomology as follows:

$$\mathbb{A}_{cris} := H^0_{cris}((\mathcal{O}_C/p)/\mathbb{Z}_p), \quad B^+_{cris} := \mathbb{A}_{cris}[1/p]. \tag{1.9}$$

By functoriality, there exists a natural Frobenius φ on $\mathbb{A}_{cris}$. It turns out that B^+_{cris} embeds into B^+_{dR} with image stable by G_K and containing $t = \log([\epsilon])$. Finally,

$$B_{cris} := B^+_{cris}[1/t]$$

and $\varphi(t) = p \cdot t$.

More abstractly, one can ask how one can pass from a continuous G_K-representation on a finite dimensional $\mathbb{Q}_p$-vector space to a finite dimensional K_0-vector space with a Frobenius and a filtration (over K). The answer is given by the following functors

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_p}(G_K) & \rightarrow & \{\text{filtered } \varphi\text{-modules}\} \\ V & \mapsto & D_{cris}(V) := (V \otimes_{\mathbb{Q}_p} B_{cris})^{G_K} \\ V_{cris}(D) = Fil^0(D \otimes_{K_0} B_{cris})^{\varphi=1} & \leftarrow & D \end{array}$$

A representation is called crystalline if it is B_{cris}-admissible, i.e. $\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(D_{cris}(V))$.

Theorem 1.7. *The functors D_{cris}, V_{cris} restrict to equivalences*

$$\{\textit{crystalline } G_K\textit{-representations}\} \overset{1:1}{\leftrightarrow} \{\textit{weakly admissible filtered } \varphi\textit{-modules over } K\},$$

i.e. weakly admissible implies admissible.

The condition "weakly admissible" is related to the statement that the "Newton polygon lies above the Hodge polygon".

Definition 1.8. Let F be a field and $0 \neq (D, Fil^i(D)) \in Fil_K$. Let $i_0 < \cdots < i_n$ be the different i with $gr^i(D) \neq 0$. The Hodge polygon is the konvex hull of the contour starting from $(0,0)$ on the left and then having $\dim_F gr^{i_j}$ Segments of slope i_j. Where i_j increase from the left to the right.

The Newton polygon is harder to define. While the Hodge-Polygon depends on the filtration, the Newton Polygon represents properties of the Frobenius operator.

Note that a similar statement for

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_p}(G_K) & \rightarrow & Fil_K \\ V & \mapsto & D_{dR}(V) := (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} \end{array}$$

where Fil_K denotes the category of filtered vector spaces does *not* hold! In particular, the functor is not fully faithful.

References

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