Notes on Electro-Magnetic Duality with view toward the Geometric Langlands Correspondence

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Abstract

These notes were written as a final project for the course "Topics in supersymmetry and super gravity" taught by Prof. Mauricio Romo in the Fall Term 2019/2020 at Tsinghua University. The goal of these notes was for me to understand the fundamentals of the Kaspustin-Witten Theory connecting electric-magnetic duality [9]. These notes don't aim to be a comprehensive treatment of this theory, rather I tried to understand the basic mathematical and physical requirements that are necessary to study this theory. These notes document some parts of this study, but exclude any treatment of the classical and geometric Langlands Program. A great reference of this topic is [1]. We start with rather simple mathematical preliminaries which will lead us to give a definition of the *Langlands dual group*. The next important concept is the introduction of *Super Yang-Mills Theory*. In the special case of $\mathcal{N} = 4$ super Yang-Mills theory in 4 dimensions, after applying Montonen-Olive duality one finds that the *Geometric Langlands Correspondence* naturally shows up. I do not claim any of the results in this report to be my own, I merely tried to connect ideas from different sources, such that they make sense for me.

1 Lie Algebras

The purpose of this chapter is to recall in particular the notion of root systems and the associated dual root system and define the Langlands dual group, since it will show up later in the discussion of the Montonen-Olive duality and also when studying the connection between Wilson line operators and 't Hooft operators. The main reference for Lie Algebras used here is [6].

1.1 Definition and main examples

For completeness we first give the definition of a Lie algebra, although the reader should already be familiar with this. Let K, be a field. For simplicity we assume that char $K = 0$. A *Lie algebra* $\mathfrak{g}$ over K is a non-unitary, non-associative K-algebra, with alternating composition law, satisfying the Jacobi identity. The composition law is usually written as $[\cdot, \cdot]$ and we call it *Lie bracket*. Note that $\mathfrak{g}$ is in particular a K-vector space.
A *Lie subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace that is closed under the Lie bracket. An *ideal* $\mathfrak{i} \subseteq \mathfrak{g}$ is a subalgebra, such that $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$. A morphism of Lie algebras is defined in the obvious way.
We define the *centralizer* of a subset $S \subseteq \mathfrak{g}$ as $C_{\mathfrak{g}}(S) := \{x \in \mathfrak{g} \,|\, [x, S] = 0\}$. The *center* of $\mathfrak{g}$ is defined as $Z(\mathfrak{g}) := C_{\mathfrak{g}}(\mathfrak{g})$. Similarly the *normalizer subalgebra* is $N_{\mathfrak{g}}(S) := \{x \in \mathfrak{g} \,|\, [x, S] \subseteq S\}$ and $N(\mathfrak{g}) := N_{\mathfrak{g}}(\mathfrak{g})$.
The most important example of a Lie algebra is $\mathfrak{gl}(V) := End(V)$ the endomorphism Ring of a finite dimensional K-vector space V with the Lie bracket given by $[\psi, \phi] = \psi \circ \phi - \phi \circ \psi$. For $V = F^n$ we also use the notation $\mathfrak{gl}(n, K)$ or $\mathfrak{gl}_n(K)$ and identify $End(V)$ with $Mat_{n \times n}(K)$ and similar for subalgebras of $\mathfrak{g}$. One very important subalgebra of $\mathfrak{gl}(V)$ is the *special linear Lie algebra* $\mathfrak{sl}(V)$ consisting of the endomorphisms with vanishing trace.
A basis of $\mathfrak{sl}(2, K)$ as a vector space is give by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

The Lie algebra structure is completely determined by $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$.
A representation of a Lie algebra L is a homomorphism $\psi : L \to \mathfrak{gl}(V)$. A very important representation is the adjoint representation defined by $x \mapsto ad_x := [x, -]$.

1.2 Basic definitions Lie algebras

A Lie algebra L is called *simple* if L it is non-abelian and has no ideal except itself and 0.

The derived series is defined by $L^{(0)} = L$, $L^{(1)} = [L, L]$, ... $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$. A Lie algebra is solvable, if $L^{(n)} = 0$ for some n. Ever Lie algebra L has a maximal solvable ideal $Rad(L)$, the *radical* of L. If $Rad(L) = 0$, L is called *semisimple*.
The descending central series of L is defined by $L^0 = L$, $L^1 = [L, L]$, ..., $L^i = [L, L^{i-1}]$. L is called nilpotent if $L^n = 0$ for some n. Since $L^{(i)} \subseteq L^i$ for all i, nilpotent algebras are solvable.
For L to be nilpotent means that for all $x_1, ..., x_n$ in L, $ad_{x_1} ad_{x_2} ... ad_{x_n}(y) = 0$. In particular $(ad_x)^n = 0$. For an arbitrary Lie algebra an element x with is called *ad-nilpotent* if ad_x is nilpotent.

Theorem: *If all elements of L are ad-nilpotent, then L is nilpotent.*

From linear algebra Let L be a non-nilpotent Lie algebra. Hence L has non ad-nilpotent elements.
We call $x \in End(V)$ *semisimple* if the roots of its minimal polynomial are all distinct. For $x \in End(V)$ there exist unique $x_s, x_n \in End(V)$ such that $x = x_s + x_n$ and x_s is semisimple and x_n nilpotent and x_n and x_s commute. This composition is called *Jordan decomposition*. Consider now an arbitrary semi-simple Lie algebra L. The adjoint representation of L is injective. We can find the Jordan decomposition of $ad_x = (ad_x)_s + (ad_x)_n$. This induces a decomposition $x = x_s + x_n$. This is called the abstract Jordan decomposition. An element $x \in L$ is called semisimple if $x_n = 0$. A subalgebra of a Lie algebra L that consists of semisimple elements is called *toral*. Toral subalgebras are abelian.
Fix a maximal toral subalgebra H of L. For $\mathfrak{sl}(n, K)$ we can choose the set of all diagonal matrices (of trace 0). Since H is abelian, ad_H is a commuting family of semisimple endomorphisms and thus all elements can be diagonalized at the same time. That means to say, that L is the direct sum of the subspaces $L_\alpha = \{x \in L | [h, x] = \alpha(h)x \text{ for all } h \in H\}$ where $\alpha \in H^*$. In particular $L_0 = C_L(H)$ and since H is abelian $H \subseteq L_0$. The set of all non-zero $\alpha \in H^*$ for which $L_\alpha \neq 0$ is denoted by Φ and the elements of Φ are called roots of L.
A *Cartan subalgebra* of a Lie algebra L is a nilpotent subalgebra which equals its normalizer in L ($N_L(H) = H$).

1.3 Root system and the Weyl Group

Let E be a fixed Euclidean space. That is an finite dimensional $\mathbb{R}$ vector space, with a scalar product (α, β). To any $\alpha \in E$ we can associate the orthogonal hyperplane $P_\alpha = \{\beta \in E | (\beta, \alpha)\}$. The orthogonal reflection at P_α is given by $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{\alpha,\alpha}\alpha$. It turns out to be handy to define $\langle \alpha, \beta \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. A subset Φ of the Euclidean space E is called a *root system* in E if the following axioms are satisfied:

(R1) Φ is finite, spans E, and does not contain 0.

(R2) If $\alpha \in \Phi$, the only multiple of α in Φ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

(R4) If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

We can define a symmetric bilinear form on L, called the Killing form, given by $\kappa(x, y) = Tr(ad_x ad_y)$. It is true that a Lie algebra is semisimple if and only if the Killing form is non-degenerate. In particular the restriction of the Killing form to a toral subalgebra H of a Lie algebra L is non-degenerate. A non-degenerate bilinear form induces an isomorphism $H^* \cong H$ sending $\phi \in H^*$ to $t_\phi \in H$ satisfying $\phi(h) = \kappa(t_\phi, h)$ for all $h \in H$. Then the set of roots of a Lie algebra Φ is a root system in E the $\mathbb{R}$ span of Φ in H^* with bilinear form $(\gamma, \delta) = \kappa(t_\gamma, t_\delta)$.
The *Weyl group* $\mathcal{W}$ is the subgroup of $GL(E)$ generated by σ_α for all $\alpha \in \Phi$. Due to (R2), we can identify $\mathcal{W}$ with a subgroup of the symmetric group on Φ.

1.4 Weight lattice

Let V be a representation of a Lie algebra L, H a Cartan subalgebra of L and let $\lambda \in H^*$. Then the *weight space* of V with weight λ is the subspace $V_\lambda := \{v \in V | h \cdot v = \lambda(h)v \text{ for all } h \in H\}$. A weight of the representation V is $\lambda \in H^*$ such that V_λ is nonzero. Consider now $0 \neq V \in V_\lambda$ for λ a weight of the representation V and $x \in L_\alpha$, then

$$h \cdot (x \cdot v) \, [(\lambda + \alpha)(h)](x \cdot v), \quad \text{for all } h \in H^* \tag{2}$$

Thus, the action of x maps the weight space with weight λ into the weight space with weight $\lambda + \alpha$. The weights which satisfy $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$ are called integral. The integral weights constitute a lattice in E.

1.5 Dual root system and coroots

If Φ is a root system in E, the *coroot* $\alpha^\vee$ of a root α is defined by

$$\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha. \tag{3}$$

The set of coroots also forms a root system $\Phi^\vee$ in E, called the *dual root system*. The lattice spanned by $\Phi^\vee$ is called the coroot lattice. Both Φ and $\Phi^\vee$ have the same Weyl group W and for $s \in W$ we have $(s\alpha)^\vee = s(\alpha^\vee)$.

1.6 Langlands Dual Group

Let T be an reductive and abelian algebraic group. Then $T \cong GL(1)^n$. That is what we call a torus in this section.
For a torus T, we define the character lattice $X^*(T) = \Lambda_T^\vee$ as the set of homomorphisms (as abelian groups) $T \to GL(1)$. Obviously we have

$$\Lambda_T^\vee \cong \mathrm{Hom}(GL(1)^n, GL(1)) \cong \mathrm{Hom}(GL(1), GL(1))^n \cong \mathbb{Z}^n \tag{4}$$

Similarly we define the cocharacter lattice /1-parameter subgroups $X_*(T) = \Lambda_T^\vee$ of T as $Hom(GL(1), T) \cong \mathbb{Z}^n$. From a reductive algebraic group over a separably closed field K we can construct its root datum $(X^*, \Delta, X_*, \Delta^\vee)$ where X^* is the lattice of characters of a maximal torus, X_* the dual lattice, Δ the roots, and $\Delta^\vee$ the coroots. The *dual root datum* is defined as $(X_*, \Delta^\vee, X^*, \Delta)$, i.e. switching the characters with the 1-parameter subgroups and switching the roots with the coroots.
If G is a connected reductive algebraic group over an algebraically closed field K, then its *Langlands dual group* ${}^LG$ is the unique connected reductive group over K whose root datum is dual to that of G. More abstractly we can define the Langlands dual group for a torus in terms of algebraic geometry. One can show that $T = \mathrm{Spec}\,\mathbb{C}[\Lambda_T^\vee] = \Lambda_T \otimes \mathbb{G}_m$. Where $\mathbb{G}_m$ denotes the complex multiplicative group scheme, which is just $GL(1)$. (The first equality means, that the $\mathbb{C}$-points of that scheme are isomorphic to the Torus). We define the Langlands dual torus as

$$T^\vee := \mathrm{Spec}\,\mathbb{C}[\Lambda_T] = \Lambda_T^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m \tag{5}$$

The philosophy is the following: For every simple Lie algebra $\mathfrak{g}$, there is a unique simple Lie group G whose Lie algebra is $\mathfrak{g}$ and which has trivial center.

This is the information we get from the roots. However, for the construction of the Langlands dual group we want to preserve the center of the group in an appropriate way. The information about the center is contained about the center is contained in the character lattice. That means, for centerless groups, the dual group just depends on the root system, for abelian group only on the character lattice. For general algebraic groups, we need both ingredients. We list some examples for the Langlands dual group below:

G	${}^L G$
$U(N)$	$U(N)$
$SU(N)$	$PSU(N) = SU(N)/\mathbb{Z}_N$
$Spin(2n)$	$SO(2n)/\mathbb{Z}_2$
$Spin(2n+1)$	$Sp(n)/\mathbb{Z}_2$
$SO(2n+1)$	$Sp(n)$
G_2	G_2
E_8	E_8

2 Some notes on Supersymmetry

This section is by no means a comprehensive treatment of supersymmetry and might even be useless for most readers. We assume however that most readers are already familiar with this topic anyway. The reason, this section is included here, is that I reviewed these topics when writing this report. For that reason I also did choose not to put it in an appendix, but here since I wanted to order the material in a way that represents how in my understanding, the topics built up on each other.

2.1 Charge Conjugation

Take $k = \mathbb{R}$ or $\mathbb{C}$.

Definition: *Let $\mathbb{S}$ a k-representation of Spin(p,q). We define a* ***charge conjugation*** *on $\mathbb{S}$ to be a Spin(p,q)-invariant non-degenerate bilinear form $C : \mathbb{S} \otimes \mathbb{S} \longrightarrow k$.*

Definition: *Let Cl(V) be a Clifford algebra, W Clifford k-module. We can define two Clifford structures on the dual space:*

$$W^\vee : (x \cdot \phi)(w) = \phi(x^t \cdot w), \quad w \in W, x \in Cl(V). \tag{6}$$

$$W^{\bar{\vee}} : (x \cdot \phi)(w) = \phi(\hat{x}^t \cdot w), \quad w \in W, x \in Cl(V) \tag{7}$$

where $\hat{(\cdot)}$ *denotes the anti-isomorphism induced by the isomorphism of* V $v \mapsto -v$. *A charge conjugation* C *on* W *is a Clifford module isomorphism between* W *and* $W^{\vee}$ *or* $W^{\bar{\vee}}$.

For $\mathbb{S}_{2m}$ a complex spin representation there exist

$$C_+ : \mathbb{S}_{2m} \longrightarrow \mathbb{S}_{2m}^{\vee}, \quad C_- : \mathbb{S}_{2m} \longrightarrow \mathbb{S}_{2m}^{\bar{\vee}}. \tag{8}$$

Let $V_{\mathbb{C}} = \mathbb{C}^{2m}$. We can extend the charge conjugation to a pairing

$$\Gamma^k_{\pm} : \mathbb{S}_{2m} \otimes \mathbb{S}_{2m} \longrightarrow \bigwedge{}^k V_{\mathbb{C}}^{\vee}. \tag{9}$$

given by

$$\Gamma^k_{\pm}(s_1, s_2)(\alpha) = C_{\pm}(s_1, \alpha \cdot s_2) \tag{10}$$

where $\alpha \cdot s_2$ is the Clifford action. For $n = 2m + 1$, there will be only one charge conjugation. We have C_+ for m even and C_- for m odd. Just denote both of them by C.
When we define $V_{\mathbb{C}} = \mathbb{C}^{2m+1}$ we get again a pairing

$$\Gamma^k : \mathbb{S}_{2m+1} \otimes \mathbb{S}_{2m+1} \longrightarrow \bigwedge{}^k V_{\mathbb{C}}^{\vee}. \tag{11}$$

2.2 Super Poincaré algebra

Let S be a representation of $\mathrm{Spin}(p, q)$ together with a $\mathrm{Spin}(p, q)$-equivariant pairing

$$\Gamma : S \otimes S \longrightarrow V \tag{12}$$

This defines a super Lie algebra structure on $V \bigoplus S$ via

$$[v_1 \oplus s_1, v_2 \oplus s_2] = -\Gamma(s_1, s_2) \tag{13}$$

The corresponding super Lie group is just $V \times \Pi S$ with group law

$$(v_1, s_1) \cdot (v_2, s_2) = (v_1 + v_2 - [s_1, s_2], s_1 + s_2). \tag{14}$$

It carries a natural $\mathrm{Spin}(p, q)$-action. Hence we can define the **Super Poincaré algebra** (also called susy algebra) by

$$Poin_S(V) = (V \times \Pi S) \rtimes \mathrm{Spin}(p, q) \tag{15}$$

The susy algebra can be extended in various ways. Consider the decomposition

$$Sym^2(S) \cong V \oplus \mathbb{R} \otimes \bigoplus_i \wedge_i^p V. \tag{16}$$

The first component gives our Poincaré algebra. The second component corresponds to central extensions. Their values on a susy representation are called central charges. The third component does not correspond to a central extension. They play the role of central charges when coupled with D-branes. Outer automorphisms of the super Poincaré algebra that commutes with the Poincaré algebra that commutes with the Poincaré subalgebra are called R-symmetries.

3 Connections on principal fiber bundles

Let P be a smooth manifold and G a Lie group acting on P, then we have the canonical surjection $\rho : P \longrightarrow P/G$ onto the sets of orbits . Let M be another smooth manifold. We call $\xi : P \longrightarrow M$ a principal fiber bundle with structure group G if it is locally trivial and there is a free, smooth right G-action on P, s.t

1. ξ and ρ are isomorphic, i.e we have a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "\rho"'] \arrow[dr, "\xi"] & \\ P/G \arrow[rr, "\cong"] & & M \end{tikzcd}$$

2. the fiber is diffeomorphic to G

3. the local trivializations respect the group action on P

Let $\pi : P \longrightarrow M$ be a principle fiber bundle. We define the vertical subspace at a point p of P to be $V_p = \{X \in T_pP | \pi_* X = 0\}$.
A *connection* is a smooth, fiberwise linear section σ of π_*, such that $\sigma_{pg} = g_* \sigma_p$. Equivalently, one can define a connection to be a choice of a subspace H_p of T_p such that $V_p = V_p \oplus H_p$ and $H_p g = g_* H_p$.
Next, we want to define the connection form. Consider the maps $\theta_p : G \to P_p$ sending g to pg and $\psi_p : P_p \to G$ the unique function such that $\theta_p(\psi_p(p')) =$

p'. This maps induce linear isomorphisms between V_p and $\mathfrak{g}$. We now define $\omega_p(u) = \psi_{p^*}(u)$ for $u \in V_p$ and $\omega_p(u) = 0$ if $u \in H_p$. The connection form satisfies

$$(g^*\omega)_p(u) = \omega_{pg}(g_* u) + Ad(g^{-1})(\omega_p(u)) \tag{17}$$

Conversely, if we have a Lie algebra valued form on the total space, i.e. $\omega \in \Gamma(P, T^*P \otimes \mathfrak{g}) = \Omega(P, \mathfrak{g})$, satisfying this property, then we can define a unique connection by $H_p = \ker\, \omega_p$.

4 Electric-magnetic duality

In this section we want to examine the electro-magnetic duality in Maxwell theory. We start with reviewing the Dirac quantization condition. Consider the Abelian gauge theory with field strength

$$F = dt \wedge E + \star(dt \wedge B) \tag{18}$$

where E and B are 1-forms. The Hodge star operator in 4-dim Minkowski space satisfies $\star^2 = -1$. The classical equations of motion for the Maxwell theory are

$$\begin{aligned} dF &= J_m \\ d \star F &= J_e \end{aligned}$$

J_e and J_m are the electric and the magnetic currents respectively. The currents J_e and J_m are closed 3-forms and hence by Poincaré duality we get 1-cycles corresponding to these currents. We can interpret them as paths in our Minkowski space and denote them by γ_e and γ_m. With that we can write our currents as

$$J_e = q_e \delta(\gamma_e), \quad J_m = q_m \delta(\gamma_m). \tag{19}$$

Here q_e is the electric charge and q_m is the magnetic charge. Clearly we have a symmetry of the theory given by

$$F \leftrightarrow \star F, \quad J_e \leftrightarrow -J_m \tag{20}$$

We have a look at the example of a static electron. Static means that the x_i coordinates are fixed and the electron is only moving along the time direction. Hence it produces a current that is supported on this path. From this we

see, that the δ in 19 can really be seen as a delta function.
Geometrically, the Maxwell theory away from γ_m describes a $U(1)$-bundle with connection 1-form such that $F = dA$ is given as the curvature of the connection. In the case that $J_m = 0$ the action functional is of the form

$$-\frac{1}{4}\int F \wedge \star F + 2\pi q_e \int_{\gamma_e} A. \qquad (21)$$

The expression $2\pi q_e \int_{\gamma_e}$ is the physics notation for holonomy of A along the curve γ_e.
Let us now assume that we have a monopole at the origin o of $\mathbb{R}^3$. And a electric current moving away from the origin. Then he theory is described by a $U(1)$-connection $2\pi q_e A$ on $\mathbb{R}^4 - \{\gamma_m\} = \mathbb{R}_t \times (\mathbb{R}^3 - o) \simeq S^2$. The irreducible representation of $U(1)$ on $\mathbb{C}$ gives an associated line bundle E on S^2. It's a fact that

$$2\pi \int_{S^2} q_e F = 2\pi c_1(E). \qquad (22)$$

Here $c_1(E)$ denotes the first Chern class of the line bundle E which is an integral cohomology class. Hence we find that

$$\int_{S^2} q_e F \in \mathbb{Z} \qquad (23)$$

On the other hand, from the Maxwell equations we get

$$\int_{S^2} F = \int_B dF = \int_B q_m \delta(\gamma_m) = q_m. \qquad (24)$$

All together we find the Dirac quantization condition

$$q_e q_m \in \mathbb{Z} \qquad (25)$$

So far we only looked at the classical theory. Next we want to understand the duality in the quantum theory. We can add a purely topological term to the action. The full action reads as

$$S = -\frac{1}{4g^2}\int F \wedge \star F + \frac{theta}{32\pi^2}\int F \wedge F. \qquad (26)$$

The θ-term is represents the second Chern class of an appropriate bundle and thus is integral. Hence We introduce the complex coupling constant

$$\tau = \frac{\theta}{32\pi^2} + \frac{4\pi i}{G^2}. \qquad (27)$$

Then we can write

$$-\frac{1}{32\pi}\,\mathrm{Im}\int \tau(F+i\tilde{F})\wedge(F+i\tilde{F}), \tag{28}$$

where we used $\tilde{F}=\star F$. We have the path integral

$$Z=\int \mathcal{D}A\; e^{iS} \tag{29}$$

Since the θ-term is integral, there is a discrete symmetry of the system given by

$$T:\tau\to\tau+1 \tag{30}$$

The constraint $dF=0$ can be implemented by introducing a Lagrangian multiplier A_D and modify the action to become

$$\begin{aligned}
&-\frac{1}{32\pi}\,\mathrm{Im}\int \tau(F+i\tilde{F})\wedge(F+i\tilde{F})+\frac{1}{8\pi}\int F_D\wedge F\\
=&-\frac{1}{32\pi}\,\mathrm{Im}\int \tau(F+i\tilde{F}))\wedge(F+i\tilde{F})+\frac{1}{16\pi}\,\mathrm{Im}\int i(F_D+i\tilde{F}_D)\wedge(F+i\tilde{F})
\end{aligned}$$

By completing the square and integrating out F, we find the dual action.

$$S_D=-\frac{1}{32\pi}\,\mathrm{Im}\int \frac{-1}{\tau}(F_D+i\tilde{F}_D)\wedge\star(F_D+i\tilde{F}_D) \tag{31}$$

Note that in the partition function there appears an integration over A_D from the introduction of the Lagrangian multiplier. Altogether we find

$$\int \mathcal{D}A\; e^{iS}=\int \mathcal{D}A_D\; e^{iS_D} \tag{32}$$

In particular, S and S_D describe equivalent physics, and we find the following duality S-transformation

$$S:\tau\to\frac{-1}{\tau} \tag{33}$$

In the case $\theta=0$, this gives $g\to\frac{4\pi}{g}$ and hence gives a duality between a strongly coupled system and a weakly coupled system. The transformations T and S generate the group $PSL(2,\mathbb{Z})$ which acts on τ by

$$\tau\to\frac{a\tau+b}{c\tau+d}. \tag{34}$$

The electric and magnetic charge (q_e, q_m) transform under the $SL(2,\mathbb{Z})$ transformation as

$$\begin{pmatrix} q_e & q_m \end{pmatrix} \to \begin{pmatrix} q_e & q_m \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \tag{35}$$

With the 2-form

$$G = -8\pi \frac{\delta S}{\delta F} = \frac{4\pi}{g^2} \star F - \frac{\theta}{2\pi} F \tag{36}$$

we can write the action as

$$-\frac{1}{16\pi} \int F \wedge G. \tag{37}$$

In that case the S-transformation acts as

$$S : \begin{pmatrix} F & G \end{pmatrix} \to \begin{pmatrix} F_D & G_D \end{pmatrix} \begin{pmatrix} F & G \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \tag{38}$$

and the T-transformation as

$$T : \begin{pmatrix} F & G \end{pmatrix} \to \begin{pmatrix} F & G \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \tag{39}$$

So the a general $SL(2,\mathbb{Z})$-transformation γ acts as

$$\begin{pmatrix} F & G \end{pmatrix} \to \begin{pmatrix} F & G \end{pmatrix} \gamma^{-1} \tag{40}$$

The derivation of the electric-magnetic duality above is valid only when X is topologically trivial. For the case of general 4-manifolds we refer to [14].

4.1 Montonen-Olive Duality

We can generalize the above analysis of the electro-magnetic duality to general gauge theories with gauge-groups other the $U(1)$. We introduce the gauge connection $A = A_\mu dx^\mu$. This is a connection 1-form on a principal G-bundle E over a four-dimensional manifold X. The curvature of the connection is given by $F_A = dA + A \wedge A$. The Yang-Mills in Euclidean signature is given by

$$S_E(A) = \int_X \left(\frac{1}{2e^2} Tr\, F \wedge \star F - i \frac{\theta}{8\pi^2} Tr\, F \wedge F \right) \tag{41}$$

On the Quantum level one is mainly interested in correlators given by

$$\langle \mathcal{O}_1 ... \mathcal{O}_k \rangle = \sum_E \int \mathcal{D}A e^{-S_E(A)} \mathcal{O}_1(A)\mathcal{O}_2(A)...\mathcal{O}_k(A) \tag{42}$$

where the sum is again taken over isomorphism class of principal G-bundles and $\mathcal{O}_i$ are gauge-invariant functions of A called observables.
One family of observables that play an important role in the Langlands duality are the Wilson loops associated to a closed curve γ in X. They are defined by

$$\mathcal{O}(A) = W_R(\gamma) = Tr\, R(Hol(A, \gamma)), \tag{43}$$

where $Hol(A, \gamma)$ is the holonomy of γ with respect to the connection A and R is an irreducible representation of G. Since the holonomy group is always a subgroup of the structure group, the expression is well defined.
From a physical point of view, inserting $W_R(\gamma)$ into the path-integral corresponds to inserting an electrically charged source ("quark") in the representation R whose worldline is γ. In the case of $G = U(1)$, we saw, that the theory enjoy a symmetry

$$F \mapsto \hat{F} = -ie^2 \star F, \quad \tau \mapsto \hat{\tau} = -\tau^{-1} \tag{44}$$

We now try to establish an electric-magnetic duality in the non-abelian case. A static magnetic source in Yang-Mills theory should create a field of the form

$$\Phi = \star_3 d \left(\frac{\mu}{2r} \right). \tag{45}$$

Here μ is an element of the Lie algebra $\mathfrak{g}$ of G defined up to adjoint action of G and $\star_3$ is the Hodge star operator in $\mathbb{R}^3$. The value of μ is not arbitrary, but quantized, i.e. using gauge freedom, we can assume that μ lies in a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and then it turns out, that μ lies in the root lattice of G, which is by definition the same as the weight lattice of ${}^L G$. Furthermore, μ is defined up to an action of the Weyl group, so possible values of μ are in one-to-one correspondence with highest weights of ${}^L G$. Based on this observation C. Montonen and D. Olive conjectured that Yang-Mills theories with gauge groups G and ${}^L G$ are isomorphic on the quantum level [10] . We will refer to this duality as Montonen-Olive duality or sometimes also as S-duality or electric-magnetic duality. For further reading recommended references are [2] and [4].

5 Towards the Kapustin-Witten Theory

This section is the main part of this notes. The aim is to give an idea how the geometric Langlands correspondence arise naturally in the context of $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions by applying Montonen-Olive duality. Some side topics are included such as stable vector bundles and Higgs bundles, a short discussion of abelian varieties and the Fourier-Mukai transform. The purpose is that the Hitchin-Moduli spaces of stable Higgs bundles arising naturally as the target space of a topological sigma model derived from the $\mathcal{N} = 4$ super Yang-Mills theory are related by mirror Symmetry. As discussed in the lecture, SYZ mirror symmetry acts on varieties with toric fibrations as a Fourier-Mukai transform on the fibers. We know that every compact group manifold is a torus. An analog statement holds for abelian varieties. Thus if we have a fibration by abelian varieties, this is in particular a toric fibration, which gives more tools to work with. For instance, one can explicitly construct the dual $A^\vee$ of an abelian variety A and proof that one has a equivalence of categories

$$D(A) \simeq D(A^\vee) \tag{46}$$

of the derived category of coherent sheaves. We start with discussing $\mathcal{N} = 1$ Super Yang-Mills Theory and use this theory to obtain the $\mathcal{N} = 4$ theory by dimensional reduction.

The main reference for the Kapustin-Witten theory are [9] and [8]. For SYZ mirror symmetry the main reference is the original paper [13]. For abelian variety I highly recommend [5]. A more standard reference is [11]. For the Fourier-Mukai good references are [7] and [12].

5.1 N=1 Super Yang-Mills Theory

Let $V = \mathbb{R}^{1,3}$ be flat Minkowski space.

$$P \longrightarrow V \tag{47}$$

a principal G-bundle for G a semi-simple Lie group. We have a charge conjugation pairing $\langle\ ,\ \rangle$, the Killing form on the Lie algebra $\mathfrak{g}$ and the Minkowski metric on V. With the Killing form and the Minkowski metric we can define an invariant product on $\Omega^*(V, \mathfrak{g})$. Together with the charge conjugation we get a non-degenerate bilinear form on $\Omega^*(V, S_+ \otimes \mathfrak{g})$. The field space for N=1 super Yang-Mills theory is given by the super space:

$$\mathcal{E} = \Omega^1(V, \mathfrak{g}) \oplus \Omega^0(V, S_+ \otimes \mathfrak{g}) \tag{48}$$

where the decomposition represents the grading of the super space. $A \in \Omega^1(V, \mathfrak{g})$ represents a connection 1-form and $\psi \in \Omega^0(V, S_+ \otimes \mathfrak{g})$ a fermion. S_+ acts on $\mathcal{E}$ as follows. For $\epsilon \in S_+$ we get

$$\Omega^0(V, S_+ \otimes \mathfrak{g}) \longrightarrow A \in \Omega^1(V, \mathfrak{g}), \quad \epsilon : \phi \mapsto \Gamma(\epsilon, \psi) \tag{49}$$

and

$$A \in \Omega^1(V, \mathfrak{g}) \longrightarrow \Omega^0(V, S_+ \otimes \mathfrak{g}), \quad \epsilon : A \mapsto \frac{1}{2} F_A \cdot \epsilon \tag{50}$$

Here $F_A = dA = \frac{1}{2}[A, A]$ is the curvature 2-form. Note that denoting the map ϵ is a slight abuse of notation so we use δ_ϵ to denote the action. The action maps $\Omega^0(V, S_+ \otimes \mathfrak{g})$ to $\Omega^1(V, \mathfrak{g})$ and vice versa. We define the Lagrangian density on $\mathcal{E}$ by

$$\mathcal{L}[A, \psi] = \frac{1}{4}\langle F_A, F_A \rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi \rangle \tag{51}$$

Where $\not{D}_A = \not{D} + A$ and $\not{D} = \gamma^\mu \partial_\mu$ is the Dirac operator. The super Yang-Mills functional is defined by

$$S^{N=1}_{SYM}[A, \psi] = \int_V \mathcal{L}[A, \psi]. \tag{52}$$

The functional is invariant under the induced S_+-action. Next we introduce the gauge transformation

$$\delta^G_\lambda A = d_A \lambda, \quad \delta^G_A \psi = [\lambda, \psi], \quad \lambda \in \Omega^0(V, \mathfrak{g}) \tag{53}$$

For and ϵ_1, ϵ_2 it holds that

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = -\mathcal{L}_\lambda A - \delta^G_{\iota_\lambda A} A, \quad [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi = -\mathcal{L}_\lambda \psi - \delta^G_{\iota_\lambda A} + \mathcal{O}(\not{D}_A \psi) \tag{54}$$

Then we find that the solution space of equation of motions of $S^{N=1}_{SYM}$ modulo gauge $Crit(S^{N=1}_{SYM})/Gauge$ carries a representation of the $N = 1$ super symmetry algebra.

5.2 N=4 Super Yang-Mills Theory

To obtain $\mathcal{N}=4$ super Yang-Mills in $d=4$ space time dimensions, we start from $\mathcal{N}=1$ super Yang-Mills in $d=10$ and dimensionally reduce to $d=4$. The Lagrangian in ten dimensions take a very easy form and furthermore is essentially unique up to choice of some coupling constants. As a reminder, the action for $d=10$ $\mathcal{N}=1$ super Yang-Mills is as follows:

$$\mathcal{L}[A.\psi] = \frac{1}{4}\langle F_A, F_A\rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi\rangle \tag{55}$$

Here $A \in \Omega^1(V, \mathfrak{g})$ is the connection form, which is a even, while $\psi \in \Omega^0(V, S_+ \otimes \mathfrak{g})$ is odd and represents a chiral fermion. Chiral just means that it transforms under the representation S_+ of $\mathrm{Spin}(9,1)$. This group is isomorphic to $SL(2,\mathbb{O})$. Note that $SL(2,\mathbb{O})$ should be regarded as the subgroup of $SL(16,\mathbb{R})$ consisting of the elements that preserve the algebra structure on $\mathbb{O}$, in particular, since $SL(16,\mathbb{R})$ is associative, so is $SL(2,\mathbb{O})$, even the octonions the $\mathbb{O}$ themselves are non-associative. Then we can identify S_+ with $\mathbb{O}^2$. So far $V=\mathbb{R}^{9,1}$. We consider the dimensional reduction what means that we write $\mathbb{R}^{9,1} = \mathbb{R}^{3,1}\times\mathbb{R}^6$ and replace the fields A and ψ to vary only along $\mathbb{R}^{3,1}$, but to be constant along the $\mathbb{R}^6$ direction. The connection 1-form in $d=10$ decomposes into a connection form $A_m u$ in $d=4$ and six scalars ϕ_i on which the R-symmetry acts by the adjoint representation. The spin group is reduced to $\mathrm{Spin}(3,1)\times$ $\mathrm{Spin}(6)$. $\mathrm{Spin}(3,1)$ plays the role of the Spin group in the theory in $d=4$ dimensions, while $\mathrm{Spin}(6) \cong SU(4)_R$ acts as an R-symmetry. Under this reduction, the chiral spinor S_+ is decomposed into

$$S_+ \otimes_{\mathbb{R}} \mathbb{C} = (S\otimes \bar{\mathbf{4}}) \oplus (\bar{S}\otimes \mathbf{4}) \tag{56}$$

Her S and $\bar{S}$ are the Weyl spinor and its complex conjugate, $\bar{\mathbf{4}}$ is the defining representation of $SU(4)$, and $\mathbf{4}$ its complex conjugate. Therefore the chiral fermion is reduced to four Weyl fermions λ^a_α, $\alpha=1,2$, $a=1,...,4$ where the R-symmetry acts on the a index by the defining representation. The Yang-Mills action becomes

$$\int d^4x \mathrm{Tr}\left(\frac{1}{4}\langle F,F\rangle + \frac{1}{2}\sum_{i=1}^{6} D_\mu\phi_i D^\mu\phi_i + \frac{1}{4}\sum_{i,j=1}^{6}[\phi_i,\phi_j]^2\right) \tag{57}$$

The fermionic part becomes

$$\int d^4\frac{1}{2}\langle\bar{\lambda}^a, \not{D}_A\lambda^a\rangle + \sum_{a=1}^{4}\langle\lambda^a,[\psi,\lambda^a]\rangle + \sum_{a=1}^{4}\langle\bar{\lambda}^a,[\psi,\bar{\lambda^a}]\rangle \tag{58}$$

Where $[\phi, \lambda^a]$ involves a bracket in $\mathfrak{g}$ as well as a Clifford multiplication in the $SU(4)_R$ factor. The susy relations are now reduced to

$$\left[Q^a_\alpha, \bar{Q}^b_{\dot{\beta}}\right] = -2\delta^{ab}\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu, \; [Q, Q] = [\bar{Q}, \bar{Q}] = 0 \tag{59}$$

Where the Pauli matrices are given by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{60}$$

5.3 Geometric Langlands twist

The idea of twisting is, that want to change to Poincaré symmetry by a global symmetry in order to formulate the theory on a general curved manifold whole preserving some of the global supercharges Q. The procedure of twisting is as follows:

(1) Consider a theory on V with symmetry $Spin(V) \times G_R$, where G_R is the R-symmetry group. Next we choose a morphism

$$\rho : \mathrm{Spin}(V) \longrightarrow G_R \tag{61}$$

(2) Find an odd operator Q (this is one reason we need susy) such that $[Q, Q] = 0$ (note that this bracket for odd elements denotes the anti-commutator rather than the commutator.

In this case, we can redefine the action of $\mathrm{Spin}(V)$ on the super Poincaré algebra to act via $id \times \rho$ where id denotes the original action of $\mathrm{Spin}(V)$ by $id \times \rho$. We want to consider observable that are Q closed. Moreover, Q-exact observables will drop out naturally in this sector. Since Q is invariant under the new action of $Spin(V)$, it is a scalar of the theory. For that reason it is often possible to formulate the theory on a nontrivial manifold M, while Q survives as a globally defined fermionic symmetry.

(3) If we have that $T_{\mu\nu}$ is Q-exact, then the twisted theory is independent of the metric.

We are interested in geometric applications of twisted theories. For that lets consider $N = 4$ super Yang-Mills theory in $d = 4$ *Euclidean* space with global symmetry

$$\mathrm{Spin}(4) \times \mathrm{Spin}(6) \cong SU(2)_l \times SU(2)_r \times SU(4)_R. \tag{62}$$

We denote by $\mathbf{d}$ the dimension of the representation of $SU(N)$ and by $\bar{\mathbf{d}}$ it's complex conjugate. The supercharges $Q^a_\alpha, \bar{Q}_{\dot{\alpha},a}$ lie in $(\mathbf{2},\mathbf{1},\bar{\mathbf{4}}) \oplus (\mathbf{1},\mathbf{2},\mathbf{4})$. A for a topological twist, we must specify a morphism $\rho : SU(2)_l \times SU(2)_r \to SU(4)$. Therefore we need to understand how $\mathbf{4}_R$ is decomposed into a representation of $SU(2)_l \times SU(2)_r$. There are three non-equivalent twists of the theory:

(1) $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{1})$. This is the Vafa-Witten twist with two scalar supercharges.

(2) $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})$. This is the "half-twisted theory" which posses only one scalar supercharge.

(3) $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$. This is the GL twist with two scalar supercharges. Kapustin-Witten have shown that this twist is related to the geometric Langlands program.

For the GL twist, we have $(A, B) \mapsto A \oplus B$ for $A, B \in SU(2)$. This is the obvious embedding map. This embedding is compatible with the following embedding of $U(1)$ into $SU(4)$: $e^{i\theta} \mapsto diag((e^{i\theta}, e^{i\theta}), (e^{-i\theta}, e^{-i\theta}))$.
We write the decomposition of $\mathbf{4}$ as $(\mathbf{2},\mathbf{1})^1 \oplus (\mathbf{1},\mathbf{2})^{-1}$. Where the superscript specifies the represents the $U(1)$-charge. Let us work out the field contents in the twist:

(1) The supercharges $Q \in (\mathbf{2},\mathbf{1},\bar{\mathbf{4}})$ transform in the new $Spin(4) \times U(1)$ as

$$(\mathbf{2},\mathbf{1})^0 \otimes \left((\mathbf{2},\mathbf{1})^{-1} \oplus (\mathbf{1},\mathbf{2})^1\right) = (\mathbf{1},\mathbf{1})^{-1} \oplus (\mathbf{3},\mathbf{1})^{-1} \oplus (\mathbf{2},\mathbf{2})^1 \tag{63}$$

Hence we find a scalar supercharge $Q_l \in (\mathbf{1},\mathbf{1})^{-1}$. Similarly we find $Q_r \in (\mathbf{1},\mathbf{1})^{-1}$.

(2) A_μ is central for the Spin(6) action, hence it remains as a connection 1-form in the twisted theory.

(3) ϕ_i lies in $\mathbf{6}$ (the adjoint representation of $SU(4)_R$. Since

$$\mathbf{6} = \bigwedge^2 \mathbf{4} \to \bigwedge^2 \left((\mathbf{2},\mathbf{1})^1 \oplus (\mathbf{1},\mathbf{2})^{-1}\right) = (\mathbf{2},\mathbf{2})^0 \oplus (\mathbf{1},\mathbf{1})^2 \oplus (\mathbf{1},\mathbf{1})^{-2} \tag{64}$$

we see that four components of ϕ_i becomes a 1-form ψ_μ, while two other scalars combine into a complex scalar σ with $U(1)$ charge 2 and its complex conjugate $\bar{\sigma}$.

(4) λ behave the same as supercharges. It decomposes into $(\psi_\mu, \tilde{\psi}_\mu, \chi_{\mu\nu}, \eta, \tilde{\eta})$ where $\psi, \tilde{\psi} \in (\mathbf{2}, \mathbf{2})^1$ are 1-forms, $\chi^+ \in (\mathbf{3}, \mathbf{1})^{-1}$ and $\chi^- \in (\mathbf{1}, \mathbf{3})^{-1}$ are self-dual and anti-selfdual parts of a two-form χ and $\eta \in \chi^+ \in (\mathbf{1}, \mathbf{1})^{-1}$ and $\tilde{\eta}(\mathbf{1}, \mathbf{1})^{-1}$ are two zero-forms.

Since Q_l and Q_r both lie in $(\mathbf{1}, \mathbf{1})^{-1}$ we have a $\mathbb{P}^1$-family of topological twisted theories with

$$Q = uQ_l + vQ_r, \quad [u : v] \in \mathbb{P}^1. \tag{65}$$

We can write the action of the Q_t-twisted theory on a arbitrary manifold M in the form

$$S = Q_t V + \frac{i\Psi}{4\ pi} \int_M \mathrm{Tr} F \wedge F \tag{66}$$

for some local action V. We set $t = v/u$, $Q_t = Q_l + tQ_r$ and

$$\Psi = \frac{\tau + \bar{\tau}}{2} + \frac{\tau - \bar{\tau}}{2} \left(\frac{t - t^{-1}}{t + t^{-1}} \right) \tag{67}$$

Where we again set

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi}{e^2} \tag{68}$$

This expression shows that S is $Q_t-closed$ and the the variation of the metric is Q_t-exact. Moreover, the topological theory only depends on the coefficient Ψ, which we call *canonical parameter*. The $SL(2, \mathbb{Z})$ duality transformation acts on Ψ via

$$\Psi \mapsto \frac{a\Psi + b}{c\Psi + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{69}$$

In the Q_t-twisted theory, the path integral will usually be localized to subspace where the Q_t-variation of fermions vanishes. This becomes the equations

$$\begin{aligned} (F - \phi \wedge \phi + tD\psi)^+ &= 0 \\ \left(F - \phi \wedge \phi + t^{-1} D\psi\right)^- &= 0 \\ D^*\phi + t^{-1}[\bar{\sigma}, \sigma] &= 0 \\ D^*\psi - t[\bar{\sigma}, \sigma] &= 0 \\ D\sigma + t[\phi, \sigma] &= 0 \\ D\sigma - t^{-1}[\phi, \sigma] &= 0 \end{aligned}$$

We will call this the *topological equations* for the twist t. For real t it is analogous to a family of a two-dimensional A-model, while for $t = \pm i$ it is analogous to a two-dimensional B-model.

5.4 Higgs bundle and Hitchin Moduli

Let C be a compact Riemann surface, G a compact Lie group and $P \to C$ a principal G-bundle. We consider pairs (A, ϕ) where A is a unitary connection on P and $\phi \in \Omega^{1,0}(C, \mathfrak{g}_{\mathbb{C}})$. Here $\mathfrak{g}$ denotes the adjoint bundle ad P. We are interested in pairs satisfying the *Hitchin equation*

$$F_A + [\phi, \phi^*] = 0, \quad \bar{\partial}_A \phi = 0. \tag{70}$$

As always F_A denotes the curvature of the connection A. $\bar{\partial}_A$ is the $(0,1)$-part of the covariant derivative with respect to A. We will denote the moduli space of solutions of Hitchin's equation modulo gauge transformations.

Next we want to study how there arises a hyperkähler structure on this moduli space. For that we first have to look at the construction of *symplectic reduction*. Let (M, ω) be a symplectic manifold. Let G be a connected Lie group with Hamiltonian action on M.

Remark: Any $\xi \in \mathfrak{g}$ induces a vector field $\rho(\xi)$ on M describing the infinitesimal action of ξ. More precisely:

$$\rho(\xi)_x = \frac{d}{dt}\Big|_{t=0} exp(t\xi) \cdot x \tag{71}$$

The G-action is said to be Hamiltonian if and only if the following condition holds. First, for every $\xi \in \mathfrak{g}$ the one-form $\iota_{\rho(\xi)}\omega$ if exact, meaning that it equals dH_ξ for for some smooth function $H_\xi : M \to \mathbb{R}$. If this holds, then one may choose the H_ξ to make $\xi \mapsto H_\xi$ linear. The second requirement for the G-action to be Hamiltonian is that the map $\xi \mapsto H_\xi$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the algebra of smooth functions on M under the Poisson bracket.

The *Poisson bracket* $\{\cdot, \cdot\}$ on (M, ω) is a bilinear operation on differentiable functions, defined by $\{f, g\} = \omega(X_f, X_g)$ where X_f, X_g are Hamiltonian vector fields, i.e. if we set $\Omega_{df} = \omega^{-1}(df)$, we set X_f to be Ω_{dH}. The we have $\{f, g\} = \omega(X_f, X_g) = \omega(\Omega_{df}, X_g) = (\iota_{\Omega_{df}}\omega)(X_g) = df(X_g) = X_g f = \mathcal{L}_{X_g} f$.

If we now have a Hamiltonian action of G on (M, ω) is Hamiltonian in this sense, then a momentum map is a map $\mu : M \to \mathfrak{g}^*$ such that writing $H_\xi = \langle \mu, \xi \rangle$ defines a Lie algebra homomorphism $\xi \mapsto H_\xi$ satisfying $\rho(\xi) = X_{H_\xi}$.

No take $\mu : M \to \mathfrak{g}^*$ be a moment map for the Hamiltonian action above. The classical Marsden-Weinstein theorem says that

$$M//G := \mu^{-1}(0)/G \tag{72}$$

is again a symplectic manifold, whose symplectic form is induced by restricting ω to $\mu^{-1}(0)$.
We briefly review the definition of a Kähler manifold.

Definition: *M is a Kähler manifold if it is endowed with the triple (g, J, ω) where g is a metric, ω is a symplectic form and J is an integrable complex structure (i.e. for all vector fields X, Y we have $-J^2[X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] = 0$), such that that the following compatibility condition holds*

$$\omega(\cdot,\cdot) = g(J\cdot,\cdot). \tag{73}$$

Let (M, g, J, ω) be a Kähler manifold with a Hamiltonian isometric action by a connected Lie group G. Then the symplectic reduction $M//G$ is a Kähler manifold with the naturally induced Kähler structure.

Definition: *A Riemannian manifold (M, g) is called hyperkähler if it is equipped with three complex structures I, J, K each of which defines a Kähler structure with g, and which satisfy the quaternion relations*

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K, JK = -KJ = I, KI = -IK = J. \tag{74}$$

Each complex structures I, J, K induces a symplectic form $\omega_I, \omega_J, \omega_K$ respectively.

Let M carry a G isometric action which is Hamiltonian with respect to all symplectic structures. Organize the three momentum maps into a single one

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{R} \tag{75}$$

Then the quotient $\mu^{-1}(0)/G$ is naturally a hyperkähler manifold. This construction is called hyperkähler quotient.
The next step is to understand the hyperkähler structure on the Hitchin moduli space. First we consider the infinite dimensional space $\mathcal{W} = \mathcal{A} \times \Omega$ where $\mathcal{A}$ denotes the space of unitary connections on P, and $\Omega = \Omega^1(C, \mathfrak{g}_\mathbb{C})$. The we get

$$T\mathcal{W} = T\mathcal{A} \oplus T\Omega = \Omega^1(C, \mathfrak{g}_\mathbb{C}) \oplus \Omega^1(C, \mathfrak{g}_\mathbb{C}) \tag{76}$$

On this space we find the following three complex structures

$$I(\delta A, \delta\phi) = (\delta A, - \star \delta\phi) \tag{77}$$
$$J(\delta A, \delta A) = (-\delta\phi, \delta A) \tag{78}$$
$$K(\delta A, \delta\phi) = (- \star \delta\phi, -\delta A). \tag{79}$$

Where $(\delta A, \delta\phi)$ is a infinitesimal variation on $\mathcal{W}$ and $\star$ is the Hodge star on 1-forms. This complex structures defines a hyperkähler structure with respect to the metric

$$ds^2 = \int_C Tr(\delta A \wedge \star\delta A + \delta\phi \wedge \star\delta\phi). \tag{80}$$

The three corresponding symplectic forms are

$$\omega_I = \frac{1}{2}\int_C Tr\delta A \wedge \star\delta A + \delta\phi \wedge \delta\phi) \tag{81}$$
$$\omega_J = \int_C Tr(-\delta A \wedge \star\delta\phi) \tag{82}$$
$$\omega_K = \int_C Tr(\delta A \wedge \delta\phi) \tag{83}$$
$$. \tag{84}$$

We consider the gauge transformation $\mathcal{G}$ sending (A, ϕ) to $(gAG^{-1}+gdg^{-1}, g\phi g^{-1})$ whose infinitesimal form is

$$\delta(A, \phi) = (D_A u, [\phi, u]), \quad u \in \Omega^0(C, g). \tag{85}$$

One can check, that this action is Hamiltonian with respect to all three symplectic forms. The moment maps are then

$$\mu_I = F_A - \frac{1}{2}[\phi, \phi] \tag{86}$$
$$\mu_J = D_A \star \phi \tag{87}$$
$$\mu_K = -D_A\phi \tag{88}$$

The vanishing locus is given by

$$F_A - \frac{1}{2}[\phi, \phi] = 0. \quad D_A\phi = D_A^*\phi = 0. \tag{89}$$

These are just Hitchin's equations. Thus the Hitchin moduli space $\mathcal{M}_H(C, G)$ inherits a hyperkähler structure. We can describe this moduli in terms of holomorphic data.

(1) We choose the complex structure I and consider the holomorphic symplectic form $\Omega_I = \omega_J + i\omega_K$. The holomorphic moment map is

$$\mu_J + i\mu_K = \bar{\partial}_A \varphi. \tag{90}$$

Here $\varphi \in \Omega^{1,0}(C, \mathfrak{g}_{\mathbb{C}})$ such that $\phi = \varphi + \varphi^\dagger$. The vanishing of the holomorphic moment map says that $\bar{\partial}_A \varphi = 0$. Hence φ determines an element in $H^1(C, \mathfrak{g}_{\mathbb{C}})$. By Serre-duality this gives rise to a holomorphic section $H^0(C, K_C \otimes \mathfrak{g}_{\mathbb{C}})$, which is called a *Higgs field*. The pair $(\bar{\partial}_A, \phi)$ of a holomorphic bundle with a Higgs field is also called a *Higgs bundle*. The other equation $F_A - [\varphi, \varphi^\dagger] = 0$ is the stability condition. From this perspective, the Hitchin moduli space describes the stable Higgs bundles.

We want the be more precise about this. Consider $G = SU(2)$. Let R be the principal G-bundle on C. We can interpret it as a rank two holomorphic vector bundle over C and φ as a holomorphic map $\phi : C \longrightarrow \mathfrak{g}_{\mathbb{C}} \otimes K_C$. A line bundle $\mathcal{L} \subset E$ is called ϕ-invariant if $\phi(\mathcal{L}) \subset \mathcal{L} \otimes K_C$. A Hitchin pair (P, ϕ) is called stable if every ϕ-invariant line bundle $\mathcal{L} \subset P$ has negative first Chern class. It is called semistable if each such $\mathcal{L}$ has non-positive first Chern class. For general G, one must consider ϕ-invariant reductions of the structure group of P to a maximal parabolic subgroup P of $G_{\mathbb{C}}$. The pair (P, ϕ) is called stable (or semistable) if for every such reduction the first Chern class is negative (or nonpositive). A pair that is semistable but non stable is called strictly semistable. The stable pairs (P, ϕ) correspond to smooth points of the Hitchin moduli space, whereas the strictly semistable pairs generally lead to singularities.

(2) We choose the complex structure J and consider the holomorphic symplectic form $\Omega_J = \omega_K + i\omega_I$. The holomorphic moment map is

$$\mu_K + i\mu_I = i\mathcal{F} \tag{91}$$

where $\mathcal{F}$ is the curvature of the complex connection $A + i\phi$. The vanishing of the holomorphic moment map says that $A + i\phi$ defines a flat connection. As we have seen above, that is equivalent to the datum of a local system and also to a group homomorphism

$$\pi_1(C) \longrightarrow G_{\mathbb{C}} \tag{92}$$

The vanishing of μ_J is again about stability condition.

(3) For a general complex structure

$$I_\xi = \frac{1-\bar{\xi}\xi}{1+\bar{\xi}\xi} I + \frac{\xi - i(\bar{\xi})}{1+\bar{\xi}\xi} J + \frac{\xi+\bar{\xi}}{1+\bar{\xi}\xi} K \tag{93}$$

For $\xi \neq 0, \xi$, an analog holomorphic moment map gives the vanishing

$$\left[\bar{\partial}_A + \xi\varphi^\dagger, \partial_A - \xi^{-1}\varphi\right] = 0. \tag{94}$$

They are equivalent to the case when $I_\xi = J$.

Next we want to discuss the *Hitchin fibration*. If E is a stable G-bundle, the the pair $(E, 0)$ is a stable Hitchin pair, where 0 indicates that we choose ϕ to be the zero-section of the vector bundle E. Thus there is a natural embedding of the moduli space of stable G-bundles on C into the Hitchin moduli space $\mathcal{M}_H(C, G)$. The image is a holomorphic submanifold in the complex structure I. In the complex structures J and K, $\mathcal{M}$ is not holomorphic. Instead, it is Lagrangian, since $\phi = 0$ implied $\delta\phi = 0$ and hence $\mu_J = \mu_K = 0$. First consider the case of $G = SU(2)$. In complex structure I, we can consider gauge invariant polynomials in φ. This simply means that we consider the quadratic Casimir operator $\omega = Tr\,\varphi^2$ where the trace is taken over the adjoint representation. We have $\bar{\partial}_A\varphi = 0$ and thus $\bar{\partial}\omega = 0$. Hence ω defines an element in $\mathcal{B} = H^0(C, K_C^2) \cong \mathbb{C}^{3g-3}$. In this case the Hitchin fibration is the map $\pi : \mathcal{M}_H(C, G) \to \mathcal{B}$ which sends (P, φ) to $\omega = Tr\,\phi^2$.
For general G, the Hitchin fibration is defined similarly, except, that one characterizes φ by all of its independent Casimirs. For example, assume that $G = SU(N)$. We define the Hitchin base to be

$$\mathcal{B} = \bigoplus_{n=2}^{N} H^0(C, K_C^n). \tag{95}$$

When we define $\omega_n = Tr\,\varphi^n$, $n = 2, ..., N$, the Hitchin fibration is then the map

$$\pi : \mathcal{M}_H(C, G) \longrightarrow \mathcal{B}, \quad (P, \phi) \mapsto (\omega_2, \omega_3, ..., \omega_n) \tag{96}$$

The dimension of $\mathcal{B}$ in general is $(g-1)\dim(G)$ which equals the dimension of $\mathcal{M}$ and one half of the dimension of $\mathcal{M}_H(C, G)$. A hence, the dimension of the typical fiber F is $(g-1)\dim(G)$ as well. Let us give an heuristic argument, why the Hitchin fibration is surjective. Take $G = SU(2)$. Pick a stable $SU(2)$ bundle P. Consider the equations $Tr\,\varphi^2 = \omega$, where φ varies in the $(3g-3)$-dimensional space $H^0(C, K_C \otimes ad(E))$ and ω is a fixed element in $\mathcal{B} = H^0(C, K_C)$. This is a system of $(3g-3)$ quadratic equations for $(3g-3)$ complex variables. The number of solutions is generically 2^{3g-3}. A similar counting can be made for other G. The fibration π gives a complete integrable system for the holomorphic form $\omega_J + \omega_K$ on $\mathcal{M}_H$.

5.5 Stable vector bundles

Let C be a smooth complex projective curve of genus $g > 1$. We fix $n > 0$ and $d \in \mathbb{Z}$. Assume $gcd(d, n) = 1$. We call a vector bundle E *semistable* if for every subbundle F

$$\mu(F) = \frac{deg(F)}{rk(F)} \leq \mu(E) = \frac{deg(E)}{rk(E)} \tag{97}$$

is satisfied. $\mu(E)$ is called the slope of the vector bundle. The vector bundle is called *stable* if the inequality above is strict.

In physics one thinks of vector bundles as being *Chan-Paton gauge fields* on *D-branes*. Then the rank of the vector bundle is proportional to the mass density of a family of coincident D-branes, while the degree, being the Chern-class, is a measure for the *RR-charge* carried by the D-branes. Thus, we can interpret *slope of a vector bundle* to be the *charge density* of the corresponding D-brane configuration.
A D-brane state is stable, if it is a *BPS-state*. This is a generalization of the concept of a charged black hole with maximal charge. Hence stable D-branes are those which maximize their charge density, hence the slope of their Chan-Paton vector bundles. Hence, the condition that every sub-bundle has smaller slope means that smaller branes can increase their charge density by forming bound states into the larger, stable object. Hence slope-stability of vector bundles and coherent sheaves is the BPS stability condition on charged D-branes.

The assumption $gcd(d, n) = 1$ ensures, that the moduli space of rank n semistable vector bundles of degree d on C is smooth projective, which we shell denote $\mathcal{N}^d$. For $n = 1$ this moduli space is just the degree d component of the Picard variety $Pic^d(C)$ associated to C, since line bundles are trivially stable. To a vector bundle of any rank we can associate the determinant line bundle. This gives a morphism

$$det : \mathcal{N}^d \longrightarrow Pic^d(C) \tag{98}$$

We choose $\Lambda \in \mathrm{Pic}^d(C)$ and define $\check{\mathcal{N}}^\Lambda := det^{-1}(\Lambda)$. We can think of points in $\check{\mathcal{N}}^\Lambda := det^{-1}(\Lambda)$ as "twisted SL_n-bundles". An SL_n bundle $E \in \mathcal{N}^0$ is a bundle with unit determinant, i.e. $det(E) = \mathcal{O}_C$. $Pic(C)$ act det-equivariantly on the moduli space of all stable vector bundles and on $Pic(C)$. Such that we can obtain $\check{\mathcal{N}}^\Lambda$ by twisting $\check{\mathcal{N}}^{\mathcal{O}} \subset \mathcal{N}^0$ with an line bundle in $Pic^d(C)$. In particular, $\check{\mathcal{N}}^\Lambda$ does not depend (up to isomorphism) on the choice of $\Lambda \in Jac^d(C)$. For that reason, by abuse of notation we write $\check{\mathcal{N}}^d$ for $\check{\mathcal{N}}^\Lambda$. The action of the abelian variety $Pic^0(C)$ on
$mathcalN^d$ is given by tensoring. We define

$$\mathcal{N}^d := \mathcal{N}^d / Pic^0(C) \tag{99}$$

If we take $\Gamma := Pic^0(C)[n]$ the n-torsion points of the Jacobian, then one can show that $\hat{\mathcal{N}} := \check{\mathcal{N}}^\Lambda/\Gamma$. Hence $\mathcal{N}^d$ is a projective orbifold. It holds that $H^*(\check{\mathcal{N}}^d) = H^*(\hat{\mathcal{N}}^d)$.

5.6 The Hitchin system

Fix some n and some d as above and write $\mathcal{N}$ for $\mathcal{N}^d$. The cotangent bundle $\mathrm{T}^*\mathcal{N}^d$ has the structure of an algebraic symplectic variety. The Ring of regular functions $\mathcal{O}_{\mathrm{T}^*\mathcal{N}}(\mathrm{T}^*\mathcal{N})$ is finitely generated $\mathbb{C}-algebra$. Then we can consider the affinization map

$$\chi : \mathrm{T}^*\mathcal{N} \longrightarrow \mathrm{Spec}(\Gamma(\mathrm{T}^*\mathcal{N}, \mathcal{O}_{\mathrm{T}^*\mathcal{N}})) \tag{100}$$

This map is the so called Hitchin map. We want to describe this map more explicitly. For a point $E \in \mathcal{N}$ we find that $T_E\mathcal{N} = H^1(C, \mathrm{End}(E))$. By Serre duality we find that

$$\mathrm{T}^*_E\mathcal{N} = H^0(C, \mathrm{End}(E) \otimes K_C) \tag{101}$$

This is the space of Higgs fields, that will appear again below. For any $(E, \varphi) \in \mathrm{T}^*\mathcal{N}$ we can consider the characteristic polynomial of the Higgs field which is of the form $t^n + a_1 t^{n-1} + ... + a_n$ where $a_i \in H^0(K^i)$. For example $a_n \in H^0(K^n)$ is the determinant of the Higgs field. The Hitchin map has then the explicit description

$$\chi : \mathrm{T}^*\mathcal{N} \longrightarrow \mathcal{A} := \bigoplus_{i=1}^{n} H^0(K^i) \tag{102}$$

The affine space $\mathcal{A}$ is called the *Hitchin base*. If we consider $hat\mathcal{N}^\Lambda$ the bundles have fixed determinants. Heuristically, since the trace is the derivative of the determinant the endomorphisms in the fiber must be traceless, i.e. they lie in the Lie algebra $\mathfrak{sl}_n$. It holds that $T^*_E\hat{\mathcal{N}}^\Lambda = H^0(\mathrm{End}_0(E) \otimes K)$, hence $\hat{\mathcal{A}} = \mathcal{B} = \bigoplus_{i=2}^{n} H^0(K^i)$. The following is a theorem by Htichin

Theorem: *If $\psi_i, \psi_j \in \Gamma(\mathrm{T}^*\mathcal{N}, \mathcal{O}_{\mathrm{T}^*\mathcal{N}})$, then they Poisson commute, i.e. $\{\psi_i, \psi_j\} = 0$. Then we have $dim\mathcal{A} = \dim\mathcal{N}$ and the generic fibers of the ψ are open subsets of abelian varieties. Therefore we have an algebraically completely integrable Hamiltonian system.*

Next we introduce the notion of a *Higgs bundle*.

Definition: *A Higgs bundle is a pair (E, φ) where E is a vector bundle and $\varphi \in H^0(C, End(E) \otimes K)$ is a Higgs field.*

The notion of semistability and stability for Higgs-bundles is similar as for vector bundles except we only consider φ-invariant subbundles. The moduli space of semi-stable Higgs bundles is denoted by $\mathcal{M}^d$, it is a non-singular, quasi-projective variety. The (semi-)stable bundles include into $\mathcal{M}$ as $E \mapsto (E, 0)$, furthermore, if E is stable, so is ever pair (E, φ) for φ a Higgs field. Hence $\mathrm{T}^*\mathcal{N}$ is a subvariety of $\mathcal{M}^d$. The Hitchin map extends to a map $\chi : \mathcal{M}^d \to \mathcal{A}$.

Theorem: *χ is a proper algebraically complete integrable Hamiltonian system. Its generic fibers are abelian varieties.*

That is also true for the moduli spaces $\hat{\mathcal{M}}^d$ and $\check{\mathcal{M}}^d$ of Higgs bundles with fixed determinants and traceless Higgs fields (possibly moding out a discrete group Γ.) These Moduli spaces carry a hyperkähler structure as we have seen in the physical discussion. The Hitchin maps is can also be extends to maps $\hat{\chi}$ and $\check{\chi}$ respectively.

5.7 Abelian Varieties and T-Duality

We consider an algebraically closed field $k = \bar{k}$.

Definition: *An abelian variety A is a group scheme $/k$, such that A is an algebraic variety $/k$ that is complete (i.e. the structure morphism $A \to Spec(k)$ is proper).*
That is A is an integral, separated group scheme of finite type over $/k$, such that $A \to Spec(k)$ is proper.

Abelian varieties are always smooth and abelian as a group scheme.

Example:

- An elliptic curve is an abelian variety.
- If $\mathbb{Z}^{2g} \cong \Lambda \subset \mathbb{C}^g$ is a lattice, then the complex torus $\mathbb{C}^g/\Lambda$ is a complex manifold with the structure of a group. If it happens to be the $\mathbb{C}$-points of a variety, than that variety is abelian.
- If C/k is a curve, the Jacobian $\mathrm{Jac}(C)$ is a projective abelian variety$/k$

Actually all abelian varieties are projective.

A *Appell-Humbert datum* is a pair (H, α) where H is a Hermitian form on V whose imaginary part E is integral and α is a map from U to the unit circle such that $\alpha(u+v) = e^{i\pi E(u,v)}\alpha(u)\alpha(v)$.For every H there exist exactly 2^2g such maps α ($g = dim_{\mathbb{C}}$)). The Set $\mathcal{P}(\Lambda)$ of all Appell-Humbert data together with the addition $(H_1, \alpha_1) + (H_2, \alpha_2) = (H_1 + H_2, \alpha_1\alpha_2)$ form an abelian group.
To every (H, α), and very $\lambda \in \Lambda$, we get a holomorphic function $e_\lambda(v)$ on V given by

$$e_\lambda(v) = \alpha(\lambda)e^{\pi H(\lambda,v)+\frac{\pi}{2}H(\lambda,\lambda)} \tag{103}$$

We can define a action of the Group Λ on the trivial line bundle $V \times \mathbb{C} \to V$ via addition: For $\lambda \in \Lambda$, $(v, z) + \lambda := (v + \lambda, e_\lambda(v)z)$. Then $L(H, \alpha) = V \times \mathbb{C}/\Lambda \to V/\Lambda = T$ is a holomorphic line bundle over T. Furthermore it is true that $L(H_1 + H_2, \alpha_1\alpha_2) \cong L(H_1, \alpha_1) \otimes L(H_2, \alpha_2)$.
The Theorem of Appell-Humbert states that isomorphy classes of holomorphic line bundles on $T = V/\Lambda$ correspond bijectively to $\mathcal{P}(\Lambda)$.
Let $\mathrm{Pic}(X)$ be the Picard group, i.e. the group of isomorphy classes of line

bundles on X and $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ be the kernel of the Chern class. We have an exact sequence:

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0. \tag{104}$$

NS is the so called Neron-Severi group and is defined as the image of $c_1 : \mathrm{H}^1(X, \mathcal{O}_X^\times) \to \mathrm{H}^2(X, \mathbb{Z})$, which is precisely the group of Hermitian forms H on V having imaginary part which takes integral values on $\Lambda \times \Lambda$. The group Pic^0 coincides with the group of unitary characters on $\Lambda, \mathrm{Hom}(\Lambda, U(1))$.

Remark: A line bundle $\mathcal{L}$ over a scheme $X_{/S}$ is called *very ample* if there is an embedding $i : X \hookrightarrow \mathbb{P}^n_S$ for some n, such that the pullback of the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n_s$ is isomorphic to L, i.e. $i^*(\mathcal{O}(1)) \cong L$. A line bundle $\mathcal{L}$ is called ample, if $\mathcal{L}^{\otimes n}$ is ample for $n \gg 0$.
It is a non-trivial fact, that abelian varieties are projective, in particular we can find a ample line bundle on them. A *polarized abelian variety* is a pair of an abelian variety and an ample line bundle.

We now discuss the theorem of the cube:

Theorem: *Let X,Y be a complete varieties, Z any variety, and x_0,y_0 and z_0 k-points. Any line bundle L on $X \times Y \times Z$ whose restriction to each $X \times Y \times z_0$, $X \times y_0 \times Z$, $x_0 \times Y \times Z$ is trivial is itself trivial.*

We call the points x_0, y_0, z_0 base points. Over $\mathbb{C}$ we can use the exponential sequence to prove this result. We assume that the cohomology of all X, Y, Z respectively has no torsion. Then the Künneth theorem tells us that the natural map

$$H^*(X, \mathbb{Z}) \otimes H^*(Y, \mathbb{Z}) \otimes H^*(Z, \mathbb{Z}) \longrightarrow H^*(X \times Y \times Z, \mathbb{Z}) \tag{105}$$

is an isomorphism. We define the inclusions ι_i and ι_{ij} including the i and j factors using the basepoints. For instance

$$\iota_1 : X \times y_0 \times z_0 \to X \times Y \times Z, \quad \iota_{12} : X \times Y \times z_0 \to X \times Y \times Z \tag{106}$$

The isomorphism 105 in degree 2 says that for any class $\alpha \in H^2(X \times Y \times Z, \mathbb{Z})$, $\alpha = \alpha_{12} + \alpha_{13} + \alpha_{23} - \alpha_1 - \alpha_2 - \alpha_3$ where $\alpha_1 2 = \iota_{12}^* \alpha \otimes 1$, $\alpha_1 = \iota_1^* \alpha \otimes 1 \otimes 1$ and the other terms similar. In particular this means if $\iota_{ij}^* \alpha = 0$ for all i,j, then $\alpha = 0$. The exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \tag{107}$$

induces a long exact sequence on cohomology

$$H^1(X \times Y \times Z, \mathcal{O}) \xrightarrow{exp} H^1(X \times Y \times Z, \mathcal{O}^*) \xrightarrow{c_1} H^2(X \times Y \times Z, \mathbb{Z}) \quad (108)$$

The middle group is just the Picard group. Given a line bundle L on $X \times Y \times Z$ this corresponds to an element in the middle group. By assumption $\iota_{ij}^* c_1(L) = c_1(\iota_{ij}^* L) = 0$ for all i, j, so $C_1(L) = 0$. By exactness we have that $L = exp(A)$ for some $A \in H^1(X \times Y \times Z, \mathcal{O})$. The Künneth isomorphism tells us

$$H^1(X, \mathcal{O}) \oplus H^1(y, \mathcal{O}) \oplus H^1(Z, \mathcal{O}) \xrightarrow{\cong} H^1(X \times Y \times Z, \mathcal{O}) \quad (109)$$

is an isomorphism, and the hypotheses imply that $\iota_i^* exp(A) = exp(\iota_i^* A) = 0$ for each i. Thus, $L = exp\left(\sum_i \iota_i^* A\right) = 0$. Thus L is the trivial line bundle. The proof is only valid over $\mathbb{C}$ but the theorem holds in general. An immediate corollary is that

$$L \cong p_{12}^* L \otimes p_{13}^* L \otimes p_{23}^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \otimes p_3^* L^{-1} \quad (110)$$

The following theorem also follows from the theorem of the cube and is called the theorem of the square:
For any k-point $x \in X(k)$, there is a translation map $t_x : X \to X$ given by $t_x(y) = y + x$.

Theorem: *For any line bundle L on an abelian variety X and any two k-points, $x, y \in X(k)$,*

$$t_{x+y}^* L \otimes L \cong t_x^* L \otimes t_y^* L \quad (111)$$

The theorem in fact tells us, that for any line bundle L, the map

$$\varphi_L : X \to \mathrm{Pic}(X), \quad x \mapsto t_x^* L \otimes L^{-1} \quad (112)$$

is a homomorphism of abelian varieties.

5.7.1 The Poincaré bundle

Let A be an abelian variety over an algebraically closed field k of dimension g and let $L \in \mathrm{Pic}(A)$ be ample. The *Mumford line bundle* is defined as

$$\Lambda(L) = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \in \mathrm{Pic}(A \times A) \quad (113)$$

where p_i, $i = 1, 2$ denote the obvious projection maps. We define

$$K(L)(k) = \{x \in A(k) \mid \Lambda(L)|_{A\times\{x\}} \text{ is trivial}\}. \tag{114}$$

We can view this as the k-rational points of a scheme $K(L)$. This scheme is a subgroup scheme of A. The quotient scheme $A/K(L)$ exists and is an abelian variety/k with the same dimension as A. This quotient is the *dual abelian variety* $A^\vee$ of A. By construction $A^\vee(k) = \mathrm{Pic}^0(X)$ and the quotient morphism $A(k) \to A^\vee(k)$ is φ_L.
The dual abelian variety $A^\vee$ has the following universal property:
There exist a uniquely determined line bundle $\mathcal{P}$ on $X \times X^\vee$, called the *Poincaré bundle*, such that

(1) $\mathcal{P}|_{X\times\{y\}} \in Pic^0(X \times \{y\})$ for all $y \in X^\vee$,

1. (2) $\mathcal{P}|_{\{0\}\times X^\vee}$ is trivial,

and if Z is a scheme with a line bundle $\mathcal{R}$ on $X \times Z$ satisfying (1) and (2), then there is a unique morphism $f : Z \to X^\vee$ such that $(id \times f)^*\mathcal{P} = \mathcal{R}$.

5.7.2 The Fourier-Mukai Transform

Consider two smooth algebraic varieties X and Y (for instance over $\mathbb{C}$). We can consider the three categories $D_{coh}(X), D_{coh}(Y)$ and $D_{coh}(X \times Y)$ the derived categories of coherent sheaves on X, Y and $X \times Y$ respectively. We get the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. $(\pi_1)^*$ is right exact and $(\pi_2)_*$ left exact, hence they give induce derived functors on between the derived categories which we by abuse of notation denote by the same symbol. Let E be an object in $D(X \times Y)$ the we can define a functor:

$$\Phi^E : D_{coh}(X) \to D_{coh}(Y), \quad F \mapsto \Phi^E(F) = (\pi_2)_* \left((\pi_1)^* F \otimes E\right). \tag{115}$$

Here $\otimes$ denotes the derived tensor product. The functors of this find that give rise to an equivalence of categories are called Fourier-Mukai transform. In order to get a better feeling about this definition, let us see the heuristic analogy to the classical Fourier transform. Recall that for a Schwartz function $f : X \to \mathbb{R}$, the Fourier transform $g : Y \to \mathbb{R}$ is $g(y) := \int f(x)e^{2\pi ixy}dx$. We want to think of $F \in D_{coh}(X)$ as being analog to the function f on X. In the same way $E \in D_{coh}(X \times Y)$ corresponds to some function on $X \times Y \to \mathbb{R}$, $(x, y) \mapsto e^{2\pi ixy}$. The pullback of f along π_1 to $X \times Y$ is defined by the formula $((\pi_1)^*f)(x,y) = f(x)$. The tensor product in analogous to product of functions, hence here want to think of $(\pi_1)^*F \otimes E$ as analogous to $((\pi_1)^*f)(x,y)e^{2\pi ixy}$. The pushforward from functions on $X \times Y$ to Y is the given by integration along the fiber. Hence $(\pi_2)_* ((\pi_1)^*F \otimes E)$ corresponds to $g(y) = \int f(x)e^{2\pi ixy}dx$ the Fourier transform of f. Requiring that Φ^E is an equivalence of categories means in the picture of the classical Fourier transform, that the Fourier transform is an isomorphism between suitable spaces of functions.

Example: The derived direct image RF_* of a morphism $f : X \to Y$ is naturally isomorphic to $\Phi^{\mathcal{O}_{\Gamma_f}}$, the Fourier-Mukai transform with the kernel the structure sheaf of the graph $\Gamma_f \subset X \times Y$ of f. We have

$$\begin{aligned}\Phi^{\mathcal{O}_{\Gamma_f}}(F) &= (\pi_2)_* \left((\pi_1)^*F \otimes \mathcal{O}_{\Gamma_f}\right)\\ &= (\pi_2)_* ((\pi_1)^*F \otimes R\Gamma_{f*}(\mathcal{O}_X))\\ &= (\pi_2)_* R\Gamma_{f*} \left(\mathcal{O}_X \otimes L\Gamma_f^* \pi_1^*(F)\right)\\ &= Rf_*(F)\end{aligned}$$

Next, let $\mathcal{E} \in D(X)$. The functor

$$\mathcal{E} \otimes (-) : D(X) \longrightarrow D(X) \tag{116}$$

is of Fourier-Mukai type. Its kernel is $\Delta_*\mathcal{E}$ where $\Delta : X \longrightarrow X \times X$ denotes the diagonal embedding morphism. Explicitly we have:

$$\Phi_{\Delta_*\mathcal{E}}(-) = \pi_{2*}(\pi_1^*(-) \otimes \Delta_*\mathcal{E}) = \pi_{2*}\Delta_*\mathcal{E} \otimes (-) = \mathcal{E} \otimes (-) \tag{117}$$

The second equality is true due to the projection formula, since $\pi_1 = \pi_2$. In particular, the Serre functor, which is the exact equivalence $S_X(-) = (-) \otimes \omega_X[\dim X]$ where ω_X is the canonical line bundle of X, is of Fourier-Mukai type.

Let X, Y and Z be varieties. Let $K \in D(X \times Y)$ and $L \in D(Y \times Z)$. Then we have an isomorphism of functors

$$\Phi_L \circ \Phi_K \simeq \Phi_{K*L} : D(X) \to D(Z) \tag{118}$$

where we define the *convolution* of the kernels K and L.

$$K * L = \pi_{13*}(\pi_{23}^* L \otimes \pi_{12}^* K) \tag{119}$$

The π_{ij} are the natural projection maps, for instance $\pi_{12} : X \times Y \times Z \to X \times Y$.
Remark: An object $K \in D(X \times Y)$ defines a family of functors

$$\Phi_K : D(X \times S) \longrightarrow D(Y \times S) \tag{120}$$

where S is any scheme. We have natural projection morphisms $\pi_{s1} : X \times Y \times S \to X \times S$ and $\pi_{s2} : X \times Y \times S \to Y \times S$. The

$$\Phi_K(\mathcal{F}) := \pi_{s2*}(\pi_{s1}^*(\mathcal{F}) \otimes K). \tag{121}$$

We still have a natural transformation

$$\Phi_L \circ \Phi_K \simeq \Phi_{K*L} : D(X \times S) \to D(Z \times S). \tag{122}$$

Naturality here means that they commute with derived pullback functors associated to morphisms $g : S \to S'$.

5.7.3 Equivalence of categories $D(A) \simeq D(A^\vee)$

Let A be an abelian variety of dimension g, $A^\vee$ its dual abelian variety and $\mathcal{P}$ the Poincaré bundle over $A \times A^\vee$. First we want to prove
Lemma: For every scheme S, the functor

$$\Phi_{\mathcal{P}}^{-1}[g] : A \times S \to A^\vee \times S \tag{123}$$

is left adjoint to

$$\Phi_{\mathcal{P}} : A \times S \to A^\vee \times S. \tag{124}$$

Proof. Let $\mathcal{F} \in D(A^\vee \times S)$ and $\mathcal{G} \in D(A \times S)$. We have

$$\begin{aligned}
\mathrm{Hom}_{D(A^\vee \times S)}(\mathcal{F}, \Phi_{\mathcal{P}}\mathcal{G}) &= \mathrm{Hom}_{D(A^\vee \times S)}(\mathcal{F}, \pi_{s2*}(\pi_{s1}^*(\mathcal{G}) \otimes \mathcal{P})) \\
&\simeq \mathrm{Hom}_{D(A \times A^\vee \times S)}(\pi_{s2}^*\mathcal{F}, \pi_{s1}^*(\mathcal{G}) \otimes \mathcal{P}) \\
&\simeq \mathrm{Hom}_{D(A \times A^\vee \times S)}(\pi_{s2}^*\mathcal{F} \otimes \mathcal{P}^{-1}, \pi_{s1}^*(\mathcal{G})) \\
&\overset{\text{S.D.}}{\simeq} \mathrm{Hom}_{D(A \times A^\vee \times S)}(\pi_{s1}^*(\mathcal{G}), \pi_{s2}^*\mathcal{F} \otimes \mathcal{P}^{-1}[2g]))^\vee \\
&\simeq \mathrm{Hom}_{D(A \times S)}(\mathcal{G}, \pi_{s1*}(\pi_{s2}^*\mathcal{F} \otimes \mathcal{P}^-1[2g]))^\vee \\
&\overset{\text{S.D}}{\simeq} \mathrm{Hom}_{D(A \times S)}(\pi_{s1*}(\pi_{s2}^*\mathcal{F} \otimes \mathcal{P}^{-1}[2g]), \mathcal{G}[g]) \\
&\simeq \mathrm{Hom}_{D(A \times S)}(\pi_{s1*}(\pi_{s2}^*\mathcal{F} \otimes \mathcal{P}^{-1}[g]), \mathcal{G})
\end{aligned}$$

With S.D. we marked that we used Serre duality.
Next we want to compute the cohomology of the Poincaré bundle.

Proposition: Let A, $A^\vee$ and $\mathcal{P}$ be as before. Write $\pi_2 : A \times A^\vee \longrightarrow A^\vee$ for the second projection. Then we have $R^n p_{2*}\mathcal{P} = k(e_{A^\vee})$ if $n = g$ and 0 otherwise. Here $k(e_{A^\vee})$ denotes the skyscraper sheaf at $e_{A^\vee}$. Furthermore it holds that $H^n(A \times A^\vee, \mathcal{P}) = k$ if $n = g$ and 0 otherwise.

Proof. We divide the proof in three steps:

(1) For all n, we have $\mathrm{supp}(R^n\pi_{2*}(\mathcal{P})) \subset e_{A^\vee}$.

Let $\xi \neq e_{A^\vee} \in A^\vee$. Then $\mathcal{P}|_{A\times\xi}$ is a non trivial line bundle on A with class in Pic^0. We know that such sheaves have zero cohomology. By [3][Ch.III,Thm. 12.11] we get the canonical isomorphism

$$R^n\pi_{2*}\mathcal{P} \otimes k(\xi) \simeq H^n(A \times \xi, \mathcal{P}|_{A\times\xi}) = 0. \tag{125}$$

for all n. The theorem also tells as that the $R^n\pi_{2*}\mathcal{P}$ are all locally free in a neighborhood of ξ. Hence, the equation above implies that the stalk at ξ is zero. This proves the claim.

(2) For all $n \neq g$ we have $R^n\pi_{2*}\mathcal{P} = 0$ and $H^n(A \times A^\vee, \mathcal{P}) = 0$.

The identity point $e_{A^\vee}$ can be viewed as a zero dimensional subscheme of $A^\vee$. By the *vanishing theorem of Grothendieck* [3][Ch.III,Thm. 2.7] we know that $H^i(A^\vee, R^n\pi_{2*}\mathcal{P}) = 0$ for all $i \geq 1$.

Remark:[The Leray spectral sequence] Let $f : X \to Y$ be a continuous map of topological spaces. Obviously we have that $\Gamma_Y \circ f_* = \Gamma_X$. Hence we find that

$$R^i(\Gamma \circ f_*)(\mathcal{F}) = H^i(X, \mathcal{F}) \tag{126}$$

Since f_* and Γ satisfy appropriate conditions there is a spectral sequence whose second page is

$$E_2^{pq} = (R^p\Gamma \circ R^qF_*)(\mathcal{F}) = H^p(Y, R^qf_*(\mathcal{F})) \tag{127}$$

This spectral sequence converges to

$$E_\infty^{pq} = R^{p+q}(\Gamma \circ f_*)(\mathcal{F}) = H^{p+q}(X, \mathcal{F}) \tag{128}$$

This is a special case of the *Grothendieck spectral sequence.*

In the current case, the Leray spectral sequence tells us that

$$E_2^{pq} = H^p(A^\vee, R^n p_{2*}\mathcal{P}) \Rightarrow H^{p+q}(A \times A^\vee, \mathcal{P}) \tag{129}$$

This implies that

$$H^n(A \times A^\vee, \mathcal{P}) \cong H^0(A^\vee, R^n p_{2*}\mathcal{P}). \tag{130}$$

Since the dimension of $A^\vee$ is g we know that $H^0(A^\vee, R^n p_{2*}\mathcal{P}) = 0$ for all $n > g$. By [3][Ch.III,Cor. 11.2] we get the stronger result that

$$R^n p_{2*}\mathcal{P} = 0 \;\; \forall n > g. \tag{131}$$

Hence

$$H^n(A \times A^\vee, \mathcal{P}) = 0 \;\; \forall n > g. \tag{132}$$

We can consider the the isomorphisms $(id_A, \iota_{A^\vee})$, $(\iota_A, id_{A^\vee}) : A \times A^\vee \to A \times A^\vee$. It is true that $\mathcal{P}^{-1} \cong (id_A, \iota_{A^\vee})^*\mathcal{P} \cong (\iota_A, id_{A^\vee})^*\mathcal{P}$. It follows that $\mathcal{P}$ and $\mathcal{P}^{-1}$ have the same cohomology. From this by Serre duality, we get

$$H^n(A \times A^\vee, \mathcal{P}) \cong H^{2g-n}(A \times A^\vee, \mathcal{P}^{-1})^\vee \cong H^{2g-n}(A \times A^\vee, \mathcal{P})^\vee \tag{133}$$

Therefore

$$H^n(A \times A^\vee, \mathcal{P}) = 0 \;\; \forall n < g. \tag{134}$$

Finally by 130 and $\mathrm{supp}(R^n p_{2*}\mathcal{P}) \supset e_{A^\vee}$ we conclude that

$$R^n p_{2*}\mathcal{P} = 0 \;\; \forall n < g. \tag{135}$$

(3) It is left to prove that $R^g p_{2*}\mathcal{P} = k(e_{A^\vee})$.

Since $\mathrm{supp}(R^n p_{2*}\mathcal{P}) \supset e_{A^\vee}$ this is a local question. Let $R = \mathcal{O}_{A^\vee, e_{A^\vee}}$ be the local ring at $e_{A^\vee}$. By an explicit calculation using a free resolution of $(R^g p_{2*}\mathcal{P})|_{\mathrm{Spec}(R)}$ one shows that $(R^g p_{2*}\mathcal{P})|_{\mathrm{Spec}(R)} \cong Ext'(k, R) \cong k$.

□

Theorem: *The natural adjunction morphism*

$$id_{D(A^\vee)} \longrightarrow \Phi_{\mathcal{P}} \circ \Phi_{\mathcal{P}^{-1}[g]} \tag{136}$$

is an isomorphism.

Proof. By the example above where we showed, that the derived tensor product is of Fourier-Mukai type, it suffices to show that $\mathcal{P}^{-1}[g] * \mathcal{P}$ is isomorphic to $\Delta_* \mathcal{O}_{A^\vee}$. All relevant morphisms are shown in the following diagrams:

$$
\begin{tikzcd}
A \times A^\vee \times A^\vee \arrow[r, "\pi_{13}"] \arrow[d, "\pi_{12}"] & A \times A^\vee \arrow[d, "\pi_1"] \\
A \times A^\vee \arrow[r, "\pi_2"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A^\vee \times A^\vee \arrow[r, "\pi_{23}"] \arrow[d, "(id \times d)"] & A^\vee \times A^\vee \arrow[d, "d"'] \\
A \times A^\vee \arrow[r, "\pi_2"] & A^\vee \arrow[u, bend right, "\Delta"']
\end{tikzcd}
$$

Here the morphism $d : A^\vee \times A^\vee \longrightarrow A^\vee$ is the difference morphism $(a, a') \mapsto a - a'$. We have

$$
\begin{aligned}
\mathcal{P}^{-1}[g] * \mathcal{P} &= \pi_{23*}(\pi_{13}^* \mathcal{P}^{-1}[g] \otimes \pi_{12}^* \mathcal{P}) & \\
&= \pi_{23*}(d \times id)^* \mathcal{P}[g] & \text{by theorem of the cube} \\
&= d * \pi_{2*} \mathcal{P}[g] & \text{by change of coordinates} \\
&= d^* \mathcal{O}_{e_{A^\vee}} & \text{by proposition above} \\
&= \Delta_* \mathcal{O}_{A^\vee}. &
\end{aligned}
$$

□

Remark: The isomorphism above also holds for families. All together we get

$$D(A^\vee \times S) \simeq D(A \times S). \tag{137}$$

In physics, this equivalence of categories is called *T-Duality*.

5.8 Low energy effective theory

The twisted theory is super symmetric on a generic manifold. In particular, we can formulate it on the product of two Riemann surfaces $M = \Sigma \times C$. We assume that the size of σ is much bigger than that of C. We look at the low energy effective theory of the twisted theory around configurations that minimize the topological action. Apart from the topological θ-term, the bosonic part of the action is minimized by

$$\mathcal{F} = 0, \; D^* \phi = 0, \; D\sigma = [\phi, \sigma] = [\sigma, \hat{\sigma}] = 0 \tag{138}$$

Here $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the curvature of the complex connection $\mathcal{A} = A + i\phi$. Under mild assumptions, the solution of these equation is obtained by taking

$\mathcal{A}$ to be the pullback from the Curve C with $\sigma = 0$. On C the equations are

$$F - \phi \wedge \phi = 0,\ D\phi = D^{*}\phi = 0. \tag{139}$$

These are just Hitchin's equations. We denote by $\mathcal{M}_H$ the moduli space of solutions of Hitchin's equations on C up to gauge transformations. Therefore the zero energy configurations are given by constant maps $\Sigma \to \mathcal{M}_H$. Assume that the size of C is small compared to Σ, then the almost zero energy effective theory on $M = \Sigma \times C$ is given by a supersymmetric σ-model

$$\Sigma \longrightarrow \mathcal{M}_H \tag{140}$$

For $\sigma = \mathbb{R}^{1,1}$, this σ-model has induced $N = (4,4)$ susy. So the target space $\mathcal{M}_H$ must be a hyper-Kähler manifold. In two dimensions the topological twisted theory looks like

(1) At $t = \pm i$. We get a B-model for the complex structure J on $\mathcal{M}_H$

(2) At $t = \bar{t}$. We get a A-model for the complex structure I_t on $\mathcal{M}_H$

As shown by Kapustin and Witten, S-duality maps G to its Langlands dual group. In particular, we get that under S-duality the B-model in complex structure K on $\mathcal{M}_H(C, G)$ is equivalent to the A-model on $\mathcal{M}_H(C,^L G)$ in complex structure K. These spaces are mirror pairs. In the notion above these duality is just SYZ mirror symmetry between $\check{\mathcal{M}}^d$ and $\hat{\mathcal{M}}^d$, where the duality acts as T-duality (i.e. Fourier-Mukai transformation) on the fibers. An important feature of mirror symmetry is that it maps boundary conditions in the A-model to boundary conditions in the B-model. We want to investigate that further in the next section.

5.9 Boundary conditions

We denote by $Y_G(C)$ the moduli stack of flat G-bundles on C and $Z_G(C)$ for the moduli stack of holomorphic G-bundles. Roughly, the geometric Langlands correspondence asserts that the category of coherent sheaves on $Y_{^L G}(C)$ is equivalent to the category of D-modules on $Z_G(C)$. Considering boundary conditions in the duality above will relate the topological twist of $N = 4$ SYM to the geometric Langlands program.

Remark: A *monoidal category* is a category C equipped with a bifunctor $\otimes : C \times C \to C$ that is associative up to natural isomorphism, and a object

I that is both a left and right identity for $\otimes$ also up to natural isomorphism. The natural isomorphisms have to obey some extra coherence conditions. First we look at the mathematical description of *topological quantum field theories*. This was introduced by Atiyah and Segal. In this formulation a TQFT in n dimensions gives rise to a functor of monoidal categories

$$\mathbf{F} : (Bord_n, \sqcup) \rightarrow (Vect, \otimes). \tag{141}$$

Here $Bord_n$ is the category of $n - dim$ bordisms, whose objects are oriented $(n-1)$-manifolds without boundary and morphisms are oriented bordisms. Furthermore $\mathbf{F}$ must satisfy

$$F(N_1 \sqcup N_2) = \mathbf{F}(N_1) \otimes \mathbf{F}(N_2) \text{ and } \mathbf{F}(\emptyset) = \mathbb{C}. \tag{142}$$

The functor $\mathbf{F}$ is related to the path integral as follows:

(1) Let $N = \partial M$. We denote by ϕ fields on M and by ϕ_∂ the fields on ∂M. We interpret $\mathbf{F}(N) = \{\phi_\partial || \phi_\partial|_{L^2} < \infty\}$ as a L^2 Hilbert space of boundary fields. $F(M)$ gives rise to a vector $|M\rangle$, which is determined by its inner product with an arbitrary element $\phi_\partial \in \mathbf{F}(N)$ by the path integral over ϕ with fixed boundary condition ϕ_∂

$$\langle \phi_\partial | M \rangle = \int_{\phi|\partial M = \phi_\partial} D\phi e^{iS[\phi]/\hbar} \tag{143}$$

(2) The manifold $\bar{N}$ is the same as the manifold N, then $\mathbf{F}(\bar{N}) = \mathbf{F}(N)^\vee$. In particular, when $\partial M = N_1 \sqcup \bar{N}_2$ we get

$$\mathbf{F}(M) : \mathbf{F}(N_1) \longrightarrow \mathbf{F}(N_2). \tag{144}$$

For $\partial M \emptyset$, that means that $\mathbf{F}(M) \in \mathbb{C} = Hom_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$. That number is called the *partition function* on M.

This definition can be extended to the definition of extended TQFTs. Then F assigns a k-category to a $(n-k-1)$-submanifold. This definition is by know means exact, but the following example might clarify it to some extend. Consider a bordism M between two manifolds with boundaries $(N_1, \partial N_1), (N_2, \partial N_2)$. A TQFT functor will map $F(N_i)$ to an object of the category of vector spaces. Now we extend the functor by assigning to each ∂N_i a category $\mathbf{F}(\partial N_i), \mathbf{F}(N_i) \in Obj(\mathbf{F}(N_i))$ and $\mathbf{F(M)} \in Mor(\mathbf{F}(N_1), \mathbf{F}(N_2)))$.

To formulate the definition of n-categories in this context is a subtle question, so we keep the naive definition we established above.
In 2d extended TQFT the boundary conditions are called $D-branes$. They form a $\mathbb{C}$-linear category. In the topological A-model, the D-branes are Lagrangian submanifolds, in the topological B-model, the D-branes are coherent sheaves.

There is a second structure we have to consider called *defects*. Consider a TQFT on M and $L \subset M$ be a k-dim submanifold. A defect on L is a local modification of states supported on L. 0-dim defects are local operators, 1-dim defects are line operators, 2-dim defects are surface operators and etc. The point is that k-dim defects of a TQFT form a k-category. The reason is that we can view k-dimensional defects on L as a boundary condition for an effective $(k+1)$-TQFT.

5.10 More on Wilson and 't Hooft operators

We already introduced the Wilson loop operator defined by

$$Tr_R \, P\exp \int_\gamma A = Tr \, R(\mathrm{Hol}(A, \gamma)) \tag{145}$$

As before, R denotes a finite dimensional representation of the gauge group G and γ a closed loop in M. The expression $P\exp \int$ is the physical notation for holonomy. The Wilson loop is a gauge-invariant function of the connection A and therefore can be regarded as a physical observable. Inserting the Wilson loop into the path-integral is equivalent to inserting an infinitely massive particle traveling along γ and having $color - electric$ degrees of freedom described by the representation R of G. One examples arises for $G = SU(3)$. There the Wilson loop with R being a three-dimensional irreducible representation corresponds to a massive quark.
The Wilson loop is not BRST-invariant and therefore doesn't give a good observable in the twisted theory. However, for $t = \pm i$ with a small twist of the definition, we get a BRST-invariant operator:

$$W_R(\gamma) = Tr_R \, R\exp \int_\gamma (A \pm i\phi) = Tr \, R(\mathrm{Hol}(A \pm i\phi, \gamma)) \tag{146}$$

This is true, because the connection $\mathcal{A} = A \pm i\phi$ itself is BRST-invariant for these specific values of t.

We are interested in the S-dual operator (i.e. the operator corresponding to $W_R(\gamma)$ under Montonen-Olive duality. The answer can't be expressed in terms of fields and thus is rather complicated. The dual operator is called *'t Hooft operator* and is a so called *disorder operator*. This means that inserting this operator into the path-integral has the effect of changing the space of fields over which one integrates. For example, a disorder operator localized on a closed curve γ is defined by specifying the singular behavior for the fields near γ. The 't Hooft operator is defined as follows:
Let μ be an element of $\mathfrak{g}$ defined up to adjoint action of G and let us choose local coordinates in the neighborhood of a point $p \in \gamma$ so that γ is defined by the equations $x^1 = x^2 = x^3 = 0$. We require the gauge field to have singular curvature with the singularity taking the form

$$F \;\; \star_3 d\left(\frac{\mu}{2r}\right), \tag{147}$$

where r is the distance to the origin in the 123 hyper plane. As before $\star_3$ is the Hodge star operator in that plane. In the dual theory we consider $t = 1$. Then Q-invariance requires the 1-form Higgs field ϕ to be singular as well:

$$\phi \;\; \frac{\mu}{2r} dx^4 \tag{148}$$

We use the gauge freedom to conjugate μ to a particular Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. One can show that μ must lie in the coweight lattice $X_*(G) \subset \mathfrak{t}$, i.e. the lattice of homomorphisms from $U(1)$ to the maximal torus T corresponding to $\mathfrak{t}$. In addition one has to identify points of the lattice which are related by an element of the Weyl group W. We conclude that 't Hooft operators are classified by elements of X_*/W. The 't Hooft operator corresponding to the coweight μ will be denoted T_μ. By definition $X_*(G)$ is identified with the weight lattice $X^*({}^LG)$ of ${}^LG$. Elements pf $X^*({}^LG)$ are in one-to-one correspondence with irreducible representations of ${}^LG$. This suggests that MO duality maps the 't Hooft operator operator corresponding to a coweight $\mu \in X_*(G)$ to the Wilson operator corresponding to a representation ${}^LR$ with highest weight in the Weyl orbit of $\mu \in X^*({}^LG)$.

5.11 Connection to the Geometric Langlands Program

The statement of geometric Langlands correspondence is that two categories are equivalent. The physical interpretation of this was formulated by

Kapustin-Witten: The Geometric Langlands duality is related to S-duality of the GL-twisted $4d\,\mathcal{N}=4$ Super Yang-Mills on $\Sigma\times C$. In terms of effective 2d topological σ-model, coherent sheaves come from the D-brane category of topological B-model, while flat bundles/D-modules come from the D -brane category of topological A-model.

Recall that Maxwell's equations in the presence of a magnetic charge look like:

$$dF = J_m \tag{149}$$

$$d\star F = J_e \tag{150}$$

The currents J_m, J_e are supported on one-dimensional objects, i.e. 1-dim defects. J_e is coupled to the gauge connection via the *Wilson line operator*

$$\int_{J_e} A \tag{151}$$

On the other hand, J_m creates certain singularity for A, which is the *'t Hooft line operator*. Under S-duality, Wilson line and 't Hooft line operator transform to each other as we have seen above. Let S be a loop in M. As discussed before, the Wilson loop operator is connected to the holonomy of the connection along S. On the other hand, we can associate the 't Hooft operator by, $T(\rho,S)$ by specifying the singularity of the connection along a $U(1)$-component of the gauge group, i.e. a homomorphism $U(1)\to G$. This essentially is picking a representation of ${}^LR$ of the dual group ${}^LG$. S-duality matches Wilson and 't Hooft operators. This compatibilities are analogue to the once expected in the geometric Langlands correspondence.
For the twist defined by $t=\pm i$ ($\Psi=\infty$), we get an effective topological model for the complex structure J on $\mathcal{M}_H$. The Wilson loop operator can be completed into a topological operator by replacing $\mathcal{A}\to\mathcal{A}\pm i\phi$. Similarly, at $t=1$, $i\tau\in\mathbb{R}$ ($\Psi=0$), we can define topological 't Hooft operator by picking up a compatible singular behavior of ϕ along S.
In the effective $2d$ topological model, we put a topological line operator $\mathcal{H}_p$ of the form $L\times p\subset\Sigma\times C$. L is a 1-dim submanifold of Σ and p is a point on C. There are boundary conditions living on $\partial\Sigma$. If L is close to the boundary, the line operator $\mathcal{H}_p$ acts on this boundary conditions in the 2d topological theory. The presence of the extra dimensions in C make the operators commute. An action on boundary conditions, defines a functor

from the category of branes in $2d$ topological theory to itself. Since the operators commute, it makes sense to talk about joint eigenbranes for the relevant line operators. A joint eigenbrane of the Wilson line operator will be called an *electric eigenbrane*, a joint eigenbrane of the 't Hooft line operator will be called an *magnetic eigenbrane*. S-duality will map magnetic eigenbranes to electric eigenbranes and vice versa. The 't Hooft line operator at $t = 1$, $i\tau \in \mathbb{R}$ corresponds to the Hecke operators in the geometric Langlands program. The magnetic eigenbranes are the analogon of the Hecke eigensheaves.

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