Chapter 7: Inference for Distributions

MATH 560-01
Statistical Data Analysis

February 22, 2021
(Last corrected: 2/22)

These slides are based on material from *Introduction to the Practice of Statistics* by David S. Moore, George P. McCabe, and Bruce A. Craig, 9th edition.

Sections

7.1 Inference for the Mean of a Population

7.2 Comparing Two Means

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- perform *paired t-tests* for equality of the means of two populations based on a matched pairs design

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- In this chapter, we consider how to make inferences about μ in the more realistic setting where σ is unknown so that we need to use the standard deviation of the sample

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

to estimate it.

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  follows a different distribution called a t **distribution with** $n-1$ **degrees of freedom**.
- The denominator $\frac{s}{\sqrt{n}}$ is sometimes called the **standard error** of t.

7.1 Inference for the Mean of a Population

t distribution

- The density curve of the t distribution with df degrees of freedom has the form

$$f(t) = \frac{\Gamma\left(\frac{df+1}{2}\right)}{\sqrt{\pi df}\,\Gamma\left(\frac{df}{2}\right)\left(1+\frac{t^2}{df}\right)^{\frac{df+1}{2}}}, t \in \mathbb{R}.$$

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- This curve is symmetric about 0 and unimodal. When $df > 1$, the mean of this distribution is 0. (For $df = 1$, the expected value $\int_{-\infty}^{\infty} t f(t)\, dt$ does not exist.)

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- This curve is symmetric about 0 and unimodal. When $df > 1$, the mean of this distribution is 0. (For $df = 1$, the expected value $\int_{-\infty}^{\infty} t f(t)\, dt$ does not exist.)
- Sometimes, we abbreviate the t distribution with df degrees of freedom by $t(df)$.

7.1 Inference for the Mean of a Population

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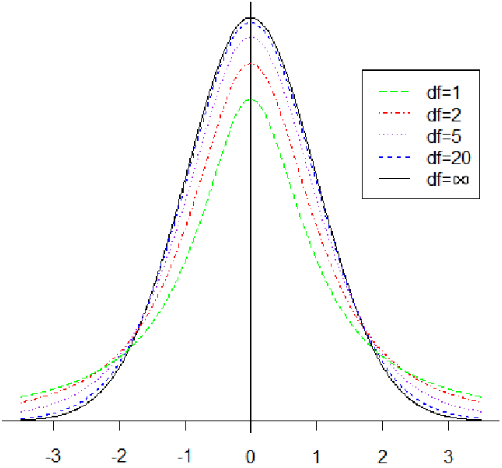
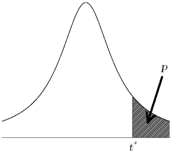


7.1 Inference for the Mean of a Population

Table for t distribution



t distribution critical values

	Upper-tail probability p								
df	.25	.20	.15	.10	.05	.025	.02	.01	.005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.90	31.82	63.66
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841
					⋮				
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581
∞	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576
	50%	60%	70%	80%	90%	95%	96%	98%	99%
	Confidence level C								

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Confidence interval for μ when σ is unknown and the population is normal

A level C confidence interval for μ is

$$\bar{x} \pm t^* \frac{s}{\sqrt{n}}$$

where t^* is the value for the $t(n-1)$ density with area C between $-t^*$ and t^*.

The confidence interval is approximately correct when n is large even when the population is not normal.

7.1 Inference for the Mean of a Population

Example 7A: Suppose we want information on the number of hours that U.S. college students who have mobile phones watch videos on their phone. Here is a SRS with the number of hours for $n = 8$ students:

$$7 \quad 9 \quad 1 \quad 6 \quad 13 \quad 10 \quad 3 \quad 5$$

Find a 95% two-sided confidence interval for the average number of hours that all U.S. college students who have mobile phones watch videos.

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- *Answer*:

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Two-sided one sample t-test for a population mean when σ is unknown and the population is normal

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One-sided (left-sided) one sample t-test for a population mean when σ is unknown and the population is normal

1. Test $H_0 : \mu = \mu_0$ vs. $H_a : \mu < \mu_0$
2. Test statistic: $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$
3. Use the t-table to find the value t^* such that $P(T < -t^*) = P(T > t^*) = \alpha$ where T follows a t distribution with $n - 1$ degrees of freedom.
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One-sided (right-sided) one sample t-test for a population mean when σ is unknown and the population is normal

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7.1 Inference for the Mean of a Population

Example 7B: A population follows a Normal distribution with unknown mean μ and unknown standard deviation σ. A SRS of size $n = 25$ is taken from this population to perform a significance test of $H_0 : \mu = 10$ versus $H_a : \mu < 10$. The observed sample statistics are $\bar{x} = 8$ and $s = 3$.
(a) Calculate an appropriate test statistic.
(b) Locate the two critical values t^* from the t-table that bracket t.
(c) Between what two values does the P-value fall?
(d) Should H_0 be rejected at level $\alpha = .01$?

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Paired t-test

- When designing experiments, comparative designs are often more useful than single sample studies.
- However, the one sample t-test procedure is useful when subjects are matched in pairs and their outcomes are compared within each matched pair.
- For example, if we take measurements on the same subjects under two different experimental conditions, then this is a **matched pairs design**.
- In analyzing data from a matched pairs design, x_i is the difference between the two measurements for the ith subject, and we use one-sample procedures (confidence intervals or t-tests to make inferences).

Example 7C: Consider the following study to compare two popular energy drinks. For each subject, a coin was flipped to determine which drink to rate first. Each drink was rated on a 0 to 100 scale, with 100 being the highest rating.

	Subject				
Drink	1	2	3	4	5
A	43	83	66	89	78
B	45	78	64	79	71

(a) Is there a difference in preferences at level $\alpha = .05$? State appropriate hypotheses, calculate an appropriate test statistic, compute the P-value or critical value, and state your conclusion.
(b) Compute a 95% two-sided confidence interval for the difference in preference between Drink A and Drink B.

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- The t procedures are usually robust against non-Normality of the population; outliers and strong skewness do affect the accuracy of the procedures, though.
- As the sample size grows, the CLT implies the distribution of $\bar{x}$ is close to normal and s will be an accurate estimate of σ.
- Except when the sample size is small, the assumption that the data are a SRS is more crucial than that the population is Normal.

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- Each group is considered to be a sample from a distinct population.
- The responses in each group are independent of those in the other group.

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- The samples from the two populations are taken independently.
- We are interested in making inferences about the difference between the population means, $\mu_1 - \mu_2$.

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- When σ_1 and σ_2 are unknown, we must replace them by estimates from the sample and use

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}.$$

7.2 Comparing Two Means

If we make the additional assumption that $\sigma_1 = \sigma_2$ and use the pooled estimate $s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$, then

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

follows a $t(n_1 + n_2 - 2)$ distribution.

7.2 Comparing Two Means

Confidence interval for $\mu_1 - \mu_2$ when $\sigma_1 = \sigma_2$ are unknown and the populations are normal

A level C confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{x}_1 - \bar{x}_2) \pm t^* s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

where t^* is the value for the $t(n_1 + n_2 - 2)$ density with area C between $-t^*$ and t^*.

The confidence interval is approximately correct when n_1 and n_2 are large even when the population is not normal.

7.2 Comparing Two Means

Two-sided two sample t-test for equality of population means when $\sigma_1 = \sigma_2$ is unknown and the population is normal

1. Test $H_0 : \mu_1 = \mu_2$ vs. $H_a : \mu_1 \neq \mu_2$
2. Test statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - 0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ where
$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$
3. Use the t-table to find the value t^* such that $P(T > t^*) = \frac{\alpha}{2}$ where T follows a t distribution with $n_1 + n_2 - 2$ degress of freedom.
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3. Use the t-table to find the value t^* such that $P(T < -t^*) = P(T > t^*) = \alpha$ where T follows a t distribution with $n_1 + n_2 - 2$ degrees of freedom.
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<u>One-sided (right-sided) two sample t-test for equality of population means when $\sigma_1 = \sigma_2$ is unknown and the population is normal</u>

1. Test $H_0 : \mu_1 = \mu_2$ vs. $H_a : \mu_1 > \mu_2$
2. Test statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - 0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ where
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3. Use the t-table to find the value t^* such that $P(T > t^*) = \alpha$ where T follows a t distribution with $n_1 + n_2 - 2$ degrees of freedom.
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7.2 Comparing Two Means

Example 7D (Book p.464): To assess the effect of calcium intake on blood pressure, an experiment was conducted on a Calcium and Placebo group, and the change in blood pressure was recorded from the beginning and end of the study for each group. Here are the summary statistics for the decrease in blood pressure:

Group	Treatment	n	$\bar{x}$	s
1	Calcium	10	5.000	8.743
2	Placebo	11	-0.273	5.901

Assume that the standard deviations of both populations are the same. Should $H_0 : \mu_1 = \mu_2$ vs. $H_a : \mu_1 > \mu_2$ be rejected at level $\alpha = .05$? Carefully show all steps of your test (i.e., calculate an appropriate test statistic, compute the P-value or critical value, and state the conclusion).

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- (Satterthwaite's Approximation) It approximately follows a $t(k)$ distribution where

$$k = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2}.$$

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- Unfortunately, the exact distribution of the random variable t is not as simple when $\sigma_1 \neq \sigma_2$.
- (Satterthwaite's Approximation) It approximately follows a $t(k)$ distribution where

$$k = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2}.$$

- A more conservative approach uses $k = \min\{n_1 - 1, n_2 - 1\}$.

7.2 Comparing Two Means

Confidence interval for $\mu_1 - \mu_2$ when $\sigma_1 \neq \sigma_2$ are unknown and the populations are normal

A level C confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{x}_1 - \bar{x}_2) \pm t^* \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where t^* is approximated by the $t\left(\frac{\left(\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2+\frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2} \right)$ density with area C between $-t^*$ and t^*.

The confidence interval is approximately correct when n_1 and n_2 are large even when the population is not normal.

7.2 Comparing Two Means

Two-sided two sample t-test for equality of population means when $\sigma_1 \neq \sigma_2$ is unknown and the population is normal

1. Test $H_0 : \mu_1 = \mu_2$ vs. $H_a : \mu_1 \neq \mu_2$
2. Test statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$
3. Use the t-table to find the value t^* such that $P(T > t^*) = \frac{\alpha}{2}$ where T approximately follows a t distribution with
$$k = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{s_2^2}{n_2}\right)^2}$$ degrees of freedom.
4. Reject H_0 if $|t| > t^*$

7.2 Comparing Two Means

One-sided (left-sided) two sample t-test for equality of population means when $\sigma_1 \neq \sigma_2$ is unknown and the population is normal

1. Test $H_0 : \mu_1 = \mu_2$ vs. $H_a : \mu_1 < \mu_2$
2. Test statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$
3. Use the t-table to find the value t^* such that $P(T < -t^*) = P(T > t^*) = \alpha$ where T approximately follows a t distribution with $k = \dfrac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2}$ degrees of freedom.
4. Reject H_0 if $t < -t^*$

7.2 Comparing Two Means

<u>One-sided (right-sided) two sample t-test for equality of population means when $\sigma_1 \neq \sigma_2$ is unknown and the population is normal</u>

1. Test $H_0 : \mu_1 = \mu_2$ vs. $H_a : \mu_1 > \mu_2$
2. Test statistic: $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$
3. Use the t-table to find the value t^* such that $P(T > t^*) = \alpha$ where T approximately follows a t distribution with
$k = \dfrac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{s_2^2}{n_2}\right)^2}$ degrees of freedom.
4. Reject H_0 if $t > t^*$

7.2 Comparing Two Means

Example 7E: Population A follows a Normal distribution with unknown mean μ_1 and unknown standard deviation σ_1, and Population B follows a Normal distribution with unknown mean μ_2 and unknown standard deviation σ_2. Here are the summary statistics for two independent SRSs

Population	n	$\bar{x}$	s
1	20	60	1
2	5	43	5

(a) Find an approximate 99% two-sided confidence interval for $\mu_1 - \mu_2$ without assuming equality of population standard deviations based on Satterthwaite's approximation.
(b) Assume that the population standard deviations are the same. Find a 99% two-sided confidence interval for $\mu_1 - \mu_2$.

7.2 Comparing Two Means

- *Answer*: