Assignment - 1

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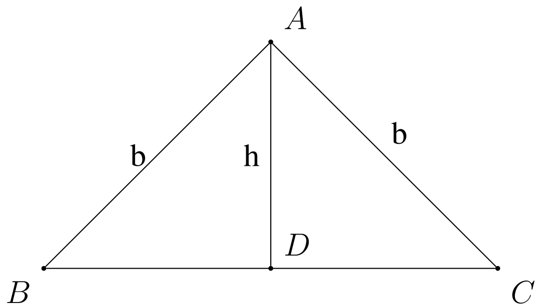
Problem

An isoceles triangle has the extremities of its base at $\begin{pmatrix}2\\5\end{pmatrix}$ and $\begin{pmatrix}-2\\2\end{pmatrix}$. Find the two possible positions of the vertex if its area is 25 sq.units



Solution

Let the vertex B be $\begin{pmatrix}2\\5\end{pmatrix}$ and vertex C be $\begin{pmatrix}-2\\2\end{pmatrix}$.

Let the other vertex A be $\begin{pmatrix}x\\y\end{pmatrix}$.

Let us find the distance of BC

$$\mathbf{B} = \begin{pmatrix}2\\5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-2\\2\end{pmatrix}$$

$$\|\mathbf{B}-\mathbf{C}\| = \left\|\begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}-2\\2\end{pmatrix}\right\| = \sqrt{4^2+3^2} = 5$$

Given, the area of the triangle= 25 sq.units
We know area of a $\triangle$ with the vertices A,B and C can be given by:

$$\boldsymbol{\Delta} = \frac{\mathbf{1}}{\mathbf{2}}\begin{vmatrix}1 & 1 & 1\\ A & B & C\end{vmatrix}$$

$$\Delta ABC = \frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ x & 2 & -2\\ y & 5 & 2\end{vmatrix}$$

Expanding along the first row,

$$\Delta ABC = \frac{1}{2}\left[1(4+10) - 1(2x+2y) + 1(5x-2y)\right]$$

$$\frac{1}{2}\left[14-(2x+2y)+(5x-2y)\right] = 25$$

$$-2x-2y+5x-2y = 50-14$$

$$3x-4y = 36 \tag{1}$$

Since ABC is an Isoceles triangle,
AD is the perpendicular bisector of BC

$$(\mathbf{A}-\mathbf{D})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{2}$$

Also,

$$\mathbf{D} = \frac{1}{2}(\mathbf{A}+\mathbf{B})$$

$$\mathbf{D} = \frac{1}{2}\left[\begin{pmatrix}2\\5\end{pmatrix} + \begin{pmatrix}-2\\2\end{pmatrix}\right]$$

$$\mathbf{D} = \begin{pmatrix}0\\3.5\end{pmatrix}$$

Now,

$$(\mathbf{A}-\mathbf{D}) = \begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}0\\3.5\end{pmatrix}$$

$$(\mathbf{A}-\mathbf{D}) = \begin{pmatrix}x\\y-3.5\end{pmatrix}$$

$$(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}-2\\2\end{pmatrix}$$

$$(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}4\\3\end{pmatrix}$$

By (2),

$$\begin{pmatrix}x\\y-3.5\end{pmatrix}^T\begin{pmatrix}4\\3\end{pmatrix} = 0$$

$$4x+3(y-3.5) = 0$$

$$4x+3y = 10.5$$

Multiply by 2,

$$\implies 8x+6y = 21 \tag{3}$$

By interchanging the columns, the area of triangle can also be expressed as,

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x & -2 & 2 \\ y & 2 & 5 \end{vmatrix}$$

Expanding along the first row,

$$\begin{aligned} \frac{1}{2}\left[1(-10-4) - 1(5x-2y) + 1(2x+2y)\right] &= 25 \\ \frac{1}{2}\left[-14 - 1(5x-2y) + 1(2x+2y)\right] &= 25 \\ -5x + 2y + 2x + 2y &= 64 \end{aligned}$$

$$\implies -3x + 4y = 64 \tag{4}$$

Let us consider the matrices representation of equations (1) and (3),

$$\begin{bmatrix} 3 & -4 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 36 \\ 21 \end{bmatrix}$$

$$\mathbf{AX} = \mathbf{B}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 8 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 36 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 6 & 4 \\ -8 & 3 \end{bmatrix}\begin{bmatrix} 36 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 300 \\ -225 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -4.5 \end{bmatrix} \tag{5}$$

Let us consider the matrices representation of equations (4) and (3),

$$\begin{bmatrix} -3 & 4 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 64 \\ 21 \end{bmatrix}$$

$$\mathbf{AX} = \mathbf{B}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 8 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 64 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-50}\begin{bmatrix} 300 \\ -575 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 11.5 \end{bmatrix} \tag{6}$$

$\therefore$ From equations (5) and (6), **The two possible vertex are** $(6, -4.5)$ **and** $(-6, 11.5)$