Assignment - 1

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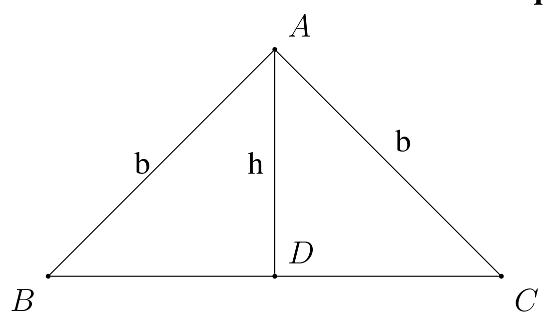
Problem

An isoceles triangle has the extremities of its base at $(2,5)$ and $(-2,2)$. Find the two possible positions of the vertex if its area is 25 sq.units



Solution

Let the vertex B be $(2,5)$ and vertex C be $(-2,2)$.

Let the other vertex A be (x,y).

Let us find the distance of BC

$$\mathbf{B} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\|\mathbf{B} - \mathbf{C}\| = \left\| \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\| = \sqrt{4^2 + 3^2} = 5$$

Given, the area of the triangle= 25 sq.units
We know area of a $\triangle$ with the vertices A,B and C can be given by:

$$\mathbf{\Delta} = \mathbf{\frac{1}{2}} \begin{vmatrix} 1 & 1 & 1 \\ A & B & C \end{vmatrix}$$

$$\Delta ABC = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x & 2 & -2 \\ y & 5 & 2 \end{vmatrix}$$

Expanding along the first row,

$$\Delta ABC = \frac{1}{2}\left[1(4+10) - 1(2x+2y) + 1(5x-2y)\right]$$

$$\frac{1}{2}\left[14 - (2x+2y) + (5x-2y)\right] = 25$$

$$-2x - 2y + 5x - 2y = 50 - 14$$

$$3x - 4y = 36 \tag{1}$$

Since ABC is an Isoceles triangle,

$$AB = AC$$

$$AB^2 = AC^2$$

$$(x-2)^2 + (y-5)^2 = (x+2)^2 + (y-2)^2$$

$$4x - 4y + 4 = -4x - 10y + 25$$

$$8x + 6y = 21 \tag{2}$$

By interchanging the columns, the area of triangle can also be expressed as,

$$\Delta ABC = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x & -2 & 2 \\ y & 2 & 5 \end{vmatrix}$$

Expanding along the first row,

$$\Delta ABC = \frac{1}{2}\left[1(-10-4) - 1(5x-2y) + 1(2x+2y)\right]$$

$$\frac{1}{2}\left[-14 - 1(5x-2y) + 1(2x+2y)\right] = 25$$

$$-5x + 2y + 2x + 2y = 64$$

$$-3x + 4y = 64 \tag{3}$$

Let us consider the matrices representation of equations (1) and (2),

$$\begin{bmatrix} 3 & -4 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 36 \\ 21 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 8 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 36 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 6 & 4 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} 36 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 300 \\ -225 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -4.5 \end{bmatrix} \tag{4}$$

Let us consider the matrices representation of equations (3) and (2),

$$\begin{bmatrix} -3 & 4 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 64 \\ 21 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 8 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 64 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-50} \begin{bmatrix} 6 & -4 \\ -8 & -3 \end{bmatrix} \begin{bmatrix} 64 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-50} \begin{bmatrix} 300 \\ -575 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 11.5 \end{bmatrix} \tag{5}$$

$\therefore$ From equations (4) and (5), **The two possible vertex are** $(6, -4.5)$ **and** $(-6, 11.5)$