EE7330: Network Information Theory **2021**

Lecture Notes 2: SEQUENCES, LIMITS AND INFINITES

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. Please email the course instructor in case of any errors.*

2.1 GIST OF LECTURE 1

In the last lecture, we have seen the introduction topics such as

1. What does a general communication system look like?

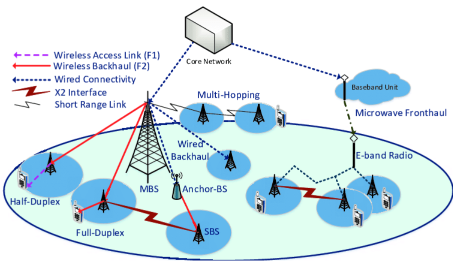

Figure 2.1: General Communication Systems

A general communication system consists of user equipments(source nodes) connected to base stations through a wireless links otherwise called channels which are noisy. These Base Stations are inturn connected to a back haul network which connects to a core network via a wired/ optical link. They are to support a large number of users.

In such systems, primary goal will be designing a noise free communciation system which require modelling of source as well as channel.

2. **Single User/ Point-to-Point System**.

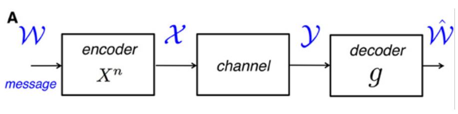

Figure 2.2: Single User/ Point-to-Point System

In a point-to-point/ single user communication system which is shown in above figure, the max rate at which the communication may take place is given by

$$\Rightarrow R = \frac{k}{n} \tag{2.1}$$

where k is number of useful message bits taken by encoder and generate codewords of n bits length.It may appear very simple. To measure the performance of the system, Rate(code rate) and P_e(bit error rate) are used. The channel is defined by the transition probability matrix i.e. $P_{\frac{Y}{X}}$. The capacity of the channel is given by

$$C = \underset{\mathbf{P_X}}{\mathrm{Max}}\, I(X;Y) \tag{2.2}$$

From Shannon Channel Coding Theorem, the design considerations can be put up together as follows:

$$\begin{aligned} \lim_{n\to\infty} R &\approx I(X;Y) \\ \lim_{n\to\infty} P_e &\approx 0 \end{aligned} \tag{2.3}$$

3. **Multi-User Systems**.

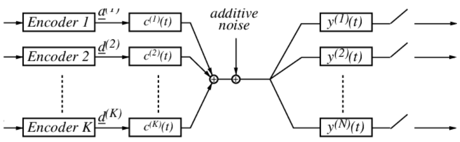

Figure 2.3: Multi-User System

In Multi-User Systems, goal is to receive the distinct number of messages without any loss or interference within the power constraints. The rate of transmission are given as below.

$$\begin{aligned} \Rightarrow R_1 &= \frac{k_1}{n} \\ \Rightarrow R_2 &= \frac{k_2}{n} ... \end{aligned} \tag{2.4}$$

(a) How to mitigate effects of Interference?
By using a type of scheduling mechanism, one can avoid the interference caused by multiple signals generated by multiple users such as TDMA(most commonly used), FDMA etc.,.

(b) How to measure performance?
Consider only 2 users are trying to communicate using a system. Here assume the noise is gaussian

i.e. $W_i \sim \mathcal{N}(\mu = 0, \sigma^2)$ In an AWGN channel, the max rate is given by

$$
\begin{aligned}
R_{max} = C = \frac{1}{2} * log_2(1 + \frac{P}{\sigma^2}) \\
where \frac{P}{\sigma^2} = SNR.
\end{aligned}
\tag{2.5}
$$

When any scheduling is done, the max rate is dependent on the time allotted or divided between the users. The max rates as per the user is given as below:

$$
\begin{aligned}
\Rightarrow R_1 \leqslant \alpha C \\
\Rightarrow R_2 \leqslant (1 - \alpha C) \\
where C = \frac{1}{2} * log_2(1 + \frac{P}{\sigma^2})
\end{aligned}
\tag{2.6}
$$

If the plot of above rates is carried out, it will look below:

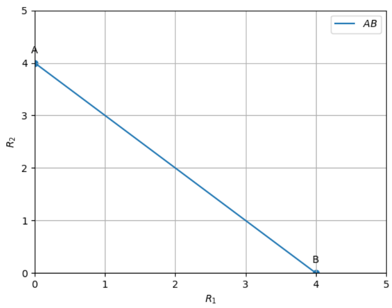

Figure 2.4: Rate Performance Graph

4. This performance is obtained by simply following simple TDMA techniques. Can we do better? The answer is Yes. The rate performance in a multi-user system can be improved by using a procedure called Successive Interference Cancellation(**SIC**). In this interference caused by another user is treated as noise and the rate of user 1 is calculated. Once the rate of user 1 is calculated, then the user 2 rate is calculated. The rates can be obtained as below:

$$
\begin{aligned}
\Rightarrow R_1 = C = \frac{1}{2} * log_2(1 + \frac{P}{P + \sigma^2}) \\
\Rightarrow R_2 = \frac{1}{2} * log_2(1 + \frac{P}{\sigma^2})
\end{aligned}
\tag{2.7}
$$

The plot of R_1 vs R_2 is shown below:

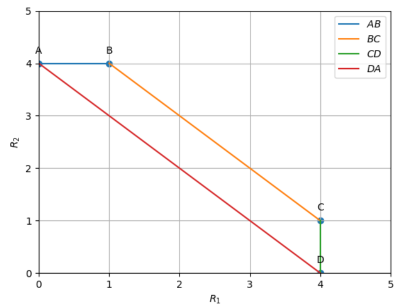

Figure 2.5: Rate Performance Graph through SIC

There are more techniques which will further enhance of multi-user systems which will be studied in this course.

2.2 Sequences

2.2.1 Commonly used notations

1. **R** - Rate
2. P_e - Probability of Error
3. X, Y, Z... - Random Varaibles
4. x,y,z..... - Deterministic Variables
5. $\mathbb{R}$ - Real Numbers, $\mathbb{Z}$ - Integers, $\mathbb{N}$ - Natural numbers
6. Pr[X=x] - Probability Distributive Function (PDF)
7. P_x - Probability Mass Function (pmf)
8. f_x - Probability Density Function (pdf)
9. $P_{\frac{x}{y}}(\frac{X}{Y})$ - Conditional pmf
10. $f_{\frac{x}{y}}(\frac{X}{Y})$ - Conditional pdf

2.2.2 Sequences

As we have seen earlier, the design considerations are

$$\begin{aligned} \lim_{n\to\infty} R &\approx C \\ \lim_{n\to\infty} P_e &\approx 0 \end{aligned} \tag{2.8}$$

i.e As the users increase, the rate at which they communicate should be as close as to capacity of channel and their communication should be error free and noise free with almost 0 probability of error.

In Information Theory, one has to deal with huge number of users, limits, Channels and their time limits. A lot of mathematics is involved and they have to be exercised vary carefully. Above are some commonly used notions to understand the further notes.Let us understand mathematics involved. Since Information Theory deal with a lot of sequential data and process, lets start with definition of sequence.

Definition of a Sequence.

Informally: A sequence is a ordered collection of numbers. These numbers may belong to sets of Natural($\mathbb{N}$) or Integer($\mathbb{Z}$) or Real numbers ($\mathbb{R}$).

Formally: A sequence over $\mathcal{A}$ is defined as a function that maps any type of integer or number from the set of Natural($\mathbb{N}$) or Integer($\mathbb{Z}$) or Real numbers ($\mathbb{R}$) to set $\mathcal{A}$ i.e. $f : \mathbb{N} \rightarrow \mathcal{A}$.

Example-1:
Let $\mathcal{A} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ Here the set $\mathcal{A} \subset \mathbb{R}$. The numbers follow a pattern and it can be represented by a funtion and it is $f = \frac{1}{n}$. So a sequence typically have a function which maps numbers from set $\mathbb{R}$ to its subset.
Example-2:
Let $\mathcal{B} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ Here the set $\mathcal{B} \subset \mathbb{R}$. The numbers follow a pattern and it can be represented by a funtion and it is $f = 2^{-(n+1)}$.

Q) Are these sequences finite or infinite ?
They are infinite in nature.As an exmaple, lets consider sequence defined by $f = \frac{1}{n}$. It is continuously increasing(in denominator) till infinity. Here the concept of limits will restrict the increment or decrement of sequence.

Q) Does every sequence have a limit ?
Based on the value of function, every sequence may or may not have a limit. The one which limits to a finite value upon n incrementing to ∞ is called a convergent sequence. The one which does not limit to a finite value or if the function value is either ∞ or $-\infty$ then it is called as Divergent Sequence. To denote the limit, it is written as

$$\lim_{nto\infty} f = \lim_{nto\infty} \frac{1}{n} = \frac{1}{\infty} = 0 \tag{2.9}$$

In the above example of $f = \frac{1}{n}$, as n $\rightarrow -\infty or \infty$ the value of f will be 0 i.e. the sequence converges to 0. If $f = n$, then as n $\rightarrow -\infty or \infty$ the value of f will be ∞ and the sequence does not converge to finite value. Hence it is a divergent sequence.

Bounded Sequences.

If any sequence contains a number of elements less than a particular value or number i.e. limit then the sequence is called a Bounded sequence. Else it is unbounded sequence.Limits are well elobarated in next section.
Formally, consider M, a rational number and $M \geqslant 0$. A finite sequence $a_1, a_2, a_3, ...a_n$ for a finite n is bounded by M iff $\mid a_i \mid \leqslant M, \forall 1 \leqslant i \leqslant n$. An infinite sequence $(a_n)_{n=1}^{\infty}$ i.e.$a_1, a_2, a_3, ...a_\infty$ is bounded by M iff $\mid a_i \mid \leqslant M, \forall i \geqslant 1$.

Combinedly, a sequence is said to be bounded iff it is bounded for some rational number M $\geqslant$ 0.

Example-3.A sequence by function $f = (-1)^n i.e.forn = 0, 1, 2, 3, 4, ...\infty$ the sequence can be written as $1, -1, 1, -1, 1, -1, 1, -1, ...$ respectively. Consider M ¿ 1, and for any value of n, $\mid a_i \mid \leqslant M$. Hence it is a bounded sequence.

Example-4.A sequence by function $f = 2^{-n} i.e.forn = 0, 1, 2, 3, 4, ...\infty$ the sequence can be written as $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...2^{-\infty}$ respectively. Consider M $\geqslant$ 1, $\mid a_i \mid \leqslant M$. Hence it is a bounded sequence.

Example-5.A sequence by function $f = -n^2 i.e.forn = 0, 1, 2, 3, 4, ...\infty$ the sequence is $0, -1, -4, -6, -9... -\infty$ respectively. , and for any value of M, $\mid a_i \mid \geqslant M$. Hence it is a unbounded sequence.

Note. Every finite sequence is a bounded sequence. For a rational sequence, such as $f = \pi$ for a finite n in decimal places, f will be a finite rational number and limit to it. As $n \to \infty$, f(n) will limit to a irrational number. Here the limits have to well-defined. Hence the concept Supremum and Infimum of a set.

Open and Closed Sets.

Bounds of a function are defined in the form of a set/range i.e (lower bound, upper bound). Any sequence will lie within the specified bounds. The type of braces/brackets used defines the closedness of a set. For $f = \frac{1}{n+1}$ i.e. sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ f(n) $\to$ 0. The sequence is defined over set $\mathcal{A}$ = (0,1). Here the sequence converges to 0 and it is not included in the set (0,1). Here $\mathcal{A}$ is called open set. By definition, **Open set** is a set in which the set of lower bound and set of upper bound does not form subset of the set i.e. for the above set $\mathcal{A}$, lower limit is 0 and upper limit is 1. $[0], [1] \nsubseteq \mathcal{A}$. If they form the subsets i.e. $[0], [1] \subseteq \mathcal{A}$ then the set is called **Closed set** and the set is represented by using square brackets (a mathematical notation to define the closed set) i.e. $\mathcal{A}$ = [0,1].

Supremum.

Supremum is defined as the least upper bound that is greater than any other number in the set. Let $\mathcal{A} \subset \mathbb{R}$ and a be the supremum of $\mathcal{A}$. Then

$$\begin{aligned} \Rightarrow a \geqslant x, &\quad \forall x \in \mathcal{A} \\ &or \\ \Rightarrow \forall \epsilon \geqslant 0, &\quad \exists x \in \mathcal{A} \\ such\ that \quad x > a - \epsilon \quad or &\quad a < x + \epsilon \end{aligned} \tag{2.10}$$

Infimum.

Infimum is defined as the greatest lower bound that is lesser than any other number in the set. Let $\mathcal{A} \subset \mathbb{R}$ and b be the infimum of $\mathcal{A}$. Then

$$\begin{aligned} \Rightarrow b \leqslant x, &\quad \forall x \in \mathcal{A} \\ &or \\ \Rightarrow \forall \epsilon \geqslant 0, &\quad \exists x \in \mathcal{A} \\ such\ that \quad x > b + \epsilon \quad or &\quad b < x - \epsilon \end{aligned} \tag{2.11}$$

It is also to be noted that sequences may converge to a point either inside or outside the set. Hence it is not necessary that both supremum and infimum lie within the set. For an open set, they lie outside the set.
Example-6:

Let $\mathcal{A}$=(0,1) be an open set. Supremum for this set can be 1 and infimum be 0.Here if x=0.99991, x $\in \mathcal{A}$, but $0, 1 \notin \mathcal{A}$.

It is to be noted that for a closed set, Supremum and Infimum lies within the set.
Supremum = max value of the set
Infimum = min value of the set.

Example-7:

Let $\mathcal{A} = (-2)^n : n \in \mathbb{Z}$ be a sequence and is defined over a closed set $[-\infty, \infty]$.

$$\begin{aligned} &\Rightarrow Sup\mathcal{A} = \infty \\ &\& Inf\mathcal{A} = -\infty. \end{aligned} \tag{2.12}$$

2.2.3 Limits of Sequences

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Though the bounds define the nature of sequence to an extent, their nature will be more evident from the limits of that sequence. The limit of a sequence is defined as the value to which a sequence $(a_n)_{n=m}^{\infty}$ converges to, for any value m and m $< \infty$. Let L be a rational number to which the sequence converges to, then the limit is written as

$$L = \lim_{n \to \infty} (a_n) \tag{2.13}$$

Distance between any two real numbers a and b is given by d(a,b) = $\mid a - b \mid$.

Limit in terms of distance:
For a function f: $\mathbb{N} \to \mathcal{A}$, with a steadyness of d in sequence i.e.($\mathcal{A}$,d), if $(f_n)_{n \geqslant 1}$ is a sequence over $\mathcal{A}$ and f_n *converges* $\to a$ then

$$\Rightarrow \lim_{n \to \infty} f_n = a \ if \ \forall \epsilon > 0, \exists \ m \in \mathbb{N} \ such \ that \ d(f_n, a) \leqslant \epsilon, \ \forall n \geqslant m \tag{2.14}$$

In **Example-3**, a sequence by function $f = (-1)^n i.e. for \ n = 0, 1, 2, 3, 4, ...\infty$ is given. The sequence is $1, -1, 1, -1, 1, -1, 1, -1, ...$ respectively. $\forall M > 1$, and for any value of n, $\mid a_i \mid \leqslant M$. Hence it is a bounded sequence.The limits of n are given as n $\to -\infty$ *or* ∞, the value of the sequence to which it converges to is -1 or 1.

$$\Rightarrow \lim_{n\to\infty}(-1)^n = either\ 1\ or\ -1.$$
$$\Rightarrow \lim_{n\to-\infty}(-1)^n = either\ 1\ or\ -1. \tag{2.15}$$

Hence it is convergent bounded sequence.

In **Example-4**, a sequence by function $f = 2^{-n} i.e. for\ n = 0, 1, 2, 3, 4, ...\infty$ is written as $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...2^{-\infty}$ for respective n. $\forall$M $\geqslant 1$, $\mid a_i \mid \leqslant M$. Hence it is a bounded sequence.The limits of n are given as n $\to -\infty\ or\ \infty$, the value of sequence to which it converges to is 0.

$$\Rightarrow \lim_{n\to\infty} 2^{-n} \approx 0.$$
$$\Rightarrow \lim_{n\to-\infty} 2^{-n} \approx 0. \tag{2.16}$$

Hence it is convergent bounded sequence.

In **Example-5**,a sequence by function $f = -(n^2) i.e. for\ n = 0, 1, 2, 3, 4, ...\infty$ is written as $0, -1, -4, -6, -9...-\infty$ respectively, and for any value of M, $\mid a_i \mid \geqslant M$. Hence it is a unbounded sequence. Here, the limits of n are given as n $\to -\infty\ or\ \infty$, the value of sequence will $\to -\infty$ and

$$\Rightarrow \lim_{n\to\infty} -(n^2) = -\infty.$$
$$\Rightarrow \lim_{n\to-\infty} -(n^2) = \infty. \tag{2.17}$$

Hence it is a divergent unbounded sequence.

Example-8. For f(n) = 2^{-n}.

$$\begin{aligned}
\lim_{n\to\infty} 2^{-n} &= 0. \\
\Rightarrow \epsilon > 0, \exists m\ such\ that\ \mid 2^{-n} - 0 \mid &\leqslant \epsilon \\
\Rightarrow 2^{-n} &\leqslant \epsilon \\
\Rightarrow n &\geqslant log_2 \frac{1}{\epsilon} \\
\Rightarrow for\ value \geqslant log_2 \frac{1}{\epsilon}, the\ sequence\ converges\ to\ 0.
\end{aligned} \tag{2.18}$$

Example-9. For f(n) = $log_2(1 + \frac{3}{n})$.

$$\begin{aligned}
\lim_{n\to\infty} log_2(1 + \frac{3}{n}) &= 0. \\
\Rightarrow log_2(1 + \frac{3}{n}) &\leqslant \epsilon \\
\Rightarrow 1 + \frac{3}{n} &\leqslant 2^{\epsilon} \\
\Rightarrow \frac{3}{n} &\leqslant 2^{\epsilon} - 1 \\
\Rightarrow n &\geqslant \frac{3}{2^{\epsilon} - 1} \\
\Rightarrow for\ value \geqslant \frac{3}{2^{\epsilon} - 1}, the\ sequence\ converges\ to\ 0.
\end{aligned} \tag{2.19}$$

Note: Limits are also called as limit points.

Limit Superior. For any sequence $(a_n)_{n=m}^{\infty}$, define a new sequence $(a_N^+)_{N=m}^{\infty}$ using the supremum of $(a_n)_{n=N}^{\infty} \Rightarrow a_N^+ = sup(a_n)_{n=N}^{\infty}$ and a_N^+ is the supremum of all elements in the sequence from a_N onwards. The limit superior of $(a_n)_{n=m}^{\infty}$ is defined as

$$\lim sup_{n\to\infty} a_n = infimum((a_N^+)_{N=m}^{\infty}) \tag{2.20}$$