SM5083 - BASICS OF PROGRAMMING

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CONTENTS

Abstract—**This paper contains solution to problem no 5 of Examples III Section of Chapter III of Analytical Geometry by Hukum Chand. Links to Python codes are available below.**

Download python codes at

https://github.com/rsgirishkumar/SM5083/
    ASSIGNMENT2

1 PROBLEM

The opposite vertices of a square are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.
Find the equations of four sides.

2 SOLUTION

Let the given points are indicated as below

$$\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \qquad (2.0.1)$$

Let the unknown vertices are indicated as **B, D**.

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2.1 FINDING VERTICES USING AFFINE TRANS-FORMATION

Lemma. If **A**and**C** are two opposite vertices of a Square ABCD, then the other two vertices are given by

$$\mathbf{B} = \mathbf{R^{-1}B'} - \mathbf{P}$$
$$\mathbf{D} = \mathbf{R^{-1}D'} - \mathbf{P}$$
$$\text{where} \qquad (2.1.1)$$
$$\mathbf{R^{-1}} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \text{ and}$$
$$\mathbf{P} = -\mathbf{A}.$$

Proof:
Lets consider a square **OFGH** with origin(**O**) as one of the vertices and x, y - axes are two sides. Let the norm of the direction vectors of sides be d.The angle between **F**(side) and **G**(diagonal) is $\theta = 45^o$. and

$$\mathbf{F} = \frac{(\mathbf{G})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\|(\mathbf{G})^T\|} \qquad (2.1.2)$$
$$\Rightarrow \mathbf{G} = \sqrt{2}(\mathbf{F})$$

The norm of diagonal direction vector is

$$\|\mathbf{OG}\| = \sqrt{2}d = \|\mathbf{FH}\|. \qquad (2.1.3)$$

The direction vectors are given by

$$\mathbf{F} = \begin{pmatrix} d \\ 0 \end{pmatrix}, \mathbf{G} - \mathbf{F} = \begin{pmatrix} 0 \\ d \end{pmatrix},$$
$$\mathbf{H} - \mathbf{G} = \begin{pmatrix} -d \\ 0 \end{pmatrix}, -\mathbf{H} = \begin{pmatrix} 0 \\ -d \end{pmatrix}$$
$$\mathbf{G} = \begin{pmatrix} d \\ d \end{pmatrix}. \qquad (2.1.4)$$
$$\mathbf{H} - \mathbf{F} = \begin{pmatrix} -d \\ d \end{pmatrix}.$$

Let the Square **ABCD** has two vertices **A** and **C** . By using inspection method, Affine transformation has to be applied for **B** and **D**.The step by step procedure involves

1) Apply affine transformation steps i.e Translation and Rotation respectively of the given vertices **A** and **C** to shift square to the **OFGH**
2) Find the translation vector **P** and rotation matrix **R**.
3) Find the diagonal/Direction vector and norm of **A$'$C$'$**.
4) Find the side direction vector and using norm and inspection method find the vertices **B$'$** and **D$'$**.
5) Find the points **B** and **D** by using inspection method and reverse affine transformation i.e rotation and translation respectively.

Translation and Rotation Let **P** be translation vector is given by

$$\mathbf{P} = -\mathbf{A}$$
$$\cos\theta = \frac{(\mathbf{C} - \mathbf{A})^T \ \mathbf{OG}}{\|(\mathbf{C} - \mathbf{A})^T\|} \quad (2.1.5)$$

2.2 For Rotation Matrix R

Consider a vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the norm is L. If vector is rotated anticlockwise around the origin by β degrees then the new vector be $\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}$. Call the angle between $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the x-axis as α. Then

$$x_1 = L \ \cos\alpha$$
$$y_1 = L \ \sin\alpha \quad (2.2.1)$$

So the angle between $\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}$ and the x-axis is $\alpha + \beta$.

$$x_1' = L \ \cos(\alpha + \beta) =$$
$$L \ \cos\alpha\cos\beta - L \ \sin\alpha\sin\beta$$
$$= x_1 \cos\beta - y_1 \sin\beta.$$
$$y_1' = L \ \sin(\alpha + \beta) =$$
$$L\sin\alpha\cos\beta + L \ \cos\alpha\sin\beta \quad (2.2.2)$$
$$= y_1 \cos\beta + x_1 \sin\beta.$$
$$\Rightarrow \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Here the rotation angle is θ. Hence the rotation matrix is given by

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (2.2.3)$$

The generalized affine transformation principles are as below

$$\mathbf{A}' = \mathbf{R}(\mathbf{A} + \mathbf{P}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{C}' = \mathbf{R}(\mathbf{C} + \mathbf{P}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix}.$$

$$\mathbf{B}' = \mathbf{A}'\mathbf{B}' + \mathbf{A}'$$
where $\mathbf{A}'\mathbf{B}' = \|\mathbf{A}'\mathbf{B}'\|(\mathbf{F})$.
$$\mathbf{D}' = \mathbf{A}'\mathbf{D}' + \mathbf{A}'$$
where $\mathbf{A}'\mathbf{D}' = \|\mathbf{A}'\mathbf{D}'\|(\mathbf{H})$.
$$(2.2.4)$$

2.3 Reverse Affine Transformation

By doing the reverse affine transformation, the Square **ABCD** can be obtained from **A$'$B$'$C$'$D$'$** by using the below transformation rules.Given

$$\mathbf{A}' = \mathbf{R}(\mathbf{A} + \mathbf{P})$$
$$\Rightarrow \mathbf{A}'\mathbf{R}^{-1} = \mathbf{A} + \mathbf{P}$$
$$\Rightarrow (\mathbf{A}'\mathbf{R}^{-1}) - \mathbf{P} = \mathbf{A} \quad (2.3.1)$$
$$\text{Hence } \mathbf{A} = \mathbf{R}^{-1}\mathbf{A}' - \mathbf{P}$$

The generalization principle can be applied to other vertices also as below

$$\mathbf{B} = \mathbf{R}^{-1}\mathbf{B}' - \mathbf{P}$$
$$\mathbf{C} = \mathbf{R}^{-1}\mathbf{C}' - \mathbf{P}$$
$$\mathbf{D} = \mathbf{R}^{-1}\mathbf{D}' - \mathbf{P}$$
$$\text{where} \quad (2.3.2)$$
$$\mathbf{R}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \text{ and}$$
$$\mathbf{P} = -\mathbf{A}.$$

Hence proved.

2.4 SOLUTION TO PROBLEM

Diagonal(Direction Vector) $\mathbf{AC} = \mathbf{C} - \mathbf{A}$ is given by

$$\mathbf{AC} = \begin{pmatrix} 0 - 0 \\ 3 - (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}. \quad (2.4.1)$$

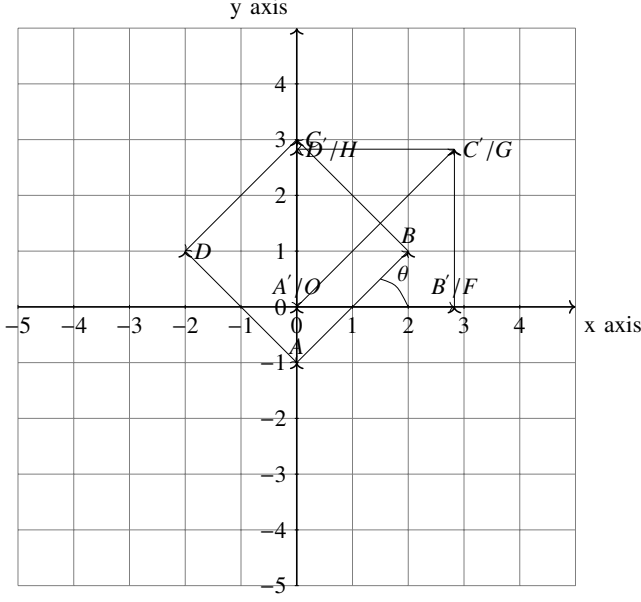Fig. 5: SQUARE ABCD and OFGH/A'B'C'D'

The line equation of $\mathbf{A}'\mathbf{C}'$ is x-y=0 or simply x=y.

$$\|\mathbf{A}'\mathbf{B}'\| = \|\mathbf{A}'\mathbf{D}'\|\frac{\|\mathbf{A}'\mathbf{C}'\|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

$$\text{Also, } \mathbf{F} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{A}'\mathbf{B}' = \|\mathbf{A}'\mathbf{B}'\|\mathbf{F} = 2\sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}. \qquad (2.4.5)$$

$$\mathbf{B}' = \mathbf{A}'\mathbf{B}' + \mathbf{A}' = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$$

$$\text{Also, } \mathbf{H} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A}'\mathbf{D}' = \|\mathbf{A}'\mathbf{D}'\|\mathbf{H} = 2\sqrt{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}. \qquad (2.4.6)$$

$$\mathbf{D}' = \mathbf{A}'\mathbf{D}' + \mathbf{A}' = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}$$

Hence the obtained vertices are

$$\mathbf{A}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B}' = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} \mathbf{C}' = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}. \mathbf{D}' = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}.$$
$$(2.4.7)$$

Translation for $\mathbf{A}$ to $\mathbf{O}$ requires a translation vector $\mathbf{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\Rightarrow \mathbf{A}' = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{C}' = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}. \qquad (2.4.2)$$

$$\mathbf{A}'\mathbf{C}' = \begin{pmatrix} 0-0 \\ 4-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Norm of $\mathbf{A}'\mathbf{C}'$

$$\|\mathbf{A}'\mathbf{C}'\| = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4 \qquad (2.4.3)$$

The rotation is by 45° clock wise. The rotation matrix is given by

$$\begin{pmatrix} \cos(-45^o) & -\sin(-45^o) \\ \sin(-45^o) & \cos(-45^o) \end{pmatrix}$$

$$\Rightarrow \mathbf{A}'\mathbf{C}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} \qquad (2.4.4)$$

$$\mathbf{C}' = \mathbf{A}'\mathbf{C}' + \mathbf{A}' = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$$

2.5 USING REVERSE AFFINE TRANSFORMATION

The rotation is by 45° counter clock wise i.e. $\theta = 45^o$. The inverse of rotation matrix $\mathbf{R}$ is given by

$$\mathbf{R^{-1}} = \frac{1}{1}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad (2.5.1)$$

and the translation vector is $-\mathbf{P}$. From above reverse affine transformation rules,

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \qquad (2.5.2)$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

2.6 LINE EQUATIONS

Coordinates are

$$A = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, D = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \quad (2.6.1)$$

In Matrix form, if $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$ & $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$ are the points given then the directional vector is given by

$$\begin{pmatrix} X_2 - X_1 \\ Y_2 - Y_1 \end{pmatrix}$$

then, the form of equation $ax+by+c=0$ can be written as

$$\begin{aligned} a &= Y_2 - Y_1, \\ b &= -(X_2 - X_1), \\ c &= \begin{pmatrix} Y_2 & -X_2 \end{pmatrix} \begin{pmatrix} X_2 - X_1 \\ Y_2 - Y_1 \end{pmatrix} \end{aligned} \quad (2.6.2)$$

By using the same, the line equation **AB** for points $A = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is as follows:

$$AB = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$a = 2, \ b = 2, \quad (2.6.3)$$

$$c = \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2$$

Line equation for **AB**

$$\Rightarrow \ 2x + 2y + 2 = 0 \ or \ x + y = -1. \quad (2.6.4)$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} X = -1 \quad (2.6.5)$$

The line equation **BC** for points $B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is as follows:

$$BC = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$a = 2, \ b = -2, \quad (2.6.6)$$

$$c = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 6$$

Line equation for **BC**

$$\Rightarrow \ 2x - 2y + 6 = 0 \ or \ x - y = -3. \quad (2.6.7)$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} X = -3 \quad (2.6.8)$$

The line equation **CD** for points $C = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, D = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is as follows:

$$CD = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$a = -2, \ b = -2, \quad (2.6.9)$$

$$c = \begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 6$$

Line equation for **CD**

$$\Rightarrow \ -2x - 2y + 6 = 0 \ or \ x + y = 3. \quad (2.6.10)$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} X = 3 \quad (2.6.11)$$

The line equation **DA** for points $D = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, A = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is as follows:

$$DA = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

$$a = -2, \ b = 2, \quad (2.6.12)$$

$$c = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} = 2$$

Line equation for **DA**

$$\Rightarrow \ -2x + 2y + 2 = 0 \ or \ x - y = 1. \quad (2.6.13)$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} X = 1 \quad (2.6.14)$$
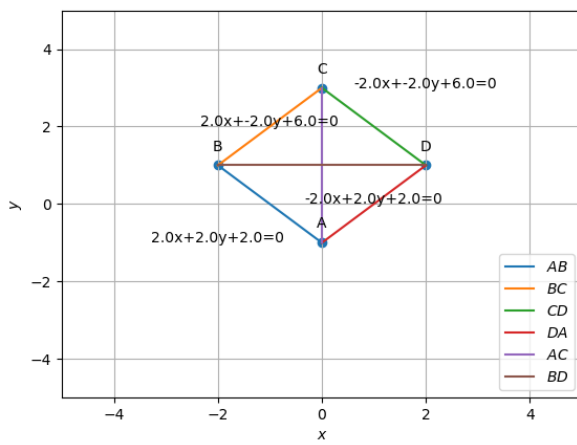
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The plotted graph is shown as below.

Fig. 5: Square ABCD