SM5083 - BASICS OF PROGRAMMING

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Contents

***Abstract*—This paper contains solution to problem no 5 of Examples III Section of Chapter III of Analytical Geometry by Hukum Chand. Links to Python codes are available below.**

Download python codes at

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https://github.com/rsgirishkumar/SM5083/
    ASSIGNMENT2
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1 Problem

The opposite vertices of a square are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. Find the equations of four sides.

2 Solution

Let the given points are indicated as below

$$\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \tag{2.0.1}$$

Let the unknown vertices are indicated as $\mathbf{B}, \mathbf{D}$.

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2.1 FINDING VERTICES USING AFFINE TRANSFORMATION

Lemma. If **A**and**C** are two opposite vertices of a Square ABCD, then the other two vertices are given by

$$\begin{aligned} \mathbf{B} &= \mathbf{R^{-1}B'} - \mathbf{P} \\ \mathbf{D} &= \mathbf{R^{-1}D'} - \mathbf{P} \\ &\text{where} \\ \mathbf{R^{-1}} &= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \text{ and} \\ \mathbf{P} &= -\mathbf{A}. \end{aligned} \tag{2.1.1}$$

Proof:
Lets consider a square **OFGH** with origin(**O**) as one of the vertices and x, y - axes are two sides. Let the norm of the direction vectors of sides be d.The angle between **F**(side) and **G**(diagonal) is $\theta = 45^o$. and

$$\begin{aligned} \mathbf{F} &= \frac{(\mathbf{G})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\|(\mathbf{G})^T\|} \\ \implies \mathbf{G} &= \sqrt{2}(\mathbf{F}) \end{aligned} \tag{2.1.2}$$

The norm of diagonal direction vector is

$$\|\mathbf{OG}\| = \sqrt{2}d = \|\mathbf{FH}\|. \tag{2.1.3}$$

The direction vectors are given by

$$\begin{aligned} \mathbf{F} &= \begin{pmatrix} d \\ 0 \end{pmatrix}, \mathbf{G} - \mathbf{F} = \begin{pmatrix} 0 \\ d \end{pmatrix}, \\ \mathbf{H} - \mathbf{G} &= \begin{pmatrix} -d \\ 0 \end{pmatrix}, -\mathbf{H} = \begin{pmatrix} 0 \\ -d \end{pmatrix} \\ \mathbf{G} &= \begin{pmatrix} d \\ d \end{pmatrix}. \\ \mathbf{H} - \mathbf{F} &= \begin{pmatrix} -d \\ d \end{pmatrix}. \end{aligned} \tag{2.1.4}$$

Let the Square **ABCD** has two vertices **A** and **C** . By using inspection method, Affine transformation has to be applied for **B** and **D**.The step by step procedure involves

1) Apply affine transformation steps i.e Translation and Rotation respectively of the given vertices $\mathbf{A}$ and $\mathbf{C}$ to shift square to the **OFGH**
2) Find the translation vector $\mathbf{P}$ and rotation matrix $\mathbf{R}$.
3) Find the diagonal/Direction vector and norm of $\mathbf{A}'\mathbf{C}'$.
4) Find the side direction vector and using norm and inspection method find the vertices $\mathbf{B}'$ and $\mathbf{D}'$.
5) Find the points $\mathbf{B}$ and $\mathbf{D}$ by using inspection method and reverse affine transformation i.e rotation and translation respectively.

Translation and Rotation Let $\mathbf{P}$ be translation vector is given by

$$\begin{aligned} \mathbf{P} &= -\mathbf{A} \\ \cos\theta &= \frac{(\mathbf{C}-\mathbf{A})^T \ \mathbf{OG}}{\|(\mathbf{C}-\mathbf{A})^T\|} \end{aligned} \tag{2.1.5}$$

2.2 For Rotation Matrix R

Consider a vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the norm is L. If vector is rotated anticlockwise around the origin by β degrees then the new vector be $\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}$. Call the angle between $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the x-axis as α. Then

$$\begin{aligned} x_1 &= L \ \cos\alpha \\ y_1 &= L \ \sin\alpha \end{aligned} \tag{2.2.1}$$

So the angle between $\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}$ and the x-axis is $\alpha+\beta$.

$$\begin{aligned} x_1' &= L \ \cos(\alpha+\beta) = \\ L \ \cos\alpha\cos\beta &- L \ \sin\alpha\sin\beta \\ &= x_1\cos\beta - y_1\sin\beta. \\ y_1' &= L \ \sin(\alpha+\beta) = \\ L\sin\alpha\cos\beta &+ L \ \cos\alpha\sin\beta \\ &= y_1\cos\beta + x_1\sin\beta. \\ \implies \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} &= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \end{aligned} \tag{2.2.2}$$

Here the rotation angle is θ. Hence the rotation matrix is given by

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.2.3}$$

The generalized affine transformation principles are as below

$$\begin{aligned} \mathbf{A}' &= \mathbf{R}(\mathbf{A}+\mathbf{P}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \mathbf{C}' &= \mathbf{R}(\mathbf{C}+\mathbf{P}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix}. \\ \mathbf{B}' &= \mathbf{A}'\mathbf{B}' + \mathbf{A}' \\ \text{where } \mathbf{A}'\mathbf{B}' &= \|\mathbf{A}'\mathbf{B}'\|(\mathbf{F}). \\ \mathbf{D}' &= \mathbf{A}'\mathbf{D}' + \mathbf{A}' \\ \text{where } \mathbf{A}'\mathbf{D}' &= \|\mathbf{A}'\mathbf{D}'\|(\mathbf{H}). \end{aligned} \tag{2.2.4}$$

2.3 Reverse Affine Transformation

By doing the reverse affine transformation, the Square **ABCD** can be obtained from $\mathbf{A}'\mathbf{B}'\mathbf{C}'\mathbf{D}'$ by using the below transformation rules.Given

$$\begin{aligned} \mathbf{A}' &= \mathbf{R}(\mathbf{A}+\mathbf{P}) \\ \implies \mathbf{A}'\mathbf{R}^{-1} &= \mathbf{A}+\mathbf{P} \\ \implies (\mathbf{A}'\mathbf{R}^{-1}) - \mathbf{P} &= \mathbf{A} \\ \text{Hence } \mathbf{A} &= \mathbf{R^{-1}}\mathbf{A}' - \mathbf{P} \end{aligned} \tag{2.3.1}$$

The generalization principle can be applied to other vertices also as below

$$\begin{aligned} \mathbf{B} &= \mathbf{R^{-1}}\mathbf{B}' - \mathbf{P} \\ \mathbf{C} &= \mathbf{R^{-1}}\mathbf{C}' - \mathbf{P} \\ \mathbf{D} &= \mathbf{R^{-1}}\mathbf{D}' - \mathbf{P} \\ &\text{where} \\ \mathbf{R^{-1}} &= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \ and \\ \mathbf{P} &= -\mathbf{A}. \end{aligned} \tag{2.3.2}$$

Hence proved.

2.4 SOLUTION TO PROBLEM

Diagonal(Direction Vector) $\mathbf{AC} = \mathbf{C} - \mathbf{A}$ is given by

$$\mathbf{AC} = \begin{pmatrix} 0-0 \\ 3-(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}. \tag{2.4.1}$$

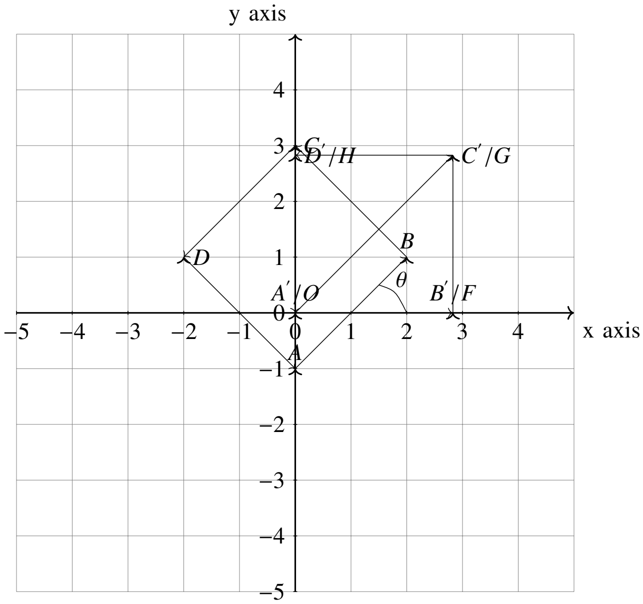

Fig. 5: SQUARE ABCD and OFGH/A'B'C'D'

Translation for $\mathbf{A}$ to $\mathbf{O}$ requires a translation vector $\mathbf{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\Rightarrow \mathbf{A}' = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{C}' = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}. \tag{2.4.2}$$

$$\mathbf{A}'\mathbf{C}' = \begin{pmatrix} 0-0 \\ 4-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Norm of $\mathbf{A}'\mathbf{C}'$

$$\|\mathbf{A}'\mathbf{C}'\| = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4 \tag{2.4.3}$$

The rotation is by 45^o clock wise. The rotation matrix is given by

$$\begin{pmatrix} \cos(-45^o) & -\sin(-45^o) \\ \sin(-45^o) & \cos(-45^o) \end{pmatrix}$$

$$\Rightarrow \mathbf{A}'\mathbf{C}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} \tag{2.4.4}$$

$$\mathbf{C}' = \mathbf{A}'\mathbf{C}' + \mathbf{A}' = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$$

The line equation of $\mathbf{A}'\mathbf{C}'$ is x-y=0 or simply x=y.

$$\|\mathbf{A}'\mathbf{B}'\| = \|\mathbf{A}'\mathbf{D}'\| = \frac{\|\mathbf{A}'\mathbf{C}'\|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

$$\text{Also, } \mathbf{F} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{A}'\mathbf{B}' = \|\mathbf{A}'\mathbf{B}'\|\mathbf{F} = 2\sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}. \tag{2.4.5}$$

$$\mathbf{B}' = \mathbf{A}'\mathbf{B}' + \mathbf{A}' = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$$

$$\text{Also,} \mathbf{H} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A}'\mathbf{D}' = \|\mathbf{A}'\mathbf{D}'\|\mathbf{H} = 2\sqrt{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}. \tag{2.4.6}$$

$$\mathbf{D}' = \mathbf{A}'\mathbf{D}' + \mathbf{A}' = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}$$

Hence the obtained vertices are

$$\mathbf{A}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B}' = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} \mathbf{C}' = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} . \mathbf{D}' = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}. \tag{2.4.7}$$

2.5 USING REVERSE AFFINE TRANSFORMATION

The rotation is by 45^o counter clock wise i.e. $\theta = 45^o$. The inverse of rotation matrix $\mathbf{R}$ is given by

$$\mathbf{R^{-1}} = \frac{1}{1}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.5.1}$$

and the translation vector is $-\mathbf{P}$. From above reverse affine transformation rules,

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \tag{2.5.2}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

2.6 LINE EQUATIONS

Coordinates are

$$\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{2.6.1}$$

In Matrix form, if $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$ & $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$ are the points given then the directional vector is given by
$\begin{pmatrix} X_2 - X_1 \\ Y_2 - Y_1 \end{pmatrix}$
then, the form of equation ax+by+c=0 can be written as

$$\begin{aligned} a &= \mathbf{Y_1} - \mathbf{Y_2}, \\ b &= \mathbf{X_2} - \mathbf{X_1}, \\ c &= \begin{pmatrix} -\mathbf{Y_1} & \mathbf{X_1} \end{pmatrix} \begin{pmatrix} \mathbf{X_2} - \mathbf{X_1} \\ \mathbf{Y_2} - \mathbf{Y_1} \end{pmatrix} \end{aligned} \tag{2.6.2}$$

By using the same, the line equation **AB** for points $\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is as follows:

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$a = -2, \ b = 2, \tag{2.6.3}$$

$$c = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2$$

Line equation for **AB**

$$\Rightarrow \ -2x + 2y - 2 = 0 \ or \ x - y = 1. \tag{2.6.4}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \ \mathbf{X} = 1 \tag{2.6.5}$$

The line equation **BC** for points $\mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is as follows:

$$\mathbf{BC} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$a = -2, \ b = -2, \tag{2.6.6}$$

$$c = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 6$$

Line equation for **BC**

$$\Rightarrow \ -2x - 2y + 6 = 0 \ or \ x + y = 3. \tag{2.6.7}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \ \mathbf{X} = 3 \tag{2.6.8}$$

The line equation **CD** for points $\mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is as follows:

$$\mathbf{CD} = \mathbf{D} - \mathbf{C} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

$$a = 2, \ b = -2, \tag{2.6.9}$$

$$c = \begin{pmatrix} -3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} = 6$$

Line equation for **CD**

$$\Rightarrow \ 2x - 2y + 6 = 0 \ or \ x - y = -3. \tag{2.6.10}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \ \mathbf{X} = -3 \tag{2.6.11}$$

The line equation **DA** for points $\mathbf{D} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is as follows:

$$\mathbf{DA} = \mathbf{A} - \mathbf{D} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$a = 2, \ b = 2, \tag{2.6.12}$$

$$c = \begin{pmatrix} -1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2$$

Line equation for **DA**

$$\Rightarrow \ 2x + 2y + 2 = 0 \ or \ x + y = -1. \tag{2.6.13}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \ \mathbf{X} = -1 \tag{2.6.14}$$

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The plotted graph is shown as below.

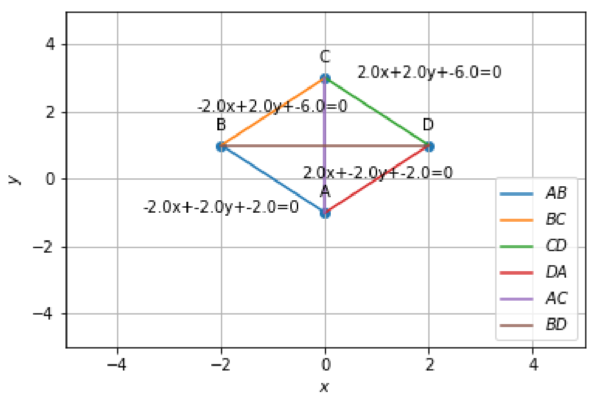

Fig. 5: Square ABCD