SM5083 - BASICS OF PROGRAMMING

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Contents

***Abstract*—This paper contains solution to problem no 5 of Examples III Section of Chapter III of Analytical Geometry by Hukum Chand. Links to Python codes are available below.**

Download python codes at

https://github.com/rsgirishkumar/SM5083/ASSIGNMENT2

1 Problem

The opposite vertices of a square are $\begin{pmatrix}0\\-1\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}$. Find the equations of four sides.

2 Solution

Let the given points are indicated as below

$$\mathbf{A}=\begin{pmatrix}0\\-1\end{pmatrix},\mathbf{C}=\begin{pmatrix}0\\3\end{pmatrix}. \tag{2.0.1}$$

Let the unknown vertices are indicated as $\mathbf{B},\mathbf{D}$.

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2.1 FINDING VERTICES USING AFFINE TRANSFORMATION

Lemma. If $\mathbf{A}=\begin{pmatrix}x_1\\y_1\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}x_2\\y_2\end{pmatrix}$ are two opposite vertices of a Square ABCD, then the other two vertices B and D can be found by using affine transformation principles(translation and rotation).

Proof:

Lets consider a square with origin($\mathbf{O}$) as a vertex and x, y - axes are two sides. Let the other vertices be $\mathbf{F},\mathbf{G},\mathbf{H}$. The direction vectors of sides are given by $\mathbf{OF},\mathbf{FG},\mathbf{GH},\mathbf{HO}$ and the diagonal direction vectors are $\mathbf{OG},\mathbf{FH}$. Let the norm of the direction vectors of sides be d. The direction vectors are given by

$$\mathbf{OF}=\begin{pmatrix}d\\0\end{pmatrix},\mathbf{FG}=\begin{pmatrix}0\\d\end{pmatrix},\mathbf{GH}=\begin{pmatrix}-d\\0\end{pmatrix},\mathbf{HO}=\begin{pmatrix}0\\-d\end{pmatrix}$$

$$\mathbf{OG}=\mathbf{OF}+\mathbf{FG}=\begin{pmatrix}d\\d\end{pmatrix}.$$

$$\mathbf{FH}=\mathbf{FO}+\mathbf{OH}=\begin{pmatrix}-d\\d\end{pmatrix}.$$

$$\|\mathbf{OF}\|=\|\mathbf{OG}\|=\|\mathbf{OH}\|=\|\mathbf{FG}\|=d \tag{2.1.1}$$

The angle between $\mathbf{OF}$(side) and $\mathbf{OG}$(diagonal) is $\theta=45^o$. and

$$\begin{aligned}\mathbf{OF}&=\mathbf{OG}cos(\theta).\\ \implies \mathbf{OG}&=\sqrt{2}.\mathbf{OF}\end{aligned} \tag{2.1.2}$$

The norm of diagonal direction vector is

$$\|\mathbf{OG}\|=\sqrt{2}d=\|\mathbf{FH}\|. \tag{2.1.3}$$

Let the Square that has to be formed with the vertices $\mathbf{A}$ and $\mathbf{C}$ denoted by $\mathbf{ABCD}$. To obtain $\mathbf{ABCD}$ from $\mathbf{OFGH}$ or viceversa, Affine transformation has to be applied.This eases out the process of finding the vertices.The step by step procedure involves

1) Apply affine transformation steps i.e Translation and Rotation respectively of the given vertices $\mathbf{A}$ and $\mathbf{C}$ to shift square to the $\mathbf{OFGH}$ for the ease of solution.

2) Find the translation vector **P** and rotation matrix **R**.
3) Find the diagonal/Direction vector and norm of $\mathbf{A}'\mathbf{C}'$.
4) Find the side direction vector and using norm and inspection method find the vertices $\mathbf{B}'$ and $\mathbf{D}'$.
5) Find the points **B** and **D** by using inspection method and reverse affine transformation i.e rotation and translation respectively.

Translation and Rotation

Let **P** be translation vector is given by

$$\mathbf{P} = \mathbf{O} - \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -y_1 \end{pmatrix} \tag{2.1.4}$$

and the angle to be rotated be θ for **AC** to align with **OG**, then the angle θ and rotation matrix **R** is given by

$$\begin{aligned} cos(\theta) &= \frac{(\mathbf{C} - \mathbf{A})^T.1}{\|(\mathbf{C} - \mathbf{A})^T\|} \\ \mathbf{C} - \mathbf{A} &= \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \\ \|\mathbf{C} - \mathbf{A}\| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned} \tag{2.1.5}$$

If a vector has to be rotated then it may be decreased or increased in length. Let's say we have a vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.The norm of the vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ has length L. We rotate this vector anticlockwise around the origin by β degrees. The rotated vector has coordinates $\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}$.The rotated vector must also have length L. Call the angle between $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the x-axis as α. Then

$$\begin{aligned} x_1 &= Lcos(\alpha) \\ y_1 &= Lsin(\alpha) \end{aligned} \tag{2.1.6}$$

We rotate $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ by angle β to get $\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}$. So the angle between $\begin{pmatrix} x_1' \\ y_1' \end{pmatrix}$ and the x-axis is $\alpha + \beta$.

$$\begin{aligned} x_1' &= Lcos(\alpha + \beta) = \\ & Lcos(\alpha)cos(\beta) - Lsin(\alpha)sin(\beta) \\ &= x_1 cos(\beta) - y_1 sin(\beta) \\ y_1' &= Lsin(\alpha + \beta) = \\ & Lsin(\alpha)cos(\beta) + Lcos(\alpha)sin(\beta) \\ &= y_1 cos(\beta) + x_1 sin(\beta) \\ \Rightarrow \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} &= \begin{pmatrix} cos(\beta) & -sin(\beta) \\ sin(\beta) & cos(\beta) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \end{aligned} \tag{2.1.7}$$

Here the rotation angle is θ. Hence the rotation matrix is given by

$$\mathbf{R} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \tag{2.1.8}$$

The points can be obtained by using the generalized affine transformation principle as below

$$\begin{aligned} \mathbf{A}' &= \mathbf{R}(\mathbf{A} + \mathbf{P}) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{O} \\ \mathbf{C}' &= \mathbf{R}(\mathbf{C} + \mathbf{P}) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix} \end{aligned} \tag{2.1.9}$$

The length of $\mathbf{A}'\mathbf{B}'$ based on projection on to $\mathbf{A}'\mathbf{C}'$ and the projection of $\mathbf{A}'\mathbf{C}'$ on to $\mathbf{A}'\mathbf{B}'$ is given by

$$\begin{aligned} \|\mathbf{A}'\mathbf{B}'\| &= \|\mathbf{A}'\mathbf{C}'\| cos(\theta) \\ \mathbf{A}'\mathbf{B}' &= \|\mathbf{A}'\mathbf{B}'\| \mathbf{OF} \\ \mathbf{B}' &= \mathbf{A}'\mathbf{B}' + \mathbf{A}' \end{aligned} \tag{2.1.10}$$

In the similar way,

$$\begin{aligned} \|\mathbf{A}'\mathbf{D}'\| &= \|\mathbf{A}'\mathbf{C}'\| cos(\theta) \\ \mathbf{A}'\mathbf{D}' &= \|\mathbf{A}'\mathbf{D}'\| \mathbf{OH} \\ \mathbf{D}' &= \mathbf{A}'\mathbf{D}' + \mathbf{A}' \end{aligned} \tag{2.1.11}$$

By doing the reverse affine transformation, the Square **ABCD** can be obtained from $\mathbf{A}'\mathbf{B}'\mathbf{C}'\mathbf{D}'$ by using the below transformation rules.Given

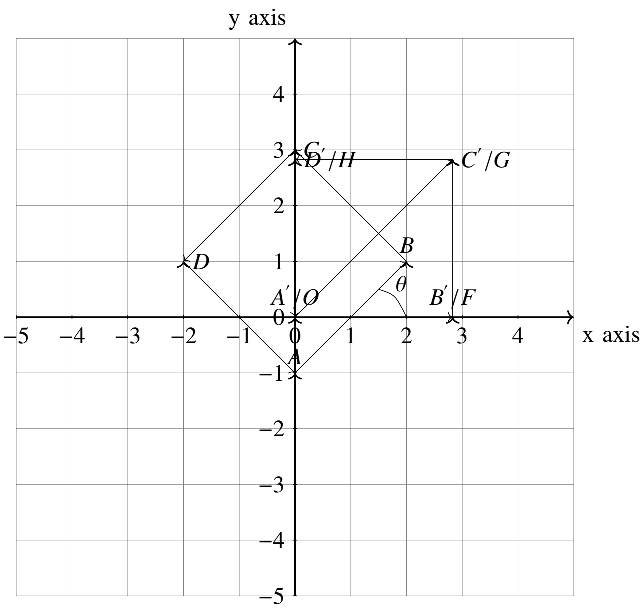

Fig. 5: SQUARE ABCD and OFGH/A'B'C'D'

$$
\begin{aligned}
\mathbf{A}' &= \mathbf{R}(\mathbf{A}+\mathbf{P})\\
\Rightarrow \frac{\mathbf{A}'}{\mathbf{R}} &= \mathbf{A}+\mathbf{P}\\
\Rightarrow \mathbf{A}'\mathbf{R}^{-1} &= \mathbf{A}+\mathbf{P}\\
\Rightarrow (\mathbf{A}'\mathbf{R}^{-1})-\mathbf{P} &= \mathbf{A}
\end{aligned}
\tag{2.1.12}
$$

Hence

$$
\mathbf{A} = \mathbf{R^{-1}}\mathbf{A}' - \mathbf{P} \tag{2.1.13}
$$

In the similar way, since it is a square and all vertices make same angle of rotation, the generalization principle can be applied and it gives

$$
\begin{aligned}
\mathbf{B} &= \mathbf{R^{-1}}\mathbf{B}' - \mathbf{P}\\
\mathbf{C} &= \mathbf{R^{-1}}\mathbf{C}' - \mathbf{P}\\
\mathbf{D} &= \mathbf{R^{-1}}\mathbf{D}' - \mathbf{P}\\
&\text{where}\\
\mathbf{R^{-1}} = \begin{pmatrix} cos(-\theta) & -sin(-\theta)\\ sin(-\theta) & cos(-\theta)\end{pmatrix} &\ and\\
\mathbf{P} &= \mathbf{O}-\mathbf{A}.
\end{aligned}
\tag{2.1.14}
$$

Hence proved.

2.2 *SOLUTION TO PROBLEM*

Diagonal(Direction Vector) $\mathbf{AC} = \mathbf{C} - \mathbf{A}$ is given by

$$
\mathbf{AC} = \begin{pmatrix} 0-0\\ 3-(-1)\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}. \tag{2.2.1}
$$

Translation for $\mathbf{A}$ to $\mathbf{O}$ requries a translation vector $\mathbf{P} = \begin{pmatrix}0\\1\end{pmatrix}$.

$$
\begin{aligned}
\Rightarrow \mathbf{A}' &= \begin{pmatrix}0\\-1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\\
\mathbf{C}' &= \begin{pmatrix}0\\3\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}.\\
\mathbf{A}'\mathbf{C}' &= \begin{pmatrix}0-0\\4-0\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}.
\end{aligned}
\tag{2.2.2}
$$

Norm of $\mathbf{A}'\mathbf{C}'$

$$
\|\mathbf{A}'\mathbf{C}'\| = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4 \tag{2.2.3}
$$

The rotation is by 45^o clock wise. The rotation matrix is given by

$$
\begin{aligned}
&\begin{pmatrix} cos(-45^o) & -sin(-45^o)\\ sin(-45^o) & cos(-45^o)\end{pmatrix}\\
\Rightarrow \mathbf{A}'\mathbf{C}' &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\4\end{pmatrix} = \begin{pmatrix}2\sqrt{2}\\2\sqrt{2}\end{pmatrix}\\
\mathbf{C}' &= \mathbf{A}'\mathbf{C}' + \mathbf{A}' = \begin{pmatrix}2\sqrt{2}\\2\sqrt{2}\end{pmatrix}
\end{aligned}
\tag{2.2.4}
$$

The line equation of $\mathbf{A}'\mathbf{C}'$ is x-y=0 or simply x=y. By inspection method, using length of a side, the vertices $\mathbf{B}'$ and $\mathbf{D}'$ can be found.

$$
\begin{aligned}
\|\mathbf{A}'\mathbf{B}'\| &= \frac{\|\mathbf{A}'\mathbf{C}'\|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.\\
&Also, \mathbf{OF} = \begin{pmatrix}1\\0\end{pmatrix}\\
\mathbf{A}'\mathbf{B}' &= \|\mathbf{A}'\mathbf{B}'\|\mathbf{OF} = 2\sqrt{2} * \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\sqrt{2}\\0\end{pmatrix}.\\
\mathbf{B}' &= \mathbf{A}'\mathbf{B}' + \mathbf{A}' = \begin{pmatrix}2\sqrt{2}\\0\end{pmatrix}
\end{aligned}
\tag{2.2.5}
$$

$$\|\mathbf{A}'\mathbf{D}'\| = \frac{\|\mathbf{A}'\mathbf{C}'\|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

$$\begin{aligned} Also, \mathbf{OH} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \mathbf{A}'\mathbf{D}' = \|\mathbf{A}'\mathbf{D}'\|\mathbf{OH} &= 2\sqrt{2} * \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}. \\ \mathbf{D}' = \mathbf{A}'\mathbf{D}' + \mathbf{A}' &= \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} \end{aligned} \tag{2.2.6}$$

Hence the obtained vertices are

$$\mathbf{A}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B}' = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} \mathbf{C}' = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} . \mathbf{D}' = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}. \tag{2.2.7}$$

2.3 USING REVERSE AFFINE TRANSFORMATION

The rotation is by 45^o counter clock wise i.e. $\theta = 45^o$. The inverse of rotation matrix $\mathbf{R}$ is given by

$$\mathbf{R^{-1}} = \frac{1}{1}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.3.1}$$

and the translation vector is $-\mathbf{P}$. From above reverse affine transformation rules,

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \mathbf{B} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \mathbf{C} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \\ \mathbf{D} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \end{aligned} \tag{2.3.2}$$

2.4 LINE EQUATIONS

Coordinates are

$$\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{2.4.1}$$

In Matrix form, if $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$ & $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$ are the points given then the directional vector is given by $\begin{pmatrix} X_2 - X_1 \\ Y_2 - Y_1 \end{pmatrix}$ then, the form of equation ax+by+c=0 can be written as

$$\begin{aligned} a &= \mathbf{Y_2} - \mathbf{Y_1}, \\ b &= -(\mathbf{X_2} - \mathbf{X_1}), \\ c &= \begin{pmatrix} \mathbf{Y_2} & -\mathbf{X_2} \end{pmatrix}\begin{pmatrix} \mathbf{X_2} - \mathbf{X_1} \\ \mathbf{Y_2} - \mathbf{Y_1} \end{pmatrix} \end{aligned} \tag{2.4.2}$$

By using the same, the line equation $\mathbf{AB}$ for points $\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -2 \\ 2 \end{pmatrix} \\ a &= 2,\ b = 2, \\ c &= \begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2 \end{aligned} \tag{2.4.3}$$

Line equation for $\mathbf{AB}$

$$\Rightarrow\ 2x + 2y + 2 = 0 \ or\ x + y = -1. \tag{2.4.4}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = -1 \tag{2.4.5}$$

The line equation $\mathbf{BC}$ for points $\mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{BC} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ a &= 2,\ b = -2, \\ c &= \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 6 \end{aligned} \tag{2.4.6}$$

Line equation for $\mathbf{BC}$

$$\Rightarrow\ 2x - 2y + 6 = 0 \ or\ x - y = -3. \tag{2.4.7}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{X} = -3 \tag{2.4.8}$$

The line equation $\mathbf{CD}$ for points $\mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is as follows:

$$\mathbf{CD} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$
$$a = -2,\ b = -2, \tag{2.4.9}$$
$$c = \begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 6$$

Line equation for **CD**

$$\implies -2x - 2y + 6 = 0 \ or \ x + y = 3. \tag{2.4.10}$$

In vector form

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = 3 \tag{2.4.11}$$

The line equation **DA** for points $\mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is as follows:

$$\mathbf{DA} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$
$$a = -2,\ b = 2, \tag{2.4.12}$$
$$c = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} = 2$$

Line equation for **DA**

$$\implies -2x + 2y + 2 = 0 \ or \ x - y = 1. \tag{2.4.13}$$

In vector form

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{X} = 1 \tag{2.4.14}$$

.
The plotted graph is shown as below.

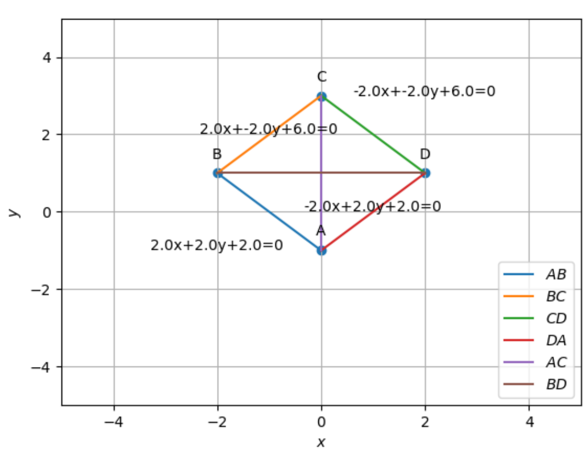

Fig. 5: Square ABCD