SM5083 - BASICS OF PROGRAMMING

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CONTENTS

Abstract—**This paper contains solution to problem no 5 of Examples III Section of Chapter III of Analytical Geometry by Hukum Chand. Links to Python codes are available below.**

Download python codes at

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https://github.com/rsgirishkumar/SM5083/
    ASSIGNMENT2
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1 PROBLEM

The opposite vertices of a square are $\begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$. Find the equations of four sides.

2 SOLUTION

Let the given points are indicated as below

$$\mathbf{A} = \begin{pmatrix}0\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}0\\3\end{pmatrix}. \tag{2.0.1}$$

Let the unknown vertices are indicated as $\mathbf{B}, \mathbf{D}$. The step by step procedure involves

1) By doing affine transformation steps i.e Translation and Rotation of the given vertices $\mathbf{A}$ and $\mathbf{C}$ to origin, those provide $\mathbf{A}'$ and $\mathbf{C}'$ for the ease of solution
2) Find the diagonal/Direction vector $\mathbf{AC}$.
3) Find the norm of $\mathbf{AC}$.
4) Find the points $\mathbf{B}$ and $\mathbf{D}$ by using inspection method and reverse affine transformation.
5) Form the equations of lines using vertices.

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2.1 USING AFFINE TRANSFORMATION

Translation and Rotation Lets consider a square with origin($\mathbf{O}$) as a vertex and x, y - axes are two sides. Let the other vertices be $\mathbf{F}, \mathbf{G}, \mathbf{H}$. For the Square that has to be formed with the vertices given be denoted by $\mathbf{ABCD}$. To obtain $\mathbf{ABCD}$ from $\mathbf{OFGH}$ or viceversa, Affine transformation has to be applied.This eases out the process of finding the vertices.

Let $\mathbf{P}$ be translation vector is given by $\mathbf{P} = \mathbf{O} - \mathbf{A}$, and the angle to be rotated be θ for $\mathbf{AC}$ to align with $\mathbf{OG}$, then the angle θ and rotation matrix $\mathbf{R}$ is given by

$$\begin{aligned} cos(\theta) &= \frac{(\mathbf{C}-\mathbf{A})^T.1}{\|(\mathbf{C}-\mathbf{A})^T\|} \\ \mathbf{R} &= \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \end{aligned} \tag{2.1.1}$$

The points can be obtained by using the generalized affine transformation principle as below

$$\begin{aligned} \mathbf{A}' &= \mathbf{R}(\mathbf{A}+\mathbf{P}) \\ \mathbf{C}' &= \mathbf{R}(\mathbf{C}+\mathbf{P}) \end{aligned} \tag{2.1.2}$$

In general $\mathbf{A}' = \mathbf{O}$.
The length of $\mathbf{A}'\mathbf{B}'$ based on projection on to $\mathbf{A}'\mathbf{C}'$ and the projection of $\mathbf{A}'\mathbf{C}'$ on to $\mathbf{A}'\mathbf{B}'$ is given by

$$\begin{aligned} \|\mathbf{A}'\mathbf{B}'\| &= \frac{\|\mathbf{A}'\mathbf{C}'\|}{cos(\theta)} \\ \mathbf{A}'\mathbf{B}' &= \|\mathbf{A}'\mathbf{B}'\|\mathbf{OF} \\ \mathbf{B}' &= \mathbf{A}'\mathbf{B}' + \mathbf{A}' \end{aligned} \tag{2.1.3}$$

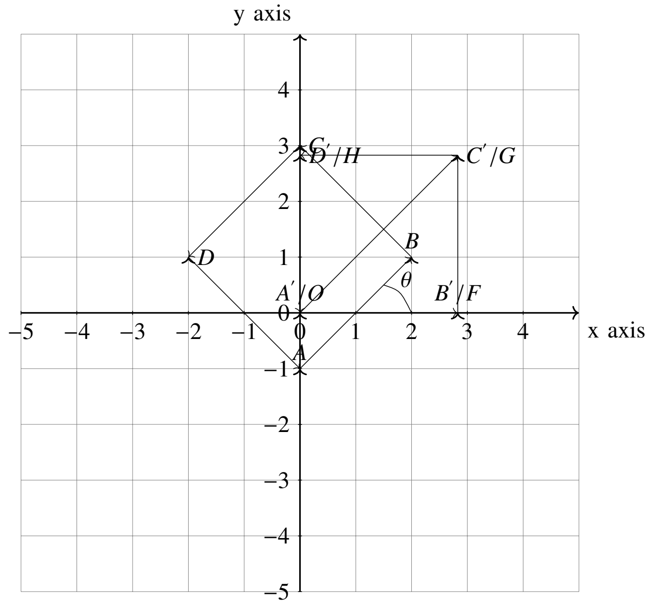

Fig. 5: SQUARE ABCD and OFGH/A'B'C'D'

In the similar way,

$$\begin{aligned}\|\mathbf{A'D'}\| &= \frac{\|\mathbf{A'C'}\|}{cos(\theta)}\\ \mathbf{A'D'} &= \|\mathbf{A'D'}\|\mathbf{OH}\\ \mathbf{D'} &= \mathbf{A'D'} + \mathbf{A'}\end{aligned} \tag{2.1.4}$$

By doing the reverse affine transformation, the Square **ABCD** can be obtained from $\mathbf{A'B'C'D'}$ by using the below transformation principles.

$$\begin{aligned}\mathbf{A} &= \mathbf{R^{-1}A'} - \mathbf{P}\\ \mathbf{B} &= \mathbf{R^{-1}B'} - \mathbf{P}\\ \mathbf{C} &= \mathbf{R^{-1}C'} - \mathbf{P}\\ \mathbf{D} &= \mathbf{R^{-1}D'} - \mathbf{P}\\ &\quad where\\ \mathbf{R^{-1}} &= \begin{pmatrix} cos(-\theta) & -sin(-\theta)\\ sin(-\theta) & cos(-\theta)\end{pmatrix} \; and\\ \mathbf{P} &= \mathbf{O} - \mathbf{A}.\end{aligned} \tag{2.1.5}$$

Diagonal(Direction Vector) $\mathbf{AC} = \mathbf{C} - \mathbf{A}$ is given by

$$\mathbf{AC} = \begin{pmatrix} 0-0\\ 3-(-1)\end{pmatrix} = \begin{pmatrix} 0\\ 4\end{pmatrix}. \tag{2.1.6}$$

Translation for **A** to **O** requries a translation vector $\mathbf{P} = \begin{pmatrix} 0\\ 1\end{pmatrix}$.

$$\begin{aligned}\Rightarrow \mathbf{A'} &= \begin{pmatrix} 0\\ -1\end{pmatrix} + \begin{pmatrix} 0\\ 1\end{pmatrix} = \begin{pmatrix} 0\\ 0\end{pmatrix}\\ \mathbf{C'} &= \begin{pmatrix} 0\\ 3\end{pmatrix} + \begin{pmatrix} 0\\ 1\end{pmatrix} = \begin{pmatrix} 0\\ 4\end{pmatrix}.\\ \mathbf{A'C'} &= \begin{pmatrix} 0-0\\ 4-0\end{pmatrix} = \begin{pmatrix} 0\\ 4\end{pmatrix}.\end{aligned} \tag{2.1.7}$$

Norm of $\mathbf{A'C'}$

$$\|\mathbf{A'C'}\| = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4 \tag{2.1.8}$$

The rotation is by 45^o clock wise. The rotation matrix is given by

$$\begin{aligned}&\begin{pmatrix} cos(-45^o) & -sin(-45^o)\\ sin(-45^o) & cos(-45^o)\end{pmatrix}\\ \Rightarrow \mathbf{A'C'} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix} 0\\ 4\end{pmatrix} = \begin{pmatrix} 2\sqrt{2}\\ 2\sqrt{2}\end{pmatrix}\\ \mathbf{C'} &= \mathbf{A'C'} + \mathbf{A'} = \begin{pmatrix} 2\sqrt{2}\\ 2\sqrt{2}\end{pmatrix}\end{aligned} \tag{2.1.9}$$

The line equation of $\mathbf{A'C'}$ is x-y=0 or simply x=y. By inspection method, using length of a side, the vertices $\mathbf{B'}$ and $\mathbf{D'}$ can be found.

$$\begin{aligned}\|\mathbf{A'B'}\| &= \frac{\|\mathbf{A'C'}\|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.\\ Also, \mathbf{OF} &= \begin{pmatrix} 1\\ 0\end{pmatrix}\\ \mathbf{A'B'} &= \|\mathbf{A'B'}\|\mathbf{OF} = 2\sqrt{2} * \begin{pmatrix} 1\\ 0\end{pmatrix} = \begin{pmatrix} 2\sqrt{2}\\ 0\end{pmatrix}.\\ \mathbf{B'} &= \mathbf{A'B'} + \mathbf{A'} = \begin{pmatrix} 2\sqrt{2}\\ 0\end{pmatrix}\end{aligned} \tag{2.1.10}$$

$$\begin{aligned}\|\mathbf{A'D'}\| &= \frac{\|\mathbf{A'C'}\|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.\\ Also, \mathbf{OH} &= \begin{pmatrix} 0\\ 1\end{pmatrix}\\ \mathbf{A'D'} &= \|\mathbf{A'D'}\|\mathbf{OH} = 2\sqrt{2} * \begin{pmatrix} 0\\ 1\end{pmatrix} = \begin{pmatrix} 0\\ 2\sqrt{2}\end{pmatrix}.\\ \mathbf{D'} &= \mathbf{A'D'} + \mathbf{A'} = \begin{pmatrix} 0\\ 2\sqrt{2}\end{pmatrix}\end{aligned} \tag{2.1.11}$$

Hence the obtained vertices are

$$\mathbf{A}^{'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B}^{'} = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} \mathbf{C}^{'} = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} . \mathbf{D}^{'} = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}. \tag{2.1.12}$$

2.2 USING REVERSE AFFINE TRANSFORMATION

The rotation is by 45^o counter clock wise i.e. $\theta = 45^o$. The inverse of rotation matrix $\mathbf{R}$ is given by

$$\mathbf{R^{-1}} = \frac{1}{1} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.2.1}$$

and the translation vector is $-\mathbf{P}$. From above reverse affine transformation principles,

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \mathbf{B} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \mathbf{C} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \\ \mathbf{D} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \end{aligned} \tag{2.2.2}$$

2.3 LINE EQUATIONS

Coordinates are

$$\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{2.3.1}$$

In Matrix form, if $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \& \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$ are the points given then the directional vector is given by $\begin{pmatrix} X_2 - X_1 \\ Y_2 - Y_1 \end{pmatrix}$ then, the form of equation ax+by+c=0 can be written as

$$\begin{aligned} a &= \mathbf{Y_2} - \mathbf{Y_1}, \\ b &= -(\mathbf{X_2} - \mathbf{X_1}), \\ c &= \begin{pmatrix} \mathbf{Y_2} & -\mathbf{X_2} \end{pmatrix} \begin{pmatrix} \mathbf{X_2} - \mathbf{X_1} \\ \mathbf{Y_2} - \mathbf{Y_1} \end{pmatrix} \end{aligned} \tag{2.3.2}$$

By using the same, the line equation $\mathbf{AB}$ for points $\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -2 \\ 2 \end{pmatrix} \\ a &= 2, \ b = 2, \\ c &= \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2 \end{aligned} \tag{2.3.3}$$

Line equation for $\mathbf{AB}$

$$\Rightarrow \ 2x + 2y + 2 = 0 \ or \ x + y = -1. \tag{2.3.4}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = -1 \tag{2.3.5}$$

The line equation $\mathbf{BC}$ for points $\mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{BC} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ a &= 2, \ b = -2, \\ c &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 6 \end{aligned} \tag{2.3.6}$$

Line equation for $\mathbf{BC}$

$$\Rightarrow \ 2x - 2y + 6 = 0 \ or \ x - y = -3. \tag{2.3.7}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{X} = -3 \tag{2.3.8}$$

The line equation $\mathbf{CD}$ for points $\mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{CD} &= \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ a &= -2, \ b = -2, \\ c &= \begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 6 \end{aligned} \tag{2.3.9}$$

Line equation for $\mathbf{CD}$

$$\Rightarrow \ -2x - 2y + 6 = 0 \ or \ x + y = 3. \tag{2.3.10}$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = 3 \tag{2.3.11}$$

The line equation **DA** for points $\mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is as follows:

$$\mathbf{DA} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$
$$a = -2, \; b = 2, \tag{2.3.12}$$
$$c = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} = 2$$

Line equation for **DA**

$$\implies -2x + 2y + 2 = 0 \; or \; x - y = 1. \tag{2.3.13}$$

In vector form

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{X} = 1 \tag{2.3.14}$$

.
The plotted graph is shown as below.

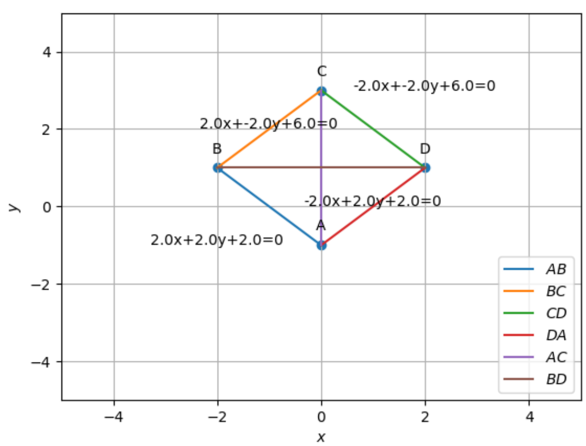

Fig. 5: Square ABCD