

SM5083 - BASICS OF PROGRAMMING

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CONTENTS

Abstract—This paper contains solution to problem no 5 of Examples III Section of Chapter III of Analytical Geometry by Hukum Chand. Links to Python codes are available below.

Download python codes at

<https://github.com/rsgirishkumar/SM5083/ASSIGNMENT2>

1 PROBLEM

The opposite vertices of a square are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.
Find the equations of four sides.

2 SOLUTION

Let the given points are indicated as below

$$\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}. \quad (2.0.1)$$

Let the unknown vertices are indicated as \mathbf{B}, \mathbf{D} .

2.1 FINDING VERTICES USING AFFINE TRANSFORMATION

Lemma. If \mathbf{A} and \mathbf{C} are two opposite vertices of a Square ABCD, then the other two vertices are given by

$$\begin{aligned} \mathbf{B} &= \mathbf{R}^{-1}\mathbf{B}' - \mathbf{P} \\ \mathbf{D} &= \mathbf{R}^{-1}\mathbf{D}' - \mathbf{P} \end{aligned} \quad \text{where} \quad (2.1.1)$$

$$\mathbf{R}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \text{ and } \mathbf{P} = -\mathbf{A}.$$

Proof:

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Lets consider a square **OFGH** with origin(**O**) as one of the vertices and x, y - axes are two sides. Let the norm of the direction vectors of sides be d . The angle between $\mathbf{F}-\mathbf{O}$ (side) and $\mathbf{G}-\mathbf{O}$ (diagonal) is $\theta = 45^\circ$. and

$$\begin{aligned} \mathbf{F} - \mathbf{O} &= \frac{(\mathbf{G} - \mathbf{O})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\|(\mathbf{G} - \mathbf{O})^T\|} \\ \Rightarrow \mathbf{G} - \mathbf{O} &= \sqrt{2}(\mathbf{F} - \mathbf{O}) \end{aligned} \quad (2.1.2)$$

The norm of diagonal direction vector is

$$\|\mathbf{OG}\| = \sqrt{2}d = \|\mathbf{FH}\|. \quad (2.1.3)$$

The direction vectors are given by

$$\begin{aligned} \mathbf{F} - \mathbf{O} &= \begin{pmatrix} d \\ 0 \end{pmatrix}, \mathbf{G} - \mathbf{F} = \begin{pmatrix} 0 \\ d \end{pmatrix}, \\ \mathbf{H} - \mathbf{G} &= \begin{pmatrix} -d \\ 0 \end{pmatrix}, \mathbf{O} - \mathbf{H} = \begin{pmatrix} 0 \\ -d \end{pmatrix} \\ \mathbf{G} - \mathbf{O} &= \begin{pmatrix} d \\ d \end{pmatrix}. \\ \mathbf{H} - \mathbf{F} &= \begin{pmatrix} -d \\ d \end{pmatrix}. \end{aligned} \quad (2.1.4)$$

Let the Square **ABCD** has two vertices \mathbf{A} and \mathbf{C} . By using inspection method, Affine transformation has to be applied for \mathbf{B} and \mathbf{D} . The step by step procedure involves

- 1) Apply affine transformation steps i.e Translation and Rotation respectively of the given vertices \mathbf{A} and \mathbf{C} to shift square to the **OFGH**
- 2) Find the translation vector \mathbf{P} and rotation matrix \mathbf{R} .
- 3) Find the diagonal/Direction vector and norm of $\mathbf{A}'\mathbf{C}'$.
- 4) Find the side direction vector and using norm and inspection method find the vertices \mathbf{B}' and \mathbf{D}' .
- 5) Find the points \mathbf{B} and \mathbf{D} by using inspection method and reverse affine transformation i.e rotation and translation respectively.

Translation and Rotation Let \mathbf{P} be translation vector is given by

$$\cos \theta = \frac{\mathbf{P} = -\mathbf{A}}{\|(\mathbf{C} - \mathbf{A})^T \mathbf{OG}\|} \quad (2.1.5)$$

The generalized affine transformation principles are as below

$$\mathbf{A}' = \mathbf{R}(\mathbf{A} + \mathbf{P}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{O}.$$

$$\mathbf{C}' = \mathbf{R}(\mathbf{C} + \mathbf{P}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix}.$$

$$\mathbf{B}' = \mathbf{A}'\mathbf{B}' + \mathbf{A}'$$

where $\mathbf{A}'\mathbf{B}' = \|\mathbf{A}'\mathbf{B}'\|(\mathbf{F} - \mathbf{O})$.

$$\mathbf{D}' = \mathbf{A}'\mathbf{D}' + \mathbf{A}'$$

where $\mathbf{A}'\mathbf{D}' = \|\mathbf{A}'\mathbf{D}'\|(\mathbf{H} - \mathbf{O})$.

(2.2.4)

2.2 For Rotation Matrix R

Consider a vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the norm is L. If vector is rotated anticlockwise around the origin by β degrees then the new vector be $\begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix}$. Call the angle between $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the x-axis as α . Then

$$\begin{aligned} x_1 &= L \cos \alpha \\ y_1 &= L \sin \alpha \end{aligned} \quad (2.2.1)$$

So the angle between $\begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix}$ and the x-axis is $\alpha + \beta$.

$$\begin{aligned} x'_1 &= L \cos(\alpha + \beta) = \\ &= L \cos \alpha \cos \beta - L \sin \alpha \sin \beta \\ &= x_1 \cos \beta - y_1 \sin \beta. \\ y'_1 &= L \sin(\alpha + \beta) = \\ &= L \sin \alpha \cos \beta + L \cos \alpha \sin \beta \\ &= y_1 \cos \beta + x_1 \sin \beta. \end{aligned} \quad (2.2.2)$$

$$\Rightarrow \begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Here the rotation angle is θ . Hence the rotation matrix is given by

$$\mathbf{R} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (2.2.3)$$

2.3 Reverse Affine Transformation

By doing the reverse affine transformation, the Square \mathbf{ABCD} can be obtained from $\mathbf{A'B'C'D'}$ by using the below transformation rules. Given

$$\begin{aligned} \mathbf{A}' &= \mathbf{R}(\mathbf{A} + \mathbf{P}) \\ \Rightarrow \mathbf{A}'\mathbf{R}^{-1} &= \mathbf{A} + \mathbf{P} \\ \Rightarrow (\mathbf{A}'\mathbf{R}^{-1}) - \mathbf{P} &= \mathbf{A} \end{aligned} \quad (2.3.1)$$

Hence $\mathbf{A} = \mathbf{R}^{-1}\mathbf{A}' - \mathbf{P}$

The generalization principle can be applied to other vertices also as below

$$\begin{aligned} \mathbf{B} &= \mathbf{R}^{-1}\mathbf{B}' - \mathbf{P} \\ \mathbf{C} &= \mathbf{R}^{-1}\mathbf{C}' - \mathbf{P} \\ \mathbf{D} &= \mathbf{R}^{-1}\mathbf{D}' - \mathbf{P} \end{aligned} \quad \text{where} \quad (2.3.2)$$

$$\mathbf{R}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \text{ and } \mathbf{P} = -\mathbf{A}.$$

Hence proved.

2.4 SOLUTION TO PROBLEM

Diagonal(Direction Vector) $\mathbf{AC} = \mathbf{C} - \mathbf{A}$ is given by

$$\mathbf{AC} = \begin{pmatrix} 0 - 0 \\ 3 - (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}. \quad (2.4.1)$$

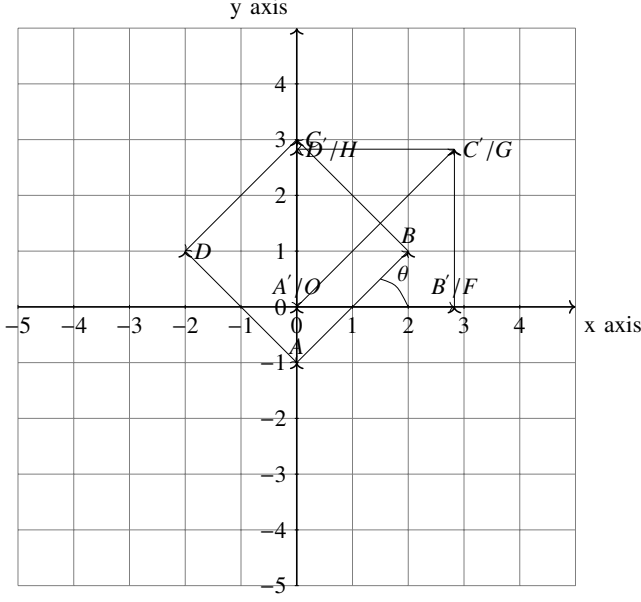


Fig. 5: SQUARE ABCD and OFGH/A'B'C'D'

The line equation of $A'C'$ is $x-y=0$ or simply $x=y$.

$$\|A'B'\| = \|A'D'\| \frac{\|A'C'\|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

$$\text{Also, } F - O = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A'B' = \|A'B'\|(F - O) = 2\sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}.$$

$$B' = A'B' + A' = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} \quad (2.4.5)$$

$$\text{Also, } H - O = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A'D' = \|A'D'\|(H - O) = 2\sqrt{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}.$$

$$D' = A'D' + A' = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} \quad (2.4.6)$$

Hence the obtained vertices are

$$A' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, B' = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}, C' = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}, D' = \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix}. \quad (2.4.7)$$

Translation for A to O requires a translation vector

$$P = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\Rightarrow A' = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$C' = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}. \quad (2.4.2)$$

$$A'C' = \begin{pmatrix} 0-0 \\ 4-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Norm of $A'C'$

$$\|A'C'\| = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4 \quad (2.4.3)$$

The rotation is by 45° clock wise. The rotation matrix is given by

$$\begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{pmatrix} \Rightarrow A'C' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} \quad (2.4.4)$$

$$C' = A'C' + A' = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$$

2.5 USING REVERSE AFFINE TRANSFORMATION

The rotation is by 45° counter clock wise i.e. $\theta = 45^\circ$. The inverse of rotation matrix R is given by

$$R^{-1} = \frac{1}{1} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (2.5.1)$$

and the translation vector is $-P$. From above reverse affine transformation rules,

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.5.2)$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

$$D = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

2.6 LINE EQUATIONS

Coordinates are

$$\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \quad (2.6.1)$$

In Matrix form, if $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$ & $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$ are the points given then the directional vector is given by $\begin{pmatrix} X_2 - X_1 \\ Y_2 - Y_1 \end{pmatrix}$ then, the form of equation $ax+by+c=0$ can be written as

$$\begin{aligned} a &= Y_2 - Y_1, \\ b &= -(X_2 - X_1), \\ c &= (Y_2 - Y_1)(X_2 - X_1) \end{aligned} \quad (2.6.2)$$

By using the same, the line equation **AB** for points $\mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -2 \\ 2 \end{pmatrix} \\ a &= 2, \quad b = 2, \\ c &= (-1 - 0) \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2 \end{aligned} \quad (2.6.3)$$

Line equation for **AB**

$$\Rightarrow 2x + 2y + 2 = 0 \text{ or } x + y = -1. \quad (2.6.4)$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = -1 \quad (2.6.5)$$

The line equation **BC** for points $\mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{BC} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ a &= 2, \quad b = -2, \\ c &= (1 - 2) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 6 \end{aligned} \quad (2.6.6)$$

Line equation for **BC**

$$\Rightarrow 2x - 2y + 6 = 0 \text{ or } x - y = -3. \quad (2.6.7)$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{X} = -3 \quad (2.6.8)$$

The line equation **CD** for points $\mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{CD} &= \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ a &= -2, \quad b = -2, \\ c &= (3 - 0) \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 6 \end{aligned} \quad (2.6.9)$$

Line equation for **CD**

$$\Rightarrow -2x - 2y + 6 = 0 \text{ or } x + y = 3. \quad (2.6.10)$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X} = 3 \quad (2.6.11)$$

The line equation **DA** for points $\mathbf{D} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is as follows:

$$\begin{aligned} \mathbf{DA} &= \begin{pmatrix} -2 \\ -2 \end{pmatrix} \\ a &= -2, \quad b = 2, \\ c &= (1 - 2) \begin{pmatrix} -2 \\ -2 \end{pmatrix} = 2 \end{aligned} \quad (2.6.12)$$

Line equation for **DA**

$$\Rightarrow -2x + 2y + 2 = 0 \text{ or } x - y = 1. \quad (2.6.13)$$

In vector form

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{X} = 1 \quad (2.6.14)$$

The plotted graph is shown as below.

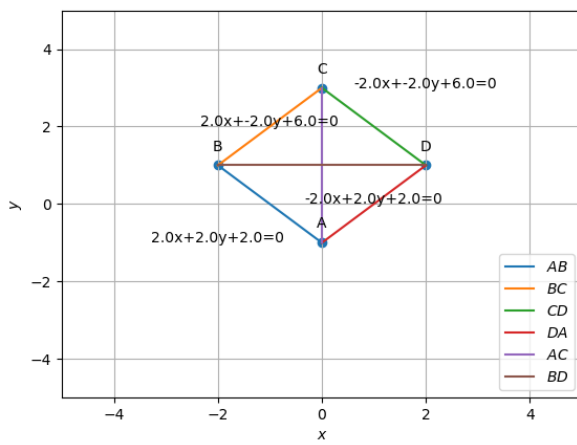


Fig. 5: Square ABCD