EE 120: Signals and Systems

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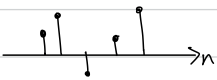
Lecture 1, 8/25/2022

[TODO]

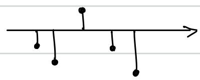
Lecture 2, 8/30/2022

Review

- Signals are functions
  - CT: $x: \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ — real CT signal / complex CT signal
  - DT: $x: \mathbb{Z} \to \mathbb{R}$ or $\mathbb{C}$
- Systems are functions



$$x \xrightarrow{X} \boxed{H} \xrightarrow{Y} y = H(x)$$



X: input signal space
Y: output signal space

- If $X = [\mathbb{R} \to \mathbb{R}]$ and $Y = [\mathbb{R} \to \mathbb{R}]$, we say that H is a CT system
  ↑ set of all real-valued CT signals
- If $X = [\mathbb{Z} \to \mathbb{R}]$ and $Y = [\mathbb{Z} \to \mathbb{R}]$, we say that H is a DT system
  ↑ set of all real-valued DT signals
- Delta Impulse $\delta(n)$, Unit Step $u(n)$
- Every DT signal can be decomposed into a linear combo of shifted impulses
  ↳ $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$

Time-Invariant (LTI)

$$x(t) \to \underset{\text{system}}{\boxed{H}} \to y(t)$$

If $\hat{x}(t) = x(t-T)$ yields $\hat{y}(t) = H(\hat{x}) = y(t-T)$ for all $T \in \mathbb{R}$ and all inputs $x \in X$, we say that H is time invariant

Linearity

(Superposition)

$$x \xrightarrow{X} \boxed{H} \xrightarrow{Y} y$$

For any $x_1, x_2 \in X$ and $\alpha, \alpha_1, \alpha_2 \in \mathbb{C}$, H is linear if the following hold:

- Scaling: $H(\alpha x_1) = \alpha H(x_1)$
- Additivity: $H(x_1 + x_2) = H(x_1) + H(x_2)$

$\Big\}\ H(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 H(x_1) + \alpha_2 H(x_2)$

Linear, Time-Invariant System

$$x(n) \longrightarrow \underset{}{\overset{\text{LTI}}{\boxed{H}}} \longrightarrow y(n)$$

An LTI system satisfies both linearity and time-invariance

Claim: $x(n) = \delta(n) \longrightarrow \overset{\text{DT-LTI}}{\boxed{H}} \longrightarrow y(n) = h(n)$ (Impulse Response)

Knowing the impulse response h enables us to determine the output corresponding to any arbitrary input x

↳ **Proof:** $h(n) = H(\delta(n))$

$h(n-k) = H(\delta(n-k))$

$x(k)\,h(n-k) = x(k)\,H(\delta(n-k)) = H(x(k)\,\delta(n-k))$

Thus, $\displaystyle\sum_{k=-\infty}^{\infty} x(k)\,h(n-k) = \sum_{k=-\infty}^{\infty} H(x(k)\,\delta(n-k)) = H\left(\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right)$

Convolution

Discrete-Time: $\displaystyle (x*h)(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$

Continuous-Time: $\displaystyle (x*h)(t) = \int_{-\infty}^{\infty} x(u)\,h(t-u)\,du$

* Commutative: $x*h = h*x$

↳ Why care?

$$x(n)=\delta(n) \longrightarrow \overset{H}{\boxed{\underset{\text{DT-LTI}}{\boxed{f}} \longrightarrow \underset{\text{DT-LTI}}{\boxed{g}}}} \longrightarrow h(n) = (f*g)(n)$$

Cascade (Series)
Interconnection of LTI System

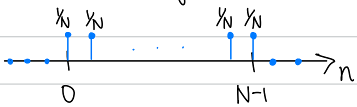
$h(n) = (f*g)(n) = (g*f)(n)$ by commutativity (cascade order doesn't matter)

Ex. Simple Moving Average

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(n-k)$$

Determine $h(n)$ and plot it.

Let $x(n) = \delta(n)$. Then $y(n) = h(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(n-k)$



Lecture 3, 9/1/2022

Time Invariance (TI)

Visualization of Time Invariance

$x(t) \rightarrow \boxed{H} \rightarrow y(t) \rightarrow \boxed{D_T} \rightarrow y(t-T)$

$x(t) \rightarrow \boxed{D_T} \rightarrow \hat{x}(t) = x(t-T) \rightarrow \boxed{H} \rightarrow \hat{y}(t)$

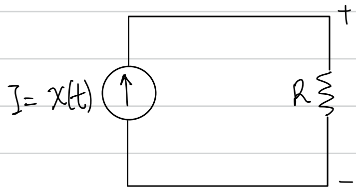
} Must equal $\forall T$ for TI to hold

H = System operator

D_T = Shift by T time

Basically, Time Invariance holds iff H & D_T are commutative

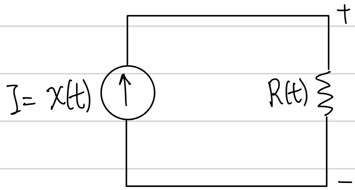
Ex 1.



$V = y(t) = Rx(t)$

$\left.\begin{array}{l}\hat{y}(t) = Rx(t-T) \\ y(t-T) = Rx(t-T)\end{array}\right\} \checkmark$ Time Invariant

Ex 2.



$V = y(t) = R(t)\,x(t)$

$\left.\begin{array}{l}\hat{y}(t) = R(t)\,x(t-T) \\ y(t-T) = R(t-T)\,x(t-T)\end{array}\right\} \times$

Time Variant

Ex 3. Two-Pointer Moving Average

$$x(n) \rightarrow \boxed{H} \rightarrow y(n) = \frac{x(n) + x(n-1)}{2}$$

- Linear? $\hat{x}(n) = \alpha x_1(n) + \beta x_2(n) \rightarrow \hat{y}(n) = \dfrac{\hat{x}(n) + \hat{x}(n-1)}{2} = \dfrac{(\alpha x_1(n) + \beta x_2(n)) + (\alpha x_1(n-1) + \beta x_2(n-1))}{2}$

  $= \alpha y_1(n) + \beta y_2(n)$ ✓

- Time Invariant? $\hat{x}(n) = x(n-N) \rightarrow \left.\begin{array}{l}\hat{y}(n) = \dfrac{\hat{x}(n) + \hat{x}(n-1)}{2} = \dfrac{x(n-N) + x(n-1-N)}{2} \\ y(n-N) = \dfrac{x(n-N) + x(n-N-1)}{2}\end{array}\right\} \checkmark$

Linear, Time-Invariant (LTI) Systems

$$x(n) \longrightarrow \boxed{H} \longrightarrow y(n)$$

LTI

<u>Claim:</u> If I know the response of H to the input $x(n) = \delta(n)$ $\forall n \in \mathbb{Z}$ (i.e. the impulse response $h(n) = H(x(n))$) then I know the output for any arbitrary input signal x.

↳ <u>Proof/Explanation:</u> We can express any input signal $x(n)$ as follows:

$$x(n) = \sum_{m=-\infty}^{\infty} x(m)\,\delta(n-m)$$

and thus any output signal $y(n)$ as follows (using LTI properties):

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\,h(n-m) = (x \star h)(n) = (h \star x)(n)$$

↑ Convolutions are commutative

<u>Cascading Systems:</u> we can interchange the order of LTI systems in a cascade (series) interconnection

$$x \longrightarrow \boxed{f} \xrightarrow{r} \boxed{g} \longrightarrow y$$

$$x \longrightarrow \boxed{g} \xrightarrow{q} \boxed{f} \longrightarrow y$$

↳ $y = x \star f \star g = x \star g \star f$, though $r \neq q$

Convolutions

<u>Def:</u> $(v \star u)(n) = \sum_{k=-\infty}^{\infty} v(n)\,u(n-k) = \sum_{m=-\infty}^{\infty} v(n-m)\,u(m)$

<u>Properties:</u>

- Commutative: $v \star u = u \star v$
- Identity Element: $x \star \underline{\delta} = x$

Complex Exponentials

$$x(t)=e^t \longleftrightarrow \begin{cases} x(0)=1 \\ \dot{x}(t) \triangleq \frac{dx(t)}{dt} = x(t) \\ \ddot{x}(t) \triangleq \frac{d^2x(t)}{dt^2} = x(t) \end{cases}$$

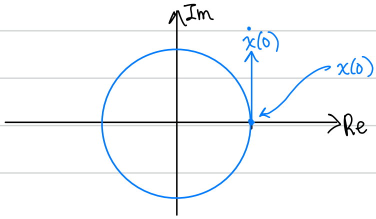
We use the properties of complex exponentials to solve interesting problems:

Ex. $x(t)=e^{it}$ = the instantaneous position of a particle on the complex plane

$x(0)=1$

$\dot{x}(t)=ie^{it}=ix(t)$

$\dot{x}(0)=i$



Claim: the particle moves on the unit circle at a constant speed counter-clockwise

↳ Proof: $x(t)=e^{it}=a(t)+ib(t)$ $\quad a:\mathbb{R}\to\mathbb{R},\ b:\mathbb{R}\to\mathbb{R}$

$\dot{x}(t) = \dot{a}(t)+i\dot{b}(t)$

$\dot{x}(t)=ix(t)=-b(t)+ia(t)$

↳ $\begin{cases} \dot{a}(t)=-b(t) \\ a(t)=\dot{b}(t) \end{cases}$

↳ $a(t)\dot{a}(t)=-b(t)\dot{b}(t)$

↳ $2a(t)\dot{a}(t)+2b(t)\dot{b}(t)=0$

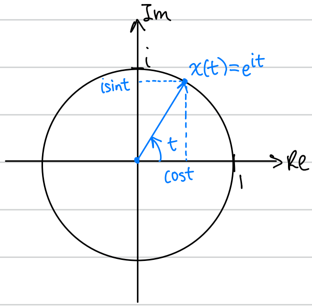
↳ $\frac{d}{dt}\left[a^2(t)+b^2(t)\right]=0$

↳ $a^2(t)+b^2(t)=C=1 \quad \forall t\in\mathbb{R}$ $\therefore$ Moves in circle of radius 1

$|\dot{x}(t)|=|ix(t)|=|x(t)|=1$ $\therefore$ Moves at a constant speed at 1 rad/sec

Euler's Formula

$$e^{it} = \cos(t) + i\sin(t) \quad \forall t \in \mathbb{R}$$



Lecture 4, 9/6/2022

Euler's Formula

$e^{it} = \cos t + i \sin t$

$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$ where ω = rotating speed in rad/sec $\begin{cases} > 0 \rightarrow \text{counterclockwise} \\ < 0 \rightarrow \text{clockwise} \end{cases}$

Inverse Euler Formulas:

$$\begin{cases} \cos(\omega t) = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2} \\ \sin(\omega t) = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i} \end{cases}$$

Note on different ways to measure rate: $\omega = \text{rad/sec} = 2\pi f$

rad/cycle ↑ ↑ cycles/sec

Periodicity

Main Idea:

$$\begin{tikzcd} R\uparrow \arrow[r] & J\downarrow \arrow[d] \\ J\uparrow \arrow[u] & R\downarrow \arrow[l] \end{tikzcd}$$

A signal/system is periodic if it is cyclic, i.e. it eventually repeats

CT Periodicity

Suppose we have a CT signal $x: \mathbb{R} \rightarrow \mathbb{R}$ or $\mathbb{C}$. If $x(t+T) = x(t)$ $\forall t \in \mathbb{R}$ for some $T \in \mathbb{R}_+$, we say that x is periodic w/ period T. If T is the smallest positive value that satisfies the relation, then T is the fundamental period.

Ex. $x(t) = \cos(\frac{2\pi}{5} t)$

$x(t+T) = \cos(\frac{2\pi}{5} t + \frac{2\pi}{5} T) = x(t) = \cos(\frac{2\pi}{5} t)$

↳ $\frac{2\pi}{5} T = 2\pi k$ where $k \in \{1, 2, 3, \ldots\}$

↳ $T = 5$ sec

DT Periodicity

Suppose we have a DT signal $x: \mathbb{Z} \to \mathbb{R}$ or $\mathbb{C}$. If $x(n+N) = x(n)\ \forall\, n \in \mathbb{Z}$ for some $N \in \mathbb{Z}_+$, we say that x is periodic w/ period N. If N is the smallest such positive integer, we call N the fundamental period of x.

Ex. $x(n) = C \quad \forall n \in \mathbb{Z}$

$\hookrightarrow N = 1$, $\omega_0 =$ fundamental frequency $= \frac{2\pi}{N} = 2\pi$

Ex. $x(n) = e^{in} \quad \forall n \in \mathbb{Z}$

$x(n+N) = e^{i(n+N)} = e^{in} e^{iN} = x(n) = e^{in}$

$\hookrightarrow e^{iN} = 1$

$\hookrightarrow N = 2\pi \notin \mathbb{Z}$

Thus, the fundamental period and frequency do not exist. This is called a Quasi-periodic signal.

Ex. $x(n) = e^{i\frac{\pi}{4}n} \quad \forall\, n \in \mathbb{Z}$

$x(n+N) = e^{i\frac{\pi}{4}(n+N)} = e^{i\frac{\pi}{4}n} \cdot e^{i\frac{\pi}{4}N} = x(n) = e^{i\frac{\pi}{4}n}$

$\hookrightarrow e^{i\frac{\pi}{4}N} = 1$

$\hookrightarrow N = \frac{4}{\pi} \cdot 2\pi = 8$, $\omega_0 = \frac{2\pi}{N} = \frac{\pi}{4}$

Periodicity of $x(n) = e^{i\omega n}$

Necessary & Sufficient conditions for $e^{i\omega n}$ to be periodic in n.

$e^{i\omega(n+N)} = e^{i\omega n}$

$e^{i\omega n} \cdot e^{i\omega N} = e^{i\omega n}$

$e^{i\omega N} = 1$

$\omega N = 2\pi k$

$\omega = \frac{2k}{N} \cdot \pi$

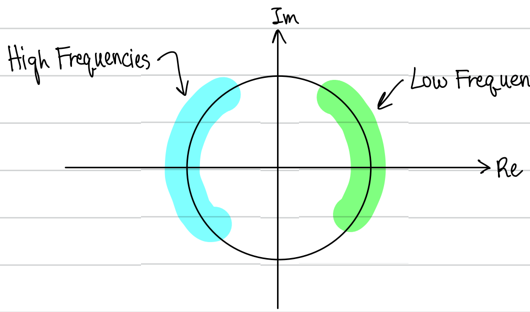
$\hookrightarrow \omega$ must be a rational multiple of $\pi \Leftrightarrow N = \frac{2\pi k}{\omega} \in \mathbb{Z}_+$ can be found

CT Oscillation Behavior

$x(t)=e^{i\omega t}$. The sky is not a limit for ω, i.e. ω can go up to ∞. As $\omega \to \infty$, oscillations become progressively fast. The slowest frequency is $\omega=0$, which corresponds to a constant signal.

DT Oscillation Behavior

$x(n)=e^{i\omega n}$. The highest frequency is $\omega=\pi$, which corresponds to a "flip-flopping" signal. The slowest frequency is $\omega=0$, which corresponds to a constant signal.



Frequency Response of a DT-LTI System

$$x(n)=\delta(n) \longrightarrow \underset{\text{DT-LTI}}{\boxed{H}} \longrightarrow y(n)=\underbrace{h(n)}_{\text{Impulse Response}}$$

$$x(n)=e^{i\omega n} \longrightarrow \boxed{H} \longrightarrow y(n)=\sum_m h(m)x(n-m)$$

$$=\sum_m h(m)e^{i\omega(n-m)}$$

$$=\left(\sum_m h(m)e^{-i\omega m}\right)e^{i\omega n}$$

$$=H(\omega)e^{i\omega n}$$

$$=\underbrace{H(\omega)}_{\text{Frequency Response}}x(n)$$

* $H(e^{i\omega n})=H(\omega)e^{i\omega n}$ is analogous to $Av=\lambda v$, and it's called the <u>Eigenfunction Property</u> of Complex Exponentials with respect to DT-LTI System.

* LTI can get rid of frequency in the input signal, but it <u>cannot</u> create a new frequency in the output signal.

Ex. $x(n) \longrightarrow \boxed{H=(\cdot)^2} \longrightarrow y(n) = x^2(n)$

$x(n) = e^{i\omega n} \longrightarrow \qquad \longrightarrow y(n) = e^{i(2\omega)n}$

Thus, this system cannot be LTI because it creates a new frequency (i.e. 2ω). It is time-invariant, but not linear.

Ex. $x(n) \longrightarrow \boxed{H} \longrightarrow y(n) = e^{i\omega_1 n} x(n)$

$x(n) = e^{i\omega_0 n} \longrightarrow \qquad \longrightarrow y(n) = e^{i\omega_1 n} e^{i\omega_0 n} = e^{i(\omega_0+\omega_1)n}$

Thus, the system cannot be LTI because it creates a new frequency (e.g. $\omega_0+\omega_1$). It is linear, but time-varying.

Property of any LTI System

$$\alpha_0 e^{i\omega_0 n} + \alpha_1 e^{i\omega_1 n} \longrightarrow \underset{\text{LTI}}{\boxed{H}} \longrightarrow \alpha_0 H(\omega_0) e^{i\omega_0 n} + \alpha_1 H(\omega_1) e^{i\omega_1 n}$$

Frequency Response of the Two-Point Moving Average Filter

$$x(n) \longrightarrow \boxed{H} \longrightarrow y(n) = \frac{x(n) + x(n-1)}{2}$$

Ways to find the frequency response $H(\omega)$:

Method I: $H(\omega) = \sum_m h(m) e^{-i\omega m} = h(0)e^{-i\omega(0)} + h(1)e^{-i\omega(1)} = \frac{1+e^{-i\omega}}{2}$

Method II: Next time!

Lecture 5, 9/8/2022

Overview

- DT Frequency Response
  - 2 point Moving Average Filter
  - Recursive Filter

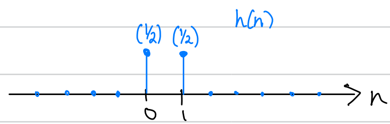
DT Frequency Response

$$x(n)=e^{i\omega n} \rightarrow \boxed{H} \rightarrow y(n)=H(\omega)e^{i\omega n}$$

$$H(\omega)=\sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n} = \text{Freq Response of Filter } H$$

2-Point Moving Average Filter

$$x(n) \rightarrow \boxed{\begin{matrix} H \\ h(n),\ H(\omega) \end{matrix}} \rightarrow y(n)=\frac{x(n)+x(n-1)}{2}$$



$h(n)$: let $x(n)=\delta(n) \rightarrow y(n)=h(n)$

$$h(n)=\frac{\delta(n)+\delta(n-1)}{2}$$

Computing $H(\omega)$ (using 2-pt moving average as example)

<u>Method I:</u> Use definition of $H(\omega)$

$$H(\omega)=\sum_n h(n)e^{-i\omega n}$$

$$H(\omega)=h(0)+h(1)e^{-i\omega}=\frac{1+e^{-i\omega}}{2}$$

<u>Method II:</u> Use eigenfunction property of complex exponentials wrt LTI systems

Note: this method assumes that the frequency response exists

Let $x(n)=e^{i\omega n} \rightarrow y(n)=H(\omega)e^{i\omega n}$. Then we have:

$$y(n)=\frac{x(n)+x(n-1)}{2} \quad \text{(Linear Constant-Coefficient Difference Equation (LCCDE))}$$

$$H(\omega)e^{i\omega n}=\frac{e^{i\omega n}+e^{i\omega(n-1)}}{2}=\frac{1+e^{-i\omega}}{2}e^{i\omega n}$$

$$H(\omega)=\frac{1+e^{-i\omega}}{2}$$

Plotting Frequency Response (using 2-pt moving average as example)

$$H(\omega) = \underbrace{|H(\omega)|}_{\text{Magnitude Response}} e^{i \overbrace{\measuredangle H(\omega)}^{\text{Phasor Response}}}$$

$$|H(\omega)| = \sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\cos(\omega)\right)^2 + \left(-\tfrac{1}{2}\sin(\omega)\right)^2} \quad \leftarrow \text{One Way}$$

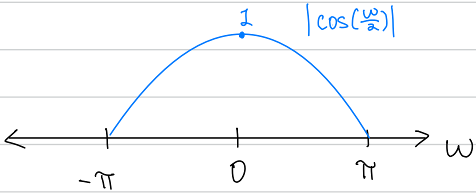
$$= \left|\tfrac{1}{2}e^{i(0)\omega} + \tfrac{1}{2}e^{i(-1)\omega}\right| \quad \leftarrow \text{Better Way (Exploit Symmetry)}$$

Balancing the Exponents

$$= \left|\frac{e^{i\omega/2}+e^{-i\omega/2}}{2}\, e^{-i\omega/2}\right|$$

$$= \left|\cos\left(\tfrac{\omega}{2}\right) e^{-i\omega/2}\right|$$

$$= \left|\cos\left(\tfrac{\omega}{2}\right)\right|$$



Low Pass

* We only consider the range $[-\pi, \pi]$ because those are limits of a DT filter

Note: $|H(\omega)| = |H(\omega+2\pi)|$ holds for all DT filter frequency responses

2π-Periodicity of DT-LTI Frequency Response

$$H(\omega) = \sum_n h(n) e^{-i\omega n}$$

$$H(\omega+2\pi) = \sum_n h(n) e^{-i(\omega+2\pi)n}$$

$$= \sum_n h(n) e^{-i\omega n} \underbrace{e^{-i(2\pi)n}}_{=1 \ \forall n \in \mathbb{Z}}$$

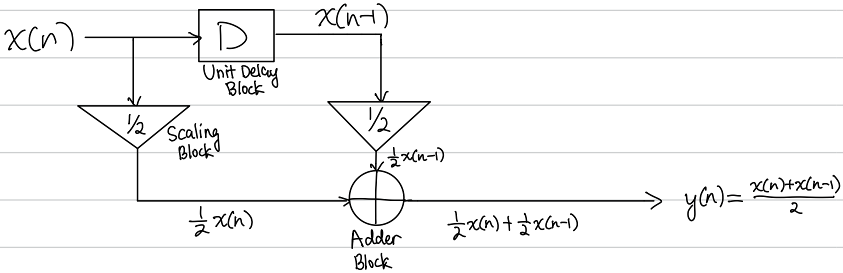
$$= \sum_n h(n) e^{-i\omega n}$$

$$= H(\omega)$$

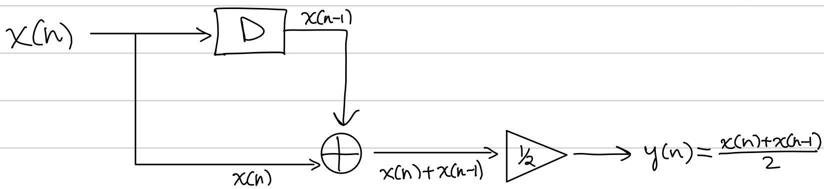
Delay-Adder-Gain Block Diagram Implementation

Normal:

DAG:

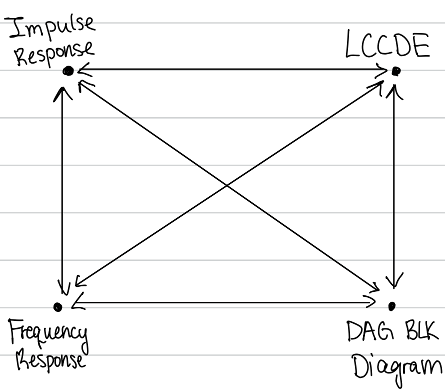


Alternate DAG:
(More Efficient)



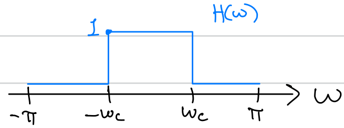
The Big Picture

Moving between various characterizations of LTI systems

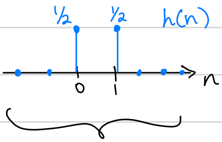


DT-LTI System w/o LCCDE Representation

$$h(n) = \frac{\sin(\omega_c n)}{\omega_c n} \quad \forall n \in \mathbb{Z} \quad \longleftrightarrow$$



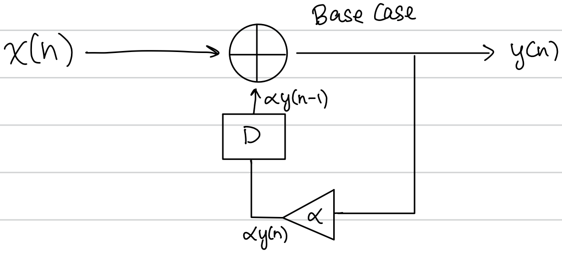
Finite-Duration Impulse Response (FIR) Filter



Finite Region of Support

Infinite-Duration Impulse Response (IIR) Filter

$$y(n) = \alpha y(n-1) + x(n), \quad \underbrace{y(-1)=0}_{\text{Base Case}}, \quad |\alpha| < 1$$



Determine $h(n)$: $h(-1) = 0$

$$h(n) = \alpha h(n-1) + \delta(n)$$

$$h(0) = \alpha h(-1) + \delta(0) = 1$$

$$h(1) = \alpha h(0) + \delta(1) = \alpha$$

$$h(2) = \alpha h(1) + \delta(2) = \alpha^2$$

$$h(3) = \alpha h(2) + \delta(3) = \alpha^3$$

$$\vdots$$

$$h(n) = \alpha^n \quad \forall \; n \geq 0$$

$\hookrightarrow h(n) = \alpha^n u(n)$ where $u(n)$ = unit step

Frequency Response for Recursive Filter

$y(n) = \alpha y(n-1) + x(n)$, $y(-1) = 0$, $|\alpha| < 1$

$h(n) = \alpha^n u(n)$

<u>Method I:</u> $H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}$

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n) e^{-i\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-i\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-i\omega})^n$$

Note: $|\alpha e^{-i\omega}| = |\alpha||e^{-i\omega}| = |\alpha| < 1$

$$= \frac{1}{1-\alpha e^{-i\omega}}$$

<u>Method II:</u> $x(n) = e^{i\omega n} \rightarrow y(n) = H(\omega) e^{i\omega n}$

$$y(n) = \alpha y(n-1) + x(n)$$

$$= \alpha H(\omega) e^{i\omega(n-1)} + e^{i\omega n}$$

$$= (\alpha H(\omega) e^{-i\omega} + 1) e^{i\omega n}$$

$$= H(\omega) e^{i\omega n}$$

$$H(\omega) = \alpha H(\omega) e^{-i\omega} + 1$$

$$1 = (1 - \alpha e^{-i\omega}) H(\omega)$$

$$H(\omega) = \frac{1}{1-\alpha e^{-i\omega}}$$

Lecture 6, 9/15/2022

Overview

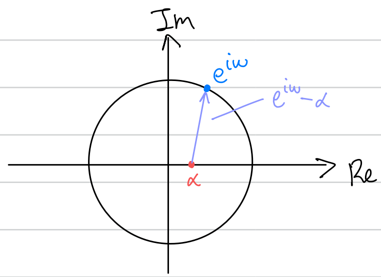
- IIR Filter Frequency Response Plot
- System Properties
  - Causality
  - BIBO Stability

First-Order IIR

$$y(n) = \alpha y(n-1) + x(n), \quad y(-1) = 0, \quad |\alpha| < 1$$

$$x(n) = e^{i\omega n} \rightarrow y(n) = H(\omega) e^{i\omega n}$$

$$y(n-1) = H(\omega) e^{i\omega(n-1)} = H(\omega) e^{-i\omega} e^{i\omega n}$$



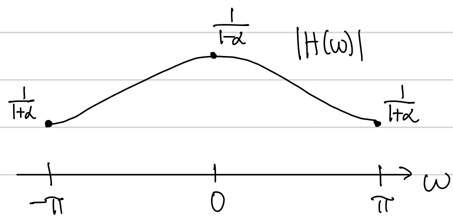
$$H(\omega) e^{i\omega n} = \alpha H(\omega) e^{-i\omega} e^{i\omega n} + e^{i\omega n}$$

$$(1 - \alpha e^{-i\omega}) H(\omega) = 1$$

$$H(\omega) = \frac{1}{1 - \alpha e^{-i\omega}} = |H(\omega)| e^{i \measuredangle H(\omega)}$$

$$= \frac{e^{i\omega}}{e^{i\omega} - \alpha}$$

$$|H(\omega)| = \left| \frac{e^{i\omega}}{e^{i\omega} - \alpha} \right| = \frac{1}{\| \nearrow \|}$$



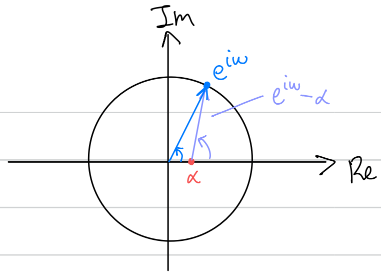
$$h(n) = \alpha^n u(n)$$

Lowpass Filter

*We can make $H(\omega)$ a high-pass filter by moving α to become negative i.e. $-1 < \alpha < 0$

In fact, we can make $H(\omega)$ peak at any angle θ if we let $\alpha = \beta e^{i\theta}$ where $|\beta| < 1$.

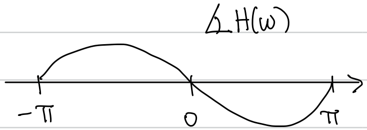
First Order IIR (cont)



$$\measuredangle H(\omega) = \measuredangle \frac{e^{i\omega} - 0}{e^{i\omega} - \alpha}$$

$$= \measuredangle (e^{i\omega} - 0) - \measuredangle (e^{i\omega} - \alpha)$$

$$= \measuredangle \nearrow - \measuredangle \nearrow$$



Note: for DT-LTI systems, only integer shifts are directly related to phase (w/o having to worry about bounding)

Causality

Causality: doesn't peek ahead in time

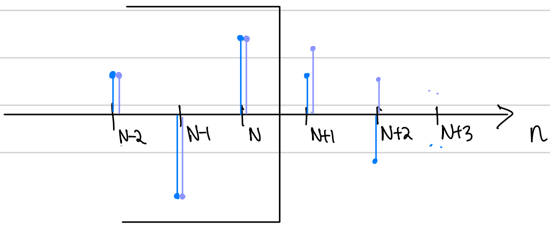
$$x \xrightarrow{X} \boxed{H} \xrightarrow{Y} y$$

H is causal if for every integer N, if it is the case that $x_1, x_2 \in X$ are equal up to and including $n=N$:

$$x_1(n) = x_2(n) \quad \forall n \leq N$$

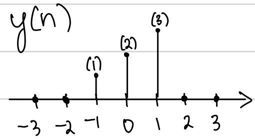
then we have that

$$y_1(n) = y_2(n) \quad \forall n \leq N$$



Causality (cont)

Ex: $x(n) = u(n) \longrightarrow \overset{\text{Linear}}{\boxed{H}} \longrightarrow$



Select the most correct statement:

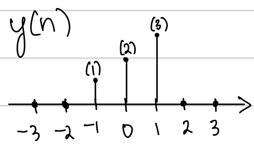
(I) H must be causal.

(II) Can't tell. Need more info.

(III) H cannot be causal.

Proof: let $x_2(n) = 0 \ \forall n \in \mathbb{Z}$. Then $x_2(n) = \overset{x_1(n)}{u(n)} \ \forall n \leq -1$. However, $y_2(-1) = 0 \neq 1 = y_1(-1)$. This contradicts the definition of causality.

Ex: $x(n) = u(n) \longrightarrow \overset{\text{TI}}{\boxed{H}} \longrightarrow$



Select the most correct statement:

(I) H must be causal.

(II) Can't tell. Need more info.

(III) H cannot be causal

Proof: Suppose $x_2(n) = \overset{x_1(n-1)}{u(n-1)}$. Then $x_2(n) = u(n) \ \forall n \leq -1$. However, $y_2(-1) = 0 \neq 1 = y_1(-1)$. This contradicts the definition of causality.

Causality for LTI

$x \rightarrow \overset{\text{LTI}}{\boxed{H}} \rightarrow y$

$h(n) = 0 \ \forall n < 0$ iff H is causal

Bounded-Input Bounded-Output (BIBO) Stability

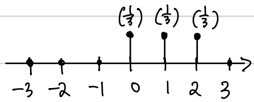
We say a signal x is <u>bounded</u> if there exists $0 < B_x < \infty$ such that $|x(n)| \leq B_x \ \forall n \in \mathbb{Z}$.

$$x \xrightarrow{X} \boxed{H} \xrightarrow{Y} y$$

We say H is <u>BIBO stable</u> if every bounded input produces a bounded output.

In other words, $\forall x \in X$ s.t. $|x(n)| \leq B_x \ \forall n \in \mathbb{Z}$ for some $0 < B_x < \infty$, then $\exists \ 0 < B_y < \infty$ s.t. $|y(n)| \leq B_y \ \forall n \in \mathbb{Z}$

<u>Ex:</u> 3-point Moving Average



$$y(n) = \frac{x(n) + x(n-1) + x(n-2)}{3}$$

Causal!!!

$$\begin{aligned}
|y(n)| &= \left| \frac{x(n) + x(n-1) + x(n-2)}{3} \right| \\
&= \frac{1}{3} \left| x(n) + x(n-1) + x(n-2) \right| \\
&\leq \frac{1}{3} \left(|x(n)| + |x(n-1)| + |x(n-2)| \right) \\
&\leq \frac{1}{3} (B_x + B_x + B_x) \\
&= B_x \\
&= B_y
\end{aligned}$$

Triangle Inequality

Thus, the 3-pt Moving Average is BIBO Stable.

BIBO DT-LTI System

$$x(n) \longrightarrow \underset{H}{\overset{\text{DT-LTI}}{\boxed{h(n)}}} \longrightarrow y(n)$$

We say H is BIBO iff $\displaystyle\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

BIBO DT-LTI System (cont)

If $\sum_{n=-\infty}^{\infty} |h(n)| < \infty \implies$ BIBO Stable

↳ <u>Proof:</u> $y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$

Let $|x(n)| \leq B_x \;\forall\, n \in \mathbb{Z}$. Then we have that

$$
\begin{aligned}
|y(n)| &= \left| \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right| \\
&\leq \sum_{k=-\infty}^{\infty} |h(k)\, x(n-k)| \\
&\leq \sum_{k=-\infty}^{\infty} |h(k)|\, B_x \\
&\leq B_x \sum_{k=-\infty}^{\infty} |h(k)| < \infty \\
&= B_y
\end{aligned}
$$

for some $B_y < \infty$.

Lecture 7, 9/20/2022

Overview

- BIBO Stability (cont.)
  - Sufficiency (done last time)
  - Necessity (today)
- LCCDEs & Frequency Response
- LCCDEs & State-Space Response

BIBO Stability of DT-LTI Systems

$$\sum_n |h(n)| < \infty \underset{\text{Today}}{\overset{\text{Last Time}}{\iff}} H \text{ is BIBO stable}$$

We want to show that H is BIBO stable $\Rightarrow \sum_n |h(n)| < \infty$ $(h \in \ell^1)$

↳ Proof: We proceed via a proof by contraposition:

$h \notin \ell^1 \Rightarrow H$ not BIBO stable

(h is not abs. summable) (∃ a bounded input that produces an unbounded output)

* We say $h \in \ell^1$ if $\sum_n |h(n)| < \infty$

where ℓ^1 = space of all absolute summable functions

Let $x(n) = \text{sign}(h(-n))$, $|x(n)| \leq 1 \;\; \forall n$.

Then: $y(n) = \sum_k x(k)\, h(n-k)$

$$y(0) = \sum_k x(k)\, h(-k)$$
$$= \sum_k \frac{h(-k)}{|h(-k)|} \cdot h(-k)$$
$$= \sum_k \frac{(h(-k))^2}{|h(-k)|}$$
$$= \sum_k |h(-k)|$$
$$= +\infty$$

Meaning H is not BIBO stable.

Thus, $h \in \ell^1 \Rightarrow H$ BIBO stable.

BIBO stability of DT-LTI Systems

Ex: $y(n) = \alpha y(n-1) + x(n)$

$h(n) = \alpha^n u(n)$

System is stable iff $|\alpha| < 1$

Ex: $h(n) = u(n)$

$$x(n) \longrightarrow \boxed{h(n)=u(n)} \longrightarrow y(n) = \sum_k x(k)\,h(n-k)$$

$$= \sum_k x(k)\,u(n-k)$$

$$= \sum_{k=-\infty}^{n} x(k) \quad \text{i.e. Cumulative Sum}$$

System is not BIBO stable

BIBO Stability & Frequency Response

If $h \in \ell^1$ (i.e. $\sum_n |h(n)| < \infty$) $\rightarrow$ (1) $|H(\omega)| < \infty \ \forall \omega$

$\rightarrow$ (2) $H(\omega)$ is continuous in ω

Proof of (1): $H(\omega) = \sum_n h(n) e^{-i\omega n}$

$$|H(\omega)| = \left|\sum_n h(n) e^{-i\omega n}\right|$$

$$\leq \sum_n \left|h(n) e^{-i\omega n}\right|$$

$$= \sum_n |h(n)|$$

$$< \infty$$

Proof of (2): TBD

* We say $h \in \ell^2$ if $\sum_n |h(n)|^2 < \infty$

where ℓ^2 = closed ℓ_2-norm space

Ex: $h(n) = G_0 \dfrac{\sin(\omega_c n)}{\pi n}$

$h \notin \ell^1$ since $\sum_n |h(n)| = \infty$

$h \in \ell^2$ since $\sum_n |h(n)|^2 < \infty$

LCCDEs & Frequency Responses

$$a_0 y(n) + a_1 y(n-1) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m) \qquad (\star)$$

* $\max(N,M)$ = order of the system = # of delay blocks to represent the system via a DAG Block Diagram

Derivation of Frequency Response ($H(\omega)$)

Let $x(n) = e^{i\omega n}$, $x(n-k) = e^{i\omega(n-k)} = e^{i\omega n} e^{-i\omega k}$

$y(n) = H(\omega) e^{i\omega n}$, $y(n-k) = H(\omega) e^{i\omega(n-k)} = H(\omega) e^{i\omega n} e^{-i\omega k}$

Plug into ($\star$):

$$\sum_{k=0}^{N} a_k H(\omega) e^{i\omega n} e^{-i\omega k} = \sum_{m=0}^{M} b_m e^{i\omega n} e^{-i\omega m}$$

$$\left(\sum_{k=0}^{N} a_k e^{-i\omega k}\right) H(\omega) = \sum_{m=0}^{M} b_m e^{-i\omega m}$$

$$H(\omega) = \frac{\sum_{m=0}^{M} b_m e^{-i\omega m}}{\sum_{k=0}^{N} a_k e^{-i\omega k}}$$

Rational in $e^{i\omega}$

LCCDEs & State-Space Representations of DT-LTI Systems

$q(n+1) = Aq(n) + Bx(n)$ — State-Evolution Equation

$y(n) = Cq(n) + Dx(n)$ — Output Equation

where

$q(n) = [q_1(n) \cdots q_N(n)]^T \in \mathbb{R}^N$	State Vector
$x(n) \in \mathbb{R}$	Input Signal
$y(n) \in \mathbb{R}$	Output Signal
$A \in \mathbb{R}^{N\times N}$	State-Transition Matrix (Time-Invariant)
$B \in \mathbb{R}^{N\times 1}$	
$C \in \mathbb{R}^{1\times N}$	
$D \in \mathbb{R}$	

LCCDEs & State-Space Representations of DT-LTI Systems

<u>Ex:</u> $y(n) + a_1 y(n-1) + a_2 y(n-2) = x(n)$

No delay term in x

$y(n) = q_1(n+1)$ $\quad q_1(n) = y(n-1)$ $\quad q_2(n) = y(n-2)$

$q_2(n+1) = y((n+1)-2)$

$= y(n-1)$

$= q_1(n)$

$$\underbrace{\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix}}_{q(n+1)} = \underbrace{\begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix}}_{q(n)} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{x(n)}_{x(n)}$$

$$\underbrace{y(n)}_{y(n)} = \underbrace{\begin{bmatrix} -a_1 & -a_2 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix}}_{q(n)} + \underbrace{1}_{D} \cdot \underbrace{x(n)}_{x(n)}$$

Lecture 8, 9/22/2022

Overview

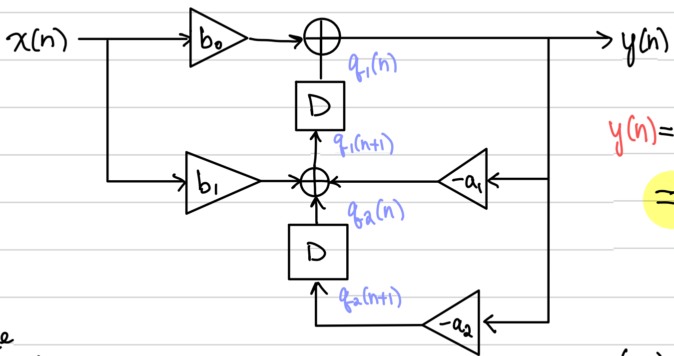
- LCCDEs & State-Space Reps (cont.)
- Dirac Delta (CT Impulse)
- CT Convolution
- CT Frequency Response

LCCDEs

Ex: $y(n)+a_1y(n-1)+a_2y(n-2)=b_0x(n)+b_1x(n-1)$

We wish to draw a DAG block diagram w/ minimum # of delay blocks

$y(n)=-a_1y(n-1)-a_2y(n-2)+b_0x(n)+b_1x(n-1)$



$y(n)=q_1(n)+b_0x(n)$

$$=\underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\underbrace{\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix}}_{q(n)}+\underbrace{b_0}_{D}x(n)$$

State-Space Representation

$q(n+1)=Aq(n)+Bx(n)$

$y(n)=Cq(n)+Dx(n)$

$q_1(n+1)=-a_1y(n)+q_2(n)+b_1x(n)$

$=-a_1(q_1(n)+b_0x(n))+q_2(n)+b_1x(n)$

$=\begin{bmatrix}-a_1 & 1\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix}+\begin{bmatrix}-a_1b_0+b_1\end{bmatrix}x(n)$

$q_2(n+1)=-a_2y(n)$

$=-a_2(q_1(n)+b_0x(n))$

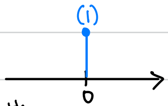
$=\begin{bmatrix}-a_2 & 0\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix}+\begin{bmatrix}-a_2b_0\end{bmatrix}x(n)$

$$q(n+1)=\begin{bmatrix}q_1(n+1)\\ q_2(n+1)\end{bmatrix}=\underbrace{\begin{bmatrix}-a_1 & 0\\ -a_2 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix}}_{q(n)}+\underbrace{\begin{bmatrix}-a_1b_0+b_1\\ -a_2b_0\end{bmatrix}}_{B}x(n)$$

9/22/2022

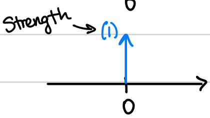
Dirac Delta (CT Impulse)

DT

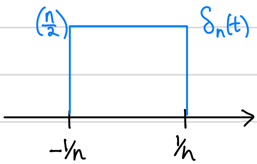


$$\delta(n)=\begin{cases}1 & n=0\\ 0 & e/w\end{cases}$$

CT



$$\delta(t)=\begin{cases}\infty & t=0\\ 0 & e/w\end{cases}$$ ← Kind of a useless description



$$\int_{-\infty}^{\infty}\delta_n(t)\,dt=1 \quad \forall n\in\{1,2,\dots\}$$

For sake of developing intuition, we can define Dirac Delta as

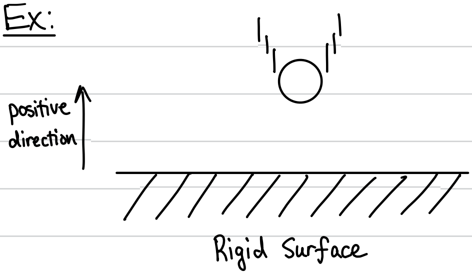
$$\delta(t)=\lim_{n\to\infty}\delta_n(t)$$ ← technically mathematically shaky

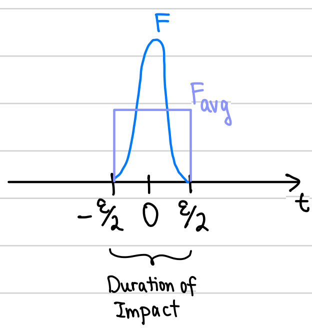
For any signal x, we have that:

Sifting Property: $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$, $\lim_{n\to\infty}\int_{-\infty}^{\infty} x(t)\,\delta_n(t)\,dt = x(0)$, $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$

Sampling Property: $x(t)\,\delta(t-t_0)=x(t_0)\,\delta(t-t_0)$, $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0) = x(t_0) = \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt = x(t_0)\underbrace{\int_{-\infty}^{\infty}\delta(t-t_0)\,dt}_{=1}$

Ex:





ε = duration of impact $= 10^{-3}$ s

$m = 0.2$ kg

$v(-\frac{\varepsilon}{2}) = -8$ m/s

$v(\frac{\varepsilon}{2}) = 8$ m/s

Newton's 2nd Law: $F(t)=\frac{d}{dt}p(t)$ where $p(t)=m(t)\,v(t)$

We solve for F_{avg}:

$$\begin{aligned}F_{avg} &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2} F(t)\,dt\\ &= \frac{1}{t_2-t_1}\left(p(t_2)-p(t_1)\right)\\ &= \frac{1}{\frac{\varepsilon}{2}-(-\frac{\varepsilon}{2})}\left(p(\tfrac{\varepsilon}{2})-p(-\tfrac{\varepsilon}{2})\right)\\ &= \frac{1}{\varepsilon}\left(mv(\tfrac{\varepsilon}{2})-mv(-\tfrac{\varepsilon}{2})\right)\\ &= 3200\text{ N}\end{aligned}$$

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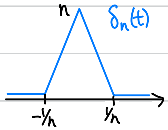
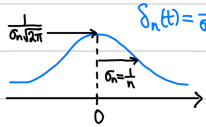
Dirac Delta (continued)

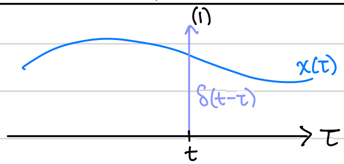
Alternative ways to get to $\delta(t)$:





$\delta_n(t) = \frac{1}{\sigma_n\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_n^2}}$

Every x(t) can be expressed as an integral of shifted impulses:



$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$ (CT counterpart of $x(n) = \sum_m x(m)\,\delta(n-m)$)

CT-LTI

$\delta(t) \longrightarrow \boxed{H} \longrightarrow h(t) =$ CT impulse response

$\delta(t-\tau) \longrightarrow \quad \longrightarrow h(t-\tau)$

$x(\tau)\,\delta(t-\tau) \longrightarrow \quad \longrightarrow x(\tau)\,h(t-\tau)$

$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \quad \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$

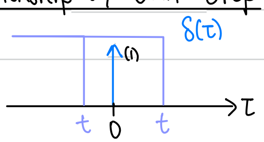
Thus:

CT-LTI

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \boxed{H} \longrightarrow y(t) = \underbrace{\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}_{\text{Convolution Integral}}$$

$y(t) = (x * h)(t) = (h * x)(t)$

Relationship w/ Unit Step



$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t \geq 0 \end{cases}$

$\frac{d}{dt} u(t) = \delta(t)$

Lecture 9, 9/27/2022

Overview

- DT-LTI Systems & Internal Stability
- CT-LTI Systems & Frequency Response

DT-LTI Systems & Internal Stability

$q(n+1) = Aq(n) + Bx(n)$ State Evolution Equation

$y(n) = Cq(n) + Dx(n)$ Output Equation

Internal (Asymptotic) Stability: $q(n+1) = Aq(n)$ "Zero-Input Response (ZIR)"

$y(n) = Cq(n)$

↳ System is internally stable if $q(n) \to 0$ as $n \to \infty$ regardless of $q(0)$ (init state)

↳ System is internally stable if $|\lambda_i(A)| < 1 \ \forall i \in [n]$

$q(n) = A^n q(0) = V\Lambda^n V^{-1} q(0) \to 0$ iff $|\lambda_i(A)| < 1 \ \forall i \in [n]$

CT-LTI Systems & Frequency Responses

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau = (x * h)(t) = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda)\, d\lambda = (h * x)(t)$$

$x(t) = e^{i\omega t}$

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\tau$$

$$= \underbrace{\int_{-\infty}^{\infty} h(\lambda)\, e^{-i\omega\lambda}\, d\lambda}_{H(\omega)} \cdot e^{i\omega t}$$

↳ $H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$ = Frequency Response (for CT-LTI systems)

DT-LTI	CT-LTI
$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}$	$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt$
$H(\omega + 2\pi) = H(\omega) \ \forall \omega \in \mathbb{R}$	No periodicity here

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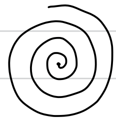
CT-LTI Systems & Frequency Responses

Ex: $x(t) \longrightarrow \boxed{g(t)} \longrightarrow y(t)$

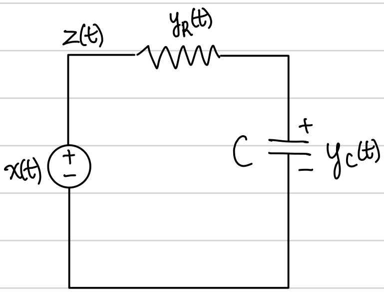
$g(t) = \beta e^{-\alpha t} u(t), \quad \beta > 0 \quad \alpha > 0$

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \beta e^{-\alpha t} u(t) e^{-i\omega t}\, dt \\
&= \int_{0}^{\infty} \beta e^{-t(\alpha + i\omega)}\, dt \\
&= \beta \left. \frac{e^{-t(\alpha+i\omega)}}{-\alpha - i\omega} \right|_0^{\infty} \\
&= \frac{\beta}{\alpha + i\omega}
\end{aligned}
$$

Note: $\lim_{t \to \infty} e^{-(\alpha+i\omega)t} = \lim_{t\to\infty} e^{-\alpha t} e^{-i\omega t} = 0$



Ex:



$x(t) \longrightarrow \boxed{h_c(t)} \longrightarrow y_c(t)$

$$
\begin{cases}
y_c(t) = \dfrac{Q_c(t)}{C} = \dfrac{1}{C} \displaystyle\int_{-\infty}^{t} z(\tau)\, d\tau \\
z(t) = \dfrac{y_R(t)}{R} = \dfrac{x(t) - y_c(t)}{R}
\end{cases}
$$

$\hookrightarrow y_c(t) = \frac{1}{RC} \int_{-\infty}^{t} [x(\tau) - y_c(t)]\, d\tau$

$\dot{y}_c(t) = \frac{1}{RC}[x(t) - y_c(t)]$

$\hookrightarrow RC\,\dot{y}_c(t) + y_c(t) = x(t)$ LCCDE

$x(t) = e^{i\omega t}$

$y_c(t) = H_c(\omega) e^{i\omega t}$

$\dot{y}_c(t) = i\omega H_c(\omega) e^{i\omega t}$

$\hookrightarrow RC\, i\omega H_c(\omega) e^{i\omega t} + H_c(\omega) e^{i\omega t} = e^{i\omega t}$

Freq Resp $\left\{ H_c(\omega) = \dfrac{1}{1 + RC i\omega} = \dfrac{1/RC}{1/RC + i\omega} \right.$

Impulse Resp $\left\{ h(t) = \beta e^{-\alpha t} u(t) = \dfrac{1}{RC} e^{-\frac{1}{RC} t} u(t) \right.$

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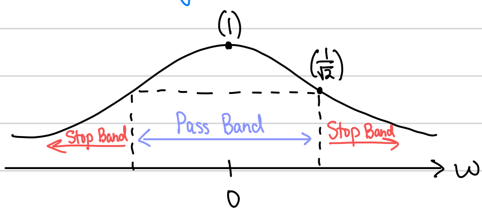
L9

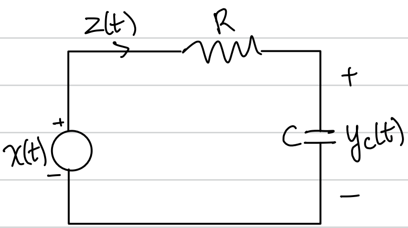
Plotting Frequency Response



Lecture 10, 9/29/2022

Overview

- CT-LTI Freq Response (cont)
  - RC-CkT
  - Mass-Spring Damper



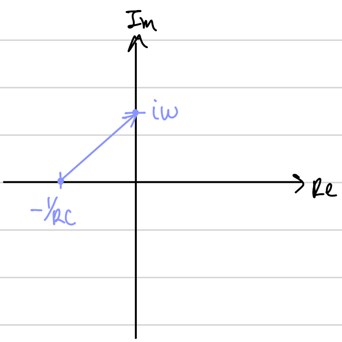
$$RC\dot{y}_c(t) + y_c(t) = x(t)$$

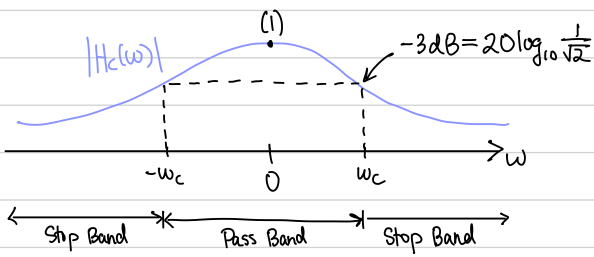
Let $x(t) = e^{i\omega t} \rightarrow y(t) = H_c(\omega)e^{i\omega t}$

Solving for $H_c(\omega)$, we get $H_c(\omega) = \frac{1/RC}{i\omega + 1/RC}$

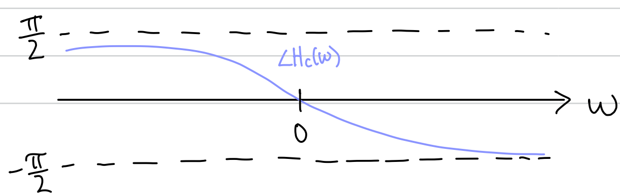
(High-pass)

$$|H_c(\omega)| = \frac{1/RC}{\|\nearrow\|}$$





$$\angle H_c(\omega) = \angle\frac{1}{RC} - \angle\left(i\omega + \frac{1}{RC}\right)$$
$$= 0 - \angle\left(i\omega + \frac{1}{RC}\right)$$
$$= -\tan^{-1}\left(\frac{\omega}{1/RC}\right)$$



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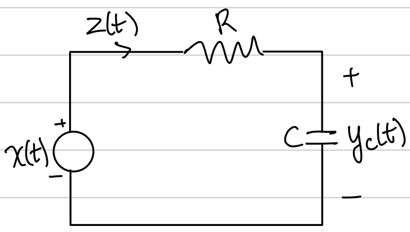
Impulse Response (Revisted)



$$h_c(t) = \frac{1}{RC} e^{-t/RC}$$

$$g(t) = \beta e^{-\alpha t} u(t) \iff G(\omega) = \frac{\beta}{i\omega + \alpha}$$

where $\alpha > 0$

$$\hookrightarrow h_c(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} \iff H_c(\omega) = \frac{\overbrace{\frac{1}{RC}}^{\beta}}{i\omega + \underbrace{\frac{1}{RC}}_{\alpha}}$$

Another way to derive $h_c(t)$:

$$RC\dot{y}_c(\tau) + y_c(\tau) = x(\tau)$$

$$RC\dot{h}_c(\tau) + h_c(\tau) = \delta(\tau)$$

Sampling Prop.

$$RC\dot{h}_c(\tau)e^{\tau/RC} + h_c(\tau)e^{\tau/RC} = \delta(\tau)e^{\tau/RC} = \delta(\tau)$$

$$\dot{h}_c(\tau)e^{\tau/RC} + \frac{1}{RC}h_c(\tau)e^{\tau/RC} = \frac{1}{RC}\delta(\tau)$$

$$\frac{d}{d\tau}\left(h(\tau)e^{\tau/RC}\right) = \frac{1}{RC}\delta(\tau)$$

$$\int_{-\infty}^{t} \frac{d}{d\tau}\left(h(\tau)e^{\tau/RC}\right) d\tau = \int_{-\infty}^{t} \frac{1}{RC}\delta(\tau)\, d\tau$$

$$h(t)e^{t/RC} = \frac{1}{RC}u(t)$$

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t)$$

Integrator-Adder-Gain Block Diagram

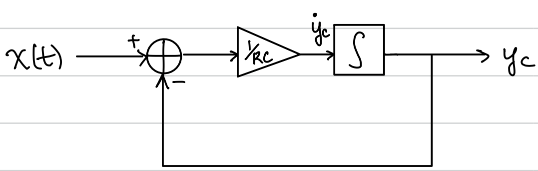
*We use integration instead differentiation because differentiation is numerically unstable.

$$RC\dot{y}_c + y_c = x$$

$$RC\, y_c + \int y_c = \int x$$

$$y_c = \frac{1}{RC}\int(-y_c) + \frac{1}{RC}\int x$$

$$= \frac{1}{RC}\int(-y_c + x)$$



CT-LTI Systems Described by LCCDEs

$$a_N \frac{d^N}{dt^N} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_M \frac{d^M}{dt^M} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{m=0}^{M} b_k \frac{d^m}{dt^m} x(t)$$

System initially at rest.

Deriving $H(\omega)$: Let $x = e^{i\omega t} \rightarrow y = H(\omega) e^{i\omega t}$

$$\frac{d^m}{dt^m} x(t) = (i\omega)^m e^{i\omega t}, \quad \frac{d^k}{dt^k} y(t) = H(\omega)(i\omega)^k e^{i\omega t}$$

$$\hookrightarrow \sum_{k=0}^{N} a_k (i\omega)^k H(\omega) e^{i\omega t} = \sum_{m=0}^{M} b_m (i\omega)^m e^{i\omega t}$$

$$H(\omega) = \frac{\sum_{m=0}^{M} b_m (i\omega)^m}{\sum_{k=0}^{N} a_k (i\omega)^k} \in \mathbb{Q} \text{ in } i\omega$$

<u>Example:</u> $\underbrace{RC}_{a_1}\dot{y}(t) + \underbrace{y_c(t)}_{a_0=1} = \underbrace{x(t)}_{b_0=1}$

$$H_c(\omega) = \frac{b_0}{a_0 + a_1(i\omega)} = \frac{1}{1 + RCi\omega} = \frac{1/RC}{1/RC + i\omega}$$ (same as what we computed previously)

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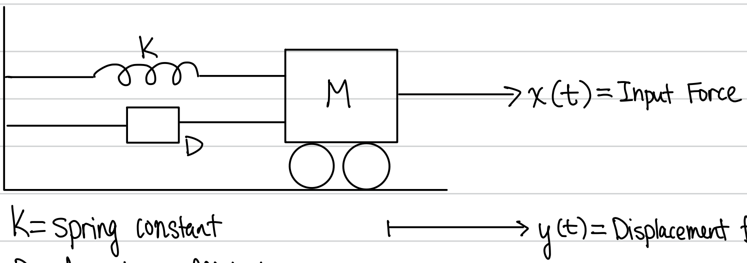
2nd-Order Mechanical System



K = Spring constant
D = damping coefficient

System Equation: $M\frac{d^2}{dt^2}y(t) = -D\frac{d}{dt}y(t) - Ky(t) + F(t)$

$$\underbrace{M}_{a_2}\frac{d^2}{dt^2}y(t) + \underbrace{D}_{a_1}\frac{d}{dt}y(t) + \underbrace{K}_{a_0}y(t) = \underbrace{F(t)}_{b_0=1}$$

Frequency Response: $H(\omega) = \frac{1}{M(i\omega)^2 + D(i\omega) + K}$

↳ Let $s = i\omega$, $\hat{H}(s) = \frac{1}{Ms^2 + Ds + K}$ *Roots of denominator are natural frequencies of the system

State-Space Derivation: $M\underbrace{\ddot{y}(t)}_{\dot{q}_2(t)} + D\underbrace{\dot{y}(t)}_{q_2(t)=\dot{q}_1(t)} + K\underbrace{y(t)}_{q_1(t)} = x(t)$

We want to determine: $\dot{q}(t) = Aq(t) + Bx(t)$ State-Evolution Eq.
$y(t) = Cq(t) + Dx(t)$ Output Eq.

↳ $\dot{q}_1(t) = q_2(t)$
$\ddot{y}(t) = -\frac{D}{M}\dot{y}(t) - \frac{K}{m}y(t) + \frac{1}{M}x(t)$
$\dot{q}_2(t) = -\frac{D}{M}q_2(t) - \frac{K}{m}q_1(t) + \frac{1}{M}x(t)$

$$\underbrace{\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix}}_{\dot{q}(t)} = \underbrace{\begin{bmatrix}0 & 1\\ -\frac{K}{M} & -\frac{D}{M}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}}_{q(t)} + \underbrace{\begin{bmatrix}0\\ \frac{1}{M}\end{bmatrix}}_{B}x(t)$$

$y(t) = q_1(t) \longrightarrow$ $$y(t) = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\underbrace{\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}}_{q(t)} + \underbrace{0}_{D}\cdot x(t)$$

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2^{nd}-Order Mechanical System—Undamped Case

Let D = damping coeff = 0

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & 0 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ \frac{K}{M} & \lambda \end{bmatrix}$$

$\det(\lambda I - A) = \lambda^2 + \frac{K}{M} = 0 \rightarrow \lambda = \pm i\sqrt{\frac{K}{M}}$ ($\omega_0 = \sqrt{\frac{K}{M}}$ = resonant/natural frequency of system)

Differential Equations for Scalars & Matrices

$\dot{q}(t) = aq(t)$, w/ initial state $q(0)$

↳ Has solution $q(t) = a^t q(0)$

We now also want to derive the solution to $\dot{q}(t) = Aq(t)$, w/ initial state $q(0)$

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$
$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$
$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + (At) + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

$$\frac{d}{dt} e^{At} = Ae^{At}$$

Thus, $\dot{q}(t) = Aq(t)$ w/ initial state $q(0)$ has solution $q(t) = e^{At} q(0)$

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Differential Matrix Equations

Let A have distinct eigenvalues (w/ linearly independent eigenvectors)
↳ Suppose 2 eigen-pairs (λ_1, v_1) & (λ_2, v_2)

$$q(0) = \alpha_1 v_1 + \alpha_2 v_2$$
$$q(t) = e^{At} q(0) = e^{At}(\alpha_1 v_1 + \alpha_2 v_2)$$
$$= \alpha_1 e^{At} v_1 + \alpha_2 e^{At} v_2$$
$$= \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2$$

* $e^{At} v = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k v = \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k v = \left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\right) v = e^{\lambda t} v$

Thus, we can express any state $q(t)$ in terms of the initial state components.

<u>Example:</u> for the 2nd order mechanical system from earlier, the state space can be expressed as

$$q(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 = \alpha_1 e^{i\omega_0 t} v_1 + \alpha_2 e^{i\omega_0 t} v_2$$

where $\omega_0 = \sqrt{\frac{K}{M}}$

2nd-Order Mechanical System - Damped Case

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ \frac{K}{M} & \lambda + \frac{D}{M} \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^2 + \frac{D}{M}\lambda + \frac{K}{M} = 0$$

*Recall $H(\omega) = \frac{1}{M(i\omega)^2 + D(i\omega) + K} = \frac{\frac{1}{M}}{(i\omega)^2 + \frac{D}{M}(i\omega) + \frac{K}{M}}$

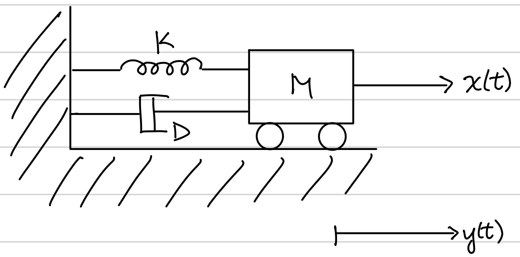
↳ We now see that the solutions to the denominator are indeed the eigenvalues (and thus the natural frequencies) of the system

Lecture 11, 10/4/2022

Overview

- Mass-Spring-Damper System (continued)
- Interconnections of LTI Systems
  - ↳ Cascade
  - ↳ Parallel
  - ↳ Feedback

Mass-Spring-Damper System



$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -K/M & D/M \end{bmatrix}}_{A} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/M \end{bmatrix}}_{B} x(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

$$M\ddot{y}(t) + D\dot{y}(t) + Ky(t) = x(t)$$

$$\det(\lambda I - A) = \lambda^2 + \frac{D}{M}\lambda + \frac{K}{M} = 0$$

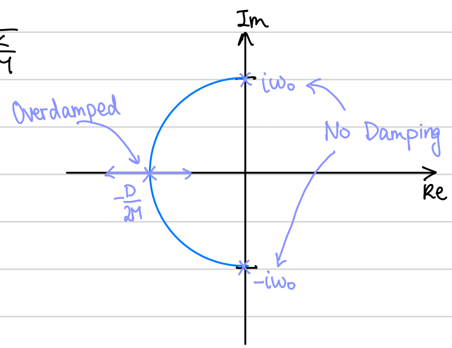
$$\hookrightarrow \lambda = -\frac{D}{2M} \pm \frac{1}{2}\sqrt{\left(\frac{D}{M}\right)^2 - \frac{4K}{M}}$$
$$= -\frac{D}{2M} \pm \sqrt{\left(\frac{D}{2M}\right)^2 - \frac{K}{M}}$$
$$= -\frac{D}{2M} \pm \frac{D}{2M}\sqrt{1 - \frac{K}{M}\left(\frac{2M}{D}\right)^2}$$

Case 1: $D = 0 \longrightarrow \lambda = \pm\sqrt{-\frac{K}{M}} = \pm i\omega_0, \; \omega_0 = \sqrt{\frac{K}{M}}$

Case 2: $\left(\frac{D}{2M}\right)^2 - \frac{K}{M} = 0 \rightarrow \lambda_1 = \lambda_2 = -\frac{D}{2M}$

Case 3: $\left(\frac{D}{2M}\right)^2 - \frac{K}{M} > 0 \rightarrow \lambda_1 \neq \lambda_2 \in \mathbb{R}$

Case 4: $\left(\frac{D}{2M}\right)^2 - \frac{K}{M} < 0 \rightarrow \lambda_1 \neq \lambda_2 \in \mathbb{C}$



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Mass-Spring-Damper System (cont)

$\dot{q}(t) = Aq(t)$, $q(0)$ initial position & velocity

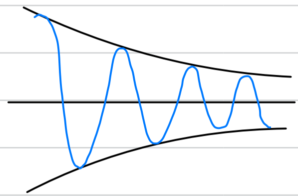
$\hookrightarrow$ Has solution $q(t) = e^{At}q(0)$

Note that if we let $V = [v_1 \; v_2] = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ be the eigenvector matrix of A, we have that

$$q(0) = \alpha_1 v_1 + \alpha_2 v_2$$
$$q(t) = \alpha_1 e^{At} v_1 + \alpha_2 e^{At} v_2 = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2$$

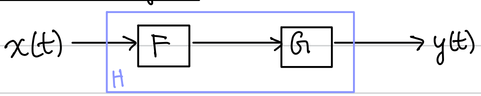
If $D=0$: $\quad q(t) = \alpha_1 e^{i\omega_0 t} v_1 + \alpha_2 e^{-i\omega_0 t} v_2$ (purely oscillatory)

If $\left(\frac{D}{2M}\right)^2 - \frac{K}{M} > 0$:
$$q_1(t) = \text{position} = \alpha_1 v_{11} e^{\lambda_1 t} + \alpha_2 v_{12} e^{\lambda_2 t}$$
$$= \alpha_1 v_{11} e^{(\sigma + i\omega_1)t} + \alpha_2 v_{12} e^{(\sigma - i\omega_1)t}$$
$$= \alpha_1 v_{11} e^{\sigma t} e^{i\omega_1 t} + \alpha_2 v_{12} e^{\sigma t} e^{-i\omega_1 t}$$
$$= e^{\sigma t}\left(\alpha_1 v_{11} e^{i\omega_1 t} + \alpha_2 v_{12} e^{-i\omega_1 t}\right)$$



Interconnections of LTI Systems

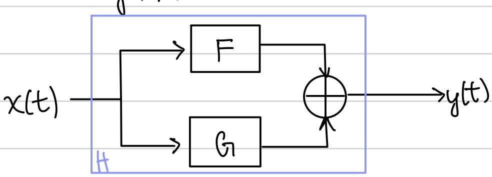
Cascade (Series) System



$h(t) = (f * g)(t)$

$H(\omega) = G(\omega)F(\omega)$

Parallel System



$h(t) = (f+g)(t)$

$H(\omega) = F(\omega) + G(\omega)$

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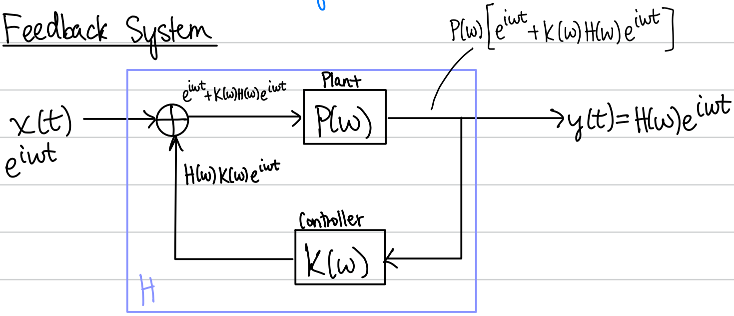
Interconnections of LTI Systems

Feedback System



$$P(\omega)\left[1+K(\omega)H(\omega)\right]e^{i\omega t}=H(\omega)e^{i\omega t}$$

$$P(\omega)+K(\omega)H(\omega)P(\omega)=H(\omega)$$

$$\left[1-K(\omega)P(\omega)\right]H(\omega)=P(\omega)$$

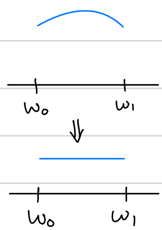
$$\hookrightarrow H(\omega)=\frac{P(\omega)}{1-K(\omega)P(\omega)}=\frac{\text{Forward Gain}}{1-\text{Loop Gain}} \quad \text{(Black's Formula)}$$

↳ **Application I:** Compensation for non-ideal element

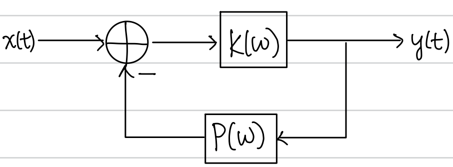
Let $|K(\omega)P(\omega)| \gg 1$ for $\underbrace{\omega\in[\omega_1,\omega_2]}_{\text{Freq. Range of Interest}}$. Then, we have

$$H(\omega)=\frac{P(\omega)}{1-(-K(\omega)P(\omega))}=\frac{P(\omega)}{1+K(\omega)P(\omega)}\approx\frac{1}{K(\omega)}$$

→ Could build flat $K(\omega)=K_0$ for $K_0<1$



↳ **Application II:** Inverse System Design (Negative Feedback)



If $|K(\omega)P(\omega)| \ll 1$ over $\omega\in[\omega_1,\omega_2]$. Then, we have

$$H(\omega)\approx\frac{K(\omega)}{K(\omega)P(\omega)}=\frac{1}{P(\omega)}$$

$$H(\omega)=\frac{K(\omega)}{1+K(\omega)P(\omega)}$$

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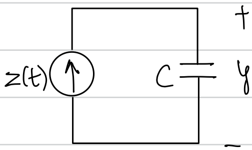
Interconnections of LTI Systems

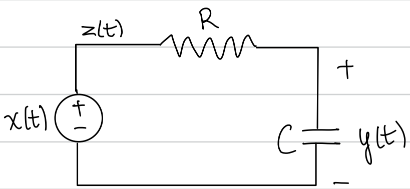
↳ <u>Application III</u>: Stabilization of Unstable Systems

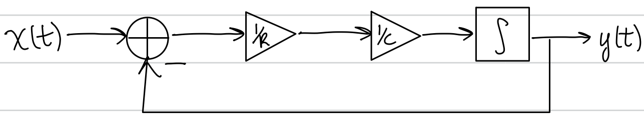


$$y(t) = \frac{1}{C}\int_{-\infty}^{t} z(\tau)\,d\tau$$

$h(t) = \frac{1}{C}\int_{-\infty}^{t} \delta(\tau)\,d\tau = \frac{1}{C}u(t)$ ⟶ Capacitor is <u>NOT</u> BIBO stable!

Let's add a resistor:





$$y(t) = \frac{1}{C}\int_{-\infty}^{t} z(\tau)\,d\tau = \frac{1}{C}\int_{-\infty}^{t} \frac{x(\tau)-y(\tau)}{R}\,d\tau$$

Lecture 12, 10/6/2022

Overview

- Feedback (Last application)
- Fourier Analysis

Feedback System

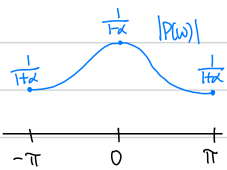
↳ <u>Application IV</u>:

$$P(\omega) = \frac{\overset{\swarrow b_0}{1}}{\underset{\uparrow a_0}{1} - \underset{\uparrow a_1}{\alpha} e^{-i\omega}}, \quad 0<\alpha<1$$

$$a_0 y(n) + a_1 y(n-1) = b_0 x(n)$$

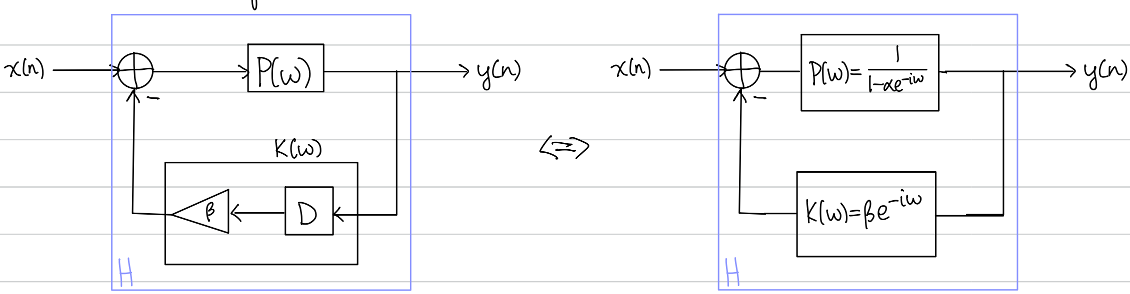
$$y(n) - \alpha y(n-1) = x(n)$$

$$y(n) = \alpha y(n-1) + x(n)$$



Low-Pass Filter

Can convert to High-Pass Filter as follows:



$$H(\omega) = \frac{P(\omega)}{1+K(\omega)P(\omega)} = \frac{\frac{1}{1-\alpha e^{-i\omega}}}{1+\frac{\beta e^{-i\omega}}{1-\alpha e^{-i\omega}}} = \frac{1}{1-(\alpha-\beta)e^{-i\omega}} \leftrightarrow y(n) = (\alpha-\beta)y(n-1) + x(n)$$

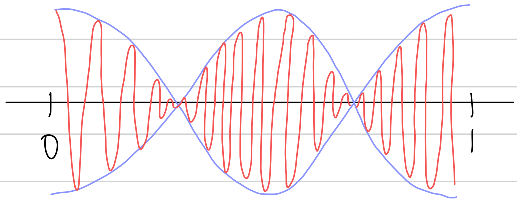
↳ Pick β such that $\alpha<\beta<\alpha+1$ to make a High-Pass filter

10/6/2022

Fourier Analysis

$$x(t) = 2\cos(2\pi t)\cos(200\pi t)$$
$$= 2\cos(2\pi(1)t)\cos(2\pi(100)t)$$
$$= 2\cos(\omega_1 t)\cos(\omega_2 t)$$

where $\omega_1 = 2\pi$ rad/s, $\omega_2 = 200\pi$ rad/s
$f_1 = 1$ Hz $\quad f_2 = 100$ Hz



"Beating Effect"

↳ Fourier Analysis: About breaking down (decomposing) a signal into its constituent frequencies

$$e^{i\omega_1 t} \longrightarrow \boxed{H(\omega)} \longrightarrow H(\omega_1)e^{i\omega_1 t}$$

$$e^{i\omega_2 t} \longrightarrow \boxed{H(\omega)} \longrightarrow H(\omega_2)e^{i\omega_2 t}$$

$$x(t) = \alpha_1 e^{i\omega_1 t} + \alpha_2 e^{i\omega_2 t} \longrightarrow \boxed{H(\omega)} \longrightarrow \alpha_1 H(\omega_1)e^{i\omega_1 t} + \alpha_2 H(\omega_2)e^{i\omega_2 t}$$

Computing constituent frequencies:

$$x(t) = 2\cos(2\pi t)\cos(200\pi t)$$

Using the identity $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$= \cos(\underbrace{2\pi \cdot 99t}_{99\text{ Hz}}) + \cos(\underbrace{2\pi \cdot 101t}_{101\text{ Hz}})$$

$$= \frac{1}{2}e^{i2\pi\cdot 99t} + \frac{1}{2}e^{-i2\pi\cdot 99t} + \frac{1}{2}e^{i2\pi\cdot 101t} + \frac{1}{2}e^{-i2\pi\cdot 101t}$$

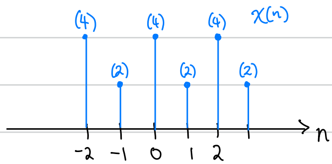
10/6/2022

Fourier Analysis

	DT	CT
Periodic	DT FS →	CT FS
Aperiodic	DT FT →	CT FT

DT FS: DT Fourier Series
CT FS: CT Fourier Series
DT FT: DT Fourier Transform
CT FT: CT Fourier Transform

DTFS



$P=2$, $\omega_0=\frac{2\pi}{P}=\frac{2\pi}{2}=\pi$, so we can express $x=\begin{bmatrix}4\\2\end{bmatrix}$ $\begin{matrix}\leftarrow x(p)=x(0)\\ \leftarrow x(p-1)=x(1)\end{matrix}$

↳ Let $\phi_0=\begin{bmatrix}1\\0\end{bmatrix}$, $\phi_1=\begin{bmatrix}0\\1\end{bmatrix}$, then $x=\begin{bmatrix}4\\2\end{bmatrix}=4\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}=4\phi_0+2\phi_1$

↳ Let $\psi_0=\begin{bmatrix}1\\1\end{bmatrix}$, $\psi_1=\begin{bmatrix}1\\-1\end{bmatrix}$, then $x=\begin{bmatrix}4\\2\end{bmatrix}=X_0\psi_0+X_1\psi_1=X_0e^{i0n}+X_1e^{i\pi n}$

↳ $\psi_0(n)=e^{i0n}$ ↳ $\psi_1(n)=e^{i\pi n}$ * $\psi_0\perp\psi_1$

To determine X_0, project x onto ψ_0: $x\cdot\psi_0=(X_0\psi_0+X_1\psi_1)\cdot\psi_0$ $\quad \psi_0\cdot\psi_1=0$

$=X_0\psi_0\cdot\psi_0$

Thus, $X_0=\frac{x\cdot\psi_0}{\psi_0\cdot\psi_0}$. Similarly, $X_1=\frac{x\cdot\psi_1}{\psi_1\cdot\psi_1}$. Finally, we have:

$$x=X_0\psi_1+X_1\psi_1=3\psi_1+\psi_1=3e^{i0n}+e^{i\pi n}$$

↳ For any P-periodic DT signal, we have that

$$x(n)=X_0e^{i\cdot 0\cdot\omega_0 n}+X_1e^{i\omega_0 n}+X_2e^{i\omega_0 2n}+\dots+X_{P-1}e^{i(P-1)\omega_0 n}$$

and only the following frequencies contribute to x: $0, \omega_0, 2\omega_0, \dots, k\omega_0, (P-1)\omega_0$

Lecture 13, 10/11/2022

Review

$x(n+2) = x(n)$, $P=2$, $\omega_0 = \frac{2\pi}{P} = \pi$

$x(n) = X_0 \psi_0(n) + X_1 \psi_1(n)$ where $\psi_k(n) = e^{ik\omega_0 n}$

$\hookrightarrow \psi_0(n) = 1 \; \forall n$, $\psi_1(n) = e^{i\pi n} = (-1)^n \; \forall n$

X_0, X_1 = Fourier Coefficients, $X_k = \frac{x^T \psi_k}{\psi_k^T \psi_k}$

DTFS

Key Properties of ψ_k:

- $\psi_k(n+p) = e^{ik\omega_0(n+p)} = e^{ik\omega_0 n} e^{ik\omega_0 p} = e^{ik\omega_0 n} e^{i2\pi} = e^{ik\omega_0 n} = \psi_k(n)$
- $\psi_{k+p}(n) = e^{i(k+p)\omega_0 n} = e^{ik\omega_0 n} e^{ip\omega_0 n} = e^{ik\omega_0 n} e^{i2\pi} = e^{ik\omega_0 n} = \psi_k(n)$

Theorem: given periodic signal $x(n+p) - x(n) \; \forall n, \exists p \in \mathbb{N}$, we can express x as follows:

$$\boxed{\begin{gathered} x(n) = X_0 \psi_0(n) + X_1 \psi_1(n) + \cdots + X_k \psi_k(n) + \cdots + X_{p-1} \psi_{p-1}(n) \\ \text{where } \psi_k(n) = e^{ik\omega_0 n}, \; \omega_0 = \frac{2\pi}{p} \end{gathered}}$$

and we have that $X_0, \dots, X_k, \dots, X_{p-1}$ are the only frequencies present in a p-period signal and form an orthogonal basis.

Inner Product: given complex vectors $f, g \in \mathbb{C}^P$, we can define inner product: $\boxed{\langle f, g \rangle \triangleq f^T g^*}$

$\hookrightarrow$ Properties:

- $\langle f, g \rangle = \langle g, f \rangle^*$
- $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$
- $\langle f, \beta g \rangle = \beta^* \langle f, g \rangle$

10/11/2022

DTFS (continued)

$$x = X_0\psi_0 + \cdots + X_k\psi_k + \cdots + X_{p-1}\psi_{p-1}$$

Assume for now that $(\psi_i)_{i\in[p-1]}$ are all mutually orthogonal.

To determine X_k, we project x onto ψ_k as follows:

$$\begin{aligned}\langle x, \psi_k\rangle &= \left\langle \sum_{\ell=0}^{p-1} X_\ell\psi_\ell, \psi_k\right\rangle \\ &= \sum_{\ell=0}^{p-1}\langle X_\ell\psi_\ell, \psi_k\rangle \\ &= \sum_{\ell=0}^{p-1} X_\ell\langle\psi_\ell,\psi_k\rangle \quad \text{Since } \langle\psi_\ell,\psi_k\rangle\ \forall \ell\neq k \\ &= X_k\langle\psi_k,\psi_k\rangle\end{aligned}$$

Thus: $\boxed{X_k = \frac{\langle x, \psi_k\rangle}{\langle\psi_k,\psi_k\rangle}}$

Show that $\langle\psi_k,\psi_k\rangle = p$:

$$\langle\psi_k,\psi_k\rangle = \psi_k^T\psi_k^* = \sum_{n=0}^{p-1}\psi_k(n)\psi_k^*(n) = \sum_{n=0}^{p-1} e^{ik\omega_0 n}e^{-ik\omega_0 n} = \sum_{n=0}^{p-1} 1 = p$$

Show that $\psi_k \perp \psi_\ell\ \forall\ k\neq \ell$:

$$\begin{aligned}\langle\psi_k,\psi_\ell\rangle &= \psi_k^T\psi_\ell^* \\ &= \sum_{n=0}^{p-1}\psi_k(n)\psi_\ell^*(n) \\ &= \sum_{n=0}^{p-1} e^{ik\omega_0 n}e^{-i\ell\omega_0 n} \\ &= \sum_{n=0}^{p-1} e^{i(k-\ell)\omega_0 n} \\ &= \sum_{n=0}^{p-1}\left[e^{i(k-\ell)\omega_0}\right]^n \\ &= \frac{e^{i(k-\ell)\omega_0 p} - e^{i(k-\ell)\omega_0\cdot 0}}{e^{i\omega_0}-1} \\ &= 0\end{aligned}$$

$$\hookrightarrow \langle\psi_k,\psi_\ell\rangle = \begin{cases} p & \text{if } k=\ell \\ 0 & \text{if } k\neq\ell\end{cases}$$
$$= p\,\delta(k-\ell)$$

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DTFS (continued)

$$x=\sum_{k=0}^{p-1} X_k \psi_k \quad \Rightarrow \quad x(n)=\sum_{k=0}^{p-1} X_k \psi_k(n)=\sum_{k=0}^{p-1} X_k e^{ik\omega_0 n} \quad \text{(Synthesis Equation)}$$

$$X_k=\frac{\langle x,\psi_k\rangle}{\langle \psi_k,\psi_k\rangle}=\frac{1}{p}\langle x,\psi_k\rangle \Rightarrow X_k=\frac{1}{p}\sum_{n=0}^{p-1} x(n)\psi_k^*(n)=\frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ik\omega_0 n} \quad \text{(Analysis Equation)}$$

Characterization of a DTFS:

$$\boxed{\begin{aligned} x(n)&=\sum_{k=0}^{p-1} X_k e^{ik\omega_0 n} && \text{Synthesis Eq.} \\ X_k&=\frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ik\omega_0 n} && \text{Analysis Eq.} \end{aligned}}$$

Interpretation of X_0: $X_0=\frac{1}{p}\sum_{n=0}^{p-1} x(n)=$ Average of x over one period

Relationships between Time / Frequency Domains:

Coefficient	X_k	$=$	X_{k+p}
	$\updownarrow$		$\updownarrow$
Complex Exponent	$\psi_k(n)$	$=$	$\psi_{k+p}(n)$
	$\updownarrow$		$\updownarrow$
Frequency	$k\omega_0$	$=$	$(k+p)\omega_0$

Then, we can relax the DTFS Equations to be more general:

$$\boxed{\begin{aligned} x(n)&=\sum_{k\in\langle p\rangle} X_k e^{ik\omega_0 n} && \text{Synthesis} \\ X_k&=\frac{1}{p}\sum_{k\in\langle p\rangle} x(n)e^{-ik\omega_0 n} && \text{Analysis} \end{aligned}}$$

Where $\langle p\rangle \triangleq$ a set of p contiguous integers

Ex. $x(n)=\cos\left(\frac{2\pi}{3}n\right)\ \forall n\in\mathbb{Z}$

↳ $p=3,\ \omega_0=\frac{2\pi}{p}=\frac{2\pi}{3}$. Instead of directly computing X_0,X_1,X_2, it's easier to just compute $\overset{=X_2}{X_{-1}},X_0,X_1$.

$x(n)=\underbrace{\frac{1}{2}e^{-i\frac{2\pi}{3}n}}_{X_{-1}=X_2}+\underbrace{\frac{1}{2}e^{i\frac{2\pi}{3}n}}_{X_1}$, $X_0=0$ $\Rightarrow \boxed{X_0=0,X_1=\frac{1}{2},X_2=\frac{1}{2}}$

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DTFS (continued)

<u>Ex.</u> $x(n) = \cos(n)$

↳ Not periodic, and thus has no DTFS expansion
(Period not rational multiple of π)

Lecture 14, 10/13/2022

Review

$$x(n) = \sum_{k\in\langle p\rangle} X_k e^{ik\omega_0 n} \quad \text{Synthesis Eqn}$$

$$X_k = \frac{1}{p}\sum_{k\in\langle p\rangle} x(n) e^{-ik\omega_0 n} \quad \text{Analysis Eqn}$$

DTFS (continued)

Ex. $x(n) = \cos^2(\frac{2\pi}{3}n)$

↳ $x(n) = \frac{1+\cos(\frac{4\pi}{3}n)}{2} = \frac{1}{2} + \frac{1}{2}\cos(\frac{4\pi}{3}n) = \underbrace{\frac{1}{4}e^{-i\frac{4\pi}{3}n}}_{X_{-2}=X_1} + \underbrace{\frac{1}{2}}_{X_0} + \underbrace{\frac{1}{4}e^{i\frac{4\pi}{3}n}}_{X_2}$

$p=3,\ \omega_0 = \frac{2\pi}{3}$

Ex: $x(n) = \sum_{\ell=-\infty}^{\infty} \delta(n-\ell p) = \cdots + \delta(n-p) + \delta(n) + \delta(n+p)$

↳ Period p, $\omega_0 = \frac{2\pi}{p}$

$X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n) e^{-ik\omega_0 n}$ ⇒ All terms are zero, except $n=0$ b/c $x(1)=x(2)=\cdots=x(p-1)=0$

$X_k = \frac{1}{p}x(0) = \frac{1}{p} \quad \forall k \in \{0,1,\ldots,p-1\}$

↳ $x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n} = \frac{1}{p}\sum_{k=0}^{p-1} e^{ik\omega_0 n}$

$$\boxed{x(n) = \sum_{\ell=-\infty}^{\infty}\delta(n-\ell p) = \frac{1}{p}\sum_{k=0}^{p-1} e^{ik\omega_0 n} \quad \text{(Poisson's Identity)}}$$

Let us prove Poisson's Identity:

- Case I: $n \bmod p = 0$, $n = mp$

  ↳ $\frac{1}{p}\sum_{k=0}^{p-1} e^{ik(\frac{2\pi}{p})mp} = \frac{1}{p}\sum_{k=0}^{p-1} e^{ik2\pi m} = \frac{1}{p}\sum_{k=0}^{p-1} 1 = 1$

- Case II: $n \bmod p \neq 0$

  ↳ $\sum_{k=0}^{p-1} e^{ik\omega_0 n} = \sum_{k=0}^{p-1}[e^{i\omega_0 n}]^k = \frac{e^{i\omega_0 pn}-1}{\lambda - 1} = 0$

Properties:

- $x(n) = x(n+p) \quad \forall n\in\mathbb{Z}, \exists p \in \mathbb{N}$
- $x(n) \in \mathbb{R}\ \forall n \iff X_k^* = X_{-k}$ (Conjugate Symmetry Property)
- $x(n) \in \mathbb{R}\ \forall n,\ x(n) = x(-n) \longrightarrow X_k^* = X_{-k} = X_k$

  (signal real & even) (Coeff real & even)

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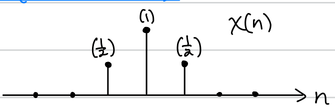
DTFS (continued)

Ex.



↳ Create a periodic extension of x: $\tilde{x}(n) = \sum_{\ell=-\infty}^{\infty} x(n+\ell p) = \cdots + x(n-p) + x(n) + x(n+p) + \cdots$
$p=3$

$$= \cdots + x(n-3) + x(n) + x(n+3) + \cdots$$

$$x(n) = \begin{cases} \tilde{x}(n) & n=-1,0,1 \\ 0 & \text{e/w} \end{cases}$$

$p=3$, $\omega_0 = \frac{2\pi}{3}$

$$\tilde{X}_k = \frac{1}{3}\sum_{n=-1}^{1} \tilde{x}(n) e^{-ik\frac{2\pi}{3}n} = \frac{1}{3}\sum_{n=-1}^{1} x(n) e^{-ik\frac{2\pi}{3}n}$$

Since x is real & even, we know that $X_k = X_{-k} \in \mathbb{R}$. Thus, we can compute:

$$\tilde{X}_0 = \text{Avg of } x(n) \text{ over } n \in \{-1,0,1\} = \frac{1}{3}(2) = \frac{2}{3}$$

$$\tilde{X}_1 = \frac{1}{3}\left(x(-1)e^{i\frac{2\pi}{3}} + x(0) + x(1)e^{-i\frac{2\pi}{3}}\right) = \frac{1}{3}\left(1 + \cos\left(\tfrac{2\pi}{3}\right)\right) = \frac{1}{6}$$
$$= \tilde{X}_{-1}$$

Thus, since $X_i = \tilde{X}_i$, we have that $x(n) = \begin{cases} \frac{2}{3} + \frac{1}{6}e^{-i\frac{2\pi}{3}n} + \frac{1}{6}e^{i\frac{2\pi}{3}n} & \text{if } n=-1,0,1 \\ 0 & \text{e/w} \end{cases}$

Matrix-Vector Formulation (DTFS)

$$x(n) = \sum_{k=0}^{p-1} X_k \underbrace{e^{ik\omega_0 n}}_{\psi_k(n) = e^{ik\omega_0 n}} = \sum_{k=0}^{p-1} X_k \psi_k(n), \quad n \in [p-1]$$

$$\begin{bmatrix} x(0) \\ \vdots \\ x(n) \\ \vdots \\ x(p-1) \end{bmatrix} = X_0 \underbrace{\begin{bmatrix} \psi_0(0) \\ \vdots \\ \psi_n(0) \\ \vdots \\ \psi_{p-1}(0) \end{bmatrix}}_{\Psi_0 = \mathbb{1}} + \cdots + X_k \underbrace{\begin{bmatrix} \psi_k(0) \\ \vdots \\ \psi_k(n) \\ \vdots \\ \psi_k(p-1) \end{bmatrix}}_{\Psi_k} + \cdots + X_{p-1} \underbrace{\begin{bmatrix} \psi_{p-1}(0) \\ \vdots \\ \psi_{p-1}(n) \\ \vdots \\ \psi_{p-1}(p-1) \end{bmatrix}}_{\Psi_{p-1}}$$

↳ $x = \underbrace{[\Psi_0 \cdots \Psi_k \cdots \Psi_{p-1}]}_{\Psi} \underbrace{\begin{bmatrix} X_0 \\ \vdots \\ X_k \\ \vdots \\ X_{p-1} \end{bmatrix}}_{X} = \Psi X$ where $\Psi = \begin{bmatrix} \psi_0(0) & \cdots & \psi_k(0) & \cdots & \psi_{p-1}(0) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_0(n) & \cdots & \psi_k(n) & \cdots & \psi_{p-1}(n) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_0(p-1) & \cdots & \psi_k(p-1) & \cdots & \psi_{p-1}(p-1) \end{bmatrix} = [\psi_k(n)] = [e^{ik\omega_0 n}]$

Different Notations

10/13/2022

Matrix-Vector Formulation (DTFS)

$\underline{x} = \Psi \underline{X}$ where Ψ is orthogonal

Note that:

- $\Psi^T \Psi^* = pI$
- $\Psi^H \Psi = pI$
- $\underbrace{\left(\frac{1}{p}\Psi^H\right)}_{\Psi^{-1}} \Psi = I$

↳ $\boxed{F = \Psi^{-1} = \frac{1}{p}\Psi^H \quad \text{(Fourier Matrix)}}$

$\underline{x} = \Psi \underline{X} \rightarrow \Psi^{-1}\underline{x} = \underline{X}$

The fourier transform converts a signal in the time domain into its coefficients in the freq domain!

↳ $\boxed{\underline{X} = F\underline{x}}$

Parseval's Theorem (for DTFS)

$$\frac{1}{N}\sum_{n\in\langle N\rangle} |x(n)|^2 = \sum_{k\in\langle N\rangle} |X_k|^2$$

Lecture 15, 10/18/2022

Overview

- DTFS Wrap-Up
- CTFS

DTFS

$$\hat{x}(n) = x(n+p) = x(n) \longrightarrow \boxed{\begin{matrix} h(n) \\ H(\omega) \end{matrix}}_{\text{DT-LTI}} \longrightarrow \hat{y}(n) = y(n+p) = y(n)$$

(x is p-periodic) (y is p-periodic)

* If x is p-periodic, then y is periodic, and its period cannot exceed p

Since y is periodic it has a DTFS expansion — what is it?

$$e^{ik\omega_0 n} \longrightarrow \boxed{H(\omega)} \longrightarrow H(k\omega_0)e^{ik\omega_0 n}$$

$$X_k e^{ik\omega_0 n} \longrightarrow \boxed{H(\omega)} \longrightarrow X_k H(k\omega_0) e^{ik\omega_0 n}$$

$$x(n) = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} \longrightarrow \boxed{H(\omega)} \longrightarrow \boxed{y(n) = \sum_{k \in \langle p \rangle} \underbrace{X_k H(k\omega_0)}_{Y_k} e^{ik\omega_0 n} = \sum_{k \in \langle p \rangle} Y_k e^{ik\omega_0 n}}$$

CTFS

$x: \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$

$x(t+p) = x(t)$ $\forall t$ for some $p > 0$

If p is the smallest possible real value that satisfies $x(t+p) = x(p)$ $\forall t$, then it is the fundamental period, and we have $\omega_0 = \frac{2\pi}{p}$.

Lecture 15

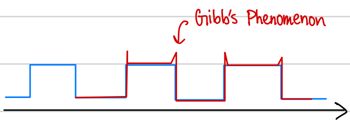
CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \underbrace{e^{ik\omega_0 t}}_{\psi_k(t)} = \sum_{k=-\infty}^{\infty} X_k \psi_k(t)$$ (Synthesis Equation)

The frequencies that contribute to x are: $\cdots$ $-2\omega_0$ $\;-\omega_0$ $\;0$ $\;\omega_0$ $\;2\omega_0$ $\cdots$

$$\begin{array}{ccccc} -2\omega_0 & -\omega_0 & 0 & \omega_0 & 2\omega_0 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ X_{-2} & X_1 & X_0 & X_1 & X_2 \end{array}$$

Example:



$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$ → will always produce some "spikes" as shown in the red. This is known as "Gibb's Phenomenon"

In computers, instead of using a perfectly continuous function $x(t)$, we use a finite sum approximation like $x_N(t) = \sum_{k=-N}^{N} X_k e^{ik\omega_0 t}$

Determining the Coefficients X_k:

Assume:
- We have a properly-defined inner product
- ψ_k are mutually orthogonal, i.e. $\psi_k \perp \psi_\ell \;\; \forall k \neq \ell$

Then, to determine X_ℓ, we project $x = \sum_k X_k \psi_k$ onto ψ_ℓ as follows:

$$\langle x, \psi_\ell \rangle = \langle \sum_k X_k \psi_k, \psi_\ell \rangle$$
$$= \sum_k X_k \langle \psi_k, \psi_\ell \rangle$$
$$= X_k \langle \psi_\ell, \psi_\ell \rangle$$
$$\hookrightarrow \; X_k = \frac{\langle x, \psi_\ell \rangle}{\langle \psi_\ell, \psi_\ell \rangle}$$

What is a good inner product? In DT, we defined $\langle f, g \rangle \triangleq \sum_{n \in \langle p \rangle} f(n) g^*(n)$

↳ In CT, we define $\boxed{\langle v, u \rangle \triangleq \int_{\langle p \rangle} v(t)\, u^*(t)\, dt}$

Lecture 15

CTFS

More on Inner Products:

$$||x||^2 = \langle x, x \rangle = \int_{\langle p \rangle} x(t)\, x^*(t)\, dt = \int_{\langle p \rangle} |x(t)|^2\, dt$$

$$||\psi_\ell||^2 = \langle \psi_\ell, \psi_\ell \rangle = \int_{\langle p \rangle} \psi_\ell(t)\, \psi_\ell^*(t)\, dt = \int_{\langle p \rangle} e^{i\ell\omega_0 t} e^{-i\ell\omega_0 t}\, dt = \int_{\langle p \rangle} 1\, dt = p$$

$$\langle \psi_k, \psi_\ell \rangle = \int_{\langle p \rangle} \psi_k(t)\, \psi_\ell^*(t)\, dt = \int_{\langle p \rangle} e^{ik\omega_0 t} e^{-i\ell\omega_0 t}\, dt = \int_{\langle p \rangle} e^{i(k-\ell)\omega_0 t}\, dt = \underbrace{\int_{\langle p \rangle} \cos[(k-\ell)\omega_0 t]\, dt}_{0} + i \underbrace{\int_{\langle p \rangle} \sin[(k-\ell)\omega_0 t]\, dt}_{0} = 0$$

where $k \neq \ell$

$$= \frac{e^{i(k-\ell)\omega_0 t}}{i(k-\ell)\omega_0}\Bigg|_0^p = \frac{e^{i(k-\ell)\omega_0 p} - 1}{i(k-\ell)\omega_0} = \frac{1-1}{i(k-\ell)\omega_0} = 0$$

Synthesis Equation: $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$

Analysis Equation: $X_k = \frac{\langle x, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle} = \frac{1}{p}\int_{\langle p \rangle} x(t)\, \psi_k^*(t)\, dt = \frac{1}{p}\int_{\langle p \rangle} x(t)\, e^{-ik\omega_0 t}\, dt$

DT vs CT

DT: $\langle p \rangle = [m, m+p-1]$

e.g. $m=0 \rightarrow [0, p-1]$

CT: $\langle p \rangle = [\tau, \tau+p)$

$[\tau, \tau+p]$

$(\tau, \tau+p)$

$(\tau, \tau+p]$

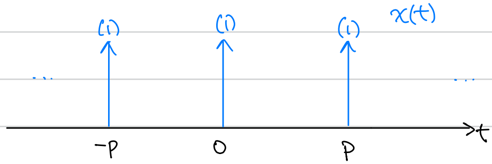
Typically, we chose $(0,p)$, $[0,p]$, or $[-\frac{p}{2}, \frac{p}{2}]$

↳ *We choose an interval where the bounds are not at any dirac deltas, so we don't have to worry about open/closed

Lecture 15

CTFS

Ex:



$$x(t) = \sum_{\ell=-\infty}^{\infty} \delta(t-\ell p)$$

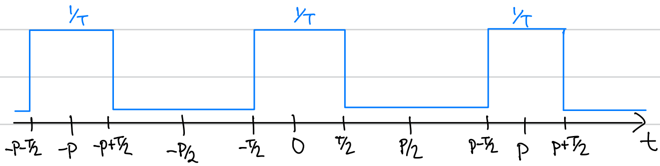
$$X_k = \frac{1}{P}\int_{-P/2}^{P/2} \delta(t) e^{-ik\omega_0 t}\,dt = \frac{1}{P}(1) = \frac{1}{P}$$

* Cannot possibly be a pointwise equality

$$\underbrace{x(t) = \sum_{\ell=-\infty}^{\infty} \delta(t-\ell p) = \sum_{k=-\infty}^{\infty} X_k \psi_k(t) = \frac{1}{P}\sum_{k=-\infty}^{\infty} e^{ik\omega_0 t}}_{\text{Poisson's Identity}}$$

$$\hookrightarrow \int \sum_{\ell} \delta(t-\ell p)\,\phi(t)\,dt = \int \frac{1}{P}\sum_{k} e^{ik\omega_0 t}\,\phi(t)\,dt$$

Ex:



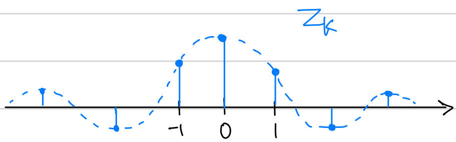
$$z(t) = \sum_{k=-\infty}^{\infty} Z_k e^{ik\omega_0 t}$$

$$Z_k = \frac{1}{P}\int_{-P/2}^{P/2} z(t) e^{-ik\omega_0 t}\,dt$$

$$= \frac{1}{PT}\int_{-P/2}^{P/2} e^{-ik\omega_0 t}\,dt$$

$$= \frac{1}{PT}\cdot \frac{e^{-ik\omega_0 t}}{-ik\omega_0}\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{P}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0 T/2}$$



Lecture 16, 10/25/2022

Overview

	DT	CT
Periodic	DTFS & DFT	CTFS
Aperiodic	DTFT ☆	CTFT

☆ You are here!

DTFT

$$h(n) \xrightarrow{} \underbrace{H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}}_{\text{DTFT of } h}$$

Today

$h: \mathbb{Z} \to \mathbb{R}$ or $\mathbb{C}$

We want to derive the synthesis equation for $x(n)$ where $\underbrace{X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n}}_{\text{Spectrum of } x}$

↳ 2 Methods!

Method I

Leverage our knowledge of CTFS

$x: \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$

$x(t+p) = x(t)$ $\forall t \in \mathbb{R}$ for some $p > 0$

$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}$

$H: \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$

$H(\omega + 2\pi) = H(\omega)$ $\forall \omega \in \mathbb{R}$

$x(t) \longleftrightarrow H(\omega)$

$t \longleftrightarrow \omega$

$p \longleftrightarrow 2\pi$

$\omega_0 = \frac{2\pi}{p} \longleftrightarrow \Omega_0 = \frac{2\pi}{2\pi} = 1$

$$
\begin{aligned}
H(\omega) &= \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} && \text{let } k = -n \\
&= \sum_{k=-\infty}^{\infty} h(-k) e^{ik \cdot 1 \cdot \omega} && \text{let } H_k = h(-k) \\
&= \sum_{k=-\infty}^{\infty} H_k e^{ik\Omega_0 \omega} \\
x(t) &= \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}
\end{aligned}
$$

Lecture 16 [DTFT]

Method I (cont)

$$X_k = \frac{1}{p}\int_{\langle p\rangle} x(t)e^{-ik\omega_0 t}\,dt$$

$$H_k = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} H(\omega)e^{-ik\Omega_0\omega}\,d\omega = h(-k)$$

let $n=-k$

$$\underbrace{h(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} H(\omega)e^{i\omega n}\,d\omega}_{\text{Synthesis Equation!}}$$

DTFT Equations:

$$\begin{cases} x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)e^{i\omega n}\,d\omega & \text{Synthesis Equation} \\ X(\omega) = \sum_n x(n)e^{-i\omega n} & \text{Analysis Equation} \end{cases}$$

↳ Interpretation: rewrite $x(n) = \int_{\langle 2\pi\rangle} \underbrace{\frac{d\omega}{2\pi} X(\omega)} e^{i\omega n}$

$$\updownarrow$$

(CTFS) $x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{ik\omega_0 t}$

↳ $X(\omega)$ indicates how much frequency ω ($e^{i\omega n}$) contributes to the signal x

The larger $|X(\omega)|$ is for an arbitrary frequency ω relative to other frequencies, the larger the contribution of frequency ω to the signal x

Lecture 16 [DTFT]

Method II — Direct Derivation via Orthogonal Expansion

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) \underbrace{e^{-i\omega n}}_{\phi_n(\omega)} = \sum_n h(n)\,\phi_n$$

(We start with the Analysis Equation from LTI system)

$\phi_n(\omega) = e^{-i\omega n}$

$H(\omega+2\pi) = H(\omega) \quad \forall \omega \in \mathbb{R}$

$\phi_n(\omega+2\pi) = e^{-i(\omega+2\pi)n} = e^{-i\omega n} e^{-i2\pi n} = e^{-i\omega n} = \phi_n(\omega) \quad \forall \omega \in \mathbb{R}$

How to find $h(\ell)$? Project H onto ϕ_ℓ!

Assume $\phi_k \perp \phi_\ell$ for $k \neq \ell$

↳ $\langle H, \phi_\ell \rangle = \langle \sum_n h(n)\phi_n, \phi_\ell \rangle = \sum_n h(n) \langle \phi_n, \phi_\ell \rangle = h(\ell) \langle \phi_\ell, \phi_\ell \rangle$

$$\boxed{h(\ell) = \frac{\langle H, \phi_\ell \rangle}{\langle \phi_\ell, \phi_\ell \rangle}}$$

How to define an appropriate inner product?

↳ Let F, G belong to our vector space $F, G: \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ and $F(\omega+2\pi) = F(\omega)$, $G(\omega+2\pi) = G(\omega)$ $\forall \omega \in \mathbb{R}$. Then, let us define $\boxed{\langle F, G \rangle \triangleq \int_{\langle 2\pi \rangle} F(\omega)\, G^*(\omega)\, d\omega}$

Now, let us prove that $\langle \phi_n, \phi_n \rangle = 2\pi$ and $\langle \phi_k, \phi_\ell \rangle = 0$ for $k \neq \ell$

↳ $\langle \phi_n, \phi_n \rangle = \int_{\langle 2\pi \rangle} \phi_n(\omega)\, \phi_n^*(\omega)\, d\omega = \int_{\langle 2\pi \rangle} e^{-i\omega n} e^{i\omega n}\, d\omega = 2\pi$

$\langle \phi_k, \phi_\ell \rangle = \int_{\langle 2\pi \rangle} \phi_k(\omega)\, \phi_\ell^*(\omega)\, d\omega = \int_{\langle 2\pi \rangle} e^{-i\omega k} e^{i\omega \ell}\, d\omega = \int_\lambda^{\lambda+2\pi} e^{i\omega(\ell-k)}\, d\omega = \left. \frac{e^{i\omega(\ell-k)}}{i(\ell-k)} \right|_\lambda^{\lambda+2\pi} = 0$

↳ $\boxed{\langle \phi_k, \phi_\ell \rangle = 2\pi\,\delta(k-\ell)}$

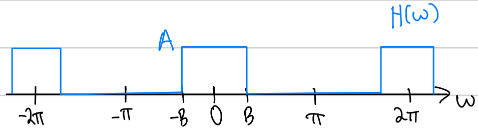
Thus, we finally have

$$h(n) = \frac{\langle H, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_{\langle 2\pi \rangle} H(\omega)\, \phi_n^*(\omega)\, d\omega}{2\pi} = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} H(\omega)\, e^{i\omega n}\, d\omega \qquad \text{(Synthesis Equation)}$$

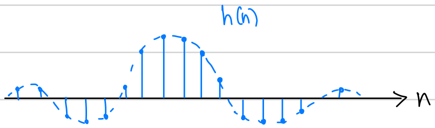
$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n} \qquad \text{(Analysis Equation)}$$

Lecture 16 [DTFT]

Ex 1



$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega) e^{i\omega n}\, d\omega = \frac{A}{2\pi}\int_{-B}^{B} e^{i\omega n}\, d\omega = \frac{A}{\pi n}\,\frac{e^{i\omega n}}{2i}\Big|_{-B}^{B} = \frac{A}{\pi n}\,\frac{e^{iBn}-e^{-iBn}}{2i} = \frac{A}{\pi n}\sin(Bn)$$



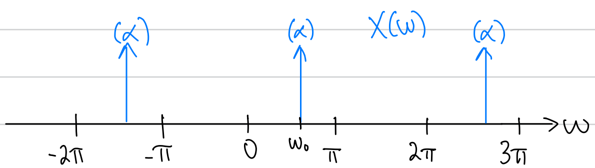
$h(n)$ decays away from $n=0$ on the order of $\frac{1}{n}$, so $\sum_n |h(n)| = \infty$, and thus not BIBO stable.
$h(n)$ is also not causal because there exist nonzero $h(n)$ where $n<0$

* If $h \notin \ell^1$ but $h \in \ell^2$ (i.e. h is square summable), then we cannot use the analysis equation to obtain the frequency response (i.e. obtain the DTFT)

Ex 2 $x(n) = e^{i\omega_0 n}$ $\quad 0<\omega_0<\pi$

Can we compute $X(\omega)$? Yes; $x \notin \ell^1$ and $x \notin \ell^2$, but x doesn't grow (or decay).

* As long as x doesn't grow faster than polynomially, $X(\omega)$ exists. However, we cannot use the analysis equation and $X(\omega)$ will be impulsive in nature (i.e. has Dirac delta(s))



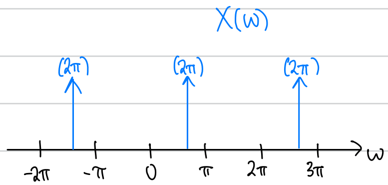
in $[-\pi, \pi]$, $X(\omega) = \alpha\,\delta(\omega-\omega_0)$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\alpha\,\delta(\omega-\omega_0)}_{X(\omega)} e^{i\omega n}\, d\omega = \frac{\alpha}{2\pi} e^{i\omega_0 n} = e^{i\omega_0 n} \longrightarrow \alpha = 2\pi$$

Lecture 16 [DTFT]

Ex 2 (cont)

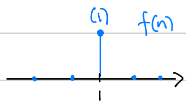
$x(n) = e^{i\omega_0 n} \xleftrightarrow{\mathcal{F}}$



$$X(\omega) = \underbrace{2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi\ell) = \sum_{n=-\infty}^{\infty} e^{-i(\omega-\omega_0)n}}_{\text{Another application of Poisson's Identity}}$$

Ex 3 $F(\omega) = e^{-i\omega}$

$f(n) = \delta(n-1)$



$\xleftrightarrow{\mathcal{F}} F(\omega) = e^{-i\omega}$

$$F(\omega) = \sum_n f(n) e^{-i\omega n} = f(1) e^{-i\omega} = e^{-i\omega}$$

Ex 4 $G(\omega) = e^{-i\omega N}$ $\quad N \in \mathbb{Z}$

$g(n) = \delta(n-N)$

Ex 5

$$H(\omega) = \begin{cases} e^{-i\omega/2} & |\omega| \le \pi \\ \text{2π-periodically replicates outside} \end{cases} \longrightarrow \cancel{h(n) = \delta(n-\tfrac{1}{2})}$$

Half Sample Delay Filter

This is nonsense since $-\frac{1}{2}$ may not be an integer!

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\omega/2} e^{i\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(n-1/2)}\, d\omega$$

$$= \frac{1}{\pi(n-1/2)} \cdot \frac{e^{i\omega(n-1/2)}}{2i}\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n-1/2)} \cdot \frac{e^{i\pi(n-1/2)} - e^{-i\pi(n-1/2)}}{2i}$$

$$= \frac{1}{\pi(n-1/2)} \sin[\pi(n-1/2)]$$

$$= \mathrm{Sinc}(n-1/2)$$

* $\mathrm{Sinc}\,\alpha \triangleq \dfrac{\sin(\pi\alpha)}{\pi\alpha}$

Lecture 17, 10/27/2022: More DTFT

Review (DTFT)

$$x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{i\omega n} \, d\omega \qquad \text{(Synthesis)}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n} \qquad \text{(Analysis)}$$

Time-Shift Property:

$$x(n) \xleftrightarrow{\mathcal{F}} X(\omega)$$

$$\hat{x}(n) = x(n-N) \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = X(\omega) e^{i\omega N}$$

Today: More DTFT

Time Reversal Property

$$x(n) \xleftrightarrow{\mathcal{F}} X(\omega)$$

$$\boxed{\hat{x}(n) = x(-n) \xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = X(-\omega)}$$

$$\begin{aligned}
\hat{X}(\omega) &= \sum_n \hat{x}(n) e^{-i\omega n} \\
&= \sum_n x(-n) e^{-i\omega n} \\
&= \sum_\ell x(\ell) e^{i\omega \ell} \\
&= X(-\omega)
\end{aligned}$$

If $x(n) \in \mathbb{R}$, then $\overbrace{X^*(\omega) = X(-\omega)}^{\text{Conjugate Symmetry}}$

If $x(n) = x(-n) \in \mathbb{R}$, then $X^*(\omega) = X(\omega)$, i.e. $X(\omega) \in \mathbb{R}$

Convolution in Time to Frequency

$$\underset{x(n)}{X(\omega)} \longrightarrow \boxed{\overset{H(\omega)}{h(n)}} \longrightarrow y(n) = (x * h)(n) = \sum_k x(k) h(n-k)$$

$$Y(\omega) = \sum_n y(n) e^{-i\omega n} = \sum_n \overbrace{\left[\sum_k x(k) h(n-k) \right]}^{y(n)} e^{-i\omega n} = \sum_k x(k) \sum_n h(n-k) e^{-i\omega n}$$

$$= \sum_k x(k) \sum_\ell h(\ell) e^{-i\omega(\ell + k)} = \underbrace{\left(\sum_k x(k) e^{-i\omega k} \right)}_{X(\omega)} \underbrace{\left(\sum_\ell h(\ell) e^{-i\omega \ell} \right)}_{H(\omega)} = X(\omega) H(\omega)$$

Lecture 17: More DTFT

Convolution in Time to Frequency

$$\boxed{y(n) = (x * h)(n) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = X(\omega)H(\omega)}$$

↳ * Convolution in Time Domain ⟷ Multiplication in Frequency Domain

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$$

$$\sum_{k=0}^{N} a_k \mathcal{F}\{y(n-k)\} = \sum_{m=0}^{M} b_m \mathcal{F}\{x(n-m)\}$$

$$\left[\sum_{k=0}^{N} a_k e^{-i\omega k}\right] Y(\omega) = \left[\sum_{m=0}^{M} b_m e^{-i\omega m}\right] X(\omega)$$

↳ $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{\sum_{m=0}^{M} b_m e^{-i\omega m}}{\sum_{k=0}^{N} a_k e^{-i\omega k}}$

Frequency-Shift Property

$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$\hat{x}(n) = x(n)e^{i\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} \hat{X}(\omega) = X(\omega - \omega_0)$$ (Amplitude Modulation)

$$\hat{x}(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} \hat{X}(\omega) e^{i\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega-\omega_0) e^{i\omega n}\, d\omega$$

$\lambda = \omega - \omega_0$, $d\lambda = d\omega$

$$= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\lambda) e^{i(\lambda+\omega_0)n}\, d\lambda$$

$$= \left[\frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\lambda) e^{i\lambda n}\, d\lambda\right] e^{i\omega_0 n}$$

$$= x(n)e^{i\omega_0 n}$$

Overview of Dual Properties

$x(n-N) \longleftrightarrow X(\omega)e^{-i\omega N}$

$x(n)e^{i\omega_0 n} \longleftrightarrow X(\omega-\omega_0)$

Lecture 17: More DTFT

Ex 1 $x(n) \in \mathbb{R} \ \forall n \xleftrightarrow{\mathcal{F}} X(\omega)$

Note that $x(n) = x_e(n) + x_o(n) \rightarrow x_e(n) = \frac{x(n)+x(-n)}{2}$

$x_o(n) = \frac{x(n)-x(-n)}{2}$

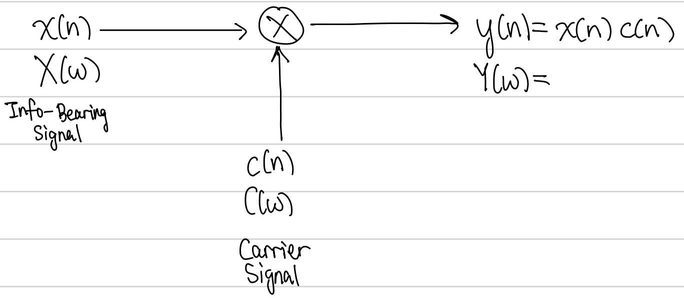
$$\mathcal{F}\{x_e(n)\} = \frac{X(\omega)+X(-\omega)}{2}$$
$$= \frac{X(\omega)+X^*(\omega)}{2}$$ b/c $x(n) \in \mathbb{R}$
$$= \mathrm{Re}\{X(\omega)\}$$

$$\mathcal{F}\{x_o(n)\} = i\,\mathrm{Im}\{X(\omega)\}$$

$\hookrightarrow$ $x_e(n) \xleftrightarrow{\mathcal{F}} \mathrm{Re}\{X(\omega)\}$

$x_o(n) \xleftrightarrow{\mathcal{F}} i\,\mathrm{Im}\{X(\omega)\}$

Modulation Property (Dual of Convolution Property $(f * g)(n) \xleftrightarrow{\mathcal{F}} F(\omega)\,G(\omega)$)



$$Y(\omega) = \sum_n y(n) e^{-i\omega n}$$
$$= \sum_n x(n) c(n) e^{-i\omega n}$$
$$= \frac{1}{2\pi} \sum_n x(n) \int_{\langle 2\pi \rangle} C(\lambda) e^{i(\lambda-\omega)n} \, d\lambda$$

$c(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} C(\lambda) e^{i\lambda n} \, d\lambda$

$$= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} C(\lambda) \left[\sum_n x(n) e^{-i(\omega-\lambda)n}\right] d\lambda$$
$$= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} C(\lambda) X(\omega-\lambda) \, d\lambda$$
$$= \frac{1}{2\pi} (C \circledast X)(\omega)$$

$^*(F \circledast G)(\omega) = \int_{\langle 2\pi \rangle} F(\lambda) G(\omega-\lambda) \, d\lambda$

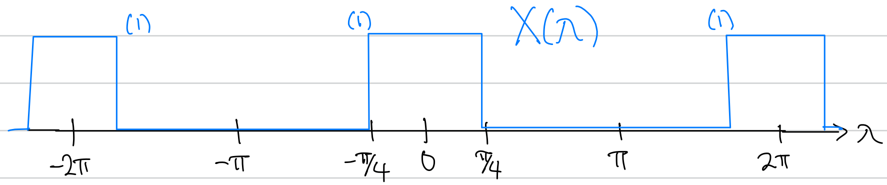
"Circular Convolution"

Lecture 17: More DTFT

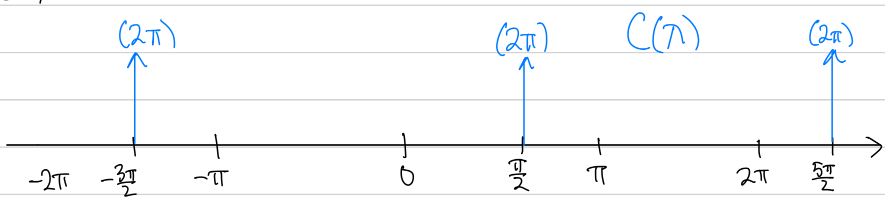
Modulation Property

$$y(n) = x(n)c(n) \xleftrightarrow{\mathcal{F}} Y(\omega) = \frac{1}{2\pi}(C \circledast X)(\omega) = \frac{1}{2\pi}(X \circledast C)(\omega)$$

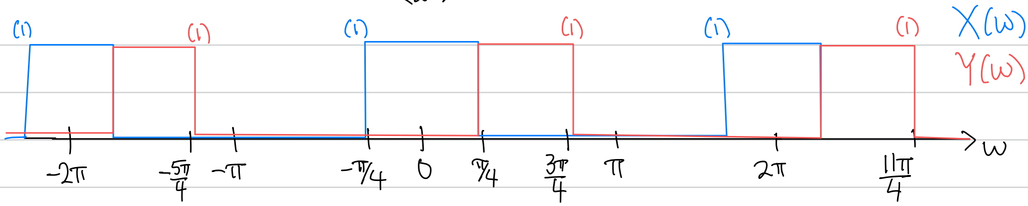
Ex 2



$$c(n) = e^{i\frac{\pi}{2}n}$$



$$Y(\omega) = \frac{1}{2\pi}(X \circledast C)(\omega) = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} X(\lambda)C(\omega-\lambda)\, d\lambda$$



Basically, since $C(\omega)$ is a bunch of impulses, to get $Y(\omega)$ we just shift $X(\omega)$ to the right by $\pi/2$

Performing Circular Convolution

1. Keep the more complicated function in place
2. For the other function, delete all replicas
3. Carry out normal convolution (within a single period)
4. Copy to all other periods

Lecture 17: More DTFT

Rayleigh-Plancherel Identity

$$\boxed{\langle x, y \rangle = \frac{1}{2\pi} \langle X, Y \rangle}$$

Special Case: $x = y$

↳ $\boxed{\text{Parseval's Identity: } \langle x, x \rangle = \frac{1}{2\pi} \langle X, X \rangle, \ \|x\|^2 = \frac{1}{2\pi} \|X\|^2}$

Recall $\left.\begin{aligned} X(\omega+2\pi) &= X(\omega) \\ Y(\omega+2\pi) &= Y(\omega) \end{aligned}\right\} \boxed{\langle X, Y \rangle \triangleq \int_{\langle 2\pi \rangle} X(\omega) Y^*(\omega)\, d\omega}$

How do we define inner product for aperiodic DT-Signals?

$$\langle x, y \rangle \triangleq \sum_{n=-\infty}^{\infty} x(n)\, y^*(n)$$

Lecture 18, 11/01/2022: CTFT

Overview

	DT	CT
Periodic	DTFS DFT	CTFS
Aperiodic	DTFT	Today! CTFT

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} \, dt \qquad \text{(Analysis)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \, d\omega \qquad \text{(Synthesis)}$$

CTFT Equations

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} \, dt \qquad \text{(Analysis)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \, d\omega \qquad \text{(Synthesis)}$$

Instead of rad/s (ω), we can also use hertz (f)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} \, dt \qquad \text{(Analysis)}$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} \, df \qquad \text{(Synthesis)}$$

However, in this class we will be using the CTFT equations w.r.t rad/s (ω)

Interpretation of Synthesis Equation

$$x(t) = \underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} X(\omega)}_{\text{Linear Combo}} e^{i\omega t}$$

We are looking at an uncountable linear combination of $e^{i\omega t}$, where $X(\omega)$ represents the contribution of the corresponding $e^{i\omega t}$.

↳ Any frequency within $[0, +\infty)$ can have a contribution to the frequency representation since x isn't periodic

Lecture 18: CTFT

Derivation Sketch of Synthesis Equation

$$X(\omega) = \int_{-\infty}^{\infty} dt \; x(t) \underbrace{e^{-i\omega t}}_{\phi_t(\omega)} \rightarrow X = \int_{-\infty}^{\infty} dt \; x(t) \; \phi_t$$

Assuming $\phi_t \perp \phi_\tau$ for $t \neq \tau$, then $\langle X, \phi_\tau \rangle = \langle \int dt \; x(t) \; \phi_t, \phi_\tau \rangle$

↳ $\langle X, \phi_\tau \rangle = \langle \int dt \; x(t) \; \phi_t, \phi_\tau \rangle$ Fubini's Theorem (Given certain conditions, allows exchange of integral ordering)

$$= \int dt \; x(t) \; \langle \phi_t, \phi_\tau \rangle$$

$$= 2\pi \int_{-\infty}^{\infty} x(t) \; \delta(t-\tau) \; dt$$ Sifting Property

$$= 2\pi \, x(\tau)$$

$$\boxed{\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \; d\omega}$$

↳ $x(\tau) = \frac{1}{2\pi} \langle X, \phi_\tau \rangle$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \, e^{i\omega\tau} \; d\omega \quad \checkmark$$

$$\boxed{\begin{aligned} \langle \phi_t, \phi_\tau \rangle &= \int_{-\infty}^{\infty} \phi_t(\omega) \, \phi_\tau^*(\omega) \; d\omega \\ &= \int_{-\infty}^{\infty} e^{-i\omega t} e^{i\omega\tau} \; d\omega \\ &= 2\pi \, \delta(t-\tau) \end{aligned}}$$

Inner Product for CTFT

For aperiodic functions of a continuous variable:

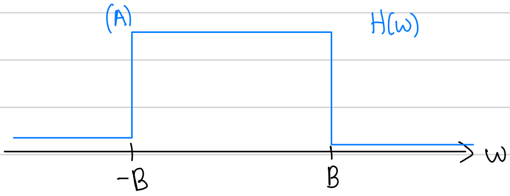
f, g are functions of time t

F, G are functions of angular freq ω

↳ $$\boxed{\begin{aligned} \langle f, g \rangle &\triangleq \int_{-\infty}^{\infty} f(t) \, g^*(t) \; dt \\ \langle F, G \rangle &\triangleq \int_{-\infty}^{\infty} F(\omega) \, G^*(\omega) \; d\omega \end{aligned}}$$

Lecture 18: CTFT

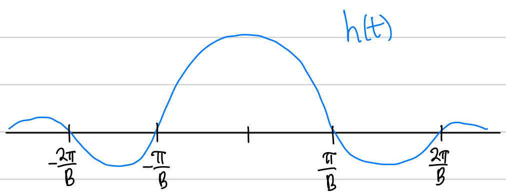
Ex 1



$$h(t) = \frac{1}{2\pi}\int_{-B}^{B} A e^{iwt}\, dw$$
$$= \frac{A}{2\pi}\cdot \frac{e^{iwt}}{it}\Big|_{-B}^{B}$$
$$= \frac{A}{\pi t}\cdot \frac{e^{iBt}-e^{-iBt}}{2i}$$
$$= \frac{A}{\pi t}\sin(Bt)$$

↳ Not Causal b/c has nonzero values for $t<0$

↳ Not BIBO stable b/c it decays according to $1/t$, which diverges



Method I

$$h(0) = \frac{A}{\pi}\lim_{t\to 0}\frac{\sin(Bt)}{t}$$
$$= \frac{A}{\pi}\cdot B\cos(B\cdot 0)$$
$$= \frac{AB}{\pi}$$

Method 2

$$h(0) = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} H(w)\, dw}_{\text{Area of Rectangle} = AB}$$

↳ $h(0) = \frac{1}{2\pi}(AB) = \frac{AB}{2\pi}$

$\int_{-\infty}^{\infty} h(t)\, dt = H(0) = A$ (i.e. Area under $h(t)$ is the evaluation of $H(w)$ at $w=0$)

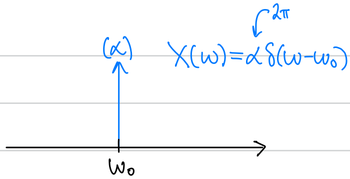
Lecture 18: CTFT

Ex 2

$x(t) = e^{i\omega_0 t}$

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{i\omega_0 t} e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{i(\omega_0-\omega)t}\, dt \\
&= 2\pi\, \delta(\omega_0-\omega) \\
&= 2\pi\, \delta(\omega-\omega_0)
\end{aligned}
$$

$\delta(\omega)$ is even

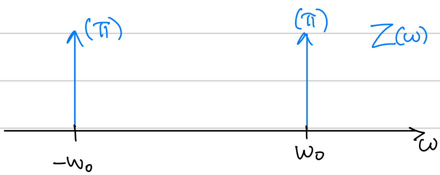


Note: $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, dt$

$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega$

Ex 3

$z(t) = \cos(\omega_0 t) = \frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t}$



Time-Shift Property

$$
\begin{aligned}
x(t) &\xleftrightarrow{\mathcal{F}} X(\omega) \\
\hat{x}(t) = x(t-t_0) &\xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = X(\omega)e^{-i\omega t_0}
\end{aligned}
$$

$$
\begin{aligned}
\hat{X}(\omega) &= \int_{-\infty}^{\infty} x(t-t_0) e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} x(\tau) e^{-i\omega(\tau+t_0)}\, d\tau \\
&= \left[\int_{-\infty}^{\infty} x(\tau) e^{-i\omega\tau}\right] e^{-i\omega t_0} \\
&= X(\omega) e^{-i\omega t_0}
\end{aligned}
$$

$\tau = t - t_0$

Frequency Shift Property (Amplitude Modulation)

$$
\begin{aligned}
x(t) &\xleftrightarrow{\mathcal{F}} X(\omega) \\
\hat{x}(t) = x(t)e^{i\omega_0 t} &\xleftrightarrow{\mathcal{F}} \hat{X}(\omega) = X(\omega-\omega_0)
\end{aligned}
$$

$$
\begin{aligned}
\hat{x}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\omega_0) e^{i\omega t}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda) e^{i(\lambda+\omega_0)t}\, d\lambda \\
&= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda) e^{i\lambda t}\, d\lambda\right] e^{i\omega_0 t} \\
&= x(t) e^{i\omega_0 t}
\end{aligned}
$$

$\lambda = \omega - \omega_0$

Lecture 18: CTFT

Modulation Property

$y(t) = x(t)\,c(t)$

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$
$$c(t) \xleftrightarrow{\mathcal{F}} C(\omega)$$

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty} y(t)\, e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} x(t)\, c(t)\, e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} C(\lambda)\, e^{i\lambda t}\, d\lambda\right] e^{-i\omega t}\, dt \qquad \text{Fubini's} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, C(\lambda) \int_{-\infty}^{\infty} x(t)\, e^{i\lambda t} e^{-i\omega t}\, dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, C(\lambda) \int_{-\infty}^{\infty} x(t)\, e^{-i(\omega-\lambda)t}\, dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\lambda)\, X(\omega-\lambda)\, d\lambda \\
&= \frac{1}{2\pi}(C \star X)(\omega) \\
&= \frac{1}{2\pi}(X \star C)(\omega)
\end{aligned}$$

Thus, we have

$$\boxed{y(t) = x(t)c(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = \frac{1}{2\pi}(C \star X)(\omega) = \frac{1}{2\pi}(X \star C)(\omega)}$$

Also note:

Convolution in Time Domain	$\xleftrightarrow{\mathcal{F}}$	Multiplication in Freq Domain
Multiplication in Freq Domain	$\xleftrightarrow{\mathcal{F}}$	Convolution in Freq Domain

Lecture 19, 11/03/2022: More CTFT & AM

Overview

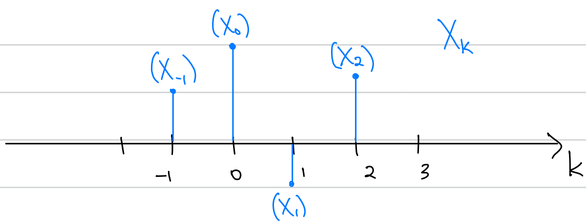
- CTFT of Periodic Signals
- Amplitude Modulation (AM)

P-periodic CT Signals and CTFT

$x(t+p) = x(t) \quad \forall t \in \mathbb{R}, \exists p > 0 \in \mathbb{R}$

$\omega_0 = \frac{2\pi}{p}$

$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$

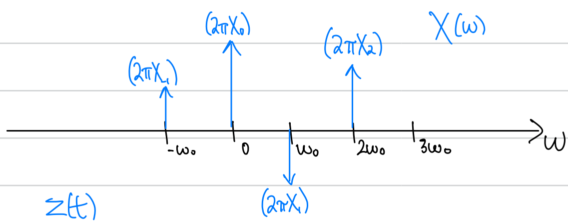


$e^{i\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\omega-\omega_0)$

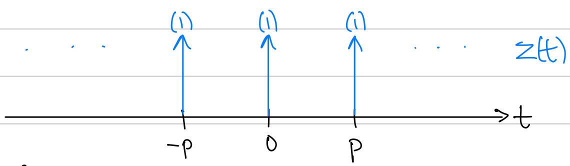
$e^{ik\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\omega-k\omega_0)$

$X_k e^{ik\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi X_k\,\delta(\omega-k\omega_0)$

$\underbrace{\sum_k X_k e^{ik\omega_0 t}}_{x(t)} \xleftrightarrow{\mathcal{F}} \underbrace{2\pi \sum_k X_k\,\delta(\omega-k\omega_0)}_{X(\omega) = \mathcal{F}\{x(t)\}}$



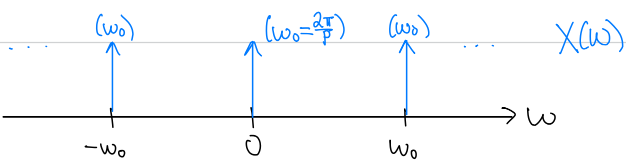
Ex 1



$z(t) = \sum_{l=-\infty}^{\infty} \delta(t - lp) = \sum_{k=-\infty}^{\infty} Z_k e^{ik\omega_0 t}$

$Z_k = \frac{1}{p} \underbrace{\int_{\langle p \rangle} z(t) e^{-ik\omega_0 t}\, dt}_{1} = \frac{1}{p}$

$Z(\omega) = 2\pi \sum_k X_k\,\delta(\omega - k\omega_0) = \frac{2\pi}{p} \sum_k \delta(\omega - k\omega_0) = \omega_0 \sum_k \delta(\omega - k\omega_0)$

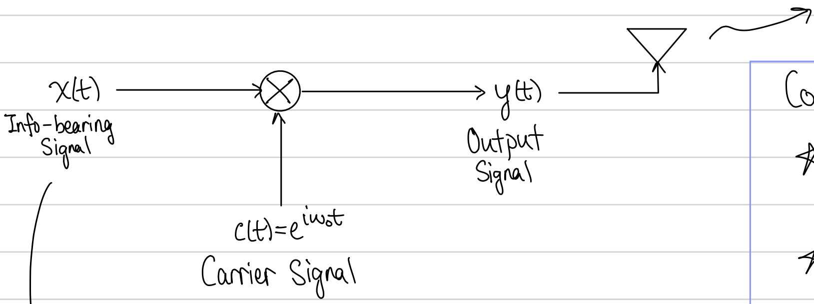


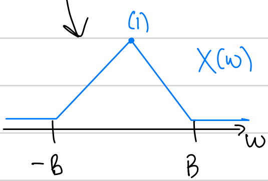
Lecture 19: More CTFT & AM

Amplitude Modulation (AM)

Modulation: Alter a property (parameter) of a signal in proportion to another signal

Amplitude Modulation: Alter the amplitude of a carrier signal in proportion to an info-bearing signal



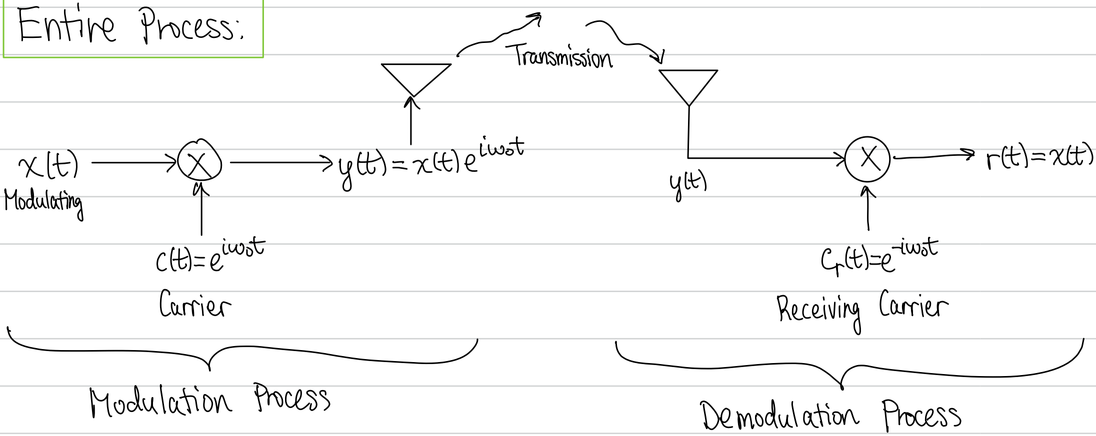


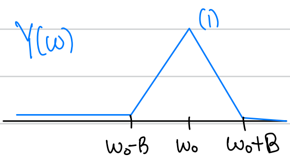
Usually, $B = 2\pi \cdot 3.3\times10^3$ rad/s

Cool Trivia:

- ☆ $c = f\lambda$ — 3×10^8 m/s, 3.3×10^3 Hz, 100 km
- ☆ Efficient Antenna Length $\geq \frac{\lambda}{4}$ $= 25$ km
- ☆ Common Frequencies:
  - AM Radio: 540 kHz – 1700 kHz
  - Satellite: 100 MHz – 40 GHz
  - Microwave: 300 MHz – 300 GHz

Entire Process:





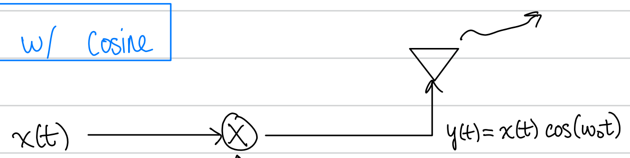
Generally, $B \ll \omega_0$

*Assumptions Made:

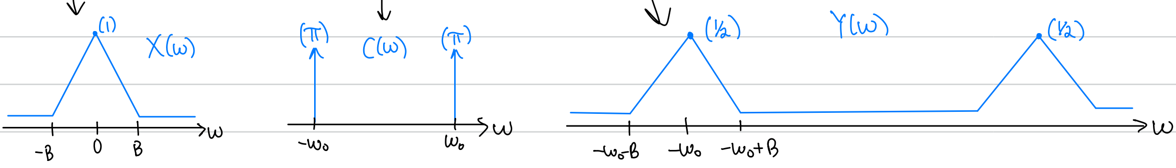
- No degradation in y during transmission (Received = Transmitted)
- No frequency overlapping other signals
- Transmitter & Receiver oscillators produce identical signals
  - ↳ Local $\omega_0^{(r)}$ = Transmitted ω_0 (No frequency drift)
- There's no phase difference ($\theta = 0$ in $e^{i(\omega_0 t + \theta)}$)

Lecture 19: More CTFT & AM

$\boxed{\text{AM w/ Cosine}}$

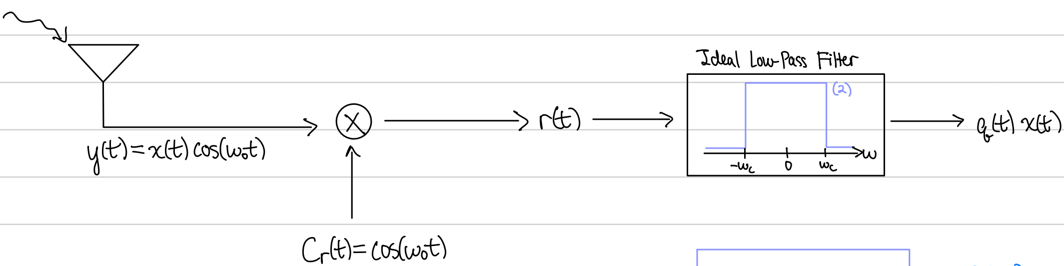


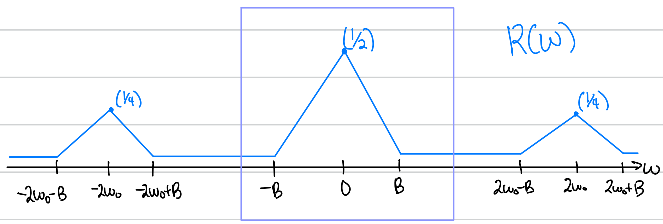
$$c(t) = \cos(\omega_0 t) = \frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t}$$

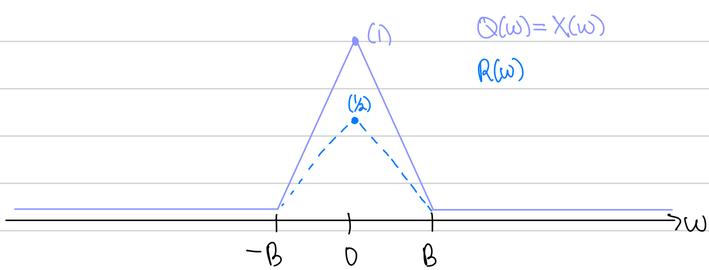


$B >> \omega_0$

$$Y(\omega) = \frac{1}{2\pi}(X \circledast C)(\omega)$$

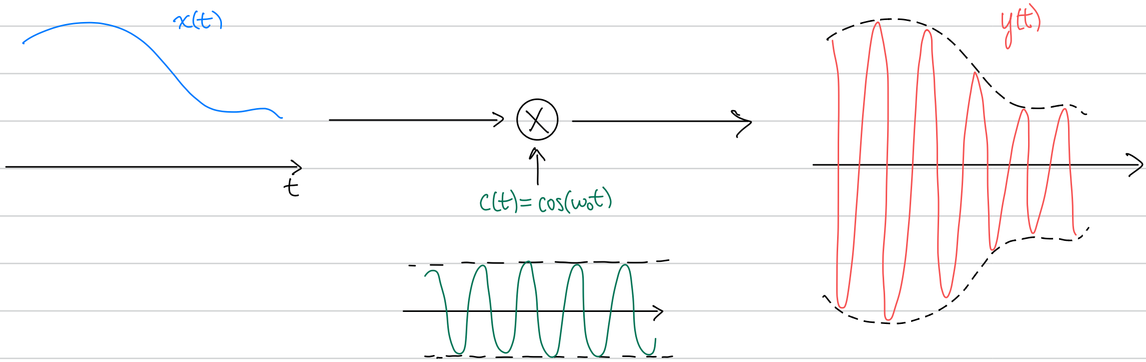






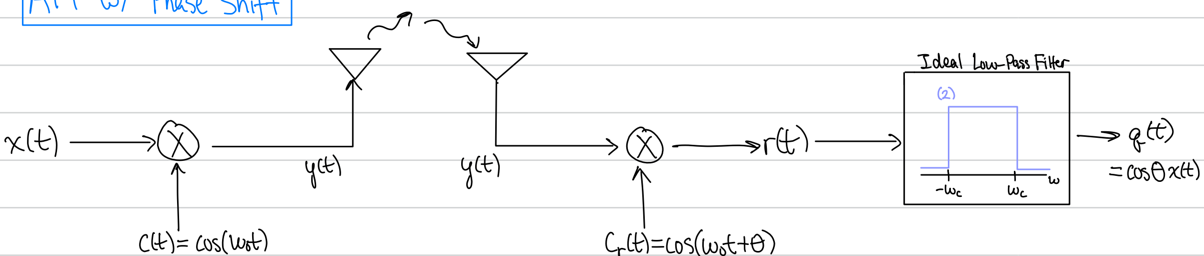
Lecture 19: More CTFT & AM

AM w/ Cosine



$$r(t) = y(t)\, C_r(t)$$
$$= x(t) \cos^2(\omega_0 t)$$
$$= \frac{x(t)}{2}\left[1 + \cos(2\omega_0 t)\right]$$
$$= \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(2\omega_0 t)$$

AM w/ Phase Shift



* $\theta = \pm\frac{\pi}{2}$ will ruin this scheme

$$r(t) = x(t)\cos(\omega_0 t)\cos(\omega_0 t + \theta)$$
$$= \frac{1}{2}x(t)\left[\cos(2\omega_0 t + \theta) + \cos\theta\right]$$
$$= \underbrace{\frac{1}{2}x(t)\cos(2\omega_0 t + \theta)}_{} + \frac{1}{2}x(t)\cos\theta$$

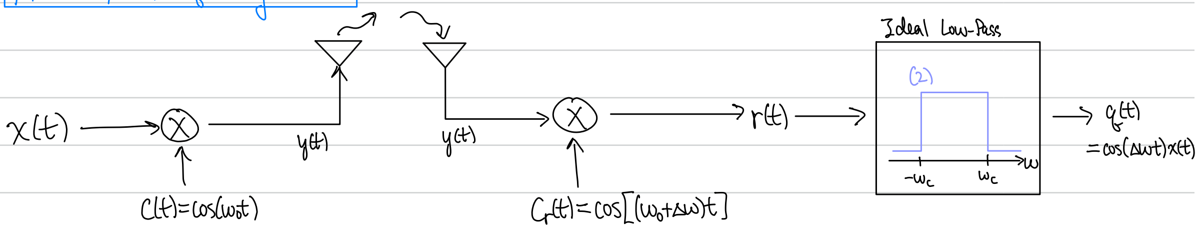
Use low-pass filter to get rid of this part

Example of Scheme Failure

Suppose frequency $f_0 = 1$ MHz, so period $T_0 = 1\ \mu s$. Then, since $\theta = \frac{\pi}{2}$ corresponds to a quarter of a cycle, if our local oscillator is even $\frac{T_0}{4} = \frac{1}{4}\ \mu s$ out of phase from the transmission oscillator, then we are basically screwed.

Lecture 19: More CTFT & AM

AM w/ Frequency Drift

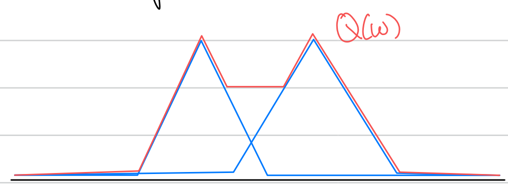


$r(t) = x(t)\cos(\omega_0 t)\cos[(\omega_0+\Delta\omega)t]$

$= \frac{1}{2}x(t)\cos[(2\omega_0+\Delta\omega)t] + \frac{1}{2}\cos(\Delta\omega t)\,x(t)$

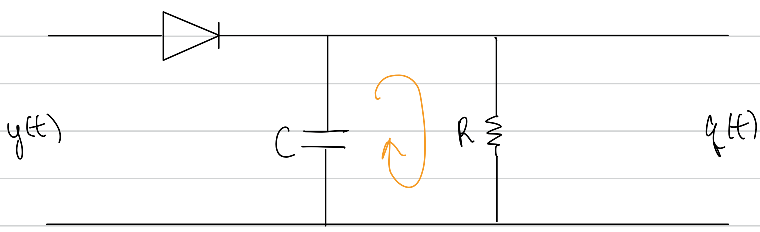
Example of Scheme Failure

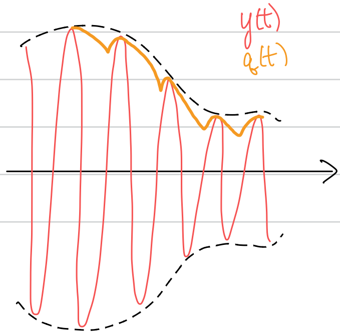
Suppose $f_0 = 1$ MHz, $\Delta f = 1$ Hz (i.e. 1 ppM drift). Then, $\Delta\omega = 2\pi$ rad/s. Thus, this makes it so that $q(t)$ decreases to 0 every 0.5 seconds, which is very suboptimal.



Asynchronous Demodulation

(No local oscillator)



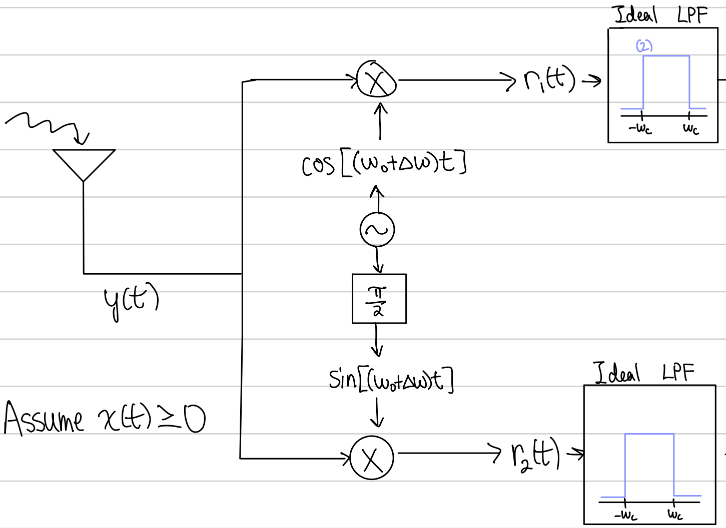


Lecture 19: More CTFT & AM

Demodulation Scheme to Deal w/ Freq Drift

$*\ \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$



$q_{f1}(t) = \cos(\Delta\omega t)\,x(t)$

$q_{f2}(t) = \sin(\Delta\omega t)\,x(t)$

Assume $x(t) \geq 0$

$$\sqrt{q_{f1}^2(t)+q_{f2}^2(t)} = \sqrt{\cos^2(\Delta\omega t)x^2(t)+\sin^2(\Delta\omega t)x^2(t)}$$

$$= \sqrt{\left[\cos^2(\Delta\omega t)+\sin^2(\Delta\omega t)\right]x^2(t)}$$

$$= x(t)$$