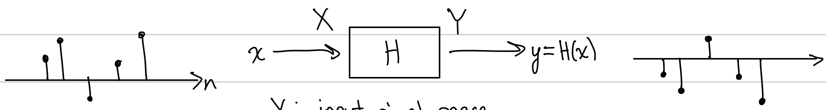
Lecture 1, 8/25/2022

[TODO]

Lecture 2, 8/30/2022

Review

- Signals are functions
  - CT: $x : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ — $\mathbb{R}$: real CT signal; $\mathbb{C}$: complex CT signal
  - DT: $x : \mathbb{Z} \to \mathbb{R}$ or $\mathbb{C}$
- Systems are functions



X: input signal space
Y: output signal space

- If $X = [\mathbb{R} \to \mathbb{R}]$ and $Y = [\mathbb{R} \to \mathbb{R}]$, we say that H is a CT system
  - ↑ set of all real-valued CT signals
- If $X = [\mathbb{Z} \to \mathbb{R}]$ and $Y = [\mathbb{Z} \to \mathbb{R}]$, we say that H is a DT system
  - ↑ set of all real-valued DT signals
- Delta Impulse $\delta(n)$, Unit Step $u(n)$
- Every DT signal can be decomposed into a linear combo of shifted impulses
  - ↳ $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$

Time-Invariant (LTI)

system

$$x(t) \to \boxed{H} \to y(t)$$

If $\hat{x}(t) = x(t-T)$ yields $\hat{y}(t) = H(\hat{x}) = y(t-T)$ for all $T \in \mathbb{R}$ and all inputs $x \in X$, we say that H is time invariant

Linearity

(Superposition)

$$x \xrightarrow{X} \boxed{H} \xrightarrow{Y} y$$

For any $x_1, x_2 \in X$ and $\alpha, \alpha_1, \alpha_2 \in \mathbb{C}$, H is linear if the following hold:

- Scaling: $H(\alpha x_1) = \alpha H(x_1)$
- Additivity: $H(x_1 + x_2) = H(x_1) + H(x_2)$

$\Big\}\ H(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 H(x_1) + \alpha_2 H(x_2)$

Linear, Time-Invariant System

$$x(n) \longrightarrow \underset{}{\overset{\text{LTI}}{\boxed{H}}} \longrightarrow y(n)$$

An LTI system satisfies both linearity and time-invariance

Claim: $x(n)=\delta(n) \longrightarrow \overset{\text{DT-LTI}}{\boxed{H}} \longrightarrow y(n)=h(n)$
(Impulse Response)

Knowing the impulse response h enables us to determine the output corresponding to any arbitrary input x

↳ Proof: $h(n)=H(\delta(n))$

$h(n-k)=H(\delta(n-k))$

$x(k)\,h(n-k)=x(k)\,H(\delta(n-k))=H(x(k)\,\delta(n-k))$

Thus, $$\sum_{k=-\infty}^{\infty} x(k)\,h(n-k)=\sum_{k=-\infty}^{\infty} H(x(k)\,\delta(n-k))=H\left(\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right)$$

Convolution

Discrete-Time: $(x*h)(n)=\sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$

Continuous-Time: $(x*h)(t)=\int_{-\infty}^{\infty} x(u)\,h(t-u)\,du$

* Commutative: $x*h=h*x$

↳ Why care?

$$x(n)=\delta(n) \longrightarrow \overset{H}{\boxed{\underset{\text{DT-LTI}}{\boxed{f}} \longrightarrow \underset{\text{DT-LTI}}{\boxed{g}}}} \longrightarrow h(n)=(f*g)(n)$$

Cascade (Series)
Interconnection of LTI System

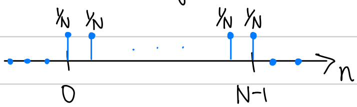
$h(n)=(f*g)(n)=(g*f)(n)$ by commutativity (cascade order doesn't matter)

Ex. Simple Moving Average

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(n-k)$$

Determine $h(n)$ and plot it.

Let $x(n) = \delta(n)$. Then $y(n) = h(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(n-k)$



Lecture 3, 9/1/2022

Time Invariance (TI)

Visualization of Time Invariance

$x(t) \rightarrow \boxed{H} \rightarrow y(t) \rightarrow \boxed{D_T} \rightarrow y(t-T)$

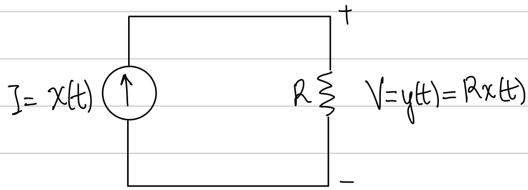
$x(t) \rightarrow \boxed{D_T} \rightarrow \hat{x}(t) = x(t-T) \rightarrow \boxed{H} \rightarrow \hat{y}(t)$

} Must equal $\forall T$ for TI to hold

H = System operator
D_T = Shift by T time

Basically, Time Invariance holds iff H & D_T are commutative

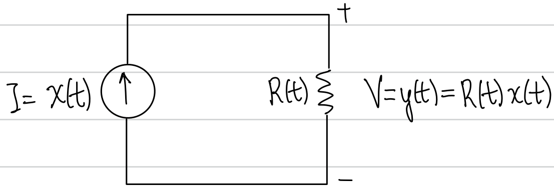
Ex 1.



$$\left.\begin{aligned} \hat{y}(t) &= Rx(t-T) \\ y(t-T) &= Rx(t-T) \end{aligned}\right\} \checkmark$$

Time Invariant

Ex 2.



$$\left.\begin{aligned} \hat{y}(t) &= R(t)\,x(t-T) \\ y(t-T) &= R(t-T)\,x(t-T) \end{aligned}\right\} \times$$

Time Variant

Ex 3. Two-Pointer Moving Average

$$x(n) \rightarrow \boxed{H} \rightarrow y(n) = \frac{x(n)+x(n-1)}{2}$$

• Linear? $\hat{x}(n) = \alpha x_1(n) + \beta x_2(n) \rightarrow \hat{y}(n) = \dfrac{\hat{x}(n)+\hat{x}(n-1)}{2} = \dfrac{(\alpha x_1(n)+\beta x_2(n)) + (\alpha x_1(n-1)+\beta x_2(n-1))}{2}$

$= \alpha y_1(n) + \beta y_2(n)$ ✓

• Time Invariant? $\hat{x}(n) = x(n-N) \rightarrow$

$$\left.\begin{aligned} \hat{y}(n) &= \frac{\hat{x}(n)+\hat{x}(n-1)}{2} = \frac{x(n-N)+x(n-1-N)}{2} \\ y(n-N) &= \frac{x(n-N)+x(n-N-1)}{2} \end{aligned}\right\} \checkmark$$

Linear, Time-Invariant (LTI) Systems

$$x(n) \longrightarrow \overset{LTI}{\boxed{H}} \longrightarrow y(n)$$

<u>Claim</u>: If I know the response of H to the input $x(n) = \delta(n)\ \forall n \in \mathbb{Z}$ (i.e. the impulse response $h(n) = H(x(n))$) then I know the output for any arbitrary input signal x.

↳ <u>Proof/Explanation</u>: We can express any input signal $x(n)$ as follows:

$$x(n) = \sum_{m=-\infty}^{\infty} x(m)\,\delta(n-m)$$

and thus any output signal $y(n)$ as follows (using LTI properties):

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\,h(n-m) = (x \ast h)(n) = (h \ast x)(n)$$

↑ Convolutions are commutative

<u>Cascading Systems</u>: we can interchange the order of LTI systems in a cascade (series) interconnection

$$x \longrightarrow \boxed{f} \xrightarrow{r} \boxed{g} \longrightarrow y$$

$$x \longrightarrow \boxed{g} \xrightarrow{q} \boxed{f} \longrightarrow y$$

↳ $y = x \ast f \ast g = x \ast g \ast f$, though $r \neq q$

Convolutions

<u>Def</u>: $(v \ast u)(n) = \sum_{k=-\infty}^{\infty} v(n)\,u(n-k) = \sum_{m=-\infty}^{\infty} v(n-m)\,u(m)$

<u>Properties</u>:
- Commutative: $v \ast u = u \ast v$
- Identity Element: $x \ast \underline{\delta} = x$

Complex Exponentials

$$x(t) = e^{t} \longleftrightarrow \begin{cases} x(0) = 1 \\ \dot{x}(t) \triangleq \dfrac{dx(t)}{dt} = x(t) \\ \ddot{x}(t) \triangleq \dfrac{d^2x(t)}{dt^2} = x(t) \end{cases}$$

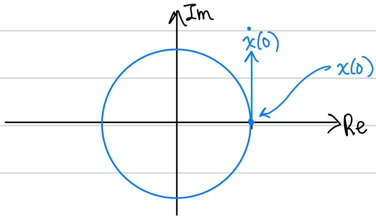
We use the properties of complex exponentials to solve interesting problems:

Ex. $x(t) = e^{it}$ = the instantaneous position of a particle on the complex plane

$x(0) = 1$

$\dot{x}(t) = ie^{it} = ix(t)$

$\dot{x}(0) = i$



Claim: the particle moves on the unit circle at a constant speed counter-clockwise

↳ Proof: $x(t) = e^{it} = a(t) + ib(t)$ $a: \mathbb{R} \to \mathbb{R}$, $b: \mathbb{R} \to \mathbb{R}$

$\dot{x}(t) = \dot{a}(t) + i\dot{b}(t)$

$\dot{x}(t) = ix(t) = -b(t) + ia(t)$

↳ $\begin{cases} \dot{a}(t) = -b(t) \\ a(t) = \dot{b}(t) \end{cases}$

↳ $a(t)\dot{a}(t) = -b(t)\dot{b}(t)$

↳ $2a(t)\dot{a}(t) + 2b(t)\dot{b}(t) = 0$

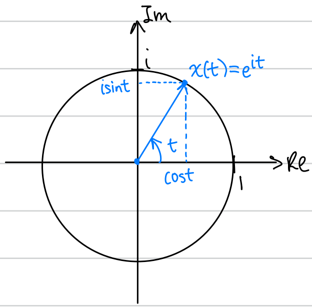
↳ $\frac{d}{dt}\left[a^2(t) + b^2(t)\right] = 0$

↳ $a^2(t) + b^2(t) = C = 1 \quad \forall t \in \mathbb{R}$ $\therefore$ Moves in circle of radius 1

$|\dot{x}(t)| = |ix(t)| = |x(t)| = 1$ $\therefore$ Moves at a constant speed at 1 rad/sec

Euler's Formula

$$e^{it} = \cos(t) + i\sin(t) \quad \forall t \in \mathbb{R}$$



L4

Lecture 4, 9/6/2022

Euler's Formula

$e^{it} = \cos t + i \sin t$

$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$ where ω = rotating speed in rad/sec $\begin{cases} > 0 \rightarrow \text{counterclockwise} \\ < 0 \rightarrow \text{clockwise} \end{cases}$

Inverse Euler Formulas:

$$\begin{cases} \cos(\omega t) = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2} \\ \sin(\omega t) = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i} \end{cases}$$

Note on different ways to measure rate: $\omega = \text{rad/sec} = \underbrace{2\pi}_{\text{rad/cycle}} \underbrace{f}_{\text{cycles/sec}}$

Periodicity

Main Idea:

$$\begin{array}{ccc} R\uparrow & \longrightarrow & J\downarrow \\ \uparrow & & \downarrow \\ J\uparrow & \longleftarrow & R\downarrow \end{array}$$

A signal/system is periodic if it is cyclic, i.e. it eventually repeats

CT Periodicity

Suppose we have a CT signal $x: \mathbb{R} \rightarrow \mathbb{R}$ or $\mathbb{C}$. If $x(t+T) = x(t)$ $\forall t \in \mathbb{R}$ for some $T \in \mathbb{R}_+$, we say that x is periodic w/ <u>period</u> T. If T is the smallest positive value that satisfies the relation, then T is the <u>fundamental period</u>.

<u>Ex.</u> $x(t) = \cos(\frac{2\pi}{5} t)$

$x(t+T) = \cos(\frac{2\pi}{5} t + \frac{2\pi}{5} T) = x(t) = \cos(\frac{2\pi}{5} t)$

$\hookrightarrow \frac{2\pi}{5} T = 2\pi k$ where $k \in \{1, 2, 3, \dots\}$

$\hookrightarrow T = 5$ sec

DT Periodicity

Suppose we have a DT signal $x: \mathbb{Z} \to \mathbb{R}$ or $\mathbb{C}$. If $x(n+N) = x(n)\ \forall\, n \in \mathbb{Z}$ for some $N \in \mathbb{Z}_+$, we say that x is periodic w/ period N. If N is the smallest such positive integer, we call N the fundamental period of x.

<u>Ex.</u> $x(n) = C \quad \forall n \in \mathbb{Z}$

↳ $N = 1$, $\omega_0 =$ fundamental frequency $= \frac{2\pi}{N} = 2\pi$

<u>Ex.</u> $x(n) = e^{in} \quad \forall n \in \mathbb{Z}$

$x(n+N) = e^{i(n+N)} = e^{in} e^{iN} = x(n) = e^{in}$

↳ $e^{iN} = 1$

↳ $N = 2\pi \notin \mathbb{Z}$

Thus, the fundamental period and frequency do not exist. This is called a <u>Quasi-periodic signal</u>.

<u>Ex.</u> $x(n) = e^{i\frac{\pi}{4}n} \quad \forall\, n \in \mathbb{Z}$

$x(n+N) = e^{i\frac{\pi}{4}(n+N)} = e^{i\frac{\pi}{4}n} \cdot e^{i\frac{\pi}{4}N} = x(n) = e^{i\frac{\pi}{4}n}$

↳ $e^{i\frac{\pi}{4}N} = 1$

↳ $N = \frac{4}{\pi} \cdot 2\pi = 8$, $\omega_0 = \frac{2\pi}{N} = \frac{\pi}{4}$

Periodicity of $x(n) = e^{i\omega n}$

Necessary & Sufficient conditions for $e^{i\omega n}$ to be periodic in n.

$e^{i\omega(n+N)} = e^{i\omega n}$

$e^{i\omega n} \cdot e^{i\omega N} = e^{i\omega n}$

$e^{i\omega N} = 1$

$\omega N = 2\pi k$

$\omega = \frac{2k}{N} \cdot \pi$

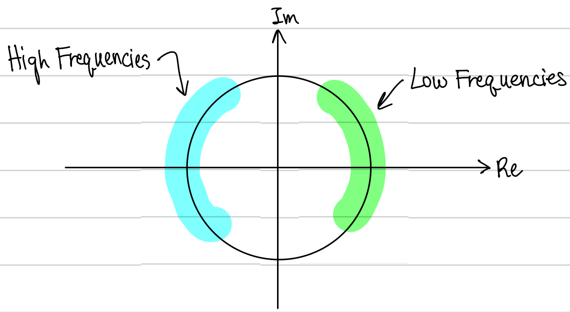
↳ ω must be a rational multiple of $\pi \Leftrightarrow N = \frac{2\pi k}{\omega} \in \mathbb{Z}_+$ can be found

CT Oscillation Behavior

$x(t)=e^{i\omega t}$. The sky is not a limit for ω, i.e. ω can go up to ∞. As $\omega \to \infty$, oscillations become progressively fast. The slowest frequency is $\omega=0$, which corresponds to a constant signal.

DT Oscillation Behavior

$x(n)=e^{i\omega n}$. The highest frequency is $\omega=\pi$, which corresponds to a "flip-flopping" signal. The slowest frequency is $\omega=0$, which corresponds to a constant signal.



Frequency Response of a DT-LTI System

DT-LTI

$x(n)=\delta(n) \longrightarrow \boxed{H} \longrightarrow y(n)=\underbrace{h(n)}_{\text{Impulse Response}}$

$x(n)=e^{i\omega n} \longrightarrow \boxed{H} \longrightarrow y(n)=\sum_m h(m)x(n-m)$

$$= \sum_m h(m)e^{i\omega(n-m)}$$

$$= \left(\sum_m h(m)e^{-i\omega m}\right)e^{i\omega n}$$

$$= H(\omega)e^{i\omega n}$$

$$= \underbrace{H(\omega)}_{\text{Frequency Response}}x(n)$$

* $H(e^{i\omega n}) = H(\omega)e^{i\omega n}$ is analogous to $Av=\lambda v$, and it's called the <u>Eigenfunction Property</u> of Complex Exponentials with respect to DT-LTI System.

* LTI can get rid of frequency in the input signal, but it <u>cannot</u> create a new frequency in the output signal.

Ex. $x(n) \longrightarrow \boxed{H=(\cdot)^2} \longrightarrow y(n)=x^2(n)$

$x(n)=e^{i\omega n} \longrightarrow \qquad \longrightarrow y(n)=e^{i(2\omega)n}$

Thus, this system cannot be LTI because it creates a new frequency (i.e. 2ω). It is time-invariant, but not linear.

Ex. $x(n) \longrightarrow \boxed{H} \longrightarrow y(n)=e^{i\omega_1 n}x(n)$

$x(n)=e^{i\omega_0 n} \longrightarrow \qquad \longrightarrow y(n)=e^{i\omega_1 n}e^{i\omega_0 n}=e^{i(\omega_0+\omega_1)n}$

Thus, the system cannot be LTI because it creates a new frequency (e.g. $\omega_0+\omega_1$). It is linear, but time-varying.

Property of any LTI System

$$\alpha_0 e^{i\omega_0 n}+\alpha_1 e^{i\omega_1 n} \longrightarrow \underset{\text{LTI}}{\boxed{H}} \longrightarrow \alpha_0 H(\omega_0)e^{i\omega_0 n}+\alpha_1 H(\omega_1)e^{i\omega_1 n}$$

Frequency Response of the Two-Point Moving Average Filter

$$x(n) \longrightarrow \boxed{H} \longrightarrow y(n)=\frac{x(n)+x(n-1)}{2}$$

Ways to find the frequency response $H(\omega)$:

Method I: $H(\omega)=\sum_m h(m)e^{-i\omega m}=h(0)e^{-i\omega(0)}+h(1)e^{-i\omega(1)}=\frac{1+e^{-i\omega}}{2}$

Method II: Next time!

Lecture 5, 9/8/2022

Overview

- DT Frequency Response
  - 2 point Moving Average Filter
  - Recursive Filter

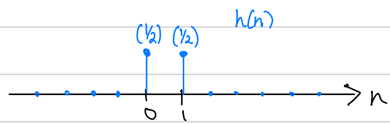
DT Frequency Response

$$x(n)=e^{i\omega n} \rightarrow \boxed{H} \rightarrow y(n)=H(\omega)e^{i\omega n}$$

$$H(\omega)=\sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n} = \text{Freq Response of Filter } H$$

2-Point Moving Average Filter

$$x(n) \longrightarrow \boxed{\begin{matrix} H \\ h(n),\ H(\omega)\end{matrix}} \longrightarrow y(n)=\frac{x(n)+x(n-1)}{2}$$



$h(n)$: let $x(n)=\delta(n) \rightarrow y(n)=h(n)$

$$h(n)=\frac{\delta(n)+\delta(n-1)}{2}$$

Computing $H(\omega)$ (using 2-pt moving average as example)

Method I: Use definition of $H(\omega)$

$$H(\omega)=\sum_n h(n)e^{-i\omega n}$$

$$H(\omega)=h(0)+h(1)e^{-i\omega}=\frac{1+e^{-i\omega}}{2}$$

Method II: Use eigenfunction property of complex exponentials wrt LTI systems

Note: this method assumes that the frequency response exists

Let $x(n)=e^{i\omega n} \rightarrow y(n)=H(\omega)e^{i\omega n}$. Then we have:

$$y(n)=\frac{x(n)+x(n-1)}{2} \quad \text{(Linear Constant-Coefficient Difference Equation (LCCDE))}$$

$$H(\omega)e^{i\omega n}=\frac{e^{i\omega n}+e^{i\omega(n-1)}}{2}=\frac{1+e^{-i\omega}}{2}e^{i\omega n}$$

$$H(\omega)=\frac{1+e^{-i\omega}}{2}$$

Plotting Frequency Response (using 2-pt moving average as example)

$$H(\omega) = \underbrace{|H(\omega)|}_{\text{Magnitude Response}} e^{i \overbrace{\measuredangle H(\omega)}^{\text{Phasor Response}}}$$

$$|H(\omega)| = \sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\cos(\omega)\right)^2 + \left(-\tfrac{1}{2}\sin(\omega)\right)^2} \quad \leftarrow \text{One Way}$$

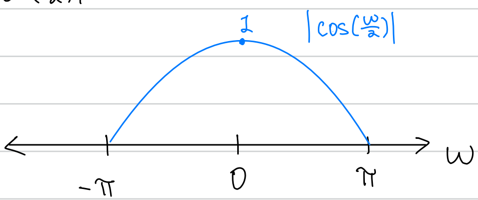
$$= \left|\tfrac{1}{2}e^{i(0)\omega} + \tfrac{1}{2}e^{i(-1)\omega}\right| \quad \leftarrow \text{Better Way (Exploit Symmetry)}$$

Balancing the Exponents

$$= \left|\frac{e^{i\omega/2} + e^{-i\omega/2}}{2}\, e^{-i\omega/2}\right|$$

$$= \left|\cos\left(\tfrac{\omega}{2}\right) e^{-i\omega/2}\right|$$

$$= \left|\cos\left(\tfrac{\omega}{2}\right)\right|$$



Low Pass

* We only consider the range $[-\pi, \pi]$ because those are limits of a DT filter

Note: $|H(\omega)| = |H(\omega+2\pi)|$ holds for all DT filter frequency responses

2π-Periodicity of DT-LTI Frequency Response

$$H(\omega) = \sum_n h(n) e^{-i\omega n}$$

$$H(\omega+2\pi) = \sum_n h(n) e^{-i(\omega+2\pi)n}$$

$$= \sum_n h(n) e^{-i\omega n} \underbrace{e^{-i(2\pi)n}}_{=1 \ \forall n \in \mathbb{Z}}$$

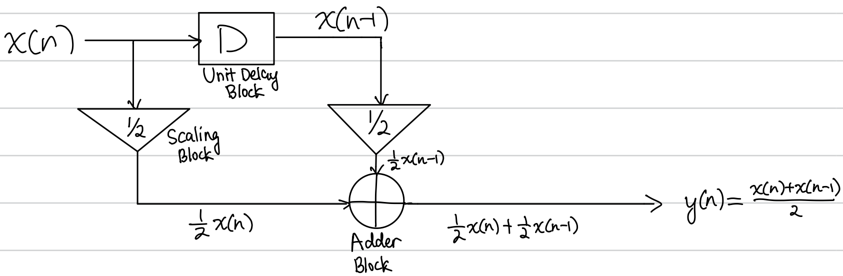
$$= \sum_n h(n) e^{-i\omega n}$$

$$= H(\omega)$$

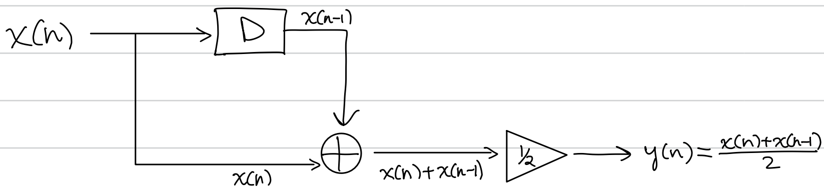
Delay-Adder-Gain Block Diagram Implementation

Normal:

DAG:

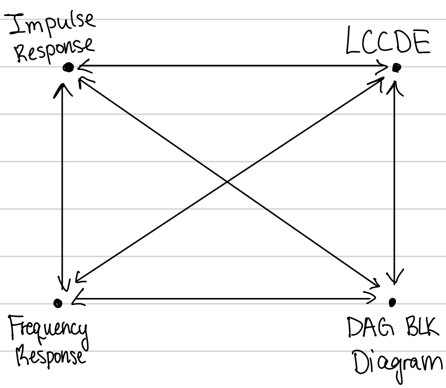


Alternate DAG:
(More Efficient)



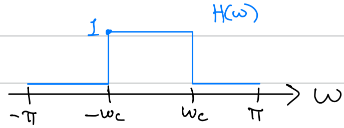
The Big Picture

Moving between various characterizations of LTI systems

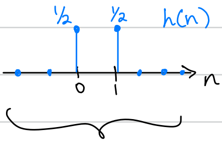


DT-LTI System w/o LCCDE Representation

$$h(n) = \frac{\sin(w_c n)}{w_c n} \quad \forall n \in \mathbb{Z} \quad \longleftrightarrow$$



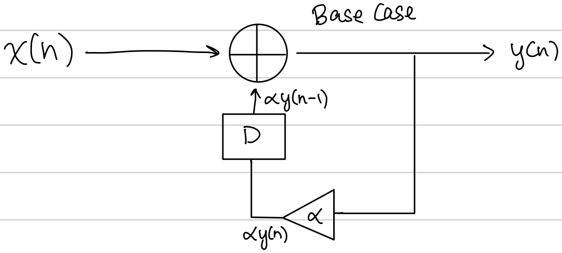
Finite-Duration Impulse Response (FIR) Filter



Finite Region of Support

Infinite-Duration Impulse Response (IIR) Filter

$$y(n) = \alpha y(n-1) + x(n), \quad \underbrace{y(-1) = 0}_{\text{Base Case}}, \quad |\alpha| < 1$$



Determine $h(n)$: $h(-1) = 0$

$h(n) = \alpha h(n-1) + \delta(n)$

$h(0) = \alpha h(-1) + \delta(0) = 1$

$h(1) = \alpha h(0) + \delta(1) = \alpha$

$h(2) = \alpha h(1) + \delta(2) = \alpha^2$

$h(3) = \alpha h(2) + \delta(3) = \alpha^3$

$\vdots$

$h(n) = \alpha^n \quad \forall \; n \geq 0$

$\hookrightarrow h(n) = \alpha^n u(n)$ where $u(n)$ = unit step

Frequency Response for Recursive Filter

$y(n) = \alpha y(n-1) + x(n)$, $y(-1) = 0$, $|\alpha| < 1$

$h(n) = \alpha^n u(n)$

Method I: $H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}$

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n) e^{-i\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-i\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-i\omega})^n$$

$$= \frac{1}{1-\alpha e^{-i\omega}}$$

Note: $|\alpha e^{-i\omega}| = |\alpha||e^{-i\omega}| = |\alpha| < 1$

Method II: $x(n) = e^{i\omega n} \to y(n) = H(\omega) e^{i\omega n}$

$$y(n) = \alpha y(n-1) + x(n)$$

$$= \alpha H(\omega) e^{i\omega(n-1)} + e^{i\omega n}$$

$$= (\alpha H(\omega) e^{-i\omega} + 1) e^{i\omega n}$$

$$= H(\omega) e^{i\omega n}$$

$$H(\omega) = \alpha H(\omega) e^{-i\omega} + 1$$

$$1 = (1 - \alpha e^{-i\omega}) H(\omega)$$

$$H(\omega) = \frac{1}{1-\alpha e^{-i\omega}}$$