(b) dimension on the outside...

$$h[X] = n \ln\left(\det\left(\sqrt{2\pi e}\,(\Sigma^{1/2})^{1/n}\right)\right) = n \ln\left(\sqrt{2\pi e}\,\det(\Sigma^{1/2})^{1/n}\right)$$
$$\approx n \ln\left(4 \det(\Sigma^{1/2})^{1/n}\right)$$

(since $\ln(x) = \frac{n}{n} \ln(x) = n \ln(x^{1/n})$)

- Interpretation: $\det(\Sigma^{1/2}) = \prod_{j=1}^{n} \sigma_j$ so $\det(\Sigma^{1/2})^{1/n} = \left(\prod_{j=1}^{n} \sigma_j\right)^{1/n} = \bar{\sigma}$

where $\bar{\sigma}$ = geometric average of the principal s.d.'s

then: $$h[X] = n \ln(\sqrt{2\pi e}\,\bar{\sigma}) \approx \underset{\text{dimension}}{n} \ln(\underset{\text{standard normal}}{4}\,\overset{\text{scale}}{\bar{\sigma}})$$

[
- $X \sim N(\mu, \overset{\Sigma}{\cancel{\sigma}}{}^2)$ is as uncertain as
  - (i) n i.i.d. draws from uniform on interval length $4\bar{\sigma}$
  - (ii) n i.i.d. draws from normal w/ s.d. $\bar{\sigma}$
]

Computing Mutual Information - Example II: Correlated Gaussian Variables

- Aim: Compute $I[X;Y]$ for:
  1. X \$ $Y \sim N$ univariate
  2. X \$ $Y \sim N$ multivariate

1. Univariate: $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu, \sigma_Y^2)$, $\mathrm{Corr}[X,Y] = r_{xy}$

- What is $I[X;Y]$?

$$I[X;Y] = \begin{cases} h[X] - h[X|Y] \\ h[Y] - h[Y|X] \\ h[X] + h[Y] - h[X,Y] \end{cases}$$

$X|Y=y$ is N $\forall y$
$Y|X=x$ is N $\forall x$

could compute
would need formula for conditionals

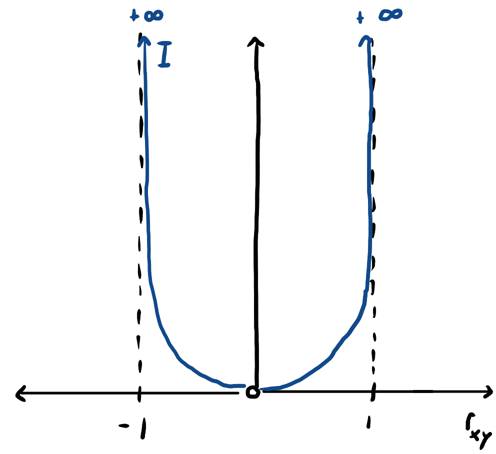
← only need formula for entropy of N's

$$I[X;Y] = h[X] + h[Y] - h[X,Y]$$

$$= \frac{1}{2}\ln(2\pi e\,\sigma_x^2) + \frac{1}{2}\ln(2\pi e\,\sigma_y^2) - \frac{1}{2}\ln\Big((2\pi e)^2 \det\Big(\underbrace{\begin{bmatrix}\sigma_x^2 & \sigma_x\sigma_Y r_{xy}\\ \sigma_x\sigma_y r_{xy} & \sigma_y^2\end{bmatrix}}_{\Sigma \text{ for } \left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right]}\Big)\Big)$$

$$= \Big(\frac{1}{2}\ln(2\pi e) + \frac{1}{2}\ln(2\pi e) - \frac{2}{2}\ln(2\pi e)\Big)$$
$$+ \frac{1}{2}\ln(\sigma_x^2) + \frac{1}{2}\ln(\sigma_Y^2) - \frac{1}{2}\ln(\sigma_x^2\sigma_Y^2 - \sigma_x^2\sigma_Y^2 r_{xY}^2)$$

$$= \ln(\sigma_x) + \ln(\sigma_Y) - \ln(\sigma_x\sigma_Y\sqrt{1-r_{xy}^2})$$
$$= \ln\Big(\frac{\sigma_x\sigma_y}{\sigma_x\sigma_y\sqrt{1-r_{xy}^2}}\Big) = -\frac{1}{2}\ln(1-r_{xy}^2)$$



so: $I[X;Y] = -\frac{1}{2}\ln(1-r_{xy}^2)$ if $X \sim N$, $Y \sim N$ (univariate) and $Corr[X,Y] = r_{xy}$

the more correlated
the larger the mutual information

- notice: (i) if $r_{xy} = 0$, $X \perp\!\!\!\perp Y$, $I[X;Y] = 0$
  (ii) if $|r_{xy}| \to 1$, Y predicts X exactly, $I \to \infty$
  (iii) I is independent of μ_x, μ_y and σ_x, σ_Y
  ... scale invariant, depends only on correlation...
  } intrinsic? or just scale invariant?

2. Multivariate: $X \sim N$ n-dimensional, $Y \sim N$ m-dimensional

$$\begin{bmatrix}X\\Y\end{bmatrix} \sim N\left(\begin{bmatrix}\mu_X\\\mu_Y\end{bmatrix}, \begin{bmatrix}\Sigma_{XX} & \Sigma_{Xy}\\ \Sigma_{YX} & \Sigma_{YY}\end{bmatrix}\right)$$

where
$$\Sigma = \begin{array}{cc} & \begin{array}{cc} \overset{\leftarrow n \rightarrow}{} & \overset{\leftarrow m \rightarrow}{} \end{array} \\ \begin{array}{c} n \\ m \end{array} & \left[\begin{array}{c|c} \Sigma_{XX} & \Sigma_{XY} \\ \hline \Sigma_{YX} & \Sigma_{YY} \end{array}\right] \end{array}$$

$\Sigma_{XX} = Cov[X]$, $\Sigma_{YY} = Cov[Y]$
$\Sigma_{XY} = Cov[X,Y] = \Sigma_{YX}^T$.

then: $I[X;Y] = h[X] + h[Y] - h[X;Y]$

$$= \frac{1}{2}\Big(\ln(\det(\Sigma_{XX})) + \ln(\det(\Sigma_{YY})) - \ln(\det(\Sigma))\Big)$$

can we reduce?

- let $Var[X] = [\mathbb{V}[X_1], \mathbb{V}[X_2], \dots \mathbb{V}[X_n]]$

  $Var[Y] = [\mathbb{V}[Y_1], \mathbb{V}[Y_2], \dots \mathbb{V}[Y_m]]$

  "variance of"

- let $D_v = diag(v_1, v_2, \dots)$

- then: $$\Sigma_{XX} = D_{Var[X]}^{1/2} R_{XX} D_{Var[X]}^{1/2} = \begin{bmatrix} SD[X_1] & & \\ & SD[X_2] & \\ & & \ddots \\ & & & SD[X_n] \end{bmatrix} \begin{array}{cc} & \begin{array}{cccc} \ & \ & j & \ \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \end{array} & \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & 0 & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{bmatrix} \end{array} \begin{bmatrix} SD[X_1] & & \\ & SD[X_2] & \\ & & \ddots \\ & & & SD[X_n] \end{bmatrix}$$

  $r_{ij} = Corr[X_i, X_j]$

  $$\Sigma_{YY} = \underbrace{D_{Var[Y]}^{1/2}}_{SD} \underbrace{R_{YY}}_{Corr} \underbrace{D_{Var[Y]}^{1/2}}_{SD}$$

and: $$\Sigma = \begin{bmatrix} D_{Var[X]}^{1/2} & \\ & D_{Var[Y]}^{1/2} \end{bmatrix} \left[\begin{array}{c|c} R_{XX} & R_{XY} \\ \hline R_{YX} & R_{XX} \end{array}\right] \begin{bmatrix} D_{Var[X]}^{1/2} & \\ & D_{Var[Y]}^{1/2} \end{bmatrix}$$

$R_{YX_{ij}} = Corr[Y_i, X_j]$

- now: $\log(\det(\Sigma)) = \log(\det(D_v^{1/2})^2 \det(R)) = \log(\det(D_v)) + \log(\det(R))$

so: $\log(\det(\Sigma_{XX})) + \log(\det(\Sigma_{YY})) - \log(\det(\Sigma))$

$$= \Big[\log(\det(D_{Var[X]})) + \log(\det(D_{Var[Y]})) - \log\left(\det\left(\begin{bmatrix} D_{Var[X]} & \\ & D_{Var[Y]} \end{bmatrix}\right)\right)$$

$$+ \log(\det(R_{XX})) + \log(\det(R_{YY})) - \log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY} \end{bmatrix}\right)\right)\Big]$$

$\hookrightarrow = \log\left(\prod_{i=1}^{n} \mathbb{V}[X_i] \prod_{j=1}^{m} \mathbb{V}[Y_j]\right) - \log\left(\prod_{i=1}^{n} \mathbb{V}[X_i] \prod_{j=1}^{m} \mathbb{V}[Y_j]\right) = 0$ ... all var. terms cancel out

$$= \log(\det(R_{XX}) \det(R_{YY})) - \log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY} \end{bmatrix}\right)\right)$$

so: $$I[X;Y] = \frac{1}{2}\left[\log(\det(R_{XX}) \det(R_{YY})) - \log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY} \end{bmatrix}\right)\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\det\left(\begin{bmatrix} R_{XX} & 0 \\ 0 & R_{YY} \end{bmatrix}\right)\right) - \log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY} \end{bmatrix}\right)\right)\right]$$

$$= -\frac{1}{2}\left[\log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY} \end{bmatrix}\right)\right) - \log\left(\det\left(\begin{bmatrix} R_{XX} & 0 \\ 0 & R_{YY} \end{bmatrix}\right)\right)\right]$$

$$= -\frac{1}{2}\left[\log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY} \end{bmatrix}\right)\right) + \log\left(\det\left(\begin{bmatrix} R_{XX}^{-1} & 0 \\ 0 & R_{YY}^{-1} \end{bmatrix}\right)\right)\right]$$

so: $I[X;Y] = \frac{1}{2}\left[\log(\det(R_{XX})\det(R_{YY})) - \log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY}\end{bmatrix}\right)\right)\right]$

$$= \frac{1}{2}\left[\log\left(\det\left(\begin{bmatrix} R_{XX} & 0 \\ 0 & R_{YY}\end{bmatrix}\right)\right) - \log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY}\end{bmatrix}\right)\right)\right]$$

$$= -\frac{1}{2}\left[\log\left(\det\left(\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY}\end{bmatrix}\right)\right) - \log\left(\det\left(\begin{bmatrix} R_{XX} & 0 \\ 0 & R_{YY}\end{bmatrix}\right)\right)\right]$$

$$= -\frac{1}{2}\left[\log(\quad \dots \quad) + \log\left(\det\left(\begin{bmatrix} R_{XX}^{-1} & 0 \\ 0 & R_{YY}^{-1}\end{bmatrix}\right)\right)\right]$$

$$= -\frac{1}{2}\left[\log(\quad \dots \quad) + \log\left(\det\left(\begin{bmatrix} R_{XX}^{-1/2} & 0 \\ 0 & R_{YY}^{-1/2}\end{bmatrix}\right)^{2}\right)\right]$$

$$= -\frac{1}{2}\log\left(\det\begin{bmatrix} R_{XX}^{-1/2} & 0 \\ 0 & R_{YY}^{-1/2}\end{bmatrix}\det\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY}\end{bmatrix}\det\begin{bmatrix} R_{XX}^{-1/2} & 0 \\ 0 & R_{YY}^{-1/2}\end{bmatrix}\right)$$

$$= -\frac{1}{2}\log\left(\det\left(\begin{bmatrix} R_{XX}^{-1/2} & 0 \\ 0 & R_{YY}^{-1/2}\end{bmatrix}\begin{bmatrix} R_{XX} & R_{XY} \\ R_{YX} & R_{YY}\end{bmatrix}\begin{bmatrix} R_{XX}^{-1/2} & 0 \\ 0 & R_{YY}^{-1/2}\end{bmatrix}\right)\right)$$

$I[X;Y] = -\frac{1}{2}\log\left(\det\left(\begin{bmatrix} I_{n\times n} & M \\ M^T & I_{m\times m}\end{bmatrix}\right)\right)$ where $M = R_{XX}^{-1/2} R_{XY} R_{YY}^{-1/2}$

- so $I[X;Y]$ is independent of all X & Y variances (scale independent), and depends "M"

to simplify, use the block determinant formula

$$\det\left(\begin{bmatrix} A & B \\ C & D\end{bmatrix}\right) = \det(A)\det(D - CA^{-1}B) \quad \text{if } A \ \& \ D \text{ are square...}$$

so:

singular values of M

$$\det\left(\begin{bmatrix} I & M \\ M^T & I\end{bmatrix}\right) = \det(I)\det(I - M^T I M) = \det(I - M^T M) = \prod_{j=1}^{m}(1 - \sigma_j(M)^2)$$

so: given $X, Y \sim N$ w/ Correlations R_{XX}, R_{XY}, R_{YY}

$$I[X;Y] = -\frac{1}{2}\log(\det(I - M^T M)) = -\frac{1}{2}\log\left(\prod_{j=1}^{m}(1 - \sigma_j(M)^2)\right) \quad \text{where } M = R_{XX}^{-1/2} R_{XY} R_{YY}$$

- fully generalizes the 1D case

$$I[X;Y] = -\frac{1}{2}\log(1 - r_{xy}^2)$$

- in both cases I only depends on the correlations, is scale independent.