EE120-F22
Lec. 22 Nov

- Z Transform (Cont.)
- Steady-State / Transient Response
- Zero-Input Response / Zero-State Response
- Equalization

Steady-State / Transient Response

$$y(n) = \alpha y(n-1) + x(n) \qquad |\alpha| < 1$$

$x(n) = u(n)$ system is initially at rest.

$$\hat{X}(z) \longrightarrow \boxed{\hat{H}(z)} \longrightarrow \hat{Y}(z) = \hat{X}(z)\hat{H}(z)$$

$$\hat{Y}(z) = \alpha z^{-1}\hat{Y}(z) + \hat{X}(z) \Longrightarrow (1-\alpha z^{-1})\hat{Y}(z) = \hat{X}(z)$$

$$\hat{H}(z) = \frac{\hat{Y}(z)}{\hat{X}(z)} = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \qquad |\alpha| < |z|$$

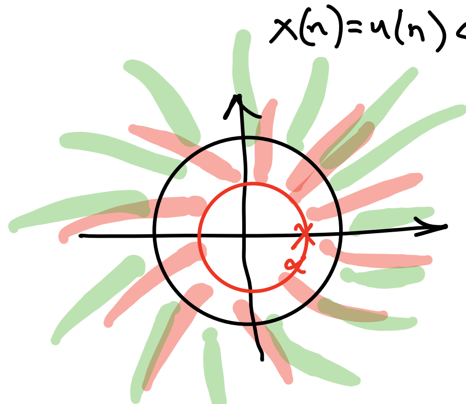
$$x(n) = u(n) \overset{Z}{\longleftrightarrow} \hat{X}(z) = \frac{z}{z-1}$$

$$1 < |z|$$

$$\hat{Y}(z) = \hat{X}(z)\hat{H}(z)$$

$$\hat{Y}(z) = \frac{z}{z-1}\frac{z}{z-\alpha}$$

$$= z\left[\frac{z}{(z-1)(z-\alpha)}\right]$$



$\underline{\underline{1 < |z|}}$

$$\frac{z}{(z-1)(z-\alpha)} = \frac{A}{z-1} + \frac{B}{z-\alpha}$$

$$A = \cancel{(z-1)} \frac{z}{\cancel{(z-1)}(z-\alpha)}\Bigg|_{z=1} = \frac{z}{z-\alpha}\Bigg|_{z=1} = A + \cancel{\frac{B(z-1)}{z-\alpha}}\Bigg|_{z=1}$$

0

$$A = \frac{z}{z-\alpha}\Bigg|_{z=1} = \frac{1}{1-\alpha}$$

$$\frac{z}{(z-1)(z-\alpha)} = \underbrace{\frac{1}{1-\alpha}}_{A} \frac{1}{z-1} - \underbrace{\frac{\alpha}{1-\alpha}}_{B} \frac{1}{z-\alpha}$$

$$B = \frac{z}{z-1}\Bigg|_{z=\alpha} = \frac{\alpha}{\alpha-1}$$

$$\hat{Y}(z) = z\left[\frac{z}{(z-1)(z-\alpha)}\right] = \frac{1}{1-\alpha}\,\frac{z}{z-1} - \frac{\alpha}{1-\alpha}\,\frac{z}{z-\alpha}$$

B

$$1 < |z|$$

$$\hat{Y}(z) = \frac{1}{1-\alpha}\,\frac{1}{1-z^{-1}} - \frac{\alpha}{1-\alpha}\,\frac{1}{1-\alpha z^{-1}}$$

$$\boxed{\beta^n u(n) \longleftrightarrow \frac{1}{1-\beta z^{-1}} \qquad |\beta| < |z|}$$

$$Z^{-1} \downarrow$$

$$y(n) = \underbrace{\frac{1}{1-\alpha} u(n)}_{y_{ss}(n)} - \underbrace{\frac{\alpha}{1-\alpha} \alpha^n u(n)}_{y_{TR}(n)}$$

persists

$$\lim_{n\to\infty} y(n) = \frac{1}{1-\alpha}$$

$$\lim_{n\to\infty} y_{TR}(n) = 0$$

Q: What's the response of the system to $x(n) = 1 \quad \forall n \in \mathbb{Z}$?

Recall $z_0^n \longrightarrow \boxed{\hat{H}(z)} \longrightarrow \hat{H}(z_0)\, z_0^n \qquad z_0 = R_0 e^{i\omega_0}$

$$z_0^n = 1^n \longrightarrow \underset{|\alpha|<|z|}{\boxed{\hat{H}(z) = \frac{z}{z-\alpha}}} \longrightarrow \hat{H}(1)\,1^n = \frac{1}{1-\alpha}$$

What's the response to the input

$$x(n) = \cos(\omega_0 n)u(n)\ ?$$

$$= \frac{1}{2}e^{i\omega_0 n}u(n) + \frac{1}{2}e^{-i\omega_0 n}u(n)$$

$$\underbrace{\beta^n u(n)}_{} \longrightarrow \boxed{h(n) = \alpha^n u(n)} \longrightarrow \underbrace{A\beta^n u(n)}_{} + \underbrace{B\alpha^n u(n)}_{}$$

Zero-Input Response (ZIR) /
Zero-State Resp. (ZSR

$$y(n) = \alpha y(n-1) + x(n) \qquad |\alpha| < 1$$

Causal
BIBO
Stable
System

$$y(-1) \neq 0 \qquad x(n) = 0 \qquad n \geq 0$$

$$y(0) = \alpha y(-1)$$

$$y(1) = \alpha y(0) = \alpha^2 y(-1)$$

$$y(2) = \alpha y(1) = \alpha^3 y(-1)$$

$$\underset{\text{ZIR}}{y(n)} = \alpha^{n+1} y(-1)$$

Now let $y(-1) = 0$, $x(n) = u(n)$

$$\boxed{y_{ZSR}(n) = \frac{1}{1-\alpha}u(n) - \frac{\alpha}{1-\alpha}\alpha^n u(n)}$$

What's the response if

$y(-1) \neq 0$ & $x(n) = u(n)$

$$y(n) = y_{ZIR}(n) + y_{ZSR}(n)$$

$$= \underbrace{\alpha^{n+1} y(-1)}_{y_{ZIR}(n)} + \underbrace{\frac{1}{1-\alpha} u(n) - \frac{\alpha}{1-\alpha} \alpha^n u(n)}_{y_{ZSR}(n)}$$

$$y(n) = \alpha y(n-1) + x(n)$$

$$y(n)u(n) = \alpha y(n-1)u(n) + x(n)u(n)$$

$$\sum_{n=-\infty}^{\infty} y(n)u(n)z^{-n} = \alpha \sum_{n=-\infty}^{\infty} y(n-1)u(n)z^{-n} + \sum_{n=-\infty}^{\infty} x(n)u(n)z^{-n}$$

$$\underbrace{\sum_{n=0}^{\infty} y(n)z^{-n}}_{\hat{y}(z)} = \alpha \underbrace{\sum_{n=0}^{\infty} y(n-1)z^{-n}}_{?} + \underbrace{\sum_{n=0}^{\infty} x(n)z^{-n}}_{\hat{X}(z)}$$

Unilateral Z Transform

What is

$$\sum_{n=0}^{\infty} y(n-1)z^{-n} = \sum_{m=-1}^{\infty} y(m)z^{-(m+1)} = \left[y(-1) + z^{-1}\underbrace{\sum_{m=0}^{\infty} y(m)z^{-m}}_{\hat{y}(z)}\right]$$

let $m = n-1 \rightarrow n = m+1$

$$\hat{y}(z) = \alpha\left[y(-1) + z^{-1}\hat{y}(z)\right] + \hat{x}(z)$$

$$(1-\alpha z^{-1})\hat{y}(z) = \alpha y(-1) + \hat{x}(z)$$

$$\hat{y}(z) = \frac{\alpha y(-1)}{1-\alpha z^{-1}} + \frac{\hat{x}(z)}{1-\alpha z^{-1}}$$

$$= \alpha y(-1)\frac{1}{1-\alpha z^{-1}} + \hat{x}(z)\frac{1}{1-\alpha z^{-1}}$$

$$\hat{y}(z) = \alpha y(-1)\hat{H}(z) + \underbrace{\hat{x}(z)\hat{H}(z)}$$

$y(n) = \alpha y(-1)h(n) +$ Done Before

$$\hat{y}(z) = \sum_{n=0}^{\infty} y(n) z^{-n}$$

If $y(n) = 0 \quad n < 0$

$$\Rightarrow \hat{Y}(z) = \hat{y}(z)$$

$$\hat{x}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

If $x(n) = 0 \quad \forall n < 0$

then $\hat{x}(z) = \hat{X}(z)$

$$\boxed{y(n) = \alpha y(-1)\underbrace{\alpha^{n} u(n)}_{h(n)} + \frac{1}{1-\alpha}u(n) - \frac{\alpha}{1-\alpha}\alpha^{n} u(n)}$$

$$x(n) \overset{UZ}{\longleftrightarrow} \hat{X}(z) \triangleq \sum_{n=0}^{\infty} x(n) z^{-n}$$

Unilateral Z Transform

$$x(n-1) \overset{UZ}{\longleftrightarrow} \sum_{n=0}^{\infty} x(n-1) z^{-n} = \sum_{m=-1}^{\infty} x(m) z^{-(m+1)}$$

let $m = n-1$

$n = m+1$

$$= x(-1) + \underbrace{\left[\sum_{m=0}^{\infty} x(m) z^{-m}\right]}_{\hat{X}(z)} z^{-1}$$

$$\boxed{\begin{aligned} x(n-1) &\overset{UZ}{\longleftrightarrow} x(-1) + z^{-1}\hat{X}(z) \\ y(n-1) &\overset{UZ}{\longleftrightarrow} y(-1) + z^{-1}\hat{Y}(z) \\ y(n-2) &\overset{UZ}{\longleftrightarrow} y(-2) + z^{-1}\, UZ\{y(n-1)\} \\ &\quad y(-2) + z^{-1}\left[y(-1) + z^{-1}\hat{Y}(z)\right] \\ &\quad y(-2) + z^{-1}y(-1) + z^{-2}\hat{Y}(z) \end{aligned}}$$

$$x(n) \overset{Z}{\longleftrightarrow} \hat{X}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) \overset{UZ}{\longleftrightarrow} \hat{X}(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) u(n) z^{-n}$$

$$\hat{X}(z) = Z\{x(n) u(n)\}$$

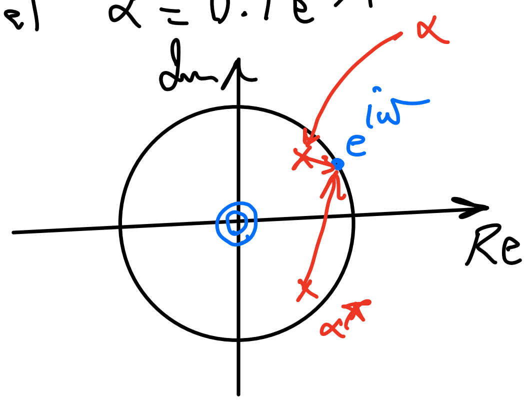
Equalization:

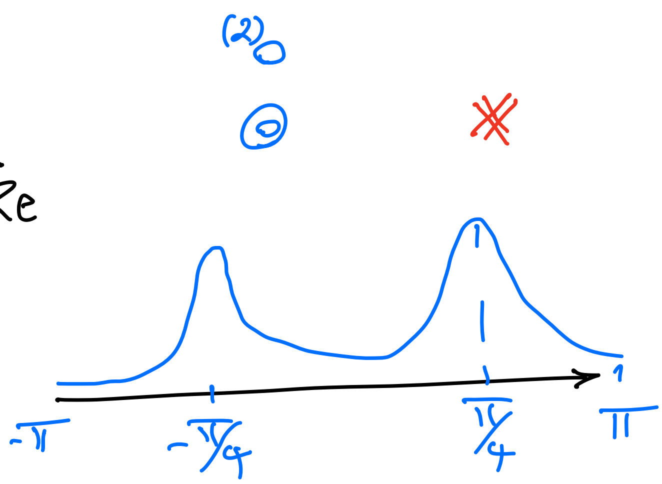
$$\hat{H}(z) = \frac{z^2}{(z-\alpha)(z-\alpha^*)}$$

Model of Room Acoustics.

let $\alpha = 0.9e^{i\pi/4}$

Causal $0.9 = |\alpha| < |z|$





$$H(\omega) = \hat{H}(z)\Big|_{z=e^{i\omega}}$$

$$H(\omega) = \frac{e^{i2\omega}}{(e^{i\omega}-\alpha)(e^{i\omega}-\alpha^*)} \to |H(\omega)| = \frac{1}{|e^{i\omega}-\alpha||e^{i\omega}-\alpha|}$$

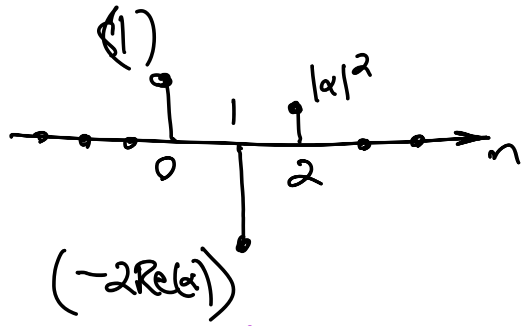
$$x(n) \to \boxed{\hat{H}(z)} \to \boxed{\hat{H}^{-1}(z)} \to y(n)$$

$$\hat{G}(z) = \hat{H}^{-1}(z) = \frac{1}{\hat{H}(z)} = \frac{(z-\alpha)(z-\alpha^*)}{z^2} = \frac{z^2 - (\alpha+\alpha^*)z + \alpha\alpha^*}{z^2}$$

$$= \frac{z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2}{z^2} = 1 - 2\mathrm{Re}(\alpha)z^{-1} + |\alpha|^2 z^{-2}$$

$$g(n) = \delta(n) - 2\mathrm{Re}(\alpha)\delta(n-1) + |\alpha|^2\delta(n-2)$$

$g(n)$



causal
BIBO stable.

What if $\hat{F}(z) = \dfrac{z}{(z-\alpha)(z-\alpha^*)}$ Room Acoustics

$\alpha = 0.9e^{i\pi/4}$

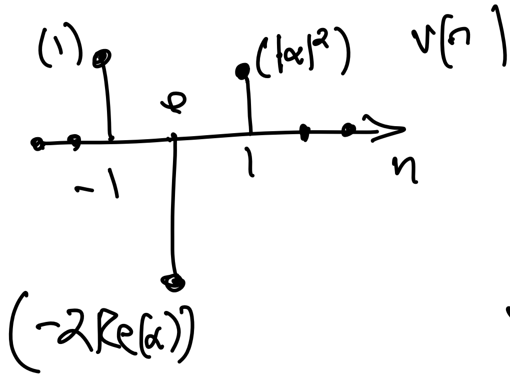
$$x(n) \longrightarrow \boxed{\hat{F}(z)} \longrightarrow \boxed{\hat{F}^{-1}(z)} \longrightarrow y(n)$$

Does a causal BIBO stable inverse exist?

$$\hat{F}^{-1}(z) = \frac{1}{\hat{F}(z)} = \frac{(z-\alpha)(z-\alpha^*)}{z} = \frac{z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2}{z}$$

$$\hat{V}(z) = \hat{F}^{-1}(z) = z - 2\mathrm{Re}(\alpha) + |\alpha|^2 z^{-1}$$

$$v(n) = \delta(n+1) - 2\mathrm{Re}(\alpha)\delta(n) + |\alpha|^2 \delta(n-1)$$

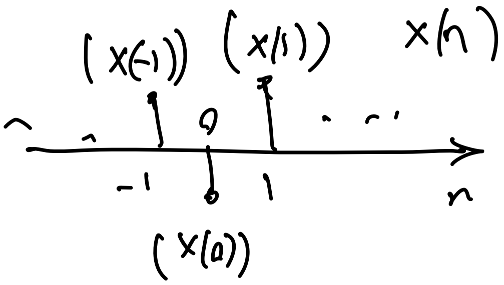


BIBO stable
not causal.

$v(n) = g(n+1)$
$g(n) = v(n-1)$

We can use $g(n)$ instead of $v(n)$ and tolerate a one-sample delay.

$$\delta(n+1) \overset{\mathcal{Z}}{\longleftrightarrow} z$$



$$x(n-N) \longleftrightarrow z^{-N}\hat{X}(z)$$

$N=-1$

$$x(n+1) \longleftrightarrow z^{1}\hat{X}(z)$$

$$x(n)=\delta(n) \longleftrightarrow \hat{X}(z)=1$$

$$x(n+1)=\delta(n+1) \longleftrightarrow z\hat{X}(z)=z$$

$$\hat{X}(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n} = \underbrace{\cdots}_{0} + \underbrace{x(-1)}_{1}z + \underbrace{x(0)}_{0} + \underbrace{x(1)}_{0}z^{-1} + \underbrace{\cdots}_{0}$$

if $x(n)=\delta(n+1)$