EE 120: Signals and Systems
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FORMULAS & TABLES

Discrete Fourier Series (DFS) Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) \, e^{-ik\omega_0 n} \ ,$$

where $\omega_0 = \dfrac{2\pi}{p}$ and $\langle p \rangle$ denotes a suitable discrete interval of length p (i.e., an interval containing p contiguous integers). For example, $\displaystyle\sum_{k=\langle p \rangle}$ may denote $\displaystyle\sum_{k=0}^{p-1}$ or $\displaystyle\sum_{k=1}^{p}$.

Continuous-Time Fourier Series (FS) Complex exponential Fourier series synthesis and analysis equations for a periodic continuous-time signal having period p:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{ik\omega_0 t} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) \, e^{-ik\omega_0 t} dt \ ,$$

where $\omega_0 = \dfrac{2\pi}{p}$ and $\langle p \rangle$ denotes a suitable continuous interval of length p. For example, $\displaystyle\int_{\langle p \rangle}$ can denote $\displaystyle\int_0^p$.

Transfer Function and Frequency Response of a DT LTI System Consider a real, discrete-time LTI system having impulse response $h : \mathbb{Z} \to \mathbb{R}$. The transfer function $\widehat{H} : \mathbb{C} \to \mathbb{C}$ of the system is given by:

$$\widehat{H}(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}\,, \quad \forall z \in \mathsf{RoC}(h).$$

If the system is stable, its frequency response $H : \mathbb{R} \to \mathbb{C}$ is given by:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}\,, \quad \forall \omega \in \mathbb{R}\,.$$

The impulse response of the system is given by:

$$h(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} H(\omega)\, e^{i\omega n}\, d\omega\,.$$

Transfer Function and Frequency Response of a CT LTI System Consider a real, continuous-time LTI system having impulse response $h : \mathbb{R} \to \mathbb{R}$.

If the system is stable, its frequency response $H : \mathbb{R} \to \mathbb{C}$ is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt, \quad \forall \omega \in \mathbb{R}\,.$$

The impulse response of the system is given by:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, d\omega\,.$$

Time domain	**Frequency domain**
$\forall\, n \in \mathbb{Z}, \quad x(n)$ is real	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = X^*(-\omega)$
$\forall\, n \in \mathbb{Z}, \quad x(n) = x^*(-n)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega)$ is real
$\forall\, n \in \mathbb{Z}, \quad y(n) = x(n-N)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = e^{-i\omega N} X(\omega)$
$\forall\, n \in \mathbb{Z}, \quad y(n) = e^{i\omega_1 n} x(n)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = X(\omega - \omega_1)$
$\forall\, n \in \mathbb{Z},$ $y(n) = \cos(\omega_1 n) x(n)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) + X(\omega + \omega_1))/2$
$\forall\, n \in \mathbb{Z},$ $y(n) = \sin(\omega_1 n) x(n)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) - X(\omega + \omega_1))/2i$
$\forall\, n \in \mathbb{Z},$ $x(n) = a x_1(n) + b x_2(n)$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = a X_1(\omega) + b X_2(\omega)$
$\forall\, n \in \mathbb{Z}, \quad y(n) = (h * x)(n)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = H(\omega) X(\omega)$
$\forall\, n \in \mathbb{Z}, \quad y(n) = x(n) p(n)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = \frac{1}{2\pi} \int\limits_0^{2\pi} X(\Omega) P(\omega - \Omega) d\Omega$
$\forall\, n \in \mathbb{Z},$ $y(n) =$ $\begin{cases} x(n/N) & n \text{ is a multiple of } N \\ 0 & \text{otherwise} \end{cases}$	$\forall\, \omega \in \mathbb{Z},$ $Y(\omega) = X(N\Omega)$

Table 1: Properties of the DTFT.

Signal	**DTFT**
$\forall\, n \in \mathbb{Z}, \quad x(n) = \delta(n)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = 1$
$\forall\, n \in \mathbb{Z},$ $x(n) = \delta(n - N)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = e^{-i\omega N}$
$\forall\, n \in \mathbb{Z}, \quad x(n) = K$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = 2\pi K \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$
$\forall\, n \in \mathbb{Z},$ $x(n) = a^n u(n), \quad \lvert a \rvert < 1$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \frac{1}{1 - ae^{-i\omega}}$
$\forall\, n \in \mathbb{Z},$ $x(n) = \begin{cases} 1 & \text{if } \lvert n \rvert \leq M \\ 0 & \text{otherwise} \end{cases}$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \frac{\sin(\omega(M + 0.5))}{\sin(\omega/2)}$
$\forall\, n \in \mathbb{Z},$ $x(n) = \frac{\sin(Wn)}{\pi n}, \quad 0 < W < \pi$	$\forall\, \omega \in [-\pi, \pi],$ $X(\omega) = \begin{cases} 1 & \text{if } \lvert \omega \rvert \leq W \\ 0 & \text{otherwise} \end{cases}$

Table 2: Discrete time Fourier transforms of key signals. The function u is the unit step.

Time domain	**Frequency domain**
$\forall\, t \in \mathbb{R}, \quad x(t)$ is real	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = X^*(-\omega)$
$\forall\, t \in \mathbb{R}, \quad x(t) = x^*(-t)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega)$ is real
$\forall\, t \in \mathbb{R}, \quad y(t) = x(t-\tau)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = e^{-i\omega\tau} X(\omega)$
$\forall\, t \in \mathbb{R}, \quad y(t) = e^{i\omega_1 t} x(t)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = X(\omega - \omega_1)$
$\forall\, t \in \mathbb{R},$ $y(t) = \cos(\omega_1 t) x(t)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) + X(\omega + \omega_1))/2$
$\forall\, t \in \mathbb{R},$ $y(t) = \sin(\omega_1 t) x(t)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) - X(\omega + \omega_1))/2i$
$\forall\, t \in \mathbb{R},$ $x(t) = a x_1(t) + b x_2(t)$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = a X_1(\omega) + b X_2(\omega)$
$\forall\, t \in \mathbb{R}, \quad y(t) = (h * x)(t)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = H(\omega) X(\omega)$
$\forall\, t \in \mathbb{R}, \quad y(t) = x(t) p(t)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(\Omega) P(\omega - \Omega) d\Omega$
$\forall\, t \in \mathbb{R},$ $y(t) = x(at)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = \frac{1}{\lvert a \rvert} X(\Omega/a)$

Table 3: Properties of the CTFT.

Signal	**CTFT**
$\forall\, t \in \mathbb{R}, \quad x(t) = \delta(t)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = 1$
$\forall\, t \in \mathbb{R}, \quad x(t) = \delta(t-\tau),\ \tau \in \mathbb{R}$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = e^{-i\omega\tau}$
$\forall\, t \in \mathbb{R}, \quad x(t) = K$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = 2\pi K \delta(\omega)$
$\forall\, t \in \mathbb{R},$ $x(t) = a^t u(t), \quad 0 < a < 1$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \dfrac{1}{j\omega - \ln(a)}$
$\forall\, t \in \mathbb{R},$ $x(t) = \begin{cases} \pi/a & \text{if } \lvert t\rvert \le a \\ 0 & \text{otherwise} \end{cases}$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \dfrac{2\pi \sin(a\omega)}{a\omega}$
$\forall\, t \in \mathbb{R},$ $x(t) = \dfrac{\sin(\pi t/T)}{\pi t/T},$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \begin{cases} T & \text{if } \lvert\omega\rvert \le \pi/T \\ 0 & \text{otherwise} \end{cases}$

Table 4: Continuous time Fourier transforms of key signals. The function u is the unit step.

Time Domain $\forall\, n \in \mathbb{Z}$	**Transform Domain** $\forall\, z \in \mathsf{RoC}$	**RoC**	**Name**
$w(n) = ax(n) + by(n)$	$\hat{W}(z) = a\hat{X}(z) + b\hat{Y}(z)$	$\mathsf{RoC}(w) \supset$ $\mathsf{RoC}(x) \cap \mathsf{RoC}(y)$	**Linearity**
$y(n) = x(n-N)$	$\hat{Y}(z) = z^{-N}\hat{X}(z)$	$\mathsf{RoC}(y) = \mathsf{RoC}(x)$	**Delay**
$y(n) = (x * h)(n)$	$\hat{Y}(z) = \hat{X}(z)\hat{H}(z)$	$\mathsf{RoC}(y) \supset$ $\mathsf{RoC}(x) \cap \mathsf{RoC}(h)$	**Convolution**
$y(n) = x^*(n)$	$\hat{Y}(z) = [\hat{X}(z^*)]^*$	$\mathsf{RoC}(y) = \mathsf{RoC}(x)$	**Conjugation**
$y(n) = x(-n)$	$\hat{Y}(z) = \hat{X}(z^{-1})$	$\mathsf{RoC}(y) =$ $\{z \mid z^{-1} \in \mathsf{RoC}(x)\}$	**Time reversal**
$y(n) = nx(n)$	$\hat{Y}(z) = -z\frac{d}{dz}\hat{X}(z)$	$\mathsf{RoC}(y) = \mathsf{RoC}(x)$	**Scaling by n**
$y(n) = a^{-n}x(n)$	$\hat{Y}(z) = \hat{X}(az)$	$\mathsf{RoC}(y) =$ $\{z \mid az \in \mathsf{RoC}(x)\}$	**Exponential scaling**
$x(n) = 0,\ \forall n < 0$	$\lim_{z\to\infty} \hat{X}(z) = x(0)$	Outside the outermost pole, out to, and including, $+\infty$	**Initial Value Theorem**

Table 5: Properties of the Z transform. In this table, a, b are complex constants, and N is an integer constant.

Discrete-time signal $\forall\, n \in \mathbb{Z}$	**Z transform** $\forall\, z \in \mathsf{RoC}(x)$	$\mathsf{RoC}(x) \subset \mathbb{C}$
$x(n) = \delta(n - M)$	$\hat{X}(z) = z^{-M}$	$\mathbb{C}$
$x(n) = u(n)$	$\hat{X}(z) = \frac{z}{z-1}$	$\{z \mid \|z\| > 1\}$
$x(n) = a^n u(n)$	$\hat{X}(z) = \frac{z}{z-a}$	$\{z \mid \|z\| > \|a\|\}$
$x(n) = a^n u(-n)$	$\hat{X}(z) = \frac{1}{1-a^{-1}z}$	$\{z \mid \|z\| < \|a\|\}$
$x(n) = \cos(\omega_0 n) u(n)$	$\hat{X}(z) = \frac{z^2 - z\cos(\omega_0)}{z^2 - 2z\cos(\omega_0) + 1}$	$\{z \mid \|z\| > 1\}$
$x(n) = \sin(\omega_0 n) u(n)$	$\hat{X}(z) = \frac{z\sin(\omega_0)}{z^2 - 2z\cos(\omega_0) + 1},$	$\{z \mid \|z\| > 1\}$
$x(n) = \frac{1}{(N-1)!}(n-1)\cdots(n-N+1)\, a^{n-N} u(n-N)$	$\hat{X}(z) = \frac{1}{(z-a)^N}$	$\{z \mid \|z\| > \|a\|\}$
$x(n) = \frac{(-1)^N}{(N-1)!}(N-1-n)\cdots(1-n)\, a^{n-N} u(-n)$	$\hat{X}(z) = \frac{1}{(z-a)^N}$	$\{z \mid \|z\| < \|a\|\}$

Table 6: Z transforms of key signals. The signal u is the unit step, δ is the Kronecker delta, a is any complex constant, ω_0 is any real constant, M is any integer constant, and $N > 0$ is any integer constant.