Numerical continuation scheme for tracing the double bounded homotopy for analysing nonlinear circuits

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***Abstract*— A numerical continuation for tracing the double bounded homotopy for obtaining DC solutions of nonlinear circuits is proposed. The *double bounded homotopy* is used to find multiple DC solutions with the advantage of having a stop criterion which is based on the property of having a double bounded trajectory. The key aspects of the implementation of the numerical continuation. Besides, in order to trace and applied the stop criterion some blocks of the numerical continuation are modified and explained.**

I. INTRODUCTION

The homotopy [1][2][3] has an important role in the analysis of multiple operating points circuits. Although, homotopy methods are able to find more than one solution to the equilibrium equation of a circuit, they still exhibit several lacks. Among them, it is worthy to mention the **stop criterion**. There are two types of paths of solutions, open and closed paths, the main problem is when to stop searching for more solutions?. For closed paths, this can be solved by testing whether a new solution has previously been found. For open paths, this is a serious drawback, because there is no reasonable and reliable stop criterion to decide when to stop seeking for more solutions.

The double bounded Homotopy (DBH) [4][5] is proposed in order to circumvent the problem of stop criterion. This homotopy lies in the main property of having two symmetrical branches bounded by a double bounding solution line. The symmetry and bounding properties of the homotopy trajectory of DBH are depicted in Figure 1. This properties are useful in order to implement a stop criterion.

The numerical continuation methods (also called path following and path tracking) are a numerical tool used to trace the homotopy trajectories. They are mixture of a variety of numerical methods, focused on designing a path in order to accomplish specific needs of a homotopy formulation. Then, it is necessary to design an appropriated numerical continuation method in order to trace the DBH implementing the stop criterion. The next section is dedicate to explain the traditional numerical continuation methods.

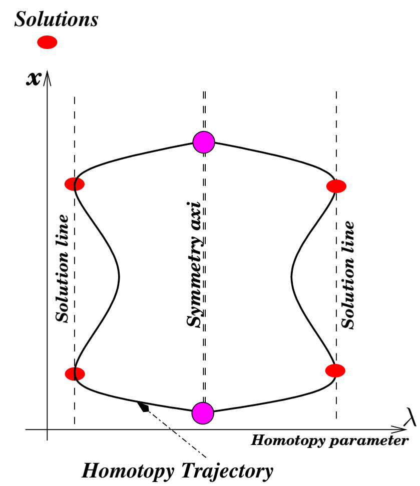

Fig. 1. Symmetry and bounding of DBH

II. NUMERICAL CONTINUATION METHODS

Some homotopies have global convergence when are applied to solve the equilibrium equation of certain type of circuits. However, without an appropriated numerical continuation method is not possible to ensure that all the solutions of the equilibrium equation will be found. There are some reasons for this problem, one of them is the predictor corrector steps, if the coeficients of this steps are not well choice the numerical continuations fails and lost the homotopy trajectory. The other reason, it is that once the numerical continuation cross the solution line [] some times the algorithm fails to determine the solution because it diverges. Then, it is important to study the characteristics of the numerical continuation in order to use them appropriately.

1) Predictor: The predictor point for (x^j, λ_j) is given by:

$$(\bar{\boldsymbol{x}}^{j+1}, \bar{\lambda}_{j+1}) = (\boldsymbol{x}^j, \lambda_j) + h * \boldsymbol{t}$$

where h is an appropriate step length and $\boldsymbol{t}$ is a normalised vector tangent to the homotopy trajectory (see Figure 2). This predictor can be considered as a step of the Euler method for solving the differential equation that describes the homotopy trajectory (continuation path). Predictor steps are usually based on tangent predictions but there are different alternatives like the secant predictor [6], interpolation predictor [7], Taylor polynomial predictor [7].

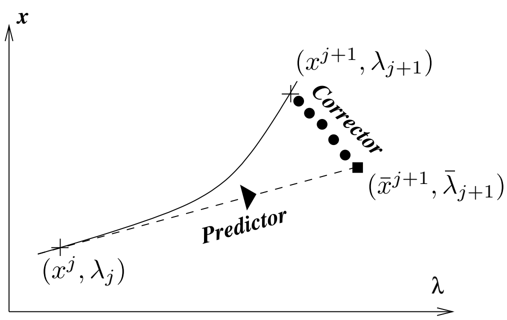

Fig. 2. Predictor-corrector steps

2) Corrector: When the predictor step finishes, it is necessary to fit the homotopy trajectory by using a corrector step. This step solves the homotopy formulation by starting from $(\bar{x}^{j+1}, \bar{\lambda}_{j+1})$ (see Figure 2). The common way to solve this equation is by using the Newton-Gauss method [7], which can solved for systems of the type $R^{N+1} \Rightarrow R^N$.

3) Step Control: The Predictor-Corrector scheme can be optimized for tracing acceleration by using a step length control. A small constant step length can trace the curve successfully but not efficiently, because this process involves too many steps along "flat" branches. Therefore, it is necessary to adapt the step length to the convergence behavior at each predictor-corrector step. The basic criterion is to control the step by observing the convergence quality of the corrector step. A change of the number of iterations in the corrector step produces a compensation factor ζ, which affects the step length as follows:

$$h_{j+2} = h_{j+1}\zeta$$

4) Find Zero Strategy: Without an efficient finding zero strategy the numerical continuation is incomplete and the homotopy could fail to converge some operating points (maybe not any solution at all). The tracing of the homotopy trajectory begins in $\lambda = 0$ and it "ends"[1] when $\rho = 1$. Therefore, when the tracing is near to $\lambda = 1$ the *find Zero Strategy* begins. The most simple example of strategy is to use $[\boldsymbol{x}_f, \lambda_f]$ (of the last iteration) like the initial point to solve the equilibrium equation $\boldsymbol{f}(\boldsymbol{x})$ with Newton method.

Nevertheless, as Newton method has local convergence it still could fail to find the solution. In [8] are reported some techniques to do the *find Zero Strategy* accurately and reliable. The basic idea is to use two points ($p_1 < 1$ and $p_2 \geq 1$) around to $\lambda = 1$, next, interpolate the point in $\lambda = 1$ in order to obtain a point nearly to the real solution and use a like Newton method to find the solution of the original system $\boldsymbol{f}(\boldsymbol{x})$.

5) Stop criterion: In fact, there isn't stop criterion in traditional numerical continuation methods (when applied to homotopy trajectory tracing). The only practical way to stop tracing the trajectory it is to used a maximum number (k^*) of predictor-corrector steps (k) without finding any solution, then, the numerical continuation stop seeking for more solutions. This technique is inefficient because can fail to find some (or all) solutions lied to the homotopy trajectory.

[1]In fact if it is wanted to find multiple solutions the numerical continuation should follow beyond $\lambda = 1$.

Finally, all the blocks of the numerical continuation methods are depicted in Fiigure 3. The next section will explain the particular modifications to this scheme in order to trace the double bounded homotopy.

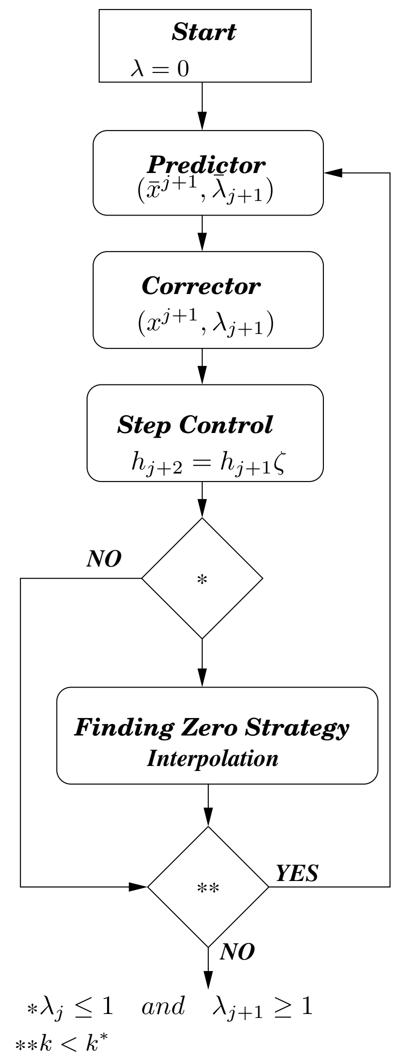

Fig. 3. Numerical continuation scheme

III. Modifiied Numerical continuation

The numerical continuation scheme consist of the next blocks: predictor, corrector, step control, find zero strategy. In order to trace the DBH it is necessary to modify the find zero strategy and add the stop criterion as a new fundamental block of the numerical continuation method.

1) **Find Zero Strategy**. The double bounded homotopy has the characteristic of never cross the $\lambda = 1$, hence, the findig zero strategy should start when bouncing in the bounding lines. A good way to achieve this process is

to monitor the change of sign of the $\Delta\lambda$ produced in the predictor step. This can be done by multipplaying the $\Delta\lambda$ of two consecutive predictor steps.

$$sign(\triangle\lambda_{j+1}\triangle\lambda_j) \neq -1$$

This procedure is depicted in Figure 4. In this figure the sign of $\Delta\lambda$ change in the bouncing points A and B. Besides, in order to apply a quadratic interpolation the algorithm needs three points (A, B, C).

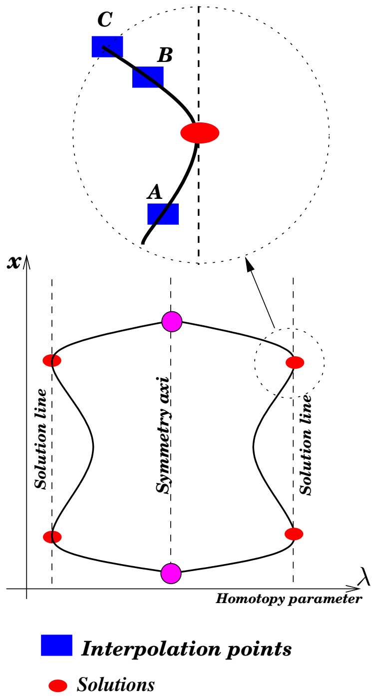

Fig. 4. Find Zero Strategy

2) **Stop criterion**. The stop criterion for this homotopy is depicted in Figure 5. The homotopy trajectory starts in the symmetry axi of the homotopy trajectory, then, it trace the half of the trayectory (called symmetrical branch) and stop when it returns to the symmetry axi. The advantage of the stop criterion is that the numerical continuation has only to trace the half side of the trayectory.

Finally, the modified numerical continuation is depicted in Figure 6 where the dashed blocks are the specific characteristics added to the procedure in order to include a stop criterion. The scheme is explained as follows: First, the predictor calculates the tangent to (x^j, λ_j) and using a step length calculates the $(\bar{x}^{j+1}, \bar{\lambda}_{j+1})$ over the tangent. Secondly, the corrector uses the solution of the predictor in order to obtain a new point on the homotopy trajectory, given by (x^{j+1}, λ_{j+1}). Next, the step control is applied in order to accelerate the tracing. Next, the so-called end-zero strategy is applied, it is triggered (by each solution) when the homotopy trajectory bounces in $\lambda = a$ or $\lambda = b$ depending on the branch where the solution lies. Finally, the numerical continuation stops tracing when the trajectory returns to the symmetry axi.

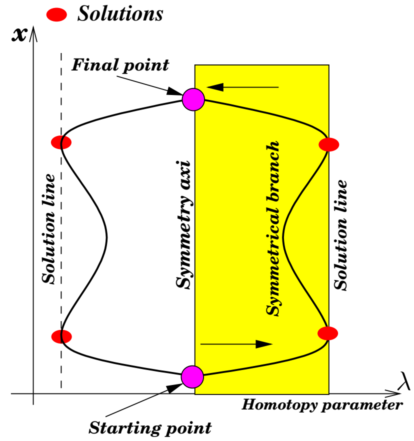

Fig. 5. Stop criterion

IV. Examples

In order to illustrate the use of the double bounded homotopy, it is applied to the next system of equations:

$$f_1(x_1, x_2) = (x_2 - 1)(x_2 - 4)(x_2 - 6) + x_1 = 0$$
$$f_2(x_1, x_2) = (x_1 - 3)(x_1 - 6)(x_1 - 9) + x_2 = 0$$

The graphic solution of the system is shown in Figure 7.

The homotopy formulation from Equation **??** results in:

$$H_1(f_1, \lambda) = 100Q + e^Q \ln(0.001 f_1^2 + 1) = 0$$
$$H_2(f_2, \lambda) = 100Q + e^Q \ln(0.001 f_2^2 + 1) = 0$$

where $Q = \lambda(\lambda - 1)$; i.e. $a = 0$ and $b = 1$.

The homotopy trajectories are depicted in Figure 8. The starting points lie on the plane defined by $\lambda = 0.5$, while, the solutions are obtained when λ reaches the value of 1.

The double bounded homotopy is applied to the latch circuit [9] of Figure 9, which contains two NMOS transistors (M_1 and M_2), two linear resistors (R_1 and R_2) and a voltage source (E). The model of the transistors is the unified MOS model reported in [10] which is a modified version of the well-known BSIM model.

The Figure 10 show the graph of the equilibrium equation and homotopy trajectory of the circuit in the space of (v_1, v_2, I_E). The homotopy find the three solutions of the circuit.

Figure 12 depict the chua's benchmark circuit of nine solutions. This circuit has 4 bipolar transistors modeled by the half-side Ebers-Moll model.

The equations system formulation is the same of [11] which is based on the branch voltages v_1, v_2, v_3, v_4. Figure 11 depict the homotopy trajectory and the six solutions found versus the branch voltage v_1.

V. Conclusions

In this work presented a numerical continuation for tracing a double bounded homotopy to obtain the operating points of nonlinear circuits having multiple solutions. The main blocks of the numerical continuation scheme were depicted. Besides, the stop criterion proposed was implement modifying some

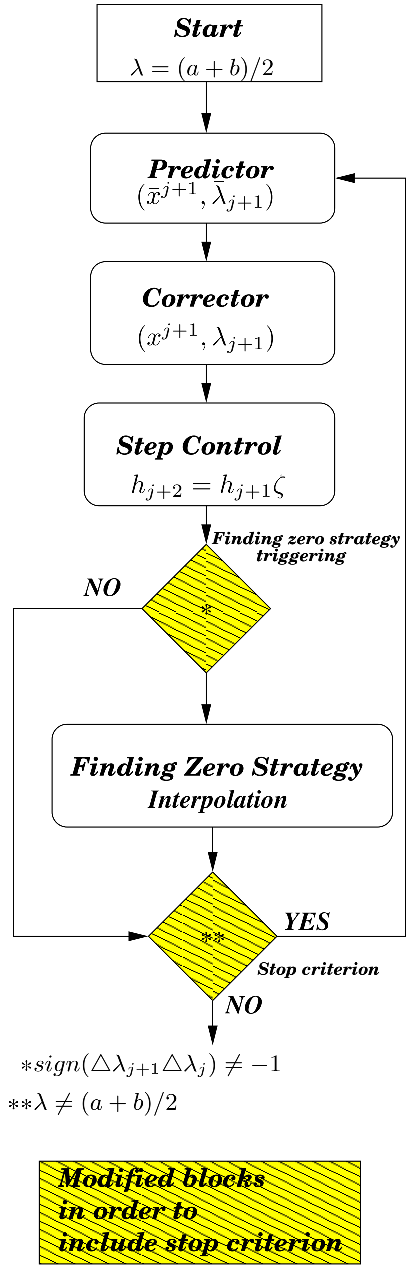

Fig. 6. Block diagram of the modified numerical continuation

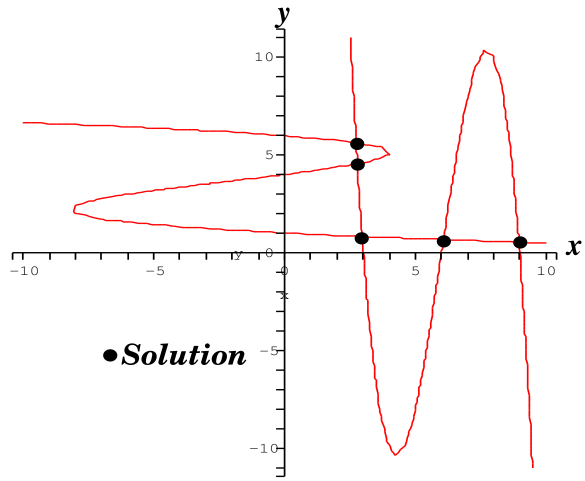

Fig. 7. System of five solutions

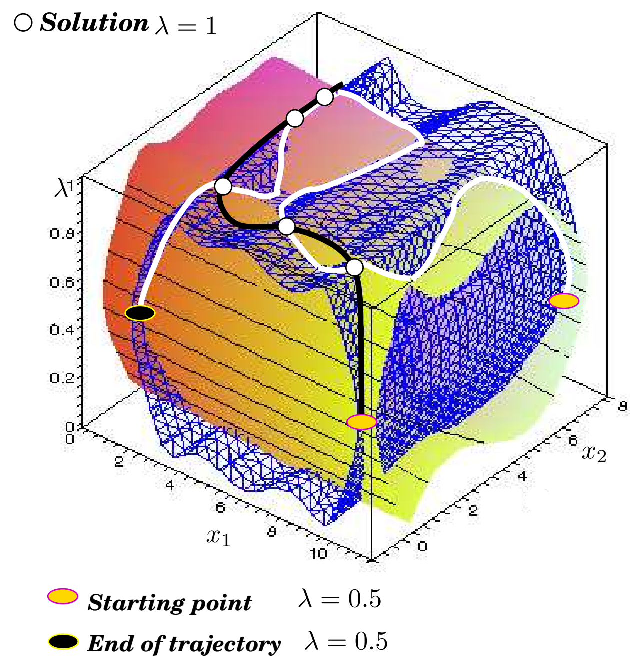

Fig. 8. Homotopy trajectory

blocks of the numerical continuation scheme. Finally, some examples applying the stop criterion were depicted.

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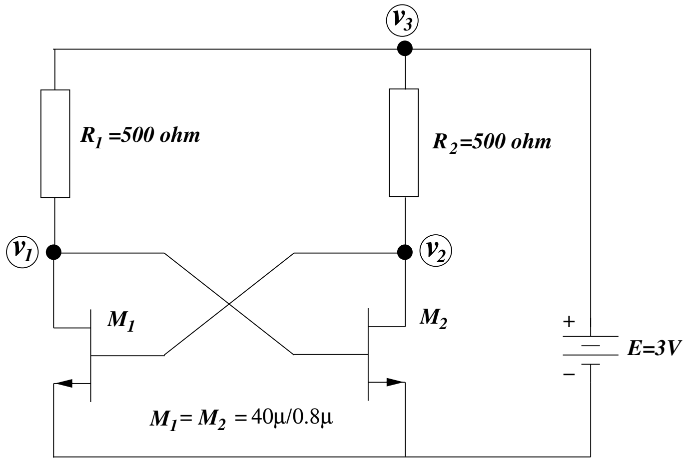

Fig. 9. Example circuit

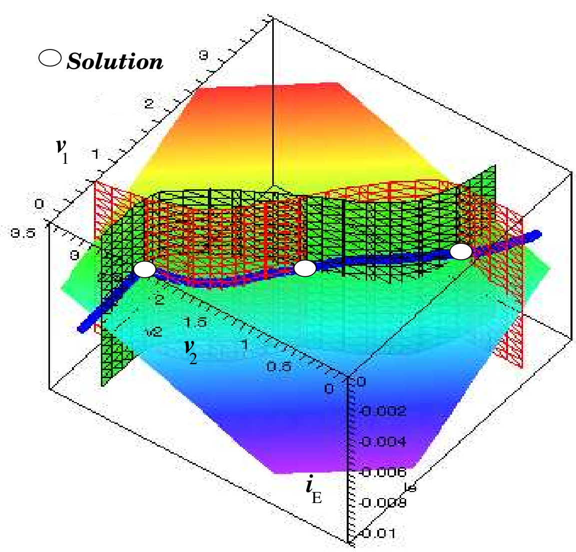

Fig. 10. Graphic of the equilibrium equation of the circuit and homotopy trajectory

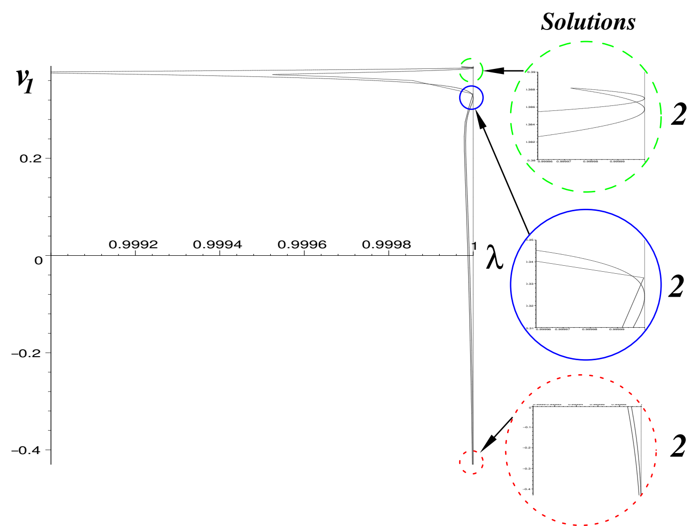

Fig. 11. Solution of the Chua's circuit

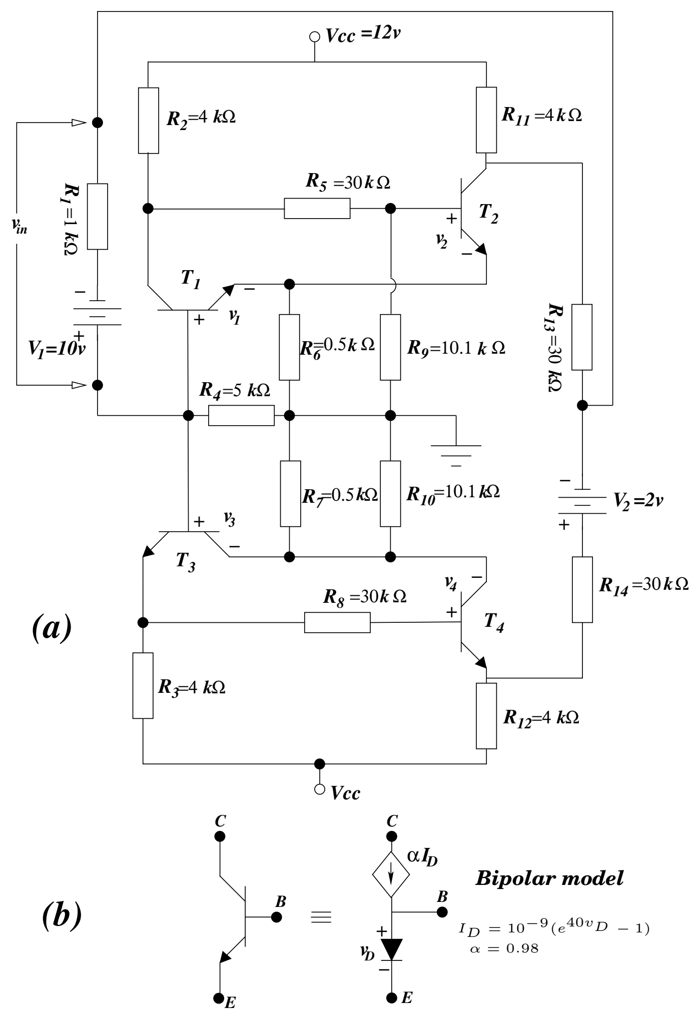

Fig. 12. Chua's circuit with nine solutions

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