Numerical continuation scheme for tracing the double bounded homotopy for analysing nonlinear circuits

Héctor Vázquez-Leal, Luis Hernández-Martínez, Arturo Sarmiento-Reyes and Roberto Castañeda-Sheissa
National Institute for Astrophysics, Optics and Electronics
Electronics Department, CAD Group
P.O. Box 51, 72000, Puebla, Pue., Mexico
E-mail *hleal@inaoep.mx*

***Abstract*— A numerical continuation for tracing the double bounded homotopy for obtaining DC solutions of nonlinear circuits is proposed. The *double bounded homotopy* is used to find multiple DC solutions with the advantage of having a stop criterion which is based on the property of having a double bounded trajectory. The key aspects of the implementation of the numerical continuation are presented in this paper. Besides, in order to trace and apply the stop criterion some blocks of the numerical continuation are modified and explained.**

I. Introduction

Homotopy methods [1][2][3] have an important role in the analysis of circuits exhibiting multiple operating points. Although, homotopy methods are able to find more than one solution to the equilibrium equation of the circuit, they still show several problems. Among them, it is worthy to mention the **stop criterion**.

It is well known that there are two types of paths of solutions, open and closed paths, the main problem is when to stop searching for more solutions. For closed paths, this can be solved by testing whether a new solution is not indeed a previesly found solution. For open paths, this is a serious drawback, because there is no reasonable and reliable stop criterion to decide when to stop seeking for more solutions.

The Double Bounded Homotopy (DBH) [4][5] has been proposed in order to circumvent the problem of the stop criterion. This homotopy possesses symmetrical branches bounded by a double bounding solution line. The symmetry and bounding properties of the trajectory of the DBH are depicted in Figure 1. These properties are useful in order to implement a reliable stop criterion.

The numerical continuation methods (also called path following and path tracking) are numerical tools used to trace the homotopy trajectories. They are a combination of a variety of numerical methods, focussed on drawing a path in order to accomplish specific needs of a particular homotopy formulation.

Due to the specific features of the DBH, it becomes necessary to devise a well-suited numerical continuation method in order to trace the homotopy trajectory having a robust stop criterion. The next section is dedicate to explain the traditional numerical continuation methods.

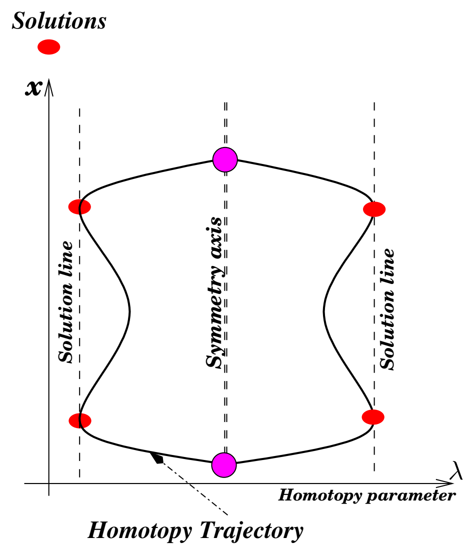

Fig. 1. Symmetry and bounding of DBH

II. Numerical continuation methods

Some homotopies have global convergence when are applied to solve the equilibrium equation of certain type of circuits. However, without an appropriated numerical continuation method is not possible to ensure that all the solutions of the equilibrium equation will be found. There are some reasons for this problem, one of them is related with the predictor-corrector steps, if the coefficients of these steps are not properly selected, then the numerical continuation fails and losts the homotopy trajectory. The other reason is that once the numerical continuation crosses the solution line, the algorithm fails to determine the solution because it diverges. It clearly results that it is important to study the characteristics of the numerical continuation in order to use them appropriately. The numerical continuation scheme consists of a predictor, a corrector, a step control, a find zero strategy and a stop criterion.

1) Predictor: The predictor point for (x^j, λ_j) is given by:

$$(\bar{\boldsymbol{x}}^{j+1}, \bar{\lambda}_{j+1}) = (\boldsymbol{x}^j, \lambda_j) + h * \boldsymbol{t}$$

where h is an appropriate step length and $\boldsymbol{t}$ is a normalised tangent vector to the homotopy trajectory (see Figure 2). This predictor can be considered as a step of the Euler method (or some numerical integration method) for solving the differential equation that describes the homotopy trajectory (continuation path). Predictor steps are usually based on tangent predictions but there are several alternatives like the secant predictor [6], interpolation predictor [7], Taylor polynomial predictor [7].

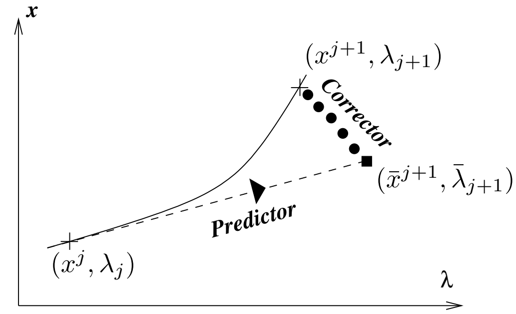

Fig. 2. Predictor-corrector steps

2) Corrector: When the predictor step finishes, it is necessary to fit the homotopy trajectory by using a corrector step. This step solves the homotopy formulation by starting from $(\bar{x}^{j+1}, \bar{\lambda}_{j+1})$ (see Figure 2). The common way to solve this equation is by using the Newton-Gauss method [7], which can be solved for systems of the type $R^{N+1} \Rightarrow R^N$.

3) Step Control: The Predictor-Corrector scheme can be optimized for tracing acceleration by using a step length control. A small constant step length can trace the curve successfully but not efficiently, because this process involves too many steps along "flat" branches. Therefore, it is necessary to adapt the step length to the convergence behavior at each predictor-corrector step. The basic criterion is to control the step by observing the convergence quality of the corrector step. A change on the number of iterations in the corrector step produces a compensation factor ζ, which affects the step length as follows:

$$h_{j+2} = h_{j+1}\zeta$$

4) Find Zero Strategy: Without an efficient finding zero strategy the numerical continuation is incomplete and the homotopy could fail to converge to some solutions. The tracing of the homotopy trajectory begins at $\lambda = 0$ and it ends[1] at $\lambda = 1$. When the tracing is close to $\lambda = 1$, the *find Zero Strategy* starts. The simplest example of strategy is to use $[\boldsymbol{x}_f, \lambda_f]$ (the last iteration) as the initial point to solve the equilibrium equation $\boldsymbol{f}(\boldsymbol{x})$ with a Newton-like method.

[1]In fact if it is wanted to find multiple solutions the numerical continuation should follow beyond $\lambda = 1$.

Because the Newton method possesses local convergence, it still could fail to find the solution. In [8] some techniques are reported that implement the *find Zero Strategy* accurately and reliably. The basic idea is to use two points ($p_1 < 1$ and $p_2 \geq 1$) around to $\lambda = 1$, and interpolate the point at $\lambda = 1$ in order to obtain a point closely to the real solution and use a Newton-like method to find the solution of the original system $\boldsymbol{f}(\boldsymbol{x})$.

5) Stop criterion: In fact, there is not stop criterion in the traditional numerical continuation methods when applied to homotopy trajectory tracing. The most common way to stop tracing the trajectory is to set a maximum allowed number ($ITMAX$) of predictor-corrector steps without finding any solution. This technique is inefficient because it can fail to find some solutions on the homotopy trajectory.

All the blocks of the numerical continuation method are depicted in Figure 3.

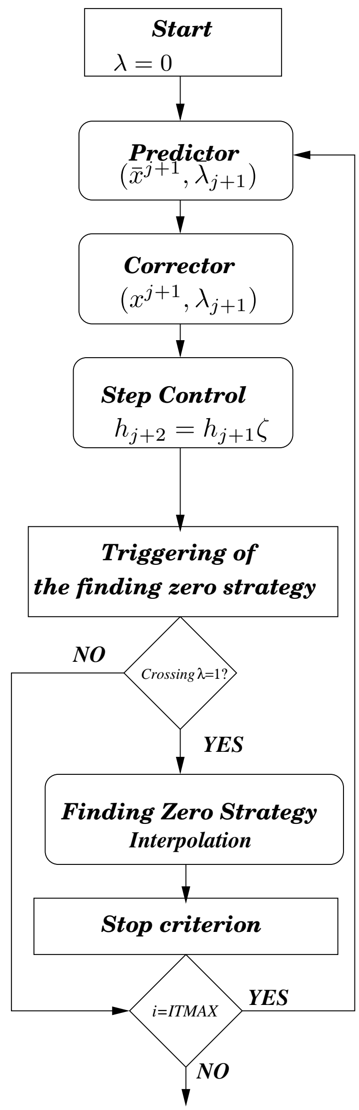

Fig. 3. Numerical continuation scheme

III. Modifiied Numerical continuation

This section explains the modifications accomplished on the scheme above with the idea of providing a reliable stop criterion to the DBH. The modifications act on both the Find Zero Strategy and the stop criterion itself.

1) **Find Zero Strategy**. The double bounded homotopy has the characteristic of never crossing $\lambda = 1$ [4], hence, the findig zero strategy should starts after the trajectory bounces on the bounding line. An efficient way to achieve this process is by monitoring the change of sign of $\Delta\lambda$ produced in the predictor step. This can be done by multiplying $\Delta\lambda$ of two consecutive predictor steps.

$$sign(\triangle\lambda_{j+1}\triangle\lambda_j) \neq -1$$

This procedure is depicted in Figure 4, where the sign of $\Delta\lambda$ changes in the bouncing points A and B. Besides, in order to apply a quadratic interpolation the algorithm needs three points (A, B, C).

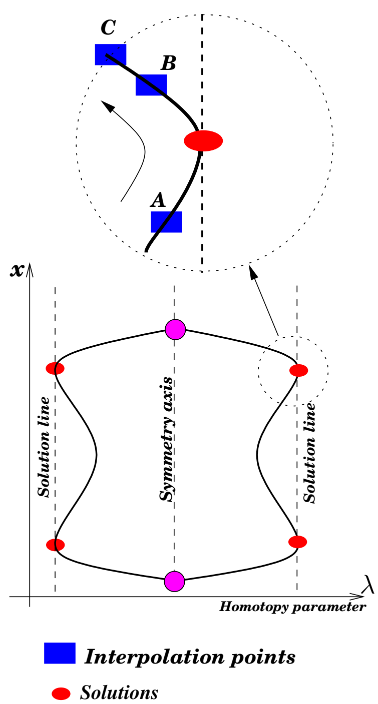

Fig. 4. Find Zero Strategy

2) **Stop criterion**. The stop criterion for this homotopy is depicted in Figure 5. The homotopy trajectory starts at the symmetry axis of the homotopy trajectory. Then it traces the half of the trayectory (the symmetrical branch) and stops when it returns to the symmetry axis. The advantage of the stop criterion is that the numerical continuation needs only to trace the half side of the trayectory.

The modified numerical continuation is depicted in Figure 6 where the dashed blocks are the specific characteristics added to the procedure. The scheme is explained as follows:

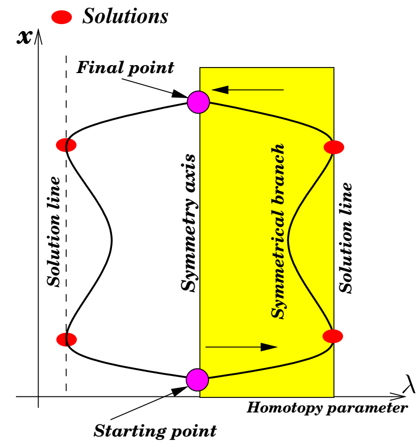

Fig. 5. Stop criterion

It starts when the predictor calculates the tangent to (x^j, λ_j) and using a step length calculates the $(\bar{x}^{j+1}, \bar{\lambda}_{j+1})$ over the tangent. Then, the corrector uses the solution of the predictor in order to obtain a new point on the homotopy trajectory, given by (x^{j+1}, λ_{j+1}). At this point, the step control is applied in order to accelerate the tracing. Next, the so-called end-zero strategy is applied which is triggered at each solution when the trajectory bounces on the solution line. Finally, the numerical continuation stops tracing when the trajectory returns to the symmetry axis.

IV. Examples

In order to illustrate the use of the DBH with the modifications, a first example is used to solve the system of equations given as:

$$f_1(x_1, x_2) = (x_2 - 1)(x_2 - 4)(x_2 - 6) + x_1 = 0$$
$$f_2(x_1, x_2) = (x_1 - 3)(x_1 - 6)(x_1 - 9) + x_2 = 0$$

The graphic solution of the system is shown in Figure 7.

The DBH formulation yields:

$$H_1(f_1, \lambda) = 100Q + e^Q \ln(0.001 f_1^2 + 1) = 0$$
$$H_2(f_2, \lambda) = 100Q + e^Q \ln(0.001 f_2^2 + 1) = 0$$

where $Q = \lambda(\lambda - 1)$; i.e. $a = 0$ and $b = 1$.

The homotopy trajectories are depicted in Figure 8. The starting points lie on the plane defined by $\lambda = 0.5$, while, the solutions are obtained when λ reaches the value of 1.

A second example is given by the latch circuit of Figure 9, which contains two NMOS transistors (M_1 and M_2), two linear resistors (R_1 and R_2) and a voltage source (E). The model of the transistors is the unified MOS model reported in [9] which is a modified version of the well-known BSIM model.

The Figure 10 shows the graph of the equilibrium equation and the homotopy trajectory of the circuit in the space (v_1, v_2, I_E). The homotopy finds all three solutions of the circuit.

A last example is the well-known benchmark circuit reported in [10]. This circuit has 4 bipolar transistors modeled by the half-sided Ebers-Moll model.

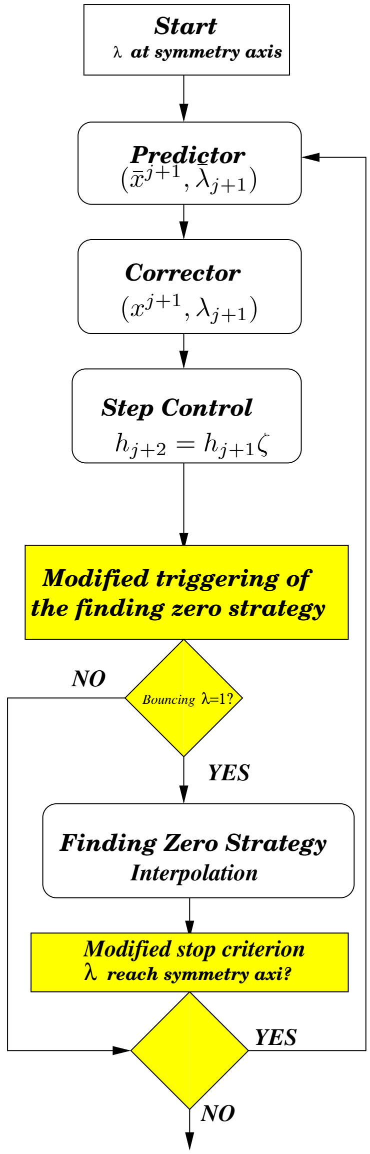

Fig. 6. Block diagram of the modified numerical continuation

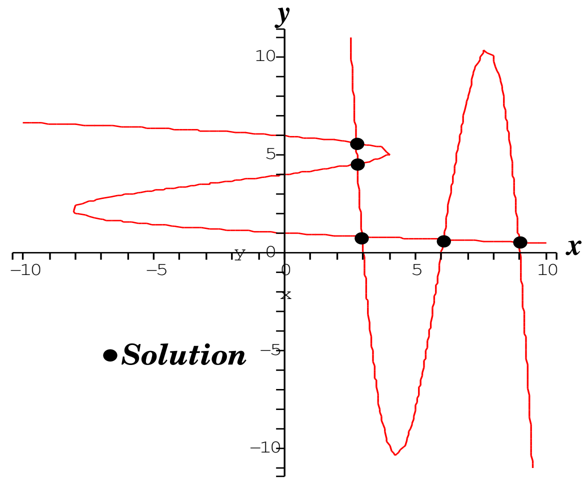

Fig. 7. System of five solutions

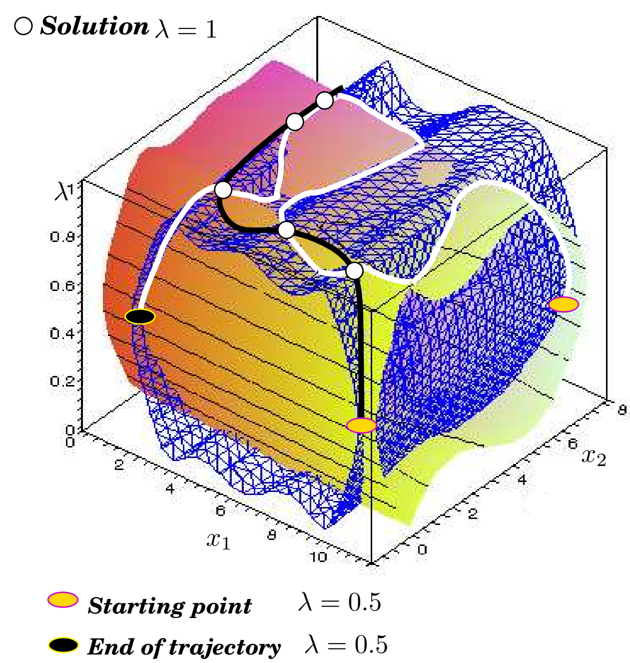

Fig. 8. Homotopy trajectory

The equations system formulation is the same of [10] which is based on the branch voltages v_1, v_2, v_3, v_4. Figure 11 depicts the homotopy trajectory and the six solutions found versus v_1.

V. Conclusions

A numerical continuation for tracing a double bounded homotopy has been presented. The numerical continuation scheme exhibits an improved performance regarding stop criterion and finding zero strategy. Some examples illustrating the aplication of the scheme to nonlinear resistive circuits were also presented.

Acknowledgements

Hector Vazquez Leal is holder of a scholarship from CONACyT México under contract 143907. This work has been partially supported by a CONACyT México research project under grant 42588-Y.

References

[1] R. C. Melville and L. Trajković, “Artificial parameter homotopy methods for the dc operating point problem,” *IEEE transactions on computer-aided design of integrated circuits and systems*, vol. 12, no. 6, pp. 861–877, June 1997.

[2] Denise M. Wolf and Seth R. Sanders, “Multiparameter homotopy methods for finding dc operating points of nonlinear circuits,” *IEEE transactions on circuits and systems-I: fundamental theory and aplications*, vol. 43, no. 10, pp. 824–837, October 1996.

[3] Lieven Vandenberghe and Joos Vandewalle, “Variable dimension algorithms for solving resistive circuits,” *International Journal of Circuit Theory and Applications*, vol. 18, no. 5, pp. 443–474, September 1990.

[4] Hector Vázquez-Leal, Luis Hernández-Martínez, and Arturo Sarmiento-

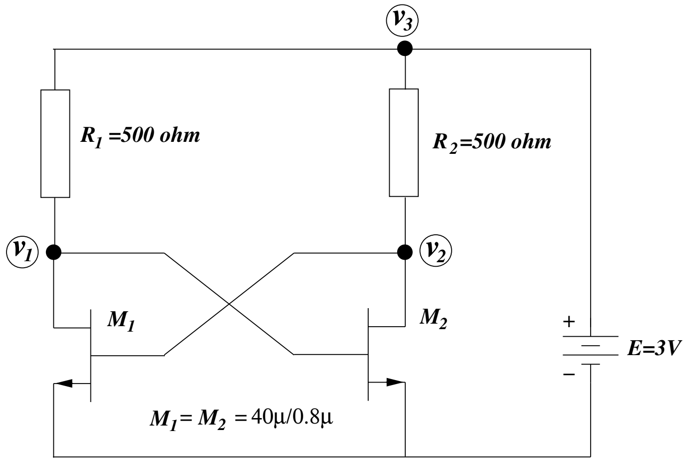

Fig. 9. Example circuit

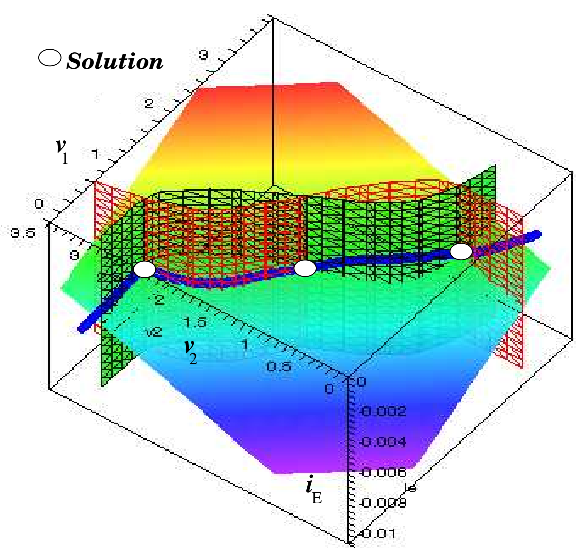

Fig. 10. Graphic of the equilibrium equation of the circuit and homotopy trajectory

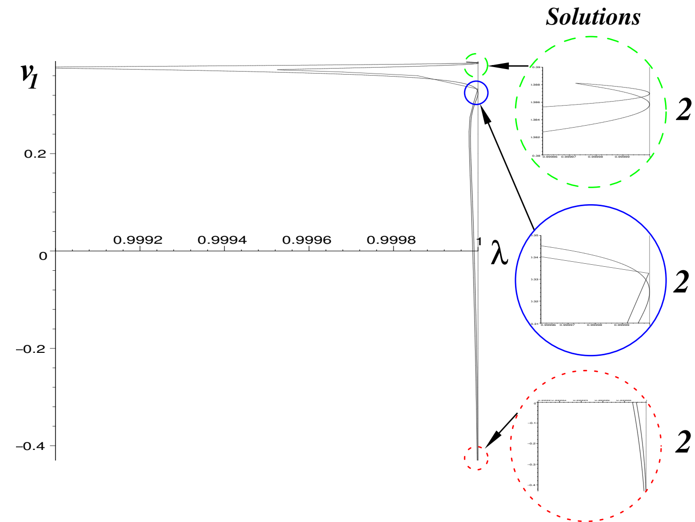

Fig. 11. Solution of the Chua's circuit

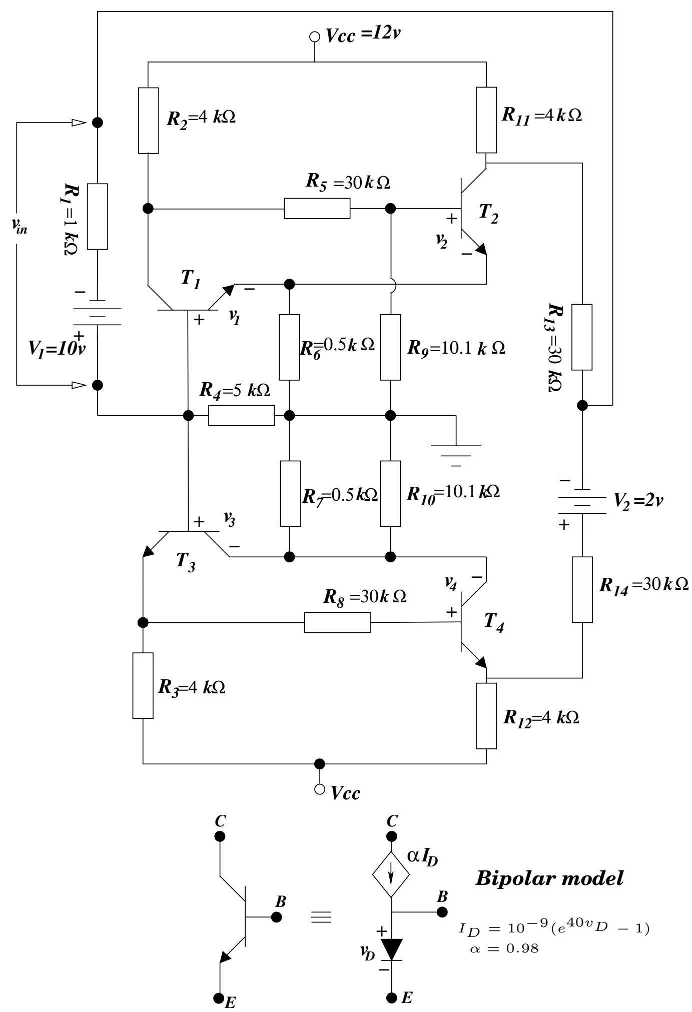

Fig. 12. Chua's circuit

Reyes, "Double-bounded homotopy for analysing nonlinear resistive circuits," *Accepted in ISCAS'05*, May 2005.

[5] Hector Vázquez-Leal, Luis Hernández-Martínez, and Arturo Sarmiento-Reyes, "Numerical path following for double bounded homotopy scheme for analysing nonlinear resistive circuits," *IBERCHIP*, 2003.

[6] R. Seydel, "Tracing branches," *World of Bifurcation*, June 1999.

[7] Eugene L. Allgower and Kurt Georg, *Numerical Path Following*, 1994.

[8] Maria Sosonkina, Layne T. Watson, and David E. Stewart, "Note on the end game in homotopy zero curve tracking," *ACM Transantions On Mathematical Software*, vol. 22, no. 3, pp. 281–287, September 1996.

[9] S.H. Jen, B.J. Sheu, and Y. Oshima, "A unified approach to submicron dc mos transistor modeling for low-voltage ics," *Analog Integrated Circuits and Signal Processing*, vol. 12, pp. 107–118, 1997.

[10] A. Ushida and L. O. Chua, "Tracing solution curves of non-linear equations with sharp turning points," *Circuit Theory and Applications*, vol. 12, pp. 1–21, 1984.