using

$$\frac{dB}{dt} - \alpha B = -r \qquad (1)$$

with initial conditions

$B(0) = B_0$ and $B(T) = 0$

where

> $B(t)$ is the outstanding mortgage balance at time t

> α is the constant interest rate (per unit time)

> r is the repayment rate (per unit time)

Constant interest rate

Solve using integrating factor method $\frac{dB}{dt} + P(t)B = Q(t)$

rewrite as

$$\frac{dB}{dt} + (-\alpha)B = -r$$

let $P(t) = -\alpha$ and $Q(t) = -r$

our integrating factor is

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int -\alpha\,dt} = e^{-\alpha t}$$

next multiply both sides of ODE by $\mu(t) = e^{-\alpha t}$

$$e^{-\alpha t}\frac{dB}{dt} - \alpha e^{-\alpha t}B = -re^{-\alpha t}$$

the L.H.S is derivative of $B(t)e^{-\alpha t}$

$$\frac{d}{dt}\left(B(t)e^{-\alpha t}\right) = -re^{-\alpha t}$$

$$\int \frac{d}{dt}\left(B(t)e^{-\alpha t}\right)dt = \int -re^{-\alpha t}\,dt$$

$$B(t)e^{-\alpha t} = \int -re^{-\alpha t}\,dt$$

Since r is const r.h.s is

$$\int -re^{-\alpha t}\,dt = -r\int e^{-\alpha t}\,dt = -r\left(\frac{e^{-\alpha t}}{-\alpha}\right) = \frac{r}{\alpha}e^{-\alpha t}$$

add constant C

$$B(t)e^{-\alpha t} = \frac{r}{\alpha}e^{-\alpha t} + C$$

multiply both sides by $e^{\alpha t}$

$$B(t) = \frac{r}{\alpha} + Ce^{\alpha t}$$

Apply initial condition $B(0) = B_0$

$$B(0) = \frac{r}{\alpha} + Ce^{0} = r/\alpha + C = B_0$$

$\Rightarrow \quad C = B_0 - r/\alpha$

so solution becomes

$$B(t) = \frac{r}{\alpha} + \left(B_0 - \frac{r}{\alpha}\right)e^{\alpha t} = B_0 e^{\alpha t} - \frac{r}{\alpha}\left(e^{\alpha t} - 1\right)$$

determining repayment rate r using $B(T) = 0$
mortgage is fully repaid at $t = T$

$$B(T) = B_0 e^{\alpha T} - \frac{r}{\alpha}\left(e^{\alpha T} - 1\right) = 0$$

Solve for r:

$$B_0 e^{\alpha T} = r/\alpha \left(e^{\alpha T} - 1\right)$$

$$r = \frac{\alpha B_0 e^{\alpha T}}{e^{\alpha T} - 1}$$

Variable interest rate solution

let $\alpha = \alpha(t)$ and rewrite (1) as

$$\frac{dB}{dt} - \alpha(t) B = -r \qquad (2)$$

1st order linear ODE
its of the form $\dfrac{dB}{dt} + P(t) B = Q(t)$

integrating factor is $\mu(t) = e^{\int P(t)dt}$

rewrite (2) $\dfrac{dB}{dt} + \left(-\alpha(t)\right) B = -r$

so $P(t) = -\alpha t$

$\therefore$

$$\mu(t) = e^{-\int_0^t \alpha(s)ds}$$

multiply by $\mu(t)$

$$e^{-\int_0^t \alpha(s)ds}\frac{dB}{dt} - \alpha(t)e^{-\int_0^t \alpha(s)ds} B = -re^{-\int_0^t \alpha(s)ds}$$

$$\frac{d}{dt}\left(B(t)e^{-\int_0^t \alpha(s)ds}\right) = -re^{-\int_0^t \alpha(s)ds}$$

$$B(t)e^{-\int_0^t \alpha(s)ds} - B(0) = -r\int_0^t e^{-\int_0^s \alpha(u)du}ds$$

using $B(0) = B_0$

$$B(t)e^{-\int_0^t \alpha(s)ds} = B_0 - r_0 \int^t e^{-\int_0^s \alpha(u)du} ds$$

multiply both sides by $e^{\int_0^t \alpha(s)ds}$

$$B(t) = e^{\int_0^t \alpha(s)ds}\left[B_0 - r_0 \int^t e^{-\int_0^s \alpha(u)du} ds\right]$$

NOW $B(T) = 0$

sub $t = T$

$$0 = e^{\int_0^T \alpha(s)ds}\left[B_0 - r_0 \int^T e^{-\int_0^s \alpha(u)du} ds\right]$$

$$B_0 = r_0 \int^T e^{-\int_0^s \alpha(u)du} ds.$$

rearrange $\int_0^T e^{-\int_0^s \alpha(u)du} ds = \frac{B_0}{r}$