Contraction properties and differentiability of p-energy forms with applications to nonlinear potential theory on self-similar sets

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Abstract

We introduce new contraction properties called the *generalized p-contraction property* for p-energy forms as generalizations of many well-known inequalities, such as Clarkson's inequalities, the strong subadditivity and the "Markov property" in the theory of nonlinear Dirichlet forms, and show that any p-energy form satisfying Clarkson's inequalities is Fréchet differentiable. We also verify the generalized p-contraction property for p-energy forms constructed by Kigami [*Mem. Eur. Math. Soc.* **5** (2023)] and by Cao–Gu–Qiu [*Adv. Math.* **405** (2022), no. 108517]. As a general framework of p-energy forms taking into consideration the generalized p-contraction property, we introduce the notion of *p-resistance form* and investigate fundamental properties for p-harmonic functions with respect to p-resistance forms. In particular, some new estimates on scaling factors of p-energy forms are obtained by establishing Hölder regularity estimates for harmonic functions, and the p-walk dimensions of the generalized Sierpiński carpets and D-dimensional level-l Sierpiński gasket are shown to be strictly greater than p.

Keywords: generalized p-contraction property, p-resistance form, p-harmonic function, p-energy measure, self-similar p-energy form

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1 Introduction

In the field of 'analysis on fractals', on a large class of self-similar sets, it is an established result that there exists a nice Dirichlet form, which is an analog of the pair of the Dirichlet 2-energy $\int |\nabla u|^2 \, dx$ and the associated $(1,2)$-Sobolev space $W^{1,2}$ on a differentiable space. A natural L^p-analog of this Dirichlet form, namely a p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ playing the role of $\int |\nabla u|^p \, dx$ and the associated $(1,p)$-Sobolev space, have been constructed in several works [HPS04, Kig23, Shi24, CGQ22, MS23+], most of which are very recent. (In [Kig23, Shi24], we need to assume that p is strictly greater than the Ahlfors regular conformal dimension of the underlying metric space. See [Kig20, p. 6] for the definition of this dimension.) Compared with the case $p = 2$, where the theory of symmetric Dirichlet forms is applicable, very little has been established to deal with the p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ in a general framework. In particular, there are two missing pieces in known results of p-energy forms: first, useful contraction properties of $(\mathcal{E}_p, \mathcal{F}_p)$, and secondly, the (Fréchet) differentiability of $\mathcal{E}_p$. We shall explain in more detail in the next two paragraphs.

The first missing piece is contraction properties of p-energy forms. Any p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ constructed in the previous studies satisfies the following *unit contractivity:*

$$u^+ \wedge 1 \in \mathcal{F}_p \text{ for any } u \in \mathcal{F}_p \text{ and } \mathcal{E}_p(u^+ \wedge 1) \le \mathcal{E}_p(u). \tag{1.1}$$

In the case $p = 2$, by using some helpful expressions of $\mathcal{E}_2$, e.g. [FOT, Lemma 1.3.4 and (3.2.12)], (1.1) can be improved to the following *normal contractivity* (see [MR, Chapter I, Theorem 4.1.2] for example): if $n \in \mathbb{N}$, $T\colon \mathbb{R}^n \to \mathbb{R}$ satisfy $|T(x)| \le \sum_{k=1}^{n} |x_k|$ and $|T(x) - T(y)| \le \sum_{k=1}^{n} |x_k - y_k|$ for any $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, then for any $\boldsymbol{u} = (u_1, \dots, u_n) \in \mathcal{F}_2^n$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}_2 \quad \text{and} \quad \mathcal{E}_2(T(\boldsymbol{u}))^{1/2} \le \sum_{k=1}^{n} \mathcal{E}_2(u_k)^{1/2}. \tag{1.2}$$

It is natural to expect that $(\mathcal{E}_p, \mathcal{F}_p)$ also has a similar property to (1.2) since $\mathcal{E}_p(u)$ is an analog of $\int |\nabla u|^p \, dx$; nevertheless, it is not clear whether (1.1) can be improved in such a way without going back to the constructions of $(\mathcal{E}_p, \mathcal{F}_p)$ in the previous studies. Not only (1.2) but also many useful inequalities, such as the following *strong subadditivity* and *p-Clarkson's inequalities*, were not mentioned in [HPS04, Kig23, Shi24, CGQ22, MS23+]:

(Strong subadditivity) For any $u, v \in \mathcal{F}_p$, we have $u \wedge v, u \vee v \in \mathcal{F}_p$ and

$$\mathcal{E}_p(u \wedge v) + \mathcal{E}_p(u \vee v) \le \mathcal{E}_p(u) + \mathcal{E}_p(v). \tag{1.3}$$

(p-Clarkson's inequality) For any $u, v \in \mathcal{F}_p$,

$$\begin{cases} \mathcal{E}_p(u+v)^{1/(p-1)} + \mathcal{E}_p(u-v)^{1/(p-1)} \le 2\bigl(\mathcal{E}_p(u) + \mathcal{E}_p(v)\bigr)^{1/(p-1)} & \text{if } p \in (1,2], \\ \mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \le 2\bigl(\mathcal{E}_p(u)^{1/(p-1)} + \mathcal{E}_p(v)^{1/(p-1)}\bigr)^{p-1} & \text{if } p \in (2,\infty). \end{cases} \tag{1.4}$$

The other missing piece is the differentiability of $\mathcal{E}_p$, which should be useful to study *p-harmonic functions* with respect to $\mathcal{E}_p$. (See [KM23, Problem 7.7] and [MS23+, Conjecture

10.9] for some motivations to investigate p-harmonic functions on fractals.) In [HPS04, Shi24, CGQ22], p-harmonic functions are defined as functions minimizing $\mathcal{E}_p$ under some fixed boundary conditions. However, it is still unclear how to give an equivalent definition of p-harmonic function in a weak sense due to the lack of 'two-variable version' $\mathcal{E}_p(u;\varphi)$ [Kig23, Problem 2 in Section 6.3]. We shall recall the Euclidean case to explain the importance of this object. Let $D \in \mathbb{N}$ and $U \subseteq \mathbb{R}^D$ a domain. A function $u \in W^{1,p}(\mathbb{R}^D)$ is said to be p-harmonic on U in the weak sense if

$$\int_{\mathbb{R}^D} |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla \varphi(x) \rangle_{\mathbb{R}^D} \, dx = 0 \quad \text{for every } \varphi \in C_c^\infty(U), \tag{1.5}$$

where $\langle \,\cdot\,, \,\cdot\, \rangle_{\mathbb{R}^D}$ is the inner product of $\mathbb{R}^D$. It is well known that (1.5) is equivalent to

$$\int_{\mathbb{R}^D} |\nabla u(x)|^p \, dx = \inf\left\{ \int_{\mathbb{R}^D} |\nabla v(x)|^p \, dx \;\middle|\; v \in W^{1,p}(\mathbb{R}^D), u - v \in W_0^{1,p}(U) \right\}. \tag{1.6}$$

An issue to consider an analog of (1.5) in terms of $\mathcal{E}_p$ is that we do not have a satisfactory counterpart, $\mathcal{E}_p(u;\varphi)$, of $\int |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \, dx$ associated with $\mathcal{E}_p$. As mentioned in [SW04, (2.1)], the ideal definition of $\mathcal{E}_p(u;\varphi)$[1] is

$$\mathcal{E}_p(u;\varphi) \coloneqq \frac{1}{p} \frac{d}{dt} \mathcal{E}_p(u + t\varphi) \Big|_{t=0}, \tag{1.7}$$

but the existence of this derivative is unclear[2] because the constructions of $\mathcal{E}_p$ in [HPS04, Kig23, Shi24, CGQ22, MS23+] include many steps such as the operation of taking a subsequential scaling limit of discrete p-energies.

To overcome this situation, in this paper, we will introduce the following notion of *generalized p-contraction property* as a candidate of the strongest possible form of contraction properties of $(\mathcal{E}_p, \mathcal{F}_p)$.

Definition 1.1 (Generalized p-contraction property; see also Definition 2.1)**.** Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$ and $q_2 \in [p, \infty]$. We say that $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies the *generalized p-contraction property* if $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfies $T(0) = 0$ and $\|T(x) - T(y)\|_{\ell^{q_2}} \le \|x - y\|_{\ell^{q_1}}$ for any $x, y \in \mathbb{R}^{n_1}$, then for any $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}_p^{n_1}$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}_p^{n_2} \quad \text{and} \quad \left\| \left(\mathcal{E}_p(T_l(\boldsymbol{u}))^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\mathcal{E}_p(u_k)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{1.8}$$

Note that the case $(p, n_1, n_2, q_1, q_2) = (2, n, 1, 1, p)$ is the same as (1.2) for symmetric Dirichlet forms and that (1.8) includes (1.3) and (1.4) (see Proposition 2.2). Furthermore, we will see that we can still modify or verify the existing constructions of p-energy forms in the previous studies so as to get ones satisfying (1.8). (See also [KS.a] for an approach,

[1] Strichartz and Wong [SW04] have proposed an approach based on *subderivative* instead of (1.7), i.e., $\mathcal{E}_p(u;\varphi)$ is defined as the interval $\left[\mathcal{E}_p^-(u;\varphi), \mathcal{E}_p^+(u;\varphi)\right]$ where $\frac{d^\pm}{dt}\mathcal{E}_p(u + t\varphi)\Big|_{t=0} =: \mathcal{E}_p^\pm(u;\varphi)$.

[2] The case $p = 2$ is special because of the parallelogram law. Indeed, $\mathcal{E}_2$ is known to be a quadratic form and hence $\mathcal{E}_2(u,v) \coloneqq 4^{-1}(\mathcal{E}_2(u+v) - \mathcal{E}_2(u-v))$ is a symmetric form satisfying (1.7).

which is based on Korevaar–Schoen p-energy forms, to obtain p-energy forms satisfying (1.8).) We also make a simple observation that (1.4) is enough for the differentiability in (1.7) (see Theorem 3.6); accordingly, it suffices to take account of (1.8) for our purpose.

As a general theory of p-energy forms taking into consideration (1.8), we will introduce *p-resistance forms* (Definition 6.1), which can be regarded as a natural L^p-analog of the theory of resistance forms developed by Kigami [Kig01, Kig12]. By using (1.8) and (1.7), we investigate p-harmonic functions with respect to p-resistance forms, which should correspond to a part of *nonlinear potential theory* where each point has a positive p-capacity. As remarkable points, we prove fundamental results on the operation of traces (Subsection 6.2) and give sharp Hölder regularity estimates of p-harmonic functions (Theorem 6.27). We also show that $R_{\mathcal{E}_p}(\,\cdot\,,\,\cdot\,)^{1/(p-1)}$, which we call the *$p$-resistance metric*, defines a metric on the underlying space (Corollary 6.28), where

$$R_{\mathcal{E}_p}(x,y) \coloneqq \sup\left\{ \frac{|u(x)-u(y)|^p}{\mathcal{E}_p(u)} \;\middle|\; u \in \mathcal{F}_p, \mathcal{E}_p(u) > 0 \right\}.$$

This metric in the case $p = 2$ is well known and is called the *resistance metric* (see [Kig01, Theorem 2.3.4] for example). The proof of the triangle inequality for the p-resistance metric is done independently by [Her10, ACFP19] for finite weighted graphs (see also [Shi21] for infinite graphs). Our result is the first result including continuous settings.

Let us move to applications of such a general theory of p-resistance forms. In forthcoming papers [KS.b, KS.c], the authors will heavily use this theory to make some essential progress in the setting of post-critically finite self-similar structures. See [KS23+] for a survey of these results in the case of the Sierpiński gasket. Here we shall discuss another application for strict estimates on *p-walk dimensions* of some special classes of fractals, namely generalized Sierpiński carpets and D-dimensional level-l Sierpiński gasket (see Figure 1.1). For such a nice self-similar set K, as shown in the previous studies, we can construct $\mathcal{E}_p$ so as to satisfy the following *self-similarity*: there exists $\sigma_p \in (0,\infty)$ (which we call the *weight* of $(\mathcal{E}_p, \mathcal{F}_p)$) such that

$$\mathcal{E}_p(u) = \sigma_p \sum_{i \in S} \mathcal{E}_p(u \circ F_i), \quad u \in \mathcal{F}_p, \tag{1.9}$$

where S is a finite set and $\{F_i\}_{i \in S}$ is a family of similitudes associated with K such that $K = \bigcup_{i \in S} F_i(K)$ and $|F_i(x) - F_i(y)| = r_* \,|x-y|$ for some $r_* \in (0,1)$. Then the p-walk dimension $d_{\mathrm{w},p}$ of K is defined by

$$d_{\mathrm{w},p} \coloneqq \frac{\log\bigl((\#S)\sigma_p\bigr)}{\log r_*^{-1}},$$

which coincides with the walk-dimension if $p = 2$. As shown in [MS23+, Theorem 7.1], the value $d_{\mathrm{w},p}$ plays a role of the space-scaling exponent in the following sense:

$$\mathcal{E}_p(u) \asymp \limsup_{r \downarrow 0} \int_K \fint_{|x-y|<r} \frac{|u(x)-u(y)|^p}{r^{d_{\mathrm{w},p}}}\, m(dy)m(dx), \quad u \in \mathcal{F}_p,$$

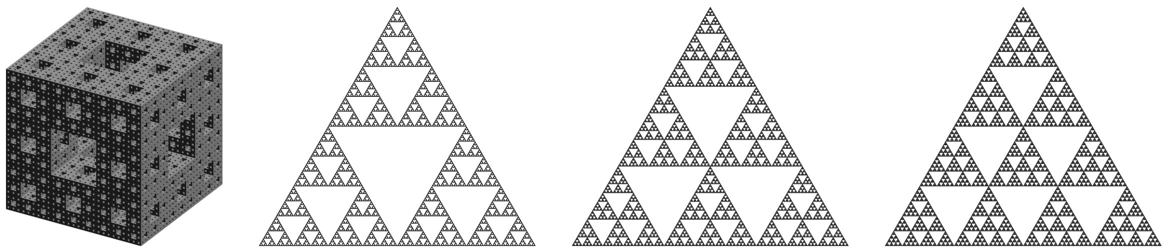

Figure 1.1: From the left, a non-planar generalized Sierpiński carpet (Menger Sponge) and 2-dimensional level-l Sierpiński gaskets ($l = 2, 3, 4$)

where m is the $\log(\#S)/\log r_*^{-1}$-dimensional Hausdorff measure on (K, d) with $m(K) = 1$. In the case $p = 2$, the strict inequality $d_{\mathrm{w},2} > 2$ has been verified for many self-similar sets, which implies a number of anomalous features of the diffusion associated with $(\mathcal{E}_2, \mathcal{F}_2)$. See [Kaj23] and the references therein for further details. Compared with the case $p = 2$, a class of self-similar sets where $d_{\mathrm{w},p} > p$ is shown is limited to the *planar* generalized Sierpiński carpets due to the lack of counterparts of many useful tools in the case $p = 2$ (see [Shi24, Theorem 2.27]). As an application of the differentiability in (1.7), we will extend this result to *any* generalized Sierpiński carpet by following the argument in [Kaj23]. We also prove $d_{\mathrm{w},p} > p$ for any D-dimensional level-l Sierpiński gasket, where the argument in [Kaj23] does not work.

We would also like to mention a geometric role of σ_p appearing in (1.9). As done in [Kig20, Kig23], the constant σ_p is determined by seeking the behavior of *conductance constants* (see [Kig23, Definition 2.17]) on approximating graphs of K. (See Theorem 8.12 for details.) A remarkable fact is that the behavior of σ_p as a function in p is deeply related to the notion of *Ahlfors regular conformal dimension*; indeed, $\sigma_p > 1$ if and only if $p > \dim_{\mathrm{ARC}}(K)$ (see, e.g., [Kig20, Theorem 4.7.6]), where $\dim_{\mathrm{ARC}}(K)$ denotes the Ahlfors regular conformal dimension of K (see Definition 8.5-(4) for the definition of $\dim_{\mathrm{ARC}}(K)$). Therefore, knowing properties of the function $p \mapsto \sigma_p$ is very important to understand the Ahlfors regular conformal dimension and related geometric information. Nevertheless, we do not know anything other than the following:

(Continuity; [Kig20, Proposition 4.7.5]) σ_p is continuous in p.

(Simple monotonicity; [Kig20, Proposition 4.7.5]) σ_p is non-decreasing in p.

(Hölder-type monotonicity; [Kig20, Lemma 4.7.4]) $d_{\mathrm{w},p}/p$ is non-increasing in p.

(Relation with $\dim_{\mathrm{ARC}}$; [Kig20, Theorem 4.7.6]) $\sigma_p > 1$ if and only if $p > \dim_{\mathrm{ARC}}(K)$.

As another application of our theory of p-resistance forms, we present the following new monotonicity behavior on σ_p (in a suitable general setting including all of the self-similar sets in Figure 1.1):

$$(\dim_{\mathrm{ARC}}(K), \infty) \ni p \mapsto \sigma_p^{1/(p-1)} \in (0, \infty) \text{ is non-decreasing in } p, \tag{1.10}$$

which is good evidence that properties of $p \mapsto \sigma_p^{1/(p-1)}$ are also important to deepen our understanding of $(\mathcal{E}_p, \mathcal{F}_p)$ and, possibly, of $\dim_{\mathrm{ARC}}(K)$.

Let us conclude the introduction by clarifying a difference between our theory and some related literatures [BBR23+, Kuw23+], where p-energy forms based on a (strongly local regular) symmetric Dirichlet form are considered. In the settings of [BBR23+, Kuw23+], *p-energy measures* $\{\mu^p_{\langle u\rangle}\}_{u\in\mathcal{F}_p}$, which can be regarded as localized versions of p-energy functionals playing the role of $|\nabla u|^p\,dx$ in the Euclidean case, can be explicitly defined by using the "density" corresponding to $|\nabla u|$ without depending on p (see Example 4.2-(3)) whereas it is almost impossible to find a priori such a density on fractals. (We can naturally define p-energy measures by using (1.9). See Section 5 for details.) In [KS.c], the authors will show that $\mu^p_{\langle u_p\rangle}$ and $\mu^q_{\langle u_q\rangle}$ are mutually singular with respect to each other for any $p, q \in (1,\infty)$ with $p \neq q$ and any $(u_p, u_q) \in \mathcal{F}_p \times \mathcal{F}_q$ for some post-critically finite self-similar sets by establishing the strict version of (1.10). This phenomenon on the singularity of energy measures *never* happens in the settings of [BBR23+, Kuw23+]. This point also motivates us to develop a general theory of p-energy forms in an abstract setting in order to deal with fractals.

This paper is organized as follows. In Section 2, we introduce the generalized p-contraction property and prove basic results on these properties. Section 3 will deal with the differentiability (1.7) and its consequences for p-energy forms satisfying p-Clarkson's inequalities. Similar results for p-energy measures are shown in Section 4 under the existence of p-energy measures. In Section 5, we give standard notations on self-similar structures and discuss the self-similarity of p-energy forms and p-energy measures. Section 6 is devoted to the study of fundamental nonlinear potential theory for p-resistance forms. In particular, we prove fundamental results on the operation of taking traces of p-resistance forms, weak comparison principles and sharp Hölder regularity estimate for harmonic functions, and the triangle inequality for the p-resistance metric. We further investigate the theory of p-resistance forms in the self-similar case in Section 7. In Section 8, the generalized p-contraction property is verified for p-energy/p-resistance forms constructed in [Kig23, CGQ22]. In Section 9, we prove $d_{\mathrm{w},p} > p$ for generalized Sierpiński carpets and D-dimensional level-l Sierpiński gasket by using properties of harmonic functions developed in Section 6.

Notation. Throughout this paper, we use the following notation and conventions.

(1) For $[0,\infty]$-valued quantities A and B, we write $A \lesssim B$ to mean that there exists an implicit constant $C \in (0,\infty)$ depending on some unimportant parameters such that $A \leq CB$. We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$.
(2) For a set A, we let $\#A \in \mathbb{N}\cup\{0,\infty\}$ denote the cardinality of A.
(3) We set $\sup\emptyset := 0$ and $\inf\emptyset := \infty$. We write $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$ and $a^+ := a \vee 0$ for $a, b \in [-\infty,\infty]$, and we use the same notation also for $[-\infty,\infty]$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty,\infty]$-valued.
(4) Let $n \in \mathbb{N}$. For $x = (x_k)_{k=1}^n \in \mathbb{R}^n$, we set $\|x\|_{\ell^p} := (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $p \in (0,\infty)$, $\|x\|_{\ell^\infty} := \max_{1\leq k\leq n}|x_k|\,t$ and $\|x\| := \|x\|_{\ell^2}$. For $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ which is differentiable on $\mathbb{R}^n$ and for $k \in \{1,\dots,n\}$, its first-order partial derivative in the k-th coordinate is denoted by $\partial_k\Phi$.
(5) Let X be a non-empty set. We define $\mathrm{id}_X\colon X \to X$ by $\mathrm{id}_X(x) := x$, $\mathbb{1}_A = \mathbb{1}_A^X \in \mathbb{R}^X$

for $A \subseteq X$ by $\mathbb{1}_A(x) \coloneqq \mathbb{1}_A^X(x) \coloneqq \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$ and set $\|u\|_{\sup} \coloneqq \|u\|_{\sup,X} \coloneqq \sup_{x\in X}|u(x)|$ for $u\colon X \to [-\infty,\infty]$. Also, set $\operatorname{osc}_X[u] \coloneqq \sup_{x,y\in X}|u(x)-u(y)|$ for $u\colon X \to \mathbb{R}$ with $\|u\|_{\sup} < \infty$.

(6) Let X be a topological space. The Borel σ-algebra of X is denoted by $\mathcal{B}(X)$, the closure of $A \subseteq X$ in X by $\overline{A}^X$, and we say that $A \subseteq X$ is *relatively compact* in X if and only if $\overline{A}^X$ is compact. We set $C(X) \coloneqq \{u \in \mathbb{R}^X \mid u \text{ is continuous}\}$, $\operatorname{supp}_X[u] \coloneqq \overline{X \setminus u^{-1}(0)}^X$ for $u \in C(X)$, $C_b(X) \coloneqq \{u \in C(X) \mid \|u\|_{\sup} < \infty\}$, and $C_c(X) \coloneqq \{u \in C(X) \mid \operatorname{supp}_X[u] \text{ is compact}\}$.

(7) Let X be a topological space having a countable open base. For a Borel measure m on X and a Borel measurable function $f\colon X \to [-\infty,\infty]$ or an m-equivalence class f of such functions, we let $\operatorname{supp}_m[f]$ denote the support of the measure $|f|\,dm$, that is, the smallest closed subset F of X such that $\int_{X\setminus F}|f|\,dm = 0$.

(8) Let (X,d) be a metric space. We set $B_d(x,r) \coloneqq \{y \in X \mid d(x,y) < r\}$ for $(x,r) \in X \times (0,\infty)$, $(A)_{d,r} \coloneqq \bigcup_{x\in A} B_d(x,r)$ and $\operatorname{dist}_d(A,B) \coloneqq \inf\{d(x,y) \mid x \in A, y \in B\}$ for subsets A, B of X.

(9) Let $(X,\mathcal{B},m)$ be a measure space. For $f \in L^1(X,m)$ and $A \in \mathcal{B}$ with $m(A) \in (0,\infty)$, define $⨍_A f\,dm \coloneqq \frac{1}{m(A)}\int_A f\,dm$.

2 The generalized p-contraction property

In this section, we will introduce the generalized p-contraction property and establish basic results on these properties. Throughout this section, we fix $p \in (1,\infty)$, a measure space $(X,\mathcal{B},m)$, a linear subspace $\mathcal{F}$ of $L^0(X,m) \coloneqq L^0(X,\mathcal{B},m)$, where

$$L^0(X,\mathcal{B},m) \coloneqq \{\text{the } m\text{-equivalence class of } f \mid f\colon X \to \mathbb{R},\ f \text{ is } \mathcal{B}\text{-measurable}\},$$

and a p-homogeneous map $\mathcal{E}\colon \mathcal{F} \to [0,\infty)$, i.e., $\mathcal{E}(au) = |a|^p\,\mathcal{E}(u)$ for any $(a,u) \in \mathbb{R}\times\mathcal{F}$. (The pair $(\mathcal{B},m)$ is arbitrary. In the case where $\mathcal{B} = 2^X$ and m is the counting measure on X, we have $L^0(X,\mathcal{B},m) = \mathbb{R}^X$.)

Definition 2.1 (Generalized p-contraction property)**.** The pair $(\mathcal{E},\mathcal{F})$ is said to satisfy the *generalized p-contraction property*, $(\mathrm{GC})_p$ for short, if and only if the following hold: if $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy

$$T(0) = 0 \quad \text{and} \quad \|T(x)-T(y)\|_{\ell^{q_2}} \le \|x-y\|_{\ell^{q_1}} \quad \text{for any } x,y \in \mathbb{R}^{n_1}, \tag{2.1}$$

then for any $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}^{n_1}$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}^{n_2} \quad \text{and} \quad \left\|\left(\mathcal{E}(T_l(\boldsymbol{u}))^{1/p}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left\|\left(\mathcal{E}(u_k)^{1/p}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}. \tag{GC$_p$}$$

The next proposition is a collection of useful inequalities included in $(\mathrm{GC})_p$.

Proposition 2.2. *Let $\varphi \in C(\mathbb{R})$ satisfy $\varphi(0) = 0$*[3] *and $|\varphi(t) - \varphi(s)| \leq |t - s|$ for any $s, t \in \mathbb{R}$. Suppose that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_p$.*

(a) *$T(x, y) \coloneqq x + y$, $x, y \in \mathbb{R}$, satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (1, p, 2, 1)$. In particular, $\mathcal{E}^{1/p}$ is a seminorm on $\mathcal{F}$, and $\mathcal{E}$ is strictly convex on $\mathcal{F}/\mathcal{E}^{-1}(0)$, i.e., for any $\lambda \in (0, 1)$ and any $f, g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) > 0$,*

$$\mathcal{E}(\lambda f + (1 - \lambda)g) < \lambda\mathcal{E}(f) + (1 - \lambda)\mathcal{E}(g). \tag{2.2}$$

(b) *$T \coloneqq \varphi$ satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (1, p, 1, 1)$. In particular,*

$$\varphi(u) \in \mathcal{F} \text{ and } \mathcal{E}(\varphi(u)) \leq \mathcal{E}(u) \text{ for any } u \in \mathcal{F}. \tag{2.3}$$

(c) *Assume that φ is non-decreasing. Define $T = (T_1, T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1, x_2) = x_1 - \varphi(x_1 - x_2) \quad \text{and} \quad T_2(x_1, x_2) = x_2 + \varphi(x_1 - x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (p, p, 2, 2)$. In particular,

$$\mathcal{E}\bigl(f - \varphi(f - g)\bigr) + \mathcal{E}\bigl(g + \varphi(f - g)\bigr) \leq \mathcal{E}(f) + \mathcal{E}(g) \quad \text{for any } f, g \in \mathcal{F}. \tag{2.4}$$

Moreover, by considering the case $\varphi(x) = x \vee 0$, we have the following strong subadditivity*: $f \vee g, f \wedge g \in \mathcal{F}$ and*

$$\mathcal{E}(f \vee g) + \mathcal{E}(f \wedge g) \leq \mathcal{E}(f) + \mathcal{E}(g). \tag{2.5}$$

(d) *For any $a_1, a_2 > 0$, define $T^{a_1,a_2}\colon \mathbb{R}^2 \to \mathbb{R}$ by*

$$T^{a_1,a_2}(x_1, x_2) \coloneqq \bigl(\bigl[(-a_1) \vee a_2^{-1}x_1\bigr] \wedge a_1\bigr) \cdot \bigl(\bigl[(-a_2) \vee a_1^{-1}x_2\bigr] \wedge a_2\bigr), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then T^{a_1,a_2} satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (1, p, 2, 1)$. In particular, for any $f, g \in \mathcal{F} \cap L^\infty(X, m)$ we have

$$f \cdot g \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(f \cdot g)^{1/p} \leq \|g\|_{L^\infty(X,m)}\,\mathcal{E}(f)^{1/p} + \|f\|_{L^\infty(X,m)}\,\mathcal{E}(g)^{1/p}. \tag{2.6}$$

(e) *Assume that $p \in (1, 2]$. Define $T = (T_1, T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1, x_2) = 2^{-(p-1)/p}(x_1 + x_2) \quad \text{and} \quad T_2(x_1, x_2) = 2^{-(p-1)/p}(x_1 - x_2), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (p/(p - 1), p, 2, 2)$. In particular, $(\mathcal{E}, \mathcal{F})$ satisfies the following p-Clarkson's inequalities:

$$\mathcal{E}(f + g) + \mathcal{E}(f - g) \geq 2\bigl(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\bigr)^{p-1}, \tag{2.7}$$
$$\mathcal{E}(f + g) + \mathcal{E}(f - g) \leq 2\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr). \tag{2.8}$$

[3] Note that $\varphi \circ f \in L^p(X, m)$ for any $f \in L^p(X, m)$ by this condition.

(f) *Assume that $p \in [2,\infty)$. Define $T = (T_1, T_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_1(x_1,x_2) = 2^{-1/p}(x_1 + x_2) \quad \text{and} \quad T_2(x_1,x_2) = 2^{-1/p}(x_1 - x_2), \quad (x_1,x_2) \in \mathbb{R}^2.$$

Then T satisfies (2.1) with $(q_1,q_2,n_1,n_2) = (p, p/(p-1), 2, 2)$. In particular, $(\mathcal{E},\mathcal{F})$ satisfies the following p-Clarkson's inequalities:

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) \le 2\bigl(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\bigr)^{p-1}, \tag{2.9}$$

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) \ge 2\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr). \tag{2.10}$$

Remark 2.3. (1) The property (2.4) is inspired by the *nonlinear Dirichlet form theory* due to Cipriani and Grillo [CG03]. See [Cla23, Theorem 4.7] and the reference therein for further background.

(2) By using an elementary inequality $2^{q-1}(a^q + b^q) \le (a+b)^q$ for $q \in (0,1]$ and $a,b \in [0,\infty)$, we easily see that the inequality (2.7) for $(\mathcal{E},\mathcal{F})$ in the case $p \in (1,2]$ implies (2.8). Similarly, by Hölder's inequality, the inequality (2.9) for $(\mathcal{E},\mathcal{F})$ in the case $p \in [2,\infty)$ implies (2.10).

Proof. (a): It is obvious that $T(x,y) \coloneqq x+y$ satisfies (2.1) with $(q_1,q_2,n_1,n_2) = (1,p,2,1)$ and hence the triangle inequality for $\mathcal{E}^{1/p}$ holds. Since $\mathbb{R} \ni x \mapsto |x|^p$ is strictly convex, for any $\lambda \in (0,1)$ and any $f,g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) > 0$,

$$\mathcal{E}(\lambda f + (1-\lambda)g) \le \bigl(\lambda \mathcal{E}(f)^{1/p} + (1-\lambda)\mathcal{E}(g)^{1/p}\bigr)^p < \lambda\mathcal{E}(f) + (1-\lambda)\mathcal{E}(g),$$

where we used the triangle inequality for $\mathcal{E}^{1/p}$ in the first inequality.

(b): This is obvious.

(c): Let $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$. For simplicity, set $z_i \coloneqq x_i - y_i$ and $A \coloneqq \varphi(x_1 - x_2) - \varphi(y_1 - y_2)$. Then $\|T(x) - T(y)\|_{\ell^p} \le \|x - y\|_{\ell^p}$ ie equivalent to

$$|z_1 - A|^p + |z_2 + A|^p \le |z_1|^p + |z_2|^p, \tag{2.11}$$

so we will show (2.11). Note that $|A| \le |z_1 - z_2|$ since φ is 1-Lipschitz. The desired estimate (2.11) is evident when $z_1 = z_2$, so we consider the case $z_1 \neq z_2$. Suppose that $z_1 > z_2$ because the remaining case $z_1 < z_2$ is similar. Then $(x_1 - x_2) - (y_1 - y_2) = z_1 - z_2 > 0$ and thus $0 \le A \le z_1 - z_2$. Set $\psi_p(t) \coloneqq |t|^p$ ($t \in \mathbb{R}$) for brevity. If $0 \le A < \frac{z_1 - z_2}{2}$, then $z_2 \le z_2 + A < z_1 - A \le z_1$ and we see that

$$\begin{aligned} |z_1 - A|^p + |z_2 + A|^p - |z_1|^p - |z_2|^p &= \int_{z_2}^{z_2+A} \psi_p'(t)\,dt - \int_{z_1-A}^{z_1} \psi_p'(t)\,dt \\ &\le A\bigl(\psi_p'(z_2 + A) - \psi_p'(z_1 - A)\bigr) \le 0. \end{aligned}$$

If $A \ge \frac{z_1 - z_2}{2}$, then $z_2 \le z_1 - A \le z_2 + A \le z_1$ and thus

$$|z_1 - A|^p + |z_2 + A|^p - |z_1|^p - |z_2|^p = \int_{z_2}^{z_1-A} \psi_p'(t)\,dt - \int_{z_2+A}^{z_1} \psi_p'(t)\,dt$$

$$\leq (z_1 - z_2 - A)\bigl(\psi_p'(z_1 - A) - \psi_p'(z_2 + A)\bigr) \leq 0,$$

which proves (2.11). The case $\varphi(x) = x^+$ immediately implies (2.5).

(d): For any $a_1, a_2 > 0$ and $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, we see that

$$\begin{aligned}
&|T^{a_1,a_2}(x_1, x_2) - T^{a_1,a_2}(x_1, x_2)| \\
&\leq \left|(-a_1) \vee a_2^{-1}x_1 \wedge a_1\right| \left|\bigl((-a_2) \vee a_1^{-1}x_2 \wedge a_2\bigr) - \bigl((-a_2) \vee a_1^{-1}y_2 \wedge a_2\bigr)\right| \\
&\quad + \left|(-a_2) \vee a_1^{-1}y_2 \wedge a_2\right| \left|\bigl((-a_1) \vee a_2^{-1}x_1 \wedge a_1\bigr) - \bigl((-a_1) \vee a_2^{-1}y_1 \wedge a_1\bigr)\right| \\
&\leq a_1 \left|a_1^{-1}x_2 - a_1^{-1}y_2\right| + a_2 \left|a_2^{-1}x_1 - a_2^{-1}y_1\right| = |x_1 - y_1| + |x_2 - y_2|,
\end{aligned}$$

whence T^{a_1,a_2} satisfies (2.1). We get (2.6) by applying $(\mathrm{GC})_p$ with $u_1 = \|g\|_{L^\infty(X,m)} f$, $u_2 = \|f\|_{L^\infty(X,m)} g$, $a_1 = \|f\|_{L^\infty(X,m)}$, $a_2 = \|g\|_{L^\infty(X,m)}$.

(e),(f): These statements follow from p-Clarkson's inequalities for the ℓ^p-norm (see, e.g., [Cla36, Theorem 2]). □

The following corollary is easily implied by Proposition 2.2-(b),(d).

Corollary 2.4. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_p$.*

(a) *Let $u \in \mathcal{F} \cap L^\infty(X, m)$ and let $\Phi \in C^1(\mathbb{R})$ satisfy $\Phi(0) = 0$. Then*

$$\Phi(u) \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(\Phi(u)) \leq \sup\bigl\{|\Phi'(t)|^p \bigm| t \in \mathbb{R}, |t| \leq \|u\|_{L^\infty(X,m)}\bigr\}\mathcal{E}(u). \tag{2.12}$$

(b) *Let $\delta, M \in (0, \infty)$ and let $f, g \in \mathcal{F}$ satisfy $f \geq 0$, $g \geq 0$, $f + g \geq \delta$ and $f \leq M$. Then there exists $C \in (0, \infty)$ depending only on p, δ, M such that*

$$\frac{f}{f+g} \in \mathcal{F} \quad \text{and} \quad \mathcal{E}\left(\frac{f}{f+g}\right) \leq C\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr). \tag{2.13}$$

(c) *Let $n \in \mathbb{N}$, $v \in \mathcal{F}$ and $\boldsymbol{u} = (u_1, \dots, u_n) \in L^0(X, m)^n$. If there exist $q \in [1, p]$ and m-versions of $v, \boldsymbol{u}$ such that $|v(x)| \leq \|\boldsymbol{u}(x)\|_{\ell^q}$ and $|v(x) - v(y)| \leq \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^q}$ for any $x, y \in X$, then $\boldsymbol{u} \in \mathcal{F}^n$ and $\mathcal{E}(v) \leq \left\|\bigl(\mathcal{E}(v_k)^{1/p}\bigr)_{k=1}^n\right\|_{\ell^q}$.*

Proof. The statement (a) is immediate from Proposition 2.2-(b), and (b) can be shown by following [MS23+, Proposition 6.25 (ii)].

Let us show (c). The proof below is similar to [MR, Corollary I.4.13]. Fix m-versions of $v, \boldsymbol{u}$ satisfying $|v(x)| \leq \|\boldsymbol{u}(x)\|_{\ell^q}$ and $|v(x) - v(y)| \leq \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^q}$ for any $x, y \in X$. Set $\boldsymbol{u}(X) \coloneqq u_1(X) \times \cdots \times u_n(X) \subseteq \mathbb{R}^n$. We define $T_0 \colon \boldsymbol{u}(X) \to \mathbb{R}$ by setting $T_0(0) \coloneqq 0$ and $T_0(\boldsymbol{z}) \coloneqq v(x)$ for each $\boldsymbol{z} \in \boldsymbol{u}(X)$, where $x \in X$ satisfies $\boldsymbol{z} = \boldsymbol{u}(x)$. This map T_0 is well-defined since $v(x) = 0$ for any $x \in X$ with $\boldsymbol{u}(x) = 0$ and $|v(x) - v(y)| \leq \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^q} = 0$ for any $x, y \in X$ with $\boldsymbol{u}(x) = \boldsymbol{u}(y) \in \boldsymbol{u}(X)$. In addition, we easily see that $|T_0(\boldsymbol{z}_1) - T_0(\boldsymbol{z}_2)| \leq \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{\ell^q}$ for any $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \boldsymbol{u}(X) \cup \{0\}$, i.e., $T_0 \colon (\boldsymbol{u}(X) \cup \{0\}, \|\cdot\|_{\ell^q}) \to \mathbb{R}$ is 1-Lipschitz. Noting that $(\mathbb{R}^n, \|\cdot\|_{\ell^q})$ is a metric space since $q \geq 1$, we can get a 1-Lipschitz map $T \colon (\mathbb{R}^n, \|\cdot\|_{\ell^q}) \to \mathbb{R}$ satisfying $T(\boldsymbol{z}) = T_0(\boldsymbol{z})$ for any $\boldsymbol{z} \in \boldsymbol{u}(X) \cup \{0\}$ by applying the McShane–Whitney extension lemma (see, e.g., [HKST, p. 99]). Since T satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (q, p, n, 1)$ and $\mathcal{E}(T(\boldsymbol{u})) = \mathcal{E}(v)$, we obtain the desired statement by $(\mathrm{GC})_p$. □

We also notice that (GC)$_p$ includes a new variant of p-Clarkson's inequality in the case $p \in [2, \infty)$, which we call improved p-Clarkson's inequality. This result is not used in the paper, but we record it for potential future applications.

Proposition 2.5 (Improved p-Clarkson's inequality)**.** *Assume that $p \in [2, \infty)$. Define $\phi_p \colon (0, \infty) \to (0, \infty)$ and $T^s = (T_1^s, T_2^s) \colon \mathbb{R}^2 \to \mathbb{R}^2$, $s \in (0, \infty)$, by*

$$\psi_p(s) \coloneqq (1+s)^{p-1} + \operatorname{sgn}(1-s) \left|1-s\right|^{p-1}, \quad s > 0. \tag{2.14}$$

and, for $(x_1, x_2) \in \mathbb{R}^2$,

$$T_1^s(x_1, x_2) \coloneqq \psi_p(s)^{-1/p} x_1 + \psi_p(s^{-1})^{-1/p} x_2, \quad T_2^s(x_1, x_2) \coloneqq \psi_p(s)^{-1/p} x_1 - \psi_p(s^{-1})^{-1/p} x_2.$$

(a) *For any $s > 0$, the map T^s satisfies (2.1) with $(q_1, q_2, n_1, n_2) = (p, p, 2, 2)$. If $(\mathcal{E}, \mathcal{F})$ satisfies (GC)$_p$, then*

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) \le \inf_{s>0} \bigl\{ \psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g) \bigr\} \quad \textit{for any } f, g \in \mathcal{F}. \tag{2.15}$$

(b) *If $\mathcal{E}$ satisfies (2.15), then p-Clarkson's inequality, (2.9), for $\mathcal{E}$ holds.*

Proof. (a): By [BCL94, Lemma 4],

$$|x+y|^p + |x-y|^p = \inf_{s>0} \bigl\{ \psi_p(s) \left|x\right|^p + \psi_p(s^{-1}) \left|y\right|^p \bigr\} \quad \text{for any } x, y \in \mathbb{R},$$

which immediately implies that T^s, $s \in (0, \infty)$, satisfies (2.1).

(b): Let $f, g \in \mathcal{F}$ with $\mathcal{E}(f) \wedge \mathcal{E}(g) > 0$, set $a \coloneqq \mathcal{E}(f)^{1/(p-1)}$ and $b \coloneqq \mathcal{E}(g)^{1/(p-1)}$. Then,

$$\inf_{s>0} \bigl\{ \psi_p(s)\mathcal{E}(f) + \psi_p(s^{-1})\mathcal{E}(g) \bigr\} \le \psi_p(b/a)a^{p-1} + \psi_p(a/b)b^{p-1} = 2(a+b)^{p-1},$$

which together with (2.15) yields (2.9). □

The property (GC)$_p$ is stable under taking "limits" and some algebraic operations like summations. To state precise results, we recall the following definition on convergences of functionals.

Definition 2.6 ([Dal, Definition 4.1 and Proposition 8.1])**.** Let $\mathcal{X}$ be a topological space, let $F \colon \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ and let $\{F_n \colon \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}\}_{n \in \mathbb{N}}$.

(1) The sequence $\{F_n\}_{n \in \mathbb{N}}$ is said to converge pointwise to F if and only if $\lim_{n\to\infty} F_n(x) = F(x)$ for any $x \in \mathcal{X}$.

(2) Suppose that $\mathcal{X}$ is a first-countable topological space. The sequence $\{F_n\}_{n \in \mathbb{N}}$ is said to Γ-converge to F (with respect to the topology of $\mathcal{X}$) if and only if the following conditions hold for any $x \in \mathcal{X}$:
   (i) If $x_n \to x$ in $\mathcal{X}$, then $F(x) \le \liminf_{n\to\infty} F_n(x_n)$.
   (ii) There exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathcal{X}$ such that

$$x_n \to x \text{ in } \mathcal{X} \quad \text{and} \quad \limsup_{n\to\infty} F_n(x_n) \le F(x). \tag{2.16}$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying (2.16) is called a *recovery sequence of* $\{F_n\}_{n\in\mathbb{N}}$ *at* x.

We also need the following reverse Minkowski inequality (see, e.g., [AF, Theorem 2.12]).

Proposition 2.7 (Reverse Minkowski inequality)**.** *Let* $(Y, \mathcal{A}, \mu)$ *be a measure space*[4] *and let* $r \in (0, 1]$*. Then for any* $\mathcal{A}$*-measurable functions* $f, g \colon Y \to [0, \infty]$*,*

$$\left(\int_Y f^r \, d\mu\right)^{1/r} + \left(\int_Y g^r \, d\mu\right)^{1/r} \leq \left(\int_Y (f+g)^r \, d\mu\right)^{1/r}. \tag{2.17}$$

In the following definition, we introduce the set of p-homogeneous functional on $\mathcal{F}$ which satisfies (GC)_p.

Definition 2.8. Recall that $\mathcal{F}$ is a linear subspace of $L^0(X, m)$. Define

$$\mathcal{U}_p^{\mathrm{GC}}(\mathcal{F}) \coloneqq \mathcal{U}_p^{\mathrm{GC}} \coloneqq \{\mathcal{E}' \colon \mathcal{F} \to [0, \infty) \mid \mathcal{E}' \text{ is } p\text{-homogeneous, } (\mathcal{E}', \mathcal{F}) \text{ satisfies } \text{(GC)}_p\}.$$

Now we can state the desired *stability* of (GC)_p.

Proposition 2.9. (a) $a_1\mathcal{E}^{(1)} + a_2\mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$ *for any* $\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$ *and any* $a_1, a_2 \in [0, \infty)$*.*

(b) *Let* $\left\{\mathcal{E}^{(n)} \in \mathcal{U}_p^{\mathrm{GC}}\right\}_{n\in\mathbb{N}}$ *and let* $\mathcal{E}^{(\infty)} \colon \mathcal{F} \to [0, \infty)$*. If* $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{N}}$ *converges pointwise to* $\mathcal{E}^{(\infty)}$*, then* $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$*.*

(c) *Suppose that* $\mathcal{F} \subseteq L^p(X, m)$ *and let us regard* $\mathcal{F}$ *as a topological space equipped with the topology of* $L^p(X, m)$*. Let* $\left\{\mathcal{E}^{(n)} \in \mathcal{U}_p^{\mathrm{GC}}\right\}_{n\in\mathbb{N}}$ *and let* $\mathcal{E}^{(\infty)} \colon \mathcal{F} \to [0, \infty)$*. If* $\{\mathcal{E}^{(n)}\}_{n\in\mathbb{N}}$ Γ*-converges to* $\mathcal{E}^{(\infty)}$*, then* $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$*.*

Proof. The statement (b) is trivial, so we will show (a) and (c). Throughout this proof, we fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1).

(a): Let $\mathcal{E}^{(1)}, \mathcal{E}^{(2)} \in \mathcal{U}_p^{\mathrm{GC}}$. Then $a\mathcal{E}^{(1)} \in \mathcal{U}_p^{\mathrm{GC}}$ is evident for any $a \in [0, \infty)$. Set $E(f) \coloneqq \mathcal{E}^{(1)}(f) + \mathcal{E}^{(2)}(f)$, $f \in \mathcal{F}$, and let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$. It suffices to prove $\left\| \left(E(T_l(\boldsymbol{u}))^{1/p}\right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left(E(u_k)^{1/p}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}$. For simplicity, we consider the case $q_2 < \infty$ (the case $q_2 = \infty$ is similar.) Then we have

$$\begin{aligned}
&\sum_{l=1}^{n_2} E\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} \\
&= \sum_{l=1}^{n_2} \Bigl[\mathcal{E}^{(1)}\bigl(T_l(\boldsymbol{u})\bigr) + \mathcal{E}^{(2)}\bigl(T_l(\boldsymbol{u})\bigr)\Bigr]^{q_2/p}
\end{aligned}$$

[4] In the book [AF], the reverse Minkowski inequality is stated and proved only for the L^r-space over non-empty open subsets of the Euclidean space equipped with the Lebesgue measure, but the same proof works for any measure space.

$$
\leq \left(\sum_{i \in \{1,2\}} \left[\sum_{l=1}^{n_2} \mathcal{E}^{(i)}\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} \right]^{p/q_2} \right)^{q_2/p} \qquad \text{(by the triangle ineq. for } \|\cdot\|_{\ell^{q_2/p}}\text{)}
$$

$$
\overset{(\mathrm{GC})_p}{\leq} \left(\left[\sum_{k=1}^{n_1} \mathcal{E}^{(1)}(u_k)^{q_1/p} \right]^{p/q_1} + \left[\sum_{k=1}^{n_1} \mathcal{E}^{(2)}(u_k)^{q_1/p} \right]^{p/q_1} \right)^{q_2/p}
$$

$$
\overset{(2.17)}{\leq} \left(\sum_{k=1}^{n_1} \Bigl[\mathcal{E}^{(1)}(u_k) + \mathcal{E}^{(2)}(u_k) \Bigr]^{q_1/p} \right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} = \left(\sum_{k=1}^{n_1} E(u_k)^{q_1/p} \right)^{q_2/q_1}, \tag{2.18}
$$

which implies $E \in \mathcal{U}_p^{\mathrm{GC}}$.

(c): Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$ and choose a recovery sequence $\{\boldsymbol{u}_n = (u_{1,n}, \dots, u_{n_1,n}) \in \mathcal{F}^{n_1}\}_{n \in \mathbb{N}}$ of $\{\mathcal{E}^{(n)}\}_{n \in \mathbb{N}}$ at $\boldsymbol{u}$. We first show tnat $\|T_l(\boldsymbol{u}) - T_l(\boldsymbol{u}_n)\|_{L^p(X,m)} \to 0$ as $n \to \infty$. Indeed, for any $\boldsymbol{v} = (v_1, \dots, v_{n_1})$ and any $\boldsymbol{z} = (z_1, \dots, z_{n_1}) \in L^p(X,m)^{n_1}$, we see that

$$
\begin{aligned}
\max_{l \in \{1,\dots,n_2\}} \|T_l(\boldsymbol{v}) - T_l(\boldsymbol{z})\|_{L^p(X,m)}^p &\overset{(2.1)}{\leq} \int_X \|\boldsymbol{v}(x) - \boldsymbol{z}(x)\|_{\ell^{q_1}}^p \, m(dx) \\
&= \int_X \left(\sum_{k=1}^{n_1} |v_k(x) - z_k(x)|^{p \cdot \frac{q_1}{p}} \right)^{p/q_1} m(dx) \\
&\leq n_1^{(p-q_1)/q_1} \sum_{k=1}^{n_1} \|v_k - z_k\|_{L^p(X,m)}^p,
\end{aligned} \tag{2.19}
$$

where we used Hölder's inequality in the last line. Since $\max_k \|u_k - u_{k,n}\|_{L^p(X,m)} \to 0$ as $n \to \infty$, (2.19) implies the desired convergence $\|T_l(\boldsymbol{u}) - T_l(\boldsymbol{u}_n)\|_{L^p(X,m)} \to 0$.

Now we prove $(\mathrm{GC})_p$ for the Γ-limit $\mathcal{E}^{(\infty)}$ of $\{\mathcal{E}^{(n)}\}_{n \in \mathbb{N}}$ (with respect to the $L^p(X,m)$-topology). It is easy to see that $\mathcal{E}^{(\infty)}$ is p-homogeneous (see, e.g., [Dal, Proposition 11.6]). We suppose that $q_2 < \infty$ since the case $q_2 = \infty$ is similar. Then,

$$
\begin{aligned}
\sum_{l=1}^{n_2} \mathcal{E}^{(\infty)}\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} &\leq \sum_{l=1}^{n_2} \liminf_{n \to \infty} \mathcal{E}^{(n)}\bigl(T_l(\boldsymbol{u}_n)\bigr)^{q_2/p} \leq \liminf_{n \to \infty} \sum_{l=1}^{n_2} \mathcal{E}^{(n)}\bigl(T_l(\boldsymbol{u}_n)\bigr)^{q_2/p} \\
&\leq \liminf_{n \to \infty} \left(\sum_{k=1}^{n_1} \mathcal{E}^{(n)}(u_{k,n})^{q_1/p} \right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} = \left(\sum_{k=1}^{n_1} \mathcal{E}^{(\infty)}(u_k)^{q_1/p} \right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}},
\end{aligned}
$$

which proves $\mathcal{E}^{(\infty)} \in \mathcal{U}_p^{\mathrm{GC}}$. □

3 Differentiability of p-energy forms and related results

In this section, we show the existence of the derivative (1.7) for any p-energy form satisfying p-Clarkson's inequality, (2.7) or (2.9). As an application of our differentiability result, we will introduce a 'two-variable' version of a p-energy form.

Throughout this section, we fix $p \in (1, \infty)$, a measure space $(X, \mathcal{B}, m)$, and a *p-energy form* $(\mathcal{E}, \mathcal{F})$ on (X, m) in the following sense:

Definition 3.1 (p-Energy form)**.** Let $\mathcal{F}$ be a linear subspace of $L^0(X, m)$ and let $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$. The pair $(\mathcal{E}, \mathcal{F})$ is said to be a *p-energy form* on (X, m) if $\mathcal{E}^{1/p}$ is a seminorm on $\mathcal{F}$.

Note that the same argument as in the proof of Proposition 2.2-(a) implies that $\mathcal{E}$ is strictly convex on $\mathcal{F}/\mathcal{E}^{-1}(0)$ (see (2.2)).

3.1 p-Clarkson's inequalities and differentiability

In this ection, we mainly deal with p-energy forms satisfying p-Clarkson's inequality in the following sense.

Definition 3.2 (p-Clarkson's inequality)**.** The pair $(\mathcal{E}, \mathcal{F})$ is said to satisfy *p-Clarkson's inequality*, $(\mathrm{Cla})_p$ for short, if and only if for any $f, g \in \mathcal{F}$,

$$\begin{cases} \mathcal{E}(f+g)^{1/(p-1)} + \mathcal{E}(f-g)^{1/(p-1)} \leq 2\bigl(\mathcal{E}(f) + \mathcal{E}(g)\bigr)^{1/(p-1)} & \text{if } p \in (1, 2], \\ \mathcal{E}(f+g) + \mathcal{E}(f-g) \leq 2\bigl(\mathcal{E}(f)^{1/(p-1)} + \mathcal{E}(g)^{1/(p-1)}\bigr)^{p-1} & \text{if } p \in [2, \infty). \end{cases} \qquad (\mathrm{Cla})_p$$

To state a consequence of $(\mathrm{Cla})_p$ on the convexity of $\mathcal{E}^{1/p}$, let us recall the notion of uniform convexity. See, e.g., [Cla36, Definition 1]. (The notion of uniform convexity is usually defined for a Banach space in the literature. We present the definition for seminormed space because we are mainly interested in $(\mathcal{F}, \mathcal{E}^{1/p})$.)

Definition 3.3 (Uniformly convex seminormed spaces)**.** Let $(\mathcal{X}, |\cdot|)$ be a seminormed space. We say that $(\mathcal{X}, |\cdot|)$ is *uniformly convex* if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ with the property that $|f + g| \leq 2(1 - \delta)$ whenever $f, g \in \mathcal{X}$ satisfy $|f| = |g| = 1$ and $|f - g| > \varepsilon$.

It is well known that $(\mathrm{Cla})_p$ implies the uniform convexity as follows.

Proposition 3.4. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$. Then $(\mathcal{F}, \mathcal{E}^{1/p})$ is uniformly convex.*

Proof. The same argument as in [Cla36, Proof of Corollary of Theorem 2] works. □

Moreover, $(\mathrm{Cla})_p$ provides us the following quantitative estimate for the central difference, which plays a central role in this section.

Proposition 3.5. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$. Then, for any $f, g \in \mathcal{F}$,*

$$\mathcal{E}(f+g) + \mathcal{E}(f-g) - 2\mathcal{E}(f) \leq \begin{cases} 2\mathcal{E}(g) & \text{if } p \in (1, 2], \\ 2(p-1)\Bigl[\mathcal{E}(f)^{\frac{1}{p-1}} + \mathcal{E}(g)^{\frac{1}{p-1}}\Bigr]^{p-2} \mathcal{E}(g)^{\frac{1}{p-1}} & \text{if } p \in (2, \infty). \end{cases} \tag{3.1}$$

In particular, $\mathbb{R} \ni t \mapsto \mathcal{E}(f+tg) \in [0,\infty)$ is differentiable and for any $s \in \mathbb{R}$,

$$\lim_{\delta\downarrow 0} \sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1} \left| \frac{\mathcal{E}(f+(s+\delta)g)-\mathcal{E}(f+sg)}{\delta} - \frac{d}{dt}\mathcal{E}(f+tg)\Big|_{t=s} \right| = 0. \tag{3.2}$$

Proof. The desired inequality (3.1) in the case $p \in (1,2]$ is immediate from (2.8), so we suppose that $p \in (2,\infty)$. Let $f, g \in \mathcal{F}$, set $a := \mathcal{E}(f)^{1/(p-1)}$ and $b := \mathcal{E}(g)^{1/(p-1)}$. Then we have (3.1) since $(\mathrm{Cla})_p$ implies that

$$\mathcal{E}(f+g)+\mathcal{E}(f-g)-2\mathcal{E}(f) \leq 2\bigl((a+b)^{p-1}-a^{p-1}\bigr) = 2(p-1)\int_a^{a+b} s^{p-2}\,ds \leq 2(p-1)(a+b)^{p-2}b.$$

Next we show that for any $t \in \mathbb{R}$,

$$\lim_{\delta\downarrow 0} \sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1} \frac{\mathcal{E}\bigl(f+(t+\delta)g\bigr)+\mathcal{E}\bigl(f+(t-\delta)g\bigr)-2\mathcal{E}(f+tg)}{\delta} = 0, \tag{3.3}$$

Let $t \in \mathbb{R}$, $\delta \in (0,\infty)$ and set

$$D_{t,\delta}(f;g) := \mathcal{E}\bigl(f+(t+\delta)g\bigr)+\mathcal{E}\bigl(f+(t-\delta)g\bigr)-2\mathcal{E}(f+tg). \tag{3.4}$$

By (3.1), we have

$$D_{t,\delta}(f;g) \leq \begin{cases} 2\delta^p\mathcal{E}(g) & \text{if } p\in(1,2], \\ 2(p-1)\delta^{p/(p-1)}\Bigl[\mathcal{E}(f+tg)^{\frac{1}{p-1}}+\mathcal{E}(\delta g)^{\frac{1}{p-1}}\Bigr]^{p-2}\mathcal{E}(g)^{\frac{1}{p-1}} & \text{if } p\in(2,\infty). \end{cases}$$

Hence we get

$$\sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1} \frac{D_{t,\delta}(f;g)}{\delta} \leq \begin{cases} 2\delta^{p-1} & \text{if } p\in(1,2], \\ 2(p-1)\delta^{1/(p-1)}\Bigl[\bigl(\mathcal{E}(f)^{1/p}+t\bigr)^{\frac{p}{p-1}}+\delta^{\frac{p}{p-1}}\Bigr]^{p-2} & \text{if } p\in(2,\infty), \end{cases} \tag{3.5}$$

which implies

$$\limsup_{\delta\downarrow 0} \sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1} \frac{D_{t,\delta}(f;g)}{\delta} \leq 0. \tag{3.6}$$

Since $\mathcal{E}$ is convex on $\mathcal{F}$, we know that the limits

$$\lim_{\delta\downarrow 0} \frac{\mathcal{E}\bigl(f+(t+\delta)g\bigr)-\mathcal{E}(f+tg)}{\delta} \quad \text{and} \quad \lim_{\delta\downarrow 0} \frac{\mathcal{E}\bigl(f+(t-\delta)g\bigr)-\mathcal{E}(f+tg)}{-\delta}$$

exist and

$$\frac{D_{t,\delta}(f;g)}{\delta} = \frac{\mathcal{E}\bigl(f+(t+\delta)g\bigr)+\mathcal{E}\bigl(f+(t-\delta)g\bigr)-2\mathcal{E}(f+tg)}{\delta} \geq 0. \tag{3.7}$$

By combining (3.6) and (3.7), we obtain (3.3) and the differentiability of $t \mapsto \mathcal{E}(f+tg)$. From the convexity of $t \mapsto \mathcal{E}(f+tg)$ again, we have

$$\sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1} \left| \frac{\mathcal{E}(f+(s+\delta)g)-\mathcal{E}(f+sg)}{\delta} - \frac{d}{dt}\mathcal{E}(f+tg)\Big|_{t=s} \right| \leq \sup_{g\in\mathcal{F};\mathcal{E}(g)\leq 1} \frac{D_{s,\delta}(f;g)}{\delta},$$

which together with (3.6) implies (3.2). $\square$

Proposition 3.5, especially (3.2), implies the Fréchet differentiability of $\mathcal{E}$ on $\mathcal{F}/\mathcal{E}^{-1}(0)$. We record this fact and basic properties of these Fréchet derivatives in the next theorem.

Theorem 3.6. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$. Then $\mathcal{E}\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to [0,\infty)$ is Fréchet differentiable on the quotient normed space $\mathcal{F}/\mathcal{E}^{-1}(0)$. In particular, for any $f, g \in \mathcal{F}$,*

$$\text{the derivative} \quad \mathcal{E}(f;g) \coloneqq \frac{1}{p}\frac{d}{dt}\mathcal{E}(f+tg)\Big|_{t=0} \in \mathbb{R} \quad \text{exists,} \tag{3.8}$$

the map $\mathcal{E}(f;\,\cdot\,)\colon \mathcal{F} \to \mathbb{R}$ is linear, $\mathcal{E}(f;f) = \mathcal{E}(f)$ and $\mathcal{E}(f;h) = 0$ if $h \in \mathcal{E}^{-1}(0)$. Moreover, for any $f, f_1, f_2, g \in \mathcal{F}$ and any $a \in \mathbb{R}$, the following hold:

$$\mathbb{R} \ni t \mapsto \mathcal{E}(f+tg;g) \in \mathbb{R} \text{ is strictly increasing if and only if } g \notin \mathcal{E}^{-1}(0). \tag{3.9}$$

$$\mathcal{E}(af;g) = \operatorname{sgn}(a)\,|a|^{p-1}\,\mathcal{E}(f;g), \quad \mathcal{E}(f+h;g) = \mathcal{E}(f;g) \quad \text{for } h \in \mathcal{E}^{-1}(0). \tag{3.10}$$

$$|\mathcal{E}(f;g)| \le \mathcal{E}(f)^{(p-1)/p}\mathcal{E}(g)^{1/p}. \tag{3.11}$$

$$|\mathcal{E}(f_1;g) - \mathcal{E}(f_2;g)| \le C_p\bigl(\mathcal{E}(f_1) \vee \mathcal{E}(f_2)\bigr)^{(p-1-\alpha_p)/p}\mathcal{E}(f_1-f_2)^{\alpha_p/p}\mathcal{E}(g)^{1/p}, \tag{3.12}$$

where $\alpha_p = \frac{1}{p} \wedge \frac{p-1}{p}$ and some constant $C_p \in (0,\infty)$ determined solely and explicitly by p.

Remark 3.7. It seems that the Hölder continuity exponent α_p appearing in (3.12) is not optimal because this exponent can be improved to $(p-1)\wedge 1$ in the case $\mathcal{E}(f;g) = \int_{\mathbb{R}^n} |\nabla f|^{p-2}\langle \nabla f, \nabla g\rangle\,dx$. However, such an improved continuity is unclear even for concrete p-energy forms constructed in the previous works [CGQ22, Kig23, MS23+, Shi24]. We can see the desired continuity ((3.12) with $(p-1)\wedge 1$ in place of α_p) for p-energy forms constructed in [KS.a], where a direct construction of p-energy forms based on the Korevaar–Schoen type p-energy forms is presented.

Proof. The existence of $\mathcal{E}(f;g)$ in (3.8) is already proved in Proposition 3.5. The properties $\mathcal{E}(f;ag) = a\mathcal{E}(f;g)$, $\mathcal{E}(af;g) = \operatorname{sgn}(a)\,|a|^{p-1}\,\mathcal{E}(f;g)$ and $\mathcal{E}(f;f) = \mathcal{E}(f)$ are obvious from the definition. The equalities $\mathcal{E}(f+h;g) = \mathcal{E}(f+g)$ and $\mathcal{E}(f;h) = 0$ for any $h \in \mathcal{E}^{-1}(0)$ follow from the triangle inequality for $\mathcal{E}^{1/p}$, so (3.10) holds. The property (3.9) is a consequence of the strict convexity of $\mathcal{E}$ (see (2.2)) and the differentiability in (3.8).

To show that $\mathcal{E}(f;\,\cdot\,)$ is linear, it suffices to prove $\mathcal{E}(f;g_1+g_2) = \mathcal{E}(f;g_1) + \mathcal{E}(f;g_2)$ for any $g_1, g_2 \in \mathcal{F}$. For any $t > 0$, the convexity of $\mathcal{E}$ implies that

$$\begin{aligned}\frac{\mathcal{E}\bigl(f+t(g_1+g_2)\bigr) - \mathcal{E}(f)}{t} &= \frac{\mathcal{E}\bigl(\frac{1}{2}(f+2tg_1) + \frac{1}{2}(f+2tg_2)\bigr) - \mathcal{E}(f)}{t} \\ &\le \frac{\mathcal{E}(f+2tg_1) - \mathcal{E}(f)}{2t} + \frac{\mathcal{E}(f+2tg_2) - \mathcal{E}(f)}{2t}.\end{aligned} \tag{3.13}$$

Passing to the limit as $t \downarrow 0$, we get $\mathcal{E}(f;g_1+g_2) \le \mathcal{E}(f;g_1) + \mathcal{E}(f;g_2)$. We obtain the converse inequality by noting that

$$\frac{\mathcal{E}(f-tg) - \mathcal{E}(f)}{t} \to -\frac{d}{dt}\mathcal{E}(f+tg)\Big|_{t=0} = -p\mathcal{E}(f;g) \quad \text{as } t \downarrow 0,$$

and by applying (3.13) with $-g_1, -g_2$ in place of g_1, g_2 respectively.

The Hölder-type estimate (3.11) follows from the following elementary estimate:

$$|a^q - b^q| = \left| \int_{a\wedge b}^{a\vee b} qt^{q-1}\,dt \right| \leq q(a^{q-1}\vee b^{q-1})\,|a-b| \quad \text{for } q\in(0,\infty),\ a,b\in[0,\infty). \tag{3.14}$$

Indeed, by (3.14) and the triangle inequality for $\mathcal{E}^{1/p}$, for any $t>0$,

$$\left| \frac{\mathcal{E}(f+tg)-\mathcal{E}(f)}{t} \right| \leq p\bigl(\mathcal{E}(f+tg)^{1/p}\vee\mathcal{E}(f)^{1/p}\bigr)^{p-1}\mathcal{E}(g)^{1/p}. \tag{3.15}$$

We obtain (3.11) by letting $t\downarrow 0$ in (3.15). We conclude that $\mathcal{E}(f;\,\cdot\,)$ is the Fréchet derivative of $\mathcal{E}$ at f by (3.2), the linearity of $\mathcal{E}(f;\,\cdot\,)$ and (3.11).

In the rest of this proof, we prove (3.12). Our proof is partially inspired by an argument by Šmulian in [Smu40]. In this proof, $C_{p,i}$, $i\in\{1,\dots,5\}$, is a constant depending only on p. We first show an analog of (3.1) for $\mathcal{E}^{1/p}$. Using (3.14), we can show that there exists $c_*\in(0,2^{-p^3})$ depending only on p such that

$$\sup\left\{ \frac{|\mathcal{E}(f)-\mathcal{E}(f+\delta g)|}{\mathcal{E}(f)} \;\middle|\; \begin{array}{l} f,g,\in\mathcal{F}, \delta\in(0,\infty) \text{ such that} \\ 0<\delta<c_*\mathcal{E}(f)^{1/p} \text{ and } \mathcal{E}(g)=1 \end{array} \right\} \leq \frac{1}{10}. \tag{3.16}$$

Let $\psi(t) := |t|^{1/p}$ and fix $g\in\mathcal{F}$ with $\mathcal{E}(g)=1$. Then there exist $\theta_1,\theta_2,\theta\in[0,1]$ such that

$$\begin{aligned} 0 &\leq \psi(\mathcal{E}(f+\delta g)) + \psi(\mathcal{E}(f-\delta g)) - 2\psi(\mathcal{E}(f)) \\ &= \psi'(A_{1,\delta})\bigl[\mathcal{E}(f+\delta g)-\mathcal{E}(f)\bigr] - \psi'(A_{2,\delta})\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr] \\ &= \psi'(A_1(\delta))D_\delta(f;g) - \bigl(\psi'(A_{1,\delta})-\psi'(A_{2,\delta})\bigr)\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr] \\ &= \psi'(A_{1,\delta})D_\delta(f;g) - \psi''\bigl(A_{1,\delta}+\theta(A_{2,\delta}-A_{1,\delta})\bigr)(A_{2,\delta}-A_{1,\delta})\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr], \end{aligned} \tag{3.17}$$

where $D_\delta(f;g) := D_{\delta,0}(f;g)$ is the same as in (3.4) and

$$A_{1,\delta} := \mathcal{E}(f) + \theta_1\bigl[\mathcal{E}(f+\delta g)-\mathcal{E}(f)\bigr], \quad A_{2,\delta} := \mathcal{E}(f-\delta g) + \theta_2\bigl[\mathcal{E}(f)-\mathcal{E}(f-\delta g)\bigr].$$

By (3.16), we note that $|A_{1,\delta}|\wedge|A_{1,\delta}+\theta(A_{2,\delta}-A_{1,\delta})| \geq \frac{1}{2}\mathcal{E}(f)$, which together with (3.17) and (3.1) implies that for any $(\delta,f)\in(0,\infty)\times\mathcal{F}$ with $0<\delta<c_*\mathcal{E}(f)^{1/p}$,

$$\begin{aligned} 0 &\leq \psi(\mathcal{E}(f+\delta g)) + \psi(\mathcal{E}(f-\delta g)) - 2\psi(\mathcal{E}(f)) \\ &\leq C_{p,1}\Bigl(\mathcal{E}(f)^{\frac{1}{p}-1+\frac{(p-2)^+}{p-1}}\delta^{p\wedge\frac{p}{p-1}} + \mathcal{E}(f)^{\frac{1}{p}-2+\frac{2(p-1)}{p}}\delta^2\Bigr) \\ &\leq C_{p,1}\delta\cdot\delta^{(p-1)\wedge\frac{1}{p-1}}\Bigl(\mathcal{E}(f)^{\frac{1}{p}-1+\frac{(p-2)^+}{p-1}} + \mathcal{E}(f)^{\frac{1}{p}-2+\frac{2(p-1)}{p}}\Bigr). \end{aligned}$$

In particular, if $\mathcal{E}(f)=1$, then

$$\mathcal{E}(f+\delta g)^{1/p} + \mathcal{E}(f-\delta g)^{1/p} \leq 2 + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\delta \quad \text{for any } \delta\in(0,c_*). \tag{3.18}$$

Next let $f_1,f_2\in\mathcal{F}$. Then, by (3.11) and (3.14),

$$|\mathcal{E}(f_2;f_1)-\mathcal{E}(f_1)| \leq |\mathcal{E}(f_2;f_1)-\mathcal{E}(f_2)| + |\mathcal{E}(f_2)-\mathcal{E}(f_1)|$$

$$\leq \Bigl(\mathcal{E}(f_2)^{(p-1)/p} + p\Bigl(\mathcal{E}(f_2)^{(p-1)/p} \vee \mathcal{E}(f_1)^{(p-1)/p}\Bigr)\Bigr)\mathcal{E}(f_1 - f_2)^{1/p}$$
$$\leq C_{p,2}\Bigl(\mathcal{E}(f_1)^{(p-1)/p} \vee \mathcal{E}(f_2)^{(p-1)/p}\Bigr)\mathcal{E}(f_1 - f_2)^{1/p}. \tag{3.19}$$

Now, for any $f_1, f_2, g \in \mathcal{F}$ with $\mathcal{E}(f_1) = \mathcal{E}(g) = 1$ and $\delta \in (0, c_*)$ we see that

$$\begin{aligned}
&\mathcal{E}(f_1; \delta g) - \mathcal{E}(f_2; \delta g) \\
&= \mathcal{E}(f_1; f_1 + \delta g) + \mathcal{E}(f_2; f_1 - \delta g) - \mathcal{E}(f_1) - \mathcal{E}(f_2; f_1) \\
&\overset{(3.11)}{\leq} \Bigl(\mathcal{E}(f_1)^{(p-1)/p} \vee \mathcal{E}(f_2)^{(p-1)/p}\Bigr)\Bigl(\mathcal{E}(f_1 + \delta g)^{1/p} + \mathcal{E}(f_1 - \delta g)^{1/p}\Bigr) - \mathcal{E}(f_1) - \mathcal{E}(f_2; f_1) \\
&\overset{(3.14),(3.18)}{\leq} \Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2 + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\delta\Bigr) - \mathcal{E}(f_1) - \mathcal{E}(f_2; f_1).
\end{aligned}$$

Similarly, we can show

$$\begin{aligned}
&\mathcal{E}(f_1; \delta g) - \mathcal{E}(f_2; \delta g) \\
&= -\mathcal{E}(f_1; f_1 - \delta g) - \mathcal{E}(f_2; f_1 + \delta g) + \mathcal{E}(f_1) + \mathcal{E}(f_2; f_1) \\
&\geq -\Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2 + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\delta\Bigr) + \mathcal{E}(f_1) + \mathcal{E}(f_2; f_1).
\end{aligned}$$

From these estimates, we have

$$\begin{aligned}
&|\mathcal{E}(f_1; g) - \mathcal{E}(f_2; g)| = \frac{|\mathcal{E}(f_1; \delta g) - \mathcal{E}(f_2; \delta g)|}{\delta} \\
&\leq \Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1} + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\Bigr) - \delta^{-1}\mathcal{E}(f_1) - \delta^{-1}\mathcal{E}(f_2; f_1) \\
&= \Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1} + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\Bigr) - 2\delta^{-1}\mathcal{E}(f_1) + \delta^{-1}\bigl(\mathcal{E}(f_1) - \mathcal{E}(f_2; f_1)\bigr) \\
&\overset{(3.19)}{\leq} \Bigl(1 + C_{p,3}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr)\Bigl(2\delta^{-1} + C_{p,1}\delta^{(p-1)\wedge(p-1)^{-1}}\Bigr) - 2\delta^{-1} + C_{p,2}\delta^{-1}\mathcal{E}(f_1 - f_2)^{1/p} \\
&\leq C_{p,4}\Bigl(\delta^{(p-1)\wedge(p-1)^{-1}} + \delta^{-1}\mathcal{E}(f_1 - f_2)^{1/p}\Bigr).
\end{aligned}$$

If $\mathcal{E}(f_1 - f_2) < c_*^{-p^2/((p-1)\vee 1)}$, then, by choosing $\delta = \mathcal{E}(f_1 - f_2)^{((p-1)\vee 1)/p^2}$, we obtain

$$|\mathcal{E}(f_1; g) - \mathcal{E}(f_2; g)| \leq C_{p,5}\mathcal{E}(f_1 - f_2)^{((p-1)\wedge 1)/p^2}. \tag{3.20}$$

The same is clearly true if $\mathcal{E}(f_1 - f_2) \geq c_*^{-p^2/((p-1)\vee 1)}$ since $\mathcal{E}(f_2) \leq 2^{p-1}\bigl(1 + \mathcal{E}(f_1 - f_2)\bigr)$. Finally, for any $f_1, f_2, g \in \mathcal{F}$ with $\mathcal{E}(f_1) \wedge \mathcal{E}(g) > 0$, we have

$$\begin{aligned}
|\mathcal{E}(f_1; g) - \mathcal{E}(f_2; g)| &= \mathcal{E}(f_1)^{(p-1)/p}\mathcal{E}(g)^{1/p}\left|\mathcal{E}\left(\frac{f_1}{\mathcal{E}(f_1)^{1/p}}; \frac{g}{\mathcal{E}(g)^{1/p}}\right) - \mathcal{E}\left(\frac{f_2}{\mathcal{E}(f_1)^{1/p}}; \frac{g}{\mathcal{E}(g)^{1/p}}\right)\right| \\
&\overset{(3.20)}{\leq} C_{p,5}\mathcal{E}(f_1)^{(p-1)/p}\mathcal{E}(g)^{1/p}\mathcal{E}\left(\frac{f_1}{\mathcal{E}(f_1)^{1/p}} - \frac{f_2}{\mathcal{E}(f_1)^{1/p}}\right)^{((p-1)\wedge 1)/p^2} \\
&\overset{(3.20)}{\leq} C_{p,5}\bigl(\mathcal{E}(f_1) \vee \mathcal{E}(f_2)\bigr)^{(p-1-\alpha_p)/p}\mathcal{E}(g)^{1/p}\mathcal{E}(f_1 - f_2)^{\alpha_p/p}.
\end{aligned}$$

The same is clearly true if $\mathcal{E}(f_2) \wedge \mathcal{E}(g) > 0$. Since (3.12) is obvious when $g \in \mathcal{E}^{-1}(0)$ or $\mathcal{E}(f_1) \vee \mathcal{E}(f_2) = 0$, so we obtain (3.12). □

We conclude this subsection by viewing typical examples of p-energy forms.

Example 3.8. (1) Let $D \in \mathbb{N}$, let $X := \Omega \subseteq \mathbb{R}^D$ be a domain, let $\mathcal{B} := \mathcal{B}(X)$, let m be the D-dimensional Lebesgue measure on X and let $\mathcal{F} = W^{1,p}(\Omega)$ be the usual $(1,p)$-Sobolev space on Ω (see [AF, p. 60] for example). Define $\mathcal{E}(f) := \|\nabla f\|_{L^p(X,m)}^p$, $f \in \mathcal{F}$, where the gradient operator ∇ is regarded in the distribution sense. Then it is clear that $(\mathcal{E},\mathcal{F})$ is a p-energy form on (X,m) satisfying (GC)$_p$. In this case, we have

$$\mathcal{E}(f;g) = \int_\Omega |\nabla f(x)|^{p-2} \langle \nabla f(x), \nabla g(x)\rangle_{\mathbb{R}^D}\, dx, \quad f,g \in \mathcal{F},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^D}$ denotes the inner product on $\mathbb{R}^D$.

(2) In the recent work [Kig23, MS23+], a p-energy form $(\mathcal{E},\mathcal{F})$ on a compact metrizable space with some geometric assumptions is constructed via discrete approximations. See [HPS04, CGQ22] for constructions of p-energy forms on post-critically finite self-similar sets. As will be seen in more detail later in Section 8, we can prove that p-energy forms constructed in [CGQ22, Kig23, MS23+] satisfy (GC)$_p$ although even (Cla)$_p$ is not mentioned in [CGQ22, Kig23].

(3) There are various ways to define $(1,p)$-Sobolev spaces in the field of ‘Analysis on metric spaces’ (see, e.g., [HKST, Chapter 10]). In these definitions, roughly speaking, we find a counterpart of $|\nabla u|$, e.g., the minimal p-weak upper gradient $g_u \geq 0$ (see, e.g., [HKST, Chapter 6] for details), and consider a p-energy form $(\widetilde{\mathcal{E}},\mathcal{F})$ on (X,m) given by $\widetilde{\mathcal{E}}(u) := \int_X g_u^p\, dm$ and $\mathcal{F} := \{u \in L^p(X,m) \mid g_u \in L^p(X,m)\}$. Unfortunately, this p-energy form may not satisfy (Cla)$_p$ because of a lack of the linearity of $u \mapsto g_u$ (see, e.g., [HKST, (6.3.18)]). However, in a suitable setting, we can construct a functional which is equivalent to $\widetilde{\mathcal{E}}$ and satisfies (Cla)$_p$; see the p-energy form denoted by $(\mathcal{F}_p, W^{1,p})$ in [ACD15, Theorem 40]. Moreover, we can verify (GC)$_p$ for $(\mathcal{F}_p, W^{1,p})$ since $(\mathcal{F}_{\delta_k,p}, W^{1,p})$ defined in [ACD15, (7.3)] satisfies (GC)$_p$ and $\mathcal{F}_p$ is defined as a Γ-limit point of $\mathcal{F}_{\delta_k,p}$ as $k \to \infty$. (See also the proof of Theorem 8.19 later.)

3.2 p-Clarkson's inequalities and approximations in p-energy forms

In this subsection, in addition to the setting specified at the beginning of this section, we also assume that

$$\mathcal{F} \text{ is a linear subspace of } L^p(X,m). \tag{3.21}$$

We introduce a norm on $\mathcal{F}$ in the following definition.

Definition 3.9 ($(\mathcal{E},\alpha)$-norm)**.** Let $\alpha \in (0,\infty)$. We define the norm $\|\cdot\|_{\mathcal{E},\alpha}$ on $\mathcal{F}$ by

$$\|f\|_{\mathcal{E},\alpha} := \Bigl(\mathcal{E}(f) + \alpha \|f\|_{L^p(X,m)}^p\Bigr)^{1/p}, \quad f \in \mathcal{F} \tag{3.22}$$

We call $\|\cdot\|_{\mathcal{E},\alpha}$ the $(\mathcal{E},\alpha)$-*norm* on $\mathcal{F}$.

The next proposition states on the convexity of $\|\cdot\|_{\mathcal{E},\alpha}$.

Proposition 3.10. *Let $\alpha \in (0,\infty)$ and suppose that $(\mathcal{E},\mathcal{F})$ satisfies* (Cla)$_p$. *Then $(\|\cdot\|_{\mathcal{E},\alpha}^p,\mathcal{F})$ is a p-energy form on (X,m) satisfying* (Cla)$_p$, *and $(\mathcal{F},\|\cdot\|_{\mathcal{E},\alpha})$ is uniformly convex.. In particular, if $(\mathcal{F},\|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space, then it is reflexive.*

Proof. We have (Cla)$_p$ for the p-energy form $(\|\cdot\|_{\mathcal{E},\alpha}^p,\mathcal{F})$ on (X,m) by applying (2.18) for $T\colon \mathbb{R}^2 \to \mathbb{R}$ given in Proposition 2.2-(e),(f). The uniform convexity $\|\cdot\|_{\mathcal{E},\alpha}$ follows from [Cla36, Proof of Corollary of Theorem 2].

Assume that $(\mathcal{F},\|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space. Then $(\mathcal{F},\|\cdot\|_{\mathcal{E},\alpha})$ is reflexive by the Milman–Pettis theorem (see, e.g., [LT, Proposition 1.e.3]) since $(\mathcal{F},\|\cdot\|_{\mathcal{E},\alpha})$ is uniformly convex. □

We will frequently use the following Mazur's lemma, which is an elementary fact in the theory of Banach spaces.

Lemma 3.11 (Mazur's lemma; see, e.g., [HKST, p. 19])**.** *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in a normed space V converging weakly to some element $v \in V$. Then there exist a subsequence $(v_{n_k})_{k\in\mathbb{N}}$, $\{N_l\}_{l\in\mathbb{N}} \subseteq \mathbb{N}$ with $N_l > l$ and $\bigl\{\lambda_{k,l} \in [0,1] \bigm| k = l, l+1, \dots, N_l\bigr\}$ with $\sum_{k=l}^{N_l} \lambda_{k,l} = 1$ such that $\sum_{k=l}^{N_l} \lambda_{k,l} v_{n_k}$ converges strongly to v as $l \to \infty$.*

We collect some useful results on converges in $\mathcal{E}$ in the next proposition. Let us regard $\mathcal{E}$ as a $[0,\infty]$-valued functional on $L^p(X,m)$ by setting $\mathcal{E}(f) \coloneqq \infty$ for $f \in L^p(X,m) \setminus \mathcal{F}$ if $(\mathcal{F},\|\cdot\|_{\mathcal{E}})$ is a Banach space.

Proposition 3.12. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies* (Cla)$_p$ *and that $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is a Banach space.*

(a) *If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^p(X,m)$ converges to $u \in L^p(X,m)$ in $L^p(X,m)$ as $n \to \infty$, then $\mathcal{E}(u) \le \liminf_{n\to\infty} \mathcal{E}(u_n)$.*

(b) *If $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges to $u \in \mathcal{F}$ in $L^p(X,m)$ as $n \to \infty$ and $\lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}(u)$, then $u \in \mathcal{F}$ and $\lim_{n\to\infty} \|u - u_n\|_{\mathcal{E},1} = 0$.*

Proof. (a): If $\liminf_{n\to\infty} \mathcal{E}(u_n) = \infty$, then the desired statement clearly holds. So, we suppose that $\liminf_{n\to\infty} \mathcal{E}(u_n) < \infty$. Pick a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mathcal{E}(u_{n_k}) = \liminf_{n\to\infty} \mathcal{E}(u_n)$. Then $\{u_{n_k}\}_k$ is a bounded sequence in $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$. Since $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is reflexive by Proposition 3.10 and $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ is continuously embedded into $L^p(X,m)$, there exists a further subsequence, which is denoted by $\{u_{n_k}\}_{k\in\mathbb{N}}$ again, such that u_{n_k} converges weakly in $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ to $u \in \mathcal{F}$. By Mazur's lemma (see Lemma 3.11), there exist $\{N_l\}_{l\in\mathbb{N}} \subseteq \mathbb{N}$ with $N_l > l$ and $\bigl\{\lambda_{k,l} \in [0,1] \bigm| k = l, l+1, \dots, N_l\bigr\}_{l\in\mathbb{N}}$ with $\sum_{k=l}^{N_l} \lambda_{k,l} = 1$ such that $v_l \coloneqq \sum_{k=l}^{N_l} \lambda_{k,l} u_{n_k}$ converges strongly in $(\mathcal{F},\|\cdot\|_{\mathcal{E},1})$ to u as $l \to \infty$. By the triangle inequality for $\mathcal{E}^{1/p}$,

$$\mathcal{E}(g_l)^{1/p} \le \sum_{k=l}^{N_l} \lambda_{k,l} \mathcal{E}(u_{n_k})^{1/p}.$$

Letting $l \to \infty$, we get $\mathcal{E}(u)^{1/p} \le \liminf_{n\to\infty} \mathcal{E}(u_n)^{1/p}$.

(b): If $u \in \mathcal{E}^{-1}(0)$, then $\mathcal{E}(u-u_n) = \mathcal{E}(u_n) \to \mathcal{E}(u) = 0$. It suffices to consider the case $\mathcal{E}(u) = 1$. Since $u + u_n$ converges in $L^p(X,m)$ to $2u$ as $n \to \infty$, by (a),

$$\begin{aligned}2 = \mathcal{E}(2u)^{1/p} \le \liminf_{n\to\infty} \mathcal{E}\bigl(u+u_n\bigr)^{1/p} &\le \limsup_{n\to\infty} \mathcal{E}\bigl(u+u_n\bigr)^{1/p} \\ &\le \lim_{n\to\infty} \mathcal{E}(u_n)^{1/p} + \mathcal{E}(u)^{1/p} = 2,\end{aligned}$$

i.e., $\lim_{n\to\infty} \mathcal{E}(u+u_n) = 2^p$. By $(\mathrm{Cla})_p$, if $p \le 2$, then

$$\begin{aligned}\lim_{n\to\infty} \mathcal{E}(u-u_n)^{1/(p-1)} &\le 2\Bigl(\mathcal{E}(u) + \lim_{n\to\infty} \mathcal{E}(u_n)\Bigr)^{1/(p-1)} - \lim_{n\to\infty} \mathcal{E}(u+u_n)^{1/(p-1)} \\ &= 2 \cdot 2^{1/(p-1)} - 2^{p/(p-1)} = 0.\end{aligned}$$

If $p \ge 2$, then

$$\lim_{n\to\infty} \mathcal{E}(u-u_n) \le 2^{p-1}\Bigl(\mathcal{E}(u) + \lim_{n\to\infty} \mathcal{E}(u_n)\Bigr) - \lim_{n\to\infty} \mathcal{E}(u+u_n) = 2^{p-1} \cdot 2 - 2^p = 0.$$

Since u_n converges in $L^p(X,m)$ to u as $n \to \infty$, we obtain the desired convergence. □

The following convergences in $\mathcal{E}$ are also useful. These are analogs of [FOT, Theorem 1.4.2-(iii), Theorem 1.4.2-(v)].

Corollary 3.13. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{Cla})_p$ and that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space. In addition, we assume the following property: if $\varphi \in C(\mathbb{R})$ satisfies $\varphi(0) = 0$ and $|\varphi(t) - \varphi(s)| \le |t-s|$ for any $s,t \in \mathbb{R}$, then $\varphi(u) \in \mathcal{F}$ and $\mathcal{E}(\varphi(u)) \le \mathcal{E}(u)$ for any $u \in \mathcal{F}$.*

(a) *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C(\mathbb{R})$ satisfy $\lim_{n\to\infty} \varphi_n(t) = t$, $\varphi_n(0) = 0$ and $|\varphi_n(t) - \varphi_n(s)| \le |t-s|$ for any $n \in \mathbb{N}$, $s,t \in \mathbb{R}$. Then $\{\varphi_n(u)\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}(u - \varphi_n(u)) = 0$ for any $u \in \mathcal{F}$.*

(b) *Let $u \in \mathcal{F}$, $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\varphi \in C(\mathbb{R})$ satisfy $\lim_{n\to\infty} \|u-u_n\|_{\mathcal{E},1} = 0$, $\varphi(0) = 0$, $|\varphi(t) - \varphi(s)| \le |t-s|$ for any $s,t \in \mathbb{R}$ and $\varphi(u) = u$. Then $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}(u - \varphi(u_n)) = 0$.*

Remark 3.14. Let us make the same remark as [KS23+, Remark 2.21] for convenience. Typical choices of $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C(\mathbb{R})$ in Corollary 3.13-(a) are $\varphi_n(t) = (-n) \vee (t \wedge n)$ and $\varphi_n(t) = t - (-\frac{1}{n}) \vee (t \wedge \frac{1}{n})$. A typical use of Corollary 3.13-(b) is to obtain a sequence of I-valued functions converging to u in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ when $I \subseteq \mathbb{R}$ is a closed interval and $u \in \mathcal{F}$ is I-valued, by considering $\varphi \in C(\mathbb{R})$ given by $\varphi(t) \coloneqq (\inf I) \vee (t \wedge \sup I)$.

Proof. (a): We have $\varphi_n(u) \in \mathcal{F}$ by the assumption on $(\mathcal{E},\mathcal{F})$. It is immediate from the dominated convergence theorem that $\varphi_n(u)$ converges in $L^p(X,m)$ to u as $n \to \infty$. By $\mathcal{E}(\varphi_n(u)) \le \mathcal{E}(u)$ and Proposition 3.12-(a),

$$\mathcal{E}(u) \le \liminf_{n\to\infty} \mathcal{E}(u_n) \le \limsup_{n\to\infty} \mathcal{E}(u_n) \le \mathcal{E}(u),$$

which implies $\lim_{n\to\infty}\mathcal{E}(u_n)=\mathcal{E}(u)$. We have $\lim_{n\to\infty}\mathcal{E}(u-\varphi_n(u))=0$ by Proposition 3.12-(b).

(b): By the dominated convergence theorem, $\varphi(u_n)$ converges in $L^p(X,m)$ to $\varphi(u)=u$ as $n\to\infty$. We have $\varphi(u_n)\in\mathcal{F}$ by the assumption on $(\mathcal{E},\mathcal{F})$. By $\mathcal{E}(\varphi(u_n))\leq\mathcal{E}(u_n)$ and Proposition 3.12-(a),

$$\mathcal{E}(u)=\mathcal{E}(\varphi(u))\leq\liminf_{n\to\infty}\mathcal{E}(\varphi(u_n))\leq\limsup_{n\to\infty}\mathcal{E}(\varphi(u_n))\leq\lim_{n\to\infty}\mathcal{E}(u_n)=\mathcal{E}(u),$$

which implies $\lim_{n\to\infty}\mathcal{E}(\varphi(u_n))=\mathcal{E}(u)$. We have $\lim_{n\to\infty}\mathcal{E}(u-\varphi(u_n))=0$ by Proposition 3.12-(b). □

3.3 Fréchet derivative and the dual space

In many practical situations, the quotient normed space $\mathcal{F}/\mathcal{E}^{-1}(0)$ (equipped with the norm $\mathcal{E}^{1/p}$) becomes a Banach space (see Subsection 6.2). To state some basic properties of this Banach space, we recall the notion of *uniformly smoothness.*

Definition 3.15 (Uniformly smooth normed space)**.** Let $(\mathcal{X},\|\cdot\|)$ be a normed space. The normed space $\mathcal{X}$ is said to be *uniformly smooth* if and only if it satisfies

$$\lim_{\tau\to0}\tau^{-1}\sup\left\{\frac{\|u+v\|+\|u-v\|}{2}-1\;\middle|\;\|u\|=1,\|v\|=\tau\right\}=0.$$

The following duality between uniform convexity and uniform smoothness is well known. (See also [BCL94, Lemma 5] for a quantitative version of this theorem.)

Theorem 3.16 (Day's duality theorem; [LT, Proposition 1.e.2])**.** *Let $\mathcal{X}$ be a Banach space. Then $\mathcal{X}$ is uniformly convex if and only if its dual space $\mathcal{X}^*$ is uniformly smooth.*

We also recall the notion of duality mapping and its fundamental results in the next proposition (see, e.g., [Miya, Definition 2.1, Lemmas 2.1 and 2.2]).

Proposition 3.17 (Duality mapping)**.** *Let $\mathcal{X}$ be a Banach space and let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$. Let $\|\cdot\|_W$ be the norm of W for each $W\in\{\mathcal{X},\mathcal{X}^*\}$. For $(x,f)\in\mathcal{X}\times\mathcal{X}^*$, we set $\langle x,f\rangle\coloneqq f(x)$. For $x\in\mathcal{X}$, define $F\colon\mathcal{X}\to2^{\mathcal{X}^*}$ by*

$$F(x)\coloneqq\bigl\{f\in\mathcal{X}^*\;\big|\;\langle x,f\rangle=\|x\|_{\mathcal{X}}^2=\|f\|_{\mathcal{X}^*}^2\bigr\},$$

which is called the duality mapping of $\mathcal{X}$*. Then the following properties hold:*

(a) *$F(x)\neq\emptyset$ for any $x\in\mathcal{X}$.*
(b) *If $\mathcal{X}$ is reflexive, then $\bigcup_{x\in\mathcal{X}}F(x)=\mathcal{X}^*$.*
(c) *If $\mathcal{X}$ is strictly convex, i.e., $\|\lambda x+(1-\lambda)y\|_{\mathcal{X}}<\lambda\|x\|_{\mathcal{X}}+(1-\lambda)\|y\|_{\mathcal{X}}$ for any $\lambda\in(0,1)$ and any $x,y\in\mathcal{X}\setminus\{0\}$, then $\#F(x)=1$ for any $x\in\mathcal{X}$.*

Now we can state a result on the dual space of $\mathcal{F}/\mathcal{E}^{-1}(0)$.

Theorem 3.18. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (Cla)$_p$ *and that $\mathcal{F}/\mathcal{E}^{-1}(0)$ is a Banach space.*

(a) *The Banach space $\mathcal{F}/\mathcal{E}^{-1}(0)$ is uniformly convex and uniformly smooth. In particular, it is reflexive and its dual Banach spaces $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^*$ is also uniformly convex and uniformly smooth.*

(b) *The map $f \mapsto \mathcal{E}(f; \cdot)$ is a bijection from $\mathcal{F}/\mathcal{E}^{-1}(0)$ to $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^*$. In particular, $\bigl(\mathcal{F}/\mathcal{E}^{-1}(0)\bigr)^* = \{\mathcal{E}(f; \cdot\,) \mid f \in \mathcal{F}\}$.*

Proof. For simplicity, set $\mathcal{X} \coloneqq \mathcal{F}/\mathcal{E}^{-1}(0)$ and $\|u\|_{\mathcal{X}} \coloneqq \mathcal{E}(u)^{1/p}$ for any $u \in \mathcal{X}$.

(a): The uniform convexity of $\mathcal{X}$ is immediate from Proposition 3.4, whence $\mathcal{X}$ is reflexive by the Milman–Pettis theorem. Also, we easily see from (3.18) that $\mathcal{X}$ is uniformly smooth. The same properties for $\mathcal{X}^*$ follow from Theorem 3.16.

(b): Let $u \in \mathcal{X}$ and define $\mathcal{A}(u) \coloneqq \mathcal{E}(u)^{2/p-1}\mathcal{E}(u; \cdot\,) \in \mathcal{X}^*$. (We define $\mathcal{A}(u) = 0$ if $\mathcal{E}(u) = 0$.) We will show that $\mathcal{A}\colon \mathcal{X} \to \mathcal{X}^*$ is a bijection. By (3.11), we have

$$\|\mathcal{A}(u)\|_{\mathcal{X}^*} = \mathcal{E}(u)^{2/p-1} \|\mathcal{E}(u; \cdot\,)\|_{\mathcal{X}^*} = \mathcal{E}(u)^{2/p-1+(p-1)/p} = \|u\|_{\mathcal{X}}\,.$$

Then $\langle u, \mathcal{A}(u)\rangle = \mathcal{E}(u)^{2/p} = \|u\|_{\mathcal{X}}^2 = \|\mathcal{A}(u)\|_{\mathcal{X}^*}^2$ and hence

$$\mathcal{A}(u) \in \{f \in \mathcal{X}^* \mid \langle u, f\rangle = \|u\|_{\mathcal{X}}^2 = \|f\|_{\mathcal{X}^*}^2\} = F(u),$$

where $F\colon \mathcal{X} \to \mathcal{X}^*$ is the duality mapping. We see from Proposition 3.17 and (a) that $\mathcal{A}\colon \mathcal{X} \to \mathcal{X}^*$ is a surjection. Note that the mapping $F^{-1}\colon \mathcal{X}^* \to \mathcal{X}^{**} = \mathcal{X}$ defined by $F^{-1}(f) = \{u \in \mathcal{X} \mid \langle u, f\rangle = \|u\|_{\mathcal{X}}^2 = \|f\|_{\mathcal{X}^*}^2\}$ for $f \in \mathcal{X}^*$ is the duality mapping from $\mathcal{X}^*$ to $\mathcal{X}$. By Proposition 3.17 and (a) again, we conclude that $\mathcal{A}$ is injective. □

We also present a similar statement for $(\mathcal{F}, \|\cdot\|_{\mathcal{E}})$.

Corollary 3.19. *Let $\alpha \in (0, \infty)$. Assume that $\mathcal{F} \subseteq L^p(X, m)$, that $(\mathcal{E}, \mathcal{F})$ satisfies* (Cla)$_p$ *and that $\mathcal{X}_\alpha \coloneqq (\mathcal{F}, \|\cdot\|_{\mathcal{E},\alpha})$ is a Banach space.*

(a) *The Banach space $\mathcal{X}_\alpha$ is uniformly convex and uniformly smooth. In particular, it is reflexive and its dual space $\mathcal{X}_\alpha^*$ is also uniformly convex and uniformly smooth.*

(b) *For each $f \in \mathcal{F}$, define a linear map $\Psi_{p,\alpha}^f\colon \mathcal{F} \to \mathbb{R}$ by*

$$\Psi_{p,\alpha}^f(g) \coloneqq \mathcal{E}(f; g) + \alpha \int_X \operatorname{sgn}(f)\,|f|^{p-1}\,g\,dm, \quad g \in \mathcal{F}. \tag{3.23}$$

Then the map $f \mapsto \Psi_{p,\alpha}^f$ is a bijection from $\mathcal{X}_\alpha$ to $\mathcal{X}_\alpha^*$. In particular, $\mathcal{X}_\alpha^* = \{\Psi_{p,\alpha}^f \mid f \in \mathcal{F}\}$.

Proof. We define $\mathcal{E}_\alpha\colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by

$$\mathcal{E}_\alpha(u; v) \coloneqq \mathcal{E}(u; v) + \alpha \int_X \operatorname{sgn}(u)\,|u|^{p-1}\,v\,dm, \quad u, v \in \mathcal{F}.$$

and set $\mathcal{E}_\alpha(u) \coloneqq \mathcal{E}_\alpha(u; u) = \|u\|_{\mathcal{E},\alpha}^p$. Then $(\mathcal{E}_\alpha, \mathcal{F})$ is a p-energy form on (X, m) and it satisfies (Cla)$_p$ by Proposition 3.10. We have the desired result by applying Theorem 3.18 for $(\mathcal{E}_\alpha, \mathcal{F})$. □

3.4 Regularity and strong locality

In this subsection, in addition to the setting specified at the beginning of this section, similar to [FOT], we make the following topological assumptions:

$$X \text{ is a locally compact separable metrizable space.} \tag{3.24}$$

$$m \text{ is a positive Radon measure on } X \text{ with full topological support.} \tag{3.25}$$

Note that (3.25) is equivalent to saying that $m(O) > 0$ for any non-empty open subset O of X. Under this setting, the map $C(X)$ to $L^0(X, \mathcal{B}, m)$, where $\mathcal{B} = \mathcal{B}(X)$, defined by taking $u \in C(X)$ to its m-equivalence class is injective and hence gives a canonical embedding of $C(X)$ into $L^0(X, m)$ as a subalgebra, and we will consider $C(X)$ as a subset of $L^0(X, m)$ through this embedding without further notice.

The following definitions are analogs of the notions in the theory of regular symmetric Dirichlet forms (see, e.g., [FOT, p. 6]).

Definition 3.20 (Core)**.** Let $\mathscr{C}$ be a subset of $\mathcal{F} \cap C_c(X)$.

(1) A core $\mathscr{C}$ is said to be a *core* of $(\mathcal{E}, \mathcal{F})$ if and only if $\mathscr{C}$ is dense both in $(\mathcal{F}, \|\cdot\|_{\mathcal{E},,1})$ and in $(C_c(X), \|\cdot\|_{\mathrm{sup}})$.

(2) A core $\mathscr{C}$ is said to be *special* if and only if $\mathscr{C}$ is a linear subspace of $\mathcal{F} \cap C_c(X)$, $\mathscr{C}$ is a dense subalgebra of $(C_c(X), \|\cdot\|_{\mathrm{sup}})$, and for any compact subset K of X and any relatively compact open subset G of X with $K \subseteq G$, there exists $\varphi \in \mathscr{C}$ such that $\varphi \geq 0$, $\varphi = 1$ on K and $\varphi = 0$ on $X \setminus G$.

Definition 3.21 (Regularity)**.** We say that $(\mathcal{E}, \mathcal{F})$ is *regular* if and only if there exists a core $\mathscr{C}$ of $(\mathcal{E}, \mathcal{F})$.

We can show the following result on regular p-energy forms, which is an analog of [FOT, Exercise 1.4.1].

Proposition 3.22. *Suppose that $(\mathcal{E}, \mathcal{F})$ is regular and that $\mathcal{F}$ satisfies the following properties:*

$$u^+ \wedge 1 \in \mathcal{F} \quad \textit{for any } u \in \mathcal{F}, \tag{3.26}$$

$$uv \in \mathcal{F} \quad \textit{for any } uv \in \mathcal{F} \cap C_b(X). \tag{3.27}$$

Then $\mathcal{F} \cap C_c(X)$ is a special core of $(\mathcal{E}, \mathcal{F})$.

Proof. It is clear that $\mathcal{F} \cap C_c(X)$ is a core of $(\mathcal{E}, \mathcal{F})$. By (3.27), $\mathcal{F} \cap C_c(X)$ is a subalgebra of $C_c(X)$. Let K be a compact subset of X and G be a relatively compact open subset G of X with $K \subseteq G$. By Urysohn's lemma, there exists $\varphi_0 \in C_c(X)$ such that $\varphi_0 = 2$ on K and $\varphi_0 = 0$ on $X \setminus G$. Let $\varepsilon \in (0, 1/2)$. Fix $\psi \in \mathcal{F} \cap C_c(X)$ satisfying $\psi = 1$ on $\overline{G}^X$, which exists by the regularity of $(\mathcal{E}, \mathcal{F})$, the locally compactness of X and (3.26). Since $\mathcal{F} \cap C_c(X)$ is a core of $(\mathcal{E}, \mathcal{F})$, there exists $\widetilde{\varphi} \in \mathcal{F} \cap C_c(X)$ such that $\|\varphi_0 - \widetilde{\varphi}\|_{\mathrm{sup}} < \varepsilon$. Now we define $\varphi \in C_c(X)$ by $\varphi \coloneqq (\widetilde{\varphi} - \varepsilon\psi)^+ \wedge 1$. (Note that $\mathrm{supp}_X[\varphi]$ is compact since $\overline{G}^X$ is compact.) Then $\varphi \in \mathcal{F} \cap C_c(X)$ by (3.26). Clearly, $\varphi = 1$ on K and $\varphi = 0$ on $X \setminus G$, so the proof is completed. □

The proposition above ensures when there exist *cutoff* functions in $\mathcal{F}$. We also introduce the following condition stating the existence of cutoff functions in a weaker sense.

Definition 3.23. We say that a p-energy form $(\mathcal{E}, \mathcal{F})$ on (X, m) satisfies the property $(\mathrm{CF})_m$[5] if and only if, for any compact subset K of X and any open subset U of X with $K \subseteq U$, there exists $\varphi \in \mathcal{F} \cap L^\infty(X, m)$ such that $\varphi(x) = 1$ for m-a.e. $x \in K$ and $\varphi(x) = 0$ for m-a.e. $x \in X \setminus U$.

Next we introduce the notion of locality of the p-energy form $(\mathcal{E}, \mathcal{F})$ on (X, m).

Definition 3.24 (Strong locality)**.** (1) We say that $(\mathcal{E}, \mathcal{F})$ has the strongly local property (SL1) if and only if, for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - \alpha_1]$ or $\mathrm{supp}_m[f_2 - \alpha_2]$ compact and $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$ for some $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$,

$$\mathcal{E}(f_1 + f_2 + g) + \mathcal{E}(g) = \mathcal{E}(f_1 + g) + \mathcal{E}(f_2 + g). \tag{3.28}$$

(2) Suppose that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$. We say that $(\mathcal{E}, \mathcal{F})$ has the strongly local property (SL2) if and only if, for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - f_2 - \alpha]$ or $\mathrm{supp}_m[g - \beta]$ compact and $\mathrm{supp}_m[f_1 - f_2 - \alpha] \cap \mathrm{supp}_m[g - \beta] = \emptyset$ for some $\alpha, \beta \in \mathcal{E}^{-1}(0)$,

$$\mathcal{E}(f_1; g) = \mathcal{E}(f_2; g). \tag{3.29}$$

In the following propositions, we collect basic results about (SL1) and (SL2).

Proposition 3.25. *Assume that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{Cla})_p$.*

(a) *If $(\mathcal{E}, \mathcal{F})$ satisfies (SL1), then for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - \alpha_1]$ or $\mathrm{supp}_m[f_2 - \alpha_2]$ compact and $\mathrm{supp}_m[f_1 - \alpha_1] \cap \mathrm{supp}_m[f_2 - \alpha_2] = \emptyset$ for some $\alpha_1, \alpha_2 \in \mathcal{E}^{-1}(0)$,*
$$\mathcal{E}(f_1 + f_2; g) = \mathcal{E}(f_1; g) + \mathcal{E}(f_2; g). \tag{3.30}$$

(b) *If $(\mathcal{E}, \mathcal{F})$ satisfies (SL2), then for any $f_1, f_2, g \in \mathcal{F}$ with either $\mathrm{supp}_m[f_1 - f_2 - \alpha]$ or $\mathrm{supp}_m[g - \beta]$ compact and $\mathrm{supp}_m[f_1 - f_2 - \alpha] \cap \mathrm{supp}_m[g - \beta] = \emptyset$ for some $\alpha, \beta \in \mathcal{E}^{-1}(0)$,*
$$\mathcal{E}(g; f_1) = \mathcal{E}(g; f_2). \tag{3.31}$$

Proof. (a): Note that (3.28) with $g = 0$ implies that $\mathcal{E}(f_1 + f_2) = \mathcal{E}(f_1) + \mathcal{E}(f_2)$. For any $t \in (0, \infty)$, we have from (3.28) that

$$\frac{\mathcal{E}(f_1 + f_2 + tg) - \mathcal{E}(f_1 + f_2)}{t} + t^{p-1}\mathcal{E}(g) = \frac{\mathcal{E}(f_1 + tg) - \mathcal{E}(f_1)}{t} + \frac{\mathcal{E}(f_2 + tg) - \mathcal{E}(f_2)}{t}.$$

We obtain (3.30) by letting $t \downarrow 0$ in this equality.

(b): Since $\mathcal{E}(g; \cdot\,)$ is linear by Theorem 3.6, it suffices to prove $\mathcal{E}(g; f_1 - f_2) = 0$, which follows from (3.29) with $g, 0, f_1 - f_2$ in place of f_1, f_2, g. □

[5] We can consider several versions of this condition such as a version requiring $\varphi \in \mathcal{F} \cap C(K)$ in addition. Note that $(\mathrm{CF})_m$ holds if $(\mathcal{E}, \mathcal{F})$ admits a special core.

Proposition 3.26. *Assume that $(\mathcal{E},\mathcal{F})$ satisfies* $(\mathrm{Cla})_p$*.*

(a) *If $(\mathcal{E},\mathcal{F})$ satisfies (SL1), then $(\mathcal{E},\mathcal{F})$ also satisfies (SL2).*

(b) *Assume that $(\mathcal{E},\mathcal{F})$ satisfies (SL2) and the following three conditions:*

$$uv \in \mathcal{F}\cap L^{\infty}(X,m) \text{ for any } u,v\in\mathcal{F}\cap L^{\infty}(X,m). \tag{3.32}$$

$$\text{For any } u\in\mathcal{F},\ u_n \coloneqq (-n)\vee u\wedge n\in\mathcal{F} \text{ and } \lim_{n\to\infty}\mathcal{E}(u-u_n)=0. \tag{3.33}$$

$$(\mathcal{E},\mathcal{F}) \text{ satisfies } (\mathrm{CF})_m. \tag{3.34}$$

Then $(\mathcal{E},\mathcal{F})$ satisfies (SL1).

Proof. (a): Let $f_1,f_2,g\in\mathcal{F}$ and $\alpha_1,\alpha_2\in\mathcal{E}^{-1}(0)$ with either $\mathrm{supp}_m[f_1-f_2-\alpha]$ or $\mathrm{supp}_m[g-\beta]$ compact and $\mathrm{supp}_m[f_1-f_2-\alpha]\cap\mathrm{supp}_m[g-\beta]=\emptyset$. Let $t\in(0,1)$. By (3.28) with $t(f_1-f_2),g,0$ in place of f_1,f_2,g, we have

$$\mathcal{E}(t(f_1-f_2)+g)=\mathcal{E}(t(f_1-f_2))+\mathcal{E}(g),$$

whence

$$\lim_{t\downarrow 0}\frac{\mathcal{E}(g+t(f_1-f_2))-\mathcal{E}(g)}{t}=\lim_{t\downarrow 0}t^{p-1}\mathcal{E}(f_1-f_2)=0.$$

Since $\mathcal{E}(g;\,\cdot\,)$ is linear by Theorem 3.6, we get $\mathcal{E}(g;f_1)=\mathcal{E}(g;f_2)$. Similarly, by (3.28) with f_2-f_1,tg,f_1 in place of f_1,f_2,g,

$$\mathcal{E}\bigl((f_2-f_1)+tg+f_1\bigr)+\mathcal{E}(f_1)=\mathcal{E}\bigl((f_2-f_1)+f_1\bigr)+\mathcal{E}(tg+f_1),$$

which implies $\mathcal{E}(f_1;g)=\mathcal{E}(f_2;g)$.

(b): We first consider the case $g\in\mathcal{F}\cap L^{\infty}(X,m)$. Let $f_1,f_2\in\mathcal{F}$ and $\alpha_1,\alpha_2\in\mathcal{E}^{-1}(0)$ with either $\mathrm{supp}_m[f_1-\alpha_1]$ or $\mathrm{supp}_m[f_2-\alpha_2]$ compact and $\mathrm{supp}_m[f_1-\alpha_1]\cap\mathrm{supp}_m[f_2-\alpha_2]=\emptyset$. We assume that $\mathrm{supp}_m[f_1-\alpha_1]$ is compact since both cases are similar. Let U be an open neighborhood of $\mathrm{supp}_m[f_1-\alpha_1]$ such that $U\subseteq X\setminus\mathrm{supp}_m[f_2-\alpha_2]$. By (3.34) and the locally compactness of K, there exist $\varphi\in\mathcal{F}\in L^{\infty}(X,m)$ such that $\varphi(x)=1$ for m-a.e. $x\in U$, $\varphi(x)=1$ for m-a.e. $x\in X\setminus\mathrm{supp}_m[f_2-\alpha_2]$ and $\mathrm{supp}_m[\varphi]$ is compact. Note that $\varphi g\in\mathcal{F}$ by (3.32). Then we see from (SL2) that

$$\begin{aligned}
\mathcal{E}(f_1+f_2+g)+\mathcal{E}(g)&=\mathcal{E}(f_1+f_2+g;f_1)+\mathcal{E}(f_1+f_2+g;f_2)+\mathcal{E}(f_1+f_2+g;g)+\mathcal{E}(g)\\
&\overset{\text{(SL2)}}{=}\mathcal{E}(f_1+g;f_1)+\mathcal{E}(f_2+g;f_2)+\mathcal{E}(f_1+f_2+g;g)+\mathcal{E}(g)\\
&=\mathcal{E}(f_1+g;f_1)+\mathcal{E}(f_2+g;f_2)\\
&\quad+\mathcal{E}(f_1+f_2+g;(1-\varphi)g)+\mathcal{E}(f_1+f_2+g;\varphi g)+\mathcal{E}(g).
\end{aligned}\tag{3.35}$$

Since $\mathrm{supp}_m[\varphi g]$ and $\mathrm{supp}_m[f_1-\alpha_1]$ are compact, $\mathrm{supp}_m[f_1-\alpha_1]\cap\mathrm{supp}_m[(1-\varphi)g]=\emptyset$ and $\mathrm{supp}_m[f_2-\alpha_2]\cap\mathrm{supp}_m[\varphi g]=\emptyset$, by (SL2), we have the following equalities:

$$\mathcal{E}(f_1+f_2+g;(1-\varphi)g)=\mathcal{E}(f_2+g;(1-\varphi)g).$$
$$\mathcal{E}(f_1+f_2+g;\varphi g)=\mathcal{E}(f_1+g;\varphi g).$$

$$\mathcal{E}(g) = \mathcal{E}(g;(1-\varphi)g) + \mathcal{E}(g;\varphi g) = \mathcal{E}(f_1+g;(1-\varphi)g) + \mathcal{E}(f_2+g;\varphi g).$$

By combining these equalities and (3.35), we obtain

$$\begin{aligned}\mathcal{E}(f_1+f_2+g)+\mathcal{E}(g) &= \mathcal{E}(f_1+g;f_1)+\mathcal{E}(f_2+g;f_2)+\mathcal{E}(f_1+g;g)+\mathcal{E}(f_2+g;g)\\ &= \mathcal{E}(f_1+g)+\mathcal{E}(f_2+g),\end{aligned}$$

which proves (SL1) in the case $g \in \mathcal{F} \cap L^\infty(X,m)$.

Lastly, we prove (SL1) without assuming the boundedness of g. Let $g \in \mathcal{F}$ and set $g_n \coloneqq (-n) \vee (g \wedge n)$, $n \in \mathbb{N}$. Then $g_n \in \mathcal{F}$ by (3.33), and the statement proved in the previous paragraph yields that

$$\mathcal{E}(f_1+f_2+g_n)+\mathcal{E}(g_n) = \mathcal{E}(f_1+g_n)+\mathcal{E}(f_2+g_n)$$

for any $n \in \mathbb{N}$. Thanks to (3.33) and the triangle inequality for $\mathcal{E}^{1/p}$, we obtain the desired equality (3.29) by letting $n \to \infty$ in the equality above. $\square$

4 p-Energy measures and their basic properties

In this section, we discuss p-energy measures dominated by a p-energy form. Similar to the case of p-energy forms, we will introduce two-variable versions of p-energy measures and prove their basic properties.

As in the previous section, in this section, we fix $p \in (1,\infty)$, a measure space $(X,\mathcal{B},m)$ and a p-energy form $(\mathcal{E},\mathcal{F})$ on (X,m) with $\mathcal{F} \subseteq L^0(X,m)$. We also assume the existence of a family of finite measures $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ on $(X,\mathcal{B})$, which we call *p-energy measures* dominated by $(\mathcal{E},\mathcal{F})$, satisfying the following conditions:

$(\mathrm{EM1})_p$ $\Gamma\langle f\rangle(X) \le \mathcal{E}(f)$ for any $f \in \mathcal{F}$.

$(\mathrm{EM2})_p$ $\Gamma\langle\,\cdot\,\rangle(A)^{1/p}$ is a seminorm on $\mathcal{F}$ for any $A \in \mathcal{B}$.

We then see that $(\Gamma\langle\,\cdot\,\rangle(A),\mathcal{F})$ is a p-energy form on (X,m) for each $A \in \mathcal{B}$ by $(\mathrm{EM2})_p$.

Definition 4.1. We say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies p-Clarkson's inequality, $(\mathrm{Cla})_p$ for short, if and only if $(\Gamma\langle\,\cdot\,\rangle(A),\mathcal{F})$ satisfies $(\mathrm{Cla})_p$ for any $A \in \mathcal{B}$, i.e., for any $f,g \in \mathcal{F}$,

$$\begin{cases}\Gamma\langle f+g\rangle(A)^{\frac{1}{p-1}}+\Gamma\langle f-g\rangle(A)^{\frac{1}{p-1}} \le 2\bigl(\Gamma\langle f\rangle(A)+\Gamma\langle g\rangle(A)\bigr)^{\frac{1}{p-1}} & \text{if } p\in(1,2],\\ \Gamma\langle f+g\rangle(A)+\Gamma\langle f-g\rangle(A) \le 2\bigl(\Gamma\langle f\rangle(A)^{\frac{1}{p-1}}+\Gamma\langle g\rangle(A)^{\frac{1}{p-1}}\bigr)^{p-1} & \text{if } p\in(2,\infty).\end{cases} \tag{Cla}_p$$

We also say that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies the generalized p-contraction property, $(\mathrm{GC})_p$ for short, if and only if $(\Gamma\langle\,\cdot\,\rangle(A),\mathcal{F})$ satisfies $(\mathrm{GC})_p$ for any $A \in \mathcal{B}$.

Example 4.2. (1) Consider the same setting as in Example 3.8-(1). Then the measures

$$\Gamma\langle f\rangle(A) \coloneqq \int_A |\nabla f(x)|^p\,dx \quad \text{for } f \in W^{1,p}(\Omega) \text{ and } A \in \mathcal{B}(\mathbb{R}^D) \text{ with } A \subseteq \Omega,$$

are easily seen to be p-energy measures dominated by $\mathcal{E}(f) = \int_\Omega |\nabla f(x)|^p \, dx$ satisfying (EM1)$_p$, (EM2)$_p$ and (GC)$_p$. Recall that $\mathcal{E}(f;g) = \int_\Omega |\nabla f(x)|^{p-2} \langle \nabla f(x), \nabla g(x)\rangle_{\mathbb{R}^D} \, dx$. Then we can see that, by the Leibniz and chain rules for ∇, for any $u, \varphi \in W^{1,p}(\Omega) \cap C^1(\Omega)$,

$$\int_\Omega \varphi \, d\Gamma\langle u\rangle = \mathcal{E}(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}\bigl(|u|^{\frac{p}{p-1}}; \varphi\bigr). \tag{4.1}$$

(2) Although p-energies are constructed on compact metric spaces [Kig23, MS23+], we do not know how to construct the associated p-energy measures because of the lack of the density "$|\nabla u(x)|^p$". (As described in (3) below, the theory of Dirichlet forms presents 2-energy measures $\{\mu_{\langle u\rangle}\}_{u\in\mathcal{F}_2}$ associated with a given nice Dirichlet form $(\mathcal{E}_2, \mathcal{F}_2)$. On a large class of self-similar sets, it is known that $\mu_{\langle u\rangle}$ is mutually singular with respect to the natural Hausdorff measure of the underlying fractal [Hin05, KM20].) In the case of self-similar sets with suitable assumptions, *self-similar p-energy forms* are constructed in [CGQ22, Kig23, MS23+, Shi24], and we can introduce p-energy measures satisfying (EM1)$_p$, (EM2)$_p$ and (GC)$_p$ by using the self-similarity of p-energy forms. See Section 5 for details.
In [KS.a], under suitable assumptions, the authors construct a good p-energy form $\mathcal{E}_p^{\mathrm{KS}}$, which is called the *Korevaar–Shoen p-energy form*, on a locally compact separable metric space (X, d) equipped with a σ-finite Borel measure m with full topological support. As an advantage of $\mathcal{E}_p^{\mathrm{KS}}$, the right-hand side of (4.1) with $\mathcal{E}_p^{\mathrm{KS}}$ in place of $\mathcal{E}$ can be extended to a bounded positive linear functional in $\varphi \in C_c(X)$ and the p-energy measure $\Gamma_p^{\mathrm{KS}}\langle u\rangle$ associated with $\mathcal{E}_p^{\mathrm{KS}}$ is constructed as the unique Radon measure corresponding to this functional through the Riesz–Markov–Kakutani theorem. A remarkable fact is that this approach does not rely on the self-similarity of the underlying space or of the p-energy form. In [KS.a], basic properties for $\Gamma_p^{\mathrm{KS}}\langle\,\cdot\,\rangle$ like (EM1)$_p$, (EM2)$_p$ and (GC)$_p$ are also shown. See [KS.a, Sections 3 and 4] for details.

(3) The case $p = 2$ is very special. (See [FOT, Section 3.2] for details on 2-energy measures associated with regular symmetric Dirichlet forms.) If $(\mathscr{E}, D(\mathscr{E}))$ is a regular strongly local Dirichlet form on $L^2(X, m)$, where X is a locally compact separable metrizable space and m is a positive Radon measure on X with full topological support, then $\mathscr{E}(u) := \mathscr{E}(u, u)$ is a 2-energy form on (X, m) satisfying (GC)$_2$ (see Proposition A.2). The Dirichlet form theory provides us the associated 2-energy measures $\{\mu_{\langle u\rangle}\}_{u\in D(\mathscr{E})}$ through the following formula[6]:

$$\int_X \varphi \, d\mu_{\langle u\rangle} = \mathscr{E}(u, u\varphi) - \frac{1}{2}\mathscr{E}(u^2, \varphi) \quad \text{for any } \varphi \in D(\mathscr{E}) \cap C_c(X). \tag{4.2}$$

(Recall (4.1).) We easily see that $\{\mu_{\langle u\rangle}\}_{u\in D(\mathscr{E})}$ satisfies (EM1)$_2$ and the parallelogram law, which implies (EM2)$_2$ and (Cla)$_2$.

(4) Let g_u be the minimal p-weak upper gradient of $u \in N^{1,p}(X, m)$, where $N^{1,p}(X, m) := \{u \in L^p(X, m) \mid g_u \in L^p(X, m)\}$ is the Newton-Sobolev space (see [HKST, Section

[6]Precisely, the formula (4.2) is valid for $u \in D(\mathscr{E}) \cap L^\infty(X, m)$. We can extend it to any $u \in D(\mathscr{E})$ by considering the limit of $(u \wedge n) \vee (-n)$ as $n \to \infty$

7.1]). Then $\Gamma\langle u\rangle(A) := \int_A g_u^p\,dm$ defines p-energy measures satisfying $(\mathrm{EM1})_p$ and $(\mathrm{EM2})_p$. Indeed, we have $(\mathrm{EM2})_p$ by [HKST, (6.3.18)]. However, $(\mathrm{Cla})_p$ for these measures is unclear because of the lack of the linearity of $u \mapsto g_u$.

The same argument as in Proposition 3.5 yields the following result.

Proposition 4.3. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies $(\mathrm{Cla})_p$. Then, for any $f, g \in \mathcal{F}$ and any $A \in \mathcal{B}$,*

$$\begin{aligned}&\Gamma\langle f+g\rangle(A) + \Gamma\langle f-g\rangle(A) - 2\Gamma\langle f\rangle(A)\\ &\quad\le \begin{cases} 2\Gamma\langle g\rangle(A) & \text{if } p \in (1,2],\\ 2(p-1)\Bigl[\Gamma\langle f\rangle(A)^{\frac{1}{p-1}} + \Gamma\langle g\rangle(A)^{\frac{1}{p-1}}\Bigr]^{p-2}\Gamma\langle g\rangle(A)^{\frac{1}{p-1}} & \text{if } p \in (2,\infty).\end{cases}\end{aligned} \tag{4.3}$$

In particular, $\mathbb{R} \ni t \mapsto \Gamma\langle f+tg\rangle(A) \in [0,\infty)$ is differentiable and for any $s \in \mathbb{R}$,

$$\lim_{\delta\downarrow 0}\sup_{\substack{A\in\mathcal{B}, g\in\mathcal{F};\\ \mathcal{E}(g)\le 1}}\left|\frac{\Gamma\langle f+(s+\delta)g\rangle(A) - \Gamma\langle f+sg\rangle(A)}{\delta} - \frac{d}{dt}\Gamma\langle f+tg\rangle(A)\Big|_{t=s}\right| = 0. \tag{4.4}$$

Definition 4.4. Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies $(\mathrm{Cla})_p$. Let $f, g \in \mathcal{F}$. Define $\Gamma\langle f; g\rangle\colon \mathcal{B} \to \mathbb{R}$ by

$$\Gamma\langle f; g\rangle(A) := \frac{1}{p}\frac{d}{dt}\Gamma\langle f+tg\rangle(A)\Big|_{t=0} \quad \text{for } A \in \mathcal{B}(X), \tag{4.5}$$

which exists by Proposition 4.3.

The following properties of $\Gamma\langle f; g\rangle$ can be shown in a similar way as Theorem 3.6.

Theorem 4.5. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies $(\mathrm{Cla})_p$. Let $A \in \mathcal{B}$. Then $\Gamma\langle f; \cdot\,\rangle(A)$ is the Fréchet derivative of $\Gamma\langle\,\cdot\,\rangle(A)\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to [0,\infty)$ at $f \in \mathcal{F}$. In particular, the map $\Gamma\langle f; \cdot\,\rangle(A)\colon \mathcal{F} \to \mathbb{R}$ is linear, $\Gamma\langle f; f\rangle(A) = \Gamma\langle f\rangle(A)$ and $\Gamma\langle f; h\rangle(A) = 0$ if $h \in \mathcal{F}$ satisfies $\Gamma\langle h\rangle(A) = 0$. Moreover, for any $f, f_1, f_2, g \in \mathcal{F}$ and $a \in \mathbb{R}$, the following hold:*

$$\mathbb{R} \ni t \mapsto \Gamma\langle f+tg; g\rangle(A) \in \mathbb{R} \text{ is strictly increasing if and only if } \Gamma\langle g\rangle(A) > 0. \tag{4.6}$$

$$\Gamma\langle af; g\rangle = \mathrm{sgn}(a)\,|a|^{p-1}\,\Gamma\langle f; g\rangle, \quad \Gamma\langle f+h; g\rangle(A) = \Gamma\langle f; g\rangle(A) \text{ if } \Gamma\langle h\rangle(A) = 0. \tag{4.7}$$

$$|\Gamma\langle f; g\rangle(A)| \le \Gamma\langle f\rangle(A)^{(p-1)/p}\Gamma\langle g\rangle(A)^{1/p}. \tag{4.8}$$

$$|\Gamma\langle f_1; g\rangle(A) - \Gamma\langle f_2; g\rangle(A)| \le C_p\bigl(\Gamma\langle f_1\rangle(A) \vee \Gamma\langle f_2\rangle(A)\bigr)^{\frac{p-1-\alpha_p}{p}}\Gamma\langle f_1 - f_2\rangle(A)^{\frac{\alpha_p}{p}}\Gamma\langle g\rangle(A)^{\frac{1}{p}}, \tag{4.9}$$

where α_p, C_p are the same as in Theorem 3.6.

The set function $\Gamma\langle f; g\rangle$ is a signed measure as shown in the next proposition.

Proposition 4.6. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies* (Cla)$_p$. *For any $f, g \in \mathcal{F}$, the set function $\Gamma\langle f; g\rangle$ is a signed measure on $(X, \mathcal{B})$. Moreover, for any $\mathcal{B}$-measurable function $u\colon X \to [0,\infty)$ with $\|u\|_{\sup} < \infty$, $\int_X u\,d\Gamma\langle\,\cdot\,\rangle\colon \mathcal{F}/\mathcal{E}^{-1}(0) \to \mathbb{R}$ is Fréchet differentiable and has the same properties as those of $\Gamma\langle\,\cdot\,\rangle$ in Theorem 4.5 with "$\Gamma\langle g\rangle(A) > 0$" in (4.6) replaced by "$\int_X u\,d\Gamma\langle g\rangle > 0$", and for any $f, g \in \mathcal{F}$,*

$$\int_X u\,d\Gamma\langle f; g\rangle = \frac{1}{p}\,\frac{d}{dt}\int_X u\,d\Gamma\langle f + tg\rangle\bigg|_{t=0}. \tag{4.10}$$

Proof. The equalities $\Gamma\langle f; g\rangle(\emptyset) = 0$ and $|\Gamma\langle f; g\rangle(X)| = |\mathcal{E}(f; g)| < \infty$ are clear from the definition. We will show the countable additivity of $\Gamma\langle f; g\rangle$. The finite additivity of $\Gamma\langle f; g\rangle$ is obvious. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}$ be a family of disjoint measurable sets. Set $B_N \coloneqq \bigcup_{n=N+1}^{\infty} A_n$ for each $N \in \mathbb{N}$. Then we see that

$$\begin{aligned}\left|\Gamma\langle f; g\rangle\left(\bigcup_{n\in\mathbb{N}} A_n\right) - \sum_{n=1}^{N}\Gamma\langle f; g\rangle(A_n)\right| &= |\Gamma\langle f; g\rangle(B_N)| \\ &\overset{(4.8)}{\le} \Gamma\langle f\rangle(B_N)^{(p-1)/p}\Gamma\langle g\rangle(B_N)^{1/p} \xrightarrow[N\to\infty]{} 0,\end{aligned}$$

which shows that $\Gamma\langle f; g\rangle$ is a signed measure on $(X, \mathcal{B})$.

The other properties except for (4.10) can be proved by following the arguments of Theorem 3.6, so we shall prove (4.10). By the finite additivity of $\int_X u\,d\Gamma\langle f; g\rangle$ and $\frac{1}{p}\frac{d}{dt}\int_X u\,d\Gamma\langle f + tg\rangle\big|_{t=0}$ in u, we can assume that $u \ge 0$. Let $s_n = \sum_{k=1}^{l_n} a_k \mathbb{1}_{A_k}$ with $a_k \ge 0$ and $A_k \in \mathcal{B}$ be a sequence of simple functions so that $s_n \uparrow u$ m-a.e. as $n \to \infty$. Then we immediately have (4.10) with $u = s_n$. Since $\lim_{n\to\infty}\int_X s_n\,d\Gamma\langle f; g\rangle = \int_X u\,d\Gamma\langle f; g\rangle$ by the dominated convergence theorem, it suffices to prove

$$\lim_{n\to\infty}\frac{d}{dt}\int_X s_n\,d\Gamma\langle f + tg\rangle\bigg|_{t=0} = \frac{d}{dt}\int_X u\,d\Gamma\langle f + tg\rangle\bigg|_{t=0}. \tag{4.11}$$

Since (3.15) with $\int_X u\,d\Gamma\langle\,\cdot\,\rangle$ in place of $\mathcal{E}$ holds by the fact that $(\int_X u\,d\Gamma\langle\,\cdot\,\rangle, \mathcal{F})$ is a p-energy form, we know that for any $\mathcal{B}$-measurable function $v\colon X \to [0,\infty)$ with $\|v\|_{\sup} < \infty$

$$\left|\frac{d}{dt}\int_X v\,d\Gamma\langle f + tg\rangle\bigg|_{t=0}\right| \le \left(\int_X v\,d\Gamma\langle f\rangle\right)^{(p-1)/p}\left(\int_X v\,d\Gamma\langle g\rangle\right)^{1/p}. \tag{4.12}$$

By combining (4.12) with $v = u - s_n$ and the dominated convergence theorem, we obtain (4.11). □

Remark 4.7. A signed measure corresponding to $\Gamma\langle f; g\rangle$ is discussed in [BV05, Section 5], but the existence of $\Gamma\langle f; g\rangle$ as a derivative (in some uniform manner) is assumed in [BV05]. See the condition (H4) and the beginning of Section 5 of [BV05] for details. Our differentiability result, (4.4) in Proposition 4.3, combined with the constructions of (self-similar) p-energy measures in [Shi24, MS23+], presents p-energy measures $\{\Gamma\langle f; g\rangle\}_{f,g\in\mathcal{F}}$ on a wide class of self-similar sets including the generalized Sierpiński carpets. (See also

Proposition 5.8.) To the best of author's knowledge[7], this is the first result on constructing such measures on fractals.

The total variation measure $|\Gamma\langle f;g\rangle|$ also satisfies a Hölder-type estimate as shown in the following proposition.

Proposition 4.8. *Assume that $\{\Gamma\langle f\rangle\}_{f\in\mathcal{F}}$ satisfies* (Cla)$_p$. *For any $f,g\in\mathcal{F}$ and any $\mathcal{B}$-measurable functions $u,v\colon X\to[0,\infty]$,*

$$\int_X uv\,d\,|\Gamma\langle f;g\rangle| \le \left(\int_X u^{\frac{p}{p-1}}\,d\Gamma\langle f\rangle\right)^{(p-1)/p}\left(\int_X v^p\,d\Gamma\langle g\rangle\right)^{1/p}. \tag{4.13}$$

Proof. Let $X=\mathcal{P}\sqcup\mathcal{N}$ be the Hahn decomposition with respect to $\Gamma\langle f;g\rangle$ such that $\Gamma\langle f;g\rangle(A)\ge 0$ for any Borel set $A\subseteq\mathcal{P}$ and $\Gamma\langle f;g\rangle(A)\le 0$ for any Borel set $A\subseteq\mathcal{N}$. Then the total variation measure $|\Gamma\langle f;g\rangle|$ is given by

$$|\Gamma\langle f;g\rangle|\,(A)=\Gamma\langle f;g\rangle(\mathcal{P}\cap A)-\Gamma\langle f;g\rangle(\mathcal{N}\cap A)\quad\text{for any } A\in\mathcal{B}.$$

Therefore, by (4.8),

$$\begin{aligned}|\Gamma\langle f;g\rangle|\,(A) &\le \Gamma\langle f\rangle(\mathcal{P}\cap A)^{(p-1)/p}\Gamma\langle g\rangle(\mathcal{P}\cap A)^{1/p}+\Gamma\langle f\rangle(\mathcal{N}\cap A)^{(p-1)/p}\Gamma\langle g\rangle(\mathcal{N}\cap A)^{1/p}\\ &\le \bigl(\Gamma\langle f\rangle(\mathcal{P}\cap A)+\Gamma\langle f\rangle(\mathcal{N}\cap A)\bigr)^{(p-1)/p}\bigl(\Gamma\langle g\rangle(\mathcal{P}\cap A)+\Gamma\langle g\rangle(\mathcal{N}\cap A)\bigr)^{1/p}\\ &=\Gamma\langle f\rangle(A)^{(p-1)/p}\Gamma\langle g\rangle(A)^{1/p}, \end{aligned}\tag{4.14}$$

where we used Hölder's inequality in the third line.

Now we prove (4.13). First, we consider the case that u and v are non-negative simple functions, that is,

$$u=\sum_{k=1}^{N_1}\widetilde{a}_k\mathbb{1}_{A_k},\quad v=\sum_{k=1}^{N_2}\widetilde{b}_k\mathbb{1}_{B_k},\ \text{where } \widetilde{a}_k,\widetilde{b}_k\in[0,\infty) \text{ and } A_k,B_k\in\mathcal{B}.$$

Then we can assume that there exist $N\in\mathbb{N}$, $\{a_k\}_{k=1}^N,\{b_k\}_{k=1}^N\subseteq[0,\infty)$ and a disjoint family of measurable sets $\{E_k\}_{k=1}^N\subseteq\mathcal{B}$ such that $u=\sum_{k=1}^N a_k\mathbb{1}_{E_k}$ and $v=\sum_{k=1}^N b_k\mathbb{1}_{E_k}$. Since $uv=\sum_{k=1}^N a_kb_k\mathbb{1}_{E_k}$, a combination of (4.14) and Hölder's inequality yields

$$\int_X uv\,d\,|\Gamma\langle f;g\rangle| = \sum_{k=1}^N a_kb_k\,|\Gamma\langle f;g\rangle(E_k)|$$

[7] In [Cap07], Harnack inequality for p-harmonic functions on *metric fractals* is proved by using p-energy measures $\Gamma\langle u;v\rangle$ (called the *measure-valued p-Lagrangian* and denoted by $\mathcal{L}^{(p)}(u,v)$ in [Cap07]). As a concrete example, the p-energy form on the Sierpiński gasket constructed in [HPS04] is discussed in [Cap07, Section 5], and it is stated that "we can define the corresponding Lagrangian $\mathcal{L}^{(p)}(u,v)$" in p. 1315 of that paper. However we have been unable to find in the literature a rigorous proof of the existence of the derivatives in [Cap07, p. 1315] defining $\mathcal{E}_g(u,v)$ and in [Cap07, p. 1303, (L5)] defining $\mathcal{L}^{(p)}(u,v)$ for the p-energy form on the Sierpinski gasket obtained in [HPS04].

$$\leq \left(\sum_{k=1}^{N} a_k^{p/(p-1)}\Gamma\langle f\rangle(E_k)\right)^{(p-1)/p} \left(\sum_{k=1}^{N} b_k^{p}\Gamma\langle g\rangle(E_k)\right)^{1/p}.$$

Hence, for any non-negative simple functions u and v, we have

$$\int_X uv\, d\,|\Gamma\langle f; g\rangle| \leq \left(\int_X u^{p/(p-1)}\, d\Gamma\langle f\rangle\right)^{(p-1)/p} \left(\int_X v^{p}\, d\Gamma\langle g\rangle\right)^{1/p}. \tag{4.15}$$

Next, suppose that u and v are non-negative $\mathcal{B}$-measurable functions and let $\{s_{n,w}\}_{n\geq 1}$ be sequences of non-negative simple functions such that $s_{n,w} \uparrow w$ m-a.e. as $n \to \infty$ for each $w \in \{u, v\}$. Then, by (4.15), for any $n \in \mathbb{N}$,

$$\int_X s_{n,u}s_{n,v}\, d\,|\Gamma\langle f; g\rangle| \leq \left(\int_X s_{n,u}^{p/(p-1)}\, d\Gamma\langle f\rangle\right)^{(p-1)/p} \left(\int_X s_{n,v}^{p}\, d\Gamma\langle g\rangle\right)^{1/p}.$$

It is clear that $\{s_{n,u}s_{n,v}\}_{n\geq 1}$ is a sequence of non-negative simple functions and $s_{n,u}s_{n,v} \uparrow uv$. Hence letting $n \to \infty$ in the inequality above yields (4.13). □

5 p-Energy measures associated with self-similar p-energy forms

In this section, we focus on the self-similar case. We will introduce the self-similarity for p-energy forms and construct p-energy measures with respect to self-similar p-energy forms. Some fundamental properties of p-energy measures will be shown.

5.1 Self-similar structure and related notions

We first recall standard notation and terminology on self-similar structures (see [Kig01, Chapter 1] for example). Throughout this section, we fix a compact metrizable space K, a finite set S with $\#S \geq 2$ and a continuous injective map $F_i \colon K \to K$ for each $i \in S$. We set $\mathcal{L} \coloneqq (K, S, \{F_i\}_{i\in S})$.

Definition 5.1. (1) Let $W_0 \coloneqq \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_n \coloneqq S^n = \{w_1 \dots w_n \mid w_i \in S \text{ for } i \in \{1,\dots,n\}\}$ for $n \in \mathbb{N}$ and let $W_* \coloneqq \bigcup_{n\in\mathbb{N}\cup\{0\}} W_n$. For $w \in W_*$, the unique $n \in \mathbb{N}\cup\{0\}$ with $w \in W_n$ is denoted by $|w|$ and called the *length of* w. For $w, v \in W_*$, $w = w_1 \dots w_{n_1}$, $v = v_1 \dots v_{n_2}$, we define $wv \in W_*$ by $wv \coloneqq w_1 \dots w_{n_1}v_1 \dots v_{n_2}$ ($w\emptyset \coloneqq w, \emptyset v \coloneqq v$).

(2) We set $\Sigma \coloneqq S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3 \dots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology of the discrete topology on S, and define the *shift map* $\sigma \colon \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3 \dots) \coloneqq \omega_2\omega_3\omega_4 \dots$. For $i \in S$ we define $\sigma_i \colon \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3 \dots) \coloneqq i\omega_1\omega_2\omega_3 \dots$. For $\omega = \omega_1\omega_2\omega_3 \dots \in \Sigma$ and $n \in \mathbb{N}\cup\{0\}$, we write $[\omega]_n \coloneqq \omega_1 \dots \omega_n \in W_n$.

(3) For $w = w_1 \dots w_n \in W_*$, we set $F_w \coloneqq F_{w_1} \circ \cdots \circ F_{w_n}$ ($F_\emptyset \coloneqq \mathrm{id}_K$), $K_w \coloneqq F_w(K)$, $\sigma_w \coloneqq \sigma_{w_1} \circ \cdots \circ \sigma_{w_n}$ ($\sigma_\emptyset \coloneqq \mathrm{id}_\Sigma$) and $\Sigma_w \coloneqq \sigma_w(\Sigma)$.

(4) A finite subset Λ of W_* is called a *partition* of Σ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \neq v$ and $\Sigma = \bigcup_{w \in \Lambda} \Sigma_w$.

Definition 5.2. $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\chi \colon \Sigma \to K$ such that $F_i \circ \chi = \chi \circ \sigma_i$ for any $i \in S$. Note that such χ, if it exists, is unique and satisfies $\{\chi(\omega)\} = \bigcap_{n \in \mathbb{N}} K_{[\omega]_n}$ for any $\omega \in \Sigma$.

In the next definition, we recall the definition of *post-critically finite self-similar structures* introduced by Kigami in [Kig93], which is mainly dealt with in Subsection 8.3.

Definition 5.3. Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure.

(1) We define the *critical set* $\mathcal{C}_{\mathcal{L}}$ and the *post-critical set* $\mathcal{P}_{\mathcal{L}}$ of $\mathcal{L}$ by

$$\mathcal{C}_{\mathcal{L}} \coloneqq \chi^{-1}\bigl(\textstyle\bigcup_{i,j \in S,\, i \neq j} K_i \cap K_j\bigr) \qquad \text{and} \qquad \mathcal{P}_{\mathcal{L}} \coloneqq \textstyle\bigcup_{n \in \mathbb{N}} \sigma^n(\mathcal{C}_{\mathcal{L}}). \tag{5.1}$$

$\mathcal{L}$ is called *post-critically finite*, or *p.-c.f.* for short, if and only if $\mathcal{P}_{\mathcal{L}}$ is a finite set.

(2) We set $V_0 \coloneqq \chi(\mathcal{P}_{\mathcal{L}})$, $V_n \coloneqq \bigcup_{w \in W_n} F_w(V_0)$ for $n \in \mathbb{N}$ and $V_* \coloneqq \bigcup_{n \in \mathbb{N} \cup \{0\}} V_n$.

The set V_0 should be considered as the *"boundary"* of the self-similar set K; indeed, by [Kig01, Proposition 1.3.5-(2)], we have

$$K_w \cap K_v = F_w(V_0) \cap F_v(V_0) \text{ for any } w, v \in W_* \text{ with } \Sigma_w \cap \Sigma_v = \emptyset. \tag{5.2}$$

According to [Kig01, Lemma 1.3.11], $V_{n-1} \subseteq V_n$ for any $n \in \mathbb{N}$, and V_* is dense in K if $V_0 \neq \emptyset$.

The family of cells $\{K_w\}_{w \in W_*}$ describes the local topology of a self-similar structure. Indeed, $\{K_{n,x}\}_{n \geq 0}$, where $K_{n,x} \coloneqq \bigcup_{w \in W_n; x \in K_w} K_w$, forms a fundamental system of neighborhoods of $x \in K$ [Kig01, Proposition 1.3.6]. Moreover, the proof of [Kig01, Proposition 1.3.6] implies that any metric d on K giving the original topology of K satisfies

$$\lim_{n \to \infty} \max_{w \in W_n} \operatorname{diam}(K_w, d) = 0. \tag{5.3}$$

Let us recall the notion of self-similar measures.

Definition 5.4 (Self-similar measures)**.** Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure and let $(\theta_i)_{i \in S} \in (0,1)^S$ satisfy $\sum_{i \in S} \theta_i = 1$. A Borel probability measure m on K is said to be a *self-similar measure on* $\mathcal{L}$ *with weight* $(\theta_i)_{i \in S}$ if and only if the following equality (of Borel measures on K) holds:

$$m = \sum_{i \in S} \theta_i (F_i)_* m. \tag{5.4}$$

Proposition 5.5 ([Kig01, Section 1.4] and [Kig09, Theorem 1.2.7])**.** *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure and let $(\theta_i)_{i \in S} \in (0,1)^S$ satisfy $\sum_{i \in S} \theta_i = 1$. Then there exists a self-similar measure m on $\mathcal{L}$ with weight $(\theta_i)_{i \in S}$. If $K \neq \overline{V_0}^K$, then $m(K_w) = \theta_w$ and $m(F_w(\overline{V_0}^K)) = 0$ for any $w \in W_*$, where $\theta_w \coloneqq \theta_{w_1} \cdots \theta_{w_n}$ for $w = w_1 \cdots w_n \in W_*$ ($\theta_\emptyset \coloneqq 1$).*

5.2 Self-similar p-energy forms and p-energy measures

In this subsection, we introduce the self-similarity for p-energy forms on self-similar structures and define the p-energy measures associated with a given self-similar p-energy form. In the rest of this subsection, we fix a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$, a σ-algebra $\mathcal{B}$ which contains $\mathcal{B}(K)$, a measure m on $\mathcal{B}$ with $m(O) > 0$ for any non-empty open subset O of K, $p \in (1,\infty)$ and a p-energy form $(\mathcal{E},\mathcal{F})$ on (K,m) with $\mathcal{F} \subseteq L^0(K,m)$.

Definition 5.6 (Self-similar p-energy form)**.** Let $\rho = (\rho_i)_{i\in S} \in (0,\infty)^S$. A p-energy form $(\mathcal{E},\mathcal{F})$ on (K,m) is said to be *self-similar on* $(\mathcal{L},m)$ *with weight* ρ if and only if the following hold:

$$\mathcal{F} \cap C(K) = \{f \in C(K) \mid f \circ F_i \in \mathcal{F}_p \text{ for any } i \in S\}, \tag{5.5}$$

$$\mathcal{E}(f) = \sum_{i\in S} \rho_i \mathcal{E}(f \circ F_i) \quad \text{for any } u \in \mathcal{F} \cap C(K). \tag{5.6}$$

Note that for any partition Λ of Σ, (5.6) implies

$$\mathcal{E}(f) = \sum_{w\in\Lambda} \rho_w \mathcal{E}(f \circ F_w), \quad u \in \mathcal{F} \cap C(K), \tag{5.7}$$

where $\rho_w := \rho_{w_1} \cdots \rho_{w_n}$ for $w = w_1 \dots w_n \in W_*$. Indeed, (5.7) follows from an induction with respect to $\max_{w\in\Lambda} |w|$.

In the rest of this subsection, we assume that $(\mathcal{E},\mathcal{F})$ is a self-similar p-energy form on $\mathcal{L}$ with weight $\rho = (\rho_i)_{i\in S}$. We can see that the two-variable version $\mathcal{E}(f;g)$ also has the following self-similarity.

Proposition 5.7. *Assume that* $(\mathcal{E},\mathcal{F} \cap C(K))$ *satisfies* (Cla)$_p$. *Then*

$$\mathcal{E}(f;g) = \sum_{i\in S} \rho_i \mathcal{E}(f \circ F_i; g \circ F_i) \quad \textit{for any } f,g \in \mathcal{F} \cap C(K). \tag{5.8}$$

Proof. For any $f,g \in \mathcal{F} \cap C(K)$ and $t > 0$, we have

$$\frac{\mathcal{E}(f+tg) - \mathcal{E}(f)}{t} = \sum_{i\in S} \rho_i \frac{\mathcal{E}\bigl(f \circ F_i + t(g \circ F_i)\bigr) - \mathcal{E}(f \circ F_i)}{t}.$$

Letting $t \downarrow 0$ yields (5.8). □

Next we will see that p-energy measures are naturally introduced by virtue of the self-similarity of $(\mathcal{E},\mathcal{F})$ (see also [Hin05, MS23+]). For $f \in \mathcal{F} \cap C(K)$, we define a finite measure $\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f\rangle$ on $W_n = S^n$ by putting $\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f\rangle(\{w\}) := \rho_w \mathcal{E}(f \circ F_w)$ for each $w \in W_n$. Then, by (5.7), $\{\mathfrak{m}_{\mathcal{E}}^{(n)}\langle f\rangle\}_{n\geq 0}$ satisfies the consistency condition and hence Kolmogorov's extension theorem yields a measure $\mathfrak{m}_{\mathcal{E}}\langle f\rangle$ on $\Sigma = S^{\mathbb{N}}$ such that $\mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma_w) = \rho_w \mathcal{E}(f \circ F_w)$

for any $w \in W_*$. In particular, $\mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma) = \mathcal{E}(f)$. We now define a finite Borel measure $\Gamma_{\mathcal{E}}\langle f\rangle$ on K by

$$\Gamma_{\mathcal{E}}\langle f\rangle(A) \coloneqq \mathfrak{m}_{\mathcal{E}}\langle f\rangle \circ \chi^{-1}(A) \coloneqq \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\chi^{-1}(A)), \quad A \in \mathcal{B}(K) \tag{5.9}$$

where $\chi\colon \Sigma \to K$ is the map in Definition 5.2.

The next proposition states basic properties and the self-similarity of $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$.

Proposition 5.8. *Let $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ be the measures defined by (5.9).*

(a) *$\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies $\Gamma_{\mathcal{E}}\langle f\rangle(K) = \mathcal{E}(f)$, especially $(\mathrm{EM1})_p$, and $(\mathrm{EM2})_p$.*

(b) *Assume that $(\mathcal{E}, \mathcal{F}\cap C(K))$ satisfies $(\mathrm{GC})_p$ and let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$. Then for any $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1), any $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \bigl(\mathcal{F}\cap C(K)\bigr)^{n_1}$ and any Borel measurable function $\varphi\colon K \to [0,\infty]$, we have*

$$\left\| \left(\left(\int_K \varphi \, d\Gamma_{\mathcal{E}}\langle T_l(\boldsymbol{u})\rangle \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\left(\int_K \varphi \, d\Gamma_{\mathcal{E}}\langle u_k\rangle \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{5.10}$$

In particular Proposition 2.2 with $(\int_K \varphi \, d\Gamma_{\mathcal{E}}\langle\,\cdot\,\rangle, \mathcal{F}\cap C(K))$ in place of $(\mathcal{E},\mathcal{F})$ holds provided $\|\varphi\|_{\sup} < \infty$.

(c) *The following equality holds:*

$$\Gamma_{\mathcal{E}}\langle f\rangle = \sum_{i\in S} \rho_i \Gamma_{\mathcal{E}}\langle f\circ F_i\rangle \circ F_i^{-1} \quad \text{for any } f \in \mathcal{F}\cap C(K). \tag{5.11}$$

(d) *Assume that $(\mathcal{E}, \mathcal{F}\cap C(K))$ satisfies $(\mathrm{Cla})_p$. Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ also satisfies $(\mathrm{Cla})_p$ and*

$$\Gamma_{\mathcal{E}}\langle f; g\rangle = \sum_{i\in S} \rho_i \Gamma_{\mathcal{E}}\langle f\circ F_i; g\circ F_i\rangle \circ F_i^{-1} \quad \text{for any } f, g \in \mathcal{F}\cap C(K). \tag{5.12}$$

Proof. (a): We easily have $\Gamma_{\mathcal{E}}(K) = \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\chi^{-1}(K)) = \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma) = \mathcal{E}(f)$. The proof of $(\mathrm{EM2})_p$ will be included in the proof of (b) below.

(b): Assume that $(\mathcal{E},\mathcal{F})$ satisfies $(\mathrm{GC})_p$. Let us fix $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1) and $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \bigl(\mathcal{F}\cap C(K)\bigr)^{n_1}$. We first show that, for any $A \in \mathcal{B}(\Sigma)$,

$$\left\| \left(\mathfrak{m}_{\mathcal{E}}\langle T_l(\boldsymbol{u})\rangle(A)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\mathfrak{m}_{\mathcal{E}}\langle u_k\rangle(A)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}; \tag{5.13}$$

in particular, $(\mathfrak{m}_{\mathcal{E}}\langle\,\cdot\,\rangle(A), \mathcal{F}\cap C(K))$ is a p-energy form on (K,m) satisfying $(\mathrm{GC})_p$. Note that the case $A = \Sigma_w$ for some $w \in W_*$ is clearly true by $(\mathrm{GC})_p$ for $(\mathcal{E},\mathcal{F})$. By a similar argument using the reverse Minkowski inequality on $\ell^{q_1/p}$ and the Minkowski inequality on $\ell^{q_2/p}$ as in (2.18), (5.13) holds on the finitely additive class generated by $\{\Sigma_w\}_{w\in W_*}$. Hence the monotone class theorem implies that (5.13) holds for any $A \in \mathcal{B}(\Sigma)$.

In particular, by choosing $A = \chi^{-1}(B)$ where $B \in \mathcal{B}(K)$, we obtain

$$\left\| \left(\Gamma_{\mathcal{E}}\langle T_l(\boldsymbol{u})\rangle(B)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left(\Gamma_{\mathcal{E}}\langle u_k\rangle(B)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}, \quad B \in \mathcal{B}(K). \tag{5.14}$$

Again by a similar argument as in (2.18), we see that (5.10) holds for any non-negative Borel measurable simple function φ on K. We get the desired extension, (5.10) for any Borel measurable function $\varphi\colon K \to [0,\infty]$, by the monotone convergence theorem.

(c): The proof is very similar to [Shi24, Proof of Theorem 7.5]. Let $k \in \mathbb{N}$, $w = w_1 \dots w_k \in W_k$ and $n \in \mathbb{N}$. We see that

$$\begin{aligned}\sum_{i\in S}\rho_i\mathfrak{m}_{\mathcal{E}}\langle f\circ F_i\rangle(\sigma_i^{-1}(\Sigma_w)) &= \rho_{w_1}\mathfrak{m}_{\mathcal{E}}\langle f\circ F_{w_1}\rangle(\sigma_{w_1}^{-1}(\Sigma_w)) = \rho_{w_1}\mathfrak{m}_{\mathcal{E}}\langle f\circ F_{w_1}\rangle(\Sigma_{w_2\dots w_k})\\ &= \rho_{w_1}\rho_{w_2\dots w_k}\mathcal{E}((f\circ F_{w_1})\circ F_{w_2\dots w_k}) = \mathfrak{m}_{\mathcal{E}}\langle f\rangle(\Sigma_w)\end{aligned}$$

Since $w \in W_*$ is arbitrary, by Dynkin's π-λ theorem, we deduce that

$$\mathfrak{m}_{\mathcal{E}}\langle f\rangle(A) = \sum_{i\in S}\rho_i\mathfrak{m}_{\mathcal{E}}\langle f\circ F_i\rangle\circ\sigma_i^{-1}(A), \quad A \in \mathcal{B}(\Sigma).$$

We obtain (5.11) by $\chi\circ\sigma_i = F_i\circ\chi$.

(d): Assume that $(\mathcal{E},\mathcal{F})$ satisfies (Cla)$_p$. Then $\{\Gamma_{\mathcal{E}}\langle f\rangle\}_{f\in\mathcal{F}\cap C(K)}$ satisfies (Cla)$_p$ by (5.14) (see also Proposition 2.2-(e),(f)). Now we obtain (5.12) by letting $t \downarrow 0$ in

$$\Gamma_{\mathcal{E}}\langle f+tg\rangle(A) = \sum_{i\in S}\rho_i\Gamma_{\mathcal{E}}\langle f\circ F_i + t(g\circ F_i)\rangle\bigl(F_i^{-1}(A)\bigr). \qquad \square$$

The next theorem states a chain rule for the signed measure $\Gamma_{\mathcal{E}}\langle\Phi(u);\Psi(v)\rangle$. Such a chain rule is also obtained in [BV05], but we provide here a self-contained proof because the framework of [BV05] is different from ours.

Theorem 5.9 (Chain rule)**.** *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$ and that $(\mathcal{E},\mathcal{F}\cap C(K))$ satisfies (2.3), (2.6) and (Cla)$_p$. Let $n \in \mathbb{N}$, $u \in \mathcal{F}\cap C(K)$, $\boldsymbol{v} = (v_1,\dots,v_n) \in \bigl(\mathcal{F}\cap C(K)\bigr)^n$, $\Phi \in C^1(\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^n)$. Then $\Phi(u),\Psi(\boldsymbol{v}) \in \mathcal{F}$ and*

$$d\Gamma_{\mathcal{E}}\langle\Phi(u);\Psi(\boldsymbol{v})\rangle = \sum_{k=1}^{n}\operatorname{sgn}\bigl(\Phi'(u)\bigr)\left|\Phi'(u)\right|^{p-1}\partial_k\Psi(\boldsymbol{v})\,d\Gamma_{\mathcal{E}}\langle u;v_k\rangle. \tag{5.15}$$

In particular, the following Leibniz rule *holds: for any $u \in \mathcal{F}$ and any $v,w \in \mathcal{F}\cap C(K)$,*

$$d\Gamma_{\mathcal{E}}\langle u;vw\rangle = v\,d\Gamma_{\mathcal{E}}\langle u;w\rangle + w\,d\Gamma_{\mathcal{E}}\langle u;v\rangle. \tag{5.16}$$

Proof. We easily obtain $\Phi(u),\Psi(\boldsymbol{v}) \in \mathcal{F}$ by Corollary 2.4-(a) and $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$. To show (5.15), we will prove

$$\lim_{l\to\infty}\left|\rho_w\mathcal{E}\bigl(\Phi(u\circ F_w);\Psi(\boldsymbol{v}\circ F_w)\bigr) - \mathcal{S}_l(w)\right| = 0 \quad \text{for any } w \in W_*, \tag{5.17}$$

where $x_0 \in K$ is fixed and

$$\mathcal{S}_l(w) \coloneqq \sum_{\tau\in W_l}\rho_{w\tau}\mathcal{E}\Biggl(\Phi'(u\circ F_{w\tau}(x_0))\cdot(u\circ F_{w\tau});\sum_{k=1}^{n}\partial_k\Psi(v\circ F_{w\tau}(x_0))\cdot(v_k\circ F_{w\tau})\Biggr), \quad l \in \mathbb{N}\cup\{0\}.$$

We need some preparations to prove (5.17). Note that, for any $z \in W_*$ and $x \in K$,

$$\begin{aligned}
&\Phi\bigl(u(F_z(x))\bigr) - \Phi\bigl(u(F_z(x_0))\bigr) \\
&= \bigl[u(F_z(x)) - u(F_z(x_0))\bigr]\Biggl(\Phi'\bigl(u(F_z(x_0))\bigr) \\
&\qquad + \int_0^1 \Bigl[\Phi'\Bigl(u(F_z(x_0)) + t\bigl(u(F_z(x)) - u(F_z(x_0))\bigr)\Bigr) - \Phi'\bigl(u(F_z(x_0))\bigr)\Bigr]\,dt\Biggr).
\end{aligned}$$

In particular,

$$\Phi(u \circ F_z) - \widehat{u}_z = \Phi\bigl(u(F_z(x_0)\bigr) - \Phi'\bigl(u(F_z(x_0))\bigr)u(F_z(x_0)) + D_z I_z,$$

where $\widehat{u}_z, D_z, I_z \in C(K)$ are given by

$$\begin{aligned}
\widehat{u}_z(x) &:= \Phi'\bigl(u(F_z(x_0))\bigr) \cdot (u \circ F_z)(x), \\
D_z(x) &:= u(F_z(x)) - u(F_z(x_0)), \\
I_z(x) &:= \int_0^1 \Bigl[\Phi'\bigl(u(F_z(x_0)) + tD_z(x)\bigr) - \Phi'\bigl(u(F_z(x_0))\bigr)\Bigr]\,dt, \quad x \in K.
\end{aligned}$$

Hence we have $\bigl|\rho_w \mathcal{E}\bigl(\Phi(u \circ F_w); \Psi(v \circ F_w)\bigr) - \mathcal{S}_l(w)\bigr| \le A_{1,l} + A_{2,l}$, where

$$\begin{aligned}
\widehat{v}_z(x) &:= \sum_{k=1}^{n} \partial_k \Psi\bigl(\boldsymbol{v}(F_z(x_0))\bigr) \cdot (v_k \circ F_z)(x) \quad \text{for } z \in W_*, x \in K, \\
A_{1,l} &:= \sum_{\tau \in W_l} \rho_{w\tau} \left|\mathcal{E}\bigl(\Phi(u \circ F_{w\tau}); \Psi(\boldsymbol{v} \circ F_{w\tau})\bigr) - \mathcal{E}\bigl(\Phi(u \circ F_{w\tau}); \widehat{v}_{w\tau}\bigr)\right|, \\
A_{2,l} &:= \sum_{\tau \in W_l} \rho_{w\tau} \left|\mathcal{E}\bigl(\Phi(u \circ F_{w\tau}); \widehat{v}_{w\tau}\bigr) - \mathcal{E}\bigl(\widehat{u}_{w\tau}; \widehat{v}_{w\tau}\bigr)\right|.
\end{aligned}$$

(Note that $\widehat{u}_z, \widehat{v}_z \in \mathcal{F}$ by (5.5).) Next we show $\lim_{l\to\infty} A_{i,l} = 0$ to obtain (5.17). By Corollary 2.4-(a), $I_z \in \mathcal{F}$ and there exists a constant $C_{u,\Phi} \in (0,\infty)$ depending only on $p, \|u\|_{\sup}, \|\Phi'\|_{\sup,[-2\|u\|_{\sup},2\|u\|_{\sup}]}$ such that $\mathcal{E}(I_z) \le C_{u,\Phi}\mathcal{E}(u \circ F_z)$ and $\mathcal{E}\bigl(\Phi(u \circ F_z)\bigr) \le C_{u,\Phi}\mathcal{E}(u \circ F_z)$. Therefore, for any $l \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}\bigl(\Phi(u \circ F_{w\tau}) - \widehat{u}_w\bigr) \\
&= \sum_{\tau \in W_l} \rho_{w\tau} \mathcal{E}(D_w I_w) \\
&\le 2^{p-1} \sum_{\tau \in W_l} \rho_{w\tau} \Bigl(\|I_w\|_{\sup}^p \mathcal{E}(D_w) + \|D_w\|_{\sup}^p \mathcal{E}(I_w)\Bigr) \\
&\le 2^{p-1} \left(\max_{\tau' \in W_l} \|I_{w\tau'}\|_{\sup}^p + \max_{\tau' \in W_l} \|D_{w\tau'}\|_{\sup}^p\right) \sum_{\tau \in W_l} \rho_{w\tau} \Bigl(\mathcal{E}(D_{w\tau}) + C_{u,\Phi}\mathcal{E}(u \circ F_{w\tau})\Bigr)
\end{aligned}$$

$$\leq 2^{p-1}(1 + C_{u,\Phi})\mathcal{E}(u)\biggl(\max_{\tau' \in W_l} \|I_{w\tau'}\|_{\sup}^p + \max_{\tau' \in W_l} \|D_{w\tau'}\|_{\sup}^p\biggr).$$

Since u and Φ are uniformly continuous on K, we have from (5.3) that both $\max_{\tau' \in W_l} \|I_{w\tau'}\|_{\sup}$ and $\max_{\tau' \in W_l} \|D_{w\tau'}\|_{\sup}$ converge to 0 as $l \to \infty$, and hence

$$\lim_{l \to \infty} \sum_{\tau \in W_l} \rho_{w\tau}\mathcal{E}\bigl(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau}\bigr) = 0. \tag{5.18}$$

Similarly, we can show that

$$\lim_{l \to \infty} \sum_{\tau \in W_l} \rho_{w\tau}\mathcal{E}\bigl(\Psi(\boldsymbol{v} \circ F_{w\tau}) - \widehat{v}_{w\tau}\bigr) = 0. \tag{5.19}$$

Then, by (3.11), (3.12) and Hölder's inequality, we have

$$A_{1,l} \lesssim \mathcal{E}(u \circ F_w)^{(p-1)/p}\left(\sum_{\tau \in W_l} \rho_{w\tau}\mathcal{E}(\Psi(\boldsymbol{v} \circ F_{w\tau}) - \widehat{v}_{w\tau})\right)^{1/p},$$

and

$$\begin{aligned}
A_{2,l} &\lesssim \sum_{\tau \in W_l} \rho_{w\tau}\mathcal{E}(u \circ F_{w\tau})^{(p-1-\alpha_p)/p}\mathcal{E}\bigl(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau}\bigr)^{\alpha_p/p}\mathcal{E}\bigl(\widehat{v}_{w\tau}\bigr)^{1/p} \\
&\leq \mathcal{E}(u \circ F_w)^{(p-1-\alpha_p)/p}\left(\sum_{\tau \in W_l} \rho_{w\tau}\mathcal{E}(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau})\right)^{\alpha_p/p}\left(\sum_{\tau \in W_l} \rho_{w\tau}\mathcal{E}\bigl(\widehat{v}_{w\tau}\bigr)\right)^{1/p} \\
&\lesssim \mathcal{E}(u \circ F_w)^{(p-1-\alpha_p)/p}\left(\sum_{\tau \in W_l} \rho_{w\tau}\mathcal{E}(\Phi(u \circ F_{w\tau}) - \widehat{u}_{w\tau})\right)^{\alpha_p/p}\max_{k \in \{1,\dots,n\}} \mathcal{E}(v_k \circ F_w)^{1/p}.
\end{aligned}$$

Combining these estimates with (5.18) and (5.19), we obtain $\lim_{l \to \infty} A_{i,l} = 0$ and thus (5.17) holds.

To complete the proof, recall that for $f, g \in \mathcal{F}$ and $A \in \mathcal{B}(\Sigma)$,

$$\mathfrak{m}_{\mathcal{E}}\langle f; g\rangle(A) \coloneqq \frac{1}{p}\frac{d}{dt}\mathfrak{m}_{\mathcal{E}}\langle f + tg\rangle(A)\biggr|_{t=0}, \tag{5.20}$$

which exists by Proposition 3.5 and (5.13). Then $\mathfrak{m}_{\mathcal{E}}\langle f; g\rangle$ is a signed measure on $(\Sigma, \mathcal{B}(\Sigma))$ by Proposition 4.6. By (5.9), (4.5) and (5.20), we immediately have $\mathfrak{m}_{\mathcal{E}}\langle f; g\rangle \circ \chi^{-1} = \Gamma_{\mathcal{E}}\langle f; g\rangle$ and $\mathfrak{m}_{\mathcal{E}}\langle f; g\rangle(\Sigma_w) = \rho_w\mathcal{E}(f \circ F_w; g \circ F_w)$ for any $w \in W_*$. In particular, to show (5.15), it suffices to prove

$$d\mathfrak{m}_{\mathcal{E}}\langle \Phi(u); \Psi(\boldsymbol{v})\rangle = \sum_{k=1}^{n} \operatorname{sgn}\bigl(\Phi'(u \circ \chi)\bigr)\,|\Phi'(u \circ \chi)|^{p-1}\,\partial_k\Psi(\boldsymbol{v} \circ \chi)\,d\mathfrak{m}_{\mathcal{E}}\langle u; v_k\rangle. \tag{5.21}$$

Note that

$$\lim_{l \to \infty}\left|\sum_{k=1}^{n}\int_{\Sigma_w} \operatorname{sgn}\bigl(\Phi'(u \circ \chi)\bigr)\,|\Phi'(u \circ \chi)|^{p-1}\,\partial_k\Psi(\boldsymbol{v} \circ \chi)\,d\mathfrak{m}_{\mathcal{E}}\langle u; v_k\rangle - \mathcal{S}_l(w)\right| = 0,$$

whence (5.17) together with the Dynkin class theorem yields (5.21).

The Leibniz rule (5.16) is immediate from $vw = \frac{1}{4}\bigl[(v+w)^2 - (v-w)^2\bigr]$ and (5.15). The proof is completed. □

In the case $n = 1$, $\Psi = \Phi$ and $v_1 = u$ in the theorem above, by noting that the proof of (5.15) does not need $(\mathrm{Cla})_p$ for $(\mathcal{E}, \mathcal{F})$, we get the following corollary.

Corollary 5.10. *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$ and that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies (2.3) and (2.6). Let $u \in \mathcal{F} \cap C(K)$ and $\Phi \in C^1(\mathbb{R})$. Then $\Phi(u) \in \mathcal{F}$ and*

$$d\Gamma_{\mathcal{E}}\langle \Phi(u) \rangle = |\Phi'(u)|^p \, d\Gamma_{\mathcal{E}}\langle u \rangle. \tag{5.22}$$

We also have the following representation formula (see also [Cap03, Theorem 4.1]).

Proposition 5.11 (Representation formula)**.** *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$ and that $(\mathcal{E}, \mathcal{F})$ satisfies (2.3), (2.6) and $(\mathrm{Cla})_p$. For any $u, \varphi \in \mathcal{F} \cap C(K)$,*

$$\int_X \varphi \, d\Gamma_{\mathcal{E}}\langle u \rangle = \mathcal{E}(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}\bigl(|u|^{\frac{p}{p-1}}; \varphi\bigr). \tag{5.23}$$

Proof. Define $\Phi \in C^1(\mathbb{R})$ by $\Phi(x) \coloneqq |x|^{p/(p-1)}$. Note that $\Phi'(x) = \frac{p}{p-1}\operatorname{sgn}(x)\,|x|^{1/(p-1)}$. By Theorem 5.9, we see that

$$\begin{aligned}
&\mathcal{E}(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}(\Phi(u); \varphi) \\
&= \int_K u \, d\Gamma_{\mathcal{E}}\langle u; \varphi \rangle + \int_K \varphi \, d\Gamma_{\mathcal{E}}\langle u \rangle - \left(\frac{p-1}{p}\right)^{p-1} \int_K \operatorname{sgn}\bigl(\Phi'(u)\bigr)\, |\Phi'(u)|^{p-1} \, d\Gamma_{\mathcal{E}}\langle u; \varphi \rangle \\
&= \int_K u \, d\Gamma_{\mathcal{E}}\langle u; \varphi \rangle + \int_K \varphi \, d\Gamma_{\mathcal{E}}\langle u \rangle - \left(\frac{p-1}{p}\right)^{p-1} \left(\frac{p}{p-1}\right)^{p-1} \int_K \operatorname{sgn}(u)\, |u| \, d\Gamma_{\mathcal{E}}\langle u; \varphi \rangle \\
&= \int_K \varphi \, d\Gamma_{\mathcal{E}}\langle u \rangle.
\end{aligned}$$

□

The following *image density property* is an important consequence of the chain rule (5.22).

Theorem 5.12 (Image density property)**.** *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$, that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies (2.3) and (2.6), and that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Then, for any $u \in \mathcal{F} \cap C(K)$, the Borel measure $\Gamma_{\mathcal{E}}\langle u \rangle \circ u^{-1}$ on $\mathbb{R}$ defined by $\Gamma_{\mathcal{E}}\langle u \rangle \circ u^{-1}(A) \coloneqq \Gamma_{\mathcal{E}}\langle u \rangle(u^{-1}(A))$, $A \in \mathcal{B}(\mathbb{R})$, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. Note that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is reflexive by Proposition 3.10. Then this is proved in exactly the same way as [Shi24, Proposition 7.6], which is a simple adaptation of [CF, Theorem 4.3.8]. □

The next theorem gives arguably the strongest possible forms of the strong locality.

Theorem 5.13 (Strong locality of energy measures)**.** *Assume that $\mathbb{R}\mathbb{1}_K \subseteq \mathcal{E}^{-1}(0)$, that $(\mathcal{E}, \mathcal{F} \cap C(K))$ satisfies (2.3), (2.6) and (Cla)$_p$, and that $(\mathcal{F}, \|\cdot\|_{\mathcal{E},1})$ is a Banach space. Let $u, u_1, u_2, v \in \mathcal{F} \cap C(K)$, $a, a_1, a_2, b \in \mathbb{R}$ and $A \in \mathcal{B}(K)$.*

(a) *If $A \subseteq u^{-1}(a)$, then $\Gamma_{\mathcal{E}}\langle u\rangle(A) = 0$.*

(b) *If $A \subseteq (u - v)^{-1}(a)$, then $\Gamma_{\mathcal{E}}\langle u\rangle(A) = \Gamma_{\mathcal{E}}\langle v\rangle(A)$.*

(c) *If $A \subseteq u_1^{-1}(a_1) \cup u_2^{-1}(a_2)$, then*

$$\Gamma_{\mathcal{E}}\langle u_1 + u_2 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A), \tag{5.24}$$

$$\Gamma_{\mathcal{E}}\langle u_1 + u_2; v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1; v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2; v\rangle(A). \tag{5.25}$$

(d) *If $A \subseteq (u_1 - u_2)^{-1}(a) \cup v^{-1}(b)$, then*

$$\Gamma_{\mathcal{E}}\langle u_1; v\rangle(A) = \Gamma_{\mathcal{E}}\langle u_2; v\rangle(A) \quad \text{and} \quad \Gamma_{\mathcal{E}}\langle v; u_1\rangle(A) = \Gamma_{\mathcal{E}}\langle v; u_2\rangle(A). \tag{5.26}$$

Proof. (a): This is immediate from Theorem 5.12.

(b): This follows from (a) and the triangle inequality for $\Gamma_{\mathcal{E}}\langle\,\cdot\,\rangle(A)^{1/p}$.

(c): Set $A_i \coloneqq A \cap u_i^{-1}(a_i)$, $i \in \{1, 2\}$. We see from (b) that

$$\begin{aligned}
&\Gamma_{\mathcal{E}}\langle u_1 + u_2 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle v\rangle(A)\\
&= \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A_1) + \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A_2) + \Gamma_{\mathcal{E}}\langle v\rangle(A)\\
&= \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A_1) + \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A_2) + \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A_1) + \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A_2)\\
&= \Gamma_{\mathcal{E}}\langle u_1 + v\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2 + v\rangle(A),
\end{aligned}$$

which proves (5.24). Note that $\Gamma_{\mathcal{E}}\langle u_1 + u_2\rangle(A) = \Gamma_{\mathcal{E}}\langle u_1\rangle(A) + \Gamma_{\mathcal{E}}\langle u_2\rangle(A)$ by (5.24) in the case $v = 0$. By using this equality and applying (5.24) with v replaced by tv for $t \in (0, \infty)$, we have

$$\begin{aligned}
&\frac{\Gamma_{\mathcal{E}}\langle u_1 + u_2 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_1 + u_2\rangle(A)}{t} + t^{p-1}\Gamma_{\mathcal{E}}\langle v\rangle(A)\\
&= \frac{\Gamma_{\mathcal{E}}\langle u_1 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_1\rangle(A)}{t} + \frac{\Gamma_{\mathcal{E}}\langle u_2 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_2\rangle(A)}{t},
\end{aligned}$$

which implies (5.25) by letting $t \downarrow 0$.

(d): The proof will be very similar to that of Proposition 3.26-(a). By applying (5.24) with $u_2 - u_1, tv, u_1$ for $t \in (0, \infty)$ in place of u_1, u_2, v, we have

$$\frac{\Gamma_{\mathcal{E}}\langle u_1 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_1\rangle(A)}{t} = \frac{\Gamma_{\mathcal{E}}\langle u_2 + tv\rangle(A) - \Gamma_{\mathcal{E}}\langle u_2\rangle(A)}{t},$$

which implies the former equality in (5.26) by letting $t \downarrow 0$. This equality in turn with $v, 0, u_1 - u_2$ in place of u_1, u_2, v yields the latter equality in (5.26) by the linearity of $\Gamma_{\mathcal{E}}\langle v; \cdot\rangle(A)$. $\square$

6 p-Resistance forms and nonlinear potential theory

In this section, we will introduce the notion of p-resistance form as a special class of p-energy forms, and investigate harmonic functions with respect to a p-resistance form. In particular, a theory on taking the operation of traces of p-resistance forms and weak comparison principle for harmonic function are developed. We also introduce the notion of p-resistance metric with respect to a given p-resistance form.

Throughout this section, we fix $p \in (1, \infty)$, a non-empty set X, a linear subspace $\mathcal{F}$ of $\mathbb{R}^X$ and $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$.

6.1 Basics of p-resistance forms

The next definition is a L^p-analog of the notion of a *resistance form*, which is established by Kigami [Kig01, Kig12].

Definition 6.1 (p-Resistance form)**.** The pair $(\mathcal{E}, \mathcal{F})$ of $\mathcal{F} \subseteq \mathbb{R}^X$ and $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$ is said to be a *p-resistance form* on X if and only if it satisfies the following conditions $(\mathrm{RF1})_p$-$(\mathrm{RF5})_p$:

$(\mathrm{RF1})_p$ $\mathcal{F}$ is a linear subspace of $\mathbb{R}^X$ containing $\mathbb{R}\mathbb{1}_X$ and $\mathcal{E}(\,\cdot\,)^{1/p}$ is a seminorm on $\mathcal{F}$ satisfying $\{u \in \mathcal{F} \mid \mathcal{E}(u) = 0\} = \mathbb{R}\mathbb{1}_X$.

$(\mathrm{RF2})_p$ The quotient normed space $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ is a Banach space.

$(\mathrm{RF3})_p$ If $x \neq y \in X$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

$(\mathrm{RF4})_p$ For any $x, y \in X$,

$$R_{\mathcal{E}}(x, y) \coloneqq R_{(\mathcal{E},\mathcal{F})}(x, y) \coloneqq \sup\left\{ \frac{|u(x) - u(y)|^p}{\mathcal{E}(u)} \;\middle|\; u \in \mathcal{F} \setminus \mathbb{R}\mathbb{1}_X \right\} < \infty. \tag{6.1}$$

$(\mathrm{RF5})_p$ $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_p$.

Remark 6.2. (1) The notion of 2-resistance form coincides with the original notion of resistance form (see, e.g., [Kig01, Definition 2.3.1]) although the condition $(\mathrm{RF5})_2$ is stronger than (RF5) in [Kig01, Definition 2.3.1]. Indeed, we can obtain $(\mathrm{RF5})_2$ by [Kig12, Theorem 3.14] and the explicit definition of $\mathcal{E}_{L_m}$ in [Kig12, Proposition 3.8].

(2) Let $(\mathcal{E}, \mathcal{F})$ be a p-resistance form on a finite set V. Then $\mathcal{F} = \mathbb{R}^V$ by $(\mathrm{RF1})_p$, $(\mathrm{RF3})_p$ and $(\mathrm{RF5})_p$ (see also [Kig12, Proposition 3.2]), so we say simply that $\mathcal{E}$ is a p-resistance form on V if V is a finite set.

Example 6.3. (1) Consider the same setting as in Example 3.8-(1) and suppose that Ω is a bounded domain satisfying the strong local Lipschitz condition (see [AF, Paragraph 4.9]). Then the p-energy form $(\int_\Omega |\nabla f|^p \, dx, W^{1,p}(\Omega))$ is a p-resistance form on Ω if and only if $p > D$. Indeed, $(\mathrm{RF1})_p$ and $(\mathrm{RF5})_p$ are clear from the definition (we used the boundedness of Ω to ensure $\mathbb{R}\mathbb{1}_\Omega \in L^p(\Omega)$), $(\mathrm{RF2})_p$ and $(\mathrm{RF3})_p$ follow from [AF, Theorem 3.3 and Corollary 3.4] for any $p \in (1, \infty)$. If $p > D$, then we can use the Morrey-type inequality [AF, Lemma 4.28] to verify $(\mathrm{RF4})_p$. Conversely, the supremum in (6.1) is not finite when $p \leq D$. To see it, we can assume that $x = 0 \in \Omega$. Let $\delta \in (0,\infty)$ be small enough so that $\overline{B(0,\delta)} \subseteq \Omega$ and $y \notin \overline{B(0,\delta)}$. For all large $n \in \mathbb{N}$ so that $n^{-1} < \delta$, define $u_n \in C(\Omega)$ by

$$u_n(z) \coloneqq \left(\frac{\log |z|^{-1} - \log \delta^{-1}}{\log n - \log \delta^{-1}}\right)^{+} \wedge 1, \quad z \in \Omega.$$

Then we easily see that $u_n(0) = 1$, $u_n(y) = 1$ and $u_n \in W^{1,p}(\Omega)$ with

$$\begin{aligned}\int_\Omega |\nabla u_n|^p \, dz &\leq \left|\frac{1}{\log{(n\delta)}}\right|^p \int_{B(0,\delta)\setminus B(0,n^{-1})} |z|^{-p}\, dz = |S_{D-1}| \left|\frac{1}{\log{(n\delta)}}\right|^p \int_{\frac{1}{n}}^{\delta} r^{-p+D-1}\, dr \\ &= \begin{cases} |S_{D-1}| \left|\log{(n\delta)}\right|^{-(p-1)} & \text{if } p = D, \\ \frac{|S_{D-1}|}{D-p} \left|\log{(n\delta)}\right|^{-p} \left(\delta^{D-p} - n^{-(D-p)}\right) & \text{if } p < D, \end{cases}\end{aligned}$$

where $|S_{D-1}|$ is the volume of the $(D-1)$-dimensional unit sphere. In particular, $\frac{|u_n(x)-u_n(y)|^p}{\||\nabla u_n|\|_{L^p(\Omega)}^p} \to \infty$ as $n \to \infty$, so (RF4)$_p$ does not hold.

(2) The construction of a regular p-energy form on a compact metric space (K,d) in [Kig23, Theorem 3.21] needs the assumption $p > \dim_{\mathrm{ARC}}(K,d)$, where $\dim_{\mathrm{ARC}}(K,d)$ is the Ahlfors regular conformal dimension of (K,d). (See Definition 8.5-(4) for the definition of $\dim_{\mathrm{ARC}}(K,d)$. The same condition $p > \dim_{\mathrm{ARC}}(K,d)$ is also assumed in [Shi24].) This condition $p > \dim_{\mathrm{ARC}}(K,d)$ plays the same role as $p > D$ in (1) above (see also [CCK23+, Theorem 1.1]). In Theorem 8.19, we will see that p-energy forms constructed in [Kig23, Theorem 3.21] are indeed p-resistance forms. We also show that p-energy forms on p.-c.f. self-similar sets in [CGQ22, Theorem 5.1] under the condition (**R**) in [CGQ22, p. 18] are p-resistance forms in Theorem 8.34.

(3) Here we recall typical p-resistance forms on finite sets given in [KS23+, Example 2.2-(1)] because these examples are important to construct self-similar p-resistance forms on p.-c.f. self-similar structures in Subsection 8.3. Let V be a non-empty finite set. Note that in this case $\mathcal{E}$ is a p-resistance form on V if and only if $\mathcal{E}\colon \mathbb{R}^V \to [0,\infty)$ satisfies (RF1)$_p$ and (RF5)$_p$; indeed, (RF3)$_p$ is obvious for $\mathcal{F} = \mathbb{R}^V$, (RF2)$_p$ and (RF4)$_p$ are easily implied by (RF1)$_p$ and $\dim \mathcal{F}/\mathbb{R}\mathbb{1}_V < \infty$. Now, consider any functional $\mathcal{E}\colon \mathbb{R}^V \to [0,\infty)$ of the form

$$\mathcal{E}(u) = \frac{1}{2} \sum_{x,y \in V} L_{xy} \left|u(x) - u(y)\right|^p \tag{6.2}$$

for some $L = (L_{xy})_{x,y\in V} \in [0,\infty)^{V\times V}$ such that $L_{xy} = L_{yx}$ for any $x, y \in V$. It is obvious that $\mathcal{E}$ satisfies (RF1)$_p$ if and only if the graph (V, E_L) is connected, where $E_L \coloneqq \{\{x,y\} \mid x,y \in V,\, x \neq y,\, L_{xy} > 0\}$. It is also easy to see that $\mathcal{E}$ satisfies (RF5)$_p$. It thus follows that $\mathcal{E}$ is a p-resistance form on V if and only if (V, E_L) is connected. Note that, while any 2-resistance form on V is of the form (6.2) with $p = 2$, *the counterpart of this fact for $p \neq 2$ is NOT true* unless $\#V \leq 2$.

In the rest of this section, we assume that $(\mathcal{E}, \mathcal{F})$ is a p-resistance form on X. Then the following proposition is immediate from the definition of $R_{\mathcal{E}}$ and Theorem 3.18.

Proposition 6.4. (1) *For any $u \in \mathcal{F}$ and any $x, y \in X$,*

$$|u(x) - u(y)|^p \leq R_{\mathcal{E}}(x, y)\mathcal{E}(u). \tag{6.3}$$

(2) *$R_{\mathcal{E}}^{1/p}$ is a metric on X.*
(3) *$(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ is a uniformly convex Banach space, and thus it is reflexive.*

In particular, X can be regarded as a metric space equipped with $R_{\mathcal{E}}^{1/p}$ and $\mathcal{F} \subseteq C(X, R_{\mathcal{E}}^{1/p})$, where $C(X, R_{\mathcal{E}}^{1/p})$ is the set of continuous functions on $(X, R_{\mathcal{E}}^{1/p})$.

We introduce the regularity and the strong locality of p-resistance forms as follows.

Definition 6.5 (Regularity and strong locality)**.** (1) Assume that $(X, R_{\mathcal{E}}^{1/p})$ is locally compact. $(\mathcal{E}, \mathcal{F})$ is said to be *regular* if and only if $\mathcal{F} \cap C_c(X, R_{\mathcal{E}}^{1/p})$ is dense in $C_c(X, R_{\mathcal{E}}^{1/p})$ with respect to the uniform norm.
(2) $(\mathcal{E}, \mathcal{F})$ is said to *strongly local* if and only if $\mathcal{E}(u_1; v) = \mathcal{E}(u_2; v)$ for any $u_1, u_2, v \in \mathcal{F}$ that satisfy $(u_1(x) - u_2(x) - a)(v(x) - b) = 0$ for any $x \in X$ for some $a, b \in \mathbb{R}$.

The regularity ensures the existence of cut-off functions.

Proposition 6.6. *Assume that $(X, R_{\mathcal{E}}^{1/p})$ is locally compact and that $(\mathcal{E}, \mathcal{F})$ is regular. For any open subsets U, V of X with $\overline{V}^X$ compact and $\overline{V}^X \subseteq U$, there exists $\psi \in \mathcal{F} \cap C_c(X, R_{\mathcal{E}}^{1/p})$ such that $0 \leq \psi \leq 1$, $\psi = 1$ on an open neighborhood of V and $\mathrm{supp}[\psi] \subseteq U$.*

Proof. Since $(X, R_{\mathcal{E}}^{1/p})$ is locally compact, we can pick open subsets Ω_1, Ω_2 of X such that $\overline{V}^X \subseteq \Omega_1 \subseteq \Omega_2$, $\overline{\Omega_2}^X \subseteq U$ and $\overline{\Omega_2}^X$ is compact. By Urysohn's lemma, there exists $\psi_0 \in C_c(X, R_{\mathcal{E}}^{1/p})$ satisfying $0 \leq \psi_0 \leq 1$, $\psi_0 = 1$ on Ω_1, and $\mathrm{supp}[\psi_0] \subseteq U$. Since $(\mathcal{E}, \mathcal{F})$ is regular, for any $\varepsilon > 0$ there exists $\psi_\varepsilon \in \mathcal{F} \cap C_c(X, R_{\mathcal{E}}^{1/p})$ such that $\|\psi_0 - \psi_\varepsilon\|_{\sup} < \varepsilon$. Now define $\psi \coloneqq \left[(1 - 2\varepsilon)^{-1}(\psi_\varepsilon - \varepsilon)^+\right] \wedge 1$, then $\psi \in \mathcal{F}$ by (RF1)$_p$ and Proposition 2.2-(b). The other desired properties of ψ are obvious. □

We need the following notation to define traces of a p-resistance form later.

Definition 6.7. Let B be a non-empty subset of X. Define a linear subspace $\mathcal{F}|_B$ of $\mathcal{F}$ by $\mathcal{F}|_B = \left\{ u|_B \;\middle|\; u \in \mathcal{F} \right\}$.

The following proposition is useful to discuss boundary conditions on finite sets.

Proposition 6.8. *For any subset B of X with $2 \leq \#B < \infty$, we have $\mathcal{F}|_B = \mathbb{R}^B$.*

Proof. It suffices to show that $\mathbb{1}_x^B \in \mathcal{F}|_B$ for any $x \in B$ by virtue of (RF1)$_p$. Let $x \in B$. For each $y \in B \setminus \{x\}$, by (RF1)$_p$ and (RF2)$_p$, there exists $u_y \in \mathcal{F}$ satisfying $u_y(x) = 1$ and $u_y(y) = 0$. Let $f \coloneqq \sum_{y \in B \setminus \{x\}}(u_y^+ \wedge 1)$ and $g \coloneqq \sum_{y \in B \setminus \{x\}}\bigl((1 - u_y)^+ \wedge 1\bigr)$. Then

$f, g \in \mathcal{F}$ by (RF1)$_p$ and (RF5)$_p$. Since $f(x) = \#B - 1$, $f|_{B\setminus\{x\}} \leq \#B - 2$, $g(x) = 0$ and $g|_{B\setminus\{x\}} \geq 1$, the function $h \in \mathcal{F}$ given by

$$h \coloneqq \bigl(f - (\#B - 2)(g^+ \wedge 1)\bigr)^+ \wedge 1$$

satisfies $h|_B = \mathbb{1}_x^B$ and hence $\mathbb{1}_x^B \in \mathcal{F}|_B$. □

The next definition is introduced to deal with Dirichlet-type boundary conditions.

Definition 6.9. For a non-empty subset $B \subseteq X$, define

$$\mathcal{F}^0(B) \coloneqq \{u \in \mathcal{F} \mid u(x) = 0 \text{ for any } x \in X \setminus B\}, \quad B^{\mathcal{F}} \coloneqq \bigcap_{u \in \mathcal{F}^0(X\setminus B)} u^{-1}(0).$$

For basic properties of $B^{\mathcal{F}}$, see [Kig12, Chapters 2, 5 and 6]. Here we only recall the following results, which will be used later.

Proposition 6.10 ([Kig12, Theorems 2.5 and 6.3])**.** *Let B be a non-empty subset of X.*

(a) $\mathbb{C}_{\mathcal{F}} \coloneqq \bigl\{B \bigm| B \subseteq X, B = B^{\mathcal{F}}\bigr\}$ *satisfies the axiom of closed sets and it defined a topology on X. Moreover, $\{x\} \in \mathbb{C}_{\mathcal{F}}$ for any $x \in X$.*

(b) *For any $B \subseteq X$ and $x \notin B^{\mathcal{F}}$, there exists $u \in \mathcal{F}$ such that $u \in \mathcal{F}^0(X \setminus B)$, $u(x) = 1$ and $0 \leq u \leq 1$.*

(c) *Assume that $(X, R_{\mathcal{E}}^{1/p})$ is locally compact and that $(\mathcal{E}, \mathcal{F})$ is regular. Then $B = B^{\mathcal{F}}$ for any closed set B of $(X, R_{\mathcal{E}}^{1/p})$.*

Proof. The statements (a) and (b) follow from [Kig12, Theorem 2.4 and Lemma 2.5]. The argument showing (R1) ⇒ (R2) in the proof of [Kig12, Theorem 6.3] proves (c). □

For $B \subseteq X$ and $x \notin B^{\mathcal{F}}$, we define

$$R_{\mathcal{E}}(x, B) \coloneqq R_{(\mathcal{E},\mathcal{F})}(x, B) \coloneqq \sup\left\{ \frac{|u(x)|^p}{\mathcal{E}(u)} \,\middle|\, u \in \mathcal{F}^0(X \setminus B), u(x) \neq 0 \right\} < \infty. \tag{6.4}$$

Note that $R_{\mathcal{E}}(x, \{y\}) = R_{\mathcal{E}}(x, y)$ for $y \in X \setminus \{x\}$ by Proposition 6.10-(a).

6.2 Harmonic functions and traces of p-resistance forms

In this subsection, we consider harmonic functions with respect to p-resistance forms and traces of p-resistance forms to subsets of the original domains.

The following proposition states that the variational and distributional formulations of harmonic functions coincide for p-resistance forms.

Proposition 6.11. *Let $h \in \mathcal{F}$ and $B \subseteq X$. Then the following conditions are equivalent:*

(1) $\mathcal{E}(h) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{F}, u|_B = h|_B\}$.

(2) $\mathcal{E}(h; \varphi) = 0$ *for any* $\varphi \in \mathcal{F}^0(X \setminus B)$.

Proof. Let $\varphi \in \mathcal{F}^0(X \setminus B)$ and set $E(t) := \mathcal{E}(h + t\varphi)$ for $t \in \mathbb{R}$. Then E is differentiable by Proposition 3.5. If $\mathcal{E}(h) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{F}, u|_B = h|_B\}$, then E takes its minimum at $t = 0$. Hence $p\mathcal{E}(h;\varphi) = \frac{d}{dt}E(t)\big|_{t=0} = 0$, which implies $\mathcal{E}(h;\varphi) = 0$ and proves (1) $\Rightarrow$ (2).

Conversely, suppose that $\mathcal{E}(h;\varphi) = 0$ for any $\varphi \in \mathcal{F}^0(X \setminus B)$. Let $v \in \mathcal{F}$ with $v|_B = h|_B$. Then $\mathcal{E}(h) - \mathcal{E}(h;v) = \mathcal{E}(h; h - v) = 0$. By (3.11) and Young's inequality,

$$\mathcal{E}(h) = \mathcal{E}(h;v) \leq \mathcal{E}(h)^{(p-1)/p}\mathcal{E}(v)^{1/p} \leq \frac{p-1}{p}\mathcal{E}(h) + \frac{1}{p}\mathcal{E}(v),$$

which implies $\mathcal{E}(h) \leq \mathcal{E}(v)$. Therefore, $\mathcal{E}(h) = \inf\{\mathcal{E}(u) \mid u \in \mathcal{F}, u|_B = h|_B\}$ and the implication (2) $\Rightarrow$ (1) is proved. □

Definition 6.12 ($\mathcal{E}$-harmonic functions)**.** Let $B \subseteq X$. A function $h \in \mathcal{F}$ is said to be *$\mathcal{E}$-harmonic* on $X \setminus B$ if h satisfies either (and hence both) of (1) and (2) in Proposition 6.11. We set $\mathcal{H}_{\mathcal{E},B} := \{h \in \mathcal{F} \mid h \text{ is } \mathcal{E}\text{-harmonic on } X \setminus B\}$.

$\mathcal{E}$-harmonic functions with given boundary values uniquely exist, and their energies under $\mathcal{E}$ define a new p-resistance form on the boundary set, as follows. This new p-resistance form is called the *trace* of $(\mathcal{E},\mathcal{F})$ on the boundary set.

Theorem 6.13. *Let $B \subseteq X$ be non-empty, and define $\mathcal{E}|_B \colon \mathcal{F}|_B \to [0,\infty)$ by*

$$\mathcal{E}|_B(u) := \inf\{\mathcal{E}(v) \mid v \in \mathcal{F}, v|_B = u\}, \quad u \in \mathcal{F}|_B. \tag{6.5}$$

Then $(\mathcal{E}|_B, \mathcal{F}|_B)$ is a p-resistance form on B and $R_{\mathcal{E}|_B} = R_{\mathcal{E}}|_{B\times B}$. Moreover, for any $u \in \mathcal{F}|_B$ there exists a unique $h_B^{\mathcal{E}}[u] \in \mathcal{F}$ such that $h_B^{\mathcal{E}}[u]\big|_B = u$ and $\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr) = \mathcal{E}|_B(u)$, so that $h_B^{\mathcal{E}}(\mathcal{F}|_B) = \mathcal{H}_{\mathcal{E},B}$, and

$$h_B^{\mathcal{E}}[au + b\mathbb{1}_B] = ah_B^{\mathcal{E}}[u] + b\mathbb{1}_X \quad \text{for any } u \in \mathcal{F}|_B \text{ and any } a, b \in \mathbb{R}, \tag{6.6}$$

$$\mathcal{E}|_B(u;v) = \mathcal{E}\bigl(h_B^{\mathcal{E}}[u]; h_B^{\mathcal{E}}[v]\bigr) \quad \text{for any } u, v \in \mathcal{F}|_B, \tag{6.7}$$

$$\mathcal{E}|_B(f|_B; g|_B) = \mathcal{E}(f;g) \quad \text{for any } f \in \mathcal{H}_{\mathcal{E},B} \text{ and any } g \in \mathcal{F}, \tag{6.8}$$

where $\mathcal{E}|_B(u;v) := \frac{1}{p}\frac{d}{dt}\mathcal{E}|_B(u+tv)\big|_{t=0}$ for $u, v \in \mathcal{F}|_B$ (recall (3.8)).

Remark 6.14. The map $h_B^{\mathcal{E}}[\,\cdot\,]$ does *not* satisfy either $h_B^{\mathcal{E}}[u+v] \leq h_B^{\mathcal{E}}[u] + h_B^{\mathcal{E}}[u]$ for any $u, v \in \mathcal{F}|_B$ or $h_B^{\mathcal{E}}[u+v] \geq h_B^{\mathcal{E}}[u] + h_B^{\mathcal{E}}[u]$ for any $u, v \in \mathcal{F}|_B$ in general, unless $p = 2$ or $\#B \leq 2$.

Proof. We first show the desired existence of $h_B^{\mathcal{E}}[u]$ for any $u \in \mathcal{F}|_B$. Let us fix $y_* \in B$ and let $\alpha := \inf\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ with } v|_B = u\bigr\} \in [0,\infty)$. Then there exists $\{v_n\}_{n\in\mathbb{N}}$ such that $v_n \in \mathcal{F}$, $v_n|_B = u$ and $\mathcal{E}(v_n) \leq \alpha + n^{-1}$ for any $n \in \mathbb{N}$. Note that $\frac{v_k+v_l}{2} \in \mathcal{F}$ also satisfies $\bigl(\frac{v_k+v_l}{2}\bigr)\big|_B = u$ for any $k, l \in \mathbb{N}$. In the case $p \in (1,2]$, we see that

$$\begin{aligned}\mathcal{E}(v_k - v_l)^{1/(p-1)} &\overset{(2.7)}{\leq} 2\bigl(\mathcal{E}(v_k) + \mathcal{E}(v_l)\bigr)^{1/(p-1)} - \mathcal{E}(v_k + v_l)^{1/(p-1)} \\ &\leq 2\bigl(2\alpha + k^{-1} + l^{-1}\bigr)^{1/(p-1)} - 2^{p/(p-1)}\alpha^{1/(p-1)}\end{aligned}$$

$$\xrightarrow[k\wedge l\to\infty]{} 2(2\alpha)^{1/(p-1)} - 2^{p/(p-1)}\alpha^{1/(p-1)} = 0. \tag{6.9}$$

Similarly, in the case $p \in [2,\infty)$, we have

$$\begin{aligned} \mathcal{E}(v_k - v_l) &\overset{(2.9)}{\leq} 2\bigl(\mathcal{E}(v_k)^{1/(p-1)} + \mathcal{E}(v_l)^{1/(p-1)}\bigr)^{p-1} - \mathcal{E}(v_k + v_l) \\ &\leq 2\bigl((\alpha + k^{-1})^{1/(p-1)} + (\alpha + l^{-1})^{1/(p-1)}\bigr)^{p-1} - 2^p\alpha \\ &\xrightarrow[k\wedge l\to\infty]{} 2\bigl(2\alpha^{1/(p-1)}\bigr)^{p-1} - 2^p\alpha = 0. \end{aligned} \tag{6.10}$$

Consequently, $\{v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$. By (RF2)_p, there exists $h \in \mathcal{F}$ such that $h(y_*) = u(y_*)$ and $\lim_{n\to\infty}\mathcal{E}(h - v_n) = 0$. For any $y \in B$, by (RF4)_p,

$$|h(y) - u(y)|^p = |h(y) - v_n(y)|^p = |(h - v_n)(y) - (h - v_n)(y_*)|^p \leq R_{\mathcal{E}}(y, y_*)\mathcal{E}(h - v_n) \to 0,$$

and hence $h|_B = u$. In particular, h is a minimizer of α. Suppose that $g \in \mathcal{F}$ also satisfies $g|_B = u$ and $\mathcal{E}(g) = \alpha$. Then a similar estimate to (6.9) or to (6.10) imply that $\mathcal{E}(h - g) = 0$. Since $h - g \in \mathcal{F}^0(X \setminus B)$ and $B \neq \emptyset$, we have $h = g =: h_B^{\mathcal{E}}[u]$ by (RF1)_p. The property (6.6) immediately follows from (RF1)_p for $(\mathcal{E}, \mathcal{F})$.

Next we prove that $(\mathcal{E}|_B, \mathcal{F}|_B)$ is a p-resistance form on B. It is clear that $\mathcal{E}|_B(au) = |a|^p\,\mathcal{E}|_B(u)$ for any $u \in \mathcal{F}|_B$. Let us show the triangle inequality for $\mathcal{E}|_B(\,\cdot\,)^{1/p}$, Since $\bigl(h_B^{\mathcal{E}}[u] + h_B^{\mathcal{E}}[v]\bigr)\big|_B = u + v$ for any $u, v \in \mathcal{F}|_B$, we see that

$$\mathcal{E}|_B(u+v)^{1/p} = \mathcal{E}\bigl(h_B^{\mathcal{E}}[u+v]\bigr)^{1/p} \leq \mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)^{1/p} + \mathcal{E}\bigl(h_B^{\mathcal{E}}[v]\bigr)^{1/p} = \mathcal{E}|_B(u)^{1/p} + \mathcal{E}|_B(v)^{1/p}.$$

By (6.6), we easily see that $\mathcal{F}|_B$ contains $\mathbb{R}\mathbb{1}_B$. If $u \in \mathcal{F}|_B$ satisfies $\mathcal{E}|_B(u) = 0$, then $h_B^{\mathcal{E}}[u] \in \mathbb{R}\mathbb{1}_X$ and hence $h_B^{\mathcal{E}}[u]\big|_B = u \in \mathbb{R}\mathbb{1}_B$. Thus (RF1)_p for $(\mathcal{E}|_B, \mathcal{F}|_B)$ holds. To prove (RF2)_p for $(\mathcal{E}|_B, \mathcal{F}|_B)$, let $\{u_n\} \subseteq \mathcal{F}|_B$ satisfy $\lim_{n\wedge m\to\infty}\mathcal{E}|_B(u_n - u_m) = 0$. Then, by the triangle inequality for $\mathcal{E}|_B(\,\cdot\,)^{1/p}$, we easily see that $\{\mathcal{E}|_B(u_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $[0,\infty)$. By (Cla)_p for $(\mathcal{E}, \mathcal{F})$ and a similar argument to (6.9) (or to (6.10)), we have $\lim_{n\wedge m\to\infty}\mathcal{E}\bigl(h_B^{\mathcal{E}}[u_n] - h_B^{\mathcal{E}}[u_m]\bigr) = 0$. Hence there exists $h \in \mathcal{F}$ such that $\lim_{n\to\infty}\mathcal{E}(h - h_B^{\mathcal{E}}[u_n]) = 0$ by (RF2)_p for $(\mathcal{E}, \mathcal{F})$. Then $\mathcal{E}|_B(h|_B - u_n) \leq \mathcal{E}(h - h_B^{\mathcal{E}}[u_n]) \to 0$, which proves the completeness of $\bigl(\mathcal{F}|_B/\mathbb{R}\mathbb{1}_B, \mathcal{E}|_B(\,\cdot\,)^{1/p}\bigr)$. The condition (RF3)_p for $\mathcal{F}|_B$ is clear from that of $\mathcal{F}$. The inequality $R_{\mathcal{E}|_B} \leq R_{\mathcal{E}}|_{B\times B}$ (and hence (RF4)_p for $(\mathcal{E}|_B, \mathcal{F}|_B)$) follows from the following estimate:

$$\frac{|u(x) - u(y)|^p}{\mathcal{E}|_B(u)} = \frac{\bigl|h_B^{\mathcal{E}}[u](x) - h_B^{\mathcal{E}}[u](y)\bigr|^p}{\mathcal{E}(h_B^{\mathcal{E}}[u])} \leq R_{\mathcal{E}}(x, y) \quad \text{for any } x, y \in B,\ u \in \mathcal{F}|_B.$$

To show the converse inequality $R_{\mathcal{E}|_B} \geq R_{\mathcal{E}}|_{B\times B}$, let $x, y \in B$ and let $u \in \mathcal{F} \setminus \mathbb{R}\mathbb{1}_X$. Suppose that $u(x) \neq u(y)$. Then $u|_B \in \mathcal{F}|_B \setminus \mathbb{R}\mathbb{1}_B$ and $\mathcal{E}(u) \geq \mathcal{E}|_B(u|_B) > 0$. Therefore,

$$\frac{|u(x) - u(y)|^p}{\mathcal{E}(u)} \leq \frac{|u|_B(x) - u|_B(y)|^p}{\mathcal{E}|_B(u|_B)} \leq R_{\mathcal{E}|_B}(x, y).$$

The same estimates is clear if $u(x) = u(y)$, so taking the supremum over $u \in \mathcal{F}\setminus\mathbb{R}\mathbb{1}_X$ yields $R_{\mathcal{E}}(x, y) \leq R_{\mathcal{E}|_B}(x, y)$. Lastly, we prove (RF5)_p for $(\mathcal{E}|_B, \mathcal{F}|_B)$. Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$,

$q_2 \in [p, \infty]$, $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (\mathcal{F}|_B)^{n_1}$, and suppose that $T = (T_1, \dots, T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfies (2.1). Note that $T_l(\boldsymbol{u}) = T_l\bigl(h_B^{\mathcal{E}}[u_1], \dots, h_B^{\mathcal{E}}[u_{n_1}]\bigr)\big|_B \in \mathcal{F}|_B$. Therefore, if $q_2 < \infty$, then

$$\left(\sum_{l=1}^{n_2} \mathcal{E}|_B\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p}\right)^{1/q_2} \leq \left(\sum_{l=1}^{n_2} \mathcal{E}\Bigl(T_l\bigl(h_B^{\mathcal{E}}[u_1], \dots, h_B^{\mathcal{E}}[u_{n_1}]\bigr)\Bigr)^{q_2/p}\right)^{1/q_2}$$
$$\leq \left(\sum_{k=1}^{n_1} \mathcal{E}\bigl(h_B^{\mathcal{E}}[u_k]\bigr)^{q_1/p}\right)^{1/q_1} = \left(\sum_{k=1}^{n_1} \mathcal{E}|_B(u_k)^{q_1/p}\right)^{1/q_1}.$$

The case $q_2 = \infty$ is similar, so $(\mathcal{E}|_B, \mathcal{F}|_B)$ satisfies (GC)_p.

We conclude the proof by showing (6.7) and (6.8). By Proposition 3.5, we know that

$$\lim_{t \downarrow 0} \frac{\mathcal{E}|_B(u \pm tv) - \mathcal{E}|_B(u)}{\pm t} = \frac{d}{dt}\mathcal{E}|_B(u + tv)\bigg|_{t=0},$$

and

$$\lim_{t \downarrow 0} \frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u] \pm t h_B^{\mathcal{E}}[v]\bigr) - \mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{\pm t} = p\mathcal{E}\bigl(h_B^{\mathcal{E}}[u]; h_B^{\mathcal{E}}[v]\bigr).$$

For any $t > 0$, we have

$$\frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u] - t h_B^{\mathcal{E}}[v]\bigr) - \mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{-t} \leq \frac{\mathcal{E}|_B(u - tv) - \mathcal{E}|_B(u)}{-t}$$
$$\leq \frac{\mathcal{E}|_B(u + tv) - \mathcal{E}|_B(u)}{t} \leq \frac{\mathcal{E}\bigl(h_B^{\mathcal{E}}[u] + t h_B^{\mathcal{E}}[v]\bigr) - \mathcal{E}\bigl(h_B^{\mathcal{E}}[u]\bigr)}{t},$$

and hence we obtain (6.7) by letting $t \downarrow 0$. If $f \in \mathcal{H}_{\mathcal{E},B}$, i.e., $h_B^{\mathcal{E}}[f|_B] = f$, then $\mathcal{E}(f; g) = \mathcal{E}(f; h_B^{\mathcal{E}}[g]) = \mathcal{E}|_B(f|_B; g|_B)$ since $g - h_B^{\mathcal{E}}[g|_B] \in \mathcal{F}^0(X \setminus B)$ for any $g \in \mathcal{F}$. This proves (6.8). $\square$

The next property states a compatibility of the operation taking traces.

Proposition 6.15. *Let A, B be subsets of X such that $\emptyset \neq A \subseteq B$. Then $(\mathcal{E}|_B|_A, \mathcal{F}|_B|_A) = (\mathcal{E}|_A, \mathcal{F}|_A)$ and $h_B^{\mathcal{E}} \circ h_A^{\mathcal{E}|_B} = h_A^{\mathcal{E}}$ for any $u \in \mathcal{F}|_A$. In particular, $h_A^{\mathcal{E}|_B}[u] = h_A^{\mathcal{E}}[u]\big|_B$.*

Proof. Clearly, we have $\mathcal{F}|_B|_A = \mathcal{F}|_A$. For any $u \in \mathcal{F}|_A$, we see that

$$\begin{aligned}
\mathcal{E}|_A(u) = \mathcal{E}\bigl(h_A^{\mathcal{E}}[u]\bigr) &\geq \min\bigl\{\mathcal{E}(v) \bigm| v \in \mathcal{F} \text{ such that } v|_B = h_A^{\mathcal{E}}[u]\big|_B\bigr\} \\
&= \mathcal{E}|_B\bigl(h_A^{\mathcal{E}}[u]\big|_B\bigr) \\
&\geq \min\bigl\{\mathcal{E}|_B(w) \bigm| w \in \mathcal{F}|_B \text{ such that } w|_A = h_A^{\mathcal{E}}[u]\big|_A = u\bigr\} \\
&= \mathcal{E}|_B|_A(u) = \mathcal{E}|_B\bigl(h_A^{\mathcal{E}|_B}[u]\bigr) = \mathcal{E}\Bigl(h_B^{\mathcal{E}}\bigl[h_A^{\mathcal{E}|_B}[u]\bigr]\Bigr) \\
&\geq \min\Bigl\{\mathcal{E}(v) \Bigm| v \in \mathcal{F} \text{ such that } v|_A = \bigl(h_B^{\mathcal{E}} \circ h_A^{\mathcal{E}|_B}\bigr)[u]\big|_A = u\Bigr\} = \mathcal{E}|_A(u),
\end{aligned}$$

which implies $\mathcal{E}|_A(u) = \mathcal{E}|_B|_A(u)$ and $\mathcal{E}\bigl(h_A^{\mathcal{E}}[u]\bigr) = \mathcal{E}\bigl((h_B^{\mathcal{E}} \circ h_A^{\mathcal{E}|_B})[u]\bigr)$. Since restrictions of both functions $h_A^{\mathcal{E}}[u]$ and $(h_B^{\mathcal{E}} \circ h_A^{\mathcal{E}|_B})[u]$ to A are u, the uniqueness in Theorem 6.13 implies $h_A^{\mathcal{E}}[u] = \bigl(h_B^{\mathcal{E}} \circ h_A^{\mathcal{E}|_B}\bigr)[u]$. Considering the restriction to B yields $h_A^{\mathcal{E}|_B}[u] = h_A^{\mathcal{E}}[u]\big|_B$. $\square$

The following theorem presents an expression of $(\mathcal{E},\mathcal{F})$ as the "inductive limit" of its traces $\{\mathcal{E}|_V\}_{V\subseteq X,1\leq\#V<\infty}$ to finite subsets, which is a straightforward extension of the counterpart for resistance forms given in [Kaj, Corollary 2.37]. This expression can be applied to get a few useful results on convergences of the seminorm $\mathcal{E}^{1/p}$.

Theorem 6.16. *It holds that*

$$\mathcal{F}=\left\{u\in\mathbb{R}^X \;\middle|\; \sup_{V\subseteq X;1\leq\#V<\infty}\mathcal{E}|_V(u|_V)<\infty\right\}, \tag{6.11}$$

$$\mathcal{E}(u)=\sup_{V\subseteq X;1\leq\#V<\infty}\mathcal{E}|_V(u|_V) \quad \textit{for any } u\in\mathcal{F}. \tag{6.12}$$

Proof. Let us define $(\mathcal{E}_*,\mathcal{F}_*)$ by

$$\mathcal{E}_*(u):=\sup_{V\subseteq X;1\leq\#V<\infty}\mathcal{E}|_V(u|_V), \quad u\in\mathbb{R}^X,$$

and $\mathcal{F}_*:=\{u\in\mathbb{R}^X \mid \mathcal{E}_*(u)<\infty\}$. Then $\mathcal{E}_*^{1/p}$ is clearly a seminorm on $\mathcal{F}_*$ and $\{u\in\mathcal{F}_* \mid \mathcal{E}_*(u)=0\}=\mathbb{R}\mathbb{1}_X$. We first show that, for any $V\subseteq X$ with $1\leq\#V<\infty$ and any $u\in\mathbb{R}^V$,

$$h_V^{\mathcal{E}}[u]\in\mathcal{F}_* \quad \text{and} \quad \mathcal{E}|_V(u)=\min\{\mathcal{E}_*(v)\mid v\in\mathcal{F}, v|_V=u\}=\mathcal{E}_*\bigl(h_V^{\mathcal{E}}[u]\bigr), \tag{6.13}$$

both of which are obtained by seeing that, for any $U\subseteq X$ with $1\leq\#U<\infty$,

$$\mathcal{E}|_U\bigl(h_V^{\mathcal{E}}[u]\big|_U\bigr)\leq\mathcal{E}\bigl(h_V^{\mathcal{E}}[u]\bigr)=\mathcal{E}|_V(u).$$

Indeed, taking the supremum over U, we get $\mathcal{E}_*\bigl(h_V^{\mathcal{E}}[u]\bigr)\leq\mathcal{E}|_V(u)$ and hence (6.13) holds. (The converse $\mathcal{E}|_V(u)\leq\mathcal{E}_*\bigl(h_V^{\mathcal{E}}[u]\bigr)$ is clear from the definition.) We also note that $\mathcal{E}_*$ satisfies $(\mathrm{Cla})_p$ since $(\mathcal{E}|_Y,\mathcal{F}|_Y)$ is a p-resistance form for each $Y\subseteq X$ and $\mathcal{E}|_V(u|_V)\leq\mathcal{E}|_U(u|_U)$ for any $U,V\subseteq X$ with $\emptyset\neq V\subseteq U$ and $u\in\mathbb{R}^U$.

The inclusion $\mathcal{F}\subseteq\mathcal{F}_*$ and $\mathcal{E}_*\leq\mathcal{E}$ (on $\mathcal{F}$) easily follow from the following estimate:

$$\mathcal{E}|_V(u|_V)=\mathcal{E}\bigl(h_V^{\mathcal{E}}[u|_V]\bigr)\leq\mathcal{E}(u) \quad \text{for any } u\in\mathcal{F} \text{ and } V\subseteq X \text{ with } 1\leq\#V<\infty.$$

To show $\mathcal{F}_*\subseteq\mathcal{F}$ and $\mathcal{E}\leq\mathcal{E}_*$, let $u\in\mathcal{F}_*$, let us choose a subset $V_n\subseteq X$ for each $n\in\mathbb{N}$ such that $1\leq\#V_n<\infty$ and $\mathcal{E}|_{V_n}(u|_{V_n})\geq\mathcal{E}_*(u)-n^{-1}$, and set $u_n:=h_{V_n}^{\mathcal{E}}[u|_{V_n}]$. Then

$$\mathcal{E}_*(u)-n^{-1}\leq\mathcal{E}|_{V_n}(u|_{V_n})\overset{(6.13)}{=}\mathcal{E}_*(u_n)\overset{(6.13)}{\leq}\mathcal{E}_*(u),$$

which implies that $\lim_{n\to\infty}\mathcal{E}_*(u_n)=\lim_{n\to\infty}\mathcal{E}(u_n)=\mathcal{E}_*(u)$. Using $(\mathrm{Cla})_p$ for $\mathcal{E}_*$ and $\mathcal{E}_*\bigl(\frac{u+u_n}{2}\bigr)\geq\mathcal{E}_*(u_n)$, we easily obtain $\lim_{n\to\infty}\mathcal{E}_*(u-u_n)=0$ similarly as (6.9) or (6.10). We next show that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X,\mathcal{E}^{1/p})$. From $(\mathrm{Cla})_p$ for $\mathcal{E}$, $\lim_{n\to\infty}\mathcal{E}(u_n)=\lim_{n\to\infty}\mathcal{E}_*(u_n)=\mathcal{E}_*(u)$ and

$$\mathcal{E}(u_k+u_l)\geq\mathcal{E}\bigl(h_{V_k\cup V_l}^{\mathcal{E}}[(u_k+u_l)|_{V_k\cup V_l}]\bigr)\geq 2^p\mathcal{E}|_{V_k\cup V_l}(u|_{V_k\cup V_l})\overset{(6.13)}{=}2^p\mathcal{E}_*(u_{k+l}),$$

we can obtain $\lim_{k\wedge l\to\infty}\mathcal{E}(u_k-u_l)=0$ similarly as (6.9) or (6.10). Hence, by (RF1)$_p$ for $(\mathcal{E},\mathcal{F})$, there exists $v\in\mathcal{F}$ such that $\lim_{n\to\infty}\mathcal{E}(v-u_n)=0$. By $\mathcal{E}_*\le\mathcal{E}$ on $\mathcal{F}$, we conclude that $\lim_{n\to\infty}\mathcal{E}_*(v-u_n)=0$, which together with the triangle inequality for $\mathcal{E}_*^{1/p}$ and $\lim_{n\to\infty}\mathcal{E}_*(u-u_n)=0$ implies that $\mathcal{E}_*(u-v)=0$ and thus $u-v\in\mathbb{R}\mathbb{1}_X$. In particular, $u=(u-v)+v\in\mathcal{F}_*$ and $\mathcal{E}(u)=\lim_{n\to\infty}\mathcal{E}(u_n)=\mathcal{E}_*(u)$, completing the proof. □

Corollary 6.17. *Let $u\in\mathcal{F}$ and let $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$.*

(a) *Assume that $\lim_{n\to\infty}(u_n(x)-u_n(y))=u(x)-u(y)$ for any $x,y\in X$. Then $\mathcal{E}(u)\le\liminf_{n\to\infty}\mathcal{E}(u_n)$.*

(b) *$\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$ if and only if $\limsup_{n\to\infty}\mathcal{E}(u_n)\le\mathcal{E}(u)$ and $\lim_{n\to\infty}(u_n(x)-u_n(y))=u(x)-u(y)$ for any $x,y\in X$.*

Proof. Suppose that $u,u_n\in\mathcal{F}$, $n\in\mathbb{N}$, satisfy $\lim_{n\to\infty}(u_n(x)-u_n(y))=u(x)-u(y)$ for any $x,y\in X$. For any $\varepsilon>0$, by Theorem 6.16, there exists $V\subseteq X$ with $1\le\#V<\infty$ such that $\mathcal{E}|_V(u|_V)>\mathcal{E}(u)-\varepsilon$. Then we have

$$\lim_{n\to\infty}\mathcal{E}|_V(u_n|_V)=\mathcal{E}|_V(u|_V)>\mathcal{E}(u)-\varepsilon,$$

since $\mathbb{R}^V$ is a finite-dimensional vector space, $\mathcal{E}|_V(\,\cdot\,)^{1/p}$ is a seminorm on $\mathbb{R}^V$ and $\lim_{n\to\infty}\max_{x,y\in V}|(u_n(x)-u_n(y))-(u(x)-u(y))|=0$. In particular, there exists $N_1\in\mathbb{N}$ (depending on ε) such that $\mathcal{E}(u_n)\ge\mathcal{E}|_V(u_n|_V)>\mathcal{E}(u)-\varepsilon$ for any $n\ge N_1$ and hence $\liminf_{n\to\infty}\mathcal{E}(u_n)\ge\mathcal{E}(u)$, proving (a). Next, in addition, we assume that $\limsup_{n\to\infty}\mathcal{E}(u_n)\le\mathcal{E}(u)$. Then $\lim_{n\to\infty}\mathcal{E}(u_n)=\mathcal{E}(u)$. Since $\{\frac{u+u_n}{2}\}_{n\in\mathbb{N}}$ satisfies the same conditions as $\{u_n\}_{n\in\mathbb{N}}$, we obtain $\lim_{n\to\infty}\mathcal{E}\bigl(\frac{u+u_n}{2}\bigr)=\mathcal{E}(u)$. Similar to (6.9) or (6.10), we have from (Cla)$_p$ for $\mathcal{E}$ that $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$. The converse part of (b) is clear from (6.3). □

Corollary 6.18. (a) *Let $\{\varphi_n\}_{n\in\mathbb{N}}\subseteq C(\mathbb{R})$ satisfy $\lim_{n\to\infty}\varphi_n(t)=t$ and $|\varphi_n(t)-\varphi_n(s)|\le|t-s|$ for any $n\in\mathbb{N}$, $s,t\in\mathbb{R}$. Then $\{\varphi_n(u)\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(u-\varphi_n(u))=0$ for any $u\in\mathcal{F}$.*

(b) *Let $u\in\mathcal{F}$, $\{u_n\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ and $\varphi\in C(\mathbb{R})$ satisfy $\lim_{n\to\infty}\mathcal{E}(u-u_n)=0$, $\lim_{n\to\infty}u_n(x)=u(x)$ for some $x\in X$, $|\varphi(t)-\varphi(s)|\le|t-s|$ for any $s,t\in\mathbb{R}$ and $\varphi(u)=u$. Then $\{\varphi(u_n)\}_{n\in\mathbb{N}}\subseteq\mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(u-\varphi(u_n))=0$.*

Proof. The statement (a) is immediate from Corollary 6.17 and (RF5)$_p$, so we show (b). Since, under the assumptions of (b), for any $y\in X$,

$$|u(y)-u_n(y)|\le R_{\mathcal{E}}(x,y)^{1/p}\mathcal{E}(u-u_n)^{1/p}+|u(x)-u_n(x)|\xrightarrow[n\to\infty]{}0,$$

we get $\lim_{n\to\infty}\varphi(u_n(y))=u(y)$. By (RF5)$_p$, we have $\varphi(u_n)\in\mathcal{F}$ and $\limsup_{n\to\infty}\mathcal{E}(\varphi(u_n))\le\lim_{n\to\infty}\mathcal{E}(u_n)=\mathcal{E}(u)$, so Corollary 6.17 yields $\lim_{n\to\infty}\mathcal{E}(u-\varphi(u_n))=0$. □

A standard machinery for constructing the "inductive limit" of p-energy forms on p.-c.f. self-similar structures can be stated as follows. (For details, see [CGQ22, Proposition 5.3] and [Kig01, Sections 2.2, 2.3 and 3.3].)

Definition 6.19. Let $\mathcal{S} = \{\mathcal{V}_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a increasing sequence of finite subsets of X with $\overline{\mathcal{V}_*}^X = X$, where $\mathcal{V}_* := \bigcup_{n\in\mathbb{N}\cup\{0\}}\mathcal{V}_n$. Define $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ by

$$\mathcal{F}_{\mathcal{S}} := \left\{u \in C(X, R_{\mathcal{E}}^{1/p}) \,\middle|\, \lim_{n\to\infty} \mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) < \infty\right\}, \tag{6.14}$$

$$\mathcal{E}_{\mathcal{S}}(u) := \lim_{n\to\infty} \mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) \in [0,\infty), \quad u \in \mathcal{F}_{\mathcal{S}}. \tag{6.15}$$

(Note that $\{\mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n})\}_{n\in\mathbb{N}\cup\{0\}}$ is non-decreasing since $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$.)

Proposition 6.20. *Let $\mathcal{S} = \{\mathcal{V}_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a increasing sequence of finite subsets of X with $\overline{\mathcal{V}_*}^X = X$. Then $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}}) = (\mathcal{E}, \mathcal{F})$. Moreover,*

$$\lim_{n\to\infty} \mathcal{E}\bigl(u - h^{\mathcal{E}}_{\mathcal{V}_n}[u|_{\mathcal{V}_n}]\bigr) = 0 \quad \text{for any } u \in \mathcal{F}, \quad \text{and} \tag{6.16}$$

$$\mathcal{E}(u; v) = \lim_{n\to\infty} \mathcal{E}|_{\mathcal{V}_n}(u|_{\mathcal{V}_n}; v|_{\mathcal{V}_n}) \quad \text{for any } u, v \in \mathcal{F}. \tag{6.17}$$

Proof. By Theorem 6.16, $\mathcal{E}_{\mathcal{S}} \le \mathcal{E}$ and $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{S}}$ are clear. To show the converse, let $u \in \mathcal{F}_{\mathcal{S}}$, set $u_n := h^{\mathcal{E}}_{\mathcal{V}_n}(u|_{\mathcal{V}_n}) \in \mathcal{F}$ and fix $x_0 \in \mathcal{V}_0$. We can assume that $u(x_0) = 0$ by considering $u - u(x_0)$ instead of u. A similar estimate to (6.9) or (6.10) for $\mathcal{E}$ and (RF2)_p together imply that $\lim_{n\to\infty} \mathcal{E}(v - u_n) = 0$ for some $v \in \mathcal{F}$ with $v(x_0) = 0$. Since $|v(x) - u(x)|^p \le R_{\mathcal{E}}(x, x_0)\mathcal{E}(u - u_n)$ for any $x \in \mathcal{V}_*$ and any $n \in \mathbb{N}$ with $x \in \mathcal{V}_n$ by (6.3), we have $v|_{\mathcal{V}_*} = u|_{\mathcal{V}_*}$. By $\overline{\mathcal{V}_*}^X = X$ and $u, v \in C(X, R_{\mathcal{E}}^{1/p})$ (see (6.3)), we conclude that $u = v \in \mathcal{F}$ and thus $\mathcal{F} = \mathcal{F}_{\mathcal{S}}$, $\mathcal{E}(u) = \mathcal{E}_{\mathcal{S}}(u)$ and $\lim_{n\to\infty} \mathcal{E}(u - u_n) = 0$, i.e., $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}}) = (\mathcal{E}, \mathcal{F})$ and (6.16) hold. The convergence in (6.17) is immediate from (6.16), (3.11) and (3.12). □

6.3 Weak comparison principles

In this subsection, we show some weak comparison principles in this context. The first one is an application of the strong subadditivity.

Proposition 6.21 (Weak comparison principle I)**.** *Let B be a non-empty subset of X. Then, for any $u, v \in \mathcal{F}|_B$ satisfying $u(y) \le v(y)$ for any $y \in B$, it holds that*

$$h^{\mathcal{E}}_B[u](x) \le h^{\mathcal{E}}_B[v](x) \quad \text{for any } x \in X. \tag{6.18}$$

In particular,

$$\inf_B u \le h^{\mathcal{E}}_B[u](x) \le \sup_B u \quad \text{for any } x \in X. \tag{6.19}$$

Proof. Let $f := h^{\mathcal{E}}_B[u]$ and $g := h^{\mathcal{E}}_B[v]$. We will prove $f \wedge g = f$, which immediately implies (6.18). Since $(f \wedge g)|_B = u$ and $(f \vee g)|_B = v$, we have

$$\mathcal{E}(f) \le \mathcal{E}(f \wedge g) \quad \text{and} \quad \mathcal{E}(g) \le \mathcal{E}(f \vee g).$$

By the strong subadditivity in (2.5), we obtain $\mathcal{E}(f \wedge g) = \mathcal{E}(f)$ (and $\mathcal{E}(f \vee g) = \mathcal{E}(g)$), which together with the uniqueness in Theorem 6.13, we have $f \wedge g = f$. □

We can extend the weak comparison principle above to arbitrary open subsets (see Proposition 6.25 below) if $(\mathcal{E}, \mathcal{F})$ is regular and strongly local. This version of weak comparison principle will be used to prove the *strong comparison principle* on p.-c.f. self-similar structures in a forthcoming paper [KS.b]. We begin with some preparations.

Definition 6.22. Let U be a non-empty open subset of $(X, R_{\mathcal{E}}^{1/p})$.

(1) Define

$$\mathcal{F}_{\mathrm{loc}}(U) := \left\{ f \in \mathbb{R}^U \;\middle|\; \begin{array}{l} f\mathbb{1}_V = f^{\#}\mathbb{1}_V \text{ for some } f^{\#} \in \mathcal{F} \text{ for each} \\ \text{relatively compact open subset } V \text{ of } U \end{array} \right\}. \tag{6.20}$$

(2) Assume that $(\mathcal{E}, \mathcal{F})$ is strongly local. Let $V \subseteq U$ be an open subset. A function $h \in \mathcal{F}_{\mathrm{loc}}(U)$ is said to be $\mathcal{E}$-harmonic on V if $\mathcal{E}(h^{\#}; \varphi) = 0$ for any $\varphi \in \mathcal{F}^0(V)$ with $\mathrm{supp}[\varphi]$ compact (with respect to the metric topology of $R_{\mathcal{E}}^{1/p}$), where $h^{\#} \in \mathcal{F}$ satisfies $h\mathbb{1}_{\mathrm{supp}[\varphi]} = h^{\#}\mathbb{1}_{\mathrm{supp}[\varphi]}$.

Remark 6.23. By the strong locality of $(\mathcal{E}, \mathcal{F})$, the value $\mathcal{E}(h^{\#}; \varphi)$ is independent of a particular choice of $h^{\#}$.

We need the following proposition to achieve the desired weak comparison principle.

Proposition 6.24. *Assume that $(X, R_{\mathcal{E}}^{1/p})$ is locally compact and that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local. Let U be a non-empty open subset of $(X, R_{\mathcal{E}}^{1/p})$ and let $u \in \mathcal{F}$ satisfy $u(x) = 0$ for any $x \in \partial U = \overline{U} \setminus U$. Then $u\mathbb{1}_U \in \mathcal{F}$.*

Proof. Define $\varphi_n \in C(\mathbb{R})$ by $\varphi_n(t) := t - \left(\frac{1}{n}\right) \vee \left(t \wedge \frac{1}{n}\right)$ and set $A_n := U \cap \mathrm{supp}[\varphi_n(u)]$ for each $n \in \mathbb{N}$. Since $u|_{\partial U} = 0$, $A_n = \overline{U}^X \cap \mathrm{supp}[\varphi_n(u)]$ and thus A_n is a compact subset of U. By Proposition 6.6, there exists $v_n \in \mathcal{F}$ such that $\mathbb{1}_{A_n} \le v_n \le \mathbb{1}_U$. Then we easily obtain $\varphi_n(u)\mathbb{1}_U = \varphi_n(u)v_n$, hence by Corollary 6.18-(a) and Proposition 2.2-(d) we have $\varphi_n(u)\mathbb{1}_U \in \mathcal{F}$. By the strong locality and Corollary 6.18-(a), $\{\varphi_n(u)\mathbb{1}_U\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$. Thus, by (RF2)$_p$ and (6.3), $\{\varphi_n(u)\mathbb{1}_U\}_{n\in\mathbb{N}}$ converges in norm in $(\mathcal{F}/\mathbb{R}\mathbb{1}_X, \mathcal{E}^{1/p})$ to its pointwise limit $u\mathbb{1}_U$, whence $u\mathbb{1}_U \in \mathcal{F}$. $\square$

Now we can state the desired version of the weak comparison principle.

Proposition 6.25 (Weak comparison principle II)**.** *Assume that $(X, R_{\mathcal{E}}^{1/p})$ is locally compact and that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local. Let U be non-empty open subset of $(X, R_{\mathcal{E}}^{1/p})$ such that $\overline{U}^X$ is compact. If $u, v \in C(\overline{U}^X, R_{\mathcal{E}}^{1/p}) \cap \mathcal{F}_{\mathrm{loc}}(U)$ are $\mathcal{E}$-harmonic on U and $u(x) \le v(x)$ for any $x \in \partial U$, then $u(x) \le v(x)$ for any $x \in \overline{U}^X$.*

Proof. Let $V \subseteq U$ be subsets of X such that $\overline{V}^X \subseteq U$, U is non-empty open and $\overline{U}^X$ is compact. Let $u^{\#}, v^{\#} \in \mathcal{F}$ satisfy $u\mathbb{1}_V = u^{\#}\mathbb{1}_V$ and $v\mathbb{1}_V = v^{\#}\mathbb{1}_V$. Define $f := u^{\#} - (u^{\#} - v^{\#})^{+}\mathbb{1}_{X\setminus V}$, $g := v^{\#} + (u^{\#} - v^{\#})^{+}\mathbb{1}_{X\setminus V}$. Then $f, g \in \mathcal{F}$ by $u^{\#}(x) \le v^{\#}(x)$‘ for any $x \in \partial V$, Propositions 2.2-(b) and 6.24. We also have $f, g \in \mathcal{H}_{\mathcal{E}, X\setminus V}$ by the strong locality. Since $f(x) = (u^{\#} \wedge v^{\#})(x) \le (u^{\#} \vee v^{\#})(x) = g(x)$ for any $x \in X \setminus V$, Proposition 6.21 implies that $u(x) = u^{\#}(x) = f(x) \le g(x) = v^{\#}(x) = v(x)$ for any $x \in V$. We can extend this estimate to any $x \in U$ since V is arbitrary, $\overline{U}^X$ is compact and $u, v \in C(\overline{U}^X, R_{\mathcal{E}}^{1/p})$. $\square$

6.4 Hölder regularity of harmonic functions

In this subsection, we present a sharp Hölder estimate on $\mathcal{E}$-harmonic functions and prove that $R_{\mathcal{E}}^{1/(p-1)}$ is a metric on X.

The following lemma is a kind of 'monotonicity on values of p-Laplacian'. This lemma will also play important roles in the subsequent works [KS.b, KS.c].

Lemma 6.26. *Let $u_1, u_2, v \in \mathcal{F}$ satisfy $((u_2 - u_1) \wedge v)(x) = 0$ for any $x \in X$. Then $\mathcal{E}(u_1; v) \geq \mathcal{E}(u_2; v)$.*

Proof. Let $t > 0$. Define $f, g \in \mathcal{F}$ by $f := u_1 + tv$ and $g := u_2$. Then we easily see that $f \vee g = u_2 + tv$ and $f \wedge g = u_1$. By (2.5), we have $\mathcal{E}(u_2 + tv) + \mathcal{E}(u_1) \leq \mathcal{E}(u_1 + tv) + \mathcal{E}(u_2)$, which implies that

$$\frac{\mathcal{E}(u_2 + tv) - \mathcal{E}(u_2)}{t} \leq \frac{\mathcal{E}(u_1 + tv) - \mathcal{E}(u_1)}{t}.$$

Letting $t \downarrow 0$, we get $\mathcal{E}(u_2; v) \leq \mathcal{E}(u_1; v)$. □

The next theorem gives a sharp Hölder continuity of $\mathcal{E}$-harmonic functions.

Theorem 6.27. *Let B be a non-empty subset of X. Then for any $x \in X \setminus B^{\mathcal{F}}$ and any $y \in X \setminus \{x\}$,*

$$h^{\mathcal{E}}_{B \cup \{x\}}\bigl[\mathbb{1}_x^{B \cup \{x\}}\bigr](y) \leq \frac{R_{\mathcal{E}}(x, y)^{1/(p-1)}}{R_{\mathcal{E}}(x, B)^{1/(p-1)}}. \tag{6.21}$$

Moreover, for any $h \in \mathcal{H}_{\mathcal{E}, B}$ with $\sup_B |h| < \infty$, any $x \in X \setminus B^{\mathcal{F}}$ and any $y \in X$,

$$|h(x) - h(y)| \leq \frac{R_{\mathcal{E}}(x, y)^{1/(p-1)}}{R_{\mathcal{E}}(x, B)^{1/(p-1)}} \operatorname*{osc}_B[h]. \tag{6.22}$$

Proof. To show (6.21), on one hand, we see that

$$\begin{aligned} -\mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x) &= \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_{B \cup \{x\}}) - \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x) \\ &= \mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_B) = R_{\mathcal{E}}(x, B)^{-1}. \end{aligned} \tag{6.23}$$

On the other hand,

$$\begin{aligned} -\mathcal{E}|_{B \cup \{x\}}(\mathbb{1}_B; \mathbb{1}_x) &= -\mathcal{E}\Bigl(h^{\mathcal{E}}_{B \cup \{x\}}[\mathbb{1}_B]; h^{\mathcal{E}}_{B \cup \{x, y\}}[\mathbb{1}_x]\Bigr) \\ &= -\mathcal{E}|_{B \cup \{x, y\}}\Bigl(h^{\mathcal{E}}_{B \cup \{x\}}[\mathbb{1}_B]\big|_{B \cup \{x, y\}}; \mathbb{1}_x\Bigr) \\ &\geq -\mathcal{E}|_{B \cup \{x, y\}}\Bigl(\bigl(h^{\mathcal{E}}_{B \cup \{x\}}[\mathbb{1}_B](y) \cdot h^{\mathcal{E}}_{\{x, y\}}[\mathbb{1}_y]\bigr)\big|_{B \cup \{x, y\}}; \mathbb{1}_x\Bigr) \quad \text{(by Lemma 6.26)} \\ &= -h^{\mathcal{E}}_{B \cup \{x\}}[\mathbb{1}_B](y)^{p-1} \mathcal{E}|_{B \cup \{x, y\}}\Bigl(h^{\mathcal{E}}_{\{x, y\}}[\mathbb{1}_y]\big|_{B \cup \{x, y\}}; \mathbb{1}_x\Bigr) \\ &= h^{\mathcal{E}}_{B \cup \{x\}}[\mathbb{1}_B](y)^{p-1} R_{\mathcal{E}}(x, y)^{-1}. \end{aligned} \tag{6.24}$$

Here, we used $\mathbb{1}_{\{x, y\}} - h^{\mathcal{E}}_{\{x, y\}}[\mathbb{1}_y] = h^{\mathcal{E}}_{\{x, y\}}[\mathbb{1}_x]$ (see Remark 6.14) in the last equality. We obtain (6.21) by combining (6.23) and (6.24).

Next we prove (6.22). Let $x \in X \setminus B^{\mathcal{F}}$, $y \in X$ and $h \in \mathcal{H}_{\mathcal{E},B}$ with $\sup_B |h| < \infty$. We can assume that $x \neq y$. Then we see that

$$\begin{aligned} h - h(x) &\le h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[(h-h(x))^{+}\big|_{B\cup\{x\}}\Bigr] \quad \text{(by Propositions 6.21 and 6.15)} \\ &\le h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\operatorname*{osc}_B[h]\cdot \mathbb{1}_B^{B\cup\{x\}}\Bigr] \quad \text{(by Proposition 6.21 and } (h-h(x))^{+}(x)=0) \\ &= \operatorname*{osc}_B[h]\cdot h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\mathbb{1}_B^{B\cup\{x\}}\Bigr]. \end{aligned}$$

Similarly, we have

$$h - h(x) \ge -h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[(h-h(x))^{-}\big|_{B\cup\{x\}}\Bigr] \ge -\operatorname*{osc}_B[h]\cdot h^{\mathcal{E}}_{B\cup\{x\}}\Bigl[\mathbb{1}_B^{B\cup\{x\}}\Bigr].$$

Hence, by combining with (6.21), we get (6.22). □

Using Theorem 6.27, we can show the triangle inequality for $R_{\mathcal{E}}^{1/(p-1)}$.

Corollary 6.28. *$R_{\mathcal{E}}^{1/(p-1)} \colon X \times X \to [0,\infty)$ is a metric on X.*

Definition 6.29 (p-Resistance metric)**.** We define $\widehat{R}_{p,\mathcal{E}} := R_{\mathcal{E}}^{1/(p-1)}$. We call $\widehat{R}_{p,\mathcal{E}}$ the *p-resistance metric* of $(\mathcal{E},\mathcal{F})$.

Proof. It suffices to prove $R_{\mathcal{E}}(x,z)^{1/(p-1)} \le R_{\mathcal{E}}(x,y)^{1/(p-1)} + R_{\mathcal{E}}(y,z)^{1/(p-1)}$ for any $x,y,z \in X$ with $\#\{x,y,z\} = 3$. By (6.21) with $B = \{z\}$, we have $h^{\mathcal{E}}_{\{x,z\}}\bigl[\mathbb{1}_x^{\{x,z\}}\bigr](y) \le \frac{R_{\mathcal{E}}(x,y)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}}$. By exchanging the roles of x and z, we get $h^{\mathcal{E}}_{\{x,z\}}\bigl[\mathbb{1}_z^{\{x,z\}}\bigr](y) \le \frac{R_{\mathcal{E}}(y,z)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}}$. Since $\mathbb{1}_X = h^{\mathcal{E}}_{\{x,z\}}\bigl[\mathbb{1}_x^{\{x,z\}}\bigr] + h^{\mathcal{E}}_{\{x,z\}}\bigl[\mathbb{1}_z^{\{x,z\}}\bigr]$, we have

$$1 \le \frac{R_{\mathcal{E}}(x,y)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}} + \frac{R_{\mathcal{E}}(y,z)^{1/(p-1)}}{R_{\mathcal{E}}(x,z)^{1/(p-1)}},$$

which proves the desired triangle inequality for $R_{\mathcal{E}}^{1/(p-1)}$. □

Example 6.30. Let $p \in (1,\infty)$ and $(\mathcal{E},\mathcal{F})$ be a p-resistance form on the unit open interval $(0,1)$ given by

$$\mathcal{F} := W^{1,p}(0,1) \quad \text{and} \quad \mathcal{E}(u) := \int_0^1 |\nabla u|^p \, dx.$$

(Recall Example 6.3-(1).) For any $x,y \in (0,1)$ with $0 < x < y < 1$, we easily see that $u \in W^{1,p}(0,1)$ defined by $u(t) := (y-x)^{-1}(t-x)\mathbb{1}_{[x,y]}(t)$, $t \in (0,1)$, is $\mathcal{E}$-harmonic on $(0,1) \setminus \{x,y\}$. Therefore we have $R_{\mathcal{E}}(x,y) = (y-x)^{p-1}$ and the p-resistance metric $\widehat{R}_{p,\mathcal{E}}$ coincides with the Euclidean metric on $(0,1)$. This example also show that exponent $1/(p-1)$ is sharp, that is, $R_{\mathcal{E}}^{\alpha}$ is not a metric for $\alpha > 1/(p-1)$ in general.

7 Self-similar p-resistance forms on self-similar sets

In this section, we investigate p-resistance forms by focusing on the self-similar case as in Section 5. Throughout this section, we fix $p \in (1,\infty)$ and a self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with $S \geq 2$.

We first introduce the notion of *self-similar p-resistance form.*

Definition 7.1 (Self-similar p-resistance form)**.** Let $\rho = (\rho_i)_{i\in S} \in (0,\infty)^S$ and let $(\mathcal{E},\mathcal{F})$ be a p-resistance form on K. We say that $(\mathcal{E},\mathcal{F})$ is a *self-similar p-resistance form on* $\mathcal{L}$ *with weight* ρ if and only if $\mathcal{F} \subseteq C(K)$ and $(\mathcal{E},\mathcal{F})$ satisfies (5.5) and (5.6).

In the rest of this section, we also fix a self-similar p-resistance form $(\mathcal{E},\mathcal{F})$ on $\mathcal{L}$ with weight $\rho = (\rho_i)_{i\in S} \in (0,\infty)^S$.

The following properties of the p-resistance metric are elemental.

Proposition 7.2. (1) *For any* $x, y \in K$,

$$R_{\mathcal{E}}(F_w(x), F_w(y)) \leq \rho_w^{-1} R_{\mathcal{E}}(x,y) \tag{7.1}$$

(2) *If* $\min_{i\in S}\rho_i > 1$*, then* $\widehat{R}_{p,\mathcal{E}}$ *is compatible with the original topology of* K*. In particular,* V_* *is dense in* $(K, \widehat{R}_{p,\mathcal{E}})$*.*

Proof. (1): It is immediate from (5.6). (See [Kig01, Lemma 3.3.5] for the case $p = 2$.)

(2): We can follow the argument in the proof of [Kig01, Theorem 3.3.4] by using (7.1) and $\min_{i\in S}\rho_i > 1$, and conclude that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the topology of K (see also [Kig09, Proposition B.2] for a similar statement). Since $\overline{V_*}^K = K$ by [Kig01, Lemma 1.3.11], we complete the proof. □

The next proposition presents a nice sequence of p-resistance forms on $\{V_n\}_{n\in\mathbb{N}\cup\{0\}}$ called a *compatible sequence* on $\mathcal{L}$.

Proposition 7.3. *Assume that* $\widehat{R}_{p,\mathcal{E}}$ *is compatible with the original topology of* K*. Let* $n \in \mathbb{N}\cup\{0\}$ *and let* Λ *be a partition of* Σ*. Define* $V_{n,\Lambda} \coloneqq \bigcup_{w\in\Lambda} F_w(V_n)$*. Then for any* $u \in \mathcal{F}|_{V_{n,\Lambda}}$,

$$\mathcal{E}|_{V_{n,\Lambda}}(u) = \sum_{w\in\Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w). \tag{7.2}$$

Moreover, for any $w \in \Lambda$,

$$h^{\mathcal{E}}_{V_{n,\Lambda}}(u) \circ F_w = h^{\mathcal{E}}_{V_n}(u \circ F_w). \tag{7.3}$$

In particular, for any $m \in \mathbb{N}\cup\{0\}$ *and any* $u \in \mathcal{F}|_{V_{n+m}}$,

$$\mathcal{E}|_{V_{n+m}}(u) = \sum_{w\in W_m} \rho_w \mathcal{E}|_{V_n}(u \circ F_w). \tag{7.4}$$

Proof. Note that (7.4) follows from (7.2) by choosing $\Lambda = W_m$. For any $u \in \mathcal{F}|_{V_{n,\Lambda}}$,

$$\begin{aligned}\mathcal{E}|_{V_{n,\Lambda}}(u) &= \min\{\mathcal{E}(v) \mid v \in \mathcal{F} \text{ with } v|_{V_{n,\Lambda}} = u\} \\ &\overset{(5.7)}{=} \min\left\{\sum_{w\in\Lambda} \rho_w \mathcal{E}(v \circ F_w) \,\middle|\, v \in \mathcal{F} \text{ with } v|_{V_{n,\Lambda}} = u\right\} \\ &\geq \sum_{w\in\Lambda} \rho_w \min\{\mathcal{E}(v) \mid v \in \mathcal{F} \text{ with } v|_{V_n} = u \circ F_w\} = \sum_{w\in\Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w).\end{aligned}$$

To prove the converse, let $v := \sum_{w\in\Lambda}(F_w)_*\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr)$, which satisfies $v \in \mathcal{F}$ by (5.2), $C(K) = C(K, \widehat{R}_{p,\mathcal{E}})$, (5.5) and $v|_{V_{n,\Lambda}} = u$. Then we have (7.2) by seeing that

$$\mathcal{E}|_{V_{n,\Lambda}}(u) \leq \mathcal{E}(v) \overset{(5.7)}{=} \sum_{w\in\Lambda} \rho_w \mathcal{E}(v \circ F_w) = \sum_{w\in\Lambda} \rho_w \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr) = \sum_{w\in\Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w).$$

Next we prove (7.3). We have $\mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u] \circ F_w\bigr) \geq \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr)$ for any $w \in \Lambda$. Since

$$\begin{aligned}\mathcal{E}|_{V_{n,\Lambda}}(u) = \mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u]\bigr) &= \sum_{w\in\Lambda} \rho_w \mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u] \circ F_w\bigr) \\ &\geq \sum_{w\in\Lambda} \rho_w \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr) = \sum_{w\in\Lambda} \rho_w \mathcal{E}|_{V_n}(u \circ F_w) = \mathcal{E}|_{V_{n,\Lambda}}(u),\end{aligned}$$

we obtain $\mathcal{E}\bigl(h^{\mathcal{E}}_{V_{n,\Lambda}}[u] \circ F_w\bigr) = \mathcal{E}\bigl(h^{\mathcal{E}}_{V_n}[u \circ F_w]\bigr)$ for any $w \in \Lambda$. The uniqueness in Theorem 6.13 implies $h^{\mathcal{E}}_{V_{n,\Lambda}}[u] \circ F_w = h^{\mathcal{E}}_{V_n}[u \circ F_w]$. □

The following corollary is an immediate consequence of Proposition 6.20.

Corollary 7.4. *Assume that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure and that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K. Then*

$$\mathcal{F} = \left\{u \in C(K) \,\middle|\, \lim_{n\to\infty} \mathcal{E}|_{V_n}(u|_{V_n}) < \infty\right\}. \tag{7.5}$$

$$\mathcal{E}(u; v) = \lim_{n\to\infty} \mathcal{E}|_{V_n}(u|_{V_n}; v|_{V_n}) \quad \text{for any } u, v \in \mathcal{F}. \tag{7.6}$$

We easily obtain the following characterization of $\mathcal{E}$-harmonic functions on $K \setminus V_n$.

Proposition 7.5. *Assume that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K. Let $n \in \mathbb{N} \cup \{0\}$. Then for each $h \in C\bigl(K, \widehat{R}_{p,\mathcal{E}}\bigr)$, the following two conditions are equivalent to each other:*

(1) $h \in \mathcal{H}_{\mathcal{E},V_n}$.

(2) $h \circ F_w \in \mathcal{H}_{\mathcal{E},V_0}$ *for any* $w \in W_n$.

If in addition $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure, then (1) (or (2)) is also equivalent to the following condition:

(3) *For any $m \in \mathbb{N}$ with $m > n$ and any $x \in V_m \setminus V_n$,*

$$\sum_{w \in W_m; x \in F_w(V_0)} \rho_w \mathcal{E}|_{V_0}\Bigl(h \circ F_w|_{V_0}; \mathbb{1}^{V_0}_{F_w^{-1}(x)}\Bigr) = 0. \tag{7.7}$$

Proof. To see (1) $\Rightarrow$ (2), let us fix $w \in W_n$ and let $\varphi \in \mathcal{F}^0(K \setminus V_0)$. Then $(F_w)_*\varphi \in \mathcal{F}$ by (5.5) and $(F_w)_*\varphi \in \mathcal{F}^0(K \setminus V_n)$ by (5.2). By (5.6), we have

$$0 = \mathcal{E}(h; (F_w)_*\varphi) = \rho_w \mathcal{E}(h \circ F_w; \varphi),$$

which implies $h \circ F_w \in \mathcal{H}_{\mathcal{E},V_0}$. The converse implication (2) $\Rightarrow$ (1) is obvious from (5.6).

Next we prove the equivalence of (1) and (3) for a p.-c.f. self-similar structure $\mathcal{L}$. We first show (1) $\Rightarrow$ (3). For any $m > n$ and any $\varphi \in \mathcal{F}^0(K \setminus V_n)$, we note that $h^{\mathcal{E}}_{V_m}[\varphi|_{V_m}]\big|_{V_n} = 0$. Then, for any $h \in \mathcal{H}_{\mathcal{E},V_n}$, we have from (7.4) that

$$0 = \mathcal{E}|_{V_m}(h|_{V_m}; \varphi|_{V_m}) = \sum_{w \in W_m} \rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0}; \varphi \circ F_w|_{V_0}\bigr) \quad \text{for any } \varphi \in \mathcal{F}^0(K \setminus V_0).$$

By choosing $\varphi \in \mathcal{F}^0(K \setminus V_n)$ so that $\varphi|_{V_m} = \mathbb{1}^{V_m}_x$ for $x \in V_m \setminus V_n$, we obtain (3). We next suppoes that $h \in C(K)$ satisfies (7.7) and fix $\varphi \in \mathcal{F}^0(K \setminus V_n)$ in order to show the converse implication (3) $\Rightarrow$ (1). For $m > n$, we see from (7.4), $\varphi|_{V_n} = 0$ and (7.7) that

$$\begin{aligned}
\mathcal{E}|_{V_m}(h|_{V_m}; \varphi|_{V_m}) &= \sum_{w \in W_m} \rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0}; \varphi \circ F_w|_{V_0}\bigr) \\
&= \sum_{w \in W_m} \sum_{y \in V_0} \varphi(F_w(y)) \rho_w \mathcal{E}|_{V_0}\bigl(h \circ F_w|_{V_0}; \mathbb{1}^{V_0}_y\bigr) \\
&= \sum_{x \in V_m \setminus V_n} \varphi(x) \sum_{w \in W_m; x \in F_w(V_0)} \rho_w \mathcal{E}|_{V_0}\Bigl(h \circ F_w|_{V_0}; \mathbb{1}^{V_0}_{F_w^{-1}(x)}\Bigr) = 0.
\end{aligned}$$

By letting $m \to \infty$, we obtain $\mathcal{E}(h; \varphi) = 0$ and hence $h \in \mathcal{H}_{\mathcal{E},V_n}$. □

Thanks to the self-similarity, we can get the following localized version of the weak comparison principle (Proposition 6.21).

Proposition 7.6 (A localized weak comparison principle)**.** *Assume that $\widehat{R}_{p,\mathcal{E}}$ is compatible with the original topology of K. Let $n \in \mathbb{N} \cup \{0\}$, $w \in W_n$, and let $u, v \in \mathcal{H}_{\mathcal{E},V_n}$ satisfy $u(x) \le v(x)$ for any $x \in F_w(V_0)$. Then $u(x) \le v(x)$ for any $x \in K_w$.*

Proof. This is immediate from a combination of Proposition 6.21 and the implication from (1) to (2) in Proposition 7.5. □

Next we will show a new monotonicity on the equal weight of the p-resistance form on a p.-c.f. self-similar structure in p. We need the following basic result, which is immediate from (5.2) and Proposition 2.9-(a).

Proposition 7.7. *Let $k \in \mathbb{N} \cup \{0\}$ and let E be a p-resistance form on V_k. Define $\mathcal{S}_\rho(E)\colon \mathbb{R}^{V_{k+1}} \to [0,\infty)$ by*

$$\mathcal{S}_\rho(E)(u) \coloneqq \sum_{i\in S} \rho_i E(u \circ F_i), \quad u \in \mathbb{R}^{V_k}. \tag{7.8}$$

Then $\mathcal{S}_\rho(E)$ is a p-resistance form on V_{k+1}.

The following theorem states the desired monotonicity. (See also Theorem 8.29 for a similar result in another framework including Sierpiński carpets.)

Theorem 7.8. *Let $p, q \in (1,\infty)$ with $p \le q$ and let $\rho_s \in (1,\infty)$ for each $s \in \{p,q\}$. Assume that K is connected, that $\mathcal{L}$ is a p.-c.f. self-similar structure and that $(\mathcal{E}_s, \mathcal{F}_s)$ is a self-similar s-resistance form on $\mathcal{L}$ with weight $(\rho_s)_{i\in S}$ for each $s \in \{p,q\}$. Then*

$$\rho_p^{1/(p-1)} \le \rho_q^{1/(q-1)}. \tag{7.9}$$

Proof. We start by some preparations on discrete energies. Let $s \in \{p,q\}$. For any s-resistance form E_s on V_0 and $n \in \mathbb{N}$, we define $\mathcal{S}_{\rho_s,n}(E_s)\colon \mathbb{R}^{V_n} \to [0,\infty)$ by

$$\mathcal{S}_{\rho_s,n}(E_s)(u) \coloneqq \rho_s^n \sum_{v\in W_n} E_s(u \circ F_v), \quad u \in \mathbb{R}^{V_n}.$$

Note that $\mathcal{S}_{\rho_s,1}^n = \mathcal{S}_{\rho_s,n}$ and that $\mathcal{S}_{\rho_s,n}(E)$ is also a s-resistance form on V_n by Proposition 7.7. We also define a s-resistance form $E_{s,n}$ on V_n by

$$E_{s,n}(u) \coloneqq \rho_s^n \sum_{v\in W_n} \sum_{\{x,y\}\in V_0} \left| u(F_v(x)) - u(F_v(y)) \right|^s, \quad u \in \mathbb{R}^{V_n}.$$

Then we easily see that $\mathcal{S}_{\rho_s,n}(E_{s,0}) = E_{s,n}$. Since both $E_{s,0}(\,\cdot\,)^{1/s}$ and $\mathcal{E}_s|_{V_0}(\,\cdot\,)^{1/s}$ are norms on the finite-dimensional vector space $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$, there exists a constant $C_s \ge 1$ depending only on s and $\#V_0$ such that

$$C_s^{-1} E_{s,0}(u) \le \mathcal{E}_s|_{V_0}(u) \le C_s E_{s,0}(u) \quad \text{for any } u \in \mathbb{R}^{V_0}. \tag{7.10}$$

Since $\mathcal{S}_{\rho_s,n}(\mathcal{E}_s|_{V_0}) = \mathcal{E}_s|_{V_n}$ by (7.4), we have from (7.10) that

$$C_s^{-1} E_{s,n}(u) \le \mathcal{E}_s|_{V_n}(u) \le C_s E_{s,n}(u) \quad \text{for any } n \in \mathbb{N}\cup\{0\} \text{ and any } u \in \mathbb{R}^{V_n}. \tag{7.11}$$

Now we move to the proof of (7.9). Let us fix $x_0, y_0 \in V_0$ with $x_0 \neq y_0$ and set $B \coloneqq \{x_0, y_0\}$. Then it is easy to find $w \in W_*$ so that $B \cap K_w = \emptyset$ and $h_{p,w} \coloneqq h_p \circ F_w \notin \mathbb{R}\mathbb{1}_K$, where $h_p \coloneqq h_{V_0}^{\mathcal{E}_p}[\mathbb{1}_{x_0}]$. (If $h_p \circ F_w \in \mathbb{R}\mathbb{1}_K$ for any $w \in W_*$ with $B \cap K_w = \emptyset$, then we can easily obtain a contradiction by using (6.3) and [Kig01, Theorem 1.6.2], where the connectedness of K is used.) Since $c \coloneqq \inf_{x\in K_w} R_{\mathcal{E}_p}(x, B) > 0$ and $0 \le h_p \le 1$ by (6.19), for any $n \in \mathbb{N}$,

$$\mathcal{E}_q|_{V_n}(h_{p,w}|_{V_n})$$

$$
\begin{aligned}
&\overset{(7.11)}{\leq} C_q E_{q,n}(h_{p,w}|_{V_n}) \\
&= C_q \rho_q^n \sum_{v \in W_n} \sum_{\{x,y\} \in E_0} |h_p(F_{wv}(x)) - h_p(F_{wv}(y))|^{q-p} \cdot |h_{p,w}(F_v(x)) - h_{p,w}(F_v(y))|^p \\
&\overset{(6.22)}{\leq} C_q \rho_q^n \sum_{w \in W_n} \sum_{\{x,y\} \in E_0} \left(\frac{R_{\mathcal{E}_p}(F_{wv}(x), F_{wv}(y))}{R_{\mathcal{E}_p}(F_{wv}(x), B)} \right)^{\frac{q-p}{p-1}} \cdot |h_{p,w}(F_w(x)) - h_{p,w}(F_w(y))|^p \\
&\overset{(7.1)}{\leq} \left(C_q c^{-(q-p)/(p-1)} \sup_{x,y \in K} R_{\mathcal{E}_p}(x,y)^{(q-p)/(p-1)} \right) \left(\rho_q \rho_p^{-(q-1)/(p-1)} \right)^n E_{p,n}(h_{p,w}|_{V_n}) \\
&\overset{(7.11)}{\leq} \left(C_p C_q c^{-(q-p)/(p-1)} \sup_{x,y \in K} R_{\mathcal{E}_p}(x,y)^{(q-p)/(p-1)} \right) \left(\rho_q \rho_p^{-(q-1)/(p-1)} \right)^n \mathcal{E}_p(h_{p,w}). \qquad (7.12)
\end{aligned}
$$

Since both $\mathcal{E}_p(h_{p,w})$ and $\mathcal{E}_q(h_{p,w})$ are not equal to 0, we conclude that $\rho_q \rho_p^{-(q-1)/(p-1)} \geq 1$ by letting $n \to \infty$ in (7.12). This proves the desired monotonicity (7.9). □

8 Constructions of p-energy forms satisfying generalized p-contraction property

We have established fundamental results on p-energy forms satisfying the generalized p-contraction property $(\mathrm{GC})_p$, especially p-Clarkson's inequality $(\mathrm{Cla})_p$. In this section, we would like to describe how to get a good p-energy form satisfying these properties in a few settings inspired by [Kig23] and [CGQ22]. (See also [KS.a] for another approach toward such a construction.)

8.1 Construction of p-energy forms on p-conductively homogeneous compact metric spaces

In this subsection, we verify that p-energy forms on *p-conductively homogeneous* compact metric spaces constructed in [Kig23] satisfy $(\mathrm{GC})_p$. We mainly follow the notation and terminology of [Kig23] in this and the next subsections. We refer to [Kig23, Chapter 2] and [Kig20, Chapters 2 and 3] for further details.

Throughout this subsection, we fix a locally finite, non-directed infinite tree (T, E_T) in the usual sense (see [Kig23, Definition 2.1] for example), and fix a *root* $\phi \in T$ of T. (Here T is the set of vertices and E_T is the set of edges.) For any $w \in T \setminus \{\phi\}$, we use $\overline{\phi w}$ to denote the unique simple path in T from ϕ to w.

Definition 8.1 ([Kig23, Definition 2.2]). (1) For $w \in T$, define $\pi \colon T \to T$ by

$$
\pi(w) := \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } \overline{\phi w} = (w_0, \dots, w_n), \\ \phi & \text{if } w = \phi. \end{cases}
$$

Set $S(w) \coloneqq \{v \in T \mid \pi(v) = w\} \setminus \{w\}$. Moreover, for $k \in \mathbb{N}$, we define $S^k(w)$ inductively as

$$S^{k+1}(w) = \bigcup_{v \in S(w)} S^k(v).$$

For $A \subseteq T$, define $S^k(A) \coloneqq \bigcup_{w \in A} S^k(A)$.

(2) For $w \in T$ and $n \in \mathbb{N} \cup \{0\}$, define $|w| \coloneqq \min\{n \geq 0 \mid \pi^n(w) = \phi\}$ and $T_n \coloneqq \{w \in T \mid |w| = n\}$.

(3) Define $\Sigma \coloneqq \{(\omega_n)_{n \geq 0} \mid \omega_n \in T_n \text{ and } \omega_n = \pi(\omega_{n+1}) \text{ for all } n \in \mathbb{N} \cup \{0\}\}$. For $\omega = (\omega_n)_{n \geq 0} \in \Sigma$, we write $[\omega]_n$ for $\omega_n \in T_n$. For $w \in T$, define $\Sigma_w \coloneqq \{(\omega_n)_{n \geq 0} \in \Sigma \mid \omega_{|w|} = w\}$. For $A \subseteq T$, define $\Sigma_A \coloneqq \bigcup_{w \in A} \Sigma_w$.

Let us recall the definition of a partition parametrized by a rooted tree (see [Kig20, Definition 2.2.1] and [Sas23, Lemma 3.6]).

Definition 8.2 (Partition parametrized by a tree)**.** Let K be a compact metrizable topological space without isolated points. A family of non-empty compact subsets $\{K_w\}_{w \in T}$ of K is called a *partition of K parametrized by the rooted tree* (T, E_T, ϕ) if and only if it satisfies the following conditions:

(P1) $K_\phi = K$ and for any $w \in T$, $\#K_w \geq 2$ and $K_w = \bigcup_{v \in S(w)} K_v$.

(P2) For any $w \in \Sigma$, $\bigcap_{n \geq 0} K_{[\omega]_n}$ is a single point.

In the rest of this subsection, we fix a compact metrizable topological space without isolated points K, a locally finite rooted tree (T, E_T, ϕ) satisfying $\#\{v \in T \mid \{v, w\} \in E_T\} \geq 2$ for any $w \in T$, a partition $\{K_w\}_{w \in T}$ parametrized by (T, E_T, ϕ), a metric d on K with $\operatorname{diam}(K, d) = 1$, and a Borel probability measure m on K. Now we introduce a graph approximation $\{(T_n, E_n^*)\}_{n \in \mathbb{N} \cup \{0\}}$ of K (see [Kig23, Proposition 2.8 and Definition 2.5-(3)]).

Definition 8.3. For $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$, define

$$E_n^* \coloneqq \bigl\{\{v, w\} \bigm| v, w \in T_n, v \neq w, K_v \cap K_w \neq \emptyset\bigr\},$$

and $E_n^*(A) = \bigl\{\{v, w\} \in E_n^* \bigm| v, w \in A\bigr\}$. Let d_n be the graph distance of (T_n, E_n^*). For $M \in \mathbb{N} \cup \{0\}$ and $w \in T_n$, define

$$\Gamma_M(w) \coloneqq \{v \in T_n \mid d_n(v, w) \leq M\} \quad \text{and} \quad U_M(x; n) \coloneqq \bigcup_{w \in T_n; x \in K_w} \bigcup_{v \in \Gamma_M(w)} K_v.$$

To state geometric assumptions in [Kig23], we need the following definition (see [Kig20, Definitions 2.2.1 and 3.1.15].)

Definition 8.4. (1) The partition $\{K_w\}_{w \in T}$ is said to be *minimal* if and only if $K_w \setminus \bigcup_{v \in T_{|w|} \setminus \{w\}} \neq \emptyset$ for any $w \in T$.

(2) The partition $\{K_w\}_{w \in T}$ is said to be *uniformly finite* if and only if $\sup_{w \in T} \#\Gamma_1(w) < \infty$.

We also recall the following standard notion on metric measure spaces (see, e.g., [Hei, Kig20, MT] for further background).

Definition 8.5. (1) The measure m is said to be *volume doubling* with respect to the metric d if and only if there exists $C_{\mathrm{D}} \in (0, \infty)$ such that

$$m(B_d(x, 2r)) \leq C_{\mathrm{D}} m(B_d(x, r)) \quad \text{for any } (x, r) \in K \times (0, \infty). \tag{8.1}$$

The constant C_{D} is called the doubling constant of m.

(2) Let $Q \in (0, \infty)$. The measure m is said to be *Q-Ahlfors regular* with respect to the metric d if and only if there exists $C_{\mathrm{AR}} \in [1, \infty)$ such that

$$C_{\mathrm{AR}}^{-1} r^Q \leq m(B_d(x, r)) \leq C_{\mathrm{AR}} r^Q \quad \text{for any } (x, r) \in K \times (0, \operatorname{diam}(K, d)). \tag{8.2}$$

The measure m is simply said to be *Ahlfors regular* (with respect to d) if there exists $Q \in (0, \infty)$ such that m is Q-Ahlfors regular. Also, the metric d is said to be Q-Ahlfors regular if there exists a Borel measure μ on K which is Q-Ahlfors regular with respect to d.

(3) A metric ρ on K is said to be *quasisymmetric* to d, $\rho \underset{\mathrm{QS}}{\sim} d$ for short, if and only if there exists a homeomorphism $\eta \colon [0, \infty) \to [0, \infty)$ such that

$$\frac{\rho(x, b)}{\rho(x, a)} \leq \eta\left(\frac{d(x, b)}{d(x, a)}\right) \quad \text{for any } x, a, b \in K \text{ with } x \neq a.$$

(4) The *Ahlfors regular conformal dimension* of (K, d) is the value $\dim_{\mathrm{ARC}}(K, d)$ defined as

$$\dim_{\mathrm{ARC}}(K, d) \coloneqq \inf\left\{ Q > 0 \;\middle|\; \begin{array}{l} \text{there exists a metric } \rho \text{ on } K \text{ such that} \\ \rho \underset{\mathrm{QS}}{\sim} d \text{ and } \rho \text{ is } Q\text{-Ahlfors regular} \end{array} \right\}.$$

If m is Ahlfors regular, then it is clearly volume doubling. It is well known that the existence of a Q-Ahlfors regular m on (K, d) implies that the Hausdorff dimension of (K, d) is Q.

Now we recall basic geometric conditions in [Kig23]. The conditions (1), (2) and (3) below are important to follow the rest of this paper.

Assumption 8.6 ([Kig23, Assumption 2.15])**.** There exist $M_* \in \mathbb{N}$ and $r_* \in (0, 1)$ such that the following conditions (1)-(5) hold.

(1) K_w is connected for any $w \in T$, $\{K_w\}_{w \in T}$ is minimal and uniformly finite, and $\inf_{m \geq 0} \min_{w \in T_m} \#S(w) \geq 2$.

(2) There exist $c_i \in (0, \infty)$, $i \in \{1, \ldots, 5\}$, such that the following conditions (2A)-(2C) are true.

(2A) For any $w \in T$,

$$c_1 r_*^{|w|} \leq \operatorname{diam}(K_w, d) \leq c_2 r_*^{|w|}. \tag{8.3}$$

(2B) For any $n \in \mathbb{N}$ and $x \in K$,

$$B_d(x, c_3 r_*^n) \subseteq U_{M_*}(x; n) \subseteq B_d(x, c_4 r_*^n). \tag{8.4}$$

(In [Kig20], the metric d is called M_*-*adapted* if the condition (8.4) holds.)

(2C) For any $n \in \mathbb{N}$ and $w \in T_n$, there exists $x_w \in K_w$ satisfying

$$K_w \supseteq B_d(x_w, c_5 r_*^n). \tag{8.5}$$

(3) There exist $m_1 \in \mathbb{N}$, $\gamma_1 \in (0,1)$ and $\gamma \in (0,1)$ such that

$$m(K_w) \geq \gamma m(K_{\pi(w)}) \quad \text{for any } w \in T, \tag{8.6}$$

and

$$m(K_v) \leq \gamma_1 m(K_w) \quad \text{for any } w \in T \text{ and } v \in S^{m_1}(w). \tag{8.7}$$

Furthermore, m is volume doubling with respect to d and

$$m(K_w) = \sum_{v \in S(w)} m(K_v) \quad \text{for any } w \in T. \tag{8.8}$$

(4) There exists $M_0 \geq M_*$ such that for any $w \in T$, $k \geq 1$ and $v \in S^k(w)$,

$$\Gamma_{M_*}(v) \cap S^k(w) \subseteq \left\{ v' \in T_{|v|} \,\middle|\, \begin{array}{l} \text{there exist } l \leq M_0 \text{ and } (v_0, \dots, v_l) \in S^k(w)^{l+1} \\ \text{such that } (v_{j-1}, v_j) \in E_{|v|}^* \text{ for any } j \in \{1, \dots, l\} \end{array} \right\}.$$

(5) For any $w \in T$, $\pi(\Gamma_{M_*+1}(w)) \subseteq \Gamma_{M_*}(\pi(w))$.

We record a simple consequence of (8.8) in the next proposition.

Proposition 8.7. *Assume that the Borel probability measure m satisfies (8.8). Then $m(K_v \cap K_w) = 0$ for any $v, w \in T$ with $v \neq w$ and $|v| = |w|$.*

Proof. Let $n \in \mathbb{N} \cup \{0\}$ and $v, w \in T_n$ such that $v \neq w$. Enumerate T_n as $\{z(1), z(2), \dots, z(l_n)\}$ such that $z(1) = v$ and $z(2) = w$, where $l_n = \# T_n$. Inductively, we define $\widetilde{K}_{z(j)}$ by

$$\widetilde{K}_{z(1)} = K_{z(1)}$$

and

$$\widetilde{K}_{z(j+1)} = K_{z(j+1)} \setminus \left(\bigcup_{i=1}^{k} \widetilde{K}_{z(i)} \right).$$

Then $\left\{ \widetilde{K}_{z(j)} \right\}_{j=1}^{l_n}$ is a disjoint family of Borel sets and $\bigcup_{j=1}^{l_n} \widetilde{K}_{z(j)} = K$. Therefore,

$$1 = m(K) = \sum_{j=1}^{l_n} m\left(\widetilde{K}_{z(j)} \right).$$

On the other hand,(8.8) implies that

$$1 = m(K_\phi) = \sum_{j=1}^{l_n} m\bigl(K_{z(j)}\bigr).$$

Therefore, we conclude that $m\bigl(K_{z(j)} \setminus \widetilde{K}_{z(j)}\bigr) = 0$ for any $j \in \{1, \dots, l_n\}$. In particular,

$$0 = m\Bigl(K_{z(2)} \setminus \widetilde{K}_{z(2)}\Bigr) = m\Bigl(K_w \setminus \bigl(K_w \setminus (K_v \cap K_w)\bigr)\Bigr) = m(K_v \cap K_w),$$

which completes the proof. □

Next we introduce conductance, neighbor disparity constants and the notion of p-conductive homogeneity in Definitions 8.10, 8.8 and 8.11, following [Kig23, Sections 2.2, 2.3 and 3.3]. We will state some definitions and statements below for any $p \in (0, \infty)$ or $p \in [1, \infty)$, but on each such occasion we will explicitly declare that we let $p \in (0, \infty)$ or $p \in [1, \infty)$. Our main interest lies in the case $p \in (1, \infty)$.

Definition 8.8 ([Kig23, Definitions 2.17 and 3.4])**.** Let $p \in (0, \infty)$, $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$.

(1) Define $\mathcal{E}^n_{p,A} \colon \mathbb{R}^A \to [0, \infty)$ by

$$\mathcal{E}^n_{p,A}(f) := \sum_{\{u,v\} \in E^*_n(A)} |f(u) - f(v)|^p, \quad f \in \mathbb{R}^A.$$

We write $\mathcal{E}^n_p(f)$ for $\mathcal{E}^n_{p,T_n}(f)$.

(2) For $A_0, A_1 \subseteq A$, define $\mathrm{cap}^n_p(A_0, A_1; A)$ by

$$\mathrm{cap}^n_p(A_0, A_1; A) := \inf\bigl\{\mathcal{E}^n_{p,A}(f) \bigm| f \in \mathbb{R}^A, f|_{A_i} = i \text{ for } i \in \{0, 1\}\bigr\}.$$

(3) (Conductance constant) For $A_1, A_2 \subseteq A$ and $k \in \mathbb{N} \cup \{0\}$, define

$$\mathcal{E}_{p,k}(A_1, A_2, A) := \mathrm{cap}^{n+k}_p\bigl(S^k(A_1), S^k(A_2); S^k(A)\bigr).$$

For $M \in \mathbb{N}$, define $\mathcal{E}_{M,p,k} := \sup_{w \in T} \mathcal{E}_{p,k}(\{w\}, T_{|w|} \setminus \Gamma_M(w), T_{|w|})$.

Let us recall the notion of *covering system*, which will be used to define neighbor disparity constants and the notion of conductive homogeneity.

Definition 8.9 ([Kig23, Definitions 2.26-(3) and 2.29])**.** Let $N_T, N_E \in \mathbb{N}$.

(1) Let $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$. A collection $\{G_i\}_{i=1}^k$ with $G_i \subseteq T_n$ is called a *covering of* $(A, E^*_n(A))$ *with covering numbers* (N_T, N_E) if and only if $A = \bigcup_{i=1}^k G_k$, $\max_{x \in A} \#\{i \mid x \in G_i\} \le N_T$ and for any $(u, v) \in E^*_n(A)$, there exists $l \le N_E$ and $\{w(1), \dots, w(l+1)\} \subseteq A$ such that $w(1) = u$, $w(l+1) = v$ and $(w(i), w(i+1)) \in \bigcup_{j=1}^k E^*_n(G_j)$ for any $i \in \{1, \dots, l\}$.

(2) Let $\mathscr{J} \subseteq \bigcup_{n\in\mathbb{N}\cup\{0\}}\{A \mid A \subseteq T_n\}$. The collection $\mathscr{J}$ is called a *covering system with covering number* (N_T, N_E) if and only if the following conditions are satisfied:
   (i) $\sup_{A\in\mathscr{J}} \#A < \infty$.
   (ii) For any $w \in T$ and $k \in \mathbb{N}$, there exists a finite subset $\mathscr{N} \subseteq \mathscr{J} \cap T_{|w|+k}$ such that $\mathscr{N}$ is a covering of $\bigl(S^k(w), E^*_{|w|+k}(S^k(w))\bigr)$ with covering numbers (N_T, N_E).
   (iii) For any $G \in \mathscr{J}$ and $k \in \mathbb{N}\cup\{0\}$, if $G \subseteq T_n$, then there exists a finite subset $\mathscr{N} \subseteq \mathscr{J} \cap T_{n+k}$ such that $\mathscr{N}$ is a covering of $\bigl(S^k(G), E^*_{n+k}(S^k(G))\bigr)$ with covering numbers (N_T, N_E).

   The collection $\mathscr{J}$ is simply said to be a *covering system* if and only if there exist $(N_T, N_E) \in \mathbb{N}^2$ such that $\mathscr{J}$ is a covering system with covering number (N_T, N_E).

Definition 8.10 ([Kig23, Definitions 2.26-(1),(2) and 2.29])**.** Let $p \in (0,\infty)$, $n \in \mathbb{N}$ and $A \subseteq T_n$.

(1) For $k \in \mathbb{N}\cup\{0\}$ and $f\colon T_{n+k} \to \mathbb{R}$, define $P_{n,k}f\colon T_n \to \mathbb{R}$ by

$$(P_{n,k}f)(w) \coloneqq \frac{1}{\sum_{v\in S^k(w)} m(K_v)} \sum_{v\in S^k(w)} f(v)m(K_v), \quad w \in T_n.$$

   (Note that $P_{n,k}f$ depends on the measure m.)

(2) (Neighbor disparity constant) For $k \in \mathbb{N}\cup\{0\}$, define

$$\sigma_{p,k}(A) \coloneqq \sup_{f\colon S^k(A)\to\mathbb{R}} \frac{\mathcal{E}^n_{p,A}(P_{n,k}f)}{\mathcal{E}^{n+k}_{p,S^k(A)}(f)}.$$

(3) Let $\mathscr{J} \subseteq \bigcup_{n\ge 0}\{A \mid A \subseteq T_n\}$ be a covering system. Define

$$\sigma^{\mathscr{J}}_{p,k,n} \coloneqq \max\{\sigma_{p,k}(A) \mid A \in \mathscr{J}, A \subseteq T_n\} \quad \text{and} \quad \sigma^{\mathscr{J}}_{p,k} \coloneqq \sup_{n\in\mathbb{N}\cup\{0\}} \sigma^{\mathscr{J}}_{p,k,n}.$$

Definition 8.11 ([Kig23, Definition 3.4])**.** Let $p \in [1,\infty)$. The compact metric space K (with a partition $\{K_w\}_{w\in T}$ and a measure m) is said to be *p-conductively homogeneous* if and only if there exists a covering system $\mathscr{J}$ such that

$$\sup_{k\in\mathbb{N}\cup\{0\}} \sigma^{\mathscr{J}}_{p,k}\mathcal{E}_{M_*,p,k} < \infty. \tag{8.9}$$

The next consequence of (8.9) is more important than the original definition of the p-conductive homogeneity for our purpose.

Theorem 8.12 (A part of [Kig23, Theorem 3.30])**.** *Let $p \in [1,\infty)$ and assume that Assumption 8.6 holds. If K is p-conductively homogeneous, then there exist $\alpha_0, \alpha_1 \in (0,\infty)$, $\sigma_p \in (0,\infty)$ and a covering system $\mathscr{J}$ such that for any $k \in \mathbb{N}\cup\{0\}$,*

$$\alpha_0\sigma_p^{-k} \le \mathcal{E}_{M_*,p,k} \le \alpha_1\sigma_p^{-k} \quad \text{and} \quad \alpha_0\sigma_p^{k} \le \sigma^{\mathscr{J}}_{p,k} \le \alpha_1\sigma_p^{k}. \tag{8.10}$$

In particular, the constant σ_p is determined by the following limit:

$$\sigma_p = \lim_{k\to\infty}\bigl(\mathcal{E}_{M_*,p,k}\bigr)^{-1/k}. \tag{8.11}$$

Remark 8.13. The existence of the limit in (8.11) is true without the p-conductive homogeneity. Indeed, if $(K, d, \{K_w\}_{w\in T})$ satisfies the conditions Assumption 8.6-(1),(2),(4),(5), then [Kig23, Theorem 2.23] together with Fekete's lemma implies the existence of the limit in (8.11) for *any* $p \in (0,\infty)$. For convenience, we call σ_p the *p-scaling factor* of $(K, d, \{K_w\}_{w\in T})$.

We also recall the "Sobolev space" $\mathcal{W}^p$ introduced in [Kig23, Lemma 3.13].

Definition 8.14. Let $p \in [1,\infty)$. Assume that Assumption 8.6-(1),(2),(4),(5) hold and let σ_p be the constant in (8.11).

(1) For $n \in \mathbb{N}\cup\{0\}$, define $P_n \colon L^1(K,m) \to \mathbb{R}$ by $P_n f(w) \coloneqq \fint_{K_w} f\,dm$, $w \in T_n$.

(2) Define $\mathcal{N}_p \colon L^p(K,m) \to [0,\infty]$ and a linear subspace $\mathcal{W}^p$ of $L^p(K,m)$ by

$$\mathcal{N}_p(f) \coloneqq \left(\sup_{n\in\mathbb{N}\cup\{0\}} \sigma_p^n \mathcal{E}_p^n(P_n f)\right)^{1/p}, \quad f \in L^p(K,m),$$
$$\mathcal{W}^p \coloneqq \bigl\{ f \in L^p(K,m) \bigm| \mathcal{N}_p(f) < \infty \bigr\},$$

and we equip $\mathcal{W}^p$ the norm $\|\cdot\|_{\mathcal{W}^p}$ defined by

$$\|f\|_{\mathcal{W}^p} \coloneqq \left(\|f\|_{L^p(K,m)}^p + \mathcal{N}_p(f)^p\right)^{1/p}, \quad f \in \mathcal{W}^p.$$

(3) For a linear subspace $\mathcal{D}$ of $\mathcal{W}^p$, we define

$$\mathcal{U}_p(\mathcal{D}) \coloneqq \left\{ \mathscr{E} \colon \mathcal{D} \to [0,\infty) \;\middle|\; \begin{array}{l} \mathscr{E}^{1/p} \text{ is a seminorm on } \mathcal{D}, \text{ there exist } \alpha_0, \alpha_1 \in (0,\infty) \\ \text{such that } \alpha_0 \mathcal{N}_p(f) \le \mathscr{E}(f)^{1/p} \le \alpha_1 \mathcal{N}_p(f) \text{ for any } f \in \mathcal{D} \end{array} \right\}.$$

For simplicity, set $\mathcal{U}_p \coloneqq \mathcal{U}_p(\mathcal{W}^p)$.

(4) For $n \in \mathbb{N}\cup\{0\}$ and $A \subseteq T_n$, we define $\widetilde{\mathcal{E}}_{p,A}^n \colon L^p(K,m) \to [0,\infty)$ by

$$\widetilde{\mathcal{E}}_{p,A}^n(f) \coloneqq \sigma_p^n \mathcal{E}_{p,A}^n(P_n f), \quad f \in L^p(K,m).$$

We also set $\widetilde{\mathcal{E}}_p^n(f) \coloneqq \widetilde{\mathcal{E}}_{p,T_n}^n(f)$.

We have the following property on $\mathcal{N}_p$ tanks to the connectedness of K and Assumption 8.6-(3).

Proposition 8.15. *Let $p \in [1,\infty)$. Assume that Assumption 8.6 holds. Then $\mathcal{N}_p(f) = 0$ if and only if there exists $c \in \mathbb{R}$ such that $f(x) = c$ for m-a.e. $x \in K$.*

Proof. It is clear that $\mathcal{N}_p(f) = 0$ if f is constant. Suppose that $f \in L^p(K,m)$ satisfies $\mathcal{N}_p(f) = 0$. Note that (T_n, E_n^*) is a connected graph for each $n \in \mathbb{N}\cup\{0\}$ (see [Kig23, Proposition 2.8]). Therefore, $\mathcal{N}_p(f) = 0$ implies that there exists $c_n \in \mathbb{R}$ such that $P_n f(w) = c_n$ for any $n \in \mathbb{N}\cup\{0\}$ and any $w \in T_n$. By (8.8), we have $c_n = c_{n+1}$ and hence there exists $c \in \mathbb{R}$ such that $c_n = c$ for any $n \in \mathbb{N} \cup \{0\}$. Now we let $\mathscr{L}_f \subseteq K$ denote the set of *Lebesgue points of* f, i.e.,

$$\mathscr{L}_f := \left\{ x \in K \;\middle|\; \lim_{r \downarrow 0} \fint_{B_d(x,r)} |f(x) - f(y)| \, m(dx) = 0 \right\}. \tag{8.12}$$

Then, by the volume doubling property of m and the Lebesgue differentiation theorem (see, e.g., [Hei, Theorem 1.8]), we have $\mathscr{L}_f \in \mathcal{B}(K)$ and $m(K \setminus \mathscr{L}_f) = 0$. For any $x \in \mathscr{L}_f$ and any $n \in \mathbb{N} \cup \{0\}$, by Proposition 8.7 and Assumption 8.6-(2),(3),

$$\begin{aligned} |f(x) - c| = \left| f(x) - \fint_{U_{M_*}(x;n)} f \, dm \right| &\le \frac{1}{m(U_{M_*}(x;n))} \int_{B_d(x, c_4 r_*^n)} |f(x) - f(y)| \, m(dy) \\ &\le C \fint_{B_d(x, c_4 r_*^n)} |f(x) - f(y)| \, m(dy), \end{aligned}$$

where we used (8.4) and the volume doubling property of m in the last inequality, and $C \in (0, \infty)$ is independent of x, f and n. By letting $n \to \infty$ in the estimate above, we obtain $f(x) = c$ for any $x \in \mathscr{L}_f$, which completes the proof. $\square$

As shown in [Shi24, Kig23], $\mathcal{W}^p$ is a nice Banach space embedded in $C(K)$ if K is p-conductively homogeneous and $p > \dim_{\mathrm{ARC}}$. In general, we can show the following theorem.

Theorem 8.16. *Let $p \in [1, \infty)$. Assume that $(K, d, \{K_w\}_{w \in T}, m)$ satisfies Assumption 8.6 and that K is p-conductively homogeneous. Then $\mathcal{W}^p$ is a Banach space and $\mathcal{W}^p$ is dense in $L^p(K, m)$. If $p \in (1, \infty)$, then $\mathcal{W}^p$ is reflexive and separable. Moreover, if in addition $p > \dim_{\mathrm{ARC}}(K, d)$, then $\mathcal{W}^p$ can be identified with a subspace of $C(K)$ and $\mathcal{W}^p$ is dense in $C(K)$ with respect to the uniform norm.*

Remark 8.17. By [Kig20, Theorem 4.6.9], the condition $p > \dim_{\mathrm{ARC}}(K, d)$ is equivalent to $\sigma_p > 1$.

Proof. Note that $\mathcal{W}^p$ is a Banach space by [Kig23, Lemma 3.24] and that $\mathcal{W}^p$ is dense in $L^p(K, m)$ by [Kig23, Lemma 3.28].

In the rest of this proof, we assume that $p \in (1, \infty)$. Let us show that $\mathcal{W}^p$ is reflexive. Theorem 8.12 and [Kig23, Lemma 2.27] together imply that there exists a constant $C \in (0, \infty)$ such that for any $k, l \in \mathbb{N}$, $A \subseteq T_k$ and $f \in \mathbb{R}^{S^l(A)}$,

$$\widetilde{\mathcal{E}}_{p,A}^{k}(P_{k,l} f) \le C \widetilde{\mathcal{E}}_{p,S^l(A)}^{k+l}(f). \tag{8.13}$$

The rest of the proof is very similar to the proof of [MS23+, Theorem 6.17], so we give a sketch (see also [Shi24, Theorem 5.9] and the proof of Theorem 8.19-(a) below). Let $\| \cdot \|_{p,n} := \left(\| \cdot \|_{L^p(K,m)}^p + \widetilde{\mathcal{E}}_p^n(\,\cdot\,) \right)^{1/p}$, which can be regarded as the L^p-norm on $K \sqcup E_n^*$. Also, we consider $\widetilde{\mathcal{E}}_p^n$ as a $[0, \infty]$-valued functional on $L^p(K, m)$. By [Dal, Theorem 8.5

and Proposition 11.6], we can assume that the sequence $\{\widetilde{\mathcal{E}}_p^n\}_{n\in\mathbb{N}}$ Γ-converges to some p-homogeneous functional $E_p\colon L^p(K,m)\to[0,\infty]$ as $n\to\infty$. Then $\|\cdot\|_{p,n}$ Γ-converges to $|||\cdot||| := \bigl(\|\cdot\|_{L^p(K,m)}^p + E_p\bigr)^{1/p}$ as $n\to\infty$, and hence $(|||\cdot|||^p, \mathcal{W}^p)$ is a p-energy form on (K,m) satisfying (Cla)$_p$. By using (8.13) and noting that $\lim_{k\to\infty} P_n f_k(w) = P_n f(w)$ for any $n\in\mathbb{N}\cup\{0\}$, any $w\in T_n$ and any $f, f_k\in L^p(K,m)$ with $\lim_{k\to\infty}\|f-f_k\|_{L^p(K,m)}=0$, we can show that $|||\cdot|||$ is a norm on $\mathcal{W}^p$ that is equivalent to $\|\cdot\|_{\mathcal{W}^p}$. Thus, $\mathcal{W}^p$ is reflexive by Proposition 3.4 and the Milman–Pettis theorem. The separability of $\mathcal{W}^p$ immediately follows from [AHM23, Proposition 4.1] since $L^p(K,m)$ is separable and the inclusion map of $\mathcal{W}^p$ into $L^p(K,m)$ is a continuous linear injection.

In the case $p>\dim_{\mathrm{ARC}}(K,d)$, $\mathcal{W}^p$ can be identified with a subspace of $C(K)$ and is dense in $(C(K),\|\cdot\|_{\sup})$ by [Kig23, Lemmas 3.15, 3.16 and 3.19]. $\square$

Let us introduce an important value, p-walk dimension, which will be a main topic in Section 9.

Definition 8.18 (p-Walk dimension)**.** Let $p\in(0,\infty)$. Assume that $(K,d,\{K_w\}_{w\in T})$ satisfies Assumption 8.6-(1),(2),(4),(5). Let $r_*\in(0,1)$ be the constant in (8.4), let σ_p be the p-scaling factor of $(K,d,\{K_w\}_{w\in T})$ (see (8.11) and Remark 8.13). We define $\tau_p\in\mathbb{R}$ by

$$\tau_p := \frac{\log\sigma_p}{\log r_*^{-1}}. \tag{8.14}$$

If in addition m is Ahlfors regular with respect to d, then we define $d_{\mathrm{w},p}\in\mathbb{R}$ by

$$d_{\mathrm{w},p} := d_{\mathrm{f}} + \tau_p. \tag{8.15}$$

We call $d_{\mathrm{w},p}$ the *p-walk dimension* of $(K,d,\{K_w\}_{w\in T})$.

Now we prove the main result in this subsection, which is an improvement of [Kig23, Theorem 3.21].

Theorem 8.19. *Let $p\in(1,\infty)$. Assume that $(K,d,\{K_w\}_{w\in T},m)$ satisfies Assumption 8.6 and that K is p-conductively homogeneous. Then there exist $\widehat{\mathcal{E}}_p\colon\mathcal{W}^p\to[0,\infty)$ and $c\in(0,\infty)$ such that the following hold:*

(a) *$(\widehat{\mathcal{E}}_p)^{1/p}$ is a seminorm on $\mathcal{W}^p$ and*

$$c\mathcal{N}_p(f)\le\widehat{\mathcal{E}}_p(f)^{1/p}\le\mathcal{N}_p(f)\quad\text{for any } f\in\mathcal{W}^p. \tag{8.16}$$

(b) *$(\widehat{\mathcal{E}}_p,\mathcal{W}^p)$ is a p-energy form on (K,m) satisfying* (GC)$_p$.

(c) (Invariance) *Let $\mathsf{T}\colon(K,\mathcal{B}(K),m)\to(K,\mathcal{B}(K),m)$ be Borel measurable and preserve m, i.e., $\mathsf{T}^{-1}(A)\in\mathcal{B}(K)$ and $m(\mathsf{T}^{-1}(A))=m(A)$ for any $A\in\mathcal{B}(K)$. Then $f\circ\mathsf{T}\in\mathcal{W}^p$ and $\widehat{\mathcal{E}}_p(f\circ\mathsf{T})=\widehat{\mathcal{E}}_p(f)$ for any $f\in\mathcal{W}^p$.*

(d) *If in addition $p>\dim_{\mathrm{ARC}}(K,d)$, then $(\widehat{\mathcal{E}}_p,\mathcal{W}^p)$ is a regular p-resistance form on K and there exist $C\in[1,\infty)$ such that*

$$C^{-1}d(x,y)^{\tau_p}\le R_{\widehat{\mathcal{E}}_p}(x,y)\le Cd(x,y)^{\tau_p}\quad\text{for any } x,y\in K. \tag{8.17}$$

Proof. The most part of the proof will be very similar to that in [Kig23, Theorem 3.21], but we present the details because we do not assume $p > \dim_{\mathrm{ARC}}(K, d)$ unlike [Kig23, Theorem 3.21]. Let $\widehat{\mathcal{E}}_p$ be a subsequential Γ-limit of $\{\widetilde{\mathcal{E}}_p^n\}_n$ with respect to the topology of $L^p(K, m)$ as in the proof of [Kig23, Theorem 3.21], i.e., there exists a subsequence $\{\widetilde{\mathcal{E}}_p^{n'}\}_{n'}$ Γ-converging to $\widehat{\mathcal{E}}_p$ with respect to $L^p(K, m)$ as $n' \to \infty$. (Note that such a subsequential Γ-limit exists by [Dal, Theorem 8.5].)

(a): $\widehat{\mathcal{E}}_p$ is p-homogeneous by [Dal, Proposition 11.6]. The triangle inequality for $\widehat{\mathcal{E}}_p(\,\cdot\,)^{1/p}$ will be included in the proof of (b), so we shall prove (8.16). From the definition of the Γ-convergence, it is immediate that $\widehat{\mathcal{E}}_p(f) \leq \liminf_{n\to\infty} \widetilde{\mathcal{E}}_p^n(f) \leq \mathcal{N}_p(f)^p$. Let us show the former inequality in (8.16). Let $f \in \mathcal{W}^p$ and let $\{f_{n'}\}_{n'}$ be a recovery sequence of $\{\widetilde{\mathcal{E}}_p^n\}_{n'}$ at f, i.e., $\lim_{n'\to\infty} \|f - f_{n'}\|_{L^p(K,m)} = 0$ and $\widehat{\mathcal{E}}_p(f) = \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}(f_{n'})$. Since $\lim_{n'\to\infty} P_k f_{n'}(w) = P_k f(w)$ for any $k \in \mathbb{N}$ and any $w \in T_k$, by (8.13),

$$\widetilde{\mathcal{E}}_p^k(f) = \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^k(f_{n'}) \leq C \lim_{n'\to\infty} \widetilde{\mathcal{E}}_p^{n'}(f_{n'}) = C\widehat{\mathcal{E}}_p(f),$$

where $C \in (0, \infty)$ is the constant in (8.13). We obtain the desired estimate by taking the supremum over $k \in \mathbb{N} \cup \{0\}$.

(b): Let us fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1). Define $Q_n\colon L^1(K, m) \to L^1(K, m)$ by

$$Q_n f \coloneqq \sum_{w\in T_n} P_n f(w)\mathbb{1}_{K_w} \quad \text{for } f \in L^1(K, m). \tag{8.18}$$

Note that $\|Q_n\|_{L^p(K,m)\to L^p(K,m)} \leq 1$ by (8.8) and Hölder's inequality. Let us show $\|f - Q_n f\|_{L^p(K,m)} \to 0$ as $n \to \infty$ for any $f \in L^p(K, m)$. Define the Hardy–Littlewood maximal operator $\mathscr{M}\colon L^p(K, m) \to L^0(K, m)$ by

$$\mathscr{M} f(x) = \sup_{r>0} ⨍_{B_d(x,r)} |f(y)|\, m(dy), \quad x \in K.$$

Since m is volume doubling with respect to d by Assumption 8.6-(3), [HKST, Theorem 3.5.6] implies that there exists a constant $C \in (0, \infty)$ such that $\|\mathscr{M} f\|_{L^p(K,m)} \leq C\|f\|_{L^p(K,m)}$ for any $f \in L^p(K, m)$. We also easily see that for any $f \in L^p(K, m)$ and any $x \in K$,

$$\begin{aligned} |Q_n f(x)| \leq \sum_{w\in T_n; x\in K_w} |P_n f(w)| &\leq \sum_{w\in T_n; x\in K_w} \frac{m(B_d(x, 2c_2 r_*^n))}{m(K_w)} ⨍_{B_d(x,2c_2 r_*^n)} |f|\, dm \\ &\leq \sum_{w\in T_n; x\in K_w} \frac{m(B_d(x, 2c_2 r_*^n))}{m(B_d(x_w, c_5 r_*^n))} \mathscr{M} f(x) \leq C_1 \mathscr{M} f(x), \end{aligned}$$

where $x_w \in K_w$ and c_2, c_5 are the same as in Assumption 8.6-(2) and we used the volume doubling property in the last inequality, and $C_1 \in (0, \infty)$ is a constant depending only on $\sup_{w\in T} \#\Gamma_1(w)$, c_2, c_5 and the doubling constant of m. Let $f \in L^p(K, m)$ and let

$\mathscr{L}_f \subseteq K$ denote the set of Lebesgue points of f (recall (8.12)). Then $\mathscr{L}_f \in \mathcal{B}(K)$ and $m(K \setminus \mathscr{L}_f) = 0$ by the Lebesgue differentiation theorem for a volume doubling metric measure space (see, e.g., [Hei, Theorem 1.8]). Since

$$\begin{aligned}|f(x) - Q_n f(x)| &\le \sum_{w \in T_n; x \in K_w} \fint_{K_w} |f(x) - f(y)|\, m(dy) \\ &\le C_1 \fint_{B_d(x, 2c_2 r_*^n)} |f(x) - f(y)|\, m(dy),\end{aligned}$$

we have $|f(x) - Q_n f(x)| \to 0$ as $n \to \infty$ for any $x \in \mathscr{L}_f$. Now the dominated convergence theorem implies $\|f - Q_n f\|_{L^p(K,m)} \to 0$.

Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (\mathcal{W}^p)^{n_1}$ and choose a recovery sequence $\{u_{k,n'}\}_{n'}$ of $\{\widetilde{\mathcal{E}}_p^{n'}\}_{n'}$ at u_k for each $k \in \{1, \dots, n_1\}$. For brevity, we write $\boldsymbol{u}_{n'} = (u_{1,n'}, \dots, u_{n_1,n'})$ and

$$\begin{aligned}P_{n'}\boldsymbol{u}_{n'}(v) &= \bigl(P_{n'}u_{1,n'}(v), \dots, P_{n'}u_{n_1,n'}(v)\bigr) \in \mathbb{R}^{n_1}, \quad v \in T_{n'}, \\ Q_{n'}\boldsymbol{u}_{n'}(v) &= \bigl(Q_{n'}u_{1,n'}(v), \dots, Q_{n'}u_{n_1,n'}(v)\bigr) \in \mathbb{R}^{n_1}, \quad v \in T_{n'}.\end{aligned}$$

Note that $\|u_{n'} - Q_{n'}u_{k,n'}\|_{L^p(K,m)} \to 0$ as $n' \to \infty$ by the fact proved in the previous paragraph. Similar to an argument in [Kig23, p. 46], by using $\|Q_n\|_{L^p(K,m) \to L^p(K,m)} \le 1$ and the estimate (2.19), we have

$$\|T_l(\boldsymbol{u}) - T_l(Q_{n'}\boldsymbol{u}_{n'})\|_{L^p(K,m)} \xrightarrow[n' \to \infty]{} 0 \quad \text{for any } l \in \{1, \dots, n_2\}. \tag{8.19}$$

Also, we note that

$$P_{n'}\bigl(T_l(Q_{n'}\boldsymbol{u}_{n'})\bigr) = T_l(P_{n'}\boldsymbol{u}_{n'}) \in \mathbb{R}^{T_{n'}} \quad \text{for any } l \in \{1, \dots, n_2\}. \tag{8.20}$$

With these preparations, we prove (GC)$_p$ for $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$. We suppose that $q_2 < \infty$ since the case $q_2 = \infty$ is similar. By (8.19) and (8.20), we see that

$$\begin{aligned}\sum_{l=1}^{n_2} \widehat{\mathcal{E}}_p\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} &\overset{(8.19)}{\le} \sum_{l=1}^{n_2} \liminf_{n' \to \infty} \widetilde{\mathcal{E}}_p^{n'}\bigl(T_l(Q_{n'}\boldsymbol{u}_{n'})\bigr)^{q_2/p} \\ &\overset{(8.20)}{\le} \liminf_{n' \to \infty} \sum_{l=1}^{n_2} \left[\frac{\sigma_p^{n'}}{2} \sum_{(v,w) \in E_{n'}^*} |T_l(P_{n'}\boldsymbol{u}_{n'}(v)) - T_l(P_{n'}\boldsymbol{u}_{n'}(w))|^{q_2 \cdot \frac{p}{q_2}}\right]^{q_2/p} \\ &\overset{(2.17)}{\le} \liminf_{n' \to \infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w) \in E_{n'}^*} \|T(P_{n'}\boldsymbol{u}_{n'}(v)) - T(P_{n'}\boldsymbol{u}_{n'}(v))\|_{\ell^{q_2}}^p\right)^{q_2/p} \\ &\overset{(2.1)}{\le} \liminf_{n' \to \infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w) \in E_{n'}^*} \|P_{n'}\boldsymbol{u}(v) - P_{n'}\boldsymbol{u}(v)\|_{\ell^{q_1}}^p\right)^{q_2/p} \\ &\le \liminf_{n' \to \infty} \left(\frac{\sigma_p^{n'}}{2} \sum_{(v,w) \in E_{n'}^*} \left[\sum_{k=1}^{n_1} |P_{n'}u_{k,n'}(v) - P_{n'}u_{k,n'}(w)|^{p \cdot \frac{q_1}{p}}\right]^{p/q_1}\right)^{q_2/p}\end{aligned}$$

$$\overset{(*)}{\leq} \liminf_{n'\to\infty}\left(\sum_{k=1}^{n_1}\left[\frac{\sigma_p^{n'}}{2}\sum_{(v,w)\in E^*_{n'}}|P_{n'}u_{k,n'}(v)-P_{n'}u_{k,n'}(w)|^p\right]^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}}$$

$$\leq \left(\sum_{k=1}^{n_1}\limsup_{n'\to\infty}\widetilde{\mathcal{E}}_p^{n'}(u_{k,n'})^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} \leq \left(\sum_{k=1}^{n_1}\widehat{\mathcal{E}}_p(u_k)^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}}, \tag{8.21}$$

where we used the triangle inequality for the ℓ^{p/q_1}-norm on E_n^* in $(*)$. Hence $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ satisfies (GC)$_p$.

(c): This is clear from $P_n f = P_n(f\circ \mathsf{T}) \in \mathbb{R}^{T_n}$ for any $n\in\mathbb{N}\cup\{0\}$, $f\in L^p(K,m)$.

(d): In the case $p>\dim_{\mathrm{ARC}}(K,d)$, a combination of (b), [Kig23, Lemmas 3.13, 3.16, 3.19 and Theorem 3.21] and Theorem 8.16 implies that $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ is a regular p-resistance form on K. Then the estimate (8.17) is exactly the same as [Kig23, (3.21) in Lemma 3.34], so we complete the proof. □

Remark 8.20. The construction of $\mathcal{E}_p^\Gamma$ in [MS23+, Theorem 6.22] is very similar to that of $\widehat{\mathcal{E}}_p$ in the proof above although the setting and assumption on a 'partition' in [MS23+] is slightly different from ours. Thanks to Proposition 8.7, the operators M_n and J_n defined in [MS23+, (6.8) and (6.9)] correspond to P_n and Q_n respectively. In particular, (8.19) and (8.20) for M_n and J_n are also true. Hence we can easily see that the p-energy form $(\mathcal{E}_p^\Gamma, \mathcal{F}_p)$ in [MS23+, Theorem 6.22] also satisfies (GC)$_p$.

8.2 Construction of self-similar p-energy forms on p-conductively homogeneous self-similar structures

In this subsection, we construct a self-similar p-resistance form on self-similar structures under suitable assumptions. Our main result in this subsection. Theorem 8.27, implies that self-similar p-energy forms constructed in [Kig23, Theorem 4.6] satisfy (GC)$_p$.

We start with some preparations before constructing self-similar p-resistance forms. In the next definition, we introduce a good partition parametrized by a rooted tree.

Definition 8.21 ([Kig23, Definition 4.2]). Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure, let $r\in(0,1)$ and let $(j_s)_{s\in S}\in\mathbb{N}^S$. Define

$$j(w) := \sum_{i=1}^{n} j_{w_i} \quad\text{and}\quad g(w) := r^{j(w)} \quad\text{for } w = w_1\dots w_n \in W_n.$$

Define $\widetilde{\pi}(w_1\cdots w_n) := w_1\cdots w_{n-1}$ for $w = w_1\dots w_n\in W_n$ and

$$\Lambda^g_{r^k} := \{w = w_1\cdots w_n \in W_* \mid g(\widetilde{\pi}(w)) > r^k \geq g(w)\}.$$

Set $T_k^{(r)} := \{(k,w) \mid w\in\Lambda^g_{r^k}\}$, $T^{(r)} := \bigcup_{k\in\mathbb{N}\cup\{0\}} T_k^{(r)}$ and define $\iota\colon T^{(r)}\to W_*$ as $\iota(k,w) = w$. Moreover, define $E_{T^{(r)}} \subseteq T^{(r)}\times T^{(r)}$ by

$$E_{T^{(r)}} := \left\{((k,v),(k+1,w)) \in T_k^{(r)}\times T_{k+1}^{(r)} \,\middle|\, k\in\mathbb{N}\cup\{0\}, v = w \text{ or } v = \widetilde{\pi}(w)\right\}.$$

In the rest of this subsection, we presume the following assumption on the geometry of our self-similar structure.

Assumption 8.22. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure such that $\#S \geq 2$. There exist $r_* \in (0,1)$ and a metric d giving the original topology of K with $\operatorname{diam}(K,d) = 1$ such that $(K, d, \{K_w\}_{w\in T^{(r_*)}}, m)$ satisfies Assumption 8.6, where m is the self-similar measure on K with weight $(r_*^{j_s d_{\mathrm{f}}})_{s\in S}$ and d_{f} is the unique number satisfying $\sum_{s\in S} r_*^{j_s d_{\mathrm{f}}} = 1$.

Under Assumption 8.22, we have the d_{f}-Ahlfors regularity of m as follows.

Proposition 8.23 ([Kig23, Proposition 4.5])**.** *The value d_{f} coincides with the Hausdorff dimension of (K,d) and m is d_{f}-Ahlfors regular with respect to d.*

To obtain a self-similar p-energy form on $\mathcal{L}$, we first discuss the self-similarity for $\mathcal{W}^p$ (recall (5.5)). The following lemma can be shown in exactly the same way as [Kig23, Theorem 4.6-(1)] although the condition $p > \dim_{\mathrm{ARC}}(K,d)$ is assumed in [Kig23, Theorem 4.6].

Lemma 8.24. *For any $u \in L^p(K,m)$, any $k \in \mathbb{N}\cup\{0\}$ and any $n \in \mathbb{N}\cup\{0\}$ with $n \geq \max_{w\in W_k} j(w)$,*

$$\sum_{w\in W_k} \mathcal{E}_p^{n-j(w)}(P_{n-j(w)}(u\circ F_w)) \leq \mathcal{E}_p^n(P_n u). \tag{8.22}$$

In particular, if in addition K is p-conductively homogeneous, then $u\circ F_w \in \mathcal{W}^p$ for any $u \in \mathcal{W}^p$ and any $w \in W_*$, and hence

$$\mathcal{W}^p \cap C(K) \subseteq \{u \in C(K) \mid u\circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}. \tag{8.23}$$

We will obtain a self-similar p-energy form as a fixed point of the following map $\mathcal{S}_{\sigma_p}$ as in the case $p = 2$ (see, e.g., [Kig00, KZ92]).

Definition 8.25. Let $p \in (1,\infty)$ and assume that K is p-conductively homogeneous. Recall that σ_p denotes the p-scaling factor (see (8.11)). Let $\mathcal{D}$ be a linear subspace of $\mathcal{W}^p$ and assume that

$$u\circ F_i \in \mathcal{D} \text{ for any } u \in \mathcal{D} \text{ and any } i \in S. \tag{8.24}$$

For any $n \in \mathbb{N}\cup\{0\}$ and any $E_p \in \mathcal{U}_p(\mathcal{D})$, we define a p-energy form $(\mathcal{S}_{\sigma_p,n}(E_p), \mathcal{D})$ on (K,m) by

$$\mathcal{S}_{\sigma_p,n}(E_p)(u) \coloneqq \sum_{w\in W_n} \sigma_p^{j(w)} E_p(u\circ F_w) \quad \text{for } u \in \mathcal{D}. \tag{8.25}$$

(Note that the triangle inequality for $\mathcal{S}_{\sigma_p,n}(E_p)^{1/p}$ is easy.) Set $\mathcal{S}_{\sigma_p} \coloneqq \mathcal{S}_{\sigma_p,1}$ for simplicity. Clearly, $\mathcal{S}_{\sigma_p,n} = \mathcal{S}_{\sigma_p}^n$.

We need the converse inclusion of (8.23) and uniform estimates on $\mathcal{S}_{\sigma_p,n}(E)$ for any/some $E \in \mathcal{U}_p(\mathcal{W}^p \cap C(K))$ to construct a self-similar p-energy form on $\mathcal{L}$. These conditions are true if K is p-conductively homogeneous and $p > \dim_{\mathrm{ARC}}(K,d)$. (This result is essentially proved in [Kig23, Proof of Theorem 4.6].)

Proposition 8.26. *Let $p \in (1,\infty)$ and assume that K is p-conductively homogeneous. If $p > \dim_{\mathrm{ARC}}(K,d)$, then*

$$\mathcal{W}^p = \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}, \tag{8.26}$$

and there exists $C \in [1,\infty)$ such that for any $E \in \mathcal{U}_p$, any $u \in \mathcal{W}^p$ and any $n \in \mathbb{N}$,

$$C^{-1}\mathcal{N}_p(u)^p \le \mathcal{S}_{\sigma_p,n}(E)(u) \le C\mathcal{N}_p(u)^p. \tag{8.27}$$

Proof. The uniform estimate (8.27) follows from [Kig23, (4.6) and (4.8)]. (In the proof of [Kig23], the assumption $p > \dim_{\mathrm{ARC}}(K,d)$ is used to obtain [Kig23, (4.8)].) In the rest of the proof, we prove

$$\mathcal{W}^p \supseteq \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\} =: \mathcal{W}^p_S.$$

(The converse inclusion is proved in Lemma 8.24.) We note that the following estimate in [Kig23, lines 8-9 in p. 61] is true for every $u \in \mathcal{W}^p_S$: there exists a constant $C' \in (0,\infty)$ such that

$$\widetilde{\mathcal{E}}^n_p(u) \le C' \sum_{w \in W_n} \sigma_p^{j(w)} \mathcal{N}_p(u \circ F_w)^p = C'\mathcal{S}_{\sigma_p,n}(\mathcal{N}_p^p)(u) \quad \text{for any } n \in \mathbb{N},\ u \in \mathcal{W}^p_S. \tag{8.28}$$

(We need $p > \dim_{\mathrm{ARC}}(K,d)$ to obtain (8.28) by following the argument in [Kig23, p. 61].) Taking the supremum over $n \in \mathbb{N}$ in the left-hand side of (8.28), we have $\mathcal{W}^p_S \subseteq \mathcal{W}^p$. $\square$

Now we can prove the desired improvement of [Kig23, Theorem 4.6].

Theorem 8.27. *Let $p \in (1,\infty)$. Assume that Assumption 8.22 holds, that K is p-conductively homogeneous and that the following* pre-self-similarity conditions *hold:*

$$\mathcal{W}^p \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{W}^p \text{ for any } i \in S\}. \tag{8.29}$$

$$\text{There exists } C \in [1,\infty) \text{ such that (8.27) holds for any } u \in \mathcal{W}^p \cap C(K),\ n \in \mathbb{N}. \tag{8.30}$$

Let σ_p be the constant in (8.11) and let $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ be any p-energy form on (K,m) given in Theorem 8.19. Then there exists $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that the following limit exists in $[0,\infty)$ for any $u \in \mathcal{W}^p \cap C(K)$:

$$\mathcal{E}_p(u) := \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \mathcal{S}^j_{\sigma_p}(\widehat{\mathcal{E}}_p)(u). \tag{8.31}$$

Moreover, the following properties hold:

(a) *$(\mathcal{E}_p, \mathcal{W}^p \cap C(K))$ is a self-similar p-energy form on $(\mathcal{L}, m)$ with weight $(\sigma_p^{j_s})_{s\in S}$, and there exists $\alpha_0, \alpha_1 \in (0,\infty)$ such that $\alpha_0\mathcal{N}_p(u)^p \le \mathcal{E}_p(u) \le \alpha_1\mathcal{N}_p(u)^p$ for any $u \in \mathcal{W}^p \cap C(K)$.*

(b) (Generalized p-contraction property) *$(\mathcal{E}_p, \mathcal{W}^p \cap C(K))$ satisfies $(\mathrm{GC})_p$.*

(c) (Strong locality; see also Definition 6.22-(1)) *Let* $u_1, u_2, v \in \mathcal{W}^p \cap C(K)$ *and* $a_1, a_2 \in \mathbb{R}$. *If* $\operatorname{supp}_K[u_1 - a_1\mathbb{1}_K] \cap \operatorname{supp}_K[u_2 - a_2\mathbb{1}_K] = \emptyset$, *then* $\mathcal{E}_p(u_1 + u_2 + v) + \mathcal{E}_p(v) = \mathcal{E}_p(u_1 + v) + \mathcal{E}_p(u_2 + v)$.

(d) *If in addition* $p > \dim_{\mathrm{ARC}}(K, d)$, *then* $(\mathcal{E}_p, \mathcal{W}^p)$ *is a regular self-similar* p*-resistance form on* $\mathcal{L}$ *with weight* $(\sigma_p^{j_s})_{s\in S}$ *and there exists* $\alpha_0, \alpha_1 \in (0,\infty)$ *such that*

$$\alpha_0 d(x,y)^{\tau_p} \le R_{\mathcal{E}_p}(x,y) \le \alpha_1 d(x,y)^{\tau_p} \quad \textit{for any } x, y \in K. \tag{8.32}$$

Remark 8.28. In [CGQ22], self-similar p-energy forms on p.-c.f. self-similar structures are constructed, which are p-resistance forms under a certain condition as shown in Subsection 8.3. Note that *any* $p \in (1,\infty)$ is allowed in the framework of [CGQ22] unlike that of [Kig23] (see (d) above). It is extremely hard to determine the value $\dim_{\mathrm{ARC}}(K,d)$ in general; however, $\dim_{\mathrm{ARC}}(K,d)$ for a p.-c.f. self-similar set K is typically 1. (See [CP14, Theorem 1.2] for a sufficient condition for $\dim_{\mathrm{ARC}}(K,d) = 1$.) In Appendix C, by using results in [CGQ22], we prove that the Ahlfors regular conformal dimension of any *affine nested fractal* equipped with the p-resistance metric is 1.

Proof. Let $(\widehat{\mathcal{E}}_p, \mathcal{W}^p)$ be a p-energy form on (K,m) given in Theorem 8.19. Set $\mathcal{D} := \mathcal{W}^p \cap C(K)$ and $E_n := \frac{1}{n}\sum_{j=0}^{n-1}\mathcal{S}_{\sigma_p}^{j}(\widehat{\mathcal{E}}_p)$ for $n \in \mathbb{N}$ for simplicity. By (8.30) and (8.16), there exists $\widetilde{C} \in [1,\infty)$ such that

$$\widetilde{C}^{-1}\mathcal{N}_p(u)^p \le E_n(u) \le \widetilde{C}\mathcal{N}_p(u)^p \quad \text{for any } u \in \mathcal{D},\ n \in \mathbb{N}. \tag{8.33}$$

Let $\mathscr{C}$ be a countable dense subset of $\mathcal{D}$, which exists by Theorem 3.18. By a standard diagonal procedure, there exists $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k < n_{k+1}$ for any $k \in \mathbb{N}$ such that $\{E_{n_k}(u)\}_{k\in\mathbb{N}}$ is convergent in $[0,\infty)$ for any $u \in \mathscr{C}$. Let $u \in \mathcal{D}$, $\varepsilon > 0$ and pick $u_* \in \mathscr{C}$ so that $\mathcal{N}_p(u - u_*) < \varepsilon$. Then for any $k, l \in \mathbb{N}$, by the triangle inequality for $E_n^{1/p}$ and (8.33),

$$\begin{aligned}
&\left|E_{n_k}(u)^{1/p} - E_{n_l}(u)^{1/p}\right| \\
&\le \left|E_{n_k}(u)^{1/p} - E_{n_k}(u_*)^{1/p}\right| + \left|E_{n_k}(u_*)^{1/p} - E_{n_l}(u_*)^{1/p}\right| + \left|E_{n_l}(u)^{1/p} - E_{n_l}(u_*)^{1/p}\right| \\
&\le 2\widetilde{C}^{1/p}\varepsilon + \left|E_{n_k}(u)^{1/p} - E_{n_l}(u)^{1/p}\right|,
\end{aligned}$$

whence $\limsup_{k\wedge l\to\infty}\left|E_{n_k}(u)^{1/p} - E_{n_l}(u)^{1/p}\right| \le 2\widetilde{C}^{1/p}\varepsilon$. In particular, $\{E_{n_k}(u)\}_{k\in\mathbb{N}}$ is convergent in $[0,\infty)$ for any $u \in \mathcal{D}$, so the limit in (8.31) exists. Define $\mathcal{E}_p\colon \mathcal{D} \to [0,\infty)$ by (8.31).

(a): By (8.33), $\widetilde{C}^{-1}\mathcal{N}_p(u)^p \le \mathcal{E}_p(u) \le \widetilde{C}\mathcal{N}_p(u)^p$ for any $u \in \mathcal{D}$ with $\widetilde{C}$ the same as in (8.33). For any $n \in \mathbb{N}$ and any $u \in \mathcal{D}$, we easily see that

$$\frac{1}{n}\widehat{\mathcal{E}}_p(u) + \mathcal{S}_{\sigma_p}(E_n)(u) = \frac{1}{n}\widehat{\mathcal{E}}_p(u) + \frac{1}{n}\sum_{l=0}^{n-1}\mathcal{S}_{\sigma_p}^{l+1}(\widehat{\mathcal{E}}_p)(u) = E_n(u) + \frac{1}{n}\mathcal{S}_{\sigma_p}^{n}(\widehat{\mathcal{E}}_p)(u). \tag{8.34}$$

Since $\lim_{k\to\infty}\mathcal{S}_{\sigma_p}(E_{n_k})(u) = \mathcal{S}_{\sigma_p}(\mathcal{E}_p)(u)$ by Lemma 8.24 and $\lim_{k\to\infty}\frac{1}{n_k}\mathcal{S}_{\sigma_p}^{n_k}(\widehat{\mathcal{E}}_p)(u) = 0$ by (8.33), we obtain $\mathcal{S}_{\sigma_p}(\mathcal{E}_p) = \mathcal{E}_p$ by letting $n \to \infty$ along $\{n_k\}_{k\in\mathbb{N}}$ in (8.34). Hence $(\mathcal{E}_p, \mathcal{D})$ is a self-similar p-energy form on $(\mathcal{L}, m)$ with weight $(\sigma_p^{j_s})_{s\in S}$.

(b): Let $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfy (2.1). Let $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{D}$. Then $T_l(u_k \circ F_s) = T_l(u_k) \circ F_s \in \mathcal{D}$ for any $k \in \{1,\dots,n_1\}$ and any $s \in S$ by (GC)$_p$ for $(\widehat{\mathcal{E}}_p, \mathcal{D})$ and Lemma 8.24. If $q_2 < \infty$, then by a similar estimate as (2.18),

$$
\begin{aligned}
&\sum_{l=1}^{n_2} \mathcal{S}_{\sigma_p}(\widehat{\mathcal{E}}_p)\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p} = \sum_{l=1}^{n_2}\left[\sum_{s\in S}\sigma_p^{j_s}\widehat{\mathcal{E}}_p(T_l(\boldsymbol{u})\circ F_s)\right]^{q_2/p} \\
&\leq \left(\sum_{s\in S}\sigma_p^{j_s}\left[\sum_{l=1}^{n_2}\widehat{\mathcal{E}}_p\bigl(T_l(\boldsymbol{u})\circ F_s\bigr)^{q_2/p}\right]^{p/q_2}\right)^{q_2/p} \qquad (\text{by the triangle ineq. for } \|\cdot\|_{\ell^{q_2/p}}) \\
&\overset{(\mathrm{GC})_p}{\leq} \left(\sum_{s\in S}\sigma_p^{j_s}\left[\sum_{k=1}^{n_1}\widehat{\mathcal{E}}_p(u_k\circ F_s)^{q_1/p}\right]^{p/q_1}\right)^{q_2/p} \\
&\overset{(2.17)}{\leq} \left(\sum_{k=1}^{n_1}\left[\sum_{s\in S}\sigma_p^{j_s}\widehat{\mathcal{E}}_p(u_k\circ F_s)\right]^{q_1/p}\right)^{\frac{p}{q_1}\cdot\frac{q_2}{p}} = \left(\sum_{k=1}^{n_1}\mathcal{S}_{\sigma_p}(\widehat{\mathcal{E}}_p)(u_k)^{q_1/p}\right)^{q_2/q_1},
\end{aligned}
$$

whence $\left\|\bigl(\mathcal{S}_{\sigma_p}(\widehat{\mathcal{E}}_p)(T_l(\boldsymbol{u}))^{1/p}\bigr)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \leq \left\|\bigl(\mathcal{S}_{\sigma_p}(\widehat{\mathcal{E}}_p)(u_k)^{1/p}\bigr)_{k=1}^{n_1}\right\|_{\ell^{q_1}}$. The case $q_2 = \infty$ is similar, so $(\mathcal{S}_{\sigma_p}(\widehat{\mathcal{E}}_p), \mathcal{D})$ satisfies (GC)$_p$. Similarly, $(\mathcal{S}^n_{\sigma_p}(\widehat{\mathcal{E}}_p), \mathcal{D})$ satisfies (GC)$_p$ for any $n \in \mathbb{N}$. Hence (GC)$_p$ for $(\mathcal{E}_p, \mathcal{D})$ holds by Proposition 2.9-(b).

(c): From the self-similarity of $(\mathcal{E}_p, \mathcal{D})$, (b) and Proposition 2.2, we can apply Theorem 5.13. In particular, (5.25) with K in place of A implies the desired strong locality.

(d): Recall that $\mathcal{W}^p \subseteq C(K)$ by $p > \dim_{\mathrm{ARC}}(K,d)$ (see Theorem 8.16). A similar argument as in the proof of Theorem 8.19-(d) shows that $(\mathcal{E}_p, \mathcal{W}^p)$ is a regular p-resistance form on K satisfying (8.32). This completes the proof. □

Similar to Theorem 7.8, we can obtain the monotonicity of $\sigma_p^{1/(p-1)}$ in $p \in (\dim_{\mathrm{ARC}}, \infty)$. Note that the following result is *not* restricted to p.-c.f. self-similar structures.

Theorem 8.29. *Assume that Assumption 8.22 holds Let $p, q \in (\dim_{\mathrm{ARC}}(K,d), \infty)$ with $p \leq q$. In addition, assume that K is s-conductively homogeneous for each $s \in \{p,q\}$. Then*

$$
\sigma_p^{1/(p-1)} \leq \sigma_q^{1/(q-1)}. \tag{8.35}
$$

Proof. The proof is very similar to Theorem 7.8. By Proposition 8.26, (8.29) and (8.27) with $s \in \{p,q\}$ in place of p hold. Let $(\mathcal{E}_s, \mathcal{W}^s)$ be a self-similar s-resistance form on $\mathcal{L}$ given in Theorem 8.27 for each $s \in \{p,q\}$. Fix two distinct points $x_0, y_0 \in K$, set $B \coloneqq \{x_0, y_0\}$ and define $h_p \coloneqq h_B^{\mathcal{E}_p}\bigl[\mathbb{1}_{x_0}^B\bigr] \in \mathcal{W}^p$. Then $0 \leq h_p \leq 1$ by the weak comparison principle (Proposition 6.21) and we can find $w \in W_*$ satisfying $K_w \cap B = \emptyset$ and $h_{p,w} \coloneqq h_p \circ F_w \notin \mathbb{R}\mathbb{1}_K$. Similar to (7.12), by using (6.22) and (7.1), we can show that

$$
|P_n h_{q,w}(u) - P_n h_{q,w}(v)|^{q-p} \leq C r_*^{n(d_{\mathrm{w},p}-d_{\mathrm{f}})\frac{q-p}{p-1}},
$$

where $C > 0$ is independent of n. Hence we have

$$\mathcal{E}_q^n(h_{p,w}) = \sum_{\{u,v\}\in E_n^*} |P_n h_{q,w}(u) - P_n h_{q,w}(v)|^q \le C r_*^{n(d_{\mathrm{w},p}-d_{\mathrm{f}})\frac{q-p}{p-1}} \mathcal{E}_p^n(h_{p,w}),$$

which implies that

$$\left(\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)}\right)^n \widetilde{\mathcal{E}}_q^n(h_{p,w}) \le C\widetilde{\mathcal{E}}_p^n(h_{p,w}) \le C\mathcal{N}_p(h_{p,w})^p. \tag{8.36}$$

By (8.13), there exists $C_q \in (0,\infty$ such that $\mathcal{N}_q(f)^q \le C_q \liminf_{n\to\infty} \widetilde{\mathcal{E}}_q^n(f)$ for any $f \in L^q(K,m)$. This together with (8.36) implies that

$$\mathcal{N}_q(h_{p,w})^q \limsup_{n\to\infty}\left(\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)}\right)^n \le C'\mathcal{N}_p(h_{p,w})^p < \infty.$$

Since $\mathcal{N}_q(h_{p,w}) > 0$, we obtain $\sigma_q^{-1}\sigma_p^{(q-1)/(p-1)} \le 1$, which yields (8.35). $\square$

8.3 Construction of self-similar p-resistance forms on post-critically finite self-similar structures

In this subsection, under the condition (**R**) of [CGQ22, p. 18], we see that the construction of p-energy forms on p.-c.f. self-similar structures constructed due to [CGQ22] yields p-resistance forms. The framework in [CGQ22] is focused only on p.-c.f. self-similar structure, but restrictions on weights of self-similar p-resistance forms are flexible so that non-arithmetic weights are allowed unlike the framework in Subsection 8.2. See Section B for details.

Throughout this subsection, we always suppose that K is connected and that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p.-c.f. self-similar structure with $\#S \ge 2$.

We start with the definition of renormalization operators.

Definition 8.30 (Renormalization operator; [CGQ22, Definition 3.1]). Let $\rho_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$ and $k \in \mathbb{N}\cup\{0\}$. For a p-resistance form E on V_k, define p-resistance forms $\mathcal{S}_{\rho_p}(E)\colon \mathbb{R}^{V_{k+1}} \to [0,\infty)$ and $\mathcal{R}_{\rho_p}(E)\colon \mathbb{R}^{V_k} \to [0,\infty)$[8] by

$$\mathcal{S}_{\rho_p}(E)(u) \coloneqq \sum_{i\in S}\rho_{p,i}E(u\circ F_i), \quad u \in \mathbb{R}^{V_{k+1}}, \quad \text{and} \quad \mathcal{R}_{\rho_p}(E) \coloneqq \mathcal{S}_{\rho_p}(E)\big|_{V_k}. \tag{8.37}$$

(Recall Proposition 7.7 and Theorem 6.13.) Precisely, $\mathcal{S}_{\rho_p}, \mathcal{R}_{\rho_p}$ depend on k, but we omit it for convenience. By [CGQ22, Lemma 3.2-(b)], we have $\mathcal{S}_{\rho_p}^n(E)\big|_{V_k} = \mathcal{R}_{\rho_p}^n(E)$ for any $n \in \mathbb{N}\cup\{0\}$, i.e.,

$$\mathcal{R}_{\rho_p}^n(E)(u) = \inf\left\{\sum_{w\in W_n}\rho_{p,w}E(v\circ F_w) \,\middle|\, v \in \mathbb{R}^{V_n} \text{ with } v|_{V_0} = u\right\}, \quad u \in \mathbb{R}^{V_k}.$$

[8] We use different symbols from [CGQ22].

The following key result is essentially due to [CGQ22, Theorem 4.2].

Theorem 8.31 (Existence of an eigenform)**.** *Let $\rho_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$. Assume that there exist $c \in (0,\infty)$ and a p-resistance form E_0 on V_0 such that*

$$\min_{x,y\in V_0; x\neq y} R_{\mathcal{R}^n_{\rho_p}(E_0)}(x,y) \geq c \max_{x,y\in V_0; x\neq y} R_{\mathcal{R}^n_{\rho_p}(E_0)}(x,y) \quad \text{for any } n \in \mathbb{N}\cup\{0\}. \tag{8.38}$$

Then there exist a p-resistance form $\mathcal{E}^{(0)}_p$ on V_0 and a unique number $\lambda = \lambda(\rho_p) \in (0,\infty)$ such that $\mathcal{R}_{\rho_p}(\mathcal{E}^{(0)}_p) = \lambda\mathcal{E}^{(0)}_p$ Furthermore, suppose that $\mathcal{U}_{p,*} \subseteq \{E \mid E\colon \mathbb{R}^{V_0} \to [0,\infty)\}$ satisfies the following properties:

(i) *$\emptyset \neq \mathcal{U}_{p,*} \subseteq \{E \mid E$ is a p-resistance form on $V_0\}$.*
(ii) *$a_1E_1 + a_2E_2 \in \mathcal{U}_{p,*}$ for any $a_1, a_2 \in (0,\infty)$ and any $E_1, E_2 \in \mathcal{U}_{p,*}$.*
(iii) *Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}_{p,*}$. If there exist $C \in [1,\infty)$ and a p-resistance form E' on V_0 such that $C^{-1}E_n(u) \leq E'(u) \leq CE_n(u)$ and $\{E_n(u)\}_{n\in\mathbb{N}}$ is convergent in $[0,\infty)$ for any $u \in \mathbb{R}^{V_0}$, then $E \in \mathcal{U}_{p,*}$, where $E(u) \coloneqq \lim_{n\to\infty} E_n(u)$, $u \in \mathbb{R}^{V_0}$.*
(iv) *$\mathcal{R}_{\rho_p}(E) \in \mathcal{U}_{p,*}$ for any $E \in \mathcal{U}_{p,*}$.*

Then $\mathcal{E}^{(0)}_p$ can be chosen so that $\mathcal{E}^{(0)}_p \in \mathcal{U}_{p,*}$.

Remark 8.32. If ρ_p satisfies (8.38) for some p-resistance form E_0 on V_0, then for any p-resistance form $\widetilde{E}_0$ on V_0 there exists $\tilde{c} \in (0,\infty)$ such that (8.38) with $\widetilde{E}_0, \tilde{c}$ in place of E_0, c holds by [CGQ22, Lemma 4.4-(a)]. Hence (8.38) is a condition on ρ_p.

Proof. The proof is done by a simple adaptation of [CGQ22, Theorem 4.2]. (Note that (8.38) is exactly the same as (**A**) in [CGQ22, p. 14].) We remark that the self-similar structure $\mathcal{L}$ is assumed to be $K \subseteq \mathbb{R}^D$ for some $D \in \mathbb{N}$ in [CGQ22], but this hypothesis is not essential.

Throughout this proof, we fix $\mathcal{U}_{p,*}(V_0)$ satisfying the conditions (i)-(iv). For any p-resistance forms E, E' on V_0, define

$$|E|_{\mathcal{M}_p} \coloneqq \sup\Bigl\{E(u) \Bigm| u \in \mathbb{R}^{V_0}, \max_{x,y\in V_0} |u(x)-u(y)| = 1\Bigr\},$$
$$|E-E'|_{\mathcal{M}_p} \coloneqq \sup\Bigl\{|E(u)-E'(u)| \Bigm| u \in \mathbb{R}^{V_0}, \max_{x,y\in V_0} |u(x)-u(y)| = 1\Bigr\}.$$

(See the definition of $\|\cdot\|_{\widetilde{\mathcal{M}}_p(V_0)}$ in [CGQ22, Definition 2.8].) By [CGQ22, Lemma 4.4 and Theorem 4.2], there exists a unique $\lambda \in (0,\infty)$ satisfying the following: for any p-resistance form E on V_0 there exists $C \geq 1$ such that

$$C^{-1}\lambda^n E(u) \leq \mathcal{R}^n_{\rho_p}(E)(u) \leq C\lambda^n E(u) \quad \text{for any } n \in \mathbb{N}\cup\{0\} \text{ and } u \in \mathbb{R}^{V_0}. \tag{8.39}$$

Let $E_0 \in \mathcal{U}_{p,*}$[9], and define

$$E_n \coloneqq \frac{1}{n+1}\sum_{k=0}^{n} \lambda^{-k}\mathcal{R}^k_{\rho_p}(E_0), \quad n \in \mathbb{N}.$$

[9] In many situations, the choice $E_0(u) \coloneqq \sum_{x,y\in V_0} |u(x)-u(y)|^p$, $u \in \mathbb{R}^{V_0}$, is enough.

Then E_n is a p-resistance form on V_0 with $E_n \in \mathcal{U}_{p,*}$. By (8.39), we have $C^{-1}E_0 \leq E_n \leq CE_0$ for any $n \in \mathbb{N} \cup \{0\}$ and hence $\sup_{n \in \mathbb{N}\cup\{0\}} |E_n|_{\mathcal{M}_p} < \infty$, so there exists a subsequence $\{E_{n_k}\}_{k \geq 0}$ and $E' \in \mathcal{U}_{p,*}(V_0)$ such that $|E_{n_k} - E'|_{\mathcal{M}_p} \to 0$ as $k \to \infty$ by [CGQ22, Lemma 4.3]. Next we show that $\{\lambda^{-l}\mathcal{R}^l_{\rho_p}(E')(u)\}_{l \in \mathbb{N}\cup\{0\}}$ is non-decreasing for any $u \in \mathbb{R}^{V_0}$. Indeed,

$$
\begin{aligned}
&\lambda^{-l-1}\mathcal{R}^{l+1}_{\rho_p}(E_n)(u) \\
&= \lambda^{-l} \inf\left\{ \frac{1}{n+1} \sum_{(i,w) \in S \times W_l} \rho_{p,wi} \sum_{k=0}^{n} \lambda^{-k-1}\mathcal{R}^k_{\rho_p}(E_0)(v \circ F_{wi}) \,\middle|\, v \in \mathbb{R}^{V_{l+1}}, v|_{V_0} = u \right\} \\
&\geq \lambda^{-l} \inf\left\{ \frac{1}{n+1} \sum_{w \in W_l} \rho_{p,w} \sum_{k=0}^{n} \lambda^{-k-1}\mathcal{R}^{k+1}_{\rho_p}(E_0)(v|_{V_l} \circ F_w) \,\middle|\, v \in \mathbb{R}^{V_{l+1}}, v|_{V_0} = u \right\} \\
&\geq \lambda^{-l} \inf\left\{ \sum_{w \in W_l} \rho_{p,w} E_n(v \circ F_w) - \frac{1}{n+1} \sum_{w \in W_l} \rho_{p,w} E_0(v \circ F_w) \,\middle|\, v \in \mathbb{R}^{V_l}, v|_{V_0} = u \right\} \\
&\geq \lambda^{-l}\mathcal{R}^l_{\rho_p}(E_n)(u) - \frac{\lambda^{-l}}{n+1} \left|\mathcal{S}^l_{\rho_p}(E_0)\right|_{\mathcal{M}_p} \left(\max_{x,y \in V_0} |u(x) - u(y)| \right)^p.
\end{aligned}
$$

Since n is arbitrary and $\left|\mathcal{R}_{\rho_p}(E_{n_k}) - \mathcal{R}_{\rho_p}(E')\right|_{\mathcal{M}_p} \to 0$ as $k \to \infty$ by [CGQ22, Lemma 4.3-(b)], we have $\lambda^{-l-1}\mathcal{R}^{l+1}_{\rho_p}(E')(u) \geq \lambda^{-l}\mathcal{R}^l_{\rho_p}(E')(u)$. Now define $\mathcal{E}^{(0)}_p(u) := \lim_{l \to \infty} \lambda^{-l}\mathcal{R}^l_{\rho_p}(E')(u)$, which satisfies $\mathcal{E}^{(0)}_p \in \mathcal{U}_{p,*}$ by (8.39). By [CGQ22, Lemma 4.3-(b)] again, we clearly have $\mathcal{R}_{\rho_p}(\mathcal{E}^{(0)}_p) = \lambda\mathcal{E}^{(0)}_p$. $\square$

In the rest of this subsection, we assume that there exist $\rho_p = (\rho_{p,i})_{i \in S} \in (0,\infty)^S$ and a p-resistance form $\mathcal{E}^{(0)}_p$ on V_0 satisfying $\mathcal{R}_{\rho_p}(\mathcal{E}^{(0)}_p) = \mathcal{E}^{(0)}_p$, which exist if (8.38) holds by Theorem 8.31. The next proposition is important to construct a self-similar p-resistance form as an "inductive limit" of discrete p-resistance forms as presented in [CGQ22, Proposition 5.3], which is an adaptation of the relevant pieces of the theory of resistance forms due to [Kig01, Sections 2.2, 2.3 and 3.3].

Proposition 8.33. *Define $\mathcal{E}^{(n)}_p := \mathcal{S}^n_{\rho_p}(\mathcal{E}^{(0)}_p)$, i.e.,*

$$\mathcal{E}^{(n)}_p(u) := \sum_{w \in W_n} \rho_{p,w}\mathcal{E}^{(0)}_p(u \circ F_w), \quad u \in \mathbb{R}^{V_n}. \tag{8.40}$$

Then $\mathcal{E}^{(n)}_p$ is a p-resistance form on V_n and $\mathcal{E}^{(n+m)}_p\big|_{V_n} = \mathcal{E}^{(n)}_p$ for any $n, m \in \mathbb{N} \cup \{0\}$. In particular, $\left\{\mathcal{E}^{(n)}_p(u|_{V_n})\right\}_{n \geq 0}$ is non-decreasing for any $u \in \mathbb{R}^{V_*}$.

Proof. We will show $\mathcal{E}^{(n+m)}_p\big|_{V_n} = \mathcal{E}^{(n)}_p$. (See [Kig01, Proposition 3.1.3] for the case $p = 2$.) It suffices to prove $\mathcal{E}^{(n+1)}_p\big|_{V_n} = \mathcal{E}^{(n)}_p$ for any $n \in \mathbb{N} \cup \{0\}$ by virtue of Proposition 6.15. Note that the case $n = 0$ is true by $\mathcal{R}_{\rho_p}(\mathcal{E}^{(0)}_p) = \mathcal{E}^{(0)}_p$, and that

$$\mathcal{E}^{(n+1)}_p(u) = \sum_{i \in S} \rho_{p,i}\mathcal{E}^{(n)}_p(u \circ F_i), \quad \text{for any } n \in \mathbb{N} \cup \{0\} \text{ and } u \in \mathbb{R}^{V_{n+1}}. \tag{8.41}$$

Assume that $\mathcal{E}_p^{(m)}|_{V_{m-1}} = \mathcal{E}_p^{(m-1)}$ for some $m \in \mathbb{N}$. Then for any $u \in \mathbb{R}^{V_m}$,

$$\begin{aligned}
\mathcal{E}_p^{(m)}(u) &\overset{(8.41)}{=} \sum_{i \in S} \rho_{p,i} \mathcal{E}_p^{(m-1)}(u \circ F_i) \\
&= \sum_{i \in S} \rho_{p,i} \min\Bigl\{ \mathcal{E}_p^{(m)}(v \circ F_i) \Bigm| v \in \mathbb{R}^{K_i \cap V_{m+1}}, v|_{K_i \cap V_m} = u|_{K_i} \Bigr\} \\
&\overset{(5.2)}{=} \min\Biggl\{ \sum_{i \in S} \rho_{p,i} \mathcal{E}_p^{(m)}(v \circ F_i) \Biggm| v \in \mathbb{R}^{V_{m+1}}, v|_{V_m} = u \Biggr\} \\
&\overset{(8.41)}{=} \min\Bigl\{ \mathcal{E}_p^{(m+1)}(v) \Bigm| v \in \mathbb{R}^{V_{m+1}}, v|_{V_m} = u \Bigr\} = \mathcal{E}_p^{(m+1)}\bigr|_{V_m}(u),
\end{aligned}$$

which completes the proof. □

Now we can show a special case of [CGQ22, Proposition 5.3].

Theorem 8.34. *In addition to the existence of $\rho_p = (\rho_{p,i})_{i \in S} \in (0, \infty)^S$ and a p-resistance form $\mathcal{E}_p^{(0)}$ on V_0 with $\mathcal{R}_{\rho_p}(\mathcal{E}_p^{(0)}) = \mathcal{E}_p^{(0)}$, we assume that*

$$\rho_{p,i} \in (1, \infty) \text{ for any } i \in S. \tag{8.42}$$

Define $(\mathcal{E}_p, \mathcal{F}_p)$ by

$$\mathcal{F}_p := \Bigl\{ u \in C(K) \Bigm| \lim_{n \to \infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty \Bigr\}, \quad \text{and} \tag{8.43}$$

$$\mathcal{E}_p(u) := \lim_{n \to \infty} \mathcal{E}_p^{(n)}(u), \quad u \in \mathcal{F}_p. \tag{8.44}$$

Then $(\mathcal{E}_p, \mathcal{F}_p)$ is a regular self-similar p-resistance form on $\mathcal{L}$ with weight ρ_p. Moreover, $\mathcal{E}_p|_{V_n} = \mathcal{E}_p^{(n)}$ for any $n \in \mathbb{N} \cup \{0\}$.

Remark 8.35. (1) In this article, we always assume the condition (8.42), which is the same as (**R**) in [CGQ22, p. 18]. In the case where (8.42) fails, we have to make an extra effort to identify a function $u \in \mathbb{R}^{V_*}$ satisfying $\lim_{n \to \infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty$ with a function on K. See [CGQ22, Theorem 5.2] for details.

(2) By choosing a suitable $\mathcal{U}_{p,*}$ in Theorem 8.31, we can verify nice properties, the *symmetry-invariance* for example (see (9.7)), of $\mathcal{E}_p$.

Proof. Note that the self-similarity conditions, (5.5) and (5.6), for $(\mathcal{E}_p, \mathcal{F}_p)$ are obvious from Proposition 8.33. We first show that $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on K.

$(\mathrm{RF1})_p$: Clearly, $\mathcal{F}_p$ is a linear subspace of $\mathbb{R}^K$ containing $\mathbb{R}\mathbb{1}_K$ and $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is a seminorm on $\mathcal{F}_p$. By $(\mathrm{RF1})_p$ for $\mathcal{E}_p^{(n)}$, we easily have $\mathcal{E}_p(a\mathbb{1}_K) = 0$ for any $a \in \mathbb{R}$. Conversely, suppose that $u \in \mathcal{E}_p^{-1}(0)$. Then we have from the monotonicity of $\{\mathcal{E}_p^{(n)}(u|_{V_n})\}_{n \geq 0}$ that $u|_{V_*}$ is constant. Since $\overline{V_*}^K = K$ and $u \in C(K)$, we conclude that $u \in \mathbb{R}\mathbb{1}_K$.

(RF5)$_p$: We fix $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfying (2.1). Let $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}_p^{n_1}$. Then, for any $l \in \{1,\dots,n_2\}$, (GC)$_p$ for $\mathcal{E}_p^{(n)}$ implies that

$$\begin{aligned}\mathcal{E}_p^{(n)}(T_l(\boldsymbol{u}))^{1/p} &\le \left\| \left(\mathcal{E}_p^{(n)}(T_l(\boldsymbol{u}))^{1/p}\right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \\ &\le \left\| \left(\mathcal{E}_p^{(n)}(u_k)^{1/p}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} \le \left\| \left(\mathcal{E}_p(u_k)^{1/p}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}} < \infty.\end{aligned}$$

By letting $n \to \infty$, we obtain (GC)$_p$ for $(\mathcal{E}_p, \mathcal{F}_p)$.

(RF3)$_p$ and (RF4)$_p$: By the proof of [CGQ22, Proposition 5.3], there exists $C \in (0,\infty)$ such that for any $u \in \mathcal{F}_p$, any $w \in W_*$ and any $x, y \in K_w \cap V_*$, we have $|u(x) - u(y)|^p \le C\rho_{p,w}^{-1}\mathcal{E}_p(u)$. Since $u \in C(K)$ and $\overline{F_w(V_*)}^K = K_w$, we obtain

$$|u(x) - u(y)|^p \le C\rho_{p,w}^{-1}\mathcal{E}_p(u) \quad \text{for any } u \in \mathcal{F}_p,\ w \in W_* \text{ and } x, y \in K_w. \tag{8.45}$$

A chaining argument using (8.45) implies (RF4)$_p$ for $(\mathcal{E}_p, \mathcal{F}_p)$ and $\sup_{x,y\in K} R_{\mathcal{E}_p}(x,y) < \infty$. Furthermore, (8.45), $\min_{i\in S}\rho_{p,i} > 1$ and (5.3) imply that any function $f \in \mathbb{R}^{V_*}$ with $\lim_{n\to\infty}\mathcal{E}_p^{(n)}(f|_{V_n}) < \infty$ can be uniquely extended to a continuous function $f_* \in C(K)$. Applying this fact to piecewise p-harmonic functions considered in [CGQ22, p. 22], we easily get (RF3)$_p$ for $(\mathcal{E}_p, \mathcal{F}_p)$.

(RF2)$_p$: Fix $x_* \in K$ and let $\{u_k\}_{k\in\mathbb{N}}$ be such that $u_k \in \mathcal{F}_p$, $u_k(x_*) = 0$ and $\lim_{k\wedge l\to\infty}\mathcal{E}_p(u_k - u_l) \to \infty$. By (RF4)$_p$ for $(\mathcal{E}_p, \mathcal{F}_p)$ and $\sup_{x,y\in K} R_{\mathcal{E}_p}(x,y) < \infty$, $\{u_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $C(K)$, sx there exists $u \in C(K)$ such that $\|u - u_k\|_{\sup} \to 0$ as $k \to \infty$. For any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\sup_{k,l\ge N_0}\mathcal{E}_p(u_k - u_l) < \varepsilon$. Since $\mathcal{E}_p^{(n)}(\,\cdot\,)^{1/p}$ is a norm on the finite-dimensional vector space $\mathbb{R}^{V_n}/\mathbb{R}\mathbb{1}_{V_n}$, we obtain

$$\mathcal{E}_p^{(n)}(u|_{V_n} - u_l|_{V_n}) \le \liminf_{k\to\infty}\mathcal{E}_p(u_k - u_l) \le \varepsilon \quad \text{for any } l \ge N_0 \text{ and any } n \in \mathbb{N}\cup\{0\}.$$

Since $n \in \mathbb{N}\cup\{0\}$ is arbitrary, we conclude that $u \in \mathcal{F}_p$ and $\lim_{l\to\infty}\mathcal{E}_p(u - u_l) = 0$, which proves (RF2)$_p$ for $(\mathcal{E}_p, \mathcal{F}_p)$.

Next we show that $(\mathcal{E}_p, \mathcal{F}_p)$ is regular (recall Definition 6.5). By Proposition 7.2-(2), the p-resistance metric $\widehat{R}_{p,\mathcal{E}_p}$ is compatible with K. In particular, $C(K) = C(K, \widehat{R}_{p,\mathcal{E}_p})$. Since $\mathcal{F}_p$ is an algebra by Proposition 2.2-(d), the Stone–Weierstrass theorem shows that $(\mathcal{E}_p, \mathcal{F}_p)$ is regular.

Lastly, we prove $\mathcal{E}_p|_{V_n} = \mathcal{E}_p^{(n)}$ for any $n \in \mathbb{N}\cup\{0\}$. Let $u \in \mathbb{R}^{V_n}$, and define $h \in \mathbb{R}^{V_*}$ so that $h|_{V_{n+m}} = h_{V_n}^{\mathcal{E}_p^{(n+m)}}[u]$ for any $m \in \mathbb{N}\cup\{0\}$, which is well-defined by virtue of Propositions 8.33 and 6.15. Then we have $\mathcal{E}_p^{(n)}(u) = \mathcal{E}_p^{(n+m)}(h|_{V_{n+m}})$ and hence $\lim_{m\to\infty}\mathcal{E}_p^{(n+m)}(h|_{V_{n+m}}) < \infty$. If $h_* \in C(K)$ is the unique function such that $h_*|_{V_*} = h$, then we have $h_* \in \mathcal{F}_p$, $h_*|_{V_n} = u$ and $\mathcal{E}_p^{(n)}(u) = \mathcal{E}_p(h_*) \ge \mathcal{E}_p|_{V_n}(u)$. Since $\mathcal{E}_p|_{V_n}(u) \le \mathcal{E}_p^{(n)}(u)$ is clear from the definition, we complete the proof. $\square$

9 p-Walk dimension of Sierpiński carpets/gaskets

In this section, we prove the strict inequality $d_{\mathrm{w},p} > p$ for generalized Sierpiński carpets and D-dimensional level-l Sierpiński gasket as an application of the nonlinear potential theory developed in Sections 6 and 7. In particular, we remove the *planarity* in the hypothesis of the previous result [Shi24, Theorem 2.27].

9.1 Generalized Sierpiński carpets

By following [Kaj23, Section 2], we recall the definition of generalized Sierpiński carpets.

Framework 9.1. Let $D, l \in \mathbb{N}$, $D \geq 2$, $l \geq 3$ and set $Q_0 := [0,1]^D$. Let $S \subsetneq \{0, 1, \dots, l-1\}^D$ be non-empty, define $f_i \colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) := l^{-1}i + l^{-1}x$ for each $i \in S$ and set $Q_1 := \bigcup_{i \in S} f_i(Q_0)$, so that $Q_1 \subsetneq Q_0$. Let K be the self-similar set associated with $\{f_i\}_{i \in S}$. Note that $K \subsetneq Q_0$. Set $F_i := f_i|_K$ for each $i \in S$ and $\mathrm{GSC}(D, l, S) := (K, S, \{F_i\}_{i \in S})$. Let $d \colon K \times K \to [0, \infty)$ be the Euclidean metric on K given by $d(x, y) := |x - y|$, set $d_{\mathrm{f}} := \log_l(\#S)$, and let m be the self-similar measure on $\mathrm{GSC}(D, l, S)$ with uniform weight $(1/\#S)_{i \in S}$.

Recall that d_{f} is the Hausdorff dimension of (K, d) and that m is a constant multiple of the d_{f}-dimensional Hausdorff measure on (K, d); see, e.g., [Kig01, Proposition 1.5.8 and Theorem 1.5.7]. Note that $d_{\mathrm{f}} < D$ by $S \subsetneq \{0, 1, \dots, l-1\}^D$.

The following definition is due to Barlow and Bass [BB99, Section 2], except that the non-diagonality condition in [BB99, Hypotheses 2.1] has been strengthened later in [BBKT] to fill a gap in [BB99, Proof of Theorem 3.19]; see [BBKT, Remark 2.10-1.] for some more details of this correction.

Definition 9.2 (Generalized Sierpiński carpet)**.** $\mathrm{GSC}(D, l, S)$ is called a *generalized Sierpiński carpet* if and only if the following four conditions are satisfied:

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of $\mathbb{R}^D$ with $f(Q_0) = Q_0$.

(GSC2) (Connectedness) Q_1 is connected.

(GSC3) (Non-diagonality) $\mathrm{int}_{\mathbb{R}^D}\bigl(Q_1 \cap \prod_{k=1}^{D}[(i_k - \varepsilon_k)l^{-1}, (i_k + 1)l^{-1}]\bigr)$ is either empty or connected for any $(i_k)_{k=1}^{D} \in \mathbb{Z}^D$ and any $(\varepsilon_k)_{k=1}^{D} \in \{0, 1\}^D$.

(GSC4) (Borders included) $[0, 1] \times \{0\}^{D-1} \subset Q_1$.

See [BB99, Remark 2.2] for a description of the meaning of each of the four conditions (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 9.2. To be precise, (GSC3) is slightly different from the formulation of the non-diagonality condition in [BBKT, Subsection 2.2], but they have been proved to be equivalent to each other in [Kaj10, Theorem 2.4]; see [Kaj10, Section 2] for some other equivalent formulations of the non-diagonality condition.

In this subsection, we assume that $\mathrm{GSC}(D, l, S) = (K, S, \{F_i\}_{i \in S})$ as introduced in Framework 9.1 is a generalized Sierpiński carpet as defined in Definition 9.2.

We next ensure the existence of a symmetry-invariant p-resistance form on $\mathrm{GSC}(D, l, S)$ for $p > \dim_{\mathrm{ARC}}(K, d)$ by applying Theorem 8.27.

Definition 9.3. We define

$$\mathcal{G}_0 \coloneqq \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^D,\ f(Q_0) = Q_0\}, \tag{9.1}$$

which forms a finite subgroup of the group of homeomorphisms of K by virtue of (GSC1).

Corollary 9.4. *Let $p \in (\dim_{\mathrm{ARC}}(K, d), \infty)$. Then Assumption 8.22 holds with $r_* = l^{-1}$, K is p-conductively homogeneous, and there exists a regular self-similar p-resistance form $(\mathcal{E}_p, \mathcal{W}^p)$ on $\mathrm{GSC}(D, l, S)$ with weight $(\sigma_p)_{i\in S}$ such that it satisfies the conditions (a)-(d) of Theorem 8.27. In particular, $(\mathcal{E}_p, \mathcal{W}^p)$ has the following property:*

$$\text{If } u \in \mathcal{W}^p \text{ and } g \in \mathcal{G}_0 \text{ then } u \circ g \in \mathcal{W}^p \text{ and } \mathcal{E}_p(u \circ g) = \mathcal{E}_p(u). \tag{9.2}$$

Proof. Assumption 8.22 and the p-conductive homogeneity for the generalized Sierpiński carpets in the case $p \in (d_{\mathrm{ARC}}, \infty)$ follow from [Kig23, Theorem 4.13] or [Shi24, Proposition 4.5 and Theorem 4.14]. Let $(\mathcal{E}_p, \mathcal{W}^p)$ be a self-similar p-resistance form given in Theorem 8.27. Then the desired properties except for (9.2) are already proved. The symmetric-invariance (9.2) follows Theorem 8.19-(c), (8.31) and the fact that $F_i^{-1} \circ g \circ F_i \in \mathcal{G}_0$ for any $i \in S$. □

Recall that σ_p and $d_{\mathrm{w},p}$ are defined for any $p \in (0, \infty)$ (under Assumption 8.22). We know the following monotonicity on $d_{\mathrm{w},p}/p$ in $p \in (0, \infty)$.

Proposition 9.5. *$d_{\mathrm{w},p}/p \geq d_{\mathrm{w},q}/q$ for any $p, q \in (0, \infty)$ with $p \leq q$.*

Proof. This follows from [Kig20, Lemma 4.7.4] and the fact that $d_{\mathrm{f}} = \log_l(\#S)$. □

The following definition is exactly the same as a part of [Kaj23, Definition 3.6].

Definition 9.6. (1) We set $V_0^\varepsilon \coloneqq K \cap (\{\varepsilon\} \times \mathbb{R}^{D-1})$ for each $\varepsilon \in \{0, 1\}$ and $U_0 \coloneqq K \setminus (V_0^0 \cup V_0^1)$.
(2) We define $g_\varepsilon \in \mathcal{G}_0$ by $g_\varepsilon \coloneqq \tau_\varepsilon|_K$ for each $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{0, 1\}^D$, where $\tau_\varepsilon \colon \mathbb{R}^D \to \mathbb{R}^D$ is given by $\tau_\varepsilon((x_k)_{k=1}^D) \coloneqq (\varepsilon_k + (1 - 2\varepsilon_k)x_k)_{k=1}^D$, and define a subgroup $\mathcal{G}_1$ of $\mathcal{G}_0$ by

$$\mathcal{G}_1 \coloneqq \{g_\varepsilon \mid \varepsilon \in \{0\} \times \{0, 1\}^{D-1}\}. \tag{9.3}$$

In the rest of this subsection, we fix $p \in (d_{\mathrm{ARC}}, \infty)$ and a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{W}^p)$ in Corollary 9.4. Recall Theorem 6.13 and let $h_0 \coloneqq h^{\mathcal{E}_p}_{V_0^0 \cup V_0^1}[\mathbb{1}_{V_0^1}] \in \mathcal{W}^p$. The strategy to prove $d_{\mathrm{w},p} > p$ is very similar to [Kaj23], that is, we will prove the *non-$\mathcal{E}_p$-harmonicity* on U_0 of $h_2 \coloneqq \sum_{w \in W_2}(F_w)_*(l^{-2}h_0 + q_1^w \mathbb{1}_K) \in \mathcal{W}^p$, which also satisfies $h_2|_{V_0^i} = i$ $(i = 0, 1)$. (See [Kaj23, Figures 2 and 3] for an illustration of h_0 and h_2.) Then the strict estimate $d_{\mathrm{w},p} > p$ will follow from $\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2)$ and the self-similarity for $\mathcal{E}_p$. Our arguments will be easier than that of [Kaj23] by virtue of $\mathcal{W}^p \subseteq C(K)$.

The next proposition is a key ingredient. Note that it requires our standing assumption that $S \neq \{0, 1, \ldots, l-1\}^D$, which excludes the case of $K = [0, 1]^D$ from the present framework.

Proposition 9.7. *Let $h_2 := \sum_{w\in W_2}(F_w)_*(l^{-2}h_0 + q_1^w \mathbb{1}_K) \in \mathcal{W}^p$. Then h_2 is not $\mathcal{E}_p$-harmonic on U_0 and $h_2|_{V_0^i} = i$ for each $i \in \{0,1\}$.*

Proof. The proof is a straightforward modification of [Kaj23, Proposition 3.11] thanks to Theorem 6.13. We present here a self-contained proof for the reader's convenience.

We claim that, if h_2 were $\mathcal{E}_p$-harmonic on U_0, then $h_0 \in \mathcal{W}^p$ would turn out to be $\mathcal{E}_p$-harmonic on $K \setminus V_0^0$, which would imply by combining with Proposition 6.11 that $\mathcal{E}_p(h_0) = \mathcal{E}_p(h_0; h_0) = 0$, which would be a contradiction by $(\mathrm{RF1})_p$ and $\mathcal{W}^p \subseteq C(K)$.

For each $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{1\} \times \{0,1\}^{D-1}$, set $U^\varepsilon := K \cap \prod_{k=1}^D(\varepsilon_k - 1, \varepsilon_k + 1)$ and $K^\varepsilon := K \cap \prod_{k=1}^D[\varepsilon_k - 1/2, \varepsilon_k + 1/2]$. Fix $\varphi_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ so that $\varphi_\varepsilon|_{K^\varepsilon} = \mathbb{1}_{K^\varepsilon}$, which exists by (8.17), $(\mathrm{RF1})_p$ and $(\mathrm{RF5})_p$. Let $v \in \mathcal{W}^p \cap C_c(K \setminus V_0^0)$ and, taking an enumeration $\{\varepsilon^{(k)}\}_{k=1}^{2^{D-1}}$ of $\{1\} \times \{0,1\}^{D-1}$ and recalling Proposition 2.2(c), define $v_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ for $\varepsilon \in \{1\} \times \{0,1\}^{D-1}$ by $v_{\varepsilon^{(1)}} := v\varphi_{\varepsilon^{(1)}}$ and $v_{\varepsilon^{(k)}} := v\varphi_{\varepsilon^{(k)}} \prod_{j=1}^{k-1}(\mathbb{1}_K - \varphi_{\varepsilon^{(j)}})$ for $k \in \{2, \dots, 2^{D-1}\}$. Then $v - \sum_{\varepsilon\in\{1\}\times\{0,1\}^{D-1}} v_\varepsilon = v \prod_{\varepsilon\in\{1\}\times\{0,1\}^{D-1}}(\mathbb{1}_K - \varphi_\varepsilon) \in \mathcal{W}^p \cap C_c(U_0)$, hence $\mathcal{E}_p(h_0; v) = \sum_{\varepsilon\in\{1\}\times\{0,1\}^{D-1}} \mathcal{E}_p(h_0; v_\varepsilon)$ by Proposition 6.11 (with $Y = K\setminus U_0$). Therefore the desired $\mathcal{E}_p$-harmonicity of h_0 on $K \setminus V_0^0$ would be obtained by deducing that $\mathcal{E}(h_0; v_\varepsilon) = 0$ for any $\varepsilon \in \{1\} \times \{0,1\}^{D-1}$.

To this end, set $\varepsilon^{(0)} := (\mathbb{1}_{\{1\}}(k))_{k=1}^D$, take $i = (i_k)_{k=1}^D \in S$ with $i_1 < l - 1$ and $i + \varepsilon^{(0)} \notin S$, which exists by $\emptyset \neq S \subsetneq \{0, 1, \dots, l-1\}^D$ and (GSC1), and let $\varepsilon = (\varepsilon_k)_{k=1}^D \in \{1\} \times \{0,1\}^{D-1}$. We will choose $i^\varepsilon \in S$ with $F_{ii^\varepsilon}(\varepsilon) \in F_i(K \cap (\{1\} \times (0,1)^{D-1}))$ and assemble $v_\varepsilon \circ g_w \circ F_w^{-1}$ with a suitable $g_w \in \mathcal{G}_1$ for $w \in W_2$ with $F_{ii^\varepsilon}(\varepsilon) \in K_w$ into a function $v_{\varepsilon,2} \in \mathcal{W}^p \cap C_c(U_0)$. Specifically, set $i^{\varepsilon,\eta} := \bigl((l-1)(\mathbb{1}_{\{1\}}(k) + 1 - \varepsilon_k) + (2\varepsilon_k - 1)\eta_k\bigr)_{k=1}^D$ for each $\eta = (\eta_k)_{k=1}^D \in \{0\} \times \{0,1\}^{D-1}$ and $I^\varepsilon := \{\eta \in \{0\} \times \{0,1\}^{d-1} \mid i^{\varepsilon,\eta} \in S\}$, so that $i^\varepsilon := i^{\varepsilon,\mathbf{0}_D} \in S$ by (GSC4) and (GSC1) and hence $\mathbf{0}_D \in I^\varepsilon$. Thanks to $v_\varepsilon \in \mathcal{W}^p \cap C_c(U^\varepsilon)$ and $i + \varepsilon^{(0)} \notin S$ we can define $v_{\varepsilon,2} \in C(K)$ by setting

$$v_{\varepsilon,2}|_{K_w} := \begin{cases} v_\varepsilon \circ g_\eta \circ F_w^{-1} & \text{if } \eta \in I^\varepsilon \text{ and } w = ii^{\varepsilon,\eta} \\ 0 & \text{if } w \notin \{ii^{\varepsilon,\eta} \mid \eta \in I^\varepsilon\} \end{cases} \quad \text{for each } w \in W_2. \tag{9.4}$$

Then $\mathrm{supp}_K[v_{\varepsilon,2}] \subset K_i \setminus V_0^0 \subset U_0$ by (9.4) and $i_1 < l - 1$. In addition, $v_{\varepsilon,2} \circ F_w \in \mathcal{W}^p$ for any $w \in W_2$ by (9.4), $v_\varepsilon \in \mathcal{W}^p$ and (9.2). Thus $v_{\varepsilon,2} \in \mathcal{F}_p$ by (5.5) and therefore $v_{\varepsilon,2} \in \mathcal{W}^p \cap C_c(U_0)$. Recall that $h_2 \circ F_w = l^{-2}h_0 + q_1^w \mathbb{1}_K$ for any $w \in W_2$ and note that, by the uniqueness in Theorem 6.13, $h_0 \circ g_\eta = h_0$ for any $\eta \in I^\varepsilon$. Then we have

$$\begin{aligned} \mathcal{E}_p(h_2; v_{\varepsilon,2}) &= \sum_{\eta\in I^\varepsilon} \sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0; v_\varepsilon \circ g_\eta) \\ &= \sum_{\eta\in I^\varepsilon} \sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0 \circ g_\eta; v_\varepsilon) = (\#I^\varepsilon)\sigma_p^2 l^{-2(p-1)} \mathcal{E}_p(h_0; v_\varepsilon). \end{aligned} \tag{9.5}$$

Now, supposing that h_2 were $\mathcal{E}_p$-harmonic on U_0, from (9.5), $\#I^\varepsilon > 0$, $v_{\varepsilon,2} \in \mathcal{F}_p \cap C_c(U_0)$ and Proposition 6.11, we would obtain $\mathcal{E}_p(h_0; v_\varepsilon) = \sigma_p^{-2} l^{2(p-1)} (\#I^\varepsilon)^{-1} \mathcal{E}_p(h_2; v_{\varepsilon,2}) = 0$, which would imply a contradiction as explained in the last two paragraphs. □

Theorem 9.8. *$d_{\mathrm{w},p} > p$ for any $p \in (0, \infty)$.*

Proof. It suffices to prove the case $p \in (d_{\mathrm{ARC}}, \infty)$ by Proposition 9.5. Let $h_0, h_2 \in \mathcal{W}^p$ be as in Proposition 9.7. By Proposition 9.7, we have $\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2)$. This strict inequality combined with (5.6) shows that

$$\mathcal{E}_p(h_0) < \mathcal{E}_p(h_2) = \bigl(\sigma_p(\#S)l^{-p}\bigr)^2\mathcal{E}_p(h_0),$$

whence $l^p < \sigma_p(\#S)$. Since $\sigma_p = l^{d_{\mathrm{w},p}-d_{\mathrm{f}}}$ and $d_{\mathrm{f}} = \log \#S/\log l$, we get $d_{\mathrm{w},p} = \log\bigl(\sigma_p(\#S)\bigr)/\log l > p$. $\square$

9.2 D-dimensional level-l Sierpiński gaskets

Following [Kaj13, Example 5.1], we introduce D-dimensional level-l Sierpiński gaskets.

Framework 9.9 (D-dimensional level-l Sierpiński gaskets)**.** Let $D, l \in \mathbb{N}$, $D \geq 2$, $l \geq 2$ and let $\{q_k\}_{k=0}^{D} \subseteq \mathbb{R}^D$ be the set of the vertices of a regular D-dimensional simplex so that $q_0, \dots, q_{D-1} \in \{(x_1, \dots, x_D) \in \mathbb{R}^D \mid x_1 = 0\}$ and $q_D \in \{(x_1, \dots, x_D) \in \mathbb{R}^D \mid x_1 \geq 0\}$. Further let $S \coloneqq \bigl\{(i_k)_{k=1}^{D} \bigm| i_k \in \mathbb{N} \cup \{0\}, \sum_{k=1}^{D} i_k \leq l-1\bigr\}$, and for each $i = (i_k)_{k=1}^{D} \in S$ we set $q_i \coloneqq q_0 + \sum_{k=1}^{D} l^{-1} i_k(q_k - q_0)$ and define $f_i \colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) \coloneqq q_i + l^{-1}(x - q_0)$. Let K be the self-similar set associated with $\{f_i\}_{i\in S}$ and set $F_i \coloneqq f_i|_K$. Let $\mathrm{SG}(D, l, S) = (K, S, \{F_i\}_{i\in S})$, which is a self-similar structure. Let $d \colon K \times K \to [0, \infty)$ be the Euclidean metric on K, set $d_{\mathrm{f}} \coloneqq \log_l \#S$, and let m be the self-similar measure on $\mathrm{SG}(D, l, S)$ with uniform weight $(1/\#S)_{i\in S}$.

Each $\mathrm{SG}(D, l, S)$ is called the *D-dimensional level-l Sierpiński gasket* and belongs to a class called the *nested fractals* (see [Kig01, Section 3.8] for details on nested fractals). In the rest of this subsection, we fix such a Sierpiński gasket $\mathrm{SG}(D, l, S)$ and the self-similar measure m as in Framework 9.9. We can easily verify [Kig23, Assumption 2.15] for $\mathrm{SG}(D, l, S)$. In addition, it is well known that m is d_{f}-Ahlfors regular (see [Kig23, Proposition E.7] for example). Similar to Corollary 9.4, we have a symmetry-invariant p-resistance form on $\mathrm{SG}(D, l, S)$ for any $p \in (1, \infty)$. (The Ahlfors regular conformal dimension of (K, d) is 1. See Theorem C.6.)

Definition 9.10. We define

$$\mathcal{G}_0 \coloneqq \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^D,\ f(V_0) = V_0\}, \tag{9.6}$$

which forms a finite subgroup of the group of homeomorphisms of K.

Corollary 9.11. *Let $p \in (1, \infty)$. Then Assumption 8.22 holds with $r_* = l^{-1}$, K is p-conductively homogeneous, and there exists a regular self-similar p-resistance form $(\mathcal{E}_p, \mathcal{W}^p)$ on $\mathrm{SG}(D, l, S)$ with weight $(\sigma_p)_{i\in S}$ such that it satisfies the conditions (a)-(d) in Theorem 8.27. In particular, $(\mathcal{E}_p, \mathcal{W}^p)$ has the following property:*

$$\textit{If } u \in \mathcal{W}^p \textit{ and } g \in \mathcal{G}_0 \textit{ then } u \circ g \in \mathcal{W}^p \textit{ and } \mathcal{E}_p(u \circ g) = \mathcal{E}_p(u). \tag{9.7}$$

Similar to Proposition 9.5, we have the following monotonicity of $d_{\mathrm{w},p}/p$ in p.

Proposition 9.12. *$d_{\mathrm{w},p}/p \geq d_{\mathrm{w},q}/q$ for any $p, q \in (0, \infty)$ with $p \leq q$.*

We can prove the following main result by using compatible sequences.

Theorem 9.13. *$d_{\mathrm{w},p} > p$ for any $p \in (0, \infty)$.*

Proof. Let $p \in (1, \infty)$ and let $(\mathcal{E}_p, \mathcal{W}^p)$ be a self-similar p-resistance form as given in Corollary 9.11. Define $u \in C(K)$ by $u(x_1, \dots, x_D) \coloneqq x_1$ for any $(x_1, \dots, x_D) \in K \subseteq \mathbb{R}^D$. Then $u|_{V_n} \in \mathcal{W}^p|_{V_n}$ for any $n \in \mathbb{N} \cup \{0\}$ by Proposition 6.8. We claim that if $u|_{V_1}$ were $\mathcal{E}_p|_{V_1}$-harmonic on $V_1 \setminus V_0$, then $\mathcal{E}_p|_{V_0}(u|_{V_0}) = 0$, which would contradict (RF1)$_p$.

Suppose that $\mathcal{E}_p|_{V_1}(u|_{V_1}; \varphi) = 0$ for every $\varphi \in \mathbb{R}^{V_1}$ with $\varphi|_{V_0} = 0$. Noting that $(u|_{V_1} \circ F_i)|_{V_0} = l^{-1}u|_{V_0} + c_i\mathbb{1}_{V_0}$ for some constant $c_i \in \mathbb{R}$ and using (7.4), we have

$$\mathcal{E}_p|_{V_1}(u|_{V_1}; \varphi) = \sigma_p \sum_{i \in S} \mathcal{E}_p|_{V_0}(u|_{V_1} \circ F_i; \varphi \circ F_i) = l^{-(p-1)}\sigma_p \sum_{i \in S} \mathcal{E}_p|_{V_0}(u|_{V_0}; \varphi \circ F_i). \tag{9.8}$$

It is easy to see that $(V_1 \setminus V_0) \cap \{(x_1, \dots, x_D) \in \mathbb{R}^D \mid x_1 = 0\} \neq \emptyset$. Let $z \in V_1 \setminus V_0$ with $z \in \{x_1 = 0\}$ and let $\varphi \coloneqq \mathbb{1}^{V_1}_{\{z\}} \in \mathbb{R}^{V_1}$. Since $u \circ g = u$ for any $g \in \mathcal{G}_0$ with $g(\{x_1 = 0\} \cap K) = \{x_1 = 0\} \cap K$, the $\mathcal{G}_0$-invariance (9.7) implies $\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_i\}}\bigr) = \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_j\}}\bigr)$ for any $i, j \in \{0, \dots, D-1\}$. Since $\varphi \circ F_j = 0 \in \mathbb{R}^{V_0}$ for any $j \in S$ with $z \notin K_j$, we have from (9.8) that

$$\begin{aligned} 0 = \mathcal{E}_p|_{V_1}(u|_{V_1}; \varphi) &= l^{-(p-1)}\sigma_p \sum_{i \in S; z \in K_i} \mathcal{E}_p|_{V_0}(u|_{V_0}; \varphi \circ F_i) \\ &= l^{-(p-1)}\sigma_p \cdot \bigl(\#\{i \in S \mid z \in K_i\}\bigr)\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_0\}}\bigr). \end{aligned}$$

Hence we get $\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_j\}}\bigr) = 0$ for every $j \in \{0, \dots, D-1\}$. Therefore,

$$\mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_D\}}\bigr) = \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}_{V_0}\bigr) = \sum_{j=0}^{D-1} \mathcal{E}_p|_{V_0}\bigl(u|_{V_0}; \mathbb{1}^{V_0}_{\{q_j\}}\bigr) = 0,$$

which yields $\mathcal{E}_p|_{V_0}(u|_{V_0}; v) = 0$ for every $v \in \mathbb{R}^{V_0}$. In particular, $\mathcal{E}_p|_{V_0}(u|_{V_0}) = 0$, which is a contradiction and hence we conclude that $u|_{V_1}$ is *not* $\mathcal{E}_p|_{V_1}$-harmonic on $V_1 \setminus V_0$. Combining with Proposition 6.15, we see that

$$\mathcal{E}_p|_{V_0}(u|_{V_0}) = \mathcal{E}_p|_{V_1}|_{V_0}(u|_{V_0}) = \mathcal{E}_p|_{V_1}\Bigl(h^{\mathcal{E}_p|_{V_1}}_{V_0}[u|_{V0}]\Bigr) < \mathcal{E}_p|_{V_1}(u|_{V_1}). \tag{9.9}$$

Similar to (9.8), we have $\mathcal{E}_p|_{V_1}(u|_{V_1}) = l^{-p}\sigma_p(\#S)\mathcal{E}_p|_{V_0}(u|_{V_0})$. Hence the strict inequality (9.9) yields $1 < l^{-p}l^{d_{\mathrm{w},p}-d_{\mathrm{f}}}(\#S) = l^{d_{\mathrm{w},p}-p}$, which proves $d_{\mathrm{w},p} > p$ for any $p \in (1, \infty)$. By Proposition 9.12, we complete the proof. $\square$

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A Symmetric Dirichlet forms satisfy the generalized 2-contraction property

In this section, we verify that any symmetric Dirichlet form satisfies (GC)$_2$. In this section, we fix a measure space $(X, \mathcal{B}, m)$.

Let us recall the definition of symmetric Dirichlet form. See, e.g., [CF, FOT, MR] for details on the theory of (symmetric) Dirichlet forms.

Definition A.1 (Symmetric Dirichlet form)**.** Let $\mathcal{F}$ be a dense linear subspace of $L^2(X,m)$ and let $\mathcal{E}\colon \mathcal{F}\times\mathcal{F}\to\mathbb{R}$ be a non-negative definite symmetric bilinear form on $\mathcal{F}$. The pair $(\mathcal{E},\mathcal{F})$ is said to be a *symmetric Dirichlet form* on $L^2(X,m)$ if and only if $\mathcal{F}$ equipped with the inner product $\mathcal{E}+\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(X,m)}$ is a Hilbert space and $u^+\wedge 1\in\mathcal{F}$, $\mathcal{E}(u^+\wedge 1, , u^+\wedge 1)\leq\mathcal{E}(u,u)$ for any $u\in\mathcal{F}$.

We can show that a symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ satisfies (GC)$_2$ by modifying the proof of [MR, Theorem I.4.12].

Proposition A.2. *Let $(\mathcal{E},\mathcal{F})$ be a symmetric Dirichlet form on $L^2(X,m)$. Then $(\mathcal{E},\mathcal{F})$ is a 2-energy form on $L^2(X,m)$ satisfying* (GC)$_2$.

Proof. The triangle inequality for $\mathcal{E}^{1/2}$ is clear, so we shall prove (GC)$_2$ for $(\mathcal{E},\mathcal{F})$. Let us fix $n_1,n_2\in\mathbb{N}$, $q_1\in(0,2]$, $q_2\in[2,\infty]$ and $T=(T_1,\dots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfying (2.1) with 2 in place of p. We consider the case $q_2<\infty$ (the case $q_2=\infty$ is similar). Let $\{G_\alpha\}_{\alpha>0}$ be the strongly continuous resolvent on $L^2(X,m)$ associated with $(\mathcal{E},\mathcal{F})$; see, e.g., [MR, Theorem I.2.8]. By [MR, Theorem I.2.13-(ii)], it suffices to prove that for any $\boldsymbol{u}=(u_1,\dots,u_{n_1})\in L^2(X,m)^{n_1}$ and any $\alpha\in(0,\infty)$,

$$\left(\sum_{l=1}^{n_2}\langle(1-\alpha G_\alpha)T_l(\boldsymbol{u}),T_l(\boldsymbol{u})\rangle_{L^2(X,m)}^{q_2/2}\right)^{1/q_2}\leq\left(\sum_{k=1}^{n_1}\langle(1-\alpha G_\alpha)u_k,u_k\rangle_{L^2(X,m)}^{q_1/2}\right)^{1/q_1}. \tag{A.1}$$

By the linearity of G_α and (2.1), it is enough to prove (A.1) in the case where u_k is a simple function for each $k\in\{1,\dots,n_1\}$, so we assume that

$$u_k=\sum_{i=1}^{N}\alpha_{ki}\mathbb{1}_{A_i},\quad k\in\{1,\dots,n_1\}, \tag{A.2}$$

where $N \in \mathbb{N}$, $(\alpha_{ki})_{i=1}^{N} \subseteq \mathbb{R}$, $\{A_i\}_{i=1}^{N} \subseteq \mathcal{B}(X)$ with $m(A_i) < \infty$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Fix $\alpha \in (0,\infty)$ and, for $i, j \in \{1,\dots,N\}$, we define

$$b_{i,j} \coloneqq \langle (1-\alpha G_\alpha)\mathbb{1}_{A_i}, \mathbb{1}_{A_j} \rangle_{L^2(X,m)}, \quad \lambda_i \coloneqq m(A_i) \quad \text{and} \quad a_{ij} \coloneqq \langle \alpha G_\alpha \mathbb{1}_{A_i}, \mathbb{1}_{A_j} \rangle_{L^2(X,m)}.$$

Then $b_{ij} = \lambda_i \delta_{ij} - a_{ij}$ by a simple calculation, and $a_{ij} = a_{ji}$ since G_α is a symmetric operator on $L^2(X,m)$ (see, e.g., [MR, Theorem I.2.8]). Hence for any $(z_1,\dots,z_N) \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{N} z_i z_j b_{ij} = \sum_{i<j} a_{ij}(z_i - z_j)^2 + \sum_{j=1}^{N} m_j z_j^2, \tag{A.3}$$

where $m_j \coloneqq \lambda_j - \sum_{i=1}^{N} a_{ij}$. Note that $a_{ij} \geq 0$ for any $i, j \in \{1,\dots,N\}$ since $\alpha G_\alpha \mathbb{1}_{A_i} \geq 0$ by [MR, Theorem I.4.4]. We set $A \coloneqq \bigcup_{i=1}^{N} A_i$, and then we have $\alpha G_\alpha(\mathbb{1}_A) \leq 1$ by [MR, Theorem I.4.4] and

$$\sum_{u=1}^{N} a_{ij} = \alpha \int_X \mathbb{1}_A G_\alpha(\mathbb{1}_{A_j})\, dm = \alpha \int_X G_\alpha(\mathbb{1}_A)\mathbb{1}_{A_j}\, dm \leq \int_X \mathbb{1}_{A_j}\, dm = \lambda_j,$$

whence $m_j \geq 0$. With these preparations, we show (A.1) for $\boldsymbol{u}$ defined in (A.2). Set $T_{l,i} \coloneqq T_l(\alpha_{1i},\dots,\alpha_{u_1 i})$ for each $l \in \{1,\dots,n_2\}$.

$$\begin{aligned}
&\sum_{l=1}^{n_2} \langle (1-\alpha G_\alpha) T_l(\boldsymbol{u}), T_l(\boldsymbol{u}) \rangle_{L^2(X,m)}^{q_2/2} = \sum_{l=1}^{n_2} \left(\sum_{i,j=1}^{N} T_{l,i} T_{l,j} b_{ij} \right)^{q_2/2} \\
&\overset{(A.3)}{=} \sum_{l=1}^{n_2} \left(\sum_{i<j} a_{ij} (T_{l,i} - T_{l,j})^{q_2 \cdot \frac{2}{q_2}} + \sum_{j=1}^{N} m_j T_{l,j}^{q_2 \cdot \frac{2}{q_2}} \right)^{q_2/2} \\
&\overset{(2.17)}{\leq} \left(\sum_{i<j} \left(a_{ij}^{q_2/2} \sum_{l=1}^{n_2} (T_{l,i} - T_{l,j})^{q_2} \right)^{2/q_2} + \sum_{j=1}^{N} \left(m_j^{q_2/2} \sum_{l=1}^{n_2} T_{l,j}^{q_2} \right)^{2/q_2} \right)^{q_2/2} \\
&\overset{(2.1)}{\leq} \left(\sum_{i<j} \left(a_{ij}^{q_2/2} \left(\sum_{k=1}^{n_1} (\alpha_{ki} - \alpha_{kj})^{q_1} \right)^{q_2/q_1} \right)^{2/q_2} + \sum_{j=1}^{N} \left(m_j^{q_2/2} \left(\sum_{k=1}^{n_1} \alpha_{kj}^{q_1} \right)^{q_2/q_1} \right)^{2/q_2} \right)^{q_2/2} \\
&= \left(\sum_{i<j} \left(\sum_{k=1}^{n_1} \bigl(a_{ij}(\alpha_{ki} - \alpha_{kj})^2\bigr)^{q_1/2} \right)^{2/q_1} + \sum_{j=1}^{N} \left(\sum_{k=1}^{n_1} (m_j \alpha_{kj}^2)^{q_1/2} \right)^{2/q_1} \right)^{\frac{q_1}{2} \cdot \frac{q_2}{q_1}} \\
&\overset{(*)}{\leq} \left(\left(\sum_{k=1}^{n_1} \left(\sum_{i<j} a_{ij} (\alpha_{ki} - \alpha_{kj})^2 + \sum_{j=1}^{N} m_j \alpha_{kj}^2 \right)^{q_1/2} \right)^{2/q_1} \right)^{\frac{q_1}{2} \cdot \frac{q_2}{q_1}} \\
&= \left(\sum_{k=1}^{n_1} \left(\sum_{i<j} a_{ij} (\alpha_{ki} - \alpha_{kj})^2 + \sum_{j=1}^{N} m_j \alpha_{kj}^2 \right)^{q_1/2} \right)^{q_2/q_1}
\end{aligned}$$

$$\overset{(A.3)}{=} \left(\sum_{k=1}^{n_1} \left(\sum_{i,=1}^{N} \alpha_{ki}\alpha_{kj} b_{ij} \right)^{q_1/2} \right)^{q_2/q_1} = \left(\sum_{k=1}^{n_1} \langle (1-\alpha G_\alpha)u_k, u_k \rangle_{L^2(X,m)}^{q_1/2} \right)^{\frac{2}{q_1}\cdot\frac{q_2}{2}},$$

where we used the triangle inequality for ℓ^{2/q_1}-norm in $(*)$. The proof is completed. $\square$

B Existence of p-resistance forms with non-arithmetic weights

In this section, we discuss a gap between the frameworks in Subsection 8.2 and in Subsection 8.3 for p.-c.f. self-similar structures. As in Subsection 8.3, we fix $p \in (1,\infty)$ and a p.-c.f. self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ with $\#S \geq 2$ and K connected.

The following proposition about the "eigenvalue" $\lambda(\rho_p)$ in Theorem 8.31 is a key result.

Proposition B.1. *Let $\rho_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$. Assume that ρ_p satisfies (8.38) (recall Remark 8.32).*

(a) *For any $a \in (0,\infty)$, $(a\rho_{p,i})_{i\in S}$ satisfies (8.38) and $\lambda(a\rho_p) = a\lambda(\rho_p)$.*

(b) *Let $\widetilde{\rho}_p = (\widetilde{\rho}_{p,i})_{i\in S} \in (0,\infty)^S$. If $\widetilde{\rho}_p$ satisfies (8.38) and $\rho_{p,i} \leq \widetilde{\rho}_{p,i}$ for any $i \in S$, then $\lambda(\rho_p) \leq \lambda(\widetilde{\rho}_p)$.*

Proof. Throughout this proof, we fix a p-resistance form E_0 on V_0.

(a): Set $a\rho_p := (a\rho_{p,i})_{i\in S}$ for $a \in (0,\infty)$. Since $\mathcal{R}^n_{a\rho_p}(E_0) = a\mathcal{R}^n_{\rho_p}(E_0)$ for any $n \in \mathbb{N}\cup\{0\}$, we easily see that $a\rho_p$ satisfies (8.38). Recall that $\lambda(a\rho_p) \in (0,\infty)$ is the unique number satisfying the following: there exists $C \in [1,\infty)$ such that

$$C^{-1}\lambda(a\rho_p)^n E_0(u) \leq \mathcal{R}^n_{a\rho_p}(E_0)(u) \leq C\lambda(a\rho_p)^n E_0(u) \quad \text{for any } n \in \mathbb{N}\cup\{0\},\ u \in \mathbb{R}^{V_0}.$$

(Recall (8.39).) Therefore, $\lambda(a\rho_p) = a\lambda(\rho_p)$.

(b): Since $\mathcal{R}^n_{\rho_p}(E_0)(u) \leq \mathcal{R}^n_{\widetilde{\rho}_p}(E_0)(u)$ for any $u \in \mathbb{R}^{V_0}$, by (8.39), there exists $C \in [1,\infty)$ such that for any $n \in \mathbb{N}\cup\{0\}$ and any $u \in \mathbb{R}^{V_0}$,

$$C^{-1}\lambda(\rho_p)^n E_0(u) \leq \mathcal{R}^n_{\rho_p}(E_0)(u) \leq \mathcal{R}^n_{\widetilde{\rho}_p}(E_0)(u) \leq C\lambda(\widetilde{\rho}_p)^n E_0(u).$$

Since $n \in \mathbb{N}\cup\{0\}$ is arbitrary, we conclude that $\lambda(\rho_p) \leq \lambda(\widetilde{\rho}_p)$. $\square$

Before showing the existence of p-resistance forms with non-arithmetic weights on some self-similar sets, we recall the definition of a special class of self-similar sets called *affine nested fractals*, which was introduced in [FHK94] as a generalization of nested fractals introduced by Liondstrøm [Lin90]. See, e.g., [Kig01, Section 3.8] for details.

Framework B.2. Let $D \in \mathbb{N}$ and let S be a non-empty finite set with $\#S \geq 2$. Let $\{c_i\}_{i\in S} \subseteq (0,1)$, $\{a_i\}_{i\in S} \subseteq \mathbb{R}^D$ and $\{U_i\}_{i\in S} \subseteq O(D)$, where $O(D)$ is the collection of orthogonal transformations of $\mathbb{R}^D$. Define $f_i\colon \mathbb{R}^D \to \mathbb{R}^D$ by $f_i(x) := c_iU_ix + a_i$ for each $i \in S$. Let K be the self-similar set associated with $\{f_i\}_{i\in S}$ and set $F_i := f_i|_K$ for each $i \in S$. Set $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$, which is a self-similar structure. We assume that K is connected, $M := \#(V_0) < \infty$ and $\sum_{i=1}^{M} q_i = 0$, where $q_i \in \mathbb{R}^D$ is defined so that $V_0 = \{q_i\}_{i=1}^{M}$. Let $d\colon K \times K \to [0,\infty)$ be the Euclidean metric on K given by $d(x,y) := |x-y|$.

Definition B.3 (Symmetry and affine nested fractals)**.** (1) A homeomorphism $g\colon K \to K$ is called a *symmetry of* $\mathcal{L}$ if and only if, for any $n \in \mathbb{N} \cup \{0\}$ there exists an injective map $g^{(n)}\colon W_n \to W_n$ such that $g(F_w(V_0)) = F_{g^{(n)}(w)}(V_0)$ for any $w \in W_n$. Let $\mathcal{G}(\mathcal{L}) := \mathcal{G} := \{g \in O(D) \mid g|_K \text{ is a symmetry of } \mathcal{L}\}$.

(2) For $x, y \in \mathbb{R}^D$ with $x \neq y$, let $H_{xy} := \bigl\{z \in \mathbb{R}^D \bigm| |x-z| = |y-z|\bigr\}$. Also let $g_{xy}\colon \mathbb{R}^D \to \mathbb{R}^D$ be the reflection in the hyperplane H_{xy}.

(3) $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is called an *affine nested fractal* if $g_{xy}|_K$ is a symmetry of $\mathcal{L}$ for any $x, y \in V_0$ with $x \neq y$.

The next theorem proves the existence of self-similar p-resistance forms on affine nested fractals, which can be shown by combining [CGQ22, Theorems 4.2, 5.1, 6.3 and Lemma 5.4] and Theorem 8.34.

Theorem B.4. *Assume that $\mathcal{L}$ is an affine nested fractal. Let $p \in (1,\infty)$ and $\rho_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$. If*

$$\rho_{p,i} = \rho_{p,g^{(1)}(i)} \quad \text{for any } i \in S \text{ and } g \in \mathcal{G}, \tag{B.1}$$

then there exist $\lambda = \lambda(\rho_p) \in (0,\infty)$ and a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{F}_p)$ on $\mathcal{L}$ with weight $\lambda\rho_p = (\lambda\rho_{p,i})_{i\in S}$ such that $u \circ g \in \mathcal{F}_p$ and $\mathcal{E}_p(u \circ g) = \mathcal{E}_p(u)$ for any $u \in \mathcal{F}_p$ and any $g \in \mathcal{G}$. Moreover, $\max_{i\in S} \lambda\rho_{p,i} \in (1,\infty)$.

Now we can show the existence of p-resistance forms with non-arithmetic weights on some affine nested fractals as follows.

Proposition B.5. *Let $\mathcal{L}$ be an affine nested fractal. Assume that there exists $i \in S$ such that*

$$\bigcup_{g\in\mathcal{G}} g^{(1)}(i) \neq S. \tag{B.2}$$

Then there exists $\rho_p = (\rho_{p,i})_{i\in S} \in (0,\infty)^S$ such that $\lambda(\rho_p) = 1$, $\rho_{p,i} > 1$ for any $i \in S$, ρ_p satisfies (B.1) and

$$\frac{\log \rho_{p,i}}{\log \rho_{p,j}} \notin \mathbb{Q} \quad \text{for some } i, j \in S. \tag{B.3}$$

In particular, there exists a self-similar p-resistance form $(\mathcal{E}_p, \mathcal{F}_p)$ on $\mathcal{L}$ with weight ρ_p.

Remark B.6. (1) Any weight $\rho_p = (\rho_{p,i})_{i\in S}$ of a p-energy form constructed in Theorem 8.27 must satisfy $\rho_{p,i} = \sigma_p^{n_i}$ for some $n_i \in \mathbb{N}$, where $\sigma_p \in (0,\infty)$ is the p-scaling factor. Hence constructions of self-similar p-energy forms with weight ρ_p which satisfies (B.3) are not covered by Theorem 8.27 (and by [Kig23, Theorem 4.6]).

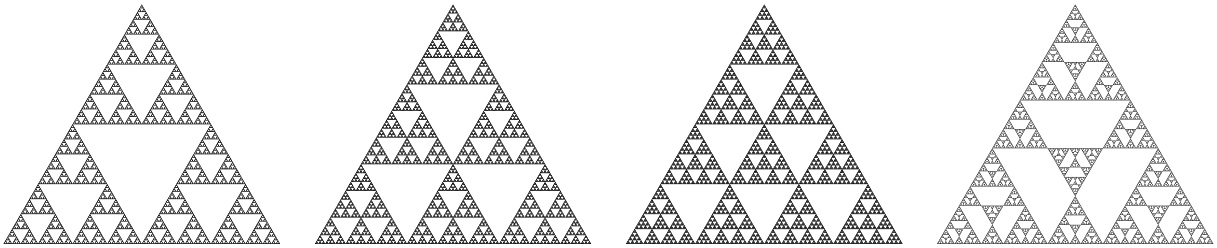

Figure B.1: The 2-dimensional level-2 Sierpiński gasket (left) does not satisfy (B.1). The other three self-similar sets are affine nested fractals satisfying (B.1)

(2) The condition (B.2) is not restrictive. See Figure B.1 for self-similar sets satisfying this condition.

Proof. Fix $i \in S$ and set $S_1 \coloneqq \bigcup_{g \in \mathcal{G}} g^{(1)}(i)$ and $S_2 \coloneqq S \setminus S_1$, which is non-empty by (B.2). For $t \in \mathbb{R}$, we define $\rho_p(t) \coloneqq (\rho_{p,s}(t))_{s \in S}$ by

$$\rho_{p,s}(t) \coloneqq 1 + t\mathbb{1}_{S_2}(s) \quad \text{for } s \in S.$$

It is easy to see that $\rho_p(t)$ satisfies (B.1). Set $\lambda_p(t) \coloneqq \lambda(\rho_p(t))$ for simplicity. By Proposition B.1, for any $t \in \mathbb{R}$, any $\delta \in (0,\infty)$ and any $s \in S$,

$$(1 - t - \delta)\lambda_p(0) \leq \lambda_p(t-\delta) \leq \lambda_p(t) \leq \lambda_p(t+\delta) \leq (1 + t + \delta)\lambda_p(0),$$

whence $\lambda_p(t)$ is continuous in t.

Fix $j \in S_2$ and define

$$r_{i,j}(t) \coloneqq \frac{\log\left(\rho_{p,i}(t)/\lambda_p(t)\right)}{\log\left(\rho_{p,j}(t)/\lambda_p(t)\right)} = \frac{-\log\left(\lambda_p(t)\right)}{\log\left(1+t\right) - \log\left(\lambda_p(t)\right)}, \quad t \in \mathbb{R}.$$

Since $r_{i,j}(0) = 1$ and $r_{i,j}(t)$ is continuous in t, there exists $t_* \in \mathbb{R}\setminus\{0\}$ such that $r_{i,j}(t_*) \notin \mathbb{Q}$. This completes the proof. □

C Ahlfors regular conformal dimension of affine nested fractals

In this section, we prove that the Ahlfors regular conformal dimension of any affine nested fractal equipped with the p-resistance metric for any $p \in (1,\infty)$ is 1. We also show that the Ahlfors regular conformal dimension with respect to the Euclidean metric is also 1 under some geometric condition,

Throughout this section, we assume that $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is an affine nested fractal (see Framework B.2 and Definition B.3). Let $c_i \in (0,1)$ be the contraction ratio of F_i for each $i \in S$. Note that $(c_i)_{i \in S} \in (0,1)^S$ must satisfy

$$c_i = c_{g^{(1)}(i)} \quad \text{for any } i \in S \text{ and any } g \in \mathcal{G}, \tag{C.1}$$

because of the symmetry of $\mathcal{L}$. For each $p \in (1,\infty)$, we also fix a self-similar p-resistance form $(\mathcal{E}_p^{\#}, \mathcal{F}_p^{\#})$ on $\mathcal{L}$ with equal weight $(\rho_{\#,p})_{i\in S} \in (1,\infty)^S$ for some $\rho_{\#,p} \in (1,\infty)$, i.e., $\mathcal{F}_p^{\#} \subseteq C(K)$ and

$$\mathcal{F}_p^{\#} = \{u \in C(K) \mid u \circ F_i \in \mathcal{F}_p^{\#} \text{ for any } i \in S\},$$
$$\mathcal{E}_p^{\#}(u) = \rho_{\#,p} \sum_{i\in S} \mathcal{E}_p^{\#}(u \circ F_i) \quad \text{for any } u \in \mathcal{F}_p^{\#}.$$

By Theorem B.4, such a self-similar p-resistance form on $\mathcal{L}$ exists and the number $\rho_{\#,p}$ is uniquely determined. Let $\widehat{R}_p^{\#}$ denote the p-resistance metric associated with $(\mathcal{E}_p^{\#}, \mathcal{F}_p^{\#})$.

The next lemma describes good geometric properties of metric balls with respect to the p-resistance metrics. (The lemma below is true for p.-c.f. self-similar structures. See [KS.a, Section 6] for details.)

Lemma C.1. *Let $p \in (1,\infty)$ and let $(\mathcal{E}_p, \mathcal{F}_p)$ be a self-similar p-resistance form on $\mathcal{L}$ with weight $\rho_p = (\rho_{p,i})_{i\in S} \in (1,\infty)^S$.*

(a) *We define $\Lambda_1 := \{\emptyset\}$,*

$$\Lambda_s := \left\{ w \;\middle|\; w = w_1 \dots w_n \in W_* \setminus \{\emptyset\}, (\rho_{p,w_1\dots w_{n-1}})^{-1/(p-1)} > s \geq \rho_{p,w}^{-1/(p-1)} \right\}$$

for each $s \in (0,1)$. (Note that $\{\Lambda_s\}_{s\in(0,1]}$ is the scale associated with the weight function $g(w) := \rho_{p,w}^{-1/(p-1)}$; see [Kig20, Definition 2.3.1].) For each $(s,x) \in (0,1] \times K$, we define $\Lambda_{s,0}(x) := \{w \in \Lambda_s \mid x \in K_w\}$, $U_0(x,s) := \bigcup_{w\in\Lambda_{s,0}(x)} K_w$, $\Lambda_{s,1}(x) := \{w \in \Lambda_s \mid K_w \cap U_0(x,s) \neq \emptyset\}$ and $U_1(x,s) := \bigcup_{w\in\Lambda_{s,1}(x)} K_w$. Then there exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that for any $(s,x) \in (0,1] \times K$,

$$B_{\widehat{R}_{p,\mathcal{E}_p}}(x, \alpha_1 s) \subseteq U_1(x,s) \subseteq B_{\widehat{R}_{p,\mathcal{E}_p}}(x, \alpha_2 s). \tag{C.2}$$

(Equivalently, $\widehat{R}_{p,\mathcal{E}_p}$ is 1-adapted to the weight function $g(w) := \rho_{p,w}^{-1/(p-1)}$; see [Kig20, Definition 2.4.1].)

(b) *Let $d_{\mathrm{f}}(\rho_p) \in (0,\infty)$ be such that $\sum_{i\in S} \rho_{p,i}^{-d_{\mathrm{f}}(\rho_p)/(p-1)} = 1$, and let m be the self-similar measure on $\mathcal{L}$ with weight $\bigl(\rho_{p,i}^{-d_{\mathrm{f}}(\rho_p)/(p-1)}\bigr)_{i\in S}$. Then there exist $c_1, c_2 \in (0,\infty)$ such that for any $(x,s) \in K \times (0, \operatorname{diam}(K, \widehat{R}_{p,\mathcal{E}_p})]$,*

$$c_1 s^{d_{\mathrm{f}}(\rho_p)} \leq m(B_{\widehat{R}_{p,\mathcal{E}_p}}(x,s)) \leq c_2 s^{d_{\mathrm{f}}(\rho_p)}. \tag{C.3}$$

In particular, $\widehat{R}_{p,\mathcal{E}_p}$ is metric doubling.

(c) *There exists $C \in (0,\infty)$ such that for any $(x,s) \in K \times (0, \operatorname{diam}(K, \widehat{R}_{p,\mathcal{E}_p})]$,*

$$\inf\left\{ \mathcal{E}_p(u) \;\middle|\; u \in \mathcal{F}_p, u|_{B_{\widehat{R}_{p,\mathcal{E}_p}}(x,\alpha_1 s)} = 1, \operatorname{supp}[u] \subseteq B_{\widehat{R}_{p,\mathcal{E}_p}}(x, 2\alpha_2 s) \right\} \leq C s^{-(p-1)}, \tag{C.4}$$

where α_1, α_2 are the constants in (C.2).

Proof. In this proof, we set $\widehat{R}_p := \widehat{R}_{p,\mathcal{E}_p}$ for simplicity.

(a): By (7.1), we have $\operatorname{diam}(K_w, \widehat{R}_p) \leq \rho_{p,w}^{-1/(p-1)} \operatorname{diam}(K, \widehat{R}_p)$ for any $w \in W_*$, which implies the latter inclusion in (C.2) with $\alpha_2 \in (2\operatorname{diam}(K, \widehat{R}_p), \infty)$ arbitrary. (In particular, $\operatorname{diam}(K_w, \widehat{R}_p) < \alpha_2 s$ for any $w \in \Lambda_s$.) We will show the former inclusion in (C.2). It suffices to prove that there exists $\alpha_1 \in (0,\infty)$ such that $\widehat{R}_p(x,y) \geq \alpha_1 s$ for any $s \in (0,1]$, any $w, v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x,y) \in K_w \times K_v$. Let $\psi_q := h_{V_0}^{\mathcal{E}_p}[\mathbb{1}_q^{V_0}]$ for any $q \in V_0$. Fix $w \in \Lambda_s$ and let $u_w \in C(K)$ be such that, for $\tau \in \Lambda_s$,

$$u_w \circ F_\tau = \begin{cases} 1 & \text{if } \tau = w, \\ \sum_{q \in V_0; F_\tau(q) \in F_w(V_0)} \psi_q & \text{if } \tau \neq w \text{ and } K_\tau \cap K_w \neq \emptyset, \\ 0 & \text{if } K_\tau \cap K_w = \emptyset. \end{cases} \tag{C.5}$$

By the self-similarity for $(\mathcal{E}_p, \mathcal{F}_p)$, we have $u_w \in \mathcal{F}_p$ and

$$\mathcal{E}_p(u_w) = \sum_{\tau \in \Lambda_s} \rho_{p,\tau} \mathcal{E}_p(u_w \circ F_\tau) = \sum_{\tau \in \Lambda_s \setminus \{w\}; K_\tau \cap K_w \neq \emptyset} \rho_{p,\tau} \mathcal{E}_p\left(\sum_{q \in V_0; F_\tau(q) \in F_w(V_0)} \psi_q \right). \tag{C.6}$$

(Note that Λ_s is a partition of Σ.) Set $\overline{\rho}_p := \max_{i \in S} \rho_{p,i} \in (1,\infty)$ and $c_1 := \max_{q \in V_0} \mathcal{E}_p(\psi_q) \in (0,\infty)$. Then $\rho_{p,\tau}^{-1} \geq (\overline{\rho}_p)^{-1} s^{p-1}$ for any $\tau \in \Lambda_s$. Since $\#\{\tau \in \Lambda_s \mid K_\tau \cap K_w \neq \emptyset\} \leq (\#\mathcal{C}_{\mathcal{L}})(\#V_0)$ by [Kig01, Lemma 4.2.3], (C.6) together with Hölder's inequality implies that

$$\mathcal{E}_p(u_w) \leq (\#\mathcal{C}_{\mathcal{L}})(\#V_0) \overline{\rho}_p s^{-p+1} (\#V_0)^{p-1} c_1 =: (\alpha_1 s)^{-(p-1)}. \tag{C.7}$$

For any $v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x,y) \in K_w \times K_v$, we clearly have $u_w(x) = 1$ and $u_w(y) = 0$. Hence

$$\widehat{R}_p(x,y) \geq \mathcal{E}_p(u)^{-1/(p-1)} \geq \alpha_1 s,$$

which proves the desired result.

(b): This is immediate from (C.2), $\#\{\tau \in \Lambda_s \mid K_\tau \cap K_w \neq \emptyset\} \leq (\#\mathcal{C}_{\mathcal{L}})(\#V_0)$ (see [Kig01, Lemma 4.2.3]) and $m(K_w) = \rho_{p,w}^{-1/(p-1)}$ (see [Kig01, Corollary 1.4.8]).

(c): Let $u_w \in \mathcal{F}_p$ be the same function as in the proof of (a) for each $w \in \Lambda_s$. Then $\varphi := \max_{w \in \Lambda_{s,1}(x)} u_w$ satisfies $\varphi|_{U_1(x,s)} = 1$. Since $\operatorname{diam}(K_w, \widehat{R}_p) < \alpha_2 s$, we see from (C.2) that $\operatorname{supp}[\varphi] \subseteq B_{\widehat{R}_p}(x, 2\alpha_2 s)$. By (2.5) for $(\mathcal{E}_p, \mathcal{F}_p)$, (C.7) and [Kig01, Lemma 4.2.3], we have $\varphi \in \mathcal{F}_p$ and

$$\mathcal{E}_p(\varphi) \leq \sum_{w \in \Lambda_{s,1}(x)} \mathcal{E}_p(u_w) \leq (\alpha_1 s)^{-(p-1)} (\#\mathcal{C}_{\mathcal{L}})(\#V_0) =: C s^{-(p-1)}. \qquad \square$$

The next proposition ensures that $\widehat{R}_p^{\#}$ is quasisymmetric to the q-resistance metric with respect to *any* self-similar q-resistance form arising from Theorem B.4. (Recall Definition 8.5-(3).)

Proposition C.2. *Let $p, q \in (1,\infty)$ and assume that $\rho_q = (\rho_{q,i})_{i \in S} \in (0,\infty)^S$ satisfies (B.1), $\rho_{q,i} > 1$ for any $i \in S$ and $\lambda(\rho_q) = 1$, where $\lambda(\rho_q) \in (0,\infty)$ is the unique number* *given in Theorem B.4. Let* $(\mathcal{E}_q, \mathcal{F}_q)$ *be a self-similar* q*-resistance form on* $\mathcal{L}$ *with weight* ρ_q*, which exists by Theorems B.4, and let* $\widehat{R}_q$ *be the* q*-resistance metric associated with* $(\mathcal{E}_q, \mathcal{F}_q)$*. Then* $\widehat{R}_{q,\mathcal{E}_q}$ *is quasisymmetric to* $\widehat{R}^{\#}_p$*.*

Proof. We will use [Kig20, Corollary 3.6.7] to show the desired statement. We first show that there exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that

$$\alpha_1 \rho_{q,w}^{-1/(p-1)} \leq \operatorname{diam}(K_w, \widehat{R}_q) \leq \alpha_2 \rho_{q,w}^{-1/(p-1)} \quad \text{for any } w \in W_*. \tag{C.8}$$

The upper estimate in (C.8) is immediate from (7.1). To prove the lower estimate in (C.8), note that we can easily find $m_0 \in \mathbb{N}$ such that for any $w \in W_*$ there exist $v^1, v^2 \in W_{|w|+m_0}$ with $v^i \leq w$, $i = 1, 2$, and $K_{v^1} \cap K_{v^2} = \emptyset$. (It is enough to choose m_0 satisfying $2(\max_{i\in S} c_i)^{m_0} < 1$.) Then, by combining the proof of Lemma C.1-(a) and $\rho_{p,v^i} \leq \rho_{q,w}(\max_{i\in S}\rho_{q,i})^{m_0}$, there exists $\alpha_1 \in (0,\infty)$ that is independent of $w \in W_*$ such that

$$\inf_{(x,y)\in K_{v^1}\times K_{v^2}} \widehat{R}_q(x,y) \geq \alpha_1 \rho_{q,w}^{-1/(p-1)},$$

which implies the desired lower estimate in (C.8).

Next we note that $\mathcal{L}$ is a rationally ramified self-similar structure by [Kig09, Proposition 1.6.12]; moreover, by combining [Kig09, Proposition 1.6.12], $K_v \cap K_w = F_v(V_0) \cap F_w(V_0)$ for any $v, w \in W_*$ with $\Sigma_v \cap \Sigma_w = \emptyset$ (see [Kig01, Proposition 1.3.5-(2)]) and the fact that each element of V_0 is a fixed point of F_i for some $i \in S_{\mathrm{fix}} \coloneqq \{i \in S \mid K_i \cap V_0 \neq \emptyset\}$, $\mathcal{L}$ is rationally ramified with a relation set

$$\mathcal{R} = \bigl\{\{(\{w(j)\}, \{v(j)\}, \varphi_j, x(j), y(j)) \mid w(j), v(j), x(j), y(j) \in W_* \setminus \{\emptyset\}\}\bigr\}_{j=1}^{k} \tag{C.9}$$

satisfying $w(j), v(j) \in S_{\mathrm{fix}}$. (See [Kig09, Sections 1.5 and 1.6 and Chapter 8] for details about rationally ramified self-similar structures.)

With these preparations, we will apply [Kig20, Corollary 3.6.7] for $\widehat{R}_{q,\mathcal{E}_q}$ and $\widehat{R}^{\#}_p$. By Lemma C.1-(a) and (C.8), $\widehat{R}_{q,\mathcal{E}_q}$ is 1-*adapted* and *exponential* (see [Kig20, Definition 2.4.7 and 3.1.15-(2)] for these definitions; see also Remark in [Kig20, p. 108]). Similarly, $\widehat{R}^{\#}_p$ is also 1-adapted and exponential. Hence, by [Kig20, Corollary 3.6.7], $\widehat{R}_{q,\mathcal{E}_q}$ is quasisymmetric to $\widehat{R}^{\#}_p$ if and only if $\widehat{R}_{q,\mathcal{E}_q}$ is *gentle* with respect to $\widehat{R}^{\#}_p$ (see [Kig20, Definition 3.3.1] for the definition of the gentleness). Define $g_q(w) \coloneqq \rho_{q,w}^{-1/(q-1)}$ and $g_{\#,p}(w) \coloneqq \rho_{\#,p}^{-|w|}$ for $w \in W_*$. Since g_q and $g_{\#,p}$ satisfy the condition (R1) in [Kig09, Theorem 1.6.6] by (B.1) and (C.9), we obtain the desired gentleness by [Kig09, Theorem 1.6.6] and (C.8). This completes the proof. $\square$

Now we can determine the Ahlfors regular conformal dimension of $(K, \widehat{R}^{\#}_p)$.

Theorem C.3. $\dim_{\mathrm{ARC}}(K, \widehat{R}^{\#}_p) = 1$*.*

Proof. We will use the characterization of the Ahlfors regular conformal dimension in [Kig20, Theorem 4.6.9]. Note that $(K, \widehat{R}^{\#}_p)$ satisfies (BF1) and (BF2) in [Kig20, Section

4.3] by Lemma C.1-(a), (C.8), [Kig09, Proposition 1.6.12, Lemmas 1.3.6 and 1.3.12]. We define a graph $G_n = (V_n, E_n)$ and q-energy $\mathcal{E}_p^{G_n}$, $q \in (1,\infty)$, on G_n by

$$E_n := \{(x,y) \mid x, y \in F_w(V_0) \text{ for some } w \in W_n\},$$

and

$$\mathcal{E}_q^{G_n}(f) := \frac{1}{2} \sum_{(x,y)\in E_n} |f(x) - f(y)|^q, \quad f \in \mathbb{R}^{V_n}.$$

Note that $\{G_n\}_{n\geq 0}$ is a *proper system of horizontal networks* with indices $(1, 2(\#V_0 - 1)\#V_0, 1, 1)$ (see [Kig20, Definition 4.6.5]). Hence, by [Kig20, Theorem 4.6.9], $\dim_{\mathrm{ARC}}(K, \widehat{R}_p^{\#}) = 1$ if and only if the following holds: for any $q \in (1,\infty)$,

$$\liminf_{k\to\infty} \sup_{w\in W_*} \inf\left\{ \mathcal{E}_q^{G_{|w|+k}}(f) \;\middle|\; f \in \mathbb{R}^{V_{|w|+k}}, f|_{F_w(V_k)} = 1, f|_{Z_{w,k}} = 0 \right\} = 0, \tag{C.10}$$

where $Z_{w,k} := \{x \in V_{|w|+n} \mid x \in F_v(V_k) \text{ for some } v \in W_{|w|} \text{ with } K_v \cap K_w = \emptyset\}$. Since both $\mathcal{E}_q^{\#}\big|_{V_0}(\,\cdot\,)^{1/q}$ and $\mathcal{E}_q^{G_0}(\,\cdot\,)^{1/q}$ are norms on the finite-dimensional vector space $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$, there exists $C \geq 1$ such that $C^{-1}\mathcal{E}_q^{\#}\big|_{V_0}(u) \leq \mathcal{E}_q^{G_0}(u) \leq C\mathcal{E}_q^{\#}\big|_{V_0}(u)$ for any $u \in \mathbb{R}^{V_0}$. Hence, by Propositions 7.2 and 7.3, we obtain $C^{-1}\mathcal{E}_q^{\#}\big|_{V_n}(u) \leq \rho_{\#,q}^{n}\mathcal{E}_q^{G_n}(u) \leq C\mathcal{E}_q^{\#}\big|_{V_n}(u)$ for any $n \in \mathbb{N}\cup\{0\}$ and any $u \in \mathbb{R}^{V_n}$. Recall that $\Gamma_1(w) = \{v \in W_{|w|} \mid K_v \cap K_w \neq \emptyset\}$ for $w \in W_*$ (see Definition 8.3). Let $h_{q,w} \in \mathcal{F}_q^{\#}$ be the unique function satisfying $h_{q,w}|_{K_w} = 1$, $h_{q,w}|_{K_v} = 0$ for any $v \in W_{|w|} \setminus \Gamma_1(w)$ and

$$\mathcal{E}_q^{\#}(h_{q,w}) = \inf\left\{ \mathcal{E}_q^{\#}(u) \;\middle|\; u|_{K_w} = 1, u|_{K_v} = 0 \text{ for any } v \in W_{|w|} \setminus \Gamma_1(w) \right\}.$$

Then we see from (C.4), (C.2) and (C.8) that

$$\begin{aligned}
&\sup_{w\in W_*} \inf\left\{ \mathcal{E}_q^{G_{|w|+k}}(f) \;\middle|\; f \in \mathbb{R}^{V_{|w|+k}}, f|_{F_w(V_k)} = 1, f|_{Z_{w,k}} = 0 \right\} \\
&\leq C\rho_{\#,q}^{-(|w|+k)} \sup_{w\in W_*} \mathcal{E}_q^{\#}\big|_{V_{|w|+k}}(h_{q,w}|_{V_{|w|+k}}) \leq C\rho_{\#,q}^{-(|w|+k)} \sup_{w\in W_*} \mathcal{E}_q^{\#}(h_{q,w}) \lesssim \rho_{\#,q}^{-k}.
\end{aligned}$$

Since $\rho_{\#,q} \in (1,\infty)$ for any $q \in (0,1)$, we obtain (C.10). The proof is completed. □

To discuss the Ahlfors regular conformal dimension of K with respect to the Euclidean metric, we need the following assumption.

Assumption C.4. We define $\Lambda_1^d := \{\emptyset\}$,

$$\Lambda_s^d := \{w \mid w = w_1 \dots w_n \in W_* \setminus \{\emptyset\}, \mathrm{diam}(K_{w_1\dots w_{n-1}}, d) > s \geq \mathrm{diam}(K_w, d)\}$$

for each $s \in (0,1)$. For $s \in (0,1]$, $M \in \mathbb{N}\cup\{0\}$ and $x \in K$, define

$$\Lambda_{s,M}^d(x) := \left\{ v \;\middle|\; \begin{array}{l} v \in \Lambda_s^d, \text{ there exists } w \in \Lambda_s^d \text{ with } x \in K_w \text{ and} \\ \{z(j)\}_{j=1}^k \subseteq \Lambda_s^d \text{ with } k \leq M+1,\ z(1) = w,\ z(k) = v \\ \text{such that } K_{z(j)} \cap K_{z(j+1)} \neq \emptyset \text{ for any } j \in \{1, \dots, k-1\} \end{array} \right\},$$

and $U^d_M(x,s) \coloneqq \bigcup_{w \in \Lambda^d_{s,M}(x)} K_w$. Then there exist $M_* \in \mathbb{N}$, $\alpha_0, \alpha_1 \in (0,\infty)$ such that

$$U^d_{M_*}(x,\alpha_0 s) \subseteq B_d(x,s) \subseteq U^d_{M_*}(x,\alpha_1 s) \quad \text{for any } (x,s) \in K \times (0,1].$$

(Equivalently, d is M_*-adapted; see [Kig20, Definition 2.4.1].)

Remark C.5. We do not know whether Assumption C.4 is true for any affine nested fractal. Even for a nested fractal, being 1-adapted with respect to the Euclidean metric is assumed in [Kig23, Assumption 4.41].

Now we can show the main result in this section under Assumption C.4.

Theorem C.6. *Assume that Assumption C.4 holds. Then* $\dim_{\mathrm{ARC}}(K,d) = 1$.

Proof. Thanks to Theorem C.3, it suffices to prove that $\widehat{R}^{\#}_p$ is quasisymmetric to d. Obviously, d is exponential since $\mathrm{diam}(K_w,d) = c_w \,\mathrm{diam}(K,d)$. By (C.1), a similar argument as in the proof of Proposition C.2 implies that $\widehat{R}^{\#}_p$ is gentle with respect to d. Hence [Kig20, Corollary 3.6.7] together with Assumption C.4 implies that $\widehat{R}^{\#}_p$ is quasisymmetric to d. □

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