# Korevaar–Schoen $p$-energy forms and associated $p$-energy measures on fractals

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**Abstract** We construct good $p$-energy forms on metric measure spaces as pointwise subsequential limits of Besov-type $p$-energy functionals under certain geometric/analytic conditions. Such forms are often called *Korevaar–Schoen $p$-energy forms* in the literature. As an advantage of our approach, the associated $p$-energy measures are obtained and investigated. We also prove that our construction is applicable to the settings of Kigami [*Mem. Eur. Math. Soc.* **5** (2023)] and Cao–Gu–Qiu [*Adv. Math.* **405** (2022), no. 108517], yields Korevaar–Schoen $p$-energy forms comparable to the $p$-energy forms constructed in these papers, and can be further modified in the case of self-similar sets to obtain self-similar $p$-energy forms keeping most of the good properties of Korevaar–Schoen ones.

**Key words:** Korevaar–Schoen $p$-energy form, $p$-energy measure, generalized $p$-contraction property, $p$-resistance form, self-similar set, self-similar $p$-energy form

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## Contents

1 Introduction . . . 2
2 $p$-Energy forms and generalized $p$-contraction property . . . 7
3 Construction and properties of Korevaar–Schoen $p$-energy forms . . . 8
4 Associated $p$-energy measures and chain rule . . . 19

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5 $p$-Energy forms on $p$-conductively homogeneous spaces . . . . . . . . . . . . . . . . . . . . . . . 33
5.1 $p$-Conductively homogeneous spaces . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 33
5.2 Localized energy estimates . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 40
5.3 Weak monotonicity and Poincaré inequality . . . . . . . . . . . . . . . . . . . . . . . . . 44
5.4 Self-similar $p$-energy forms based on Korevaar–Schoen $p$-energy forms . 47
6 $p$-Resistance forms on p.-c.f. self-similar structures . . . . . . . . . . . . . . . . . . . . . . . . . . . . 53
6.1 Geometry under the $p$-resistance metric . . . . . . . . . . . . . . . . . . . . . . . . . . . . 53
6.2 Estimates on self-similar $p$-energy measures and weak monotonicity . . . . 59
References . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 63

## 1 Introduction

In this article, assuming that $(K, d)$ is a locally compact separable metric space and that $m$ is a Radon measure (i.e., a Borel measure finite on any compact subset) on $K$ with full topological support (i.e., strictly positive on any non-empty open subset), we consider *Korevaar-Schoen*-type $p$-energy forms on $(K, d, m)$, where $p \in (1, \infty)$. Namely, we are concerned with a functional

$$E_{p,s}(u) := \limsup_{r\downarrow 0} \int_K \fint_{B_d(x,r)} \frac{|u(x) - u(y)|^p}{r^{sp}}\, m(dy)m(dx), \quad u \in L^p(K, m),$$

where $B_d(x,r) := \{y \in K \mid d(x,y) < r\}$ and $\fint_A (\,\cdot\,)\, dm := \frac{1}{m(A)} \int_A (\,\cdot\,)\, dm$ for a Borel subset $A$ of $K$ with $m(A) \in (0, \infty)$. Here $s \in (0, \infty)$ is a parameter controlling the smoothness of functions. In the classical settings, the $n$-dimensional Euclidean space $(K, d, m) = (\mathbb{R}^n, |\cdot|, dx)$ for example, the choice $s = 1$ is natural. Indeed, one can show (see, e.g., [33, Corollary 6.3] and [17, Theorem 7.13]; see also [15, Theorem 3.5] for a related result) that there exists $C \in (0, \infty)$ such that the distributional gradient $\nabla u$ of any Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$ satisfies

$$C^{-1} \int_{\mathbb{R}^n} |\nabla u|^p \, dx \le \limsup_{r\downarrow 0} \int_{\mathbb{R}^n} \fint_{|y-x|<r} \frac{|u(x) - u(y)|^p}{r^p}\, dy dx \le C \int_{\mathbb{R}^n} |\nabla u|^p \, dx.$$

In particular, the domain of the functional $E_{p,1}$ is given by the $(1, p)$-Sobolev space $W^{1,p}(\mathbb{R}^n)$ in this case. Note that the functional $E_{p,1}$ can be considered as a variant of the functional considered by Korevaar and Schoen in [31], where they constructed a $(1, p)$-Sobolev space $W^{1,p}(\Omega, X)$ of maps from a domain $\Omega$ in a Riemannian manifold to a complete metric space $X$. On the basis of an idea in [31], Koskela and MacManus [32] introduced a $(1, p)$-Sobolev space $\mathcal{L}^{1,p}$ on any metric measure space satisfying the volume doubling property and the Poincaré inequality (in terms of weak upper gradients), a so-called *PI-space*, as the domain of a functional similar to $E_{p,1}$, and showed that $\mathcal{L}^{1,p}$ coincides with the $(1, p)$-Sobolev spaces introduced by Hajłasz [16] and Hajłasz–Koskela [18]; see [32, Theorem 4.5]. For any PI-space $(K, d, m)$, one can show (see, e.g., [33, Corollary 6.3] and [19, Corollary 10.4.6]) that $\mathcal{L}^{1,p} = \{u \in L^p(K, m) \mid E_{p,1}(u) < \infty\}$, and it turns out that the exponent $s = 1$ is critical in the sense that for every $s > 1$, any function

$u \in L^p(K,m)$ with $E_{p,s}(u) < \infty$ is constant $m$-a.e. if $K$ is connected. (See [19, Chapter 10] for various ways to define $(1,p)$-Sobolev spaces on $(K,d,m)$ and relations among them.) Recently, for more general $(K,d,m)$ which may not be a PI-space, Baudoin [5] proposed to define a $(1,p)$-Sobolev space $\mathrm{KS}^{1,p}$ as the domain $\{u \in L^p(K,m) \mid E_{p,s}(u) < \infty\}$ of $E_{p,s}$ with $s = s_p$, where $s_p$ is the *critical $L^p$-Besov exponent* defined by

$$s_p := \sup\{s \in (0,\infty) \mid E_{p,s}(u) < \infty \text{ for some non-constant } u \in L^p(K,m)\},$$

and discussed some properties of $\mathrm{KS}^{1,p}$ such as Sobolev-type embeddings.

The aim of this article is to construct as nice a $p$-energy form $\mathcal{E}_p^{\mathrm{KS}}$ comparable to $E_{p,s_p}$ as possible. Such $\mathcal{E}_p^{\mathrm{KS}}$ is desired to satisfy at least the following *generalized $p$-contraction property* (see Definition 2.2): if $q_1 \in (0,p]$, $q_2 \in [p,\infty]$, $n_1, n_2 \in \mathbb{N}$ and $T = (T_1,\dots,T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfies $T(0) = 0$ and $\|T(x) - T(y)\|_{\ell^{q_2}} \leq \|x-y\|_{\ell^{q_1}}$ for any $x, y \in \mathbb{R}^{n_1}$, then for any $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in (\mathrm{KS}^{1,p})^{n_1}$,

$$T(\boldsymbol{u}) \in (\mathrm{KS}^{1,p})^{n_2} \text{ and } \left\| \left(\mathcal{E}_p^{\mathrm{KS}}(T_l(\boldsymbol{u}))^{1/p}\right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left(\mathcal{E}_p^{\mathrm{KS}}(u_k)^{1/p}\right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{1.1}$$

The property (1.1) has been introduced in [23] as arguably the strongest possible form of contraction properties of $L^p$-like energy forms. As revealed in [23], (1.1) plays important roles in developing *nonlinear potential theory* in general frameworks including typical self-similar fractals, on which one can construct $p$-energy forms via discrete approximations as established in [20, 29, 38, 7, 35] (see also [13] for a different approach). A problem with $E_{p,s}$ is that $E_{p,s}$ may not satisfy (1.1) because of the operation of taking limsup. To avoid this issue, we would like to take a limit (in some sense) of the Besov-type functionals

$$E_{p,s}(u,r) := \int_K \fint_{B_d(x,r)} \frac{|u(x)-u(y)|^p}{r^{sp}}\, m(dy)m(dx) \tag{1.2}$$

as $r \downarrow 0$. This strategy does not work for all $s \in (0,\infty)$, but does work in the critical case $s = s_p$ in the presence of the following *weak monotonicity* type estimate, which turns out to hold in many situations: there exists a constant $C \in [1,\infty)$ such that for any $u \in L^p(K,m)$ with $\sup_{r>0} E_{p,s_p}(u,r) < \infty$,

$$\sup_{r>0} E_{p,s_p}(u,r) \leq C \liminf_{r\downarrow 0} E_{p,s_p}(u,r). \tag{1.3}$$

This condition (1.3) was introduced in [5] (see Example 3.14). Our first main result, Theorem 3.8, gives a desired $p$-energy form $\mathcal{E}_p^{\mathrm{KS}}$ as a subsequential limit of $\{E_{p,s_p}(\,\cdot\,,r)\}_{r>0}$ under the assumption of (1.3). More precisely, in Theorem 3.8, we establish a subsequential limit of the energy functionals given by

$$\int_K \fint_{B_d(x,r)} \frac{\operatorname{sgn}\bigl(u(x)-u(y)\bigr)\, |u(x)-u(y)|^{p-1}\, (v(x)-v(y))}{r^{s_p p}}\, m(dy)m(dx),$$

that is, we directly construct a two-variable version $\mathcal{E}_p^{\mathrm{KS}}(u;v)$, which is the counterpart of

$$(u,v) \mapsto \int_{\mathbb{R}^n} |\nabla u|^{p-2} \langle \nabla u, \nabla v \rangle \, dx \tag{1.4}$$

in the Euclidean case, where $\langle \cdot , \cdot \rangle$ denotes the inner product on $\mathbb{R}^n$. An advantage of our construction is that we can obtain a good quantitative estimate on the continuity of $\mathcal{E}_p^{\mathrm{KS}}(u;v)$ with respect to the nonlinear part $u$. Namely, unlike our earlier result in [23] (see (2.12) below), the present construction of $\mathcal{E}_p^{\mathrm{KS}}$ allows us to achieve the best Hölder continuity exponent as expected from the formal expression (1.4), i.e., to show that there exists a constant $C \in (0,\infty)$ such that for any $u_1, u_2, v \in \mathrm{KS}^{1,p}$,

$$\left|\mathcal{E}_p^{\mathrm{KS}}(u_1;v) - \mathcal{E}_p^{\mathrm{KS}}(u_2;v)\right| \le C\left[\max_{i\in\{1,2\}} \mathcal{E}_p^{\mathrm{KS}}(u_i)\right]^{\frac{(p-2)^+}{p}} \mathcal{E}_p^{\mathrm{KS}}(u_1-u_2)^{\frac{(p-1)\wedge 1}{p}} \mathcal{E}_p^{\mathrm{KS}}(v)^{\frac{1}{p}}$$

(see (3.12)), which is not known for the $p$-energy forms constructed in the preceding works [20, 29, 38, 13, 7, 35]. See Section 3 for details.

Another superiority of our direct approach is that we can introduce the *$p$-energy measures* associated with $\mathcal{E}_p^{\mathrm{KS}}$. Roughly speaking, for each $u \in \mathrm{KS}^{1,p}$, the $p$-energy measure $\Gamma_p^{\mathrm{KS}}\langle u \rangle$ is a Radon measure on $K$ playing the same role as $|\nabla u|^p \, dx$ in the Euclidean case. Since we have no counterpart of $|\nabla u|$, it is highly non-trivial to construct $\Gamma_p^{\mathrm{KS}}\langle u \rangle$; indeed, it is not known how to construct canonical $p$-energy measures associated with a given $p$-energy form without relying on the self-similarity of the underlying space and the $p$-energy form (see [29, p. 113] and [35, Problem 12.5]). However, our construction of $\mathcal{E}_p^{\mathrm{KS}}$ allows us to employ a naive approach as described below. From the Leibniz and chain rules for the usual gradient operator $\nabla$ on $\mathbb{R}^n$, we easily see that for any $\varphi, u \in C^1(\mathbb{R}^n)$,

$$\varphi\,|\nabla u|^p = |\nabla u|^{p-2}\langle \nabla u, \nabla(u\varphi)\rangle - \left(\frac{p-1}{p}\right)^{p-1} \left|\nabla\bigl(|u|^{\frac{p}{p-1}}\bigr)\right|^{p-2} \left\langle \nabla\bigl(|u|^{\frac{p}{p-1}}\bigr), \nabla\varphi \right\rangle.$$

Since $\mathcal{E}_p^{\mathrm{KS}}(u;v)$ is expected to be the counterpart of $\int_{\mathbb{R}^n} |\nabla u|^{p-2}\langle \nabla u, \nabla v\rangle \, dx$, the $p$-energy measure $\Gamma_p^{\mathrm{KS}}\langle u \rangle$ of $u \in \mathrm{KS}^{1,p}$ associated with $\mathcal{E}_p^{\mathrm{KS}}$ should be characterized as a unique Radon measure on $K$ such that for any $\varphi \in \mathrm{KS}^{1,p} \cap C_c(K)$,

$$\int_K \varphi \, d\Gamma_p^{\mathrm{KS}}\langle u \rangle = \mathcal{E}_p^{\mathrm{KS}}(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}_p^{\mathrm{KS}}\bigl(|u|^{\frac{p}{p-1}}; \varphi\bigr) =: \Psi_{p,u}^{\mathrm{KS}}(\varphi). \tag{1.5}$$

In fact, in the case $p = 2$, this is exactly the same as the definition of energy measures in the theory of regular symmetric Dirichlet forms (see [12, (3.2.14)]). By virtue of our direct construction, we can show that $\Psi_{p,u}^{\mathrm{KS}}$ is a bounded positive linear functional on $\mathrm{KS}^{1,p} \cap C_c(K)$ and we obtain $\Gamma_p^{\mathrm{KS}}\langle u \rangle$ by applying the Riesz–Markov–Kakutani representation theorem under the assumption that $\mathrm{KS}^{1,p} \cap C_c(K)$ is dense in $C_c(K)$ with respect to the uniform norm. We also establish some basic properties of $\Gamma_p^{\mathrm{KS}}\langle u \rangle$

like the generalized $p$-contraction property and the chain rule. See Section 4 for details.

As mentioned above, our construction of $\mathcal{E}_p^{\mathrm{KS}}(u;v)$ and $\Gamma_p^{\mathrm{KS}}\langle u\rangle$ relies heavily on the assumption of the weak monotonicity estimate (1.3), and fortunately it turns out that (1.3) holds in many situations. As proved in [5, Theorem 5.1] (see also [33, Corollary 6.3]), (1.3) holds on any PI-spaces. Besides, (1.3) has been proved for the Vicsek set and for the Sierpiński gasket in [5, Theorems 6.2 and 6.4], for nested fractals in [14, 8], for generalized Sierpiński carpets with $p$ strictly greater than the Ahlfors regular conformal dimension in [39], and in a general setting including the Sierpiński carpet with any $p \in (1,\infty)$ in [35, Theorem 7.1]. See also [21] for related results for the Sierpiński gasket. As extensions of these results, we present two general settings where we can show (1.3). The first one described in Section 5 (see Assumptions 5.19 and 5.35) is based on the notion of *$p$-conductive homogeneity* due to [29], and includes the settings of [29, Theorems 3.21 and 4.6] except that we need to assume the Ahlfors regularity of $m$, which is not assumed in [29, Theorem 3.21]. (This setting is very similar to that in [35, Section 7], although there are indeed slight differences between the setting of discrete approximations of $(K,d)$ in [29] and that in [35].) In particular, *all* the examples of self-similar sets in [29, Sections 4.4–4.6] and those planned to be treated in [30] fall within the framework of our main results in Section 5 (see also Remark 5.15-(2)). The second one presented in Section 6 (see Assumption 6.1) treats the case of *post-critically finite self-similar structures*. In particular, by virtue of the work [7], this framework includes all *affine nested fractals*, which were covered only partially in [29] (see Remark 6.2-(3)).

Very recently, for any $p \in [1,\infty)$, Alonso-Ruiz and Baudoin [4] constructed $p$-energy forms and $p$-energy measures on PI-spaces as $\Gamma$-limits of $E_{p,1}$ and $\overline{\Gamma}$-limits of localized versions of $E_{p,1}$, respectively. Their framework is very different from ours although we do not deal with the case $p=1$. Indeed, $s_p = 1$ on PI-spaces while $s_p > 1$ on generalized Sierpiński carpets and some Sierpiński gaskets as proved in [23, Section 9]. Also, our construction of $p$-energy measures enables us to prove some fundamental properties of them, which were not shown in [4].

This article is organized as follows. In Section 2, we introduce the notion of $p$-energy form and the generalized $p$-contraction property and recall some basic consequences of this property, following [23]. In Section 3, we present basic notation related to the Besov-type functionals (1.2) and, under the assumptions of (1.3) and some mild conditions, we construct a good $p$-energy form $\mathcal{E}_p^{\mathrm{KS}}$ as a subsequential pointwise limit of $\{E_{p,s_p}(\,\cdot\,,r)\}_{r>0}$. We also recall the notion of $p$-resistance form and present a sufficient condition for $\mathcal{E}_p^{\mathrm{KS}}$ to be a $p$-resistance form in the end of Section 3. Section 4 is devoted to discussions on the $p$-energy measures associated with $\mathcal{E}_p^{\mathrm{KS}}$. (More precisely, we prove these results in Sections 3 and 4 in a synthetic way for a more general family of kernels.) In Section 5, we first recall from [29] the setting of $p$-conductively homogeneous compact metric spaces and then verify (1.3) for them under some geometric assumptions. In Section 6, we show (1.3) for post-critically finite self-similar structures under the assumption of the existence of nice self-similar $p$-resistance forms. In Sections 5 and 6, we also show *localized energy estimates*, some estimates on localized versions $\int_E \fint_{B_d(x,r)} \frac{|u(x)-u(y)|^p}{r^{ps_p}}\, m(dy)m(dx)$

of $E_{p,s_p}(u)$ for any Borel subset $E$ of $K$, and that our construction can be further modified in the case of self-similar sets to obtain self-similar $p$-energy forms keeping most of the good properties of Korevaar–Schoen ones.

**Notation** Throughout this paper, we use the following notation and conventions.

(1) For $[0,\infty]$-valued quantities $A$ and $B$, we write $A \lesssim B$ to mean that there exists an implicit constant $C \in (0,\infty)$ depending on some unimportant parameters such that $A \leq CB$. We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$.

(2) For a set $A$, we let $\#A \in \mathbb{N} \cup \{0,\infty\}$ denote the cardinality of $A$.

(3) We set $\sup \emptyset := 0$ and $\inf \emptyset := \infty$. We write $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$ and $a^+ := a \vee 0$ for $a, b \in [-\infty,\infty]$, and we use the same notation also for $[-\infty,\infty]$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty,\infty]$-valued.

(4) Let $X$ be a non-empty set. We define $\mathrm{id}_X \colon X \to X$ by $\mathrm{id}_X(x) := x$, $\mathbf{1}_A = \mathbf{1}_A^X \in \mathbb{R}^X$ for $A \subseteq X$ by $\mathbf{1}_A(x) := \mathbf{1}_A^X(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$ and set $\|u\|_{\sup} := \|u\|_{\sup,X} := \sup_{x \in X} |u(x)|$ for $u \colon X \to [-\infty,\infty]$.

(5) We define $\mathrm{sgn} \colon \mathbb{R} \to \mathbb{R}$ by $\mathrm{sgn}(a) := \mathbf{1}_{(0,\infty)}(a) - \mathbf{1}_{(-\infty,0)}(a)$.

(6) Let $X$ be a topological space. The Borel $\sigma$-algebra of $X$ is denoted by $\mathcal{B}(X)$, the closure of $A \subseteq X$ in $X$ by $\overline{A}^X$, and we say that $A \subseteq X$ is *relatively compact* in $X$ if and only if $\overline{A}^X$ is compact. We set $C(X) := \{u \in \mathbb{R}^X \mid u \text{ is continuous}\}$, $\mathrm{supp}_X[u] := \overline{X \setminus u^{-1}(0)}^X$ for $u \in C(X)$, $C_b(X) := \{u \in C(X) \mid \|u\|_{\sup} < \infty\}$, $C_c(X) := \{u \in C(X) \mid \mathrm{supp}_X[u] \text{ is compact}\}$, and $C_0(X) := \overline{C_c(X)}^{C_b(X)} = \{u \in C(X) \mid u^{-1}(\mathbb{R} \setminus (-\varepsilon,\varepsilon)) \text{ is compact for any } \varepsilon \in (0,\infty)\}$, where $C_b(X)$ is equipped with the uniform norm $\|\cdot\|_{\sup}$.

(7) Let $X$ be a topological space having a countable open base. For a Borel measure $m$ on $X$ and a Borel measurable function $f \colon X \to [-\infty,\infty]$ or an $m$-equivalence class $f$ of such functions, we let $\mathrm{supp}_m[f]$ denote the support of the measure $|f|\,dm$, that is, the smallest closed subset $F$ of $X$ such that $\int_{X \setminus F} |f|\,dm = 0$.

(8) Let $(X,d)$ be a metric space. We set $B_d(x,r) := \{y \in X \mid d(x,y) < r\}$ for $(x,r) \in X \times (0,\infty)$, $(A)_{d,r} := \bigcup_{x \in A} B_d(x,r)$ for $A \subseteq X$ and $r \in (0,\infty)$, and $\mathrm{dist}_d(A,B) := \inf\{d(x,y) \mid x \in A, y \in B\}$ for $A, B \subseteq X$.

(9) Let $(X,\mathcal{B},m)$ be a measure space. We set $⨍_A f\,dm := \frac{1}{m(A)} \int_A f\,dm$ for $f \in L^1(X,m)$ and $A \in \mathcal{B}$ with $m(A) \in (0,\infty)$, and set $m|_A := m|_{\mathcal{B}|_A}$ for $A \in \mathcal{B}$, where $\mathcal{B}|_A := \{B \cap A \mid B \in \mathcal{B}\}$. When $m$ is $\sigma$-finite, the product measure space of $(X,\mathcal{B},m)$ and itself is denoted by $(X \times X, \mathcal{B} \otimes \mathcal{B}, m \times m)$.

## 2 $p$-Energy forms and generalized $p$-contraction property

In this section, following [23], we recall the generalized $p$-contraction property and some basic consequences of it. Throughout this section, we fix $p \in (1, \infty)$, a measure space $(K, \mathcal{B}, m)$, a linear subspace $\mathcal{F}$ of $L^0(K, m) := L^0(K, \mathcal{B}, m)$, where

$$L^0(K, \mathcal{B}, m) := \{\text{the } m\text{-equivalence class of } u \mid u\colon K \to \mathbb{R},\, u \text{ is } \mathcal{B}\text{-measurable}\},$$

and a functional $\mathcal{E}\colon \mathcal{F} \to [0, \infty)$ which is *$p$-homogeneous*, i.e., satisfies $\mathcal{E}(au) = |a|^p\, \mathcal{E}(u)$ for any $(a, u) \in \mathbb{R} \times \mathcal{F}$. (Note that the pair $(\mathcal{B}, m)$ is arbitrary. In the case where $\mathcal{B} = 2^K$ and $m$ is the counting measure on $K$, we have $L^0(K, \mathcal{B}, m) = \mathbb{R}^K$.)

Let us recall the definitions of a $p$-energy form and the generalized $p$-contraction property introduced in [23].

**Definition 2.1 ($p$-Energy form; [23, Definition 3.1])** The pair $(\mathcal{E}, \mathcal{F})$ is said to be a *$p$-energy form* on $(K, m)$ if and only if $\mathcal{E}^{1/p}$ is a seminorm on $\mathcal{F}$.

**Definition 2.2 (Generalized $p$-contraction property; [23, Definition 2.1])** The pair $(\mathcal{E}, \mathcal{F})$ is said to satisfy the *generalized $p$-contraction property*, $(\mathrm{GC})_p$ for short, if and only if the following hold: if $n_1, n_2 \in \mathbb{N}$, $q_1 \in (0, p]$, $q_2 \in [p, \infty]$ and $T = (T_1, \dots, T_{n_2})\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfies

$$T(0) = 0 \quad \text{and} \quad \|T(x) - T(y)\|_{\ell^{q_2}} \le \|x - y\|_{\ell^{q_1}} \quad \text{for any } x, y \in \mathbb{R}^{n_1}, \tag{2.1}$$

then for any $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \mathcal{F}^{n_1}$ we have

$$T(\boldsymbol{u}) \in \mathcal{F}^{n_2} \quad \text{and} \quad \left\| \left( \mathcal{E}(T_l(\boldsymbol{u}))^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left( \mathcal{E}(u_k)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{GC$_p$}$$

See [23, Sections 2 and 3] for details on consequences of $(\mathrm{GC})_p$. Here, in Propositions 2.3, 2.4 and 2.5, we recall some results from [23] that will be used in this paper.

**Proposition 2.3 ([23, Proposition 2.2])** *Suppose that $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_p$.*

(1) *$\mathcal{E}^{1/p}$ satisfies the triangle inequality. In particular, $\mathcal{E}$ is convex on $\mathcal{F}$.*

(2) *Let $\varphi \in C(\mathbb{R})$ satisfy $\varphi(0) = 0$ and $|\varphi(t) - \varphi(s)| \le |t - s|$ for any $s, t \in \mathbb{R}$. Then $\varphi(u) \in \mathcal{F}$ and $\mathcal{E}(\varphi(u)) \le \mathcal{E}(u)$ for any $u \in \mathcal{F}$. Furthermore, for any $u \in \mathcal{F} \cap L^\infty(K, m)$ and $\Phi \in C^1(\mathbb{R})$ with $\Phi(0) = 0$, we have $\Phi(u) \in \mathcal{F}$ and*

$$\mathcal{E}(\Phi(u)) \le \sup\left\{ |\Phi'(t)|^p \;\middle|\; t \in \mathbb{R}, |t| \le \|u\|_{L^\infty(K,m)} \right\} \mathcal{E}(u). \tag{2.2}$$

(3) *For any $u, v \in \mathcal{F}$, we have $u \wedge v, u \vee v \in \mathcal{F}$ and*

$$\mathcal{E}(u \vee v) + \mathcal{E}(u \wedge v) \le \mathcal{E}(u) + \mathcal{E}(v). \tag{2.3}$$

(4) *For any $u, v \in \mathcal{F} \cap L^\infty(K, m)$, we have $uv \in \mathcal{F} \cap L^\infty(K, m)$ and*

$$\mathcal{E}(uv)^{1/p} \le \|v\|_{L^\infty(K,m)}\, \mathcal{E}(u)^{1/p} + \|u\|_{L^\infty(K,m)}\, \mathcal{E}(v)^{1/p}. \tag{2.4}$$

(5) *If* $p \in (1,2]$*, then for any* $u, v \in \mathcal{F}$*,*

$$2\bigl(\mathcal{E}(u)^{1/(p-1)}+\mathcal{E}(v)^{1/(p-1)}\bigr)^{p-1} \le \mathcal{E}(u+v)+\mathcal{E}(u-v) \le 2\bigl(\mathcal{E}(u)+\mathcal{E}(v)\bigr). \tag{2.5}$$

*If* $p \in [2,\infty)$*, then for any* $u, v \in \mathcal{F}$*,*

$$2\bigl(\mathcal{E}(u)^{1/(p-1)}+\mathcal{E}(v)^{1/(p-1)}\bigr)^{p-1} \ge \mathcal{E}(u+v)+\mathcal{E}(u-v) \ge 2\bigl(\mathcal{E}(u)+\mathcal{E}(v)\bigr). \tag{2.6}$$

**Proposition 2.4 ([23, Proposition 3.5])** *Suppose that* $(\mathcal{E},\mathcal{F})$ *satisfies* $(\mathrm{GC})_p$*. Then for any* $u, v \in \mathcal{F}$*,*

$$\mathcal{E}(u+v)+\mathcal{E}(u-v)-2\mathcal{E}(u) \le 2\bigl((p-1)\wedge 1\bigr)\Bigl(\mathcal{E}(u)^{\frac{1}{p-1}}+\mathcal{E}(v)^{\frac{1}{p-1}}\Bigr)^{(p-2)^+}\mathcal{E}(v)^{1\wedge\frac{1}{p-1}}. \tag{2.7}$$

*In particular, in view of the convexity of* $\mathcal{E}$ *from Proposition 2.3-*(1)*,* $\mathbb{R} \ni t \mapsto \mathcal{E}(u+tv) \in [0,\infty)$ *is differentiable and*

$$\lim_{s\to 0}\sup_{h\in\mathcal{F};\mathcal{E}(h)\le 1}\left|\frac{\mathcal{E}(u+sh)-\mathcal{E}(u)}{s}-\frac{d}{dt}\mathcal{E}(u+th)\Big|_{t=0}\right| = 0. \tag{2.8}$$

**Proposition 2.5 ([23, Theorem 3.6])** *Suppose that* $(\mathcal{E},\mathcal{F})$ *satisfies* $(\mathrm{GC})_p$*. For any* $u, v \in \mathcal{F}$*, we define*

$$\mathcal{E}(u;v) := \frac{1}{p}\,\frac{d}{dt}\mathcal{E}(u+tv)\Big|_{t=0}, \tag{2.9}$$

*which exists by Proposition 2.4. Then for any* $u, u_1, u_2, v \in \mathcal{F}$*,* $\mathcal{E}(u;\,\cdot\,)\colon \mathcal{F} \to \mathbb{R}$ *is linear,* $\mathcal{E}(u;u)=\mathcal{E}(u)$*,* $\mathcal{E}(au;v)=\mathrm{sgn}(a)\,|a|^{p-1}\,\mathcal{E}(u;v)$ *for any* $a \in \mathbb{R}$*,*

$$\mathcal{E}(u;h)=0 \quad \textit{and} \quad \mathcal{E}(u+h;v)=\mathcal{E}(u;v) \quad \textit{for any } h \in \mathcal{E}^{-1}(0), \tag{2.10}$$

$$|\mathcal{E}(u;v)| \le \mathcal{E}(u)^{(p-1)/p}\mathcal{E}(v)^{1/p}, \tag{2.11}$$

$$|\mathcal{E}(u_1;v)-\mathcal{E}(u_2;v)| \le c_p\left[\max_{i\in\{1,2\}}\mathcal{E}(u_i)\right]^{\frac{p-1-\alpha_p}{p}}\mathcal{E}(u_1-u_2)^{\alpha_p/p}\mathcal{E}(v)^{1/p} \tag{2.12}$$

*for* $\alpha_p := \frac{(p-1)\wedge 1}{p}$ *and some* $c_p \in (0,\infty)$ *determined solely and explicitly by* $p$*.*

**Notation** Throughout this paper, for any $p$-energy form $(\mathcal{E},\mathcal{F})$ on $(K,m)$ satisfying $(\mathrm{GC})_p$ and any $u, v \in \mathcal{F}$, we let $\mathcal{E}(u;v) \in \mathbb{R}$ denote the element given by (2.9).

## 3 Construction and properties of Korevaar–Schoen $p$-energy forms

In this section, we show the existence of *Korevaar–Schoen* $p$*-energy forms*, i.e., pointwise subsequential limits of the Besov-type $p$-energy functionals (1.2) under the assumption of the weak monotonicity estimate (1.3), and give some basic properties of the limit $p$-energy forms. To be precise, we will prove these results for a more general *family of kernels* in a synthetic way in order to apply the results in this section to construct self-similar $p$-energy forms later in Sections 5 and 6.

Throughout this section, we fix a separable metric space $(K, d)$ with $\#K \geq 2$ and a $\sigma$-finite Borel measure $m$ on $K$ with full topological support. Under this setting, the map from $C(K)$ to $L^0(K, m)$ defined by taking $u \in C(K)$ to its $m$-equivalence class is injective and hence gives a canonical embedding of $C(K)$ into $L^0(K, m)$ as a subalgebra, and we will consider $C(K)$ as a subset of $L^0(K, m)$ through this embedding without further notice.

We also fix $p \in (1, \infty)$ throughout this section unless otherwise stated. We will state some definitions and statements below for any $p \in [1, \infty)$, but on each such occasion we will explicitly declare that we let $p \in [1, \infty)$.

First, we introduce a function space determined by a family of kernels $\{k_r\}_{r>0}$.

**Definition 3.1** Let $p \in [1, \infty)$ and let $\boldsymbol{k} = \{k_r\}_{r>0}$ be a sequence of $[0, \infty]$-valued Borel measurable functions on $K \times K$ with $k_r(x, y) = k_r(y, x)$ for $m \times m$-a.e. $(x, y) \in K \times K$ and $\operatorname{ess\,sup}_{x \in K} \|k_r(x, \cdot)\|_{L^1(K,m)} < \infty$[1]. We define a linear subspace $B^{\boldsymbol{k}}_{p,\infty}$ of $L^p(K, m)$ by

$$B^{\boldsymbol{k}}_{p,\infty} := \left\{ f \in L^p(K, m) \;\middle|\; \sup_{r>0} \int_K \int_K |f(x) - f(y)|^p \, k_r(x, y) \, m(dy)m(dx) < \infty \right\} \tag{3.1}$$

and equip $B^{\boldsymbol{k}}_{p,\infty}$ with the norm $\|\cdot\|_{B^{\boldsymbol{k}}_{p,\infty}}$ defined by

$$\|f\|_{B^{\boldsymbol{k}}_{p,\infty}} := \|f\|_{L^p(K,m)} + \left( \sup_{r>0} \int_K \int_K |f(x) - f(y)|^p \, k_r(x, y) \, m(dy)m(dx) \right)^{1/p}.$$

Also, we define $J_{p,r} \colon L^p(K, m) \to [0, \infty)$ by

$$J^{\boldsymbol{k}}_{p,r}(f) := \int_K \int_K |f(x) - f(y)|^p \, k_r(x, y) \, m(dy)m(dx), \quad f \in L^p(K, m).$$

In the rest of this section, we fix a family of kernels $\boldsymbol{k} = \{k_r\}_{r>0}$ as in Definition 3.1. To state some basic properties of $J^{\boldsymbol{k}}_{p,r}$, let us recall the reverse Minkowski inequality (see, e.g., [1, Theorem 2.13]).

**Proposition 3.2 (Reverse Minkowski inequality)** *Let $(Y, \mathcal{A}, \mu)$ be a measure space*[2] *and let $r \in (0, 1]$. Then for any $\mathcal{A}$-measurable functions $f, g \colon Y \to [0, \infty]$,*

$$\left( \int_Y f^r \, d\mu \right)^{1/r} + \left( \int_Y g^r \, d\mu \right)^{1/r} \leq \left( \int_Y (f + g)^r \, d\mu \right)^{1/r}. \tag{3.2}$$

[1] Thanks to this condition, we easily see that $\int_K \int_K |f(x) - f(y)|^p \, k_r(x, y) \, m(dy)m(dx) < \infty$ for any $r \in (0, \infty)$ and any $f \in L^p(K, m)$.

[2] In the book [1], the reverse Minkowski inequality is stated and proved only for the $L^r$-space over non-empty open subsets of the Euclidean space equipped with the Lebesgue measure, but the same proof works for any measure space.

For ease of notation, we define $\gamma_p \colon \mathbb{R} \to \mathbb{R}$ by

$$\gamma_p(a) \coloneqq \operatorname{sgn}(a)\,|a|^{p-1}.$$

The following proposition is elementary.

**Proposition 3.3** *For any $r \in (0,\infty)$, $(J^{\boldsymbol{k}}_{p,r}, L^p(K,m))$ is a $p$-energy form on $(K,m)$ satisfying* $(\mathrm{GC})_p$*, and for any $f, g \in L^p(K,m)$,*

$$J^{\boldsymbol{k}}_{p,r}(f;g) = \int_K\int_K \gamma_p\bigl(f(x)-f(y)\bigr)(g(x)-g(y))k_r(x,y)\,m(dy)m(dx). \tag{3.3}$$

*Proof.* Suppose that $T = (T_1,\dots,T_{n_2}) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ satisfies (2.1) and that $q_2 < \infty$. Then for any $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in L^p(K,m)^{n_1}$ and any $r \in (0,\infty)$,

$$\begin{aligned}
&\sum_{l=1}^{n_2} J^{\boldsymbol{k}}_{p,r}\bigl(T_l(\boldsymbol{u})\bigr)^{q_2/p}\\
&\overset{(3.2)}{\leq} \left(\int_K\int_K \left[\sum_{l=1}^{n_2}|T_l(\boldsymbol{u}(xm)) - T_l(\boldsymbol{u}(y))|^{q_2}\right]^{p/q_2} k_r(x,y)\,m(dy)m(dx)\right)^{q_2/p}\\
&\overset{(2.1)}{\leq} \left(\int_K\int_K \left[\sum_{k=1}^{n_1}|u_k(x) - u_k(y)|^{q_1}\right]^{p/q_1} k_r(x,y)\,m(dy)m(dx)\right)^{q_2/p}\\
&\overset{(*)}{\leq} \left(\sum_{k=1}^{n_1}\left(\int_K\int_K |u_k(x) - u_k(y)|^{p}\, k_r(x,y)\,m(dy)m(dx)\right)^{q_1/p}\right)^{q_2/q_1}\\
&= \left(\sum_{k=1}^{n_1} J^{\boldsymbol{k}}_{p,r}(u_k)^{q_1/p}\right)^{q_2/q_1}.
\end{aligned} \tag{3.4}$$

Here we used the triangle inequality for the norm of $L^{p/q_1}(K\times K, m_r(dxdy))$ in $(*)$, where $m_r(dxdy) \coloneqq k_r(x,y)\,m(dy)m(dx)$. The proof for the case $q_2 = \infty$ is similar, so $(J^{\boldsymbol{k}}_{p,r}, L^p(K,m))$ is a $p$-energy form on $(K,m)$ satisfying $(\mathrm{GC})_p$. The equality (3.3) follows from the dominated convergence theorem. □

Similarly, we can show the next proposition.

**Proposition 3.4** *For $r \in (0,\infty)$ and $f \in L^p(K,m)$, define $N^{\boldsymbol{k}}_{p,r}(f) \coloneqq \|f\|^p_{L^p(K,m)} + J^{\boldsymbol{k}}_{p,r}(f)$. Then $(N^{\boldsymbol{k}}_{p,r}, L^p(K,m))$ is a $p$-energy form on $(K,m)$ satisfying* $(\mathrm{GC})_p$*. In particular, for any $f, g \in L^p(K,m)$ with $N^{\boldsymbol{k}}_{p,r}(f) \vee N^{\boldsymbol{k}}_{p,r}(g) \leq 1$,*

$$N^{\boldsymbol{k}}_{p,r}(f+g) \leq \left(2^{p \vee \frac{p}{p-1}} - N^{\boldsymbol{k}}_{p,r}(f-g)\right)^{(p-1)\wedge 1}. \tag{3.5}$$

*Proof.* A similar estimate as (3.4) shows that $(N^{\boldsymbol{k}}_{p,r}, L^p(K,m))$ is a $p$-energy form on $(K,m)$ satisfying $(\mathrm{GC})_p$. The desired estimate (3.5) immediately follows from Proposition 2.3-(5). $\square$

Let us introduce a couple of important conditions on $\boldsymbol{k}$.

**Definition 3.5** (1) We say that $\boldsymbol{k} = \{k_r\}_{r>0}$ is *asymptotically local* if and only if there exists $\{\delta(r)\}_{r>0} \subseteq (0,\infty)$ such that $\lim_{r\downarrow 0}\delta(r) = 0$ and

$$\lim_{r\downarrow 0}\int_K \int_{K\setminus B_d(x,\delta(r))} k_r(x,y)\, m(dy)m(dx) = 0. \tag{3.6}$$

(2) Let $p \in [1,\infty)$. We say that $(\mathrm{WM})_{p,\boldsymbol{k}}$ holds if and only if there exists $C \in [1,\infty)$ such that

$$\sup_{r>0} J^{\boldsymbol{k}}_{p,r}(f) \le C \liminf_{r\downarrow 0} J^{\boldsymbol{k}}_{p,r}(f) \quad \text{for any } f \in B^{\boldsymbol{k}}_{p,\infty}. \tag{$\mathrm{WM}$}_{p,\boldsymbol{k}}$$

The next theorem states that the normed space $B^{\boldsymbol{k}}_{p,\infty}$ equipped with $\|\cdot\|_{B^{\boldsymbol{k}}_{p,\infty}}$ is a nice Banach space. Our proof is very similar to that for the case of the $(1,p)$-Korevaar–Schoen–Sobolev space $\mathrm{KS}^{1,p}$ given in [5, Theorems 3.1 and 4.4]. We present a complete proof here to make this paper self-contained.

**Theorem 3.6** *For any $p \in [1,\infty)$, the normed space $B^{\boldsymbol{k}}_{p,\infty}$ is a Banach space. Moreover, if $p \in (1,\infty)$ and $(\mathrm{WM})_{p,\boldsymbol{k}}$ holds, then $B^{\boldsymbol{k}}_{p,\infty}$ is reflexive and separable.*

*Proof.* Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $B^{\boldsymbol{k}}_{p,\infty}$. Then there exists a $L^p$-limit $f \in L^p(K,m)$ of $\{f_n\}_{n\in\mathbb{N}}$. For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f_{n'}\|_{B^{\boldsymbol{k}}_{p,\infty}} < \varepsilon$ for any $n, n' \ge N$. By using Fatou's lemma, we see that $J^{\boldsymbol{k}}_{p,r}(f - f_n) \le \varepsilon^p$ for any $n \ge N$ and hence

$$J^{\boldsymbol{k}}_{p,r}(f)^{1/p} \le J^{\boldsymbol{k}}_{p,r}(f - f_n)^{1/p} + J^{\boldsymbol{k}}_{p,r}(f_n)^{1/p} \le \varepsilon + \sup_{n\in\mathbb{N}} \|f_n\|_{B^{\boldsymbol{k}}_{p,\infty}}.$$

Therefore, $f \in B^{\boldsymbol{k}}_{p,\infty}$ and $\{f_n\}_n$ converges to $f$ in $B^{\boldsymbol{k}}_{p,\infty}$, i.e., $B^{\boldsymbol{k}}_{p,\infty}$ is a Banach space.

Next we assume that $p \in (1,\infty)$ and that $(\mathrm{WM})_{p,\boldsymbol{k}}$ holds. Then $|||f|||_{B^{\boldsymbol{k}}_{p,\infty}} := \bigl(\|f\|^p_{L^p(K,m)} + \limsup_{r\downarrow 0} J^{\boldsymbol{k}}_{p,r}(f)\bigr)^{1/p}$ is a norm on $B^{\boldsymbol{k}}_{p,\infty}$ that is equivalent to $\|\cdot\|_{B^{\boldsymbol{k}}_{p,\infty}}$. We will show that $|||\cdot|||_{B^{\boldsymbol{k}}_{p,\infty}}$ is uniformly convex (see [10, Definition 1] for the definition) and thus $B^{\boldsymbol{k}}_{p,\infty}$ is reflexive by the Milman–Pettis theorem (see, e.g., [19, Theorem 2.49]). Let $\varepsilon > 0$ and $f, g \in B^{\boldsymbol{k}}_{p,\infty}$ with $|||f|||_{B^{\boldsymbol{k}}_{p,\infty}} \vee |||g|||_{B^{\boldsymbol{k}}_{p,\infty}} < 1$ and $|||f - g|||_{B^{\boldsymbol{k}}_{p,\infty}} > \varepsilon$. By [5, Lemma 4.11], it suffices to find $\delta \in (0,\infty)$ that is independent of $f, g$ such that $|||f + g|||_{B^{\boldsymbol{k}}_{p,\infty}} \le 2(1-\delta)$. Choose $r_0 \in (0,\infty)$ so that

$$\|f\|^p_{L^p(K,m)} + J^{\boldsymbol{k}}_{p,r}(f) < 1, \quad \|g\|^p_{L^p(K,m)} + J^{\boldsymbol{k}}_{p,r}(g) < 1 \quad \text{for any } r \in (0, r_0).$$

Since $(\mathrm{WM})_{p,\boldsymbol{k}}$ implies that

$$\begin{aligned}\varepsilon^p &< \|f-g\|_{L^p(K,m)}^p + \limsup_{r\downarrow 0} J_{p,r}^{\boldsymbol{k}}(f-g) \\ &\le C\Bigl(\|f-g\|_{L^p(K,m)}^p + \liminf_{r\downarrow 0} J_{p,r}^{\boldsymbol{k}}(f-g)\Bigr),\end{aligned}$$

there exists $r_1 \in (0,\infty)$ such that

$$\|f-g\|_{L^p(K,m)}^p + J_{p,r}^{\boldsymbol{k}}(f-g) > C^{-1}\varepsilon^p \quad \text{for any } r \in (0, r_1).$$

Hence, for any $r \in (0, r_0 \wedge r_1)$, by using (3.5), we see that

$$\|f+g\|_{L^p(K,m)}^p + J_{p,r}^{\boldsymbol{k}}(f+g) < \Bigl[2^{p \vee \frac{p}{p-1}} - C^{-1}\varepsilon^p\Bigr]^{(p-1)\wedge 1},$$

which implies $\|f+g\|_{L^p(K,m)}^p + J_{p,r}^{\boldsymbol{k}}(f+g) \le 2^p(1-\delta)$ for some $\delta \in (0,\infty)$ depending only on $p, C, \varepsilon$. The desired uniform convexity is proved.

Since $L^p(K,m)$ is separable and the inclusion map of $B_{p,\infty}^{\boldsymbol{k}}$ into $L^p(K,m)$ is a continuous linear injection, $B_{p,\infty}^{\boldsymbol{k}}$ is separable by [2, Proposition 4.1]. □

To obtain the local Hölder continuity with exponent $(p-1)\wedge 1$ of the Korevaar–Schoen $p$-energy forms (see Theorem 3.8-(d) below), we will need the following elementary inequality (see also [34, Proof of Corollary 5.8]).

**Lemma 3.7** *For any $a, b \in \mathbb{R}$,*

$$\bigl|\gamma_p(a) - \gamma_p(b)\bigr| \le \begin{cases} \bigl((p-1) \vee 2^{p-1}\bigr)\,|a-b|^{p-1} & \text{if } p \in (1,2], \\ (p-1)\bigl(|a|^{p-2} \vee |b|^{p-2}\bigr)\,|a-b| & \text{if } p \in (2,\infty). \end{cases}$$

*Proof.* The desired estimate is evident when $|a| \wedge |b| = 0$, so we can assume that $0 < |b| \le |a|$ by exchanging $a$ and $b$ if necessary. The proof is divided into the following five cases.

**Case 1:** $p \in (1,2]$ and $ab < 0$.

We can assume that $b < 0 < a$ by considering $-a, -b$ instead of $a, b$ respectively if necessary. Note that $|a| \le |a-b|$. We see that

$$\bigl|\gamma_p(a) - \gamma_p(b)\bigr| = a^{p-1} - (-b)^{p-1} \le 2\,|a|^{p-1} \le 2\,|a-b|^{p-1}.$$

**Case 2:** $p \in (1,2]$, $ab > 0$ and $|a-b| \le |b|$.

By the same reason as the previous case, we can assume that $a \ge b > 0$. Noting that $|a-b|^{p-2} \ge |b|^{p-2}$, we have

$$\begin{aligned}\bigl|\gamma_p(a) - \gamma_p(b)\bigr| = a^{p-1} - b^{p-1} &= (p-1)\int_b^a t^{p-2}\,dt \\ &\le (p-1)\,|b|^{p-2}\,|a-b| \le (p-1)\,|a-b|^{p-1}.\end{aligned}$$

**Case 3:** $p \in (1,2]$, $ab > 0$ and $|a-b| \ge |b|$.

Similar to the previous cases, we can assume that $a \geq b > 0$. Then $|a - b| \geq |b|$ is equivalent to $b \in (0, a/2]$, whence $\frac{a}{2} \leq a - b = |a - b|$. Now we see that

$$\left|\gamma_p(a) - \gamma_p(b)\right| = a^{p-1} - b^{p-1} \leq a^{p-1} \leq 2^{p-1} |a - b|^{p-1}.$$

**Case 4:** $p \in (2, \infty)$ and $ab < 0$.

In this case, we have $|b|^{p-2} \leq |a|^{p-2}$ by $p-2 \geq 0$. We can assume that $b < 0 < a$ similarly to Case 1. Then

$$\left|\gamma_p(a) - \gamma_p(b)\right| = |a|^{p-2}\, a - |b|^{p-2}\, b \leq |a|^{p-2}\, a - |a|^{p-2}\, b = |a|^{p-2} |a - b|.$$

**Case 5:** $p \in (2, \infty)$ and $ab > 0$.

Similar to Cases 2 and 3, we can assume that $a \geq b > 0$. Then

$$\left|\gamma_p(a) - \gamma_p(b)\right| = a^{p-1} - b^{p-1} = (p-1) \int_b^a t^{p-2}\, dt \leq (p-1)\, |a|^{p-2} |a - b|.$$

The above five cases complete the proof. □

Now we can state and prove the first main theorem of this paper as follows. Recall that we have fixed $p \in (1, \infty)$.

**Theorem 3.8** *Suppose that* $(\mathrm{WM})_{p,\boldsymbol{k}}$ *holds. Then any sequence* $\{\tilde{r}_n\}_{n\in\mathbb{N}} \subseteq (0, \infty)$ *with* $\tilde{r}_n \to 0$ *has a subsequence* $\{r_n\}_{n\in\mathbb{N}}$ *such that the following limit exists in* $[0, \infty)$ *for any* $f \in B^{\boldsymbol{k}}_{p,\infty}$*:*

$$\mathcal{E}^{\boldsymbol{k}}_p(f) := \lim_{n\to\infty} J^{\boldsymbol{k}}_{p,r_n}(f). \tag{3.7}$$

*Moreover, for any such* $\{r_n\}_{n\in\mathbb{N}}$*, the functional* $\mathcal{E}^{\boldsymbol{k}}_p \colon B^{\boldsymbol{k}}_{p,\infty} \to [0, \infty)$ *defined by* (3.7) *satisfies the following properties:*

(a) $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$ *is a* $p$*-energy form on* $(K, m)$ *such that*

$$C^{-1} \sup_{r>0} J^{\boldsymbol{k}}_{p,r}(f) \leq \mathcal{E}^{\boldsymbol{k}}_p(f) \leq C \liminf_{r\downarrow 0} J^{\boldsymbol{k}}_{p,r}(f) \quad \text{for any } f \in B^{\boldsymbol{k}}_{p,\infty}, \tag{3.8}$$

*where* $C \in (0, \infty)$ *is the same as in* $(\mathrm{WM})_{p,\boldsymbol{k}}$*. In particular,* $\mathbf{1}_K \in B^{\boldsymbol{k}}_{p,\infty}$ *and* $\mathcal{E}^{\boldsymbol{k}}_p(\mathbf{1}_K) = 0$ *if* $m(K) < \infty$*.*

(b) $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$ *satisfies* $(\mathrm{GC})_p$*. Furthermore, for any* $f, g \in B^{\boldsymbol{k}}_{p,\infty}$*,* $\{J^{\boldsymbol{k}}_{p,r_n}(f; g)\}_{n\in\mathbb{N}}$ *is convergent in* $\mathbb{R}$ *and*

$$\mathcal{E}^{\boldsymbol{k}}_p(f; g) = \lim_{n\to\infty} J^{\boldsymbol{k}}_{p,r_n}(f; g). \tag{3.9}$$

(c) (Function-wise generalized $p$-contraction property) *Let* $n_1, n_2 \in \mathbb{N}$*,* $q_1 \in (0, p]$*,* $q_2 \in [p, \infty]$*,* $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (B^{\boldsymbol{k}}_{p,\infty})^{n_1}$ *and* $\boldsymbol{v} = (v_1, \dots, v_{n_2}) \in L^p(K, m)^{n_2}$*. If*

$$\|\boldsymbol{v}(x) - \boldsymbol{v}(y)\|_{\ell^{q_2}} \leq \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^{q_1}} \quad \text{for } m \times m\text{-a.e. } (x, y) \in K \times K, \tag{3.10}$$

*then* $\boldsymbol{v} \in (B^{\boldsymbol{k}}_{p,\infty})^{n_2}$ *and*

$$\left\| \left( \mathcal{E}^{\boldsymbol{k}}_p(v_l)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left( \mathcal{E}^{\boldsymbol{k}}_p(u_k)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{3.11}$$

(d) (Local Hölder continuity) *There exists* $C_p \in (0,\infty)$ *determined solely and explicitly by* $p$ *such that for any* $f_1, f_2, g \in B^{\boldsymbol{k}}_{p,\infty}$,

$$\left| \mathcal{E}^{\boldsymbol{k}}_p(f_1; g) - \mathcal{E}^{\boldsymbol{k}}_p(f_2; g) \right| \leq C_p \left[ \max_{i \in \{1,2\}} \mathcal{E}^{\boldsymbol{k}}_p(f_i) \right]^{\frac{(p-2)^+}{p}} \mathcal{E}^{\boldsymbol{k}}_p(f_1 - f_2)^{\frac{(p-1)\wedge 1}{p}} \mathcal{E}^{\boldsymbol{k}}_p(g)^{\frac{1}{p}}. \tag{3.12}$$

(e) (Strong locality) *Suppose that* $\boldsymbol{k}$ *is asymptotically local.*

(i) *Let* $f_1, f_2, g \in B^{\boldsymbol{k}}_{p,\infty}$. *If* $\operatorname{supp}_m[f_1 - a_1 \mathbf{1}_K] \cap \operatorname{supp}_m[f_2 - a_2 \mathbf{1}_K] = \emptyset$ *and either* $\operatorname{supp}_m[f_1 - a_1 \mathbf{1}_K]$ *or* $\operatorname{supp}_m[f_2 - a_2 \mathbf{1}_K]$ *is compact for some* $a_1, a_2 \in \mathbb{R}$, *then*

$$\mathcal{E}^{\boldsymbol{k}}_p(f_1 + f_2 + g) + \mathcal{E}^{\boldsymbol{k}}_p(g) = \mathcal{E}^{\boldsymbol{k}}_p(f_1 + g) + \mathcal{E}^{\boldsymbol{k}}_p(f_2 + g), \tag{3.13}$$

$$\mathcal{E}^{\boldsymbol{k}}_p(f_1 + f_2; g) = \mathcal{E}^{\boldsymbol{k}}_p(f_1; g) + \mathcal{E}^{\boldsymbol{k}}_p(f_2; g). \tag{3.14}$$

(ii) *Let* $f_1, f_2, g \in B^{\boldsymbol{k}}_{p,\infty}$. *If* $\operatorname{supp}_m[f_1 - f_2 - a\mathbf{1}_K] \cap \operatorname{supp}_m[g - b\mathbf{1}_K] = \emptyset$ *and either* $\operatorname{supp}_m[f_1 - f_2 - a\mathbf{1}_K]$ *or* $\operatorname{supp}_m[g - b\mathbf{1}_K]$ *is compact for some* $a, b \in \mathbb{R}$, *then*

$$\mathcal{E}^{\boldsymbol{k}}_p(f_1; g) = \mathcal{E}^{\boldsymbol{k}}_p(f_2; g) \quad \text{and} \quad \mathcal{E}^{\boldsymbol{k}}_p(g; f_1) = \mathcal{E}^{\boldsymbol{k}}_p(g; f_2). \tag{3.15}$$

(f) (Invariance) *Let* $T \colon K \to K$ *be Borel measurable and preserve* $m$, *i.e., satisfy* $T^{-1}(A) \in \mathcal{B}(K)$ *and* $m(T^{-1}(A)) = m(A)$ *for any* $A \in \mathcal{B}(K)$. *If* $\boldsymbol{k}$ *is* $T$-invariant, *i.e.,* $k_r(T(x), T(y)) = k_r(x,y)$ *for* $m \times m$*-a.e.* $(x,y) \in K \times K$ *for each* $r \in (0,\infty)$, *then* $f \circ T \in B^{\boldsymbol{k}}_{p,\infty}$ *and* $\mathcal{E}^{\boldsymbol{k}}_p(f \circ T) = \mathcal{E}^{\boldsymbol{k}}_p(f)$ *for any* $f \in B^{\boldsymbol{k}}_{p,\infty}$.

**Definition 3.9 ($\boldsymbol{k}$-Korevaar–Schoen $p$-energy form)** Suppose that $(\mathrm{WM})_{p,\boldsymbol{k}}$ holds. For each sequence $\{r_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ as in Theorem 3.8, the $p$-energy form $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$ on $(K,m)$ defined by (3.7) is called the *$\boldsymbol{k}$-Korevaar–Schoen $p$-energy form* on $(K,m)$ along $\{r_n\}_{n\in\mathbb{N}}$.

**Remark 3.10** Advantages of our $p$-energy form $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$ on $(K,m)$ are (c) and (d). The estimate (3.12) with the Hölder continuity exponent $(p-1)\wedge 1$ is not known for the $p$-energy forms constructed in [7, 20, 29, 35, 38]. (As stated in Proposition 2.5, the existence of the derivative as in (2.9) and its local Hölder continuity (2.12) with exponent $\frac{(p-1)\wedge 1}{p}$ for $p$-energy forms satisfying $(\mathrm{GC})_p$ have been proved in [23].) We also do not know whether (c) holds for the $p$-energy forms constructed in [7, 20, 29, 35, 38].

*Proof of Theorem 3.8.* Fix a sequence of positive numbers $\{\tilde{r}_n\}_{n\in\mathbb{N}}$ with $\tilde{r}_n \to 0$. Since $B^{\boldsymbol{k}}_{p,\infty}$ is separable by Theorem 3.6, there exists a countable dense subset $\mathscr{C}$ of $B^{\boldsymbol{k}}_{p,\infty}$. A standard diagonal argument yields a subsequence $\{r_n\}_{n\in\mathbb{N}}$ of $\{\tilde{r}_n\}_{n\in\mathbb{N}}$

so that $\lim_{n\to\infty} J^{\boldsymbol{k}}_{p,r_n}(u)$ exists in $\mathbb{R}$ for any $u \in \mathscr{C}$. Let $\varepsilon > 0$, $f \in B^{\boldsymbol{k}}_{p,\infty}$ and pick $f_* \in \mathscr{C}$ satisfying $\sup_{r>0} J^{\boldsymbol{k}}_{p,r}(f - f_*)^{1/p} < \varepsilon$. Then for any $k, l \in \mathbb{N}$, by using the triangle inequality for $J^{\boldsymbol{k}}_{p,r}(\,\cdot\,)^{1/p}$,

$$\begin{aligned}
&\left| J^{\boldsymbol{k}}_{p,r_k}(f)^{1/p} - J^{\boldsymbol{k}}_{p,r_l}(f)^{1/p} \right| \\
&\leq J^{\boldsymbol{k}}_{p,r_k}(f - f_*)^{1/p} + \left| J^{\boldsymbol{k}}_{p,r_k}(f_*)^{1/p} - J^{\boldsymbol{k}}_{p,r_l}(f_*)^{1/p} \right| + J^{\boldsymbol{k}}_{p,r_l}(f - f_*)^{1/p} \\
&\leq 2\varepsilon + \left| J^{\boldsymbol{k}}_{p,r_k}(f_*)^{1/p} - J^{\boldsymbol{k}}_{p,r_l}(f_*)^{1/p} \right|.
\end{aligned}$$

Letting $k \wedge l \to \infty$ in this inequality, we obtain

$$\limsup_{k\wedge l\to\infty} \left| J^{\boldsymbol{k}}_{p,r_k}(f)^{1/p} - J^{\boldsymbol{k}}_{p,r_l}(f)^{1/p} \right| \leq 2\varepsilon,$$

which proves that $\{J^{\boldsymbol{k}}_{p,r_n}(f)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $[0,\infty)$ and hence is convergent in $[0,\infty)$. Now we define $\mathcal{E}^{\boldsymbol{k}}_p \colon B^{\boldsymbol{k}}_{p,\infty} \to [0,\infty)$ by $\mathcal{E}^{\boldsymbol{k}}_p(f) \coloneqq \lim_{n\to\infty} J^{\boldsymbol{k}}_{p,r_n}(f)$.

Clearly, $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$ is a $p$-energy form on $(K, m)$ satysfying (3.8) by $(\mathrm{WM})_{p,\boldsymbol{k}}$. Let us show (b), (c), (d) and (e) because the other properties (a) and (f) are immediate from the expression of $J^{\boldsymbol{k}}_{p,r}(\,\cdot\,)$ and the definition of $\mathcal{E}^{\boldsymbol{k}}_p$.

(b),(c): Obviously, (c) implies $(\mathrm{GC})_p$ for $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$, so we first show (c). For simplicity, we consider the case $q_2 < \infty$ (the case $q_2 = \infty$ is similar). Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (B^{\boldsymbol{k}}_{p,\infty})^{n_1}$ and $\boldsymbol{v} = (v_1, \dots, v_{n_2}) \in L^p(K, m)^{n_2}$ satisfy (3.10). Then the same argument as in (3.4) shows that for any $r \in (0, \infty)$,

$$\sum_{l=1}^{n_2} J^{\boldsymbol{k}}_{p,r}(v_l)^{q_2/p} \leq \left( \sum_{k=1}^{n_1} J^{\boldsymbol{k}}_{p,r}(u_k)^{q_1/p} \right)^{q_2/q_1}, \tag{3.16}$$

which implies that $v_l \in B^{\boldsymbol{k}}_{p,\infty}$ for any $l \in \{1, \dots, n_2\}$. Using (3.7) to take the limit of (3.16) with $r = r_n$ as $n \to \infty$, we obtain $\sum_{l=1}^{n_2} \mathcal{E}^{\boldsymbol{k}}_p(v_l)^{q_2/p} \leq \left( \sum_{k=1}^{n_1} \mathcal{E}^{\boldsymbol{k}}_p(u_k)^{q_1/p} \right)^{q_2/q_1}$. This completes the proof of (c).

Next we prove (3.9). We know that $\mathcal{E}^{\boldsymbol{k}}_p$ is Fréchet differentiable on $B^{\boldsymbol{k}}_{p,\infty}$ by (2.8) in Proposition 2.4. Also, by combining (2.7) (in Proposition 2.4) for $J^{\boldsymbol{k}}_{p,r}$ and the convexity of $t \mapsto J^{\boldsymbol{k}}_{p,r}(f + tg)$, we see that for any $t \in (0, 1)$,

$$\begin{aligned}
&\left| \frac{J^{\boldsymbol{k}}_{p,r}(f + tg) - J^{\boldsymbol{k}}_{p,r}(f)}{t} - pJ^{\boldsymbol{k}}_{p,r}(f; g) \right| \\
&= \left| \frac{J^{\boldsymbol{k}}_{p,r}(f + tg) - J^{\boldsymbol{k}}_{p,r}(f)}{t} - \frac{d}{dt} J^{\boldsymbol{k}}_{p,r}(f + tg)\Big|_{t=0} \right| \\
&\leq \frac{J^{\boldsymbol{k}}_{p,r}(f + tg) + J^{\boldsymbol{k}}_{p,r}(f - tg) - 2J^{\boldsymbol{k}}_{p,r}(f)}{t} \overset{(2.7)}{\leq} O_t(f; g),
\end{aligned}$$

where $O_t(f;g) = C_{p,f,g} t^{(p-1)\wedge\frac{1}{p-1}}$ for some constant $C_{p,f,g}$ which depends only on $p$, $|f|_{B^{\boldsymbol{k}}_{p,\infty}}$ and $|g|_{B^{\boldsymbol{k}}_{p,\infty}}$. Hence we see that

$$\begin{aligned}
&\limsup_{n\to\infty} \left|\mathcal{E}^{\boldsymbol{k}}_p(f;g) - J^{\boldsymbol{k}}_{p,r_n}(f;g)\right| \\
&\leq \lim_{n\to\infty} \left|\mathcal{E}^{\boldsymbol{k}}_p(f;g) - \frac{1}{p}\cdot\frac{J^{\boldsymbol{k}}_{p,r_n}(f+tg) - J^{\boldsymbol{k}}_{p,r_n}(f)}{t}\right| + \frac{1}{p}O_t(f;g) \\
&= \left|\mathcal{E}^{\boldsymbol{k}}_p(f;g) - \frac{1}{p}\cdot\frac{\mathcal{E}^{\boldsymbol{k}}_p(f+tg) - \mathcal{E}^{\boldsymbol{k}}_p(f)}{t}\right| + \frac{1}{p}O_t(f;g) \xrightarrow[t\downarrow 0]{} 0,
\end{aligned}$$

which shows (3.9).

(d): This is immediate from (3.3), Hölder's inequality, Lemma 3.7 and (3.9).

(e): Note that (3.13) with $f_2 - f_1, tg, f_1$ for $t \in \mathbb{R}\setminus\{0\}$ in place of $f_1, f_2, g$ and (3.9) together imply the former equality in (3.15), which in turn with $g, 0, f_1 - f_2$ in place of $f_1, f_2, g$ yields the latter by the linearity of $\mathcal{E}^{\boldsymbol{k}}_p(g;\,\cdot\,)$. Also, (3.14) follows by applying (3.13) with $g$ replaced by $tg$ for $t \in \mathbb{R}\setminus\{0\}$ and by 0, taking the differences of both sides of the resulting equalities, dividing both sides by $t$ and then letting $t \to 0$ on the basis of (3.9). (See [23, Section 3] for the details of these arguments.) Hence it suffices to prove (3.13). For simplicity, for $u \in L^p(K,m)$ and $E \in \mathscr{B}(K)$, define

$$\widetilde{J}^{\boldsymbol{k}}_{p,r}(u \mid E) := \int_E \int_{B_d(x,\delta(r))} |u(x) - u(y)|^p\, k_r(x,y)\, m(dy) m(dx),$$

and set $A_i := \operatorname{supp}_m[f_i - a_i \mathbf{1}_K]$ for $i \in \{1,2\}$. We also set $\widetilde{J}^{\boldsymbol{k}}_{p,r}(u) := \widetilde{J}^{\boldsymbol{k}}_{p,r}(u \mid K)$. Note that there exists $r_0 \in (0,\infty)$ such that $\operatorname{dist}_d(A_1, A_2) > 2\delta(r)$ for any $r \in (0, r_0)$ since either $A_1$ or $A_2$ is compact. Set $N_r := K \setminus ((A_1)_{d,\delta(r)} \cup (A_2)_{d,\delta(r)})$ for $r \in (0,\infty)$. Then for any $r \in (0, r_0)$,

$$\begin{aligned}
&\widetilde{J}^{\boldsymbol{k}}_{p,r}(f_1 + f_2 + g) + \widetilde{J}^{\boldsymbol{k}}_{p,r}(g) \\
&= \widetilde{J}^{\boldsymbol{k}}_{p,r}(f_1 + g \mid (A_1)_{d,\delta(r)}) + \widetilde{J}^{\boldsymbol{k}}_{p,r}(f_2 + g \mid (A_2)_{d,\delta(r)}) + \widetilde{J}^{\boldsymbol{k}}_{p,r}(g \mid N_r) + \widetilde{J}^{\boldsymbol{k}}_{p,r}(g) \\
&= \widetilde{J}^{\boldsymbol{k}}_{p,r}(f_1 + g \mid (A_1)_{d,\delta(r)}) + \widetilde{J}^{\boldsymbol{k}}_{p,r}(g \mid (A_2)_{d,\delta(r)} \cup N_r) \\
&\quad + \widetilde{J}^{\boldsymbol{k}}_{p,r}(f_2 + g \mid (A_2)_{d,\delta(r)}) + \widetilde{J}^{\boldsymbol{k}}_{p,r}(g \mid (A_1)_{d,\delta(r)} \cup N_r) \\
&= \widetilde{J}^{\boldsymbol{k}}_{p,r}(f_1 + g) + \widetilde{J}^{\boldsymbol{k}}_{p,r}(f_2 + g).
\end{aligned} \tag{3.17}$$

Noting that $\lim_{n\to\infty} \widetilde{J}^{\boldsymbol{k}}_{p,r_n}(u) = \mathcal{E}^{\boldsymbol{k}}_p(u)$ for any $u \in B^{\boldsymbol{k}}_{p,\infty} \cap L^\infty(K,m)$ by (3.7) and the asymptotic locality of $\boldsymbol{k}$, we obtain (3.13) by letting $r := r_n$ and $n \to \infty$ in (3.17) provided $f_1, f_2, g \in B^{\boldsymbol{k}}_{p,\infty} \cap L^\infty(K,m)$. Finally, since $(-n) \vee (u \wedge n) \in B^{\boldsymbol{k}}_{p,\infty} \cap L^\infty(K,m)$, $\lim_{n\to\infty} \mathcal{E}^{\boldsymbol{k}}_p(u - (-n) \vee (u \wedge n)) = 0$ (see [23, Section 3] for details) and $\operatorname{supp}_m[u - c\mathbf{1}_K] = \operatorname{supp}_m[(-n) \vee (u \wedge n) - c\mathbf{1}_K]$ for any $u \in B^{\boldsymbol{k}}_{p,\infty}$ and any $(n,c) \in \mathbb{N}\times\mathbb{R}$ with $n > |c|$, (3.13) extends to the remaining case $\{f_1, f_2, g\} \not\subseteq L^\infty(K,m)$ by the triangle inequality for $\mathcal{E}^{\boldsymbol{k}}_p(\,\cdot\,)^{1/p}$, completing the proof. □

Next we would like to state further properties of $\boldsymbol{k}$-Korevaar–Shoen $p$-energy forms in the "strongly $p$-recurrent" case. To this end, we recall the notion of $p$-resistance form introduced in [23] (see [25, 27] for the theory for the case $p = 2$).

**Definition 3.11 ($p$-Resistance form)** Let $K$ be a non-empty set. The pair $(\mathcal{E}, \mathcal{F})$ of $\mathcal{F} \subseteq \mathbb{R}^K$ and $\mathcal{E} \colon \mathcal{F} \to [0, \infty)$ is said to be a *$p$-resistance form* on $K$ if and only if it satisfies the following conditions $(\mathrm{RF1})_p$-$(\mathrm{RF5})_p$:

$(\mathrm{RF1})_p$ $\mathcal{F}$ is a linear subspace of $\mathbb{R}^K$ (containing $\mathbb{R}\mathbf{1}_K$) and $\mathcal{E}(\,\cdot\,)^{1/p}$ is a seminorm on $\mathcal{F}$ satisfying $\{u \in \mathcal{F} \mid \mathcal{E}(u) = 0\} = \mathbb{R}\mathbf{1}_K$.

$(\mathrm{RF2})_p$ The quotient normed space $(\mathcal{F}/\mathbb{R}\mathbf{1}_K, \mathcal{E}^{1/p})$ is a Banach space.

$(\mathrm{RF3})_p$ If $x \neq y \in K$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

$(\mathrm{RF4})_p$ For any $x, y \in K$,

$$R_{\mathcal{E}}(x, y) := R_{(\mathcal{E},\mathcal{F})}(x, y) := \sup\left\{ \frac{|u(x) - u(y)|^p}{\mathcal{E}(u)} \;\middle|\; u \in \mathcal{F} \setminus \mathbb{R}\mathbf{1}_K \right\} < \infty. \tag{3.18}$$

$(\mathrm{RF5})_p$ $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{GC})_p$.

We also need to recall the following standard notions on the metric $d$ and the measure $m$.

**Definition 3.12** Let $Q \in (0, \infty)$.

(1) The metric $d$ is said to be *metric doubling* if and only if for any $\delta \in (0, 1)$ there exists $N \in \mathbb{N}$ such that for any $(x, r) \in K \times (0, \infty)$ we can find $\{x_j\}_{j=1}^N \subseteq K$ so that $B_d(x, r) \subseteq \bigcup_{j=1}^N B_d(x_j, \delta r)$.

(2) The measure $m$ is said to be *volume doubling with growth exponent $Q$* (with respect to the metric $d$) if and only if there exists $C_{\mathrm{D}} \in (0, \infty)$ such that

$$m(B_d(x, s)) \le C_{\mathrm{D}} \left(\frac{s}{r}\right)^Q m(B_d(x, r)) < \infty \quad \text{for any } x \in K \text{ and any } 0 < r \le s. \tag{3.19}$$

(3) The measure $m$ is said to be *$Q$-Ahlfors regular* (with respect to the metric $d$) if and only if there exists $C_{\mathrm{AR}} \in [1, \infty)$ such that

$$C_{\mathrm{AR}}^{-1} s^Q \le m(B_d(x, s)) \le C_{\mathrm{AR}} s^Q \quad \text{for any } (x, s) \in K \times (0, \operatorname{diam}(K, d)). \tag{3.20}$$

It is well known that the $Q$-Ahlfors regularity of $m$ implies that the volume doubling property of $m$ with growth exponent $Q$, and that the volume doubling property of $m$ with respect to $d$ implies the metric doubling property of $d$.

Now we give a sufficient condition for a $\boldsymbol{k}$-Korevaar–Schoen $p$-energy form $(\mathcal{E}_p^{\boldsymbol{k}}, B_{p,\infty}^{\boldsymbol{k}})$ on $(K, m)$ to be a $p$-resistance form on $K$.

**Proposition 3.13** *Suppose that there exist $Q, \beta_p \in (0, \infty)$ with $\beta_p > Q$ such that the following hold:*

(i) *The measure $m$ satisfies $m(K) < \infty$ and is volume doubling with growth exponent $Q \in (0, \infty)$.*

(ii) $(\mathrm{WM})_{p,\boldsymbol{k}}$ *holds.*

(iii) $\left\{u \in B^{\boldsymbol{k}}_{p,\infty} \,\middle|\, \sup_{r>0} J^{\boldsymbol{k}}_{p,r}(u) = 0\right\} = \mathbb{R}\mathbf{1}_K$.

(iv) $B^{\boldsymbol{k}}_{p,\infty} \subseteq C(K)$*, and there exists* $C \in (0,\infty)$ *such that for any* $f \in B^{\boldsymbol{k}}_{p,\infty}$ *and any* $x, y \in K$,

$$|f(x) - f(y)| \leq C d(x,y)^{(\beta_p - Q)/p} \sup_{r>0} J^{\boldsymbol{k}}_{p,r}(f)^{1/p}, \quad x, y \in K. \tag{3.21}$$

(v) *There exists* $C \in (0,\infty)$ *such that for any* $(x,s) \in K \times (0,\infty)$ *with* $B_d(x,s) \neq K$,

$$\begin{aligned}\inf&\left\{\sup_{r>0} J^{\boldsymbol{k}}_{p,r}(f) \,\middle|\, f \in C(K), \operatorname{supp}_K[f] \subseteq B_d(x,2s), f \geq 1 \text{ on } B_d(x,s)\right\}\\ &\leq C\frac{m(B_d(x,s))}{s^{\beta_p}}.\end{aligned} \tag{3.22}$$

*Then any* $\boldsymbol{k}$*-Korevaar–Schoen* $p$*-energy form* $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$ *on* $(K,m)$*, which exists by* (ii) *and Theorem 3.8, is a* $p$*-resistance form on* $K$*. If in addition* $m$ *is* $Q$*-Ahlfors regular, then there exist* $\alpha_0, \alpha_1 \in (0,\infty)$ *such that for any such* $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$,

$$\alpha_0 d(x,y)^{\beta_p - Q} \leq R_{\mathcal{E}^{\boldsymbol{k}}_p}(x,y) \leq \alpha_1 d(x,y)^{\beta_p - Q} \quad \text{for any } x, y \in K. \tag{3.23}$$

*Proof.* Let $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$ be a $\boldsymbol{k}$-Korevaar–Schoen $p$-energy form on $(K,m)$. We shall show that $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$ is a $p$-resistance form on $K$. $(\mathrm{RF1})_p$ and $(\mathrm{RF5})_p$ are clear from Theorem 3.8 and (iii). The condition (3.22) immediately implies $(\mathrm{RF3})_p$. By (3.21) and the lower inequality in (3.8) we have $R_{\mathcal{E}^{\boldsymbol{k}}_p}(x,y) \lesssim d(x,y)^{\beta_p - Q}$ for any $x, y \in K$, whence $(\mathrm{RF4})_p$ and the upper estimate in (3.23) hold. In particular, $\sup_{x,y\in K} R_{\mathcal{E}^{\boldsymbol{k}}_p}(x,y) < \infty$. To prove $(\mathrm{RF2})_p$, we see from (3.21) that for any $f \in B^{\boldsymbol{k}}_{p,\infty}$,

$$\begin{aligned}\int_K \left| f(x) - \fint_K f\,dm \right|^p m(dx) &\leq \int_K \fint_K |f(x) - f(y)|^p \, m(dy)m(dx)\\ &\lesssim \left(\sup_{x,y\in K} R_{\mathcal{E}^{\boldsymbol{k}}_p}(x,y)\right)\mathcal{E}^{\boldsymbol{k}}_p(f)m(K).\end{aligned} \tag{3.24}$$

Let $\{f_n\}_{n\in\mathbb{N}} \subseteq B^{\boldsymbol{k}}_{p,\infty}$ be a Cauchy sequence in $(B^{\boldsymbol{k}}_{p,\infty}/\mathbb{R}\mathbf{1}_K, \mathcal{E}^{\boldsymbol{k}}_p(\,\cdot\,)^{1/p})$ with $\fint_K f_n\,dm = 0$. Then (3.24) implies that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(K,m)$, and thus $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $B^{\boldsymbol{k}}_{p,\infty}$. Since $B^{\boldsymbol{k}}_{p,\infty}$ is a Banach space by Theorem 3.6, we conclude that $(B^{\boldsymbol{k}}_{p,\infty}/\mathbb{R}\mathbf{1}_K, \mathcal{E}^{\boldsymbol{k}}_p(\,\cdot\,)^{1/p})$ is also a Banach space.

Next we show the lower estimate in (3.23) under the assumption that $m$ is $Q$-Ahlfors regular. Let $x, y \in K$ and let $s > 0$ satisfy $d(x,y) > 2s \geq 2^{-1}d(x,y)$. Then $B_d(x,s) \neq \emptyset$. By (3.22), there exists $f \in B^{\boldsymbol{k}}_{p,\infty} \cap C(K)$ such that $\operatorname{supp}_K[f] \subseteq B_d(x,s)$, $f \geq 1$ on $B_d(x,2s)$ and $\mathcal{E}^{\boldsymbol{k}}_p(f) \leq C_1 s^{Q-\beta_p}$, where $C_1 \in (0,\infty)$ depends only on $C$ in (3.22) and $C_{\mathrm{AR}}$ in (3.20). Hence we have

$$R_{\mathcal{E}^{\boldsymbol{k}}_p}(x,y) \geq \mathcal{E}^{\boldsymbol{k}}_p(f)^{-1} \geq C_1^{-1} s^{\beta_p - Q} \gtrsim d(x,y)^{\beta_p - Q}. \qquad \square$$

**Example 3.14 (Korevaar–Schoen–Sobolev space)** In addition to the setting specified at the beginning of this section, we suppose that $K$ is connected and that $m(B_d(x,r)) < \infty$ for any $(x,r) \in K \times (0,\infty)$. For $s > 0$, define $\boldsymbol{k}^s = \{k_r^s\}_{r>0}$ by

$$k_r^s(x,y) \coloneqq \frac{\mathbf{1}_{B_d(x,r)}(y)}{r^{ps} m(B_d(x,r))}, \quad x,y \in K. \tag{3.25}$$

Clearly, $\boldsymbol{k}^s$ is asymptotically local. We define the *Besov–Lipschitz space* $B^s_{p,\infty}$ by $B^s_{p,\infty} \coloneqq B^{\boldsymbol{k}^s}_{p,\infty}$. Then the *critical $L^p$-Besov exponent $s_p$* of $(K,d,m)$ is defined as

$$s_p \coloneqq \sup\bigl\{ s > 0 \bigm| B^s_{p,\infty} \text{ contains a non-constant function} \bigr\}. \tag{3.26}$$

We call $\mathrm{KS}^{1,p} \coloneqq B^{s_p}_{p,\infty}$ the $(1,p)$*-Korevaar–Schoen–Sobolev space* on $(K,d,m)$. We also write $\mathrm{KS}^{1,p}(K,d,m)$ for $\mathrm{KS}^{1,p}$ when we would like to clarify the underlying metric measure space $(K,d,m)$. If $m$ is $Q$-Ahlfors regular with respect to $d$ for some $Q \in (0,\infty)$, then $\boldsymbol{k}^{s_p,Q} = \bigl\{k_r^{s_p,Q}\bigr\}_{r>0}$ given by

$$k_r^{s_p,Q}(x,y) \coloneqq r^{-ps_p-Q} \mathbf{1}_{B_d(x,r)}(y), \quad x,y \in K,$$

which again is obviously asymptotically local, also corresponds to the $(1,p)$-Korevaar–Schoen–Sobolev space, i.e., $B^{\boldsymbol{k}^{s_p,Q}}_{p,\infty} = \mathrm{KS}^{1,p}$. If $(\mathrm{WM})_{p,\boldsymbol{k}^{s_p}}$ holds, then we write $\mathcal{E}_p^{\mathrm{KS}}$ instead of $\mathcal{E}_p^{\boldsymbol{k}^{s_p}}$ and call each $\boldsymbol{k}^{s_p}$-Korevaar–Schoen $p$-energy form $(\mathcal{E}_p^{\mathrm{KS}}, \mathrm{KS}^{1,p})$ on $(K,m)$ a *Korevaar–Schoen $p$-energy form* on $(K,d,m)$.

It is not easy to verify $(\mathrm{WM})_{p,\boldsymbol{k}}$ and (3.22) in general; see Sections 5 and 6 for some settings in which we can prove $(\mathrm{WM})_{p,\boldsymbol{k}}$ and (3.22). On the other hand, a reasonable sufficient condition for (3.21) is known. In fact, if $m$ is volume doubling with growth exponent $Q \in (0,\infty)$ and $ps_p > Q$, then (3.21) holds for $\mathrm{KS}^{1,p}$; see, e.g., [3, Theorem 5.1] or [5, Theorem 3.2].

## 4 Associated $p$-energy measures and chain rule

Next in this section, we introduce the $p$-energy measures associated with a given $\boldsymbol{k}$-Korevaar–Schoen $p$-energy form $(\mathcal{E}_p^{\boldsymbol{k}}, B^{\boldsymbol{k}}_{p,\infty})$, and show their basic properties.

Throughout this section, as in the previous section, we fix $p \in (1,\infty)$, a separable metric space $(K,d)$ with $\#K \geq 2$ and a $\sigma$-finite Borel measure $m$ on $K$ with full topological support. In addition, we suppose that $(K,d)$ is locally compact. We also fix a family of kernels $\boldsymbol{k} = \{k_r\}_{r>0}$ as in Definition 3.1, suppose that $\boldsymbol{k}$ is asymptotically local and that $(\mathrm{WM})_{p,\boldsymbol{k}}$ holds, and fix an arbitrary sequence $\{r_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ as in Theorem 3.8, so that we have the $\boldsymbol{k}$-Korevaar–Schoen $p$-energy form $(\mathcal{E}_p^{\boldsymbol{k}}, B^{\boldsymbol{k}}_{p,\infty})$ on $(K,m)$ along $\{r_n\}_{n\in\mathbb{N}}$ defined by (3.7). For ease of notation, we set

$$m_n(dxdy) \coloneqq k_{r_n}(x,y)m(dy)m(dx).$$

For each $u \in B^{\boldsymbol{k}}_{p,\infty} \cap L^\infty(K,m)$, define a linear map $\Psi^{\boldsymbol{k}}_p(u;\,\cdot\,)\colon B^{\boldsymbol{k}}_{p,\infty} \cap L^\infty(K,m) \to \mathbb{R}$ by, for each $\varphi \in B^{\boldsymbol{k}}_{p,\infty} \cap L^\infty(K,m)$,

$$\Psi^{\boldsymbol{k}}_p(u;\varphi) \coloneqq \mathcal{E}^{\boldsymbol{k}}_p(u;u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}^{\boldsymbol{k}}_p\bigl(|u|^{\frac{p}{p-1}};\varphi\bigr). \tag{4.1}$$

(Note that $u\varphi, |u|^{\frac{p}{p-1}} \in B^{\boldsymbol{k}}_{p,\infty}$ by Theorem 3.8-(b) and Proposition 2.3-(4),(2).)

**Theorem 4.1** *Let $u \in B^{\boldsymbol{k}}_{p,\infty} \cap C_b(K)$ and $\varphi \in B^{\boldsymbol{k}}_{p,\infty} \cap L^\infty(K,m)$. If $\{u,\varphi\} \cap C_c(K) \neq \emptyset$, then*

$$\Psi^{\boldsymbol{k}}_p(u;\varphi) = \lim_{n\to\infty} \int_K \int_K |u(x)-u(y)|^p\, \varphi(x) k_{r_n}(x,y)\, m(dy)m(dx), \tag{4.2}$$

$$\bigl|\Psi^{\boldsymbol{k}}_p(u;\varphi)\bigr| \leq \|\varphi\|_{L^\infty(K,m)}\, \mathcal{E}^{\boldsymbol{k}}_p(u). \tag{4.3}$$

*In particular, if in addition $\varphi \geq 0$, then $\Psi^{\boldsymbol{k}}_p(u;\varphi) \geq 0$.*

*Proof.* First, we observe that

$$\begin{aligned}
&\Psi^{\boldsymbol{k}}_{p,n}(u;\varphi) \coloneqq J^{\boldsymbol{k}}_{p,r_n}(u;u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} J^{\boldsymbol{k}}_{p,r_n}\bigl(|u|^{\frac{p}{p-1}};\varphi\bigr) \\
&= \int_{K\times K} \Biggl[ |u(x)-u(y)|^p\, \varphi(x) + \gamma_p\bigl(u(x)-u(y)\bigr)\cdot(\varphi(x)-\varphi(y))u(y) \\
&\quad - \left(\frac{p-1}{p}\right)^{p-1} \gamma_p\Bigl(|u(x)|^{\frac{p}{p-1}} - |u(y)|^{\frac{p}{p-1}}\Bigr)\cdot(\varphi(x)-\varphi(y))\Biggr] m_n(dxdy).
\end{aligned} \tag{4.4}$$

Define $F_n \in \mathcal{B}(K\times K)$ by

$$F_n \coloneqq \{(x,y) \in K\times K \mid d(x,y) < \delta(r_n), \varphi(x)-\varphi(y) \neq 0, \varphi(x) \neq 0, \varphi(y) \neq 0\},$$

and set

$$\begin{aligned}
&I^{\boldsymbol{k}}_{p,n}(u;\varphi) \\
&\coloneqq \int_{F_n} \Biggl[ |u(x)-u(y)|^p\, \varphi(x) + \gamma_p\bigl(u(x)-u(y)\bigr)\cdot(\varphi(x)-\varphi(y))u(y) \\
&\quad - \left(\frac{p-1}{p}\right)^{p-1} \gamma_p\Bigl(|u(x)|^{\frac{p}{p-1}} - |u(y)|^{\frac{p}{p-1}}\Bigr)\cdot(\varphi(x)-\varphi(y))\Biggr] m_n(dxdy).
\end{aligned}$$

Note that $\lim_{n\to\infty}\bigl(\Psi^{\boldsymbol{k}}_{p,n}(u;\varphi) - I^{\boldsymbol{k}}_{p,n}(u;\varphi)\bigr) = 0$ by (3.6) and $\|u\|_{\sup} \vee \|\varphi\|_{L^\infty(K,m)} < \infty$. Since $\overline{F_n}^{K\times K}$ is compact for sufficiently large $n \in \mathbb{N}$ when $\varphi \in B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$, $u$ is uniformly continuous on $F_n$ for such $n$. By combining this observation with the uniform continuity of $t \mapsto |t|^{1/(p-1)}\operatorname{sgn}(t)$ on $u(K)$ and (3.6), for any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\begin{aligned}&\left|\frac{p-1}{p}\Bigl(|u(x)|^{\frac{p}{p-1}}-|u(y)|^{\frac{p}{p-1}}\Bigr)-\bigl(u(x)-u(y)\bigr)\,|u(y)|^{\frac{1}{p-1}}\operatorname{sgn}\bigl(u(y)\bigr)\right|\\&\le\left|\int_{u(y)}^{u(x)}\Bigl[|t|^{\frac{1}{p-1}}\operatorname{sgn}(t)-|u(y)|^{\frac{1}{p-1}}\operatorname{sgn}\bigl(u(y)\bigr)\Bigr]\,dt\right|\le\varepsilon\,|u(x)-u(y)|\,,\end{aligned}\tag{4.5}$$

for any $(x,y)\in\bigcup_{n\ge N}F_n$. Using (4.5), Lemma 3.7 and Hölder's inequality, we can find $C_{p,u}\in(0,\infty)$ depending only on $p$, $\|u\|_{\sup}$ and $\mathcal{E}_p^{\boldsymbol{k}}(u)$ such that

$$\begin{aligned}&\sup_{n\ge N}\left|\int_{F_n}\Bigl[\gamma_p\bigl(u(x)-u(y)\bigr)\cdot(\varphi(x)-\varphi(y))u(y)\right.\\&\quad\left.-\left(\frac{p-1}{p}\right)^{p-1}\gamma_p\Bigl(|u(x)|^{\frac{p}{p-1}}-|u(y)|^{\frac{p}{p-1}}\Bigr)\cdot(\varphi(x)-\varphi(y))\Bigr]\,m_n(dxdy)\right|\\&\le C_{p,u}\varepsilon^{\frac{(p-1)\wedge1}{p}}\mathcal{E}_p^{\boldsymbol{k}}(u)^{\frac{(p-1)\wedge1}{p}}\mathcal{E}_p^{\boldsymbol{k}}(\varphi)^{\frac{1}{p}}=:C_{p,u,\varphi}\varepsilon^{\frac{(p-1)\wedge1}{p}}.\end{aligned}$$

Therefore, (4.4) implies that for any $n\ge N$,

$$\begin{aligned}&\left|\Psi_{p,n}^{\boldsymbol{k}}(u;\varphi)-\int_{K\times K}|u(x)-u(y)|^p\,\varphi(x)\,m_n(dxdy)\right|\\&\le\left|\Psi_{p,n}^{\boldsymbol{k}}(u;\varphi)-I_{p,n}^{\boldsymbol{k}}(u;\varphi)\right|+\int_{F_n^c}|u(x)-u(y)|^p\,\varphi(x)\,m_n(dxdy)+C_{p,u,\varphi}\varepsilon^{\frac{(p-1)\wedge1}{p}},\end{aligned}$$

which together with $\lim_{n\to\infty}\Psi_{p,n}^{\boldsymbol{k}}(u;\varphi)=\Psi_p^{\boldsymbol{k}}(u;\varphi)$ and (3.6) yields (4.2). Then the estimate (4.3) is clear from (4.2). □

By Theorem 4.1, we can associate to the functional $\Psi_p^{\boldsymbol{k}}(u;\,\cdot\,)$ a unique Radon measure $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$ on $K$ under the additional assumption that $B_{p,\infty}^{\boldsymbol{k}}\cap C_c(K)$ is dense in $(C_c(K),\|\cdot\|_{\sup})$, as follows.

**Theorem 4.2** *Suppose that $B_{p,\infty}^{\boldsymbol{k}}\cap C_c(K)$ is dense in $(C_c(K),\|\cdot\|_{\sup})$. Let $u\in B_{p,\infty}^{\boldsymbol{k}}\cap C_b(K)$. Then there exists a unique positive Radon measure $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$ on K such that for any $\varphi\in B_{p,\infty}^{\boldsymbol{k}}\cap C_c(K)$,*

$$\int_K\varphi\,d\Gamma_p^{\boldsymbol{k}}\langle u\rangle=\mathcal{E}_p^{\boldsymbol{k}}(u;u\varphi)-\left(\frac{p-1}{p}\right)^{p-1}\mathcal{E}_p^{\boldsymbol{k}}\bigl(|u|^{\frac{p}{p-1}};\varphi\bigr).\tag{4.6}$$

*Moreover, $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(K)\le\mathcal{E}_p^{\boldsymbol{k}}(u)<\infty$, and for any $\varphi\in C_0(K)$,*

$$\int_K\varphi\,d\Gamma_p^{\boldsymbol{k}}\langle u\rangle=\lim_{n\to\infty}\int_K\int_K|u(x)-u(y)|^p\,\varphi(x)k_{r_n}(x,y)\,m(dy)m(dx).\tag{4.7}$$

**Definition 4.3 ($p$-Energy measure associated with a $\boldsymbol{k}$-Korevaar–Schoen $p$-energy form $(\mathcal{E}_p^{\boldsymbol{k}},B_{p,\infty}^{\boldsymbol{k}})$)** Suppose that $B_{p,\infty}^{\boldsymbol{k}}\cap C_c(K)$ is dense in $(C_c(K),\|\cdot\|_{\sup})$, and let $u \in B^{\boldsymbol{k}}_{p,\infty} \cap C_b(K)$. The positive Radon measure $\Gamma^{\boldsymbol{k}}_p\langle u\rangle$ on $K$ as in Theorem 4.2 is called the *p-energy measure of u associated with* $(\mathcal{E}^{\boldsymbol{k}}_p, B^{\boldsymbol{k}}_{p,\infty})$.

*Proof of Theorem 4.2.* By virtue of (4.3), we can extend $\Psi^{\boldsymbol{k}}_p(u;\,\cdot\,)$ to a bounded linear functional on $C_0(K)$ in a standard way as follows. Let $u \in B^{\boldsymbol{k}}_{p,\infty} \cap C_b(K)$, let $\varphi \in C_0(K)$ and choose $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$ so that $\lim_{j\to\infty} \|\varphi - \varphi_j\|_{\sup} = 0$. Then $\{\Psi^{\boldsymbol{k}}_p(u;\varphi_j)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ since $\left|\Psi^{\boldsymbol{k}}_p(u;\varphi_j) - \Psi^{\boldsymbol{k}}_p(u;\varphi_{j'})\right| \le \|\varphi_j - \varphi_{j'}\|_{\sup} \mathcal{E}^{\boldsymbol{k}}_p(u)$ for any $j, j' \in \mathbb{N}$ by (4.3). Now we define $\widetilde{\Psi}^{\boldsymbol{k}}_p(u;\varphi) := \lim_{j\to\infty} \Psi^{\boldsymbol{k}}_p(u;\varphi_j)$, which does not depend on the choice of $\{\varphi_j\}_{j\in\mathbb{N}}$. Clearly, we have $\left|\widetilde{\Psi}^{\boldsymbol{k}}_p(u;\varphi)\right| \le \|\varphi\|_{\sup} \mathcal{E}^{\boldsymbol{k}}_p(u)$. If $\varphi \ge 0$, then we obtain $\widetilde{\Psi}^{\boldsymbol{k}}_p(u;\varphi) \ge 0$ by considering $\{\varphi_j^+\}_j$ instead of $\{\varphi_j\}_j$. By applying the Riesz–Markov–Kakutani representation theorem (see, e.g., [36, Theorems 2.14 and 2.18]), there exists a unique positive Radon measure $\Gamma^{\boldsymbol{k}}_p\langle u\rangle$ on $K$ satisfying

$$\widetilde{\Psi}^{\boldsymbol{k}}_p(u;\psi) = \int_K \psi \, d\Gamma^{\boldsymbol{k}}_p\langle u\rangle \quad \text{for any } \psi \in C_c(K). \tag{4.8}$$

In particular, $\Gamma^{\boldsymbol{k}}_p\langle u\rangle$ satisfies (4.6) for any $\varphi \in B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$ by (4.8) and (4.1).

Next, to show the claimed uniqueness of $\Gamma^{\boldsymbol{k}}_p\langle u\rangle$ and $\Gamma^{\boldsymbol{k}}_p\langle u\rangle(K) \le \mathcal{E}^{\boldsymbol{k}}_p(u)$, let $\mu$ be a positive Radon measure on $K$ satisfying (4.6) with $\mu$ in place of $\Gamma^{\boldsymbol{k}}_p\langle u\rangle$ for any $\varphi \in B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$. Then for any compact subset $F$ of $K$, noting (3.1) and the assumption that $B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$ is dense in $(C_c(K), \|\cdot\|_{\sup})$, we can choose $\varphi \in B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$ so that $\mathbf{1}_F \le \varphi \le \mathbf{1}_K$ on $K$, hence $\mu(F) \le \int_K \varphi \, d\mu = \Psi^{\boldsymbol{k}}_p(u;\varphi) \le \mathcal{E}^{\boldsymbol{k}}_p(u)$ by (4.3) and thus $\mu(K) \le \mathcal{E}^{\boldsymbol{k}}_p(u) < \infty$. In particular, $C_0(K) \ni \psi \mapsto \int_K \psi \, d\mu$ is a bounded linear functional on $C_0(K)$ which coincides with $\Psi^{\boldsymbol{k}}_p(u;\,\cdot\,)$ on $B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$ and thus with $\widetilde{\Psi}^{\boldsymbol{k}}_p(u;\,\cdot\,)$ on $C_0(K)$, and therefore $\mu = \Gamma^{\boldsymbol{k}}_p\langle u\rangle$ by the uniqueness of a positive Radon measure on $K$ satisfying (4.8).

Lastly, we shall prove (4.7). Note that (4.7) is true for $\varphi \in B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$ by (4.2) in Theorem 4.1. As in the first paragraph of this proof, let $\varphi \in C_0(K)$ and choose $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq B^{\boldsymbol{k}}_{p,\infty} \cap C_c(K)$ so that $\lim_{j\to\infty} \|\varphi - \varphi_j\|_{\sup} = 0$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $\|\varphi - \varphi_j\|_{\sup} \mathcal{E}^{\boldsymbol{k}}_p(u) < \varepsilon$ for any $j \ge N$. Then, for any $n \in \mathbb{N}$ and any $j \ge N$,

$$\begin{aligned}
&\left|\int_K \varphi \, d\Gamma^{\boldsymbol{k}}_p\langle u\rangle - \int_{K\times K} |u(x) - u(y)|^p \, \varphi(x) \, m_n(dxdy)\right| \\
&\le \left|\widetilde{\Psi}^{\boldsymbol{k}}_p(u;\varphi) - \widetilde{\Psi}^{\boldsymbol{k}}_p(u;\varphi_j)\right| + \left|\widetilde{\Psi}^{\boldsymbol{k}}_p(u;\varphi_j) - \Psi^{\boldsymbol{k}}_{p,n}(u;\varphi_j)\right| \\
&\quad + \|\varphi - \varphi_j\|_{\sup} \int_{K\times K} |u(x) - u(y)|^p \, m_n(dxdy) \\
&\le 2\varepsilon + \left|\Psi^{\boldsymbol{k}}_p(u;\varphi_j) - \Psi^{\boldsymbol{k}}_{p,n}(u;\varphi_j)\right|,
\end{aligned}$$

where $\Psi^{\boldsymbol{k}}_{p,n}(u;\,\cdot\,)$ is the same as in (4.4). Hence we have

$$\limsup_{n\to\infty}\left|\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle - \int_{K\times K} |u(x)-u(y)|^p\, \varphi(x)\, m_n(dxdy)\right| \le 2\varepsilon,$$

which proves (4.7). ◻

In the rest of this section, we always suppose in addition that $B^{\boldsymbol{k}}_{p,\infty}\cap C_c(K)$ is dense in $(C_c(K), \|\cdot\|_{\sup})$.

Note that both the boundedness and the continuity of $u$ are essential in Theorem 4.2; the former is required for the right-hand side of (4.6) to make sense, and the latter has been used heavily in the proof of Theorem 4.1 above. Next we would like to extend $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$ to a wider range of $u$. Let us use the following notation for simplicity.

**Definition 4.4** We define closed linear subspaces $\mathcal{D}^{\boldsymbol{k},b}_{p,\infty}$ and $\mathcal{D}^{\boldsymbol{k},c}_{p,\infty}$ of $B^{\boldsymbol{k}}_{p,\infty}$ by

$$\mathcal{D}^{\boldsymbol{k},b}_{p,\infty} \coloneqq \overline{B^{\boldsymbol{k}}_{p,\infty}\cap C_b(K)}^{B^{\boldsymbol{k}}_{p,\infty}} \quad\text{and}\quad \mathcal{D}^{\boldsymbol{k},c}_{p,\infty} \coloneqq \overline{B^{\boldsymbol{k}}_{p,\infty}\cap C_c(K)}^{B^{\boldsymbol{k}}_{p,\infty}}. \tag{4.9}$$

By virtue of the expression (4.2), we can show the generalized $p$-contraction property $(\mathrm{GC})_p$ for $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, B^{\boldsymbol{k}}_{p,\infty}\cap C_b(K))$ for any $\varphi\in C_c(K)$ with $\varphi\ge 0$, which further allows us to extend $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$ canonically to $u\in\mathcal{D}^{\boldsymbol{k},b}_{p,\infty}$.

**Theorem 4.5** *For any $u\in\mathcal{D}^{\boldsymbol{k},b}_{p,\infty}$, there exists a unique positive Radon measure $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$ on $K$ such that for any $\{u_n\}_{n\in\mathbb{N}}\subseteq B^{\boldsymbol{k}}_{p,\infty}\cap C_b(K)$ with $\lim_{n\to\infty}\mathcal{E}_p^{\boldsymbol{k}}(u-u_n)=0$ and any Borel measurable function $\varphi\colon K\to[0,\infty)$ with $\|\varphi\|_{\sup}<\infty$,*

$$\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle = \lim_{n\to\infty}\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle, \tag{4.10}$$

*and $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$ further satisfies $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(K)\le\mathcal{E}_p^{\boldsymbol{k}}(u)$. Moreover, for each such $\varphi$, $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, \mathcal{D}^{\boldsymbol{k},b}_{p,\infty})$ is a $p$-energy form on $(K,m)$ satisfying* $(\mathrm{GC})_p$.

*Proof.* First, for any $\varphi\in C_c(K)$ with $\varphi\ge 0$, we will show that $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, B^{\boldsymbol{k}}_{p,\infty}\cap C_b(K))$ satisfies $(\mathrm{GC})_p$. Throughout this proof, we fix $n_1,n_2\in\mathbb{N}$, $q_1\in(0,p]$, $q_2\in[p,\infty]$ and $T=(T_1,\dots,T_{n_2})\colon\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ satisfying (2.1). Let us consider the case $q_2<\infty$ since the proof for the case $q_2=\infty$ is similar. Let $\boldsymbol{u}=(u_1,\dots,u_{n_1})\in\bigl(B^{\boldsymbol{k}}_{p,\infty}\cap C_b(K)\bigr)^{n_1}$. Note that $T_l(\boldsymbol{u})\in B^{\boldsymbol{k}}_{p,\infty}\cap C_b(K)$ for each $l\in\{1,\dots,n_2\}$. For any $n\in\mathbb{N}$, we see that

$$\begin{aligned}
&\sum_{l=1}^{n_2}\left(\int_{K\times K}|T_l(\boldsymbol{u}(x))-T_l(\boldsymbol{u}(y))|^p\,\varphi(x)\,m_n(dxdy)\right)^{q_2/p}\\
&\overset{(3.2)}{\le}\left(\int_{K\times K}\left[\sum_{l=1}^{n_2}|T_l(\boldsymbol{u}(x))-T_l(\boldsymbol{u}(y))|^{q_2}\right]^{p/q_2}\varphi(x)\,m_n(dxdy)\right)^{q_2/p}\\
&\overset{(2.1)}{\le}\left(\int_{K\times K}\left[\sum_{k=1}^{n_1}|u_k(x)-u_k(y)|^{q_1}\right]^{p/q_1}\varphi(x)\,m_n(dxdy)\right)^{q_2/p}
\end{aligned}$$

$$\overset{(*)}{\leq} \left( \sum_{k=1}^{n_1} \left( \int_{K\times K} |u_k(x) - u_k(y)|^p \, \varphi(x) \, m_n(dxdy) \right)^{q_1/p} \right)^{q_2/q_1},$$

where we used the triangle inequality for the norm of $L^{p/q_1}(K \times K, m_n)$ in $(*)$. By letting $n \to \infty$, we obtain from (4.7) that

$$\left\| \left( \left( \int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle T_l(\boldsymbol{u})\rangle \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left( \left( \int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle u_k\rangle \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{4.11}$$

Next we will extend (4.11) to any Borel measurable function $\varphi \colon K \to [0, \infty]$. Let us start with the case $\varphi = \mathbf{1}_A$, where $A \in \mathcal{B}(K)$. By [36, Theorem 2.18], there exist sequences $\{K_n\}_{n\in\mathbb{N}}$ and $\{U_n\}_{n\in\mathbb{N}}$ such that $K_n \subseteq A \subseteq U_n$, $K_n$ is compact, $U_n$ is open and $\lim_{n\to\infty} \max_{v\in\{T_l(\boldsymbol{u})\}_l \cup \{u_k\}_k} \Gamma_p^{\boldsymbol{k}}\langle v\rangle(U_n \setminus K_n) = 0$. By Urysohn's lemma, we can pick $\varphi_n \in C_c(K)$ so that $0 \leq \varphi_n \leq 1$, $\varphi_n|_{K_n} = 1$ and $\operatorname{supp}_K[\varphi_n] \subseteq U_n$. Applying (4.11) for $\varphi_n$, we obtain

$$\left\| \left( \Gamma_p^{\boldsymbol{k}}\langle T_l(\boldsymbol{u})\rangle(K_n)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \leq \left\| \left( \Gamma_p^{\boldsymbol{k}}\langle u_k\rangle(U_n)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}.$$

By letting $n \to \infty$, we get (4.11) with $\varphi = \mathbf{1}_A$. Using the reverse Minkowski inequality on $\ell^{q_1/p}$ and the Minkowski inequality on $\ell^{q_2/p}$ (see also [23, Proof of Proposition 2.9-(a)], where $(\mathrm{GC})_p$ is shown to be stable under addition), we see that (4.11) holds also for any non-negative Borel measurable simple function $\varphi$ on $K$. We get the desired extension, (4.11) for any Borel measurable function $\varphi \colon K \to [0, \infty]$, by the monotone convergence theorem.

Now let us extend $p$-energy measures. In the rest of this proof, let $\varphi \colon K \to [0, \infty)$ be a Borel measurable function such that $\|\varphi\|_{\sup} < \infty$. Let $u \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$ satisfy $\lim_{n\to\infty} \mathcal{E}_p^{\boldsymbol{k}}(u - u_n) = 0$. By Proposition 2.3-(1) for $(\int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K))$, for any $n, n' \in \mathbb{N}$,

$$\left| \left( \int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle \right)^{1/p} - \left( \int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle u_{n'}\rangle \right)^{1/p} \right| \leq \|\varphi\|_{\sup}^{1/p} \, \mathcal{E}_p^{\boldsymbol{k}}(u_n - u_{n'})^{1/p},$$

which implies that the limit $\lim_{n\to\infty} \int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle =: I_u(\varphi)$ exists in $\mathbb{R}$ and it is independent of the choice of $\{u_n\}_n$. In addition, by letting $n' \to \infty$ in the estimate above, we have that

$$\left| \left( \int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle \right)^{1/p} - I_u(\varphi)^{1/p} \right| \leq \|\varphi\|_{\sup}^{1/p} \, \mathcal{E}_p^{\boldsymbol{k}}(u_n - u)^{1/p}. \tag{4.12}$$

Also, it is clear that $0 \leq I_u(\varphi) \leq \|\varphi\|_{\sup} \mathcal{E}_p^{\boldsymbol{k}}(u)$ and that $I_n$ is linear in the sense that $I_u\bigl(\sum_{k=1}^N a_k\varphi_k\bigr) = \sum_{k=1}^N a_k I_u(\varphi_k)$ for any $N \in \mathbb{N}$, $(a_k)_{k=1}^N \subseteq [0, \infty)$ and Borel measurable functions $\varphi_k \colon K \to [0, \infty)$ with $\|\varphi_k\|_{\sup} < \infty$, $k \in \{1, \ldots, N\}$. Now we define $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(A) \coloneqq I_u(\mathbf{1}_A) \in [0, \infty)$ for $A \in \mathcal{B}(K)$, and show that $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$ is a finite

Borel measure on $K$. Clearly, $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$ is finitely additive and $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(K) \le \mathcal{E}_p^{\boldsymbol{k}}(u) < \infty$. Hence it suffices to prove the countable additivity of $\Gamma_p^{\boldsymbol{k}}\langle u\rangle$. By (4.12), for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\sup_{A\in\mathcal{B}(K)} \left|\Gamma_p^{\boldsymbol{k}}\langle u\rangle(A)^{1/p} - \Gamma_p^{\boldsymbol{k}}\langle u_n\rangle(A)^{1/p}\right| < \varepsilon$ for any $n \ge N_0$. Let $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathcal{B}(K)$ be a sequence of disjoint Borel sets, and set $B_N := \bigcup_{k=N+1}^{\infty} A_k$ for each $N \in \mathbb{N}$. Then we see that for any $N \in \mathbb{N}$ and any $n \ge N_0$,

$$\left|\Gamma_p^{\boldsymbol{k}}\langle u\rangle\left(\bigcup_{k\in\mathbb{N}} A_k\right) - \sum_{k=1}^{N} \Gamma_p^{\boldsymbol{k}}\langle u\rangle(A_k)\right|^{1/p} = \Gamma_p^{\boldsymbol{k}}\langle u\rangle(B_N)^{1/p} \le \varepsilon + \Gamma_p^{\boldsymbol{k}}\langle u_n\rangle(B_N)^{1/p},$$

whence $\lim_{N\to\infty} \left|\Gamma_p^{\boldsymbol{k}}\langle u\rangle\left(\bigcup_{k\in\mathbb{N}} A_k\right) - \sum_{k=1}^{N} \Gamma_p^{\boldsymbol{k}}\langle u\rangle(A_k)\right| = 0$, proving the desired countable additivity.

Before showing (4.10), i.e., $I_u(\varphi) = \int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle$, we will extend (4.11) to the pair $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, \mathcal{D}_{p,\infty}^{\boldsymbol{k},b})$. To this end, we need to show that for any $\{u_n\}_{n\in\mathbb{N}} \subseteq B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$ converging weakly in $B_{p,\infty}^{\boldsymbol{k}}$ to $u \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ as $n \to \infty$,

$$\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle \le \liminf_{n\to\infty} \int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle. \tag{4.13}$$

By extracting a subsequence of $\{u_n\}_n$ if necessary, we can assume that the limit $\lim_{n\to\infty} \int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle$ exists. By Mazur's lemma (see, e.g., [19, p. 19]), there exist $N(n) \in \mathbb{N}$ and $\{\alpha_{n,k}\}_{k=n}^{N(n)} \subseteq [0,1]$ with $N(n) > n$ and $\sum_{k=n}^{N(n)} \alpha_{n,k} = 1$ for each $n \in \mathbb{N}$ such that $v_n := \sum_{k=n}^{N(n)} \alpha_{n,k} u_k$ converges to $u$ in $B_{p,\infty}^{\boldsymbol{k}}$ as $n \to \infty$. We see from $(\mathrm{GC})_p$ and Proposition 2.3-(1) for $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K))$ that

$$\left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle v_n\rangle\right)^{1/p} \le \sum_{k=n}^{N(n)} \alpha_{n,k} \left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_k\rangle\right)^{1/p},$$

which implies (4.13) by letting $n \to \infty$. With this preparation, let us show that the pair $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, \mathcal{D}_{p,\infty}^{\boldsymbol{k},b})$ satisfies $(\mathrm{GC})_p$. Let $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in (\mathcal{D}_{p,\infty}^{\boldsymbol{k},b})^{n_1}$. For each $k \in \{1, \dots, n_1\}$, fix $\{u_{k,n}\}_{n\in\mathbb{N}} \subseteq B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$ so that $\lim_{n\to\infty} \left\|u_k - u_{k,n}\right\|_{B_{p,\infty}^{\boldsymbol{k}}} = 0$. Set $\boldsymbol{u}_n := (u_{1,n}, \dots, u_{n_1,n})$. By $(\mathrm{GC})_p$ for $(\mathcal{E}_p^{\boldsymbol{k}}, B_{p,\infty}^{\boldsymbol{k}})$ (see Theorem 3.8-(b)) and (2.1), we know that $\{T_l(\boldsymbol{u}_n)\}_n$ is bounded in $B_{p,\infty}^{\boldsymbol{k}}$ and that $\lim_{n\to\infty} \|T_l(\boldsymbol{u}_n) - T_l(\boldsymbol{u})\|_{L^p} = 0$. Since $B_{p,\infty}^{\boldsymbol{k}}$ is reflexive (see Theorem 3.6) and $B_{p,\infty}^{\boldsymbol{k}}$ is continuously embedded in $L^p(K,m)$, we see that $T_l(\boldsymbol{u}) \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and that there exists a subsequence $\{T_l(\boldsymbol{u}_{n_j})\}_j$ such that $T_l(\boldsymbol{u}_{n_j})$ weakly converges to $T_l(\boldsymbol{u})$ in $B_{p,\infty}^{\boldsymbol{k}}$ as $j \to \infty$ for any $l \in \{1, \dots, n_2\}$. By (4.13), we see that

$$\left\|\left(\left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle T_l(\boldsymbol{u})\rangle\right)^{1/p}\right)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left(\sum_{l=1}^{n_2} \liminf_{j\to\infty} \left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle T_l(\boldsymbol{u}_{n_j})\rangle\right)^{1/p}\right)^{1/q_2}$$

$$\begin{aligned}
&\le \liminf_{j\to\infty}\left(\sum_{l=1}^{n_2}\left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle T_l(\boldsymbol{u}_{n_j})\rangle\right)^{1/p}\right)^{1/q_2}\\
&\le \liminf_{j\to\infty}\left(\sum_{k=1}^{n_1}\left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_{k,n_j}\rangle\right)^{1/p}\right)^{1/q_1}\\
&= \left\|\left(\left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_k\rangle\right)^{1/p}\right)_{k=1}^{n_1}\right\|_{\ell^{q_1}}
\end{aligned}$$

if $q_2 < \infty$. The case $q_2 = \infty$ is similar, so $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, \mathcal{D}_{p,\infty}^{\boldsymbol{k},b})$ satisfies $(\mathrm{GC})_p$.

Finally, we can prove (4.10). Let $\{u_n\}_{n\in\mathbb{N}} \subseteq B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$ be a sequence satisfying $\lim_{n\to\infty} \mathcal{E}_p^{\boldsymbol{k}}(u-u_n) = 0$. By Proposition 2.3-(1) for $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, \mathcal{D}_{p,\infty}^{\boldsymbol{k},b})$, we have

$$\left|\left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle\right)^{1/p} - \left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle\right)^{1/p}\right| \le \|\varphi\|_{\sup}^{1/p}\, \mathcal{E}_p^{\boldsymbol{k}}(u-u_n)^{1/p},$$

which together with (4.12) implies that

$$\begin{aligned}
&\left|I_u(\varphi)^{1/p} - \left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle\right)^{1/p}\right|\\
&\le \left|I_u(\varphi)^{1/p} - \left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle\right)^{1/p}\right| + \left|\left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_n\rangle\right)^{1/p} - \left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle\right)^{1/p}\right|\\
&\le 2\,\|\varphi\|_{\sup}^{1/p}\, \mathcal{E}_p^{\boldsymbol{k}}(u-u_n)^{1/p} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Hence we obtain (4.10). □

Thanks to Proposition 2.3-(5) for $(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, \mathcal{D}_{p,\infty}^{\boldsymbol{k},b})$, we can show the next result. See [23, Theorem 4.5 and Proposition 4.6] for further details on $\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle$ in the theorem below.

**Theorem 4.6** *Let $u, v \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$. Define $\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle \colon \mathcal{B}(K) \to \mathbb{R}$ by*

$$\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle(A) := \frac{1}{p}\,\frac{d}{dt}\Gamma_p^{\boldsymbol{k}}\langle u+tv\rangle(A)\bigg|_{t=0} \quad \text{for } A \in \mathcal{B}(K). \tag{4.14}$$

*Then $\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle$ is a signed Borel measure on $K$ and satisfies $\Gamma_p^{\boldsymbol{k}}\langle u; u\rangle = \Gamma_p^{\boldsymbol{k}}\langle u\rangle$. Moreover, for any $u, v \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and any Borel measurable functions $\varphi, \psi \colon K \to [0,\infty]$,*

$$\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u;\,\cdot\,\rangle \colon \mathcal{D}_{p,\infty}^{\boldsymbol{k},b} \to \mathbb{R} \text{ is the Fréchet derivative of } \frac{1}{p}\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle \text{ at } u \tag{4.15}$$

*provided $\|\varphi\|_{\sup} < \infty$, and*

$$\int_K \varphi\psi \, d\left|\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle\right| \le \left(\int_K \varphi^{\frac{p}{p-1}} \, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle\right)^{\frac{p-1}{p}} \left(\int_K \psi^p \, d\Gamma_p^{\boldsymbol{k}}\langle v\rangle\right)^{\frac{1}{p}}. \tag{4.16}$$

*Proof.* It is proved in [23, Theorem 4.5] that $\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle$ is a signed measure. The statements (4.15) and (4.16) follow from [23, Propositions 4.6 and 4.8]. $\square$

As an important consequence of the strong locality of $(\mathcal{E}_p^{\boldsymbol{k}}, B_{p,\infty}^{\boldsymbol{k}})$ obtained in Theorem 3.8-(e), the inequality $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(K) \le \mathcal{E}_p^{\boldsymbol{k}}(u)$ in Theorems 4.2 and 4.5 turns out to be an equality as long as $u \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},c}$. Namely, we have the following proposition, which is the counterpart for $(\mathcal{E}_p^{\boldsymbol{k}}, B_{p,\infty}^{\boldsymbol{k}})$ of the well-known equality [12, Lemma 3.2.3] for the strongly local part of a regular symmetric Dirichlet form.

**Proposition 4.7** *If $u, v \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},c}$, then $\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle(K) = \mathcal{E}_p^{\boldsymbol{k}}(u; v)$.*

*Proof.* Since $(\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle(K), \mathcal{D}_{p,\infty}^{\boldsymbol{k},b})$ and $(\mathcal{E}_p^{\boldsymbol{k}}, B_{p,\infty}^{\boldsymbol{k}})$ satisfy $(\mathrm{GC})_p$ by Theorems 4.5 and 3.8-(b), thanks to the linearity of $\mathcal{E}(u; \cdot)$, (2.11) and (2.12) from Proposition 2.5 for $(\mathcal{E}, \mathcal{F}) = (\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle(K), \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}), (\mathcal{E}_p^{\boldsymbol{k}}, B_{p,\infty}^{\boldsymbol{k}})$ it suffices to consider the case $u, v \in B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K)$. We first show that $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(K) = \mathcal{E}_p^{\boldsymbol{k}}(u)$ for any $u \in B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K)$. Since $K$ is locally compact and we assume that $B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K)$ is dense in $(C_c(K), \|\cdot\|_{\sup})$, by using Proposition 2.3-(2), we can find an open neighborhood $U$ of the compact subset $\mathrm{supp}_K[u]$ of $K$ and $\varphi \in B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K)$ so that $0 \le \varphi \le 1$ and $\varphi(x) = 1$ for any $x \in U$. Then $\mathrm{supp}_K[u] \cap \mathrm{supp}_K[\varphi - \mathbf{1}_K] = \emptyset$. By Theorem 3.8-(e), we have $\mathcal{E}_p^{\boldsymbol{k}}(u; u\varphi - u) = 0$ and $\mathcal{E}_p^{\boldsymbol{k}}\bigl(|u|^{\frac{p}{p-1}}; \varphi\bigr) = 0$. In particular, by (4.6),

$$\Gamma_p^{\boldsymbol{k}}\langle u\rangle(K) \ge \int_K \varphi \, \Gamma_p^{\boldsymbol{k}}\langle u\rangle = \mathcal{E}_p^{\boldsymbol{k}}(u),$$

whence we have $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(K) = \mathcal{E}_p^{\boldsymbol{k}}(u)$.

Next let $u, v \in B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K)$. The argument in the previous paragraph implies that for any $t \in (0, 1)$,

$$\frac{\Gamma_p^{\boldsymbol{k}}\langle u + tv\rangle(K) - \Gamma_p^{\boldsymbol{k}}\langle u\rangle(K)}{t} = \frac{\mathcal{E}_p^{\boldsymbol{k}}(u + tv) - \mathcal{E}_p^{\boldsymbol{k}}(u)}{t}.$$

By letting $t \downarrow 0$ in this equality, we have $\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle(K) = \mathcal{E}_p^{\boldsymbol{k}}(u; v)$ by (4.14) and (3.9). $\square$

We also have the following expression of $\int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle$ if $\varphi \in C_c(K)$. In particular, we can deduce the analogs of Theorem 3.8-(c),(d) for $(\int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle, \mathcal{D}_{p,\infty}^{\boldsymbol{k},b})$.

**Theorem 4.8** *For any $u, v \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and any $\varphi \in C_c(K)$,*

$$\begin{aligned}&\int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle \\ &= \lim_{n\to\infty} \int_K \int_K \gamma_p\bigl(u(x) - u(v)\bigr)(v(x) - v(y))\varphi(x) k_{r_n}(x, y)\, m(dy) m(dx).\end{aligned} \tag{4.17}$$

*In particular, the following hold:*

(a) *Let* $n_1, n_2 \in \mathbb{N}$, $q_1 \in [1, p]$, $q_2 \in [p, \infty]$, $\boldsymbol{u} = (u_1, \dots, u_{n_1}) \in \bigl(\mathcal{D}^{\boldsymbol{k},b}_{p,\infty}\bigr)^{n_1}$, $\boldsymbol{v} = (v_1, \dots, v_{n_2}) \in L^0(K, m)^{n_2}$, *and let* $\psi\colon K \to [0, \infty]$ *be Borel measurable. If there exist* $m$*-versions of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *such that* $\|\boldsymbol{v}(x)\|_{\ell^{q_2}} \le \|\boldsymbol{u}(x)\|_{\ell^{q_1}}$ *and* $\|\boldsymbol{v}(x) - \boldsymbol{v}(y)\|_{\ell^{q_2}} \le \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^{q_1}}$ *for any* $(x, y) \in K \times K$, *then* $\boldsymbol{v} \in \bigl(\mathcal{D}^{\boldsymbol{k},b}_{p,\infty}\bigr)^{n_2}$ *and*

$$\left\| \left( \left( \int_K \psi \, d\Gamma^{\boldsymbol{k}}_p \langle v_l \rangle \right)^{1/p} \right)_{l=1}^{n_2} \right\|_{\ell^{q_2}} \le \left\| \left( \left( \int_K \psi \, d\Gamma^{\boldsymbol{k}}_p \langle u_k \rangle \right)^{1/p} \right)_{k=1}^{n_1} \right\|_{\ell^{q_1}}. \tag{4.18}$$

(b) *For any* $u_1, u_2, v \in \mathcal{D}^{\boldsymbol{k},b}_{p,\infty}$ *and any Borel measurable function* $\psi\colon K \to [0, \infty)$ *with* $\|\psi\|_{\sup} < \infty$,

$$\begin{aligned} &\left| \int_K \psi \, d\Gamma^{\boldsymbol{k}}_p \langle u_1; v \rangle - \int_K \psi \, d\Gamma^{\boldsymbol{k}}_p \langle u_2; v \rangle \right| \\ &\le C_p \left[ \max_{i \in \{1,2\}} \int_K \psi \, d\Gamma^{\boldsymbol{k}}_p \langle u_i \rangle \right]^{\frac{(p-2)^+}{p}} \left( \int_K \psi \, d\Gamma^{\boldsymbol{k}}_p \langle u_1 - u_2 \rangle \right)^{\frac{(p-1)\wedge 1}{p}} \left( \int_K \psi \, d\Gamma^{\boldsymbol{k}}_p \langle v \rangle \right)^{\frac{1}{p}}, \end{aligned} \tag{4.19}$$

*where* $C_p$ *is the constant in Theorem 3.8.*

*Proof.* Throughout this proof, we fix $\varphi \in C_c(K)$. We first show (4.17) in the case $u = v$. Define

$$\mathcal{I}^n_\varphi \langle f \rangle := \int_{K \times K} |f(x) - f(y)|^p \, \varphi(x) \, m_n(dxdy) \quad \text{for } n \in \mathbb{N} \text{ and } f \in \mathcal{D}^{\boldsymbol{k},b}_{p,\infty}.$$

Fix $\{u_k\}_{k \in \mathbb{N}} \subseteq B^{\boldsymbol{k}}_{p,\infty} \cap C_b(K)$ satisfying $\lim_{k \to \infty} \|u - u_k\|_{B^{\boldsymbol{k}}_{p,\infty}} = 0$. We easily have $\bigl| \mathcal{I}^n_\varphi \langle u \rangle^{1/p} - \mathcal{I}^n_\varphi \langle u_k \rangle^{1/p} \bigr| \le \mathcal{I}^n_\varphi \langle u - u_k \rangle^{1/p} \le C^{1/p} \|\varphi\|_{\sup} \mathcal{E}^{\boldsymbol{k}}_p (u - u_k)^{1/p}$, where $C \in (0, \infty)$ is the constant in (3.8). By (4.10) and Proposition 2.3-(1) for $(\int_K \varphi \, d\Gamma^{\boldsymbol{k}}_p \langle \,\cdot\, \rangle, \mathcal{D}^{\boldsymbol{k},b}_{p,\infty})$, we see that for any $n, k \in \mathbb{N}$,

$$\begin{aligned} &\left| \left( \int_K \varphi \, d\Gamma^{\boldsymbol{k}}_p \langle u \rangle \right)^{1/p} - \mathcal{I}^n_\varphi \langle u \rangle^{1/p} \right| \\ &\le \left| \left( \int_K \varphi \, d\Gamma^{\boldsymbol{k}}_p \langle u \rangle \right)^{1/p} - \left( \int_K \varphi \, d\Gamma^{\boldsymbol{k}}_p \langle u_k \rangle \right)^{1/p} \right| + \left| \left( \int_K \varphi \, d\Gamma^{\boldsymbol{k}}_p \langle u_k \rangle \right)^{1/p} - \mathcal{I}^n_\varphi \langle u_k \rangle^{1/p} \right| \\ &\quad + \left| \mathcal{I}^n_\varphi \langle u \rangle^{1/p} - \mathcal{I}^n_\varphi \langle u_k \rangle^{1/p} \right| \\ &\le (1 + C^{1/p}) \, \|\varphi\|_{\sup}^{1/p} \, \mathcal{E}^{\boldsymbol{k}}_p (u - u_k)^{1/p} + \left| \left( \int_K \varphi \, d\Gamma^{\boldsymbol{k}}_p \langle u_k \rangle \right)^{1/p} - \mathcal{I}^n_\varphi \langle u_k \rangle^{1/p} \right|. \end{aligned}$$

Since $\lim_{n\to\infty}\left|\left(\int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u_k\rangle\right)^{1/p} - \mathcal{I}_\varphi^n\langle u_k\rangle^{1/p}\right| = 0$ by (4.2) and $k \in \mathbb{N}$ is arbitrary, we conclude that $\lim_{n\to\infty}\mathcal{I}_\varphi^n\langle u\rangle = \int_K \varphi\, d\Gamma_p^{\boldsymbol{k}}\langle u\rangle$.

Next we consider the general case $u \neq v$. By Proposition 2.4 and the convexity of $t \mapsto \mathcal{I}_\varphi^n\langle u + tg\rangle$, for any $t \in (0,1)$ and $n \in \mathbb{N}$,

$$\left|\frac{\mathcal{I}_\varphi^n\langle u + tv\rangle - \mathcal{I}_\varphi^n\langle u\rangle}{t} - \frac{d}{dt}\mathcal{I}_\varphi^n\langle u + tv\rangle\bigg|_{t=0}\right| \leq \|\varphi\|_{\sup} O_t(u;v), \tag{4.20}$$

where $O_t(u;v) = C_{p,u,v}t^{(p-1)\wedge\frac{1}{p-1}}$ for some constant $C_{p,u,v} \in (0,\infty)$ which depends only on $p$, $\mathcal{E}_p^{\boldsymbol{k}}(u)$ and $\mathcal{E}_p^{\boldsymbol{k}}(v)$. Now we obtain (4.17) by noting that

$$\frac{d}{dt}\mathcal{I}_\varphi^n\langle u + tg\rangle\bigg|_{t=0} = \int_{K\times K}\gamma_p\big(u(x) - u(v)\big)(v(x) - v(y))\varphi(x)\, m_n(dxdy)$$

and taking suitable limits in (4.20).

Let us show (a) and (b).

(a): By Theorem 4.5, $(\mathcal{E}_p^{\boldsymbol{k}}, \mathcal{D}_{p,\infty}^{\boldsymbol{k},b})$ is a $p$-energy form on $(K,m)$ satisfying $(\mathrm{GC})_p$. For each $l \in \{1,\dots,n_2\}$, by the argument in [23, Proof of Corollary 2.4-(c)], we can find a 1-Lipschitz map $T_l\colon (\mathbb{R}^{n_1}, \|\cdot\|_{\ell^{q_1}}) \to \mathbb{R}$ satisfying $T_l(0) = 0$ and $T_l(\boldsymbol{u}(x)) = v_l(x)$ for any $x \in K$. By applying $(\mathrm{GC})_p$, we have $v_l \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and hence $\boldsymbol{v} \in \big(\mathcal{D}_{p,\infty}^{\boldsymbol{k},b}\big)^{n_2}$. Then the inequality (4.18) in the case $\psi \in C_c(K)$ is immediate from (4.17), and we can further extend (4.18) to general $\psi$ in exactly the same way as the second paragraph of the proof of Theorem 4.5.

(b): The estimate (4.19) in the case $\psi \in C_c(K)$ is immediate from (4.17). We can easily extend it to the desired case since $C_c(K)$ is dense in $L^1(K,\mu)$ for the finite Borel measure $\mu$ on $K$ given by

$$\mu \coloneqq \left|\Gamma_p^{\boldsymbol{k}}\langle u_1; v\rangle\right| + \left|\Gamma_p^{\boldsymbol{k}}\langle u_2; v\rangle\right| + \Gamma_p^{\boldsymbol{k}}\langle u_1\rangle + \Gamma_p^{\boldsymbol{k}}\langle u_2\rangle + \Gamma_p^{\boldsymbol{k}}\langle u_1 - u_2\rangle + \Gamma_p^{\boldsymbol{k}}\langle v\rangle. \qquad \square$$

The next theorem states the chain rule for our $p$-energy measures.

**Theorem 4.9 (Chain rule)** *Let $n \in \mathbb{N}$, $u \in B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$, $\boldsymbol{v} = (v_1,\dots,v_n) \in \big(B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)\big)^n$, $\Phi \in C^1(\mathbb{R})$, $\Psi \in C^1(\mathbb{R}^n)$ and suppose that $\Phi(0) = \Psi(0) = 0$. Then $\Phi(u), \Psi(\boldsymbol{v}) \in B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$ and*

$$d\Gamma_p^{\boldsymbol{k}}\langle\Phi(u);\Psi(\boldsymbol{v})\rangle = \sum_{k=1}^n \gamma_p\big(\Phi'(u)\big)\partial_k\Psi(\boldsymbol{v})\, d\Gamma_p^{\boldsymbol{k}}\langle u; v_k\rangle. \tag{4.21}$$

*Proof.* It is immediate from Theorem 3.8-(b) (see also Proposition 2.3-(2)) that $\Phi(u), \Psi(\boldsymbol{v}) \in B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$. Note that

$$d\mu \coloneqq d\left|\Gamma_p^{\boldsymbol{k}}\langle\Phi(u);\Psi(\boldsymbol{v})\rangle\right| + \sum_{k=1}^n \left|\gamma_p\big(\Phi'(u)\big)\partial_k\Psi(\boldsymbol{v})\right|\, d\left|\Gamma_p^{\boldsymbol{k}}\langle u; v_k\rangle\right|$$

defines a finite Borel measure $\mu$ on $K$ by (4.16). Since $C_c(K)$ is dense in $L^1(K,\mu)$, it suffices to prove that for any $\varphi \in C_c(K)$,

$$\int_K \varphi \, d\Gamma_p^{\boldsymbol{k}}\langle \Phi(u); \Psi(v)\rangle = \sum_{k=1}^{n} \int_K \varphi \gamma_p\bigl(\Phi'(u)\bigr)\partial_k \Psi(v)\, d\Gamma_p^{\boldsymbol{k}}\langle u; v_k\rangle.$$

Let $\varphi \in C_c(K)$ and define $F_n \in \mathcal{B}(K \times K)$, $n \in \mathbb{N}$, by

$$F_n \coloneqq \{(x,y) \in K \times K \mid d(x,y) < \delta(r_n), \varphi(x) \neq 0\}.$$

Note that $\overline{F_n}^{K\times K}$ is a compact subset of $K \times K$ for sufficiently large $n \in \mathbb{N}$ since $\varphi \in C_c(K)$, $\lim_{n\to 0} \delta(r_n) = 0$ and $(K,d)$ is locally compact. Set

$$a_n \coloneqq \int_{F_n} \gamma_p\bigl(\Phi(u(x)) - \Phi(u(v))\bigr)(\Psi(v(x)) - \Psi(v(y)))\varphi(x)\, m_n(dxdy)$$

and

$$b_n \coloneqq \sum_{k=1}^{n} \int_{F_n} \gamma_p\bigl(\Phi'(u(x))\bigr)\partial_k \Psi(v(x)) \cdot \gamma_p\bigl(u(x) - u(y)\bigr)(v(x) - v(y))\varphi(x)\, m_n(dxdy).$$

By Theorem 4.8 and (3.6), it suffices to show $\lim_{n\to\infty} |a_n - b_n| = 0$. To estimate $|a_n - b_n|$, we introduce

$$c_n \coloneqq \int_{F_n} \gamma_p\bigl(\Phi'(u(x))\bigr) \cdot \gamma_p\bigl(u(x) - u(y)\bigr)(v(x) - v(y))\varphi(x)\, m_n(dxdy).$$

We will show that $\lim_{n\to\infty} |a_n - c_n| = \lim_{n\to\infty} |b_n - c_n| = 0$. Note that

$$\Phi(u(y)) - \Phi(u(x)) = \bigl[u(y) - u(x)\bigr]\bigl(\Phi'(u(x)) + e_{\Phi,u}(x,y)\bigr),$$

where we set $e_{\Phi,u}(x,y) \coloneqq \int_0^1 \bigl[\Phi'\bigl(u(x) + t(u(y) - u(x))\bigr) - \Phi'(u(x))\bigr]\, dt$. Let $\varepsilon > 0$. Since $\Phi'$ is continuous, $\|u\|_{\sup} < \infty$ and $u$ is uniformly continuous on $F_n$ for large enough $n \in \mathbb{N}$, we can find $N_1 \in \mathbb{N}$ so that $\bigl|e_{\Phi,u}(x,y)\bigr| < \varepsilon$ for any $(x,y) \in \bigcup_{n \ge N_1} F_n$. By Lemma 3.7, there exists $C_p \in (0,\infty)$ depending only on $p$ such that for any $n \ge N_1$ and $(x,y) \in \bigcup_{n\ge N_1} F_n$,

$$\begin{aligned}
&\bigl|\gamma_p\bigl(\Phi(u(x)) - \Phi(u(y))\bigr) - \gamma_p\bigl(\Phi'(u(x))\bigr) \cdot \gamma_p\bigl(u(x) - u(y)\bigr)\bigr| \\
&\le C_p \varepsilon^{(p-1)\wedge 1} A_{u,\Phi}(x,y)^{(p-2)^+} |u(x) - u(y)|^{(p-1)\wedge 1},
\end{aligned}$$

where $A_{u,\Phi}(x,y) \coloneqq |\Phi(u(y)) - \Phi(u(x))| \vee |\Phi'(u(x))(u(y) - u(x))|$. By Hölder's inequality, we have

$$\sup_{n \ge N_1} |a_n - c_n|$$

$$\leq C_p \varepsilon^{(p-1)\wedge 1} \Bigl[ C_{\Phi,u} \bigl( \|\Phi(u)\|_{B^{\boldsymbol{k}}_{p,\infty}} + \|u\|_{B^{\boldsymbol{k}}_{p,\infty}} \bigr) \Bigr]^{(p-2)^+} \|u\|^{(p-1)\wedge 1}_{B^{\boldsymbol{k}}_{p,\infty}} \|v\|_{B^{\boldsymbol{k}}_{p,\infty}},$$

where $C_{\Phi,u} := 1 + \|\Phi'\|_{\sup,[-\|u\|_{\sup},\|u\|_{\sup}]}$. In particular, we get $\lim_{n\to\infty} |a_n - c_n| = 0$.

Similarly, we can find $N_2 \in \mathbb{N}$ so that for any $(x, y) \in \bigcup_{n \geq N_2} F_n$,

$$\left| \bigl(\Psi(v(x)) - \Psi(v(y))\bigr) - \sum_{k=1}^{n} \partial_k \Psi(v(x))(v(x) - v(y)) \right| \leq \varepsilon \, |v(x) - v(y)| \, .$$

Then we easily see that

$$\sup_{n \geq N_2} |b_n - c_n| \leq \varepsilon \, \|\Phi'\|^{p-1}_{\sup,[-\|u\|_{\sup},\|u\|_{\sup}]} \|u\|^{p-1}_{B^{\boldsymbol{k}}_{p,\infty}} \|v\|_{B^{\boldsymbol{k}}_{p,\infty}},$$

whence $\lim_{n\to\infty} |b_n - c_n| = 0$. □

The following *image density property* of $p$-energy measures is a consequence of the chain rule. We note that the proof below does not rely on specific representations of $\Gamma^{\boldsymbol{k}}_p$ like (4.7) and (4.17).

**Theorem 4.10 (Image density property)** *For any $u \in B^{\boldsymbol{k}}_{p,\infty} \cap C_b(K)$, the Borel measure $\Gamma^{\boldsymbol{k}}_p\langle u\rangle \circ u^{-1}$ on $\mathbb{R}$ defined by $\Gamma^{\boldsymbol{k}}_p\langle u\rangle \circ u^{-1}(A) := \Gamma^{\boldsymbol{k}}_p\langle u\rangle(u^{-1}(A))$, $A \in \mathcal{B}(\mathbb{R})$, is absolutely continuous with respect to the 1-dimensional Lebesgue measure on $\mathbb{R}$.*

*Proof.* This is proved, on the basis of Theorem 4.9, in exactly the same way as [38, Proposition 7.6], which is a simple adaptation of [9, Theorem 4.3.8], but we present the details because in [38] the underlying topological space $K$ is assumed to be a generalized Sierpiński carpet, a self-similar compact set in the Euclidean space. It suffices to prove that $\Gamma^{\boldsymbol{k}}_p\langle u\rangle \circ u^{-1}(F) = 0$ for any $u \in B^{\boldsymbol{k}}_{p,\infty} \cap C_b(K)$ and any compact subset $F$ of $\mathbb{R}$ such that $\mathscr{L}^1(F) = 0$, where $\mathscr{L}^1$ denotes the 1-dimensional Lebesgue measure on $\mathbb{R}$. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C_c(\mathbb{R})$ satisfy $|\varphi_n| \leq 1$, $\lim_{n\to\infty} \varphi_n(x) = \mathbf{1}_F(x)$ for any $x \in \mathbb{R}$ and

$$\int_0^\infty \varphi_n(t)\, dt = \int_{-\infty}^0 \varphi_n(t)\, dt = 0 \quad \text{for any } n \in \mathbb{N}.$$

We define $\Phi_n(x) := \int_0^x \varphi_n(t)\, dt$, $x \in \mathbb{R}$, and $u_n := \Phi_n \circ u$ for any $n \in \mathbb{N}$. Then we easily see that $\Phi_n \in C^1(\mathbb{R}) \cap C_c(\mathbb{R})$, $\Phi_n(0) = 0$, and $\Phi_n' = \varphi_n$ for any $n \in \mathbb{N}$. Also, $u_n$ converges to 0 in $L^p(K, m)$ as $n \to \infty$ by the dominated convergence theorem. By Proposition 2.3-(2), we deduce that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $B^{\boldsymbol{k}}_{p,\infty}$. Since $B^{\boldsymbol{k}}_{p,\infty}$ is reflexive by Theorem 3.6 and $B^{\boldsymbol{k}}_{p,\infty}$ is continuously embedded in $L^p(K, m)$, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ weakly converging to 0 in $B^{\boldsymbol{k}}_{p,\infty}$. By Mazur's lemma, there exist $N(l) \in \mathbb{N}$ and $\{a_{l,k}\}_{k=l}^{N(l)} \subseteq [0, 1]$ with $N(l) > l$ and $\sum_{k=l}^{N(l)} a_{l,k} = 1$ for each $l \in \mathbb{N}$ such that $\sum_{k=l}^{N(l)} a_{l,k} u_{n_k}$ converges to 0 in $B^{\boldsymbol{k}}_{p,\infty}$ as $l \to \infty$. Let us define $\Psi_l \in C^1(\mathbb{R})$ by $\Psi_l := \sum_{k=l}^{N(l)} a_{l,k} \Phi_{n_k}$. Then $\Psi_l(0) = 0$, $\Psi_l' \to \mathbf{1}_F$ and, by Fatou's lemma, Theorem 4.9 and Proposition 4.7,

$$\begin{aligned}\Gamma_p^{\boldsymbol{k}}\langle u\rangle\circ u^{-1}(F)&=\int_{\mathbb{R}}\lim_{l\to\infty}\left|\Psi_l'(t)\right|^p\,\left(\Gamma_p^{\boldsymbol{k}}\langle u\rangle\circ u^{-1}\right)(dt)\\&\le\liminf_{l\to\infty}\int_K\left|\Psi_l'(u(x))\right|^p\,\Gamma_p^{\boldsymbol{k}}\langle u\rangle(dx)\\&=\liminf_{l\to\infty}\Gamma_p^{\boldsymbol{k}}\langle\Psi_l(u)\rangle(K)=\liminf_{l\to\infty}\mathcal{E}_p^{\boldsymbol{k}}\bigl(\Psi_l(u)\bigr)=0,\end{aligned}$$

which completes the proof. □

Now we can obtain the strongest possible forms of the strong locality of $\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,;\,\cdot\,\rangle$ as in the following theorem, which is an easy consequence of Theorem 4.10, the triangle inequality for $\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle^{1/p}$ and (4.14).

**Theorem 4.11 (Strong locality of $p$-energy measures)** *Let $u, u_1, u_2, v \in B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$, $a, a_1, a_2, b \in \mathbb{R}$ and $A \in \mathcal{B}(K)$.*

(a) *If $A \subseteq u^{-1}(a)$, then $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(A) = 0$.*

(b) *If $A \subseteq (u-v)^{-1}(a)$, then $\Gamma_p^{\boldsymbol{k}}\langle u\rangle(A) = \Gamma_p^{\boldsymbol{k}}\langle v\rangle(A)$.*

(c) *If $A \subseteq u_1^{-1}(a_1) \cup u_2^{-1}(a_2)$, then*

$$\Gamma_p^{\boldsymbol{k}}\langle u_1+u_2+v\rangle(A)+\Gamma_p^{\boldsymbol{k}}\langle v\rangle(A)=\Gamma_p^{\boldsymbol{k}}\langle u_1+v\rangle(A)+\Gamma_p^{\boldsymbol{k}}\langle u_2+v\rangle(A), \tag{4.22}$$

$$\Gamma_p^{\boldsymbol{k}}\langle u_1+u_2;v\rangle(A)=\Gamma_p^{\boldsymbol{k}}\langle u_1;v\rangle(A)+\Gamma_p^{\boldsymbol{k}}\langle u_2;v\rangle(A). \tag{4.23}$$

(d) *If $A \subseteq (u_1-u_2)^{-1}(a) \cup v^{-1}(b)$, then*

$$\Gamma_p^{\boldsymbol{k}}\langle u_1;v\rangle(A)=\Gamma_p^{\boldsymbol{k}}\langle u_2;v\rangle(A)\quad\textit{and}\quad\Gamma_p^{\boldsymbol{k}}\langle v;u_1\rangle(A)=\Gamma_p^{\boldsymbol{k}}\langle v;u_2\rangle(A). \tag{4.24}$$

*Proof.* This is proved, on the basis of Theorem 4.10, in exactly the same way as [23, Proof of Theorem 5.13], but we present the details because in the proof in [23] the underlying topological space $K$ is assumed to be a self-similar (compact) set.

(a): This is immediate from $\mathscr{L}^1(\{a\}) = 0$ and Theorem 4.10.

(b): This follows from (a) and the triangle inequality for $\Gamma_p^{\boldsymbol{k}}\langle\,\cdot\,\rangle(A)^{1/p}$ implied by Theorem 4.5 and Proposition 2.3-(1).

(c): Setting $A_1 := A \cap u_1^{-1}(a_1)$ and $A_2 := A \setminus A_1$, we have $A_i \subseteq u_i^{-1}(a_i)$ for $i \in \{1,2\}$ by $A \subseteq u_1^{-1}(a_1) \cup u_2^{-1}(a_2)$, and see therefore from (b) that

$$\begin{aligned}&\Gamma_p^{\boldsymbol{k}}\langle u_1+u_2+v\rangle(A)+\Gamma_p^{\boldsymbol{k}}\langle v\rangle(A)\\&=\Gamma_p^{\boldsymbol{k}}\langle u_2+v\rangle(A_1)+\Gamma_p^{\boldsymbol{k}}\langle u_1+v\rangle(A_2)+\Gamma_p^{\boldsymbol{k}}\langle v\rangle(A)\\&=\Gamma_p^{\boldsymbol{k}}\langle u_2+v\rangle(A_1)+\Gamma_p^{\boldsymbol{k}}\langle v\rangle(A_2)+\Gamma_p^{\boldsymbol{k}}\langle u_1+v\rangle(A_2)+\Gamma_p^{\boldsymbol{k}}\langle v\rangle(A_1)\\&=\Gamma_p^{\boldsymbol{k}}\langle u_2+v\rangle(A)+\Gamma_p^{\boldsymbol{k}}\langle u_1+v\rangle(A),\end{aligned}$$

proving (4.22). Then (4.23) follows, in the same way as the proof of (3.14) in Theorem 3.8, by applying (4.22) with $v$ replaced by $tv$ for $t \in \mathbb{R} \setminus \{0\}$ and by 0, taking the differences of both sides of the resulting equalities, dividing both sides by $t$ and then letting $t \to 0$ on the basis of (4.14).

(d): In the same way as the proof of (3.15) in Theorem 3.8, (4.22) with $u_2 - u_1, tv, u_1$ for $t \in \mathbb{R} \setminus \{0\}$ in place of $u_1, u_2, v$ and (4.14) together imply the former equality in (4.24), which in turn with $v, 0, u_1 - u_2$ in place of $u_1, u_2, v$ yields the latter by the linearity of $\Gamma_p^{\boldsymbol{k}}\langle v; \cdot\rangle(A)$. □

Using Theorem 4.11, we can extend Proposition 4.7 as follows.

**Corollary 4.12** *Let $u, v \in \mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$. If $\{u, v\} \cap \mathcal{D}_{p,\infty}^{\boldsymbol{k},c} \neq \emptyset$, then $\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle(K) = \mathcal{E}_p^{\boldsymbol{k}}(u; v)$.*

*Proof.* Similar to the proof of Proposotion 4.7, it suffices to consider the case $u, v \in B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$ with $\{u, v\} \cap B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K) \neq \emptyset$. Let $f, g \in \{u, v\}$ satisfy $\{f, g\} = \{u, v\}$ and $f \in B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K)$. Similar to the proof of Proposition 4.7, we can find an open neighborhood $U$ of the compact subset $\mathrm{supp}_K[f]$ of $K$ and $\varphi \in B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K)$ so that $0 \leq \varphi \leq 1$ and $\varphi(x) = 1$ for any $x \in U$. Then $\mathrm{supp}_K[f] \cap \mathrm{supp}_K[g(\varphi - \mathbf{1}_K)] = \emptyset$, so we have

$$\mathcal{E}_p^{\boldsymbol{k}}(u; v) = \begin{cases} \mathcal{E}_p^{\boldsymbol{k}}(f; g\varphi) & \text{if } f = u, \\ \mathcal{E}_p^{\boldsymbol{k}}(g\varphi; f) & \text{if } f = v, \end{cases}$$

by Theorem 3.8-(e) and

$$\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle(K) = \begin{cases} \Gamma_p^{\boldsymbol{k}}\langle f; g\varphi\rangle(K) & \text{if } f = u, \\ \Gamma_p^{\boldsymbol{k}}\langle g\varphi; f\rangle(K) & \text{if } f = v, \end{cases}$$

by Theorem 4.11-(c),(d). Since $f, g\varphi \in B_{p,\infty}^{\boldsymbol{k}} \cap C_c(K)$, we obtain $\Gamma_p^{\boldsymbol{k}}\langle u; v\rangle(K) = \mathcal{E}_p^{\boldsymbol{k}}(u; v)$ by Proposition 4.7. □

## 5 $p$-Energy forms on $p$-conductively homogeneous spaces

In this section, we verify $(\mathrm{WM})_{p,\boldsymbol{k}}$ for a family of kernels $\boldsymbol{k}$ corresponding to the $(1, p)$-Korevaar–Schoen–Sobolev space (see Example 3.14) on *$p$-conductively homogeneous* compact metric spaces equipped with Ahlfors regular measures. We also show some estimates on localized versions of Korevaar–Schoen $p$-energy forms, and construct, on the basis of Korevaar–Schoen $p$-energy forms, self-similar $p$-energy forms on $p$-conductively homogeneous self-similar sets as well. We refer to [29, Sections 4.3–4.6] for many concrete examples covered by this framework.

### 5.1 $p$-Conductively homogeneous spaces

Les us recall the notation and terminology in [28, 29] by following [23, Section 8.1]. We fix a locally finite (non-directed) infinite tree $(T, E_T)$ in the usual sense (see [29, Definition 2.1] for example), and fix a *root* $\phi \in T$ of $T$. (Here $T$ is the set of vertices and $E_T$ is the set of edges.) For any $w \in T \setminus \{\phi\}$, we use $\overline{\phi w}$ to denote the unique simple path in $T$ from $\phi$ to $w$.

**Definition 5.1 ([29, Definition 2.2])**

(1) For $w \in T$, define $\pi\colon T \to T$ by

$$\pi(w) := \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } \overline{\phi w} = (w_0, \ldots, w_n), \\ \phi & \text{if } w = \phi. \end{cases}$$

Set $S(w) := \{v \in T \mid \pi(v) = w\} \setminus \{w\}$. Moreover, for $k \in \mathbb{N}$, we define $S^k(w)$ inductively as

$$S^{k+1}(w) = \bigcup_{v \in S(w)} S^k(v).$$

For $A \subseteq T$, define $S^k(A) := \bigcup_{w \in A} S^k(A)$.

(2) For $w \in T$ and $n \in \mathbb{N} \cup \{0\}$, define $|w| := \min\{n \geq 0 \mid \pi^n(w) = \phi\}$ and $T_n := \{w \in T \mid |w| = n\}$.

(3) Define $\Sigma := \{(\omega_n)_{n\geq 0} \mid \omega_n \in T_n \text{ and } \omega_n = \pi(\omega_{n+1}) \text{ for all } n \in \mathbb{N} \cup \{0\}\}$. For $\omega = (\omega_n)_{n\geq 0} \in \Sigma$, we write $[\omega]_n$ for $\omega_n \in T_n$. For $w \in T$, define $\Sigma_w := \{(\omega_n)_{n\geq 0} \in \Sigma \mid \omega_{|w|} = w\}$. For $A \subseteq T$, define $\Sigma_A := \bigcup_{w \in A} \Sigma_w$.

We introduce a partition parametrized by a rooted tree (see [28, Definition 2.2.1] and [37, Lemma 3.6]).

**Definition 5.2 (Partition parametrized by a tree)** Let $K$ be a compact metrizable topological space without isolated points. A family of non-empty compact subsets $\{K_w\}_{w\in T}$ of $K$ is called a *partition of K parametrized by the rooted tree* $(T, E_T, \phi)$ if and only if it satisfies the following conditions:

(P1) $K_\phi = K$ and for any $w \in T$, $\#K_w \geq 2$ and $K_w = \bigcup_{v \in S(w)} K_v$.

(P2) For any $w \in \Sigma$, $\bigcap_{n\geq 0} K_{[\omega]_n}$ is a single point.

In the rest of this section, we fix a compact metrizable topological space without isolated points $K$, a locally finite rooted tree $(T, E_T, \phi)$ satisfying $\#\{v \in T \mid \{v, w\} \in E_T\} \geq 2$ for any $w \in T$, a partition $\{K_w\}_{w\in T}$ parametrized by $(T, E_T, \phi)$, a metric $d$ on $K$ with $\operatorname{diam}(K, d) = 1$, and a Borel probability measure $m$ on $K$. In the following definition, we collect some basic pieces of the notation used in [28, 29].

**Definition 5.3** For $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$, define

$$E_n^* := \bigl\{\{v, w\} \bigm| v, w \in T_n, v \neq w, K_v \cap K_w \neq \emptyset\bigr\},$$

and $E_n^*(A) = \bigl\{\{v, w\} \in E_n^* \bigm| v, w \in A\bigr\}$. Let $d_n$ be the graph distance of $(T_n, E_n^*)$. For $M \in \mathbb{N} \cup \{0\}$, $w \in T_n$ and $x \in K$, define

$$\Gamma_M(w) := \{v \in T_n \mid d_n(v, w) \leq M\} \quad \text{and} \quad U_M(x; n) := \bigcup_{w \in T_n; x \in K_w} \bigcup_{v \in \Gamma_M(w)} K_v.$$

To state geometric assumptions in [29], we need the following definition (see [28, Definitions 2.2.1 and 3.1.15].)

**Definition 5.4** (1) The partition $\{K_w\}_{w\in T}$ is said to be *minimal* if and only if $K_w \setminus \bigcup_{v\in T_{|w|}\setminus\{w\}} \neq \emptyset$ for any $w \in T$.
(2) The partition $\{K_w\}_{w\in T}$ is said to be *uniformly finite* if and only if $\sup_{w\in T} \#\Gamma_1(w) < \infty$.

We also use the following notation for simplicity.

**Definition 5.5** For $n \in \mathbb{N}\cup\{0\}$ and $U \subseteq K$, define $T_n[U] \coloneqq \{w \in T_n \mid K_w \cap U \neq \emptyset\}$.

Now we describe basic geometric conditions in [29]. The conditions (1), (2) and (5.6) in (3) below are important to follow the rest of this paper.

**Assumption 5.6 ([29, Assumption 2.15])** Let $(K, \mathcal{O})$ be a connected compact metrizable space, $\{K_w\}_{w\in T}$ a partition parametrized by the rooted tree $(T, \phi)$, $d$ a metric on $K$ that is compatible with the topology $\mathcal{O}$ and $\mathrm{diam}(K, d) = 1$ and $m$ a Borel probability measure on $K$. There exist $M_* \in \mathbb{N}$ and $r_* \in (0, 1)$ such that the following conditions (1)–(5) hold.

(1) $K_w$ is connected for any $w \in T$, $\{K_w\}_{w\in T}$ is minimal and uniformly finite, and $\inf_{m\geq 0} \min_{w\in T_m} \#S(w) \geq 2$.
(2) There exist $c_i > 0$, $i \in \{1, \dots, 5\}$, such that the following conditions (2A)–(2C) are true.

(2A) For any $w \in T$,
$$c_1 r_*^{|w|} \leq \mathrm{diam}(K_w, d) \leq c_2 r_*^{|w|}. \tag{5.1}$$

(2B) For any $n \in \mathbb{N}$ and $x \in K$,
$$B_d(x, c_3 r_*^n) \subseteq U_{M_*}(x; n) \subseteq B_d(x, c_4 r_*^n). \tag{5.2}$$

(2C) For any $n \in \mathbb{N}$ and $w \in T_n$, there exists $x \in K_w$ satisfying
$$K_w \supseteq B_d(x, c_5 r_*^n). \tag{5.3}$$

(3) There exist $m_1 \in \mathbb{N}$, $\gamma_1 \in (0, 1)$ and $\gamma \in (0, 1)$ such that
$$m(K_w) \geq \gamma m(K_{\pi(w)}) \quad \text{for any } w \in T, \tag{5.4}$$
and
$$m(K_v) \leq \gamma_1 m(K_w) \quad \text{for any } w \in T \text{ and } v \in S^{m_1}(w). \tag{5.5}$$
Furthermore, $m$ is volume doubling with respect to $d$ and
$$m(K_w) = \sum_{v\in S(w)} m(K_v) \quad \text{for any } w \in T. \tag{5.6}$$

(4) There exists $M_0 \geq M_*$ such that for any $w \in T$, $k \geq 1$ and any $v \in S^k(w)$,

$$\Gamma_{M_*}(v)\cap S^k(w) \subseteq \left\{ v' \in T_{|v|} \,\middle|\, \begin{array}{l} \text{there exist } l \le M_0 \text{ and } (v_0,\dots,v_l) \in S^k(w)^{l+1} \\ \text{such that } (v_{j-1},v_j) \in E^*_{|v|} \text{ for any } j \in \{1,\dots,l\} \end{array} \right\}.$$

(5) For any $w \in T$, $\pi(\Gamma_{M_*+1}(w)) \subseteq \Gamma_{M_*}(\pi(w))$.

Note that if a Borel probability measure $m$ on $K$ satisfies (5.6), then we have

$$m(K_v \cap K_w) = 0 \quad \text{for any } v, w \in T \text{ with } v \neq w \text{ and } |v| = |w|; \tag{5.7}$$

see [23, Proposition 8.7] for a proof of this fact.

Next we introduce conductance, neighbor disparity constants and the notion of $p$-conductive homogeneity in Definitions 5.9, 5.7 and 5.10. We also recall the notion of a covering system in Definition 5.8, which is used in the definition of neighbor disparity constants. See [29, Sections 2.2, 2.3 and 3.3] for further details on these topics. In the rest of this section, we fix $p \in (1,\infty)$ unless otherwise stated. We will state some definitions and statements below for any $p \in [1,\infty)$, but on each such occasion we will explicitly declare that we let $p \in [1,\infty)$.

**Definition 5.7 ([29, Definitions 2.17 and 3.4])** Let $p \in [1,\infty)$, $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$.

(1) Define $\mathcal{E}^n_{p,A} \colon \mathbb{R}^A \to [0,\infty)$ by

$$\mathcal{E}^n_{p,A}(f) := \sum_{\{u,v\} \in E^*_n(A)} |f(u) - f(v)|^p, \quad f \in \mathbb{R}^A.$$

We write $\mathcal{E}^n_p(f)$ for $\mathcal{E}^n_{p,T_n}(f)$.

(2) For $A_0, A_1 \subseteq A$, define $\mathrm{cap}^n_p(A_0, A_1; A)$ by

$$\mathrm{cap}^n_p(A_0, A_1; A) := \inf\bigl\{ \mathcal{E}^n_{p,A}(f) \bigm| f \in \mathbb{R}^A, f|_{A_i} = i \text{ for } i \in \{0,1\} \bigr\}.$$

(3) (Conductance constant) For $A_1, A_2 \subseteq A$ and $k \in \mathbb{N} \cup \{0\}$, define

$$\mathcal{E}_{p,k}(A_1, A_2, A) := \mathrm{cap}^{n+k}_p\bigl(S^k(A_1), S^k(A_2); S^k(A)\bigr).$$

For $M \in \mathbb{N}$, define $\mathcal{E}_{M,p,k} := \sup_{w \in T} \mathcal{E}_{p,k}(\{w\}, T_{|w|} \setminus \Gamma_M(w), T_{|w|})$.

**Definition 5.8 ([29, Definitions 2.26-(3) and 2.29])** Let $N_T, N_E \in \mathbb{N}$.

(1) Let $n \in \mathbb{N} \cup \{0\}$ and $A \subseteq T_n$. A collection $\{G_i\}_{i=1}^k$ with $G_i \subseteq T_n$ is called a *covering of* $(A, E^*_n(A))$ *with covering numbers* $(N_T, N_E)$ if and only if $A = \bigcup_{i=1}^k G_k$, $\max_{x \in A} \#\{i \mid x \in G_i\} \le N_T$ and for any $(u,v) \in E^*_n(A)$, there exists $l \le N_E$ and $\{w(1),\dots,w(l+1)\} \subseteq A$ such that $w(1) = u$, $w(l+1) = v$ and $(w(i), w(i+1)) \in \bigcup_{j=1}^k E^*_n(G_j)$ for any $i \in \{1,\dots,l\}$.

(2) Let $\mathscr{J} \subseteq \bigcup_{n \in \mathbb{N}\cup\{0\}} \{A \mid A \subseteq T_n\}$. The collection $\mathscr{J}$ is called a *covering system with covering number* $(N_T, N_E)$ if and only if the following conditions are satisfied:

(i) $\sup_{A \in \mathscr{J}} \#A < \infty$.

(ii) For any $w \in T$ and $k \in \mathbb{N}$, there exists a finite subset $\mathscr{N} \subseteq \mathscr{J} \cap T_{|w|+k}$ such that $\mathscr{N}$ is a covering of $\bigl(S^k(w), E^*_{|w|+k}(S^k(w))\bigr)$ with covering numbers $(N_T, N_E)$.

(iii) For any $G \in \mathscr{J}$ and $k \in \mathbb{N} \cup \{0\}$, if $G \subseteq T_n$, then there exists a finite subset $\mathscr{N} \subseteq \mathscr{J} \cap T_{n+k}$ such that $\mathscr{N}$ is a covering of $\bigl(S^k(G), E^*_{n+k}(S^k(G))\bigr)$ with covering numbers $(N_T, N_E)$.

The collection $\mathscr{J}$ is simply said to be a *covering system* if $\mathscr{J}$ is a covering system with covering numbers $(N_T, N_E)$ for some $(N_T, N_E) \in \mathbb{N}^2$.

**Definition 5.9 ([29, Definitions 2.26 and 2.29])** Let $p \in [1, \infty)$, $n \in \mathbb{N}$ and $A \subseteq T_n$.

(1) For $k \in \mathbb{N} \cup \{0\}$ and $f \colon T_{n+k} \to \mathbb{R}$, define $P_{n,k} f \colon T_n \to \mathbb{R}$ by

$$(P_{n,k} f)(w) := \frac{1}{\sum_{v \in S^k(w)} m(K_v)} \sum_{v \in S^k(w)} f(v) m(K_v), \quad w \in T_n.$$

(Note that $P_{n,k} f$ depends on the measure $m$.)

(2) (Neighbor disparity constant) For $k \in \mathbb{N} \cup \{0\}$, define

$$\sigma_{p,k}(A) := \sup_{f \colon S^k(A) \to \mathbb{R}} \frac{\mathcal{E}^n_{p,A}(P_{n,k} f)}{\mathcal{E}^{n+k}_{p,S^k(A)}(f)}.$$

(3) Let $\mathscr{J} \subseteq \bigcup_{n \geq 0} \{A \mid A \subseteq T_n\}$ be a covering system. Define

$$\sigma^{\mathscr{J}}_{p,k,n} := \max\{\sigma_{p,k}(A) \mid A \in \mathscr{J}, A \subseteq T_n\} \quad \text{and} \quad \sigma^{\mathscr{J}}_{p,k} := \sup_{n \in \mathbb{N} \cup \{0\}} \sigma^{\mathscr{J}}_{p,k,n}.$$

**Definition 5.10 ([29, Definition 3.4])** Let $p \in [1, \infty)$. The compact metric space $K$ (with a partition $\{K_w\}_{w \in T}$ and a measure $m$) is said to be *$p$-conductively homogeneous* if and only if there exists a covering system $\mathscr{J}$ such that

$$\sup_{k \in \mathbb{N} \cup \{0\}} \sigma^{\mathscr{J}}_{p,k} \mathcal{E}_{M_*,p,k} < \infty. \tag{5.8}$$

**Theorem 5.11 (A part of [29, Theorem 3.30])** *Let $p \in [1, \infty)$ and suppose that Assumption 5.6 holds. If $K$ is $p$-conductively homogeneous, then there exist $c_1, c_2, \sigma_p \in (0, \infty)$ and a covering system $\mathscr{J}$ such that for any $k \in \mathbb{N} \cup \{0\}$,*

$$c_1 \sigma_p^{-k} \leq \mathcal{E}_{M_*,p,k} \leq c_2 \sigma_p^{-k} \quad \text{and} \quad c_1 \sigma_p^k \leq \sigma^{\mathscr{J}}_{p,k} \leq c_2 \sigma_p^k. \tag{5.9}$$

The following weak monotonicity is a key consequence of the $p$-conductive homogeneity.

**Lemma 5.12 (Weak monotonicity)** *Let $p \in [1, \infty)$ and suppose that Assumption 5.6 holds. If $K$ is $p$-conductively homogeneous, then there exists $C \in (0, \infty)$ such that for any $k, l \in \mathbb{N}$, any $A \subseteq T_k$ and any $f \in L^1(K, m)$,*

$$\sigma_p^k \mathcal{E}^k_{p,A}(P_k f) \leq C \sigma_p^{k+l} \mathcal{E}^{k+l}_{p,S^l(A)}(P_{k+l} f), \tag{5.10}$$

*where $\sigma_p$ is the constant in* (5.9)*.*

*Proof.* This follows immediately by combining [29, Lemma 2.27] and (5.9). $\square$

We also recall the "Sobolev space" $\mathcal{W}^p$ introduced in [29, Lemma 3.13].

**Definition 5.13** Let $p \in [1, \infty)$. Suppose that Assumption 5.6 holds and that $K$ is $p$-conductively homogeneous. Let $\sigma_p$ be the constant in (5.9).

(1) For $n \in \mathbb{N} \cup \{0\}$, $w \in T_n$, $E \in \mathcal{B}(K)$ with $E \supseteq K_w$ and $f \in L^1(E, m|_E)$, define $P_n f(w) \coloneqq ⨍_{K_w} f\,dm$.

(2) We define $\mathcal{N}_p \colon L^p(K,m) \to [0,\infty]$ and $\mathcal{W}^p \subseteq L^p(K,m)$ by

$$\mathcal{N}_p(f) \coloneqq \left(\sup_{n \in \mathbb{N} \cup \{0\}} \sigma_p^n \mathcal{E}_p^n(P_n f)\right)^{1/p}, \quad f \in L^p(K,m),$$
$$\mathcal{W}^p \coloneqq \bigl\{ f \in L^p(K,m) \bigm| \mathcal{N}_p(f) < \infty \bigr\}.$$

Note that $\mathcal{N}_p(f) = 0$ if and only if $f$ is constant on $K$ (see [23, Section 8.1] for details). We also equip $\mathcal{W}^p$ with the norm $\|\cdot\|_{\mathcal{W}^p}$ defined by

$$\|f\|_{\mathcal{W}^p} \coloneqq \Bigl(\|f\|_{L^p(K,m)}^p + \mathcal{N}_p(f)^p\Bigr)^{1/p}, \quad f \in \mathcal{W}^p.$$

(3) For $n \in \mathbb{N} \cup \{0\}$, $A \subseteq T_n$, $E \in \mathcal{B}(K)$ with $E \supseteq \bigcup_{w \in A} K_w$ and $f \in L^1(E, m|_E)$, we define

$$\widetilde{\mathcal{E}}_{p,A}^n(f) \coloneqq \sigma_p^n \mathcal{E}_{p,A}^n(P_n f).$$

We also set $\widetilde{\mathcal{E}}_p^n(f) \coloneqq \widetilde{\mathcal{E}}_{p,T_n}^n(f)$ for $f \in L^1(K,m)$.

Now we can introduce a framework to construct a $p$-resistance form on $K$.

**Assumption 5.14** Let $(K, d, \{K_w\}_{w \in T}, m)$ satisfy Assumption 5.6. In addition, $(K, d, \{K_w\}_{w \in T}, m, p)$ satisfies the following conditions:

(1) The measure $m$ is Ahlfors regular with respect to $d$. (Recall (3.20).)

(2) $K$ is $p$-conductively homogeneous.

(3) $\sigma_p > 1$, where $\sigma_p$ is the constant in (5.9).

**Remark 5.15** (1) By [28, Theorem 4.6.9], Assumption 5.14-(3) is equivalent to $p > \dim_{\mathrm{ARC}}(K,d)$, where $\dim_{\mathrm{ARC}}(K,d)$ denotes the Ahlfors regular conformal dimension of $(K,d)$. (See, e.g., [29, (1.1)] for the definition of $\dim_{\mathrm{ARC}}(K,d)$.)

(2) It is highly non-trivial in general to verify that a given compact metric space $K$ is $p$-conductively homogeneous. In [29, Sections 4.3–4.6] and [30], the $p$-conductive homogeneity for $p > \dim_{\mathrm{ARC}}(K,d)$ has been proved for various large classes of self-similar sets $K$ in $\mathbb{R}^n$ equipped with the Euclidean metric $d$.

In the following theorem, we recall a fundamental result on $\mathcal{W}^p$.

**Theorem 5.16 ([29, Lemmas 3.16, 3.19, 3.24 and Theorem 3.22], [23, Theorem 8.16])** *Let $p \in [1, \infty)$. Suppose that $(K, d, \{K_w\}_{w \in T}, m)$ satisfies Assumption 5.6 and that $K$ is $p$-conductively homogeneous. Then $\mathcal{W}^p$ equipped with the norm $\| \cdot \|_{\mathcal{W}^p}$ is a Banach space. If $p > 1$, then $\mathcal{W}^p$ is reflexive and separable. If $p > \dim_{\mathrm{ARC}}(K, d)$, or equivalently $\sigma_p > 1$, then $\mathcal{W}^p \subseteq C(K)$ and $\mathcal{W}^p$ is dense in $(C(K), \| \cdot \|_{\sup})$.*

Let us introduce an important value, $p$-walk dimension, to describe the main result in this section.

**Definition 5.17** Suppose that Assumption 5.6 holds, that $m$ is Ahlfors regular and that $K$ is $p$-conductively homogeneous. Let $r_* \in (0, 1)$ be the constant in (5.1), let $\sigma_p$ be the constant in (5.9) and let $d_{\mathrm{f}}$ be the Hausdorff dimension of $(K, d)$. Define

$$d_{\mathrm{w},p} \coloneqq d_{\mathrm{f}} + \frac{\log \sigma_p}{\log r_*^{-1}}. \tag{5.11}$$

We call $d_{\mathrm{w},p}$ the *$p$-walk dimension* of $(K, d, \{K_w\}_{w \in T}, m)$.

The next lemma states a suitable capacity upper bound in this framework.

**Lemma 5.18** *Suppose that $(K, d, \{K_w\}_{w \in T}, m, p)$ satisfies Assumption 5.14. Then there exists $C \in (0, \infty)$ such that for any $(x, s) \in K \times (0, 1]$,*

$$\inf\Big\{ \mathcal{N}_p(f)^p \;\Big|\; f \in \mathcal{W}^p, f|_{B_d(x,s)} = 1, \mathrm{supp}_K[f] \subseteq B_d(x, 2s) \Big\} \le C s^{d_{\mathrm{f}} - d_{\mathrm{w},p}}. \tag{5.12}$$

*Proof.* Let $r_* \in (0, 1)$ and $M_* \in \mathbb{N}$ be the constants in Assumption 5.6. Let $s > 0$ and let $n \in \mathbb{N}$ satisfy

$$c_2(M_* + 1) r_*^n < 2s \le c_2(M_* + 1) r_*^{n-1},$$

where $c_2$ is the constant in (5.1). Let $x \in K$ and set $T_n(x, s) \coloneqq T_n[B_d(x, s)]$ for simplicity. Thanks to the metric doubling property of $(K, d)$, we have $\#T_n(x, s) \lesssim 1$. By [29, Lemma 3.18] and its proof, for any $w \in T_n(x, s)$ there exists $h_{M_*,w} \in \mathcal{W}^p$ such that $h_{M_*,w}|_{K_w} \equiv 1$, $\mathrm{supp}_K[h_{M_*,w}] \subseteq U_{M_*}(w)$ and $\mathcal{N}_p(h_{M_*,w})^p \lesssim \sigma_p^n$. Let $\psi_{x,s} \coloneqq \sum_{w \in T_n(x,s)} h_{M_*,w} \in \mathcal{W}^p$. Then $\psi_{x,s}|_{B_d(x,s)} \ge 1$, $\mathrm{supp}_K[\psi_{x,s}] \subseteq B_d(x, 2s)$ and $\mathcal{N}_p(\psi_{x,s})^p \lesssim \sigma_p^n = r_*^{n(d_{\mathrm{f}} - d_{\mathrm{w},p})} \lesssim s^{d_{\mathrm{f}} - d_{\mathrm{w},p}}$. Since $\mathcal{N}_p(\psi_{x,s} \wedge 1)^p \lesssim \mathcal{N}_p(\psi_{x,s})^p$ by [29, Theorem 3,21], we obtain (5.12). □

We also consider the following setting to deal with the case $p \le \dim_{\mathrm{ARC}}(K, d)$.

**Assumption 5.19** Let $(K, d, \{K_w\}_{w \in T}, m)$ satisfy Assumption 5.6. In addition, $(K, d, \{K_w\}_{w \in T}, m, p)$ satisfies the following conditions:

(1) The measure $m$ is Ahlfors regular with respect to $d$.
(2) $K$ is $p$-conductively homogeneous.
(3) There exists $C \in (0, \infty)$ such that for any $(x, s) \in K \times (0, 1]$,

$$\inf\Big\{ \mathcal{N}_p(f)^p \;\Big|\; f \in \mathcal{W}^p \cap C(K), f|_{B_d(x,s)} = 1, \mathrm{supp}_K[f] \subseteq B_d(x, 2s) \Big\}$$

$$\leq Cs^{d_{\mathrm{f}}-d_{\mathrm{w},p}}. \tag{5.13}$$

Note that Assumption 5.14 implies Assumption 5.19 by Lemma 5.18.

The same argument as in [35, Lemma 6.26] yields a good partition of unity under Assumption 5.19 as given in Lemma 5.20 and thus we obtain the regularity of $\mathcal{W}^p$ in Corollary 5.21.

**Lemma 5.20** *Suppose that Assumption 5.19 holds. Let $\varepsilon \in (0,1)$ and let $V$ be a maximal $\varepsilon$-net of $(K,d)$. Then there exists a family of functions $\{\psi_z\}_{z\in V}$ that satisfies the following properties:*

(i) $\sum_{z\in V}\psi_z \equiv 1$.

(ii) $\psi_z \in \mathcal{W}^p \cap C(K)$, $0 \leq \psi_z \leq 1$, $\psi_z|_{B_d(z,\varepsilon/4)} \equiv 1$ *and* $\mathrm{supp}_K[\psi_z] \subseteq B_d(z,5\varepsilon/4)$ *for any* $z\in V$.

(iii) *If* $z\in V$ *and* $z'\in V\setminus\{z\}$, *then* $\psi_{z'}|_{B_d(z,\varepsilon/4)} \equiv 0$.

(iv) *There exists* $C\in(0,\infty)$ *such that* $\mathcal{N}_p(\psi_z)^p \leq C\varepsilon^{d_{\mathrm{f}}-d_{\mathrm{w},p}}$ *for any* $z\in V$.

**Corollary 5.21** *Suppose that Assumption 5.19 holds. Then $\mathcal{W}^p \cap C(K)$ is dense in* $(C(K),\|\cdot\|_{\sup})$.

## 5.2 Localized energy estimates

In this subsection, we show localized energy estimates on Korevaar–Schoen $p$-energy forms, which will imply $(\mathrm{WM})_{p,\boldsymbol{k}}$ with the family of kernels $\boldsymbol{k}^{s_p}$ (recall (3.25)) and the equality $s_p = d_{\mathrm{w},p}/p$. Estimates in this subsection are very similar to [35, Section 7] although the setting of "partitions" in [35] is slightly different from ours.

We start with the following lemma, which gives a Poincaré-type estimate.

**Lemma 5.22** *Suppose that Assumption 5.19 holds. Then there exists a constant $C\in(0,\infty)$ such that for any $f\in L^p(K,m)$ and any $w\in T$,*

$$\int_{K_w}\left|f(x)-f_{K_w}\right|^p\, m(dx) \leq Cr_*^{|w|d_{\mathrm{w},p}}\liminf_{n\to\infty}\widetilde{\mathcal{E}}^{n+|w|}_{p,S^n(w)}(f). \tag{5.14}$$

*Proof.* Let $f\in L^p(K,m)$, $w\in T$ and set $k := |w|$. For $n\in\mathbb{N}$, define $Q_nf\in L^p(K,m)$ by

$$Q_nf := \sum_{v\in T_n} P_nf(v)\mathbf{1}_{K_v}.$$

Then we can see that $\|Q_nf - f\|_{L^p(K,m)} \to 0$ as $n\to\infty$ (see [23, Proof of Theorem 8.19]). For any $n\in\mathbb{N}$, we see from (5.7) and $\|Q_nf-f\|_{L^p(K,m)}\to 0$ that

$$\begin{aligned}
&\frac{1}{m(K_w)}\sum_{v\in S^n(w)}\left|f_{K_v}-f_{K_w}\right|^p m(K_v)\\
&=\frac{1}{m(K_w)}\sum_{v\in S^n(w)}\int_{K_v}\left|Q_{n+k}f(x)-f_{K_w}\right|^p\, m(dx)
\end{aligned}$$

$$= \fint_{K_w} \left|Q_{n+k} f(x) - f_{K_w}\right|^p \, m(dx) \xrightarrow[n\to\infty]{} \fint_{K_w} \left|f(x) - f_{K_w}\right|^p \, m(dx). \tag{5.15}$$

By [29, (5.11) in Theorem 5.11] and (5.9), there exists $C \in (0,\infty)$ which is independent of $f$ and $n$ such that

$$\frac{1}{m(K_w)} \sum_{v \in S^n(w)} \left|f_{K_v} - f_{K_w}\right|^p m(K_v) \le C r_*^{k(d_{\mathrm{w},p} - d_{\mathrm{f}})} \widetilde{\mathcal{E}}_{p,S^n(w)}^{n+k}(f). \tag{5.16}$$

We obtain (5.14) by combining (5.15), (5.16), (5.3) and the Ahlfors regularity of $m$. □

The next proposition shows an upper bound on localized Korevaar–Schoen energy functionals.

**Proposition 5.23** *Suppose that Assumption 5.19 holds. Then there exists $C \in (0,\infty)$ such that for any $E \in \mathcal{B}(K)$, any open neighborhood $E'$ of $\overline{E}^K$ and any $f \in L^p(E', m|_{E'})$,*

$$\begin{aligned} \limsup_{r\downarrow 0} \int_E \fint_{B_d(x,r)} &\frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w},p}}} \, m(dy) m(dx) \\ &\le C \limsup_{r\downarrow 0} \liminf_{n\to\infty} \widetilde{\mathcal{E}}_{p,T_n[(E)_{d,r}]}^n(f), \end{aligned} \tag{5.17}$$

*Furthermore, with $C \in (0,\infty)$ the same as in* (5.17)*, for any $f \in L^p(K,m)$,*

$$\sup_{r>0} \int_K \fint_{B_d(x,r)} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w},p}}} \, m(dy) m(dx) \le C \mathcal{N}_p(f)^p. \tag{5.18}$$

*Proof.* Let $r_* \in (0,1)$ and $M_* \in \mathbb{N}$ be the constants in Assumption 5.6. Let $r > 0$ and choose $n(r) \in \mathbb{N}$ satisfying $c_3 r_*^{n(r)+1} < r \le c_3 r_*^{n(r)}$, where $c_3$ is the constant in (5.2). Then, for any $w \in T_{n(r)}$ and $x \in K_w$, we have $B_d(x,r) \subseteq U_{M_*}(x; n(r)) \subseteq U_{M_*+1}(w)$. Let $f \in L^p(E', m|_{E'})$, where $E'$ is an open neighborhood of $\overline{E}^K$. Set $c := (M_* + 2) c_2 (c_3 r_*)^{-1} \in (0,\infty)$, where $c_2$ is the constant in (5.1). Then, by (5.1), $\bigcup_{w \in T_{n(r)}[E]} S^k(\Gamma_{M_*+1}(w)) \subseteq T_{k+n(r)}[(E)_{d,cr}]$ for any $k \in \mathbb{N}$ and there exists $r_0 \in (0,\infty)$ such that $(E)_{d,cr} \subseteq E'$ for any $r \in (0,r_0)$. By using $|f(x) - f(y)|^p \lesssim \left|f(x) - f_{K_w}\right|^p + \left|f(y) - f_{K_v}\right|^p + \left|f_{K_v} - f_{K_w}\right|^p$ and Lemma 5.22, we see that for any $r \in (0,r_0)$,

$$\begin{aligned} &\int_E \fint_{B_d(x,r)} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w},p}}} \, m(dy) m(dx) \\ &\le r_*^{-n(r) d_{\mathrm{f}}} \sum_{w \in T_{n(r)}[E], v \in \Gamma_{M_*+1}(w)} \int_{K_w} \int_{K_v} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w},p}}} \, m(dy) m(dx) \\ &\lesssim \sum_{w \in T_{n(r)}[E], v \in \Gamma_{M_*+1}(w)} \left( \liminf_{k\to\infty} \widetilde{\mathcal{E}}_{p,S^k(v)}^{k+n(r)}(f) + \liminf_{k\to\infty} \widetilde{\mathcal{E}}_{p,S^k(w)}^{k+n(r)}(f) \right) \end{aligned}$$

$$\begin{aligned}
&\quad + \sum_{w \in T_{n(r)}[E], v \in \Gamma_{M_*+1}(w)} \sigma_p^{n(r)} \left| f_{K_v} - f_{K_w} \right|^p \\
&\lesssim \sum_{w \in T_{n(r)}[E]} \liminf_{k \to \infty} \widetilde{\mathcal{E}}^{k+n(r)}_{p, S^k(\Gamma_{M_*+1}(w))}(f) + \sum_{w \in T_{n(r)}[E]} \widetilde{\mathcal{E}}^{n(r)}_{p, \Gamma_{M_*+1}(w)}(f). \qquad (5.19)
\end{aligned}$$

Since the partition $\{K_w\}_{w \in T}$ is uniformly finite, we have

$$\begin{aligned}
\sum_{w \in T_{n(r)}[E]} \liminf_{k \to \infty} \widetilde{\mathcal{E}}^{k+n(r)}_{p, S^k(\Gamma_{M_*+1}(w))}(f) &\leq \liminf_{k \to \infty} \sum_{w \in T_{n(r)}[E]} \widetilde{\mathcal{E}}^{n+n(r)}_{p, S^k(\Gamma_{M_*+1}(w))}(f) \\
&\lesssim \liminf_{k \to \infty} \widetilde{\mathcal{E}}^{k}_{p, T_k[(E)_{d,cr}]}(f). \qquad (5.20)
\end{aligned}$$

We also have from Lemma 5.12 that

$$\sum_{w \in T_{n(r)}[E]} \widetilde{\mathcal{E}}^{n(r)}_{p, \Gamma_{M_*+1}(w)}(f) \lesssim \widetilde{\mathcal{E}}^{n(r)}_{p, T_{n(r)}[(E)_{d,cr}]}(f) \lesssim \liminf_{n \to \infty} \widetilde{\mathcal{E}}^{n}_{p, T_n[(E)_{d,cr}]}(f). \qquad (5.21)$$

By (5.19), (5.20) and (5.21), there exists $C \in (0, \infty)$ (depending only on the constants associated with Assumption 5.6) such that for any $r \in (0, r_0)$,

$$\int_E \fint_{B_d(x,r)} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w},p}}} \, m(dy) m(dx) \leq C \liminf_{n \to \infty} \widetilde{\mathcal{E}}^{n}_{p, T_n[(E)_{d,cr}]}(f), \qquad (5.22)$$

whence we obtain (5.17) by letting $r \downarrow 0$ in (5.22). If $f \in L^p(K, m)$, then we have (5.18) by letting $E \coloneqq K$ in (5.22). $\square$

Before proving inequalities in the converse direction matching (5.17) and (5.18), let us introduce a localized version of $\mathcal{W}^p$.

**Definition 5.24** Let $U$ be a non-empty open subset of $K$. We define a linear subspace $\mathcal{W}^p_{\mathrm{loc}}(U)$ of $L^0(U, m|_U)$ by

$$\mathcal{W}^p_{\mathrm{loc}}(U) \coloneqq \left\{ f \in L^0(U, m|_U) \;\middle|\; \begin{array}{l} f = f^\# \ m\text{-a.e. on } V \text{ for some } f^\# \in \mathcal{W}^p \text{ for} \\ \text{each relatively compact open subset } V \text{ of } U \end{array} \right\}.$$

A lower bound on localized Korevaar–Schoen energy functionals can be shown in a similar way as [5, Theorem 5.2].

**Proposition 5.25** *Suppose that Assumption 5.19 holds. Then there exists $C \in (0, \infty)$ such that for any $E \subseteq K$, any open neighborhood $E'$ of $\overline{E}^K$ and any $u \in \mathcal{W}^p_{\mathrm{loc}}(E')$,*

$$\limsup_{n \to \infty} \widetilde{\mathcal{E}}^{n}_{p, T_n[E]}(u) \leq C \lim_{\delta \downarrow 0} \liminf_{r \downarrow 0} \int_{(E)_{d,\delta}} \fint_{B_d(x,r)} \frac{|u(x) - u(y)|^p}{r^{d_{\mathrm{w},p}}} \, m(dy) m(dx). \qquad (5.23)$$

*Furthermore, with $C \in (0, \infty)$ the same as in* (5.23)*, for any $f \in L^p(K, m)$,*

$$\mathcal{N}_p(f)^p \leq C \liminf_{r \downarrow 0} \int_K \fint_{B_d(x,r)} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w},p}}} \, m(dy) m(dx). \qquad (5.24)$$

*Proof.* Let $r \in (0,1)$, let $N_r$ be a maximal $r$-net of $(K,d)$, and let $\{\psi_{z,r}\}_{z \in N_r}$ be a partition of unity as given in Lemma 5.20. Define $A_r \colon L^p(K,m) \to \mathcal{W}^p \cap C(K)$ by $A_r f \coloneqq \sum_{z \in N_r} f_{B_d(z,r/4)} \psi_{z,r}$ for $f \in L^p(K,m)$. Then we can easily see that $\lim_{r \to 0} \lVert A_r f - f \rVert_{L^p(K,m)} = 0$ and $\sup_{r>0} \lVert A_r \rVert_{L^p(K,m) \to L^p(K,m)} < \infty$. For any large $n \in \mathbb{N}$ so that $4c_2 r_*^n < r$, where $c_2$ is the constant in (5.1), a similar argument as in [35, Lemma 7.4] shows that there exists $C_1 > 0$ depending only on the constants associated with Assumption 5.6 such that

$$\begin{aligned} &\widetilde{\mathcal{E}}^n_{p,T_n[B_d(z,5r/4)]}(A_r f) \\ &\le C_1 \sum_{w \in N_r \cap B_d(z,11r/4)} \int_{B_d(w,3r)} \fint_{B_d(x,9r)} \frac{|f(x)-f(y)|^p}{r^{d_{\mathrm{w},p}}}\, m(dy)m(dx). \end{aligned} \tag{5.25}$$

Let us fix $\delta > 0$ and define $N_r(E) \coloneqq \{z \in N_r \mid E \cap B_d(z,r) \neq \emptyset\}$. Then, for any small enough $r > 0$ so that $r < \delta/7$, we have $E \subseteq \bigcup_{z \in N_r(E)} B_d(z,5r/4)$ and

$$\bigcup_{z \in N_r(E)} \bigcup_{w \in N_r \cap B_d(z,11r/4)} B_d(w,3r) \subseteq (E)_{d,\delta},$$

whence we see that for any $f \in L^p(K,m)$,

$$\begin{aligned} &\widetilde{\mathcal{E}}^n_{p,T_n[E]}(A_r f) \\ &\le \sum_{z \in N_r(E)} \widetilde{\mathcal{E}}^n_{p,T_n[B_d(z,5r/4)]}(A_r f) \\ &\overset{(5.25)}{\le} C_1 \sum_{z \in N_r(E)} \sum_{w \in N_r \cap B_d(z,11r/4)} \int_{B_d(w,3r)} \fint_{B_d(x,9r)} \frac{|f(x)-f(y)|^p}{r^{d_{\mathrm{w},p}}}\, m(dy)m(dx) \\ &\lesssim \int_{(E)_{d,\delta}} \fint_{B_d(x,9r)} \frac{|f(x)-f(y)|^p}{r^{d_{\mathrm{w},p}}}\, m(dy)m(dx), \end{aligned} \tag{5.26}$$

where we used the metric doubling property of $(K,d)$ in the last inequality. (Here, we consider small enough $r > 0$ so that $r < \delta/7$ and large enough $n \in \mathbb{N}$ so that $4c_2 r_*^n < r$.)

To prove the desired estimate (5.23) for $u \in \mathcal{W}^p_{\mathrm{loc}}(E')$, we fix a relatively compact open subset $V$ of $E'$ and $u^\# \in \mathcal{W}^p$ satisfying $V \supseteq \overline{E}^K$, $u^\# \in \mathcal{W}^p$ and $u = u^\#$ $m$-a.e. on $V$. Also, fix a sequence $\{r_k\}_{k \in \mathbb{N}} \subseteq (0,\infty)$ such that $r_k \downarrow 0$ as $k \to \infty$ and

$$\lim_{k \to \infty} r_k^{-d_{\mathrm{w},p}} J_{p,r_k}(u^\# \mid (E)_{d,\delta}) = \liminf_{r \downarrow 0} r^{-d_{\mathrm{w},p}} J_{p,r}(u^\# \mid (E)_{d,\delta}) \le \mathcal{N}_p(u^\#) < \infty,$$

where $J_{p,r}(g \mid A) \coloneqq \int_A \fint_{B_d(x,r)} \frac{|g(x)-g(y)|^p}{r^{d_{\mathrm{w},p}}}\, m(dy)m(dx)$ for $g \in L^p(K,m)$ and $A \in \mathcal{B}(K)$. Set $u_k \coloneqq A_{r_k/9} u^\#$ for each $k \in \mathbb{N}$. By combining (5.26) with $E = K$ and (5.18), for all large $k \in \mathbb{N}$, we have

$$\mathcal{N}_p(u_k)^p \lesssim \int_K \fint_{B_d(x,r_k)} \frac{\left|u^{\#}(x)-u^{\#}(y)\right|^p}{r_k^{d_{\mathrm{w},p}}}\,m(dy)m(dx) \lesssim \mathcal{N}_p(u^{\#})^p < \infty, \quad (5.27)$$

which implies that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{W}^p$. Since $\mathcal{W}^p$ is reflexive by Theorem 5.16, we can assume that $u_k$ converges weakly in $\mathcal{W}^p$ to some function $u_\infty \in \mathcal{W}^p$ as $k \to \infty$. Since $\mathcal{W}^p$ is continuously embedded in $L^p(K,m)$, we have $u_\infty = u^{\#}$. Hence, by Mazur's lemma and (5.26), we obtain

$$\limsup_{n\to\infty} \widetilde{\mathcal{E}}^n_{p,T_n[E]}(u^{\#}) \le \lim_{\delta\downarrow 0}\liminf_{r\downarrow 0} r^{-d_{\mathrm{w},p}} J_{p,r}(u^{\#} \mid (E)_{d,\delta}). \quad (5.28)$$

Note that, by (5.1), $\bigcup_{w\in T_n[E]} K_w \subseteq V$ for all large enough $n \in \mathbb{N}$ and $(E)_{d,r+\delta} \subseteq V$ for all small enough $\delta, r \in (0,\infty)$. For such $n$, $\delta$ and $r$, we have $\widetilde{\mathcal{E}}^n_{p,T_n[E]}(u^{\#}) = \widetilde{\mathcal{E}}^n_{p,T_n[E]}(u)$ and $J_{p,r}(u^{\#} \mid (E)_{d,\delta}) = J_{p,r}(u \mid (E)_{d,\delta})$, whence we obtain (5.23).

We next consider the case $E = K$. Let $f \in L^p(K,m)$ and set $J_{p,r}(f) := J_{p,r}(f|K)$ for $r > 0$. Similar to the previous case, we assume that $\{r_k\}_{k\in\mathbb{N}}$ is a sequence of positive numbers such that $r_k \downarrow 0$ as $k \to \infty$ and

$$\lim_{k\to\infty} r_k^{-d_{\mathrm{w},p}} J_{p,r_k}(f) = \liminf_{r\downarrow 0} r^{-d_{\mathrm{w},p}} J_{p,r}(f) < \infty,$$

which together with (5.26) implies that $\{A_{r_k/9} f\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{W}^p$. A similar argument using Mazur's lemma as in the previous paragraph yields (5.24). $\square$

## 5.3 Weak monotonicity and Poincaré inequality

Now we can prove the main theorem of this section, which solves a part of [29, Section 6.3, Problem 4], as follows.

**Theorem 5.26** *Suppose that $(K, d, \{K_w\}_{w\in T}, m, p)$ satisfies Assumption 5.19, let $s_p$ and $\mathrm{KS}^{1,p}$ be as defined in Example 3.14, and define $\boldsymbol{k} = \{k_r\}_{r>0}$ by*

$$k_r(x,y) := \frac{\mathbf{1}_{B_d(x,r)}(y)}{r^{d_{\mathrm{w},p}} m(B_d(x,r))}, \quad x, y \in K. \quad (5.29)$$

*Then $s_p = d_{\mathrm{w},p}/p$, $\mathcal{W}^p = \mathrm{KS}^{1,p}$, $\mathcal{W}^p \cap C(K)$ is dense in $\mathcal{W}^p$, and $(\mathrm{WM})_{p,\boldsymbol{k}}$ holds. Moreover, there exists $C \in [1,\infty)$ such that*

$$C^{-1} \sup_{r>0} J^{\boldsymbol{k}}_{p,r}(f) \le \mathcal{N}_p(f)^p \le C \liminf_{r\downarrow 0} J^{\boldsymbol{k}}_{p,r}(f) \quad \textit{for any } f \in L^p(K,m). \quad (5.30)$$

*Proof.* By (5.18) and (5.24), we have $\mathcal{W}^p = B^{d_{\mathrm{w},p}/p}_{p,\infty}$ and (5.30). (Recall Example 3.14 for the definition of $B^s_{p,\infty}$.) In particular, $s_p \ge d_{\mathrm{w},p}/p$. To show the converse, let $s > d_{\mathrm{w},p}/p$ and let $f \in \mathcal{W}^p \setminus \mathbb{R}\mathbf{1}_K$. (Note that $\mathcal{W}^p$ contains a non-constant function by (5.12).) Let $A_r : L^p(K,m) \to \mathcal{W}^p \cap C(K)$ be the same operator as in the proof of Proposition 5.25 for each $r \in (0,1)$. Then, by (5.26) with $E = K$,

$$\frac{r^{d_{\mathrm{w},p}}}{r^{sp}} \widetilde{\mathcal{E}}_p^n(A_r f) \leq C \int_K \fint_{B_d(x,9r)} \frac{|f(x) - f(y)|^p}{r^{sp}}\, m(dy)m(dx) \tag{5.31}$$

for any $n \in \mathbb{N}$ and $r \in (0,1)$ with $4c_2 r_*^n < r$, where $c_2$ is the constant in (5.1) and $C > 0$ is a constant independent of $f$, $r$, and $n$. As in the proof of [29, Theorem 3.21], let $\{\widetilde{\mathcal{E}}_p^{n_k}\}_{k\in\mathbb{N}}$ be a $\Gamma$-converging subsequence of $\{\widetilde{\mathcal{E}}_p^n\}_{n\in\mathbb{N}}$ and define $\widehat{\mathcal{E}}_p$ as its $\Gamma$-limit. Since $\widehat{\mathcal{E}}_p$ is lower semicontinuous with respect to the $L^p(K,m)$-topology (see [11, Proposition 6.8]) and $\widehat{\mathcal{E}}_p \asymp \mathcal{N}_p(\,\cdot\,)^p$ (see [29, pp. 45–46]), we see that

$$0 < \mathcal{N}_p(f)^p \lesssim \widehat{\mathcal{E}}_p(f) \leq \liminf_{r\downarrow 0} \widehat{\mathcal{E}}_p(A_r f) \leq \liminf_{r\downarrow 0} \liminf_{k\to\infty} \widetilde{\mathcal{E}}_p^{n_k}(A_r f),$$

which together with (5.31) and $\lim_{r\to 0} r^{d_{\mathrm{w},p} - sp} = \infty$ implies that $f \notin B^s_{p,\infty}$. Since $s > d_{\mathrm{w},p}/p$ is arbitrary, we conclude that $d_{\mathrm{w},p}/p \geq s_p$. In particular, we obtain $\mathcal{W}^p = \mathrm{KS}^{1,p}$ and $(\mathrm{WM})_{p,\boldsymbol{k}}$. The inclusion $\mathcal{W}^p \subseteq \overline{\mathcal{W}^p \cap C(K)}^{\mathcal{W}^p}$ follows from (5.27) and Mazur's lemma, so we complete the proof. □

**Corollary 5.27** *Suppose that $(K, d, \{K_w\}_{w\in T}, m, p)$ satisfies Assumption 5.14. Then any Korevaar–Schoen p-energy form $(\mathcal{E}_p^{\mathrm{KS}}, \mathcal{W}^p)$ on $(K,d,m)$, which exists by Theorems 5.26 and 3.8 (recall Example 3.14), is a p-resistance form on K, and there exist $\alpha_0, \alpha_1 \in (0,\infty)$ such that for any such $(\mathcal{E}_p^{\mathrm{KS}}, \mathcal{W}^p)$,*

$$\alpha_0 d(x,y)^{d_{\mathrm{w},p} - d_{\mathrm{f}}} \leq R_{\mathcal{E}_p^{\mathrm{KS}}}(x,y) \leq \alpha_1 d(x,y)^{d_{\mathrm{w},p} - d_{\mathrm{f}}} \quad \text{for any } x, y \in K. \tag{5.32}$$

*Proof.* Define $\boldsymbol{k} = \{k_r\}_{r>0}$ by (5.29). Then by Theorem 5.26, Lemma 5.18 and [5, Theorem 3.2], the assumptions of Proposition 3.13 with $d_{\mathrm{f}}, d_{\mathrm{w},p}$ in place of $Q, \beta_p$ hold under Assumption 5.14, so $(\mathcal{E}_p^{\mathrm{KS}}, \mathcal{W}^p)$ is a $p$-resistance form on $K$. The estimate (5.32) follows from the $d_{\mathrm{f}}$-Ahlfors regularity of $m$ and Proposition 3.13. □

We also have a Poincaré-type inequality in terms of the localized versions of $(\mathcal{E}_p^{\boldsymbol{k}}, \mathcal{W}^p)$. (For the Vicsek set, such a Poincaré-type inequality was proved in [6, Corollary 4.2].)

**Proposition 5.28** *Suppose that $(K, d, \{K_w\}_{w\in T}, m, p)$ satisfies Assumption 5.19. Then there exist $C \in (0,\infty)$ and $A \in [1,\infty)$ such that for any $(z,s) \in K \times (0,1]$ and any $f \in \mathcal{W}^p_{\mathrm{loc}}(B_d(z, As))$,*

$$\begin{aligned}
&\int_{B_d(z,s)} \left|f(y) - f_{B_d(z,s)}\right|^p \, m(dy) \\
&\leq C s^{d_{\mathrm{w},p}} \liminf_{r\downarrow 0} \int_{B_d(z,As)} \fint_{B_d(x,r)} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w},p}}}\, m(dy)m(dx).
\end{aligned} \tag{5.33}$$

*Proof.* Throughout this proof, $M_* \in \mathbb{N}$ and $r_* \in (0,1)$ are the same constants as in Assumption 5.6. We assume that $f \in \mathcal{W}^p$ for simplicity. Let $(z,s) \in K \times (0,1]$ and choose $n \in \mathbb{N}$ satisfying $c_3 r_*^n \geq s > c_3 r_*^{n+1}$, where $c_3$ is the constant in (5.2). Let $f \in L^p(K, m)$ and set $\Gamma_{M_*}(z; n) \coloneqq \{v \in T \mid v \in \Gamma_{M_*}(w) \text{ for some } w \in T_n \text{ with } z \in K_w\}$. Then we see that

$$
\begin{aligned}
&\int_{U_{M_*}(z;n)} \left|f(y) - f_{U_{M_*}(x;n)}\right|^p \, m(dy) \\
&\leq \sum_{w \in \Gamma_{M_*}(z;n)} \int_{K_w} \left|f(y) - f_{U_{M_*}(x;n)}\right|^p \, m(dy) \\
&\leq 2^{p-1} \sum_{w \in \Gamma_{M_*}(z;n)} \left( \int_{K_w} \left|f(y) - f_{K_w}\right|^p \, m(dy) + m(K_w) \left|f_{K_w} - f_{U_{M_*}(x;n)}\right|^p \right) \\
&\lesssim \sum_{w \in \Gamma_{M_*}(z;n)} \left( s^{d_{\mathrm{w},p}} \liminf_{k \to \infty} \widetilde{\mathcal{E}}^{n+k}_{p, S^k(w)}(f) + s^{d_{\mathrm{f}}} \left|f_{K_w} - f_{U_{M_*}(z;n)}\right|^p \right). \qquad (5.34)
\end{aligned}
$$

Since $\min_{v \in \Gamma_{M_*}(z;n)} f_{K_v} \leq f_{U_{M_*}(z;n)} \leq \max_{v \in \Gamma_{M_*}(z;n)} f_{K_v}$, for any $w \in \Gamma_{M_*}(z; n)$ there exists $w' \in \Gamma_{M_*}(z; n) \setminus \{w\}$ such that $\left|f_{K_w} - f_{U_{M_*}(z;n)}\right| \leq \left|f_{K_w} - f_{K_{w'}}\right|$, which together with Hölder's inequality yields that

$$
\left|f_{K_w} - f_{U_{M_*}(z;n)}\right|^p \lesssim \mathcal{E}^n_{p, \Gamma_{2M_*}(w)}(f) \lesssim s^{d_{\mathrm{w},p} - d_{\mathrm{f}}} \liminf_{k \to \infty} \widetilde{\mathcal{E}}^{n+k}_{p, S^k(\Gamma_{2M_*}(w))}(f), \quad (5.35)
$$

where we used (5.9) and [29, (2.17)] in the last inequality. Note that $\sup_{v \in T} \#\Gamma_M(w) \leq L_*^M$ by (5.2) and the volume doubling property of $m$. This observation together with (5.34) and (5.35) implies that

$$
\begin{aligned}
&\int_{U_{M_*}(z;n)} \left|f(y) - f_{U_{M_*}(x;n)}\right|^p \, m(dy) \\
&\lesssim s^{d_{\mathrm{w},p}} \liminf_{k \to \infty} \sum_{w \in \Gamma_{M_*}(z;n)} \widetilde{\mathcal{E}}^{n+k}_{p, S^k(\Gamma_{2M_*}(w))}(f) \lesssim s^{d_{\mathrm{w},p}} \liminf_{k \to \infty} \widetilde{\mathcal{E}}^k_{p, T_k[B_d(z, As/2)]}(f) \\
&\overset{(5.23)}{\lesssim} s^{d_{\mathrm{w},p}} \liminf_{r \downarrow 0} \int_{B_d(z, As)} \fint_{B_d(x,r)} \frac{|f(y) - f(x)|^p}{r^{d_{\mathrm{w},p}}} \, m(dy) m(dx),
\end{aligned}
$$

which yields (5.33) in the case $f \in \mathcal{W}^p$ since

$$
\int_{U_{M_*}(z;n)} \left|f(y) - f_{U_{M_*}(x;n)}\right|^p \, m(dy) \gtrsim \int_{B_d(z,s)} \left|f(y) - f_{B_d(z,s)}\right|^p \, m(dy).
$$

The case $f \in \mathcal{W}^p_{\mathrm{loc}}(B_d(x, A's))$, where $A' > A$ (set, e.g., $A' = 2A$), is similar. $\square$

## 5.4 Self-similar $p$-energy forms based on Korevaar–Schoen $p$-energy forms

In this subsection, we construct a self-similar $p$-energy form by improving [29, Theorem 4.6]. We need some preparations before constructing such a good self-similar $p$-energy form. We first review basic notation and terminology on self-similar structures. In particular, we recall the notion of a post-critically finite self-similar structure introduced by Kigami [24], which is mainly dealt with in the next section. See [25, Section 1] and [26, Chapter 1] for further details. Throughout this section, we fix a compact metrizable space $K$, a finite set $S$ with $\#S \geq 2$ and a continuous injective map $F_i \colon K \to K$ for each $i \in S$. We set $\mathcal{L} := (K, S, \{F_i\}_{i\in S})$.

**Definition 5.29** (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_n := S^n = \{w_1 \dots w_n \mid w_i \in S \text{ for } i \in \{1, \dots, n\}\}$ for $n \in \mathbb{N}$ and let $W_* := \bigcup_{n\in\mathbb{N}\cup\{0\}} W_n$. For $w \in W_*$, the unique $n \in \mathbb{N} \cup \{0\}$ with $w \in W_n$ is denoted by $|w|$ and called the *length of* $w$.

(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3 \dots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology of the discrete topology on $S$, and define the *shift map* $\sigma \colon \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3 \dots) := \omega_2\omega_3\omega_4 \dots$. For $i \in S$ we define $\sigma_i \colon \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3 \dots) := i\omega_1\omega_2\omega_3 \dots$. For $\omega = \omega_1\omega_2\omega_3 \dots \in \Sigma$ and $n \in \mathbb{N}\cup\{0\}$, we write $[\omega]_n := \omega_1 \dots \omega_n \in W_n$.

(3) For $w = w_1 \dots w_n \in W_*$, we set $F_w := F_{w_1} \circ \cdots \circ F_{w_n}$ ($F_\emptyset := \mathrm{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \cdots \circ \sigma_{w_n}$ ($\sigma_\emptyset := \mathrm{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$.

(4) Let $w, v \in W_*$, $w = w_1 \dots w_{n_1}$, $v = v_1 \dots v_{n_2}$. We define $wv \in W_*$ by $wv := w_1 \dots w_{n_1} v_1 \dots v_{n_2}$ ($w\emptyset := w$, $\emptyset v := v$). We write $w \leq v$ if and only if $w = v\tau$ for some $\tau \in W_*$.

(5) A finite subset $\Lambda$ of $W_*$ is called a *partition* of $\Sigma$ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \neq v$ and $\Sigma = \bigcup_{w\in\Lambda} \Sigma_w$.

(6) Let $\Lambda_1, \Lambda_2$ be partitions of $\Sigma$. We say that $\Lambda_1$ is a *refinement* of $\Lambda_2$, and write $\Lambda_1 \leq \Lambda_2$, if and only if for each $w^1 \in \Lambda_1$ there exists $w^2 \in \Lambda_2$ such that $w^1 \leq w^2$.

**Definition 5.30** $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\chi \colon \Sigma \to K$ such that $F_i \circ \chi = \chi \circ \sigma_i$ for any $i \in S$. Note that such $\chi$, if it exists, is unique and satisfies $\{\chi(\omega)\} = \bigcap_{n\in\mathbb{N}} K_{[\omega]_n}$ for any $\omega \in \Sigma$.

**Definition 5.31** Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a self-similar structure.

(1) We define the *critical set* $C_{\mathcal{L}}$ and the *post-critical set* $\mathcal{P}_{\mathcal{L}}$ of $\mathcal{L}$ by

$$C_{\mathcal{L}} := \chi^{-1}\bigl(\textstyle\bigcup_{i,j\in S,\, i\neq j} K_i \cap K_j\bigr) \qquad \text{and} \qquad \mathcal{P}_{\mathcal{L}} := \textstyle\bigcup_{n\in\mathbb{N}} \sigma^n(C_{\mathcal{L}}).$$

$\mathcal{L}$ is called *post-critically finite*, *p.-c.f.* for short, if and only if $\mathcal{P}_{\mathcal{L}}$ is a finite set.

(2) We set $V_0 := \chi(\mathcal{P}_{\mathcal{L}})$, $V_n := \bigcup_{w\in W_n} F_w(V_0)$ for $n \in \mathbb{N}$ and $V_* := \bigcup_{n\in\mathbb{N}\cup\{0\}} V_n$.

The set $V_0$ should be considered as the *"boundary"* of the self-similar set $K$; indeed, $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w \cap \Sigma_v = \emptyset$ by [25, Proposition 1.3.5-(2)]. According to [25, Lemma 1.3.11], $V_{n-1} \subseteq V_n$ for any $n \in \mathbb{N}$, and $V_*$ is dense in $K$ if $V_0 \neq \emptyset$.

**Definition 5.32 (Self-similar measure)** Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure and let $(\theta_i)_{i \in S} \in (0,1)^S$ satisfy $\sum_{i \in S} \theta_i = 1$. A Borel probability measure $m$ on $K$ is said to be a *self-similar measure on* $\mathcal{L}$ *with weight* $(\theta_i)_{i \in S}$ if and only if the following equality (of Borel measures on $K$) holds:

$$m = \sum_{i \in S} \theta_i (F_i)_* m. \tag{5.36}$$

Next we introduce the notion of self-similarity for $p$-energy forms and $p$-resistance forms.

**Definition 5.33 (Self-similar $p$-energy/$p$-resistance form)** Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure and let $m$ be a Radon measure on $K$ with full topological support. Let $(\rho_{p,s})_{s \in S} \in (0, \infty)^S$ and define $\rho_{p,w} := \rho_{p,w_1} \cdots \rho_{p,w_n}$ for each $w = w_1 \dots w_n \in W_*$. A $p$-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ on $(K, m)$ (with $\mathcal{F}_p \subseteq L^p(K, m)$) is called a *self-similar $p$-energy form on* $(\mathcal{L}, m)$ *with weight* $(\rho_{p,s})_{s \in S}$ if and only if the following hold:

$$\mathcal{F}_p \cap C(K) = \{u \in C(K) \mid u \circ F_s \in \mathcal{F}_p \text{ for any } s \in S\}, \tag{5.37}$$

$$\mathcal{E}_p(u) = \sum_{s \in S} \rho_{p,s} \mathcal{E}_p(u \circ F_s) \quad \text{for any } u \in \mathcal{F}_p \cap C(K). \tag{5.38}$$

If $\mathcal{F}_p \subseteq C(K)$ and $(\mathcal{E}_p, \mathcal{F}_p)$ is a $p$-resistance form on $K$ satisfying (5.37) and (5.38), then $(\mathcal{E}_p, \mathcal{F}_p)$ is called a *self-similar $p$-resistance form on* $\mathcal{L}$ *with weight* $(\rho_{p,s})_{s \in S}$.

We will focus on self-similar structures having *rationally related contraction ratios* as in [29]. In the next definition, we introduce a good partition parametrized by a rooted tree.

**Definition 5.34 ([29, Definition 4.2])** Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure, let $r \in (0,1)$ and let $(j_s)_{s \in S} \in \mathbb{R}^S$. Define

$$j(w) := \sum_{i=1}^{n} j_{w_i} \quad \text{and} \quad g(w) := r^{j(w)} \quad \text{for } w = w_1 \dots w_n \in W_n.$$

Define $\widetilde{\pi}(w_1 \cdots w_n) := w_1 \cdots w_{n-1}$ for $w = w_1 \dots w_n \in W_n$ and

$$\Lambda^g_{r^k} := \{w = w_1 \cdots w_n \in W_* \mid g(\widetilde{\pi}(w)) > r^k \geq g(w)\}.$$

Set $T^{(r)}_k := \{(k, w) \mid w \in \Lambda^g_{r^k}\}$ and $T^{(r)} := \bigcup_{k \in \mathbb{N} \cup \{0\}} T^{(r)}_k$. Moreover, define $E_{T^{(r)}} \subseteq T^{(r)} \times T^{(r)}$ by

$$E_{T^{(r)}} := \left\{((k, v), (k+1, w)) \in T^{(r)}_k \times T^{(r)}_{k+1} \,\middle|\, k \in \mathbb{N} \cup \{0\}, v = w \text{ or } v = \widetilde{\pi}(w)\right\}.$$

We introduce the following assumption in order to construct a self-similar $p$-energy form on $(\mathcal{L}, m)$. (Recall that we have fixed $p \in (1, \infty)$.)

**Assumption 5.35** Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a self-similar structure such that $\#S \geq 2$ and $K$ is connected. There exist $r_* \in (0, 1)$, $(j_s)_{s \in S} \in \mathbb{N}^S$ and a metric $d$ giving the original topology of $K$ with $\mathrm{diam}(K, d) = 1$ such that $(K, d, \{K_w\}_{w \in T^{(r_*)}}, m, p)$ satisfies Assumption 5.19, where $d_{\mathrm{f}} \in (0, \infty)$ is such that $\sum_{s \in S} r_*^{j_s d_{\mathrm{f}}} = 1$ and $m$ is the self-similar measure on $K$ with weight $(r_*^{j_s d_{\mathrm{f}}})_{s \in S}$. (The collection $\{F_i\}_{i \in S}$ is said to have rationally related contraction ratios $(r_*^{j_s})_{s \in S}$.)

Under Assumption 5.35, we have $V_0 \neq \emptyset$ since $K$ is connected and $\#S \geq 2$ (see [25, Proposition 1.3.5-(3)] or [25, Theorem 1.6.2]). Also, we can easily show that $m$ is $d_{\mathrm{f}}$-Ahlfors regular as stated in the following proposition (see [29, Proposition 4.5]).

**Proposition 5.36** *Suppose that $\mathcal{L}$ is a self-similar structure and that there exist $r_* \in (0, 1)$, $(j_s)_{s \in S} \in \mathbb{N}^S$ and a metric $d$ giving the original topology of $K$ with $\mathrm{diam}(K, d) = 1$ such that $(K, d, \{K_w\}_{w \in T^{(r_*)}}, m)$ satisfies Assumption 5.6. Let $d_{\mathrm{f}} \in (0, \infty)$ be such that $\sum_{s \in S} r_*^{j_s d_{\mathrm{f}}} = 1$ and let $m$ be the self-similar measure on $K$ with weight $(r_*^{j_s d_{\mathrm{f}}})_{s \in S}$. Then $d_{\mathrm{f}}$ is the Hausdorff dimension of $(K, d)$ and $m$ is $d_{\mathrm{f}}$-Ahlfors regular with respect to $d$.*

To construct a self-similar $p$-energy form, we need to take care of the pre-self-similarity condition (see [35, Theorem 8.12]). We can easily verify this condition in the case $\sigma_p > 1$ by modifying [29, Proof of Theorem 4.6]; see [23, Section 8.2] for details.

**Proposition 5.37** *Suppose that Assumption 5.35 holds and that $\sigma_p > 1$. Then* (5.37) *with $\mathcal{W}^p$ in place of $\mathcal{F}_p$ holds and there exists $C \in [1, \infty)$ such that for any $n \in \mathbb{N}$ and any $u \in \mathcal{W}^p \subseteq C(K)$,*

$$C^{-1} \sum_{w \in W_n} \sigma_p^{j(w)} \mathcal{N}_p(u \circ F_w)^p \leq \mathcal{N}_p(u)^p \leq C \sum_{w \in W_n} \sigma_p^{j(w)} \mathcal{N}_p(u \circ F_w)^p.$$

Now we can present an improvement of [29, Theorem 4.6] in the following formulation.

**Theorem 5.38** *Suppose that Assumption 5.35 holds, that* (5.37) *with $\mathcal{W}^p$ in place of $\mathcal{F}_p$ holds, and that there exists $C_0 \in [1, \infty)$ such that for any $n \in \mathbb{N}$ and any $u \in \mathcal{W}^p \cap C(K)$,*

$$C_0^{-1} \sum_{w \in W_n} \sigma_p^{j(w)} \mathcal{N}_p(u \circ F_w)^p \leq \mathcal{N}_p(u)^p \leq C_0 \sum_{w \in W_n} \sigma_p^{j(w)} \mathcal{N}_p(u \circ F_w)^p. \quad (5.39)$$

*For each $n \in \mathbb{N}$, define $\boldsymbol{k}^{(n)} = \{k_r^{(n)}\}_{r > 0}$ by*

$$k_r^{(n)}(x, y) \coloneqq \frac{1}{n+1} \sum_{l=0}^{n} \sum_{w \in W_l} r_*^{-j(w) \cdot (d_{\mathrm{w},p} + d_{\mathrm{f}})} \frac{\mathbf{1}_{A_{w,r}}(x, y)}{r^{d_{\mathrm{w},p} + d_{\mathrm{f}}}}, \quad x, y \in K,$$

*where* $A_{w,r} := \{(x,y) \in K_w \times K_w \mid d(F_w^{-1}(x), F_w^{-1}(y)) < r\}$*. Then* $\boldsymbol{k}^{(n)}$ *is asymptotically local,* $(\mathrm{WM})_{p,\boldsymbol{k}^{(n)}}$ *holds,* $B_{p,\infty}^{\boldsymbol{k}^{(n)}} = \mathcal{W}^p$*, and for any sequence* $\{(\mathcal{E}_p^{\boldsymbol{k}^{(n)}}, \mathcal{W}^p)\}_{n\in\mathbb{N}}$ *with* $(\mathcal{E}_p^{\boldsymbol{k}^{(n)}}, \mathcal{W}^p)$ *a* $\boldsymbol{k}^{(n)}$*-Korevaar–Schoen* $p$*-energy form on* $(K,m)$ *for each* $n \in \mathbb{N}$*, there exists a sequence* $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ *with* $n_j < n_{j+1}$ *for any* $j \in \mathbb{N}$ *such that the following limit exists in* $[0,\infty)$ *for any* $u \in \mathcal{W}^p$*:*

$$\breve{\mathcal{E}}_p^{\mathrm{KS}}(u) := \lim_{j\to\infty} \mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u). \tag{5.40}$$

*Moreover, for any such* $\{\mathcal{E}_p^{\boldsymbol{k}^{(n)}}\}_{n\in\mathbb{N}}$ *and* $\{n_j\}_{j\in\mathbb{N}}$*, the functional* $\breve{\mathcal{E}}_p^{\mathrm{KS}} \colon \mathcal{W}^p \to [0,\infty)$ *defined by* (5.40) *satisfies the following properties:*

(a) $(\breve{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$ *is a self-similar* $p$*-energy form on* $(\mathcal{L}, m)$ *with weight* $(\sigma_p^{j_s})_{s\in S}$*.*

(b) *For any* $u \in \mathcal{W}^p$*,*

$$(CC_0)^{-1}\mathcal{N}_p(u)^p \le \breve{\mathcal{E}}_p^{\mathrm{KS}}(u) \le CC_0\mathcal{N}_p(u)^p, \tag{5.41}$$

*where* $C, C_0 \in [1,\infty)$ *are the constants in* (5.30) *and in* (5.39) *respectively.*

(c) $(\breve{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$ *satisfies* $(\mathrm{GC})_p$*. Furthermore, for any* $u, v \in \mathcal{W}^p$*,* $\{\mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u;v)\}_{j\in\mathbb{N}}$ *is convergent in* $\mathbb{R}$ *and*

$$\breve{\mathcal{E}}_p^{\mathrm{KS}}(u;v) = \lim_{j\to\infty} \mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u;v). \tag{5.42}$$

(d) *Theorem 3.8-*(c),(d),(e) *with* $(\breve{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$ *in place of* $(\mathcal{E}_p^{\boldsymbol{k}}, B_{p,\infty}^{\boldsymbol{k}})$ *hold.*

(e) *For any isometric map* $T\colon (K,d) \to (K,d)$ *preserving* $m$*,* $u \circ T \in \mathcal{W}^p$ *and* $\breve{\mathcal{E}}_p^{\mathrm{KS}}(u\circ T) = \breve{\mathcal{E}}_p^{\mathrm{KS}}(u)$ *for any* $u \in \mathcal{W}^p$*.*

(f) *If in addition* $\sigma_p > 1$*, then* $(\breve{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$ *is a* $p$*-resistance form on* $K$*, and there exist* $\alpha_0, \alpha_1 \in (0,\infty)$ *independent of particular choices of* $\{\mathcal{E}_p^{\boldsymbol{k}^{(n)}}\}_{n\in\mathbb{N}}$ *and* $\{n_j\}_{j\in\mathbb{N}}$ *such that*

$$\alpha_0 d(x,y)^{d_{\mathrm{w},p}-d_{\mathrm{f}}} \le R_{\breve{\mathcal{E}}_p^{\mathrm{KS}}}(x,y) \le \alpha_1 d(x,y)^{d_{\mathrm{w},p}-d_{\mathrm{f}}} \quad \textit{for any } x,y \in K. \tag{5.43}$$

*Proof.* Set $\boldsymbol{k} = \{k_r\}_{r>0}$ by $k_r(x,y) := r^{-d_{\mathrm{w},p}-d_{\mathrm{f}}}\mathbf{1}_{B_d(x,r)}(y)$. Recall that $B_{p,\infty}^{\boldsymbol{k}} = \mathrm{KS}^{1,p}$ since $ps_p = d_{\mathrm{w},p}$ and $m$ is $d_{\mathrm{f}}$-Ahlfors regular (see Example 3.14, Theorem 5.26 and Proposition 5.36). By using (5.2), we can easily see that $\boldsymbol{k}^{(n)}$ is asymptotically local. Let us show $(\mathrm{WM})_{p,\boldsymbol{k}^{(n)}}$. Note that for any $r > 0$ and any $u \in L^p(K,m)$, we have

$$J_{p,r}^{\boldsymbol{k}^{(n)}}(u) = \frac{1}{n+1}\sum_{l=0}^{n}\sum_{w\in W_l}\sigma_p^{j(w)} J_{p,r}^{\boldsymbol{k}}(u\circ F_w), \tag{5.44}$$

where we used $(F_w \times F_w)^{-1}(A_{w,r}) = \{(x,y) \in K\times K \mid d(x,y) < r\}$ and $m = r_*^{j(w)d_{\mathrm{f}}}(F_w)_*m$. By combining (5.44), Theorem 5.26 and (5.39), we obtain $(\mathrm{WM})_{p,\boldsymbol{k}^{(n)}}$. Moreover, for any $n \in \mathbb{N}$ and any $u \in \mathcal{W}^p$,

$$(CC_0)^{-1}\sup_{r>0} J_{p,r}^{\boldsymbol{k}^{(n)}}(u) \le \mathcal{N}_p(u)^p \le CC_0 \liminf_{r\to 0} J_{p,r}^{\boldsymbol{k}^{(n)}}(u), \tag{5.45}$$

where $C, C_0 \in [1,\infty)$ are the constants in (5.30) and in (5.39) respectively. In particular, $B_{p,\infty}^{\boldsymbol{k}^{(n)}} = \mathcal{W}^p$ and $\{\mathcal{E}_p^{\boldsymbol{k}^{(n)}}(u)\}_{n\in\mathbb{N}}$ is bounded for each $u \in \mathcal{W}^p$. Since $\mathcal{W}^p$ is separable and $\mathcal{E}_p^{\boldsymbol{k}^{(n)}} \asymp \mathcal{N}_p(\,\cdot\,)^p$ by (5.45), a standard diagonal argument implies that there exists $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_j < n_{j+1}$ such that the limit $\lim_{j\to\infty}\mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u) =: \breve{\mathcal{E}}_p^{\mathrm{KS}}(u)$ exists for any $u \in \mathcal{W}^p$. From this definition, (5.45) and Theorem 3.8-(b), we immediately see that (5.41) holds and that $(\breve{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$ satisfies $(\mathrm{GC})_p$.

(a): Since we assume that $\mathcal{W}^p$ satisfies (5.37), it suffices to show the following equality for any $u \in \mathcal{W}^p$:

$$\breve{\mathcal{E}}_p^{\mathrm{KS}}(u) = \sum_{s\in S} \sigma_p^{j_s} \breve{\mathcal{E}}_p^{\mathrm{KS}}(u \circ F_s). \tag{5.46}$$

From Theorem 3.8 together with a diagonal argument, we can choose a sequence $\{r_l\}_{l\in\mathbb{N}} \subseteq (0,\infty)$ with $\lim_{l\to\infty} r_l = 0$ such that $\mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u) = \lim_{l\to\infty} J_{p,r_l}^{\boldsymbol{k}^{(n_j)}}(u)$ for any $j \in \mathbb{N}$ and any $u \in \mathcal{W}^p$. Using (5.44), we easily see that for any $(j,l) \in \mathbb{N}^2$ and any $u \in L^p(K,m)$,

$$\begin{aligned}
&\sum_{s\in S} \sigma_p^{j_s} J_{p,r_l}^{\boldsymbol{k}^{(n_j)}}(u \circ F_s) + \frac{1}{n_j+1} J_{p,r_l}^{\boldsymbol{k}^{(n_j)}}(u) \\
&= J_{p,r_l}^{\boldsymbol{k}^{(n_j)}}(u) + \frac{1}{n_j+1} \sum_{w\in W_{n_j+1}} \sigma_p^{j(w)} J_{p,r_l}^{\boldsymbol{k}^{(n_j)}}(u \circ F_w).
\end{aligned}$$

Letting $l \to \infty$ and $j \to \infty$, we obtain (5.46) by (5.45) and (5.39).

(c): Similar to the proof of (3.9), by using Proposition 2.4 and the convexity of $t \mapsto \mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u + tv)$, we can prove (5.42).

(d): This is clear from Theorem 3.8-(c),(d),(e) for $(\breve{\mathcal{E}}_p^{\boldsymbol{k}^{(n)}}, \mathcal{W}^p)$ and (5.42).

(e): If $T\colon (K,d) \to (K,d)$ is an isometric map preserving $m$, then for any $n \in \mathbb{N}$, $\boldsymbol{k}^{(n)}$ is clearly $T$-invariant, and hence by Theorem 3.8-(f) and $B_{p,\infty}^{\boldsymbol{k}^{(n)}} = \mathcal{W}^p$ we have $u \circ T \in \mathcal{W}^p$ and $\mathcal{E}_p^{\boldsymbol{k}^{(n)}}(u \circ T) = \mathcal{E}_p^{\boldsymbol{k}^{(n)}}(u)$ for any $u \in \mathcal{W}^p$, which together with (5.40) implies that $\breve{\mathcal{E}}_p^{\mathrm{KS}}(u \circ T) = \breve{\mathcal{E}}_p^{\mathrm{KS}}(u)$ for any $u \in \mathcal{W}^p$.

(f): In the case $\sigma_p > 1$, we easily see that $(\breve{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$ is a $p$-resistance form on $K$ satisfying (5.43) by combining Proposition 3.13, $d_{\mathrm{f}}$-Ahlfors regularity of $m$, $d_{\mathrm{w},p} > d_{\mathrm{f}}$ by $\sigma_p > 1$, Theorem 5.16, Lemma 5.18 and [29, Lemma 3.34]. □

We collect properties of the $p$-energy measures associated with $(\breve{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$ in the following theorem. See also [23, Sections 4 and 5] for other basic properties. Let us emphasize that we do not know whether Theorem 5.39-(c) below holds in a more general setting of self-similar $p$-energy forms like that of [23].

**Theorem 5.39** *Suppose the same assumptions as in Theorem 5.38, let* $(\mathcal{E}_p^{\boldsymbol{k}^{(n)}}, \mathcal{W}^p)$ *be any* $\boldsymbol{k}^{(n)}$*-Korevaar–Schoen* $p$*-energy form on* $(K,m)$ *for each* $n \in \mathbb{N}$*, let*

*$\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ be any sequence as in Theorem 5.38, and let $(\breve{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$ be the $p$-energy form on $(K, m)$ defined by* (5.40)*. Then for any $u \in \mathcal{W}^p \cap C(K)$, there exists a unique positive Radon measure $\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle$ on $K$ such that*

$$\begin{aligned}&\int_K \varphi \, d\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle \\ &= \breve{\mathcal{E}}_p^{\mathrm{KS}}(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \breve{\mathcal{E}}_p^{\mathrm{KS}}\bigl(|u|^{\frac{p}{p-1}}; \varphi\bigr) \quad \textit{for any } \varphi \in \mathcal{W}^p \cap C(K). \end{aligned} \tag{5.47}$$

*Moreover, the following hold:*

(a) *For any $u \in \mathcal{W}^p$, there exists a unique positive Radon measure $\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle$ on $K$ such that for any $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{W}^p \cap C(K)$ with $\lim_{n\to\infty} \mathcal{N}_p(u - u_n) = 0$ and any Borel measurable function $\varphi \colon K \to [0, \infty)$ with $\|\varphi\|_{\sup} < \infty$,*

$$\int_K \varphi \, d\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle = \lim_{n\to\infty} \int_K \varphi \, d\breve{\Gamma}_p^{\mathrm{KS}}\langle u_n\rangle, \tag{5.48}$$

*and $\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle$ further satisfies $\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle(K) = \breve{\mathcal{E}}_p^{\mathrm{KS}}(u)$. Moreover, for each such $\varphi$, $(\int_K \varphi \, d\breve{\Gamma}_p^{\mathrm{KS}}\langle\,\cdot\,\rangle, \mathcal{W}^p)$ is a $p$-energy form on $K$ satisfying* $(\mathrm{GC})_p$*.*

(b) *Theorem 4.6, with $\mathcal{W}^p$ and $\breve{\Gamma}_p^{\boldsymbol{k}}$ in place of $\mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and $\Gamma_p^{\boldsymbol{k}}$ respectively, holds. In particular, for any $u, v \in \mathcal{W}^p$,*

$$\breve{\Gamma}_p^{\mathrm{KS}}\langle u; v\rangle(A) := \frac{1}{p} \frac{d}{dt} \breve{\Gamma}_p^{\mathrm{KS}}\langle u + tv\rangle(A)\Big|_{t=0}, \quad A \in \mathcal{B}(K), \tag{5.49}$$

*defines a signed Borel measure on $K$ such that $\breve{\Gamma}_p^{\mathrm{KS}}\langle u; v\rangle(K) = \breve{\mathcal{E}}_p^{\mathrm{KS}}(u; v)$ and $\breve{\Gamma}_p^{\mathrm{KS}}\langle u; u\rangle = \breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle$. Furthermore, for any $u, v \in \mathcal{W}^p$ and any $\varphi \in C(K)$,*

$$\int_K \varphi \, d\breve{\Gamma}_p^{\mathrm{KS}}\langle u; v\rangle = \lim_{j\to\infty} \int_K \varphi \, d\Gamma_p^{\boldsymbol{k}^{(n_j)}}\langle u; v\rangle. \tag{5.50}$$

(c) *Theorem 4.8-*(a),(b)*, with $\mathcal{W}^p$ and $\breve{\Gamma}_p^{\mathrm{KS}}$ in place of $\mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and $\Gamma_p^{\boldsymbol{k}}$ respectively, hold.*

(d) *Theorems 4.9, 4.10 and 4.11, with $\mathcal{W}^p \cap C(K)$ and $\breve{\Gamma}_p^{\mathrm{KS}}$ in place of $B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$ and $\Gamma_p^{\boldsymbol{k}}$ respectively, hold.*

*Proof.* Fix $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ so that $\breve{\mathcal{E}}_p^{\mathrm{KS}} = \lim_{j\to\infty} \mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}$. Let $u \in \mathcal{W}^p \cap C(K)$. Letting $j \to \infty$ in (4.6) with $\mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}$ in place of $\mathcal{E}_p^{\boldsymbol{k}}$ and using (5.42), we have

$$0 \le \Psi_p(u; \varphi) := \breve{\mathcal{E}}_p^{\mathrm{KS}}(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \breve{\mathcal{E}}_p^{\mathrm{KS}}\bigl(|u|^{\frac{p}{p-1}}; \varphi\bigr) \le \|\varphi\|_{\sup} \breve{\mathcal{E}}_p^{\mathrm{KS}}(u)$$

for any $\varphi \in \mathcal{W}^p \cap C(K)$ with $\varphi \ge 0$. Since $\mathcal{W}^p \cap C(K)$ is dense in $C(K)$, we can get the desired positive Radon measure $\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle$ (in the case $u \in \mathcal{W}^p \cap C(K)$) by using the Riesz–Markov–Kakutani representation theorem as done in the proof of Theorem 4.2. Also, we easily see that

$$\int_K \psi \, d\check{\Gamma}_p^{\mathrm{KS}}\langle u \rangle = \lim_{j \to \infty} \int_K \psi \, d\Gamma_p^{\boldsymbol{k}^{(n_j)}}\langle u \rangle \quad \text{for any } \psi \in C(K), \tag{5.51}$$

whence $\bigl(\int_K \psi \, d\check{\Gamma}_p^{\mathrm{KS}}\langle \,\cdot\, \rangle, \mathcal{W}^p \cap C(K)\bigr)$ is a $p$-energy form on $(K, m)$ satisfying $(\mathrm{GC})_p$. Then we can prove (a) by following the same argument as in the proof of Theorem 4.5.

The property (b) except for $\check{\Gamma}_p^{\mathrm{KS}}\langle u; v \rangle(K) = \check{\mathcal{E}}_p^{\mathrm{KS}}(u; v)$ and for (5.50) follow from [23, Theorem 4.5 and Proposition 4.6]. The equality (5.50) can be shown in the same way as the proof of (3.9) by using (5.51), Proposition 2.4 and the convexity of $t \mapsto \int_K \varphi \, d\check{\Gamma}_p^{\mathrm{KS}}\langle u + tv \rangle$. We have $\check{\Gamma}_p^{\mathrm{KS}}\langle u; v \rangle(K) = \check{\mathcal{E}}_p^{\mathrm{KS}}(u; v)$ from Proposition 4.7, (5.40) and (5.50).

The statement (c) and the chain rule (4.21) with $\check{\Gamma}_p^{\mathrm{KS}}$ in place of $\Gamma_p^{\boldsymbol{k}}$ are immediate from (5.50) and the corresponding properties of $\Gamma_p^{\boldsymbol{k}^{(n_j)}}$. Since we can follow the proofs of Theorems 4.10 and 4.11 by using the chain rule of $\check{\Gamma}_p^{\mathrm{KS}}$, we complete the proof of (d). □

**Remark 5.40** There is another way to construct the $p$-energy measures associated with $(\check{\mathcal{E}}_p^{\mathrm{KS}}, \mathcal{W}^p)$, which is based on the self-similarity (5.46); see [23, Section 5.2] for the details of this construction (see also Proposition 6.12 below). The resulting $p$-energy measures turn out to satisfy (5.47) and therefore coincide with the ones $\{\check{\Gamma}_p^{\mathrm{KS}}\langle u \rangle\}_{u \in \mathcal{W}^p}$ constructed in Theorem 5.39 (see [23, Proposition 5.11]).

# 6 $p$-Resistance forms on p.-c.f. self-similar structures

In this section, we verify $(\mathrm{WM})_{p,\boldsymbol{k}}$ for a family of kernels $\boldsymbol{k}$ corresponding to the $(1, p)$-Korevaar–Schoen–Sobolev space under the assumption of the existence of a good $p$-resistance form on a post-critically finite self-similar structure. (See [7, Theorem 4.2] or [23, Section 8.3] for the existing construction of self-similar $p$-resistance forms in this setting.)

## 6.1 Geometry under the $p$-resistance metric

We first present the setting of this section. Throughout this section, we presume the following assumption.

**Assumption 6.1** Let $p \in (1, \infty)$ and $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be a p.-c.f. self-similar structure with $\#S \geq 2$ and $K$ connected. Let $(\mathcal{E}_p, \mathcal{F}_p)$ be a self-similar $p$-resistance form on $\mathcal{L}$ with weight $(\rho_{p,i})_{i \in S} \in (0, \infty)^S$ such that

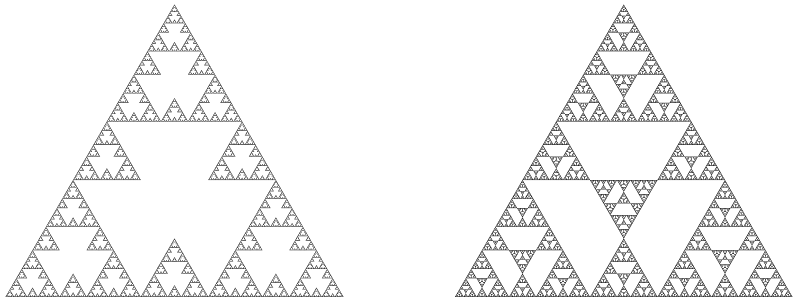

**Fig. 1** Some examples of affine nested fractals

$$\min_{i \in S} \rho_{p,i} > 1. \tag{6.1}$$

Let $d_{\mathrm{f},p} \in (0, \infty)$ be such that $\sum_{i \in S} \rho_{p,i}^{-d_{\mathrm{f},p}/(p-1)} = 1$, and let $m$ be the self-similar measure on $\mathcal{L}$ with weight $\bigl(\rho_{p,i}^{-d_{\mathrm{f},p}/(p-1)}\bigr)_{i \in S}$.

**Remark 6.2** (1) The condition (6.1) corresponds to the condition (R) in [7, p. 18].

(2) Assumption 6.1 is equivalent to the existence of a *p-eigenform* on $V_0$ with respect to the renormalization operator with weight $(\rho_{p,i})_{i \in S} \in (1, \infty)^S$, i.e., a $p$-resistance form $\mathcal{E}_p^{(0)}$ on $V_0$ such that

$$\inf\left\{ \sum_{i \in S} \rho_{p,i} \mathcal{E}_p^{(0)}(v \circ F_i) \;\middle|\; v \in \mathbb{R}^{V_1}, v|_{V_0} = u \right\} = \mathcal{E}_p^{(0)}(u) \quad \text{for any } u \in \mathbb{R}^{V_0};$$

see [23, Sections 6.2 and 8.3] for a detailed proof of this equivalence. In the case $p = 2$, this is nothing but the existence of a regular harmonic structure on $\mathcal{L}$ as defined in [25, Definition 3.1.2].

(3) Any self-similar $p$-resistance form constructed in [29, Theorem 4.6] must satisfy $\rho_{p,i} = \sigma_p^{n_i}$ for some $n_i \in \mathbb{N}$, where $\sigma_p$ is the constant in (5.9). This restriction excludes the self-similar $p$-resistance forms with weight $(\rho_{p,i})_{i \in S} \in (1, \infty)^S$ satisfying $(\log \rho_{p,i})/\log \rho_{p,j} \notin \mathbb{Q}$ for some $i, j \in S$, whereas they are covered by [7]; as proved in [23, Appendix B], they do exist abundantly on plenty of typical affine nested fractals.

(4) It is easy to see that $d_{\mathrm{f},p} \geq 1$ by using (6.1) and (6.3) below.

In this subsection, we will show the Ahlfors regularity of $m$, the capacity upper bound and the Poincaré inequality in terms of the *p-resistance metric* of $(\mathcal{E}_p, \mathcal{F}_p)$, which is defined as follows.

**Definition 6.3 ($p$-Resistance metric; [23, Definition 6.29])** We define the *$p$-resistance metric* $\widehat{R}_{p,\mathcal{E}_p} \colon K \times K \to [0, \infty)$ of $(\mathcal{E}_p, \mathcal{F}_p)$ by

$$\widehat{R}_{p,\mathcal{E}_p}(x, y) := R_{\mathcal{E}_p}(x, y)^{\frac{1}{p-1}}, \quad x, y \in K \tag{6.2}$$

(recall (3.18)). For simplicity, we write $\widehat{R}_p := \widehat{R}_{p,\mathcal{E}_p}$.

We record some properties of $R_{\mathcal{E}_p}$ and $\widehat{R}_p$.

**Proposition 6.4 ([23, Proposition 7.2 and Corollary 6.28])**

(1) *For any $w \in W_*$ and any $x, y \in K$,*

$$R_{\mathcal{E}_p}(F_w(x), F_w(y)) \leq \rho_{p,w}^{-1} R_{\mathcal{E}_p}(x, y). \tag{6.3}$$

(2) *$\widehat{R}_p$ is a metric on $K$ giving the original topology of $K$. In particular, $\overline{V_*}^{\widehat{R}_p} = K$.*

(3) *For any $u \in \mathcal{F}_p$ and $x, y \in K$,*

$$|u(x) - u(y)|^p \leq R_{\mathcal{E}_p}(x, y)\mathcal{E}_p(u).$$

*In particular, $\mathcal{F}_p \subseteq C(K)$.*

In the next definition, we introduce the *symmetry* on $K$ with respect to $(\mathcal{E}_p, \mathcal{F}_p)$.

**Definition 6.5** We define

$$\mathcal{G} := \left\{ T \;\middle|\; \begin{array}{l} T\colon K \to K,\ T \text{ is a homeomorphism preserving } m, \text{ and} \\ u \circ T, u \circ T^{-1} \in \mathcal{F}_p \text{ and } \mathcal{E}_p(u \circ T) = \mathcal{E}_p(u) \text{ for any } u \in \mathcal{F}_p \end{array} \right\}, \tag{6.4}$$

which forms a subgroup of the group of surjective isometries of $(K, \widehat{R}_p)$ by (3.18) and (6.2).

Let us introduce natural *scales* $\{\Lambda_s\}_{s\in(0,1]}$ with respect to $\widehat{R}_p$. (See [22, Definitions 6.12 and 6.13] for the case $p = 2$.)

**Definition 6.6** (1) We define $\Lambda_1 := \{\emptyset\}$,

$$\Lambda_s := \left\{ w \;\middle|\; w = w_1 \dots w_n \in W_* \setminus \{\emptyset\}, (\rho_{p,w_1\dots w_{n-1}})^{-1/(p-1)} > s \geq \rho_{p,w}^{-1/(p-1)} \right\}$$

for each $s \in (0, 1)$. (Note that $\{\Lambda_s\}_{s\in(0,1]}$ is the scale associated with the weight function $g(w) := \rho_{p,w}^{-1/(p-1)}$; see [28, Definition 2.3.1].)

(2) For each $(s, x) \in (0, 1] \times K$, we define $\Lambda_{s,0}(x) := \{w \in \Lambda_s \mid x \in K_w\}$, $U_0(x, s) := \bigcup_{w\in\Lambda_{s,0}(x)} K_w$, $\Lambda_{s,1}(x) := \{w \in \Lambda_s \mid K_w \cap U_0(x, s) \neq \emptyset\}$ and $U_1(x, s) := \bigcup_{w\in\Lambda_{s,1}(x)} K_w$.

Similar to the case $p = 2$ in [22, Section 6.1], it is easy to see that $\lim_{s\downarrow 0} \min\{|w| \mid w \in \Lambda_s\} = \infty$, that $\Lambda_s$ is a partition of $\Sigma$ for any $s \in (0, 1]$, and that $\Lambda_{s_1} \leq \Lambda_{s_2}$ for any $s_1, s_2 \in (0, 1]$ with $s_1 \leq s_2$. By [28, Proposition 2.3.7], for any $x \in K$, each of $\{U_0(x, s)\}_{s\in(0,1]}$ and $\{U_1(x, s)\}_{s\in(0,1]}$ is non-decreasing in $s$ and forms a fundamental system of neighborhoods of $x$ in $K$. Moreover, $\{U_1(x, s)\}_{s\in(0,1]}$ can be used as a replacement for the metric balls $\{B_{\widehat{R}_p}(x, s)\}_{(x,s)\in K\times(0,\mathrm{diam}(K,\widehat{R}_p))]}$ in $(K, \widehat{R}_p)$ by virtue of the following lemma, which was obtained in [22, Lemma 6.14] in the case $p = 2$.

**Lemma 6.7** *There exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that for any $(s,x) \in (0,1] \times K$,*

$$B_{\widehat{R}_p}(x, \alpha_1 s) \subseteq U_1(x,s) \subseteq B_{\widehat{R}_p}(x, \alpha_2 s). \tag{6.5}$$

*Proof.* By (5.38), we have $\operatorname{diam}(K_w, \widehat{R}_p) \le \rho_{p,w}^{-1/(p-1)} \operatorname{diam}(K, \widehat{R}_p)$ for any $w \in W_*$, which implies the latter inclusion in (6.5) with $\alpha_2 \in (2\operatorname{diam}(K, \widehat{R}_p), \infty)$ arbitrary. (In particular, $\operatorname{diam}(K_w, \widehat{R}_p) < \alpha_2 s$ for any $w \in \Lambda_s$.) We will show the former inclusion in (6.5) in the rest of this proof. To this end, it suffices to prove that there exists $\alpha_1 \in (0,\infty)$ such that $\widehat{R}_p(x,y) \ge \alpha_1 s$ for any $s \in (0,1]$, any $w, v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x,y) \in K_w \times K_v$. Let $\psi_q := h_{V_0}^{\mathcal{E}_p}\bigl[\mathbf{1}_q^{V_0}\bigr]$ for any $q \in V_0$, where $h_{V_0}^{\mathcal{E}_p}$ denotes the $\mathcal{E}_p$-harmonic extension operator from $V_0$, that is, $\psi_q$ is the unique function in $\mathcal{F}_p$ such that $\psi_q|_{V_0} = \mathbf{1}_q^{V_0}$ and $\mathcal{E}_p(\psi_q) = \min\bigl\{\mathcal{E}_p(v) \bigm| v \in \mathcal{F}_p, v|_{V_0} = \mathbf{1}_q^{V_0}\bigr\}$ (see [23, Theorem 6.13]). Fix $w \in \Lambda_s$ and let $u_w \in C(K)$ be such that, for $\tau \in \Lambda_s$,

$$u_w \circ F_\tau = \begin{cases} 1 & \text{if } \tau = w, \\ \sum_{q \in V_0; F_\tau(q) \in F_w(V_0)} \psi_q & \text{if } \tau \ne w \text{ and } K_\tau \cap K_w \ne \emptyset, \\ 0 & \text{if } K_\tau \cap K_w = \emptyset. \end{cases} \tag{6.6}$$

Since $\Lambda_s$ is a partition of $\Sigma$, we have $u_w \in \mathcal{F}_p$ by (5.37), and

$$\begin{aligned} \mathcal{E}_p(u_w) &= \sum_{\tau \in \Lambda_s} \rho_{p,\tau} \mathcal{E}_p(u_w \circ F_\tau) \\ &= \sum_{\tau \in \Lambda_s \setminus \{w\}; K_\tau \cap K_w \ne \emptyset} \rho_{p,\tau} \mathcal{E}_p\left( \sum_{q \in V_0; F_\tau(q) \in F_w(V_0)} \psi_q \right) \end{aligned} \tag{6.7}$$

by (5.38). Set $\overline{\rho}_p := \max_{i \in S} \rho_{p,i} \in (1,\infty)$ and $c_1 := \max_{q \in V_0} \mathcal{E}_p(\psi_q) \in (0,\infty)$. Then $\rho_{p,\tau}^{-1} \ge (\overline{\rho}_p)^{-1} s^{p-1}$ for any $\tau \in \Lambda_s$. Since $\#\{\tau \in \Lambda_s \mid K_\tau \cap K_w \ne \emptyset\} \le (\#C_{\mathcal{L}})(\#V_0)$ by [25, Lemma 4.2.3], (6.7) together with Hölder's inequality implies that

$$\mathcal{E}_p(u_w) \le (\#C_{\mathcal{L}})(\#V_0)\overline{\rho}_p s^{-p+1} (\#V_0)^{p-1} c_1 =: (\alpha_1 s)^{-(p-1)}. \tag{6.8}$$

For any $v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x,y) \in K_w \times K_v$, we clearly have $u_w(x) = 1$ and $u_w(y) = 0$. Hence

$$\widehat{R}_p(x,y) \ge \mathcal{E}_p(u)^{-1/(p-1)} \ge \alpha_1 s,$$

which proves the desired result. □

Now we can show that $m$ is $d_{\mathrm{f},p}$-Ahlfors regular (see [22, Lemma 6.8] for the case $p = 2$).

**Lemma 6.8** *There exist $c_1, c_2 \in (0,\infty)$ such that for any $x \in K$ and any $s \in (0, 2\operatorname{diam}(K, \widehat{R}_p)]$,*

$$c_1 s^{d_{\mathrm{f},p}} \le m(B_{\widehat{R}_p}(x,s)) \le c_2 s^{d_{\mathrm{f},p}}. \tag{6.9}$$

*Proof.* This is immediate from (6.5), $\#\{\tau \in \Lambda_s \mid K_\tau \cap K_w \neq \emptyset\} \leq (\#C_{\mathcal{L}})(\#V_0)$ (see [25, Lemma 4.2.3]) and $m(K_w) = \rho_{p,w}^{-1/(p-1)}$ (see [25, Corollary 1.4.8]). □

The proof of Lemma 6.7 includes the following capacity upper bound in terms of the $p$-resistance metric $\widehat{R}_p$.

**Proposition 6.9** *There exists $C \in (0,\infty)$ such that for any $x \in K$ and any $s \in (0, 2\operatorname{diam}(K, \widehat{R}_p)]$,*

$$\inf\left\{\mathcal{E}_p(u) \;\middle|\; u \in \mathcal{F}_p, u|_{B_{\widehat{R}_p}(x,\alpha_1 s)} = 1, \operatorname{supp}_K[u] \subseteq B_{\widehat{R}_p}(x, 2\alpha_2 s)\right\} \leq Cs^{-(p-1)}, \tag{6.10}$$

*where $\alpha_1, \alpha_2$ are the constants in* (6.5).

*Proof.* Let $u_w \in \mathcal{F}_p$ be the same function as in the proof of Lemma 6.7 for each $w \in \Lambda_s$. Then $\varphi := \max_{w \in \Lambda_{s,1}(x)} u_w$ satisfies $\varphi|_{U_1(x,s)} = 1$. Since $\operatorname{diam}(K_w, \widehat{R}_p) < \alpha_2 s$, we see from (6.5) that $\operatorname{supp}_K[\varphi] \subseteq B_{\widehat{R}_p}(x, 2\alpha_2 s)$. By (2.3) for $(\mathcal{E}_p, \mathcal{F}_p)$, (6.8) and [25, Lemma 4.2.3], we have $\varphi \in \mathcal{F}_p$ and

$$\mathcal{E}_p(\varphi) \leq \sum_{w \in \Lambda_{s,1}(x)} \mathcal{E}_p(u_w) \leq (\alpha_1 s)^{-(p-1)}(\#C_{\mathcal{L}})(\#V_0) =: Cs^{-(p-1)}. \qquad \square$$

Similar to Lemma 5.20 and Corollary 5.21, we can easily show the next lemma as a consequence of (6.10), and obtain the regularity of $\mathcal{F}_p$.

**Lemma 6.10** *Let $\varepsilon \in (0,1)$ and let $V$ be a maximal $\varepsilon$-net of $(K, \widehat{R}_p)$. Then there exists a family of functions $\{\psi_z\}_{z \in V}$ that satisfies the following properties:*

(i) $\sum_{z \in V} \psi_z \equiv 1$.

(ii) $\psi_z \in \mathcal{F}_p$, $0 \leq \psi_z \leq 1$, $\psi_z|_{B_{\widehat{R}_p}(z,\varepsilon/4)} \equiv 1$ *and* $\operatorname{supp}_K[\psi_z] \subseteq B_{\widehat{R}_p}(z, 5\varepsilon/4)$ *for any $z \in V$;*

(iii) *If $z \in V$ and $z' \in V \setminus \{z\}$, then $\psi_{z'}|_{B_{\widehat{R}_p}(z,\varepsilon/4)} \equiv 0$.*

(iv) *There exists $C \in (0,\infty)$ such that $\mathcal{E}_p(\psi_z) \leq C\varepsilon^{-(p-1)}$ for any $z \in V$.*

**Corollary 6.11** *$(\mathcal{E}_p, \mathcal{F}_p)$ is regular, i.e., $\mathcal{F}_p$ is dense in $(C(K), \|\cdot\|_{\sup})$.*

Next, in order to state a Poincaré-type inequality in this context, we introduce the associated self-similar $p$-energy measures in Proposition 6.12 and a localized version of $\mathcal{F}_p$ in Definition 6.13. Thanks to (5.38), we can define the $p$-energy measures associated with $(\mathcal{E}_p, \mathcal{F}_p)$ by using Kolmogorov's extension theorem. We recall fundamental results on the $p$-energy measures constructed in this way in the following proposition. See [35, Section 9] and [23, Section 5.2] for further details and properties of them.

**Proposition 6.12 (Self-similar $p$-energy measures)** *For each $u \in \mathcal{F}_p$, there exists a unique positive Radon measure $\Gamma_{\mathcal{E}_p}\langle u\rangle$ on $K$ satisfying*

$$\int_K \varphi \, d\Gamma_{\mathcal{E}_p}\langle u\rangle = \mathcal{E}_p(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}_p\left(|u|^{\frac{p}{p-1}}; \varphi\right) \quad \textit{for any } \varphi \in \mathcal{F}_p. \tag{6.11}$$

*Moreover, the following hold:*

(i) $\Gamma_{\mathcal{E}_p}\langle u\rangle(K_w) = \rho_{p,w}\mathcal{E}_p(u \circ F_w)$ *for any* $u \in \mathcal{F}_p$ *and any* $w \in W_*$.

(ii) $\Gamma_{\mathcal{E}_p}\langle\,\cdot\,\rangle(A)^{1/p}$ *is a seminorm on* $\mathcal{F}_p$ *for any* $A \in \mathcal{B}(K)$.

(iii) $\Gamma_{\mathcal{E}_p}\langle u\rangle(K_w \cap K_\tau) = 0$ *for any* $u \in \mathcal{F}_p$ *and any* $w, \tau \in W_*$ *with* $\Sigma_w \cap \Sigma_\tau = \emptyset$.

(iv) $\Gamma_{\mathcal{E}_p}\langle u\rangle(A) = \Gamma_{\mathcal{E}_p}\langle v\rangle(A)$ *for any* $u, v \in \mathcal{F}_p$ *and any* $A \in \mathcal{B}(K)$ *with* $(u-v)|_A \in \mathbb{R}\mathbf{1}_A$.

*Proof.* For the construction of a candidate for $\Gamma_{\mathcal{E}_p}\langle u\rangle$, see [35, Section 9] or [23, Section 5.2]. Then the properties (ii), (iii) and (iv) follow from [35, Proposition 9.3, Corollaries 9.8 and 9.9] since $\#(K_w \cap K_\tau) < \infty$ by $\#V_0 < \infty$ and [25, Proposition 1.3.5-(2)]. We obtain (i) by combining (iii) and [35, Proposition 9.4]. The equality (6.11) is proved in [23, Proposition 5.11], and the uniqueness of a positive Radon measure on $K$ satisfying (6.11) follows from Corollary 6.11 and the uniqueness part of the Riesz–Markov–Kakutani representation theorem (see, e.g., [36, Theorems 2.14 and 2.18]). □

**Definition 6.13** Let $U$ be a non-empty open subset of $K$. We define a linear subspace $\mathcal{F}_{p,\mathrm{loc}}(U)$ of $C(U)$ by

$$\mathcal{F}_{p,\mathrm{loc}}(U) := \left\{ f \in C(U) \;\middle|\; \begin{array}{l} f|_A = f^{\#}|_A \text{ for some } f^{\#} \in \mathcal{F} \text{ for each} \\ \text{relatively compact open subset } A \text{ of } U \end{array} \right\}. \tag{6.12}$$

For each $f \in \mathcal{F}_{p,\mathrm{loc}}(U)$, we further define a positive Radon measure $\Gamma_{\mathcal{E}_p}\langle f\rangle$ on $U$ as follows. We first define $\Gamma_{\mathcal{E}_p}\langle f\rangle(E) := \Gamma_{\mathcal{E}_p}\langle f^{\#}\rangle(E)$ for each relatively compact Borel subset $E$ of $U$, with $A \subseteq U$ and $f^{\#} \in \mathcal{F}_p$ as in (6.12) chosen so that $E \subseteq A$; this definition of $\Gamma_{\mathcal{E}_p}\langle f\rangle(E)$ is independent of a particular choice of such $A$ and $f^{\#}$ by Proposition 6.12-(iv). We then define $\Gamma_{\mathcal{E}_p}\langle f\rangle(E) := \lim_{n\to\infty}\Gamma_{\mathcal{E}_p}\langle f\rangle(E \cap A_n)$ for each $E \in \mathcal{B}(U)$, where $\{A_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of relatively compact open subsets of $U$ such that $\bigcup_{n\in\mathbb{N}} A_n = U$; it is clear that this definition of $\Gamma_{\mathcal{E}_p}\langle f\rangle(E)$ is independent of a particular choice of $\{A_n\}_{n\in\mathbb{N}}$, coincides with the previous one when $E$ is relatively compact in $U$, and gives a Radon measure on $U$.

Now we can prove a Poincaré-type inequality in terms of the $p$-resistance metric.

**Proposition 6.14** **($(p,p)$-Poincaré inequality)** *There exist* $C, A \in (0,\infty)$ *with* $A \geq 1$ *such that for any* $(x,s) \in K \times (0, \mathrm{diam}(K, \widehat{R}_p)]$ *and any* $u \in \mathcal{F}_{p,\mathrm{loc}}(B_{\widehat{R}_p}(x, As))$,

$$\int_{B_{\widehat{R}_p}(x,s)} \left|u(y) - u_{B_{\widehat{R}_p}(x,s)}\right|^p m(dy) \leq C s^{d_{\mathrm{f},p}+p-1}\Gamma_{\mathcal{E}_p}\langle u\rangle(B_{\widehat{R}_p}(x, As)). \tag{6.13}$$

*Proof.* For simplicity, we consider the case $u \in \mathcal{F}_p$. Note that, since $m(K_v \cap K_{v'}) = 0$ for any $v, v' \in W_*$ with $\Sigma_v \cap \Sigma_{v'} = \emptyset$ (see [25, Corollary 1.4.8]),

$$\int_{B_{\widehat{R}_p}(x,\alpha_1 s)} \left|u - u_{B_{\widehat{R}_p}(x,\alpha_1 s)}\right|^p dm \leq \sum_{w\in\Lambda_{s,1}(x)} \int_{K_w} \left|u - u_{U_1(x,s)}\right|^p dm.$$

Let $w \in \Lambda_{s,1}(x)$. For any $(y,z) \in K_w \times U_1(x,s)$, there exist $v^1, v^2, v^3 \in \Lambda_{s,1}(x)$ such that $v^1 = w$, $z \in K_{v^3}$ and $K_{v^i} \cap K_{v^{i+1}} \neq \emptyset$ for each $i \in \{1,2\}$. Let us fix $x_i \in K_{v^i} \cap K_{v^{i+1}}$ and $q_i \in V_0$ so that $x_i = F_{v^i}(q_i)$. Then

$$\begin{aligned}
|u(y) - u(z)|^p &\leq 3^{p-1}\Bigl(\bigl|u(y) - u(x^1)\bigr|^p + \bigl|u(x^1) - u(x^2)\bigr|^p + \bigl|u(x^2) - u(z)\bigr|^p\Bigr) \\
&\leq \bigl(3\,\mathrm{diam}(K, \widehat{R}_p)\bigr)^{p-1} \sum_{i=1}^{3} \rho_{p,v^i}^{-1} \Gamma_{\mathcal{E}_p}\langle u \rangle (K_{v^i}) \\
&\leq C s^{p-1} \Gamma_{\mathcal{E}_p}\langle u \rangle \left(\bigcup_{i=1}^{3} K_{v^i}\right) \leq C s^{p-1} \Gamma_{\mathcal{E}_p}\langle u \rangle (B_{\widehat{R}_p}(x, \alpha_2 s)).
\end{aligned}$$

Therefore, noting that $m(K_w) \lesssim s^{d_{\mathrm{f},p}}$ by (6.5) and (6.9), we have

$$\begin{aligned}
\int_{K_w} \bigl|u(y) - u_{U_1(x,s)}\bigr|^p \, m(dy) &\leq \int_{K_w} \fint_{U_1(x,s)} |u(y) - u(z)|^p \, m(dx) m(dy) \\
&\lesssim s^{d_{\mathrm{f},p}+p-1} \Gamma_{\mathcal{E}_p}\langle u \rangle (B_{\widehat{R}_p}(x, \alpha_2 s)),
\end{aligned}$$

which together with $\sup_{(x,s) \in K \times (0,1]} \#\Lambda_{s,1}(x) < \infty$ (see [25, Lemma 4.2.3]) yields (6.13). □

## 6.2 Estimates on self-similar $p$-energy measures and weak monotonicity

In this subsection, we show localized energy estimates on Korevaar–Schoen $p$-energy forms in terms of their associated self-similar $p$-energy measures and verify $(\mathrm{WM})_{p,\boldsymbol{k}}$. We continue to follow the setting in the previous subsection, i.e., we suppose that Assumption 6.1 holds. We consider $\mathcal{E}_p$ as a $[0,\infty]$-valued functional defined on $L^p(K,m)$ by setting $\mathcal{E}_p(f) := \infty$ for $f \in L^p(K,m) \setminus \mathcal{F}_p$.

Similar arguments as in Propositions 5.23 and 5.25 yield an upper bound on localized Korevaar–Schoen energy functionals in Proposition 6.15 and a lower bound on them in Proposition 6.16 below.

**Proposition 6.15** *There exists $C \in (0,\infty)$ such that for any $E \in \mathcal{B}(K)$, any open neighborhood $E'$ of $\overline{E}^K$ and any $u \in \mathcal{F}_{p,\mathrm{loc}}(E')$,*

$$\limsup_{s \downarrow 0} \int_E \fint_{B_{\widehat{R}_p}(x,s)} \frac{|u(x) - u(y)|^p}{s^{d_{\mathrm{f},p}+p-1}} \, m(dy) m(dx) \leq C \Gamma_{\mathcal{E}_p}\langle u \rangle \bigl(\overline{E}^K\bigr). \tag{6.14}$$

*Moreover, with $C \in (0,\infty)$ the same as in* (6.14)*, for any $f \in L^p(K,m)$,*

$$\sup_{s > 0} \int_K \fint_{B_{\widehat{R}_p}(x,s)} \frac{|f(x) - f(y)|^p}{s^{d_{\mathrm{f},p}+p-1}} \, m(dy) m(dx) \leq C \mathcal{E}_p(f). \tag{6.15}$$

*Proof.* Let $V$ be a relatively compact open subset of $E'$ with $V \supseteq \overline{E}^{K}$ and let $u^{\#} \in \mathcal{F}_p$ satisfy $u^{\#} = u$ $m$-a.e. on $V$. Similar to [35, (7.2)], by using (6.9) and (6.13), we easily see that for any $s \in (0, \infty)$,

$$\int_E \fint_{B_{\widehat{R}_p}(x,s)} \frac{\left|u^{\#}(x) - u^{\#}(y)\right|^p}{s^{d_{\mathrm{f},p}+p-1}}\, m(dy)m(dx) \leq C\Gamma_{\mathcal{E}_p}\langle u^{\#}\rangle\bigl((E)_{\widehat{R}_p,2As}\bigr), \tag{6.16}$$

where $A \in [1, \infty)$ is the constant in (6.13) and $C \in (0, \infty)$ is independent of $x$, $s$ and $f$. We get (6.14) by letting $s \downarrow 0$ since $\Gamma_{\mathcal{E}_p}\langle u^{\#}\rangle\bigl((E)_{\widehat{R}_p,2As}\bigr) = \Gamma_{\mathcal{E}_p}\langle u\rangle\bigl((E)_{\widehat{R}_p,2As}\bigr)$ for any $s \in (0, \infty)$ with $(E)_{\widehat{R}_p,2As} \subseteq V$ by Proposition 6.12-(iv). The estimate (6.15) for $f \in \mathcal{F}_p$ is easily implied by $\Gamma_{\mathcal{E}_p}\langle f\rangle(K) = \mathcal{E}_p(f)$ and (6.16) with $E = K$. For $f \in L^p(K,m) \setminus \mathcal{F}_p$, (6.15) is obvious by $\mathcal{E}_p(f) = \infty$, so the proof is completed. □

**Proposition 6.16** *There exists $C \in (0, \infty)$ such that for any $E \in \mathcal{B}(K)$, any open neighborhood $E'$ of $\overline{E}^{K}$ and any $u \in \mathcal{F}_{p,\mathrm{loc}}(E')$,*

$$\Gamma_{\mathcal{E}_p}\langle u\rangle(E) \leq C \lim_{\delta\downarrow 0} \liminf_{s\downarrow 0} \int_{(E)_{\widehat{R}_p,\delta}} \fint_{B_{\widehat{R}_p}(x,s)} \frac{|u(x) - u(y)|^p}{s^{d_{\mathrm{f},p}+p-1}}\, m(dy)m(dx). \tag{6.17}$$

*Furthermore, with $C \in (0, \infty)$ the same as in* (6.17)*, for any $f \in L^p(K,m)$,*

$$\mathcal{E}_p(f) \leq C \liminf_{s\downarrow 0} \int_K \fint_{B_{\widehat{R}_p}(x,s)} \frac{|f(x) - f(y)|^p}{s^{d_{\mathrm{f},p}+p-1}}\, m(dy)m(dx). \tag{6.18}$$

*Proof.* Let $s \in (0, 1)$ and fix a maximal $r$-net $N_s$ of $(K, \widehat{R}_p)$. Let $\{\psi_{z,s}\}_{z\in N_s}$ be a partition of unity as given in Lemma 6.10 and define $A_s \colon L^p(K,m) \to \mathcal{F}_p$ by $A_s f := \sum_{z\in N_s} f_{B_{\widehat{R}_p}(z,s/4)}\psi_{z,s}$ for $f \in L^p(K,m)$. Then we can easily see that $\lim_{r\to 0}\|A_r f - f\|_{L^p(K,m)} = 0$ and $\sup_{r>0}\|A_r\|_{L^p(K,m)\to L^p(K,m)} < \infty$. Using Proposition 6.12-(iv), we can show that there exists $C_1 > 0$ that is independent of $x$, $s$ and $f$ such that

$$\begin{aligned}&\Gamma_{\mathcal{E}_p}\langle A_s f\rangle\bigl(B_{\widehat{R}_p}(z, 5s/4)\bigr)\\ &\leq C_1 \sum_{w\in N_s\cap B_{\widehat{R}_p}(z,11s/4)} \int_{B_{\widehat{R}_p}(w,3s)} \fint_{B_{\widehat{R}_p}(x,9s)} \frac{|f(x) - f(y)|^p}{s^{d_{\mathrm{f},p}+p-1}}\, m(dy)m(dx),\end{aligned} \tag{6.19}$$

for any small enough $s > 0$. Let us fix $\delta > 0$ and define $N_s(E) := \{z \in N_s \mid E \cap B_{\widehat{R}_p}(z, s) \neq \emptyset\}$. Since $\bigcup_{z\in N_s(E)} \bigcup_{w\in N_s\cap B_{\widehat{R}_p}(z,11s/4)} B_{\widehat{R}_p}(w, 3s) \subseteq (E)_{\widehat{R}_p,\delta}$ for all small enough $s > 0$ and $(K, \widehat{R}_p)$ is metric doubling by Lemma 6.8, we have

$$\begin{aligned}&\Gamma_{\mathcal{E}_p}\langle A_s f\rangle(E)\\ &\leq \sum_{z\in N_s(E)} \Gamma_{\mathcal{E}_p}\langle A_s f\rangle\bigl(B_{\widehat{R}_p}(z, 5s/4)\bigr)\end{aligned}$$

$$\overset{(6.19)}{\le} C \int_{(E)_{\widehat{R}_p,\delta}} \fint_{B_{\widehat{R}_p}(x,9s)} \frac{|f(x)-f(y)|^p}{s^{d_{\mathrm{f},p}+p-1}}\, m(dy)m(dx), \quad f \in L^p(K,m), \tag{6.20}$$

where $C \in (0,\infty)$ is independent of $x, s$ and $f$. Once we get (6.20), the argument in the proof of Proposition 5.25 with minor modifications proves (6.17). Indeed, for $u \in \mathcal{F}_{p,\mathrm{loc}}(E')$, a relatively compact open subset $V$ of $E'$ with $V \supseteq \overline{E}^K$ and $u^\# \in \mathcal{F}_p$ satisfying $u^\# = u$ $m$-a.e. on $V$, we have from Proposition 6.12-(iv) that $\Gamma_{\mathcal{E}_p}\langle A_s u^\# \rangle(E) = \Gamma_{\mathcal{E}_p}\langle A_s u \rangle(E)$ if $s$ is sufficiently small. Then similar arguments using Mazur's lemma as in the proof of Proposition 5.25 implies (6.17) and (6.18). □

Now we can identify $\mathcal{F}_p$ as the $(1,p)$-Korevaar–Schoen–Sobolev space.

**Theorem 6.17** *Let $s_p$ and $\mathrm{KS}^{1,p}(K,\widehat{R}_p,m)$ be as defined in Example 3.14 with $\widehat{R}_p$ in place of $d$, and define $\boldsymbol{k} = \{k_r\}_{r>0}$ by*

$$k_r(x,y) := \frac{\mathbf{1}_{B_{\widehat{R}_p}(x,r)}(y)}{r^{d_{\mathrm{f},p}+p-1} m(B_{\widehat{R}_p}(x,r))}, \quad x,y \in K.$$

*Then $s_p = (d_{\mathrm{f},p}+p-1)/p$, $\mathcal{F}_p = \mathrm{KS}^{1,p}(K,\widehat{R}_p,m)$, and $(\mathrm{WM})_{p,\boldsymbol{k}}$ holds. Moreover, there exists $C \in [1,\infty)$ such that*

$$C^{-1} \sup_{r>0} J^{\boldsymbol{k}}_{p,r}(f) \le \mathcal{E}_p(f) \le C \liminf_{r\downarrow 0} J^{\boldsymbol{k}}_{p,r}(f) \quad \text{for any } f \in L^p(K,m). \tag{6.21}$$

*Proof.* We have $\mathcal{F}_p = B^{(d_{\mathrm{f},p}+p-1)/p}_{p,\infty}$ and (6.21) by (6.15) and (6.18). In particular, $s_p \ge (d_{\mathrm{f},p}+p-1)/p$. Let $s > (d_{\mathrm{f},p}+p-1)/p$ and let $f \in \mathcal{F}_p \setminus \mathbb{R}\mathbf{1}_K$, which exists by (6.10). Let $A_r \colon L^p(K,m) \to \mathcal{F}_p$ be the same operator as in the proof of Proposition 6.16 for each $r \in (0,1)$. Then, by (6.20) with $E = K$, for any $r \in (0,1)$ and $f \in L^p(K,m)$,

$$\frac{r^{d_{\mathrm{f},p}+p-1}}{r^{sp}} \mathcal{E}_p(A_r f) \le C \int_K \fint_{B_{\widehat{R}_p}(x,9r)} \frac{|f(x)-f(y)|^p}{r^{sp}}\, m(dy)m(dx), \tag{6.22}$$

where $C > 0$ is independent of $f$ and $r$. Clearly, $\sup_{r>0} \mathcal{E}_p(A_r f) > 0$ and $r^{d_{\mathrm{f},p}+p-1-sp} \to \infty$ as $r \downarrow 0$. Hence we obtain $s \ge s_p$ since $f \notin B^s_{p,\infty}$ by (6.22). This implies that $(d_{\mathrm{f},p}+p-1)/p \ge s_p$. In particular, we obtain $\mathcal{F}_p = \mathrm{KS}^{1,p}(K,\widehat{R}_p,m)$. Also, $(\mathrm{WM})_{p,\boldsymbol{k}}$ follows from (6.15) and (6.18). □

Unfortunately, it is not clear whether Korevaar–Schoen $p$-energy forms $(\mathcal{E}^{\mathrm{KS}}_p, \mathcal{F}_p)$ on $(K,\widehat{R}_p,m)$, which exist by Theorems 6.17 and 3.8 (recall Example 3.14), are self-similar or not. However, we can construct a self-similar $p$-resistance form on $\mathcal{L}$ by the same argument as in the proof of Theorem 5.38. Recall that $\mathcal{F}_p \cap C(K) = \mathcal{F}_p$ is dense both in $(C(K), \|\cdot\|_{\sup})$ and in $\mathcal{F}_p$ by Proposition 6.4-(3) and Corollary 6.11.

**Theorem 6.18** *For each $n \in \mathbb{N}$, define $\boldsymbol{k}^{(n)} = \{k_r^{(n)}\}_{r>0}$ by*

$$k_r^{(n)}(x,y) \coloneqq \frac{1}{n+1}\sum_{l=0}^{n}\sum_{w\in W_l}\rho_{p,w}^{(2d_{\mathrm{f},p}+p-1)/(p-1)}\frac{\mathbf{1}_{A_{w,r}}(x,y)}{r^{2d_{\mathrm{f},p}+p-1}}, \quad x,y \in K,$$

*where $A_{w,r} \coloneqq \left\{(x,y) \in K_w \times K_w \;\middle|\; \widehat{R}_p(F_w^{-1}(x),F_w^{-1}(y)) < r\right\}$. Then $\boldsymbol{k}^{(n)}$ is asymptotically local,* $(\mathrm{WM})_{p,\boldsymbol{k}^{(n)}}$ *holds, $B_{p,\infty}^{\boldsymbol{k}^{(n)}} = \mathcal{F}_p$, and for any sequence $\{(\mathcal{E}_p^{\boldsymbol{k}^{(n)}},\mathcal{F}_p)\}_{n\in\mathbb{N}}$ with $(\mathcal{E}_p^{\boldsymbol{k}^{(n)}},\mathcal{F}_p)$ a $\boldsymbol{k}^{(n)}$-Korevaar–Schoen $p$-energy form on $(K,m)$ for each $n \in \mathbb{N}$, there exists a sequence $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_j < n_{j+1}$ for any $j \in \mathbb{N}$ such that the following limit exists in* $[0,\infty)$ *for any $u \in \mathcal{F}_p$:*

$$\breve{\mathcal{E}}_p^{\mathrm{KS}}(u) \coloneqq \lim_{j\to\infty}\mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u). \tag{6.23}$$

*Moreover, for any such $\{\mathcal{E}_p^{\boldsymbol{k}^{(n)}}\}_{n\in\mathbb{N}}$ and $\{n_j\}_{j\in\mathbb{N}}$, the functional $\breve{\mathcal{E}}_p^{\mathrm{KS}}\colon \mathcal{F}_p \to [0,\infty)$ defined by* (6.23) *satisfies the following properties:*

(a) *$(\breve{\mathcal{E}}_p^{\mathrm{KS}},\mathcal{F}_p)$ is a self-similar $p$-resistance form on $\mathcal{L}$ with weight $(\rho_{p,i})_{i\in S}$.*

(b) *For any $u \in \mathcal{F}_p$,*

$$C^{-1}\mathcal{E}_p(u) \le \breve{\mathcal{E}}_p^{\mathrm{KS}}(u) \le C\mathcal{E}_p(u),$$

*where $C \in [1,\infty)$ is the constant in* (6.21).

(c) *For any $u,v \in \mathcal{F}_p$, $\{\mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u;v)\}_{j\in\mathbb{N}}$ is convergent in $\mathbb{R}$ and*

$$\breve{\mathcal{E}}_p^{\mathrm{KS}}(u;v) = \lim_{j\to\infty}\mathcal{E}_p^{\boldsymbol{k}^{(n_j)}}(u;v). \tag{6.24}$$

(d) *Theorem 3.8-*(c)*,*(d)*,*(e) *with $(\breve{\mathcal{E}}_p^{\mathrm{KS}},\mathcal{F}_p)$ in place of $(\mathcal{E}_p^{\boldsymbol{k}},B_{p,\infty}^{\boldsymbol{k}})$ hold.*

(e) *$\breve{\mathcal{E}}_p^{\mathrm{KS}}(u\circ T) = \breve{\mathcal{E}}_p^{\mathrm{KS}}(u)$ for any $u \in \mathcal{F}_p$ and any $T \in \mathcal{G}$ (recall* (6.4)*).*

In addition, we obtain the $p$-energy measures associated with the $p$-resistance form $(\breve{\mathcal{E}}_p^{\mathrm{KS}},\mathcal{F}_p)$ in the same way as in Theorem 5.39. (See also [23, Sections 4 and 5] for other basic properties. As mentioned before Theorem 5.39, we do not know whether Theorem 6.19-(c) below holds for general self-similar $p$-resistance forms.)

**Theorem 6.19** *Let $(\mathcal{E}_p^{\boldsymbol{k}^{(n)}},\mathcal{F}_p)$ be any $\boldsymbol{k}^{(n)}$-Korevaar–Schoen $p$-energy form on $(K,m)$ for each $n \in \mathbb{N}$, let $\{n_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$ be any sequence as in Theorem 6.18, and let $(\breve{\mathcal{E}}_p^{\mathrm{KS}},\mathcal{F}_p)$ be the $p$-resistance form on $K$ defined by* (6.23)*. Then for any $u \in \mathcal{F}_p$, there exists a unique positive Radon measure $\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle$ on $K$ such that*

$$\int_K \varphi\, d\breve{\Gamma}_p^{\mathrm{KS}}\langle u\rangle = \breve{\mathcal{E}}_p^{\mathrm{KS}}(u;u\varphi) - \left(\frac{p-1}{p}\right)^{p-1}\breve{\mathcal{E}}_p^{\mathrm{KS}}\bigl(|u|^{\frac{p}{p-1}};\varphi\bigr) \quad \textit{for any } \varphi \in \mathcal{F}_p. \tag{6.25}$$

*Moreover, the following hold:*

(a) *Let $\varphi\colon K \to [0,\infty)$ be a Borel measurable function with $\|\varphi\|_{\sup} < \infty$. Then $(\int_K \varphi\, d\breve{\Gamma}_p^{\mathrm{KS}}\langle\,\cdot\,\rangle,\mathcal{F}_p)$ is a $p$-energy form on $(K,m)$ satisfying* $(\mathrm{GC})_p$.

(b) *Theorem 4.6, with $\mathcal{F}_p$ and $\check{\Gamma}_p^{\boldsymbol{k}}$ in place of $\mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and $\Gamma_p^{\boldsymbol{k}}$ respectively, holds. In particular, for any $u, v \in \mathcal{F}_p$,*

$$\check{\Gamma}_p^{\mathrm{KS}}\langle u; v\rangle(A) := \frac{1}{p} \left.\frac{d}{dt}\check{\Gamma}_p^{\mathrm{KS}}\langle u + tv\rangle(A)\right|_{t=0}, \quad A \in \mathcal{B}(K), \tag{6.26}$$

*defines a signed Borel measure on $K$ such that $\check{\Gamma}_p^{\mathrm{KS}}\langle u; v\rangle(K) = \check{\mathcal{E}}_p^{\mathrm{KS}}(u; v)$ and $\check{\Gamma}_p^{\mathrm{KS}}\langle u; u\rangle = \check{\Gamma}_p^{\mathrm{KS}}\langle u\rangle$. Furthermore, for any $u, v \in \mathcal{F}_p$ and any $\varphi \in C(K)$,*

$$\int_K \varphi \, d\check{\Gamma}_p^{\mathrm{KS}}\langle u; v\rangle = \lim_{j\to\infty} \int_K \varphi \, d\Gamma_p^{\boldsymbol{k}^{(n_j)}}\langle u; v\rangle. \tag{6.27}$$

(c) *Theorem 4.8-*(a),(b)*, with $\mathcal{F}_p$ and $\check{\Gamma}_p^{\mathrm{KS}}$ in place of $\mathcal{D}_{p,\infty}^{\boldsymbol{k},b}$ and $\Gamma_p^{\boldsymbol{k}}$ respectively, hold.*

(d) *Theorems 4.9, 4.10 and 4.11, with $\mathcal{F}_p$ and $\check{\Gamma}_p^{\mathrm{KS}}$ in place of $B_{p,\infty}^{\boldsymbol{k}} \cap C_b(K)$ and $\Gamma_p^{\boldsymbol{k}}$ respectively, hold.*

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## References

1. R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Pure and Applied Mathematics 140, second edition, Academic Press, Amsterdam, 2013. MR2424078
2. R. Alvarado, P. Hajłasz and L. Malý, A simple proof of reflexivity and separability of $N^{1,p}$ Sobolev spaces, *Ann. Fenn. Math.* **48** (2023), no. 1, 255–275. MR4559212
3. P. Alonso Ruiz and F. Baudoin, Gagliardo-Nirenberg, Trudinger-Moser and Morrey inequalities on Dirichlet spaces, *J. Math. Anal. Appl.* **497** (2021), no. 2, Paper No. 124899. MR4196573
4. P. Alonso Ruiz and F. Baudoin, Korevaar-Schoen $p$-energies and their $\Gamma$-limits on Cheeger spaces, preprint, 2024. arXiv:2401.15699
5. F. Baudoin, Korevaar-Schoen-Sobolev spaces and critical exponents in metric measure spaces, preprint, 2022. arXiv:2207.12191
6. F. Baudoin and L. Chen, Sobolev spaces and Poincaré inequalities on the Vicsek fractal, *Ann. Fenn. Math.* **48** (2023), no. 1, 3–26. MR4494039
7. S. Cao, Q. Gu and H. Qiu, $p$-energies on p.c.f. self-similar sets, *Adv. Math.* **405** (2022), Paper No. 108517, 58 pp. MR4437617
8. D. Chang, J. Gao, Z. Yu, J. Zhang, Weak monotonicity property of Korevaar-Schoen norms on nested fractals, preprint, 2023. arXiv:2310.15060.
9. Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory*, London Mathematical Society Monographs Series, vol. 35, Princeton University Press, Princeton, NJ, 2012. MR2849840
10. J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40** (1936), no. 3, 396–414. MR1501880
11. G. Dal Maso, *An introduction to* $\Gamma$*-convergence*, Progress in Nonlinear Differential Equations and their Applications, vol. 8, Birkhäuser Boston, Inc., Boston, MA, 1993. MR1201152

12. M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Second revised and extended edition, de Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 2011. MR2778606
13. J. Gao, Z. Yu and J. Zhang, Convergence of $p$-Energy Forms on Homogeneous p.c.f Self-Similar Sets, *Potential Analysis* **59** (2023), no. 4, 1851–1874. MR4684378
14. J. Gao, Z. Yu and J. Zhang, Heat kernel-based $p$-energy norms on metric measure spaces, preprint, 2023. arXiv:2303.10414.
15. W. Górny, Bourgain-Brezis-Mironescu approach in metric spaces with Euclidean tangents. *J. Geom. Anal.* **32** (2022), no. 4, Paper No. 128, 22 pp. MR4375837
16. P. Hajłasz, Sobolev spaces on an arbitrary metric space, *Potential Anal.* **5** (1996), no. 4, 403–415. MR1401074
17. P. Hajłasz, Sobolev spaces on metric-measure spaces, in: *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, 173–218. Contemp. Math., 338 *American Mathematical Society, Providence, RI*, 2003. MR2039955
18. P. Hajłasz and P. Koskela, Sobolev meets Poincaré, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), no. 10, 1211–1215. MR1336257
19. J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson. Sobolev spaces on metric measure spaces. An approach based on upper gradients. *New Mathematical Monographs*, **27**. Cambridge, University Press, Cambridge, 2015. MR3363168
20. P. E. Herman, R. Peirone and R. S. Strichartz, $p$-energy and $p$-harmonic functions on Sierpinski gasket type fractals, *Potential Anal.* **20** (2004), no.2, 125–148. MR2032945
21. J. Hu, Y. Ji and Z. Wen, Hajłasz-Sobolev type spaces and $p$-energy on the Sierpinski gasket, *Acad. Sci. Fenn. Math.* **30** (2005), no. 1, 99–111. MR2140300
22. N. Kajino and M. Murugan, On the conformal walk dimension: quasisymmetric uniformization for symmetric diffusions, *Invent. Math.* **231** (2023), no. 1, 263–405. MR4526824
23. N. Kajino and R. Shimizu, Contraction properties and differentiability of $p$-energy forms with applications to nonlinear potential theory on self-similar sets, preprint, 2024.
24. J. Kigami, Harmonic calculus on p.c.f. self-similar sets, *Trans. Amer. Math. Soc.* **335** (1993), no. 2, 721–755. MR1076617
25. J. Kigami, Analysis on fractal, *Cambridge Tracts in Mathematics* **143**, Cambridge University Press, Cambridge, 2001. MR1840042
26. J. Kigami, Volume doubling measures and heat kernel estimates on self-similar sets, *Mem. Amer. Math. Soc.* **199**, no. 932 (2009). MR2512802
27. J. Kigami, Resistance forms, quasisymmetric maps and heat kernel estimates, *Mem. Amer. Math. Soc.* **216** (2012), no. 1015. MR2919892
28. J. Kigami, Geometry and analysis of metric spaces via weighted partitions, *Lecture Notes in Mathematics* **2265**, Springer, Cham, 2020. MR4175733
29. J. Kigami, Conductive homogeneity of compact metric spaces and construction of $p$-energy, *Mem. Eur. Math. Soc.* Vol. 5, European Mathematical Society (EMS), Berlin, 2023. MR4615714
30. J. Kigami and Y. Ota, Conductive homogeneity of regular-polygon based self-similar sets, in preparation.
31. N. J. Korevaar, R. M. Schoen, Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.* **1** (1993), no. 3-4, 561–659. MR1266480
32. P. Koskela and P. MacManus, Quasiconformal mappings and Sobolev spaces. *Studia Math.* **131** (1998), no. 1, 1–17. MR1628655
33. P. Lahti, A. Pinamonti and X. Zhou, A characterization of BV and Sobolev functions via nonlocal functionals in metric spaces, preprint, 2022. arXiv:2207.02488
34. P. Lindqvist, Notes on the $p$-Laplace equation, Rep. Univ. Jyväskylä Dep. Math. Stat., 102, *University of Jyväskylä, Jyväskylä,*, 2006. MR2242021
35. M. Murugan and R. Shimizu, First-order Sobolev spaces, self-similar energies and energy measures on the Sierpiński carpet, preprint, 2023. arXiv:2308.06232
36. W. Rudin, Real and complex analysis. Third edition. *McGraw-Hill Book Co., New York* 1987. MR0924157

37. K. Sasaya, Systems of dyadic cubes of complete, doubling, uniformly perfect metric spaces without detours, *Colloq. Math.* **172** (2023), no. 1, 49–64. MR4565993
38. R. Shimizu, Construction of $p$-energy and associated energy measures on Sierpiński carpets, *Trans. Amer. Math. Soc.* **377** (2024), no. 2, 951–1032. MR4688541
39. M. Yang, Korevaar–Schoen spaces on Sierpiński carpets, preprint, 2023. arXiv:2306.09900.