# Algorithm and abstraction in formal mathematics

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**Abstract.** I analyse differences in style between traditional prose mathematics writing and computer-formalised mathematics writing, presenting five case studies. I note two aspects where good style seems to differ between the two: in their incorporation of computation and of abstraction. I argue that this reflects a different mathematical aesthetic for formalised mathematics.

## 1 Introduction

In the last twenty years, formalisation—building up proofs as line-by-line logical deductions from the axioms of mathematics, with the help of specialised computer systems<sup>1</sup>—has seen increasing interest from mathematicians. The rapidly increasing coverage of the mathematical literature in these systems is very much a social process: their mathematical libraries are built collaboratively by hundreds of people, and code contributed by one person will be reviewed in detail by another, and often thoroughly re-worked a year later by a third.

In this kind of human and social process, culture develops spontaneously. The back-and-forth of discussion in this process includes frequent comment on a formalised proof's beauty, elegance, cleverness, and other abstract properties generally associated with mathematical aesthetics. The communities of mathematicians doing this work consider computer-formalised proofs to be, not simply utilitarian certificates for the correctness of logical claims, but a fully-fledged medium for mathematical exposition.

In this article I describe (necessarily very subjectively) some aspects of this aesthetic of computer-formalised mathematics writing. Much of this aesthetic is inherited from traditional prose mathematics writing, on which there is a vast literature [2,4,15,16,22,25,30,31,32]. I therefore focus on cases in which good style in formalised mathematics seems to differ from good style in traditional prose mathematics. I present five case studies,<sup>2</sup> grouped by theme: how to integrate computation (Section 2) and how much use to make of abstraction (Section 3).

## 2 Computation

A faithful computer-formalised translation of a traditional prose proof will commonly use computation "in the small:" a proof step which seems obvious to

<sup>&</sup>lt;sup>1</sup> Examples include Agda, Coq, Lean, HOL Light, Isabelle, Metamath and Mizar.

<sup>&</sup>lt;sup>2</sup> Disproportionately drawn from Lean's [27,26] Mathlib [8], of which I am a maintainer.

humans often represents a whole chain of strict logical reasoning, and in most modern systems automation is used to help construct such chains.

Interestingly, such a translation will sometimes also use computation "in the large:" several notable formalisations' [11,14,21] targets are theorems whose published proofs rely on the result reported by a computer program.

So what about using computation "in the middle?" In this section I explore proofs where there is no absolute need to outsource a calculation to computer—and where, in traditional writing, simple inertia would prevent one from doing this—but which are arguably improved by increased reliance on computation.

#### 2.1 Classification of wallpaper groups

My first example arises in classifying the 17 wallpaper groups. This classification is heavily dependent on case analysis, one branch of which is to consider wallpaper groups which contain translations and rotations but no reflections. These can be classified according to the orbit types of centres of symmetry. For example, one of these wallpaper groups, which in our classification we will associate to the tuple (2,3,6), has three centres of symmetry, at which the stabilisers are generated by rotations of  $\frac{2\pi}{2}$ ,  $\frac{2\pi}{3}$ , and  $\frac{2\pi}{6}$ .

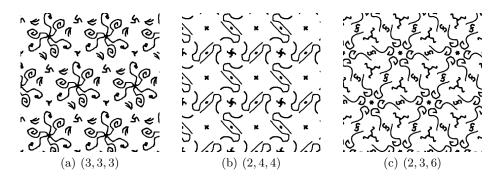


Fig. 1. The wallpaper groups containing rotations but no reflections. (Images via [10].)

The following arithmetic lemma classifies the possible tuples which can arise. The wallpaper groups associated to these tuples are depicted in Fig. 1.

**Lemma 1.** Let  $2 \le p \le q \le r$  be natural numbers, with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1. \tag{*}$$

Then (p, q, r) is one of (3, 3, 3), (2, 4, 4), (2, 3, 6).

I first present a traditional prose proof lifted from a textbook [9].

*Proof.* We get p=q=r=3 if all of  $\frac{1}{p}$ ,  $\frac{1}{q}$  and  $\frac{1}{r}$  have their mean value of  $\frac{1}{3}$ . Otherwise p must be 2.

If r and q have their mean value of  $\frac{1}{4}$ , we get p=2, q=r=4. If not, q must be 3, and r is forced to be 6, by  $(\star)$ .

Secondly, I describe a proof of this lemma that I wrote in Lean with Anne Baanen. I am trying to translate the Lean code fairly literally.

*Proof.* The inequalities  $0 < \frac{1}{r} \le \frac{1}{q} \le \frac{1}{2}$  and the equality  $(\star)$  yield that

$$\frac{1}{3} \le \frac{1}{p} \le \frac{1}{2} \tag{1}$$

$$\frac{1}{2} \left[ 1 - \frac{1}{p} \right] \le \frac{1}{q} < \frac{1}{2}$$

$$\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q}.$$
(2)

$$\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q}.\tag{3}$$

There are finitely many natural numbers p satisfying (1); case-split on these. For each of these there are finitely many natural numbers q satisfying (2); case-split on these. For each of these, r can be determined from (3).

There is an algorithm implicit in these proofs. The second (formalised) proof looks almost like a recipe for cooking the first (textbook) proof: it describes the steps to be carried out, rather than actually performing those steps visibly for the reader (i.e. documenting the available choices at each case split).

This is very typical: as mentioned, proof-writing in systems such as Lean frequently invokes "tactics," small computer programs to construct parts of proofs. But once we start to describe proofs via the recipes which would construct them, there is no need to stick to the original recipe. This was noted by Hales et al. [14]:

In the original, computer calculations were a last resort after as much was done by hand as feasible. In the [formalisation], the use of computer has been fully embraced. As a result, many laborious lemmas of the original proof can be automated or eliminated altogether.

I will argue that an aesthetically pleasing formal proof is one which has a short and simple recipe. As the next two examples will show, this is not the same thing as a proof which is itself short and simple.

#### The Kochen-Specker paradox

I next consider a theorem from quantum mechanics.

Theorem 1 (Kochen-Specker [24]). There does not exist a boolean (say red or green) colouring of the vectors in  $\mathbb{R}^3$ , such that all triples  $u, v, w \in \mathbb{R}^3$  of nonzero mutually-orthogonal vectors are coloured green, red, red in some order.

I will discuss a streamlined proof due to Peres [28]. The approach is to deduce a contradiction from the colouring of the following 33 nonzero vectors<sup>3</sup> in  $\mathbb{R}^3$ :

<sup>&</sup>lt;sup>3</sup> Down from 117 vectors in the original Kochen-Specker proof. Following Peres' notation,  $\overline{1}$  is shorthand for -1, 2 is shorthand for  $\sqrt{2}$ , and  $\overline{2}$  is shorthand for  $-\sqrt{2}$ .

The basic driver of the proof is that, once enough of the 33 vectors have been coloured, the colours of the rest can be determined greedily: a vector orthogonal to a green vector must be red, and a vector orthogonal to two orthogonal red vectors must be green.

Here is an outline of Peres' proof [28] of the impossibility.

*Proof.* We can determine the colours of some vectors without loss of generality.

By the symmetry	and by the known facts	we can assume
choice of $z$ -axis		001 green; 100, 010 red
choice of $x$ vs $-x$	010 red	$101 \text{ green; } \overline{1}01 \text{ red}$
choice of $y$ vs $-y$	100 red	$011 \text{ green}; 0\overline{1}1 \text{ red}$
choice of $x$ vs $y$	001 green, thus 110 red	$1\overline{1}2$ green; $\overline{1}12$ red

Now a suitable greedy sequence of deductions [depicted in Fig. 2, written out explicitly in the original] forces a contradiction.

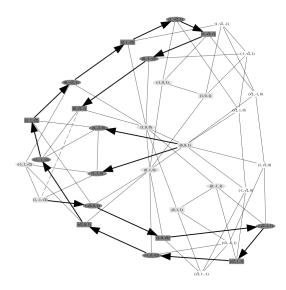


Fig. 2. Peres' argument [28] for the impossibility of a suitable boolean colouring of the 33 vectors (4). Edges connect orthogonal vectors. Oval (respectively, rectangular) labels denote vectors found to be red (resp. green). Light-shaded (resp. dark-shaded) labels denote vectors whose colour is chosen by an initial symmetry argument (resp. is forced as a consequence of adjacent vectors' colours). Arrows indicate the order in which these deductions occur. The dotted edge indicates two orthogonal vectors which are both green, contradiction.

For comparison, here is an outline of a proof formalised by John Harrison in HOL Light in 2005.<sup>4</sup> It is really a brute force search.

*Proof.* Perform the following binary search: split on a vector whose colour is not yet known; then in each case (red or green) greedily make all possible deductions. Stop if a contradiction is found. Recurse if not.

The result of this process is that every branch terminates in a contradiction.

The most notable difference from Peres' prose proof is to abandon his symmetry argument (which reduces to only one configuration on which the greedy algorithm need be run) and instead just run the greedy argument at each stage of a binary search.<sup>5</sup> In effect, both arguments amount to the implementation of a search algorithm. The search algorithm used in the formalised proof is very simple, whereas the symmetry considerations incorporated in the original proof can be considered as baroque optimisations to the search algorithm to get its "runtime" within the scale of human readability. Harrison [19, section 3] defends this choice in his formalisation on the grounds of convenience: the simpler algorithm is easier to implement. But I argue that it is also defensible on the grounds of aesthetics: the simpler algorithm is easier for the reader to grasp.<sup>6</sup>

## 2.3 Multiplication of Chebyshev polynomials

My last example on the theme of computation highlights a different kind of computation. Let  $T_n(x)$  denote the *n*-th Chebyshev polynomial of the first kind. Recall these polynomials satisfy a recurrence relation

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x).$$

Lemma 2 (Multiplication formula for Chebyshev polynomials). For all natural numbers m and k,  $2T_mT_{m+k} = T_{2m+k} + T_k$ .

A purely algebraic<sup>7</sup> proof of this lemma is necessarily an induction. The inductive step of the scheme that works has us prove a statement for m + 2,

$$\forall k : \mathbb{N}, 2T_{m+2}T_{(m+2)+k} = T_{2(m+2)+k} + T_k,$$

<sup>&</sup>lt;sup>4</sup> https://github.com/jrh13/hol-light/blob/e736197/Tutorial/Custom\_tactics.ml

<sup>&</sup>lt;sup>5</sup> A second difference is that when running the greedy algorithm from a partial colouring produces a contradiction, the formalised version does not write out the *certificate*: an explicit path of deductions leading to the concluding contradiction. But this is less controversial. In the case of the configuration in Fig. 2, when the path of deductions is written out, it does not appear to contain any particular insight. See Harrison [17, section 3.4] for a similar example.

<sup>&</sup>lt;sup>6</sup> Gonthier et al. [12, section 4.3] report a similar example, in which an appeal to a logical decision procedure produces an "intellectually more satisfying" proof than the original argument involving a detailed combinatorial case analysis.

<sup>&</sup>lt;sup>7</sup> There is an alternative approach using trigonometric identities.

given the corresponding statements for m and m+1.

The following is how we have been trained to write rigorous proofs of equalities in mathematics articles: as a transitive chain of reasoning.

*Proof.* Indeed,

$$\begin{split} 2T_{m+2}T_{m+k+2} &= 2[2xT_{m+1} - T_m]T_{m+k+2} \\ &= 2x[2T_{m+1}T_{(m+1)+(k+1)}] - 2T_mT_{m+(k+2)} \\ &= 2x[T_{2(m+1)+(k+1)} + T_{k+1}] - [T_{2m+(k+2)} + T_{k+2}] \\ &= [2xT_{2m+k+3} - T_{2m+k+2}] + [2xT_{k+1} - T_{k+2}] \\ &= T_{2m+k+4} + T_k. \end{split}$$

Halmos [15, section 16], writing long before interactive proof assistants were widespread, calls out the "proof that consists of a chain of expressions separated by equal signs" as an example of lazy writing,

unhelpful [symbolism] that merely reports the result of the act and leaves the reader to guess how they were obtained,

and advocates for replacing such proofs by a "recipe for action" (a metaphor I already borrowed in Section 2.1). Here is an alternate proof of the Chebyshev lemma which precisely consists of such a recipe. This approach follows a formalisation of mine, contributed to Mathlib.<sup>8</sup>

*Proof.* Indeed, two applications of the inductive hypothesis give

$$2T_{m+1}T_{(m+1)+(k+1)} = T_{2(m+1)+(k+1)} + T_{k+1} \tag{*}_1$$

$$2T_m T_{m+(k+2)} = T_{2m+(k+2)} + T_{k+2} \tag{*}_2$$

and three applications of the recurrence relation give

$$T_{m+2} = 2xT_{m+1} - T_m \tag{*1}$$

$$T_{(2m+k+2)+2} = 2xT_{(2m+k+2)+1} - T_{2m+k+2} \tag{*2}$$

$$T_{k+2} = 2xT_{k+1} - T_k \tag{*3}$$

A Gröbner basis computation shows that LHS – RHS of the desired result,

$$2T_{m+2}T_{m+k+2} = T_{2m+k+4} + T_k$$

is in the ideal generated by LHS – RHS of  $(\star_1)$ ,  $(\star_2)$ ,  $(*_1)$ ,  $(*_2)$ ,  $(*_3)$ .

 $<sup>^8</sup>$  Mathlib [8], RingTheory/Polynomial/Chebyshev, line 209

<sup>&</sup>lt;sup>9</sup> This ability to send a computation to the Gröbner basis algorithm is a standard offering in formalisation languages [18,6,29]. In Lean this is performed via an external call to Sage; it was implemented by Dhruv Bhatia and Rob Lewis in 2022.

In a traditional prose proof, there is a high barrier to outsourcing this kind of computation to a specialised computer algebra system. The code performing the calculation must set up (under some names) the 11 variables  $^{10}$ 

the five polynomials in these 11 variables which generate the ideal, and a sixth polynomial whose membership in the ideal is to be checked. This process is tedious and error-prone; it will demand close attention from both author and reader. By contrast, when formalising, there is no such barrier: the problem statement is already available in a suitable electronic format.

The point is not just that in formalisation the second proof becomes feasible; I argue it is also more elegant. It is easier to grasp at high level: it is clear upfront what facts are being used, and the reader can check by eye that the goal appears to be within the scope of the Gröbner basis algorithm as run on these facts. Its black-boxing of the routine algorithm also makes the ideas more transparent—in this case, the choice of specialisations of the two inductive hypotheses. <sup>11</sup>

In summary, I argue that in formalisation the threshold for switching to full automation should lower, with many "mid-sized" computations automated away.

## 3 Abstraction

I now turn to the second realm in which I argue that there is a stylistic difference between prose and formal mathematics: the question of abstraction.

The principle that every mathematical argument should be generalised to exactly its proper context dates at least to Bourbaki [5, section 2]:

$$2T_{m+1}T_{(m+1)+(k+b)} = T_{2(m+1)+(k+b)} + T_{k+b},$$
  

$$2T_mT_{m+(k+c)} = T_{2m+(k+c)} + T_{k+c}.$$

In order for there to be a nontrivial polynomial relation among these, the goal

$$2T_{m+2}T_{(m+2)+k} = T_{2(m+2)+k} + T_k,$$

and some uses of the recurrence, we need to arrange that the T-indices which appear,

$$m+\{2,1,0\},\quad k+\{0,b,c\},\quad m+k+\{2,b+1,c\},\quad 2m+k+\{4,b+2,c\},$$

are all either (a) sets of three consecutive numbers (in which case the recurrence relation provides an identity connecting them) or (b) all the same. This forces c = 2, and that forces b = 1, leading to the instantiations  $(\star_1)$ ,  $(\star_2)$  chosen.

<sup>&</sup>lt;sup>10</sup> We normalise indices before the computation.

Indeed, let k+b be the chosen instantiation of the (m+1)-inductive hypothesis and k+a that of the m-inductive hypothesis:

Where the superficial observer sees only two, or several, quite distinct theories, lending one another "unexpected support" ... [we advocate] to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them ... and to put them in their proper light.

This idea was profoundly influential. But though widely agreed on in principle, it is not followed universally in practice. For example, Halmos [15] advises writers,

The observation that a proof proves something a little more general than it was invented for can frequently be left to the reader.

The main reason is psychological: abstractions seem to be a cognitive barrier for readers. A secondary, related reason is practical: you can't expect your reader to be confident in the application of an abstraction that she has never seen before, and so it's courteous to her to specialise it.

In formalised mathematics the trade-offs are different. The practical obstruction to abstraction nearly disappears, <sup>12</sup> though the psychological one remains. Moreover, as the examples in this section will explore, the usual arguments in favour of abstraction apply somewhat more strongly than in prose mathematics writing. All told, formal mathematics favours decidedly more use of abstraction.

#### 3.1 Lax-Milgram theorem

I first consider the Lax–Milgram theorem, a functional analysis result which turns up in the standard approach to linear elliptic partial differential equations. Let H be a real Hilbert space,  $B: H \times H \to \mathbb{R}$  a bilinear form.

**Theorem 2** (Lax–Milgram). Suppose there exist constants  $\alpha, \beta > 0$  so that

- $\ (\textit{boundedness}) \ \textit{for all} \ u,v \in H, \ |B[u,v]| \leq \alpha \|u\| \|v\|$
- (coercivity) for all  $u \in H$ ,  $B[u, u] \ge \beta ||u||^2$ .

Then for each  $f \in H^*$ , there exists a unique  $u \in H$  so that for all  $v \in H$ , B[u,v] = f(v).

The proof of this theorem begins by constructing a bounded linear map  $A: H \to H$  such that for all  $u, v \in H$ , we have  $B[u, v] = \langle A(u), v \rangle$ . By the coercivity of B, we have for all u

$$\beta \|u\|^2 \le B[u, u] = \langle A(u), u \rangle \le \|A(u)\| \|u\|,$$

so (by the above if  $u \neq 0$  and trivially if u = 0)

$$\beta \|u\| \le \|A(u)\|. \tag{5}$$

It suffices to show that the operator A is bijective. I will concentrate on one step of the bijectivity argument: the step where we exploit (5) to establish that A is injective and has closed range. As usual I present two proofs.

Your reader has immediate access to a full exposition of an unfamiliar abstraction; moreover, thanks to verification, she can trust you when you state that all the preconditions hold for that abstraction to be applicable in the context at hand.

Proof ([20], slightly compressed). If  $u_1, u_2 \in H$ , then  $||A(u_1 - u_2)|| \ge \beta ||u_1 - u_2||$ , from which it's clear that A is injective.

To see that the range of A is closed in H, let  $\{u_j\}_{j=1}^{\infty} \subset H$  satisfy  $Au_j \to w$  for some  $w \in H$ . We need to show that there exists  $u \in H$  so that Au = w.

For this, we notice that

$$||u_i - u_j|| \le \beta^{-1} ||Au_i - Au_j||.$$

The sequence  $\{Au_j\}_{j=1}^{\infty}$  converges, so it must be Cauchy, so we see that  $\{u_j\}_{j=1}^{\infty}$  must be Cauchy, and so must converge to some  $u \in H$ . Since A is bounded,

$$||Au - w|| = \lim_{j \to \infty} ||Au - Au_j|| \le \alpha \lim_{j \to \infty} ||u - u_j|| = 0.$$

That is, Au = w.

A close read of this proof snippet suggests that it doesn't seem to use the Hilbert space structure very much. And indeed, it is possible to extract the work as a lemma in general metric spaces. The appropriate abstraction is the following property of a function  $f: X \to Y$  between metric spaces: that there exists a constant  $\beta > 0$  such that for all  $x_1$  and  $x_2$ ,  $\beta d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$ .

As it turns out, the same argument appears in the proof of the Contraction Mapping Theorem, in a different special case (the case Y=X). When Yury Kudryashov formalised the Contraction Mapping Theorem for Mathlib in 2020, he recognised the appropriate context for the argument, <sup>13</sup> and wrote a self-contained theory development in Mathlib for such functions, <sup>14</sup> for which he introduced the name antilipschitz maps. In fact, I would not be surprised to learn that this fairly short (1-2 pages of text) and easy theory has been rediscovered and redeveloped many times, under many names.

With that abstraction and theory development available, the snippet of the Lax–Milgram theorem we are discussing reduces simply to the following:

Proof. A is uniformly continuous and by (5) it is antilipschitz, so it is injective and has closed range.

Daniel Roca González contributed this efficient proof of the Lax–Milgram theorem to Mathlib in 2022.<sup>15</sup> (The theorem had earlier been formalised in Coq [3], following a somewhat different proof.)

In this example we see illustrated Bourbaki's original argument in favour of abstraction: deduplication. Formal mathematics is done at scale: it is written from the axioms up, so nontrivial proofs form parts of a vast corpus; writing formal mathematics is much more like writing an encyclopaedia than like writing an article. At this scale, the "two, or several theories" united by an abstraction are very likely all to turn up, and simple efficiency favours using the abstraction.<sup>16</sup>

<sup>&</sup>lt;sup>13</sup> https://github.com/leanprover-community/mathlib/pull/1859#discussion\_ r365490281

 $<sup>^{14}</sup>$  Mathlib [8], Topology/MetricSpace/Antilipschitz

<sup>15</sup> Mathlib [8], Analysis/InnerProductSpace/LaxMilgram

To believe that habitual abstraction really will avoid the repetition of proofs at large scale, you must be something of a Platonist: you must believe (as I do!) that the

#### 3.2 Smooth vector bundles

My last example (a bit more technical than the others in this article) is taken from the theory of smooth vector bundles in Lean, which is joint work of mine with Floris van Doorn in 2022–23.

In this example, the particular definition of smooth vector bundle we chose for our theory matters.

**Definition 1.** A smooth vector bundle with fibre F over a smooth manifold B consists of

- a collection of topological vector spaces indexed by B;
- a topology on the total space, i.e. on their disjoint union;
- a collection of trivialisations, each identifying the fibre-union over some open set  $U \subseteq B$  homeomorphically with  $U \times F$ , commuting via projections with the identity on U, and fibrewise an isomorphism of topological vector spaces;
- with the property that for two trivialisations in the collection the induced map  $U \cap V \to \text{End}(F)$  is smooth.

I will discuss two approaches to the proof of the following statement.

**Proposition 1.** The total space of a smooth vector bundle is a smooth manifold.

Note that the fact that this is a theorem to be proved, rather than part of the definition, is a consequence of our choice of definition.

Here is how you might prove this theorem in prose. Since I didn't find a presentation of the theory of vector bundles in the literature which started with precisely our definition, this proof is not taken directly from real life.

*Proof.* Let H be the model space for the smooth manifold B. Given a trivialisation  $\psi = (\psi_b, \psi_f) : \pi^{-1}(U) \to U \times F$  and a chart  $\varphi : V \xrightarrow{\sim} \varphi(V) \subseteq H$  for B, define a candidate chart

$$\Phi_{\psi,\varphi}: \pi^{-1}(U \cap V) \to \varphi(U \cap V) \times F,$$
  
$$\Phi_{\psi,\varphi}(p) := (\varphi(\psi_b(p)), \psi_f(p)).$$

We need to check that for any two trivialisations  $\psi_1, \psi_2$  and any two charts  $\varphi_1, \varphi_2$  the transition function  $\Phi_{\psi_2,\varphi_2} \circ \Phi_{\psi_1,\varphi_1}^{-1}$  is smooth. This works out since  $\psi_2 \circ \psi_1^{-1}, \varphi_1$  and  $\varphi_2$  are all smooth.

Our Lean formalisation uses Kobayashi–Nomizu's abstraction of a *structure groupoid* [23] for a way in which a space is modelled on another space, which is used there as the approach to the definition of smooth manifolds. Sébastien Gouëzel developed this theory in Mathlib in 2019.<sup>17</sup>

<sup>&</sup>quot;natural context" of an argument is sufficiently unambiguous that others who need it will formulate it in the same way, and thus be led to stumble across your version.

17 Mathlib [8], Geometry/Manifold/ChartedSpace

The advantage of our chosen definition of smooth vector bundle is that, following a suggestion of Gouëzel,  $^{18}$  it too can be expressed using this structure groupoid abstraction. Let H be the model space for the smooth manifold B. Let E be a smooth vector bundle over B with fibre F. We consider the sequence

$$E \xrightarrow{\text{modelled on}} B \times F \xrightarrow{\text{modelled on}} H \times F$$
:

- 1. E is modelled on  $B \times F$  with the charts being the trivialisations, and our vector bundle definition amounts to the condition that the transition functions between these charts lie in the *smooth fibrewise-linear groupoid*;
- 2.  $B \times F$  is in turn is modelled on  $H \times F$  with the charts being the usual product manifold charts, and with the transition functions between these charts lying in the usual smooth manifold structure groupoid.

In this language, here is an outline of our formalisation <sup>19</sup> of the theorem.

*Proof.* "Modellings" can be composed, so the modellings of E on  $B \times F$  and of  $B \times F$  on  $H \times F$  yield a modelling of E on  $H \times F$ . Structure groupoid properties can also be composed, so the transition functions between these induced charts lie in the smooth manifold structure groupoid for  $B \times F$ .

This composition theorem for structure groupoids was formulated by us for the project;<sup>20</sup> to our knowledge it does not appear in the literature.

This hierarchy of undigested abstractions is certainly a more obscure approach to this material than would be acceptable in a traditional prose presentation. But it has a certain elegance, and it brings organisational assistance: some work can be done cleanly at high level, and the more painful direct manipulation of partially-defined smooth functions appears only when checking the various preconditions for the abstractions to apply.

This is very much a slogan of formalisation: that it incentivises abstraction to cope with the demands of writing proofs in full detail [7]. Gonthier [11] similarly notes that his formalisation of the Four-Colour Theorem produced several abstractions, "new and rather elegant nuggets of mathematics," as a byproduct.

### 4 Conclusion

In this article I discuss only the question of, given a fixed statement, what constitutes a good proof (formal or informal) of that statement. An orthogonal question is how to best express the development of a whole mathematical theory.<sup>21</sup> This is a big question and it has produced an interesting literature [1,13].

<sup>&</sup>lt;sup>18</sup> Mathlib [8], Geometry/Manifold/ChartedSpace, line 139

<sup>19</sup> Mathlib [8], Geometry/Manifold/VectorBundle/Basic, line 486

 $<sup>^{20}</sup>$  Mathlib [8], Geometry/Manifold/LocalInvariantProperties, line 698

A crude analogy is to consider the statements of a mathematical theory as a digraph, with edges denoting easy implications (some implications are easy in both directions and their edges are bidirectional). To design a mathematical theory development, you must select a spanning tree for this digraph.

I have argued that good computer-formalised writing differs from good prose writing in two aspects: its incorporation of algorithms and of abstractions. These two aspects have an interesting commonality: in prose writing, both represent breaks in tone, or even in the very experience of reading—moments at which the reader is sent to a reference in order to read up on an unfamiliar abstraction, or to her computer to study and run a piece of code. But in formalised writing these are not breaks: prerequisites, computation and main argument form an integrated whole.

Montaño [25] argues that we experience a proof as beautiful according to the narrative experience of reading it, the "quality of its storytelling." In formalised mathematical writing, more kinds of thinking can be incorporated without causing breaks in the narrative flow. Our storytelling will be all the richer in consequence.

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