CHEP XXXXX

Field theories on the boundary of Minkowski space

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Abstract

Stuff goes here...

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1 Introduction

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2 Allowed Fine-Tunings?

Even though

Our observations

3 Acknowledgments

We thank ...

A AdS and Minkowski space

One way to define the various AdS spaces is through an embedding in a higher dimensional flat space (Lorenzian AdS in a flat space with two time directions and Euclidean AdS on Minkowski space). Specifically, one defines (Lorenzian) AdS_{d+1} as the following algebraic set

$$\mathrm{AdS}_{d+1} \coloneqq \{(X^{-1}, X^0, \dots X^d) \in \mathbb{R}^{d,2} \mid \eta_{AB} X^A X^B = -L^2\} \tag{A.1}$$

where $\eta = \mathrm{diag}(-1, -1, 1, \dots 1)$ is the canonical metric on $\mathbb{R}^{d,2}$ and L is a positive real number which sets a length scale on this space (called the AdS length scale). The set above derives its topology, smooth structure and pseudo-Riemmanian structure from that of its ambient space.

Another interesting property of this space is that it's (conformal) boundary (note that the space above is not a manifold with boundary, we speak of its conformal boundary) is a flat Minkowski space of one lower dimension. To see this, set $X^A = s\, x^A$ and take the limit $s \to \infty$. The set $\partial(\mathrm{AdS}_{d+1})$ obtained this way is

$$\partial(\mathrm{AdS}_{d+1}) = \{[x^{-1}, x^0, \dots x^d] \in \mathbb{RP}^{d+1} \mid \eta_{AB} x^A x^B = -\frac{L^2}{s^2} \to 0\} \tag{A.2}$$

In other words, the set of points with $(x^{-1})^2 + (x^0)^2 = (x^1)^2 + \dots + (x^d)^2$. In addition, there is an overall scale equivalence ($x^A \sim c\, x^A$, where, importantly, c can be negative) because of the above construction (hence $\mathbb{RP}^{d+1}$ and not $\mathbb{R}^{d,2}$). Thus, to fix this ambiguity, one can always take $(x^{-1})^2 + (x^0)^2 = (x^1)^2 + \dots + (x^d)^2 = 1$ (with the remaining identification of x^A and $-x^A$). This space is topologically $\boldsymbol{S}^1 \times \boldsymbol{S}^{d-1}/\mathbb{Z}_2$ and has closed timelike curves. It is, in fact, a compactification of Minkowski space $\mathbb{R}^{d-1,1}$. One "unfolds" the time circle to get the universal cover $\mathbb{R} \times \boldsymbol{S}^{d-1}$.

To see how A.2 is $\mathbb{R}^{d-1,1}$, write $u = X^{-1} - X^d$ and $v = X^{-1} + X^d$. The set in A.2 then is the locus $-uv + \sum_{ij} \eta_{ij} x^i x^j = 0$. Here η_{ij} is the metric on $\mathbb{R}^{d-1,1}$ with coordinates $(x^0, \ldots x^{d-1})$. If we ignore the measure zero set $v = 0$, one can set it equal to one by scale equivalence and get u as a function of of the x^i which is just Minkowski space.

Another important property of the two sets defined above is that they have a natural action of $SO(d,2)$ from the ambient space. Indeed, it is not hard to see that this is the full isometry group of AdS_{d+1} (since the other isometries of $\mathbb{R}^{d,2}$, translations, don't leave this set invariant). For the compactified Minkowski space A.2, this is the group of comformal isometries. This makes immediate sense since the boundary is a conformally defined strucure.

A.1 Coordinates on AdS

An obvious way to construct a coordinate chart for the set A.1 is

$$
\begin{aligned}
X^{-1} &= L \cosh\rho \cos\tau \\
X^{0} &= L \cosh\rho \sin\tau \\
X^{a} &= L \sinh\rho \, y^a \qquad a = 1, \ldots, d
\end{aligned}
\tag{A.3}
$$

where y^a are coordinates on the sphere $\boldsymbol{S}^{d-1}$ (e.g. ultraspherical coordinates). These are called **global** coordinates because the global (topological) structure of AdS_{d+1} is manifest in them. Note that the "time" coordiate τ is "unfolded" (takes values in all of $\mathbb{R}$). The induced metric on this space can now be easily written.

$$
\mathrm{d}s^2 = \eta_{AB} \frac{\partial X^A}{\partial u^a} \frac{\partial X^B}{\partial u^b} \mathrm{d}u^a \mathrm{d}u^b = L^2(-\cosh^2\rho \, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho \, \mathrm{d}\Omega_{d-1}^2) \tag{A.4}
$$

The boundary in these coordinates corresponds to $\rho \to \infty$. One can "bring" the boundary (not at a closer distance but at finite values of the coordinates) by changing to a **trigonometric version** of global coordinates defined by $\sinh\rho = \tan\eta$. (The intuition here is simple, $\cosh^2\rho - \sinh^2\rho = 1$ but also $\sec^2\eta - \tan^2\eta = 1$). The metric in these coordinates is

$$
\mathrm{d}s^2 = \frac{L^2}{\cos^2\eta}(-\mathrm{d}\tau^2 + \mathrm{d}\eta^2 + \sin^2\eta \, \mathrm{d}\Omega_{d-1}^2) \tag{A.5}
$$

And indeed, the boundary is now at $\eta \to \frac{\pi}{2}$. In fact, these coordinates give AdS it's usual picture as a cylinder (for AdS_3, see Figure 1)

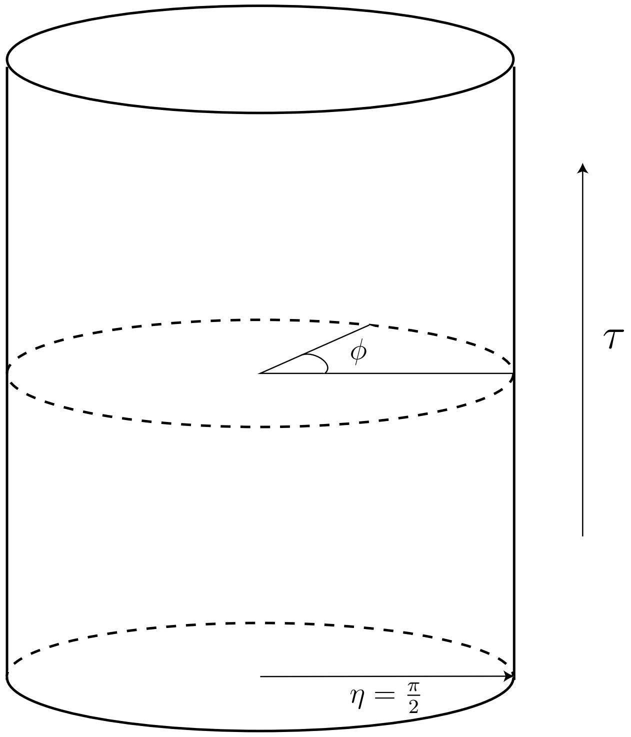

Figure 1: AdS_3 in trigonometric global coordinates

Another convenient set of coordinates is the **area gauge**. The idea is to make the volume of the $(d-1)$-sphere equal to it's usual volume. It's easy to see that $r = L \sinh \rho$ does the job. The metric in these coordinates read (with another tiny change $t = L\,\tau$)

$$\mathrm{d}s^2 = -(1 + \frac{r^2}{L^2})\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{(1 + \frac{r^2}{L^2})} + r^2\,\mathrm{d}\Omega^2_{d-1} \tag{A.6}$$

One can further define a **tortoise coordinate** r^* to simplify the radial null directions

$$r^* = \int \frac{\mathrm{d}r}{(1 + \frac{r^2}{L^2})} = L \tan^{-1}\frac{r}{L} \qquad \mathrm{d}s^2 = \sec^2\frac{r^*}{L}\,(-\mathrm{d}t^2 + \mathrm{d}r^{*2}) + L^2 \tan^2\frac{r^*}{L}\,\mathrm{d}\Omega^2_{d-1} \tag{A.7}$$

which is exactly the same as the trigonometric global coordinates above (with $\eta = \frac{r^*}{L}$).

To further simplify the null directions we can also go to forward or backwards Eddington-Finklestein coordinates or directly jump to the **Bondi** or **double-null** gauge $(u,\, v,\, y^a)$

$$u = t - r^*$$

$$v = t + r^*$$

$$\mathrm{d}s^2 = -\sec^2\frac{v-u}{2L}\,\mathrm{d}u\,\mathrm{d}v + L^2\tan^2\frac{v-u}{2L}\,\mathrm{d}\Omega^2_{d-1} \tag{A.8}$$

Another extremely useful gauge is the **Poincare patch**. Most formulations of AdS/CFT correspondence use this gauge since the Minkowski structure of the boundary is manifest in them. To derive them, one works in "light-cone" coordinates on the ambient space.

$$\begin{aligned} X^+ &= X^{-1} + X^d \\ X^- &= X^{-1} - X^d \\ X^\mu & \qquad \mu = 0, 1, \ldots d-1 \end{aligned} \tag{A.9}$$

AdS_{d+1} is now (the locus of) $-X^+X^- + \eta_{\mu\nu}X^\mu X^\nu = -L^2$. Where $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, \ldots, 1)$. After a bit of thought, one can arrive at the following parametrization

$$\begin{aligned} X^+ &= z + \frac{x^\mu x_\mu}{z} \\ X^- &= \frac{L^2}{z} \\ X^\mu &= \frac{L}{z} x^\mu \end{aligned} \tag{A.10}$$

The metric in these coordinates takes the simple form

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \mathrm{d}x^\mu \mathrm{d}x_\mu\right) \tag{A.11}$$

Note that the conformal boundary is now at $z \to 0$ which is not part of the manifold. But this forces us to choose either $z > 0$ or $z < 0$. Therefore, these coordinates divide the space into two "patches" (hence the name) joined in between by the boundary. Note how it is clearly evident that the boundary is Minkowski space with global inertial coordinates x^μ.

A.2 AdS Isometries and Killing vectors

From the argument about isometries earlier, it is clear that the Killing vectors of AdS_{d+1} are the Lorentz generators $\boldsymbol{J}_{AB} \in \mathrm{T}_p\mathbb{R}^{d,2}$ of the ambient space projected down to A.1. (I write tangent vector fields in bold)

$$\boldsymbol{J}_{AB} = X_A \boldsymbol{\partial}_B - X_B \boldsymbol{\partial}_A \tag{A.12}$$

The projection is a simple pullback

$$\boldsymbol{j}_{AB} = (\iota_* \boldsymbol{J}^\flat_{AB})^\sharp = g^{-1}(\iota_* \eta(\boldsymbol{J}_{AB}, -), -) \tag{A.13}$$

of the embedding/inclusion map $\iota\colon \mathrm{AdS}_{d+1} \to \mathbb{R}^{d,2}$. In coordinates, this is (assuming u^i are coordinates on AdS, like one of the above)

$$\boldsymbol{j}^k_{AB} = g^{ki}\frac{\partial X^C}{\partial u^i}\eta_{CD}\boldsymbol{J}^D_{AB} \tag{A.14}$$

It is straightforward to calculate this for the various coordinate systems described above. Below, I do this for AdS_3 for simplicity.

A.2.1 Global coordinates (ρ, τ, ϕ)

$$\textbf{Metric} \quad \mathrm{d}s^2 = L^2(-\cosh^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\phi^2) \tag{A.15}$$

$$\textbf{Boundary} \quad \rho \to \infty \tag{A.16}$$

Killing vector	On Bulk	On Boundary
$\boldsymbol{j}_{-10}$	$-\boldsymbol{\partial}_\tau$	$-\boldsymbol{\partial}_\tau$
$\boldsymbol{j}_{-11}$	$-\cos\tau\cos\phi\,\boldsymbol{\partial}_\rho$ $+\tanh\rho\sin\tau\cos\phi\,\boldsymbol{\partial}_\tau$ $+\coth\rho\cos\tau\sin\phi\,\boldsymbol{\partial}_\phi$	$\sin\tau\cos\phi\,\boldsymbol{\partial}_\tau$ $+\cos\tau\sin\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{-12}$	$-\cos\tau\sin\phi\,\boldsymbol{\partial}_\rho$ $+\tanh\rho\sin\tau\sin\phi\,\boldsymbol{\partial}_\tau$ $-\coth\rho\cos\tau\cos\phi\,\boldsymbol{\partial}_\phi$	$\sin\tau\sin\phi\,\boldsymbol{\partial}_\tau$ $-\cos\tau\cos\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{01}$	$-\sin\tau\cos\phi\,\boldsymbol{\partial}_\rho$ $-\tanh\rho\cos\tau\cos\phi\,\boldsymbol{\partial}_\tau$ $+\coth\rho\sin\tau\sin\phi\,\boldsymbol{\partial}_\phi$	$-\cos\tau\cos\phi\,\boldsymbol{\partial}_\tau$ $+\sin\tau\sin\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{02}$	$-\sin\tau\sin\phi\,\boldsymbol{\partial}_\rho$ $-\tanh\rho\cos\tau\sin\phi\,\boldsymbol{\partial}_\tau$ $-\coth\rho\sin\tau\cos\phi\,\boldsymbol{\partial}_\phi$	$-\cos\tau\sin\phi\,\boldsymbol{\partial}_\tau$ $-\sin\tau\cos\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{12}$	$\boldsymbol{\partial}_\phi$	$\boldsymbol{\partial}_\phi$

A.2.2 Trigonometric global coordinates (also in tortoise coordinates) (η, τ, ϕ)

$$\textbf{Metric} \quad \mathrm{d}s^2 = \frac{L^2}{\cos^2\eta}(-\mathrm{d}\tau^2 + \mathrm{d}\eta^2 + \sin^2\eta\,\mathrm{d}\phi^2) \tag{A.17}$$

$$\textbf{Boundary} \quad \eta \to \frac{\pi}{2} \tag{A.18}$$

Killing vector	On Bulk	On Boundary
$\boldsymbol{j}_{-10}$	$-\boldsymbol{\partial}_\tau$	$-\boldsymbol{\partial}_\tau$
$\boldsymbol{j}_{-11}$	$-\cos\eta\cos\tau\cos\phi\,\boldsymbol{\partial}_\rho$ $+\sin\eta\sin\tau\cos\phi\,\boldsymbol{\partial}_\tau$ $+\csc\eta\cos\tau\sin\phi\,\boldsymbol{\partial}_\phi$	$\sin\tau\cos\phi\,\boldsymbol{\partial}_\tau$ $+\cos\tau\sin\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{-12}$	$-\cos\eta\cos\tau\sin\phi\,\boldsymbol{\partial}_\rho$ $+\sin\eta\sin\tau\sin\phi\,\boldsymbol{\partial}_\tau$ $-\csc\eta\cos\tau\cos\phi\,\boldsymbol{\partial}_\phi$	$\sin\tau\sin\phi\,\boldsymbol{\partial}_\tau$ $-\cos\tau\cos\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{01}$	$-\cos\eta\sin\tau\cos\phi\,\boldsymbol{\partial}_\rho$ $-\sin\eta\cos\tau\cos\phi\,\boldsymbol{\partial}_\tau$ $+\csc\eta\sin\tau\sin\phi\,\boldsymbol{\partial}_\phi$	$-\cos\tau\cos\phi\,\boldsymbol{\partial}_\tau$ $+\sin\tau\sin\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{02}$	$-\cos\eta\sin\tau\sin\phi\,\boldsymbol{\partial}_\rho$ $-\sin\eta\cos\tau\sin\phi\,\boldsymbol{\partial}_\tau$ $-\csc\eta\sin\tau\cos\phi\,\boldsymbol{\partial}_\phi$	$-\cos\tau\sin\phi\,\boldsymbol{\partial}_\tau$ $-\sin\tau\cos\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{12}$	$\boldsymbol{\partial}_\phi$	$\boldsymbol{\partial}_\phi$

A.2.3 Area gauge (r, t, ϕ)

$$\textbf{Metric}\quad \mathrm{d}s^2 = -(1+\frac{r^2}{L^2})\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{(1+\frac{r^2}{L^2})} + r^2\,\mathrm{d}\phi^2 \tag{A.19}$$

$$\textbf{Boundary}\quad r\to\infty \tag{A.20}$$

Killing vector	On Bulk	On Boundary
$\boldsymbol{j}_{-10}$	$-L\,\boldsymbol{\partial}_t$	$-L\,\boldsymbol{\partial}_t$
$\boldsymbol{j}_{-11}$	$-\sqrt{L^2+r^2}\cos\phi\cos\frac{t}{L}\,\boldsymbol{\partial}_r + \frac{Lr\cos\phi\sin\frac{t}{L}}{\sqrt{L^2+r^2}}\,\boldsymbol{\partial}_t + \frac{\sqrt{L^2+r^2}\sin\phi\cos\frac{t}{L}}{r}\,\boldsymbol{\partial}_\phi$	$L\sin\frac{t}{L}\cos\phi\,\boldsymbol{\partial}_t + \cos\frac{t}{L}\sin\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{-12}$	$-\sqrt{L^2+r^2}\sin\phi\cos\frac{t}{L}\,\boldsymbol{\partial}_r + \frac{Lr\sin\phi\sin\frac{t}{L}}{\sqrt{L^2+r^2}}\,\boldsymbol{\partial}_t - \frac{\sqrt{L^2+r^2}\cos\phi\cos\frac{t}{L}}{r}\,\boldsymbol{\partial}_\phi$	$L\sin\frac{t}{L}\sin\phi\,\boldsymbol{\partial}_t - \cos\frac{t}{L}\cos\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{01}$	$-\sqrt{L^2+r^2}\cos\phi\sin\frac{t}{L}\,\boldsymbol{\partial}_r - \frac{Lr\cos\phi\cos\frac{t}{L}}{\sqrt{L^2+r^2}}\,\boldsymbol{\partial}_t + \frac{\sqrt{L^2+r^2}\sin\phi\sin\frac{t}{L}}{r}\,\boldsymbol{\partial}_\phi$	$-L\cos\frac{t}{L}\cos\phi\,\boldsymbol{\partial}_t + \sin\frac{t}{L}\sin\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{02}$	$-\sqrt{L^2+r^2}\sin\phi\sin\frac{t}{L}\,\boldsymbol{\partial}_r - \frac{Lr\sin\phi\cos\frac{t}{L}}{\sqrt{L^2+r^2}}\,\boldsymbol{\partial}_t - \frac{\sqrt{L^2+r^2}\cos\phi\sin\frac{t}{L}}{r}\,\boldsymbol{\partial}_\phi$	$-L\cos\frac{t}{L}\sin\phi\,\boldsymbol{\partial}_t - \sin\frac{t}{L}\cos\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{12}$	$\boldsymbol{\partial}_\phi$	$\boldsymbol{\partial}_\phi$

A.2.4 Bondi gauge (u, v, ϕ)

$$\textbf{Metric}\quad \mathrm{d}s^2 = -\sec^2\left(\frac{v-u}{2L}\right)\mathrm{d}u\,\mathrm{d}v + L^2\tan^2\left(\frac{v-u}{2L}\right)\mathrm{d}\phi^2 \tag{A.21}$$

$$\textbf{Boundary}\quad v-u \to \pm\pi L \tag{A.22}$$

Killing vector	On Bulk	On Boundary
$\boldsymbol{j}_{-10}$	$-L\left(\boldsymbol{\partial}_u+\boldsymbol{\partial}_v\right)$	$-L\left(\boldsymbol{\partial}_u+\boldsymbol{\partial}_v\right)$
$\boldsymbol{j}_{-11}$	$L\cos\phi\cos\frac{u}{L}\,\boldsymbol{\partial}_u$ $-L\cos\phi\cos\frac{v}{L}\,\boldsymbol{\partial}_v$ $-\sin\phi\cos\frac{u+v}{2L}\csc\frac{u-v}{2L}\,\boldsymbol{\partial}_\phi$	$L\cos\frac{u}{L}\cos\phi\left(\boldsymbol{\partial}_u+\boldsymbol{\partial}_v\right)$ $-\sin\frac{u}{L}\sin\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{-12}$	$L\sin\phi\cos\frac{u}{L}\,\boldsymbol{\partial}_u$ $-L\sin\phi\cos\frac{v}{L}\,\boldsymbol{\partial}_v$ $-\cos\phi\cos\frac{u+v}{2L}\csc\frac{u-v}{2L}\,\boldsymbol{\partial}_\phi$	$L\cos\frac{u}{L}\sin\phi\left(\boldsymbol{\partial}_u+\boldsymbol{\partial}_v\right)$ $+\sin\frac{u}{L}\cos\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{01}$	$L\cos\phi\sin\frac{u}{L}\,\boldsymbol{\partial}_u$ $-L\cos\phi\sin\frac{v}{L}\,\boldsymbol{\partial}_v$ $-\sin\phi\sin\frac{u+v}{2L}\csc\frac{u-v}{2L}\,\boldsymbol{\partial}_\phi$	$L\sin\frac{u}{L}\cos\phi\left(\boldsymbol{\partial}_u+\boldsymbol{\partial}_v\right)$ $+\cos\frac{u}{L}\sin\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{02}$	$L\sin\phi\sin\frac{u}{L}\,\boldsymbol{\partial}_u$ $-L\sin\phi\sin\frac{v}{L}\,\boldsymbol{\partial}_v$ $-\cos\phi\sin\frac{u+v}{2L}\csc\frac{u-v}{2L}\,\boldsymbol{\partial}_\phi$	$L\sin\frac{u}{L}\sin\phi\left(\boldsymbol{\partial}_u+\boldsymbol{\partial}_v\right)$ $-\cos\frac{u}{L}\cos\phi\,\boldsymbol{\partial}_\phi$
$\boldsymbol{j}_{12}$	$\boldsymbol{\partial}_\phi$	$\boldsymbol{\partial}_\phi$

A.2.5 Poincare patch (z,t,x)

$$\textbf{Metric}\quad \mathrm{d}s^2=\frac{L^2}{z^2}\left(\mathrm{d}z^2-\mathrm{d}t^2+\mathrm{d}x^2\right) \tag{A.23}$$

$$\textbf{Boundary}\quad z\to 0 \tag{A.24}$$

Killing vector	On Bulk	On Boundary
$\boldsymbol{j}_{-10}$	$-\frac{tz}{L}\,\boldsymbol{\partial}_z$ $-\frac{L^2+t^2+x^2+z^2}{2L}\,\boldsymbol{\partial}_t$ $-\frac{tx}{L}\boldsymbol{\partial}_x$	$-\frac{L^2+t^2+x^2}{2L}\,\boldsymbol{\partial}_t$ $-\frac{tx}{L}\boldsymbol{\partial}_x$
$\boldsymbol{j}_{-11}$	$\frac{xz}{L}\,\boldsymbol{\partial}_z+\frac{tx}{L}\,\boldsymbol{\partial}_t$ $+\frac{-L^2+t^2+x^2-z^2}{2L}\,\boldsymbol{\partial}_x$	$\frac{tx}{L}\,\boldsymbol{\partial}_t$ $+\frac{-L^2+t^2+x^2}{2L}\,\boldsymbol{\partial}_x$
$\boldsymbol{j}_{-12}$	$-z\,\boldsymbol{\partial}_z-t\,\boldsymbol{\partial}_t-x\,\boldsymbol{\partial}_x$	$-t\,\boldsymbol{\partial}_t-x\,\boldsymbol{\partial}_x$
$\boldsymbol{j}_{01}$	$-x\,\boldsymbol{\partial}_t-t\,\boldsymbol{\partial}_x$	$-x\,\boldsymbol{\partial}_t-t\,\boldsymbol{\partial}_x$
$\boldsymbol{j}_{02}$	$-\frac{tz}{L}\,\boldsymbol{\partial}_z$ $-\frac{-L^2+t^2+x^2+z^2}{2L}\,\boldsymbol{\partial}_t$ $-\frac{tx}{L}\,\boldsymbol{\partial}_x$	$-\frac{-L^2+t^2+x^2}{2L}\,\boldsymbol{\partial}_t$ $-\frac{tx}{L}\,\boldsymbol{\partial}_x$
$\boldsymbol{j}_{12}$	$\frac{xz}{L}\,\boldsymbol{\partial}_z+\frac{tx}{L}\,\boldsymbol{\partial}_t$ $+\frac{L^2+t^2+x^2-z^2}{2L}\,\boldsymbol{\partial}_x$	$\frac{tx}{L}\,\boldsymbol{\partial}_t$ $+\frac{L^2+t^2+x^2}{2L}\,\boldsymbol{\partial}_x$

On a Poincare' patch, one can also write down the exact form of the finite isometries as I menioned earlier, since the action of $SO(d-1,1)$ Lorentz subgroup of $SO(d,2)$ is manifest here on the coordinates x^μ. Not only that, the action of the **conformal group** $SO(d,2)$ of $\mathbb{R}^{d-1,1}$ is manifest here.

Translations

$$x^\mu \to x^\mu + a^\mu$$

$$z \to z$$

Lorentz transformations

$$x^\mu \to \Lambda^\mu_\nu x^\nu$$

$$z \to z$$

Dilations

$$x^\mu \to \lambda x^\mu$$

$$z \to \lambda z$$

Special conformal transformations (SCTs)

$$x^\mu \to \frac{x^\mu + b^\mu A}{1 + 2b_\mu x^\mu + b^\mu b_\mu A}$$

$$z \to \frac{z}{1 + 2b_\mu x^\mu + b^\mu b_\mu A}$$

$$\text{with} \quad A = z^2 + x^\mu x_\mu$$

Note how the boundary limit ($z \to 0$) of these transformations become exactly conformal transformations of the boundary $\mathbb{R}^{d-1,1}$

A.3 Causal structure of AdS

The causal structure of the universal cover of AdS spacetimes is evident in trigonometric global coordinates. A penrose diagram is shown below

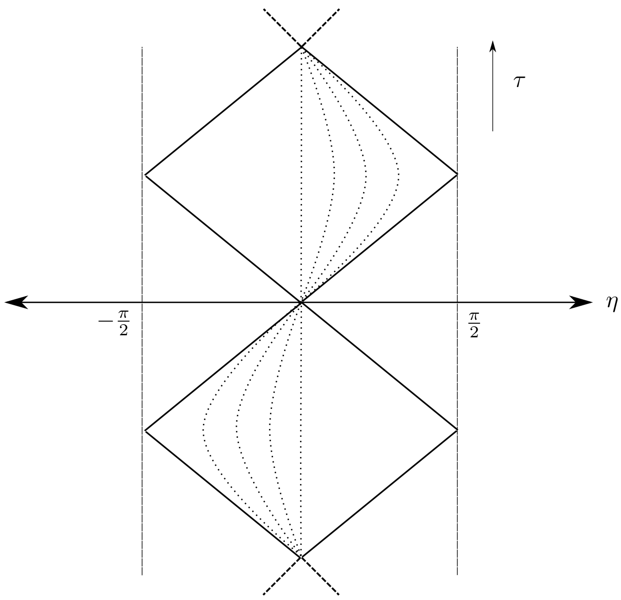

Figure 2: Penrose diagram of AdS universal covering space. Solid 45° lines are null geodesics whereas dotted lines are timelike geodesics

B Minkowski space in Bondi gauge

The Bondi or in this case the double null gauge was alluded to in the last section in the context of AdS. Here we study $\mathbb{R}^{3,1}$ in these coordinates. These are related to global intertial coordinates as follows ($r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$):

$$\begin{aligned} u &= x^0 - r \\ v &= x^0 + r \\ z &= \frac{x^1 + ix^2}{r - x^3} = \cot\frac{\theta}{2} e^{i\phi} \\ \bar{z} &= \frac{x^1 - ix^2}{r - x^3} = \cot\frac{\theta}{2} e^{-i\phi} \end{aligned} \tag{B.1}$$

$$\begin{aligned} x^0 &= \frac{u+v}{2} \\ x^1 &= \frac{(v-u)\,\mathrm{Re}\,z}{1+z\bar{z}} \\ x^2 &= \frac{(v-u)\,\mathrm{Im}\,z}{1+z\bar{z}} \\ x^3 &= \left(\frac{v-u}{2}\right)\frac{1-z\bar{z}}{1+z\bar{z}} \end{aligned} \tag{B.2}$$

The metric reads

$$\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + \left(\frac{v-u}{2}\right)^2 \frac{4\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+z\bar{z})^2} \tag{B.3}$$

Note that z is simply the inverse stereographic projection from the North pole.

References