A Trip Down Fourier Lane

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Robert S. Huston

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1 Introduction

This document consists of a collection of Fourier analysis notes I've kept around over the years. I finally took the time to put them into a single document so that I don't have to keep looking for them again and again. I use a justification approach to the mathematics instead of a formal and more rigorous approach. If you really need rigor, you can read Papoulis' book (listed in the references).

Gunga galunga. Gunga, gunga-lagunga.
- Carl Spackler (Bill Murray), Caddyshack, 1980

2 The Harmonic Complex Exponential Function

Consider the harmonic complex exponential function, $e^{j\theta}$. Euler's identity shows us that the harmonic complex exponential function is a rotating unit vector in the complex plane:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

To see why this is true, recall the power series expansions for e^x, $\cos x$, and $\sin x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The complex form of the exponential power series is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \cdots$$

Setting $z = j\theta$ gives us

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \frac{(j\theta)^7}{7!} + \cdots \\ &= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - j\frac{\theta^7}{7!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\ &= \cos\theta + j\sin\theta \end{aligned}$$

As should be expected

$$\begin{aligned} \frac{d}{d\theta}(\cos\theta + j\sin\theta) &= -\sin\theta + j\cos\theta \\ &= j\cos\theta - \sin\theta \\ &= j(\cos\theta + j\sin\theta) \\ &= je^{j\theta} \\ &= \frac{d}{d\theta}e^{j\theta} \end{aligned}$$

Observe that the harmonic complex exponential function, e^{jy}, is simply the imaginary part of the argument of the complex exponential function

$$\begin{aligned} e^z &= e^{x+jy} \\ &= e^x e^{jy} \\ &= e^x(\cos y + j\sin y) \end{aligned}$$

If we define

$$f(z) = e^z$$

and rewrite the expression in the format of

$$f(z) = u(x, y) + jv(x, y)$$

we have

$$f(z) = e^x \cos y + je^x \sin y$$

so that

$$u(x, y) = e^x \cos y$$

$$v(x, y) = e^x \sin y$$

Applying the Cauchy-Riemann evaluations (we'll use the "*x*" partial derivatives) gives

$$\begin{aligned} f'(z) &= u_x(x, y) + jv_x(x, y) \\ &= e^x \cos y + je^x \sin y \\ &= e^x(\cos y + j\sin y) \end{aligned}$$

This clearly shows that

$$f'(z) = f(z)$$

which, of course, is the differentiation property of the exponential function. So I've got that going for me, which is nice.

3 Orthogonal Functions

Two complex-valued functions $\phi_1(t)$ and $\phi_2(t)$ are said to be orthogonal over the interval $[t_1, t_2]$ if

$$\int_{t_1}^{t_2} \phi_1(t)\phi_2^*(t)dt = \int_{t_1}^{t_2} \phi_1^*(t)\phi_2(t)dt = 0$$

The members of a set of complex-valued basis functions, $\phi_n(t)$, where $n = 0, \pm 1, \pm 2, \ldots$, are said to be mutually orthogonal over the interval $[t_1, t_2]$ if

$$\int_{t_1}^{t_2} \phi_n(t)\phi_m^*(t)dt = \begin{cases} K_n & n = m \\ 0 & n \neq m \end{cases}$$

A set of basis functions $\phi_n(t)$, not necessarily orthogonal, is said to be normalized if

$$K_n = \int_{t_1}^{t_2} |\phi_n(t)|^2 dt = 1 \quad \text{for all } n$$

If a set of basis functions is both orthogonal and normalized, it is called an orthonormal set.

Now let us investigate the set of harmonic complex exponential functions over the interval $[t_1, t_2]$:

$$\phi_n(t) = e^{jn\omega_0 t} \quad n = 0, \pm 1, \pm 2, \ldots$$

where ω_0 is a constant to be determined.

Invoking the orthogonality relation gives us

$$\int_{t_1}^{t_2} \phi_n(t)\phi_m^*(t)dt = \int_{t_1}^{t_2} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt$$

$$= \int_{t_1}^{t_2} e^{j(n-m)\omega_0 t} dt$$

For $n = m$ we have

$$\int_{t_1}^{t_2} e^{j(m-m)\omega_0 t} dt = \int_{t_1}^{t_2} dt$$

$$= t_2 - t_1$$

For $n \neq m$ we have

$$\int_{t_1}^{t_2} e^{j(n-m)\omega_0 t} dt = \frac{1}{j(n-m)\omega_0}\left[e^{j(n-m)\omega_0 t_2} - e^{j(n-m)\omega_0 t_1}\right]$$

$$= \frac{1}{j(n-m)\omega_0} e^{j(n-m)\omega_0 t_1}\left[e^{j(n-m)\omega_0 (t_2 - t_1)} - 1\right]$$

In order to be an orthogonal basis, we need the bracketed term

$$\left[e^{j(n-m)\omega_0 (t_2 - t_1)} - 1\right]$$

to be equal to zero. Notwithstanding the trivial case where $t_1 = t_2$, if we note that $(n - m)$ is an integer, we can force the bracketed term to zero if

$$\omega_0 (t_2 - t_1) = 2\pi$$

Hence, the set of complex exponential functions

$$\phi_n(t) = e^{jn\omega_0 t} \quad n = 0, \pm 1, \pm 2, \ldots$$

forms a mutually orthogonal set of basis functions over the interval $[t_1, t_2]$, provided that

$$\omega_0 = \frac{2\pi}{(t_2 - t_1)}$$

because

$$\int_{t_1}^{t_2} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt = \begin{cases} (t_2 - t_1) & n = m \\ 0 & n \neq m \end{cases}$$

4 The Exponential Fourier Series

The exponential Fourier Series expansion for a function $f(t)$ defined over an arbitrary time interval $[t_1, t_2]$ is

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where

$$\omega_0 = \frac{2\pi}{(t_2 - t_1)}$$

The Fourier Series expansion expresses the function $f(t)$ over the time interval $[t_1, t_2]$ using a linear combination of the mutually orthogonal basis functions $\phi_n(t) = e^{jn\omega_0 t}$ for all integer values of n.

The coefficients, F_n, can be determined by multiplying both sides of the $f(t)$ expansion expression by $\phi_m^*(t) = e^{-jm\omega_0 t}$:

$$f(t)e^{-jm\omega_0 t} = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{-jm\omega_0 t}$$

and then integrating with respect to t over the interval $[t_1, t_2]$:

$$\int_{t_1}^{t_2} f(t)e^{-jm\omega_0 t} dt = \int_{t_1}^{t_2} \left[\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} e^{-jm\omega_0 t}\right] dt$$

$$= \sum_{n=-\infty}^{\infty} F_n \int_{t_1}^{t_2} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt$$

Because $\phi_n(t) = e^{jn\omega_0 t}$ is a mutually orthogonal basis, all of the $n \neq m$ terms in the summation vanish, and only the $n = m$ term is left:

$$\int_{t_1}^{t_2} f(t)e^{-jm\omega_0 t} dt = \sum_{n=m}^{m} F_n \int_{t_1}^{t_2} dt$$

$$= F_m(t_2 - t_1)$$

Therefore

$$F_m = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) e^{-jm\omega_0 t} dt$$

and so

$$F_n = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) e^{-jn\omega_0 t} dt$$

A function is said to be *T*-periodic if

$$f_T(t) = f_T(t + T)$$

The exponential Fourier Series expansion for a *T*-periodic function $f_T(t)$ is then

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

and the fundamental frequency is

$$\omega_0 = \frac{2\pi}{T}$$

Because $f_T(t)$ is *T*-periodic, the coefficients, F_n, are the same for every period and, therefore, completely describe the entire function.

5 The Trigonometric Fourier Series

The exponential Fourier Series expansion is a general form that imposes no restrictions on the nature of $f(t)$. Typically, $f(t)$ is a real-valued function (e.g., an audio signal). Given this restriction, it would be advantageous if we could somehow express the Fourier Series in terms of real-valued functions. In order to explore this, we first need to explore some properties of complex functions.

Any complex function, $f(t)$, can be expressed in terms of its real and imaginary components. Dropping the time component for brevity, we can express a complex function f as:

$$f = f_r + j\, f_i$$

and its complex conjugate as

$$f^* = f_r - j\, f_i$$

Combining both forms gives us

$$f_r = \frac{1}{2}(f + f^*)$$

$$f_i = \frac{1}{2j}(f - f^*)$$

If we have two functions, f_1 and f_2, their product is

$$\begin{aligned} f_1 f_2 &= \left(f_{1,r} + j\, f_{1,i}\right)\left(f_{2,r} + j\, f_{2,i}\right) \\ &= \left(f_{1,r} f_{2,r} - f_{1,i} f_{2,i}\right) + j\left(f_{1,r} f_{2,i} + f_{1,i} f_{2,r}\right) \end{aligned}$$

The real component of the product is then

$$\Re\{f_1 f_2\} = \Re\{f_1\}\Re\{f_2\} - \Im\{f_1\}\Im\{f_2\}$$

Now, let us recall the exponential Fourier Series expression

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

Expanding the exponential term using Euler's identity gives

$$f(t) = \sum_{n=-\infty}^{\infty} F_n(\cos(n\omega_0 t) + j\sin(n\omega_0 t))$$

If we impose the restriction that $f(t)$ be real-valued, then

$$f(t) = \sum_{n=-\infty}^{\infty} \Re\{F_n\}\Re\{e^{jn\omega_0 t}\} - \Im\{F_n\}\Im\{e^{jn\omega_0 t}\}$$

$$= \sum_{n=-\infty}^{\infty} \Re\{F_n\}\cos(n\omega_0 t) - \Im\{F_n\}\sin(n\omega_0 t)$$

$$= \sum_{n=-\infty}^{\infty} \Re\{F_n\}\cos(n\omega_0 t) - \sum_{n=-\infty}^{\infty} \Im\{F_n\}\sin(n\omega_0 t)$$

Recall the expression for the coefficients, F_n

$$F_n = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) e^{-jn\omega_0 t} dt$$

The complex conjugate form of the coefficients is

$$F_n^* = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) e^{jn\omega_0 t} dt \; = \; F_{-n}$$

Using the real and imaginary function relations presented above gives

$$\Re\{F_n\} = \frac{1}{2}(F_n + F_n^*) = \frac{1}{2}(F_n + F_{-n})$$

$$\Im\{F_n\} = \frac{1}{2j}(F_n - F_n^*) = \frac{1}{2j}(F_n - F_{-n})$$

Then our expression for $f(t)$ becomes

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2}(F_n + F_{-n})\cos(n\omega_0 t) - \sum_{n=-\infty}^{\infty} \frac{1}{2j}(F_n - F_{-n})\sin(n\omega_0 t)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}(F_n + F_{-n})\cos(n\omega_0 t) + \sum_{n=-\infty}^{\infty} \frac{1}{2}j(F_n - F_{-n})\sin(n\omega_0 t)$$

Define

$$a_n \triangleq (F_n + F_{-n}) = 2\,\Re\{F_n\}$$

$$b_n \triangleq j\,(F_n - F_{-n}) = -2\,\Im\{F_n\}$$

Then $f(t)$ becomes

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\, a_n \cos(n\omega_0 t) + \sum_{n=-\infty}^{\infty} \frac{1}{2}\, b_n \sin(n\omega_0 t)$$

Note that a_n is an even function of n, and that b_n is an odd function of n. Therefore, the two summations are symmetric about $n = 0$. Because of the symmetry, we can double the quantities inside the summations and then sum over positive values of n. Hence, $f(t)$ becomes

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

where $a_0 = 2F_0$.

Expressions for the a_n and b_n terms can be determined by applying real and imaginary decomposition of the expression for the coefficients, F_n, and remembering that $f(t)$ is real-valued:

$$\Re\{F_n\} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) \cos(-n\omega_0 t)\, dt$$

$$= \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) \cos(n\omega_0 t)\, dt$$

$$\Im\{F_n\} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) \sin(-n\omega_0 t)\, dt$$

$$= \frac{-1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) \sin(n\omega_0 t)\, dt$$

Then a_n and b_n are found by

$$a_n = 2\,\Re\{F_n\}$$

$$= \frac{2}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n = -2\,\Re\{F_n\}$$

$$= \frac{2}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t)\sin(n\omega_0 t)\,dt$$

So, if $f(t)$ is a real-valued function defined over an arbitrary time interval $[t_1, t_2]$, it can be expressed as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

where

$$a_0 = \frac{2}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t)dt$$

$$a_n = \frac{2}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t)\cos(n\omega_0 t)\,dt \quad ,n \neq 0$$

$$b_n = \frac{2}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t)\sin(n\omega_0 t)\,dt$$

The above relations constitute the trigonometric form of the Fourier Series expansion for a real-valued function, $f(t)$. It is this form that is usually presented first when learning Fourier Series, and the form which lends itself most easily to laboratory applications. However, the exponential form is better suited when exploring the full range of Fourier analysis concepts.

6 The Fourier Transform

In preparation for performing the $T \to \infty$ limiting operations, we define

$$\omega_n \triangleq n\omega_0$$

and

$$F(\omega_n) \triangleq TF_n$$

so that the exponential Fourier Series expansion for a T-periodic function $f_T(t)$ can be rewritten as

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega_n) e^{j\omega_n t}$$

where

$$F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt$$

The spacing between adjacent lines in the Fourier line spectrum is

$$\Delta\omega = \omega_0 = \frac{2\pi}{T}$$

and so

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} F(\omega_n) e^{j\omega_n t}$$

or

$$f_T(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \Delta\omega$$

Now, as $T \to \infty$, the periodic nature of $f_T(t)$ become aperiodic, with $f_T(t) \to f(t)$, $\omega_n \to \omega$, and $\Delta\omega \to d\omega$, and so

$$\lim_{T\to\infty} f_T(t) = \lim_{T\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \Delta\omega$$

becomes

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

and

$$\lim_{T \to \infty} F(\omega_n) = \lim_{T \to \infty} \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt$$

becomes

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

The integral expression for $F(\omega)$ is the Fourier Transform of $f(t)$, whereas the integral expression for $f(t)$ is the inverse Fourier Transform of $F(\omega)$.

7 Numerical Evaluation of the Fourier Transform Integrals

Consider the Fourier Transform integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

and the task of numerically evaluating it using rectangle approximation. If the width of the rectangles is denoted by T_s, then the integral approximation of $F(\omega)$ becomes

$$F_A(\omega) = \sum_{k=-\infty}^{\infty} f(kT_s)e^{-j\omega kT_s}T_s$$

The approximation essentially samples $f(t)$ at T_s intervals. As T_s is reduced, the approximation becomes better.

An immediate consequence of the sampling of $f(t)$ as $f(kT_s)$ is that $F_A(\omega)$ is periodic:

$$F_A(\omega) = F_A(\omega + \Omega)$$

where

$$\Omega = \frac{2\pi}{T_s}$$

because the kernel term can be expanded as

$$e^{-j\omega kT_s} = e^{-j(\omega kT_s + k2\pi)} = e^{-j(\omega + \frac{2\pi}{T_s})kT_s} = e^{-j(\omega+\Omega)kT_s}$$

Likewise, the inverse Fourier Transform integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

can be evaluated numerically using rectangle approximation. If the width of the rectangles is denoted by Ω_s, then the integral approximation of $f(t)$ becomes

$$f_A(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(n\Omega_s)e^{jn\Omega_s t}\Omega_s$$

Similar to the periodicity of $F_A(\omega)$, an immediate consequence of the sampling of $F(\omega)$ as $F(n\Omega_s)$ is that $f_A(t)$ is periodic:

$$f_A(t) = f_A(t+T)$$

where

$$T = \frac{2\pi}{\Omega_s}$$

because the kernel term can be expanded as

$$e^{jn\Omega_s t} = e^{j(n\Omega_s t + n2\pi)} = e^{j(t+\frac{2\pi}{\Omega_s})n\Omega_s} = e^{j(t+T)n\Omega_s}$$

Note that the summations for both $f_A(t)$ and $F_A(\omega)$ are infinite. But these results form the foundation for the existence of the Discrete Fourier Transform and its periodic properties, the key observation being that the introduction of the sampling of a function in one domain causes the transformed data to be periodic in the transform range.

8 The Discrete Fourier Transform

Given a sequence of N equally spaced samples of $f(t)$ over the interval $[0, NT_s)$, where T_s is the sample time spacing:

$$f_D(kT_s) = \{f(0), f(T_s), f(2T_s), \dots, f\big((N-1)T_s\big)\}$$

then $f_D(kT_s)$ can be expressed as

$$f_D(kT_s) = \frac{1}{N}\sum_{n=0}^{N-1} F_D(n\Omega_s)e^{j\Omega_s T_s kn} \quad , k = 0,1,2,\dots,N-1$$

where

$$F_D(n\Omega_s) = \sum_{k=0}^{N-1} f_D(kT_s)e^{-j\Omega_s T_s nk} \quad , n = 0,1,2,\dots,N-1$$

and where Ω_s is the frequency spacing

$$\Omega_s = \frac{2\pi}{NT_s}$$

The summation expression for $F_D(n\Omega_s)$ is the Discrete Fourier Transform (DFT) of $f_D(kT_s)$, whereas the summation expression for $f_D(kT_s)$ is the inverse DFT of $F_D(n\Omega_s)$.

It is worth noting that $f_D(kT_s)$ is periodic by NT_s:

$$f_D(kT_s) = f_D(kT_s + NT_s)$$

This can be seen by expanding its kernel term as

$$e^{j\Omega_s T_s kn} = e^{jkT_s\Omega_s n} = e^{-j(kT_s\Omega_s n + n2\pi)} = e^{-j(kT_s + \frac{2\pi}{\Omega_s})n\Omega_s} = e^{-j(kT_s + NT_s)n\Omega_s}$$

Likewise, $F_D(n\Omega_s)$ is periodic by $N\Omega_s$:

$$F_D(n\Omega_s) = F_D(n\Omega_s + N\Omega_s)$$

because its kernel term can be expanded as

$$e^{-j\Omega_s T_s nk} = e^{-jn\Omega_s T_s k} = e^{-j(n\Omega_s T_s k + k2\pi)} = e^{-j(n\Omega_s + \frac{2\pi}{T_s})kT_s} = e^{-j(n\Omega_s + N\Omega_s)kT_s}$$

Observe that NT_s is the period of one full cycle of the $f_D(kT_s)$ sequence, and so the cycle period T is

$$T = NT_s$$

and therefore the DFT frequency spacing

$$\Omega_s = \frac{2\pi}{T}$$

is the same as the fundamental frequency parameter, ω_0, in the Fourier Series expansion

$$\omega_0 = \frac{2\pi}{T}$$

We also know that $N\Omega_s$ is the period of one full cycle of the $F_D(n\Omega_s)$ sequence, and so the cycle period Ω is

$$\Omega = N\Omega_s$$

which is the radian sampling frequency

$$\Omega = \frac{2\pi}{T_s}$$

So $f_D(kT_s)$ is periodic by T:

$$f_D(kT_s) = f_D(kT_s + T)$$

and $F_D(n\Omega_s)$ is periodic by Ω:

$$F_D(n\Omega_s) = F_D(n\Omega_s + \Omega)$$

In addition, because the Fourier Transform of real-valued functions is always symmetric about $\omega = 0$, the DFT envelope exhibits a fold-over frequency that is half the periodic frequency Ω:

$$\omega_f = \frac{\Omega}{2} = \frac{N\Omega_s}{2} = \frac{\pi}{T_s}$$

To illustrate the DFT relationships between $f_D(kT_s)$ and $F_D(n\Omega_s)$, let us consider a unit pulse function that begins at $t = 0$ and is of duration 1.0 s. Hence,

$$f(t) = p_d\left(t - \frac{d}{2}\right)$$

where $d = 1.0$ is the pulse duration. The Fourier Transform of a pulse function is a sinc function, and in this case, because our pulse is time shifted, the Fourier Transform of $f(t)$ is

$$F(\omega) = d\,\mathrm{sinc}\left(\frac{\omega d}{2}\right)e^{-j\omega\frac{d}{2}} = \mathrm{sinc}\left(\frac{\omega}{2}\right)e^{-j\omega\frac{1}{2}}$$

For our illustration, we will only focus on the magnitude characteristics of $F(\omega)$, and so we have

$$|F(\omega)| = \mathrm{sinc}\left(\frac{\omega}{2}\right)$$

The delayed unit pulse and its Fourier Transform magnitude envelope are shown in Figure 1.

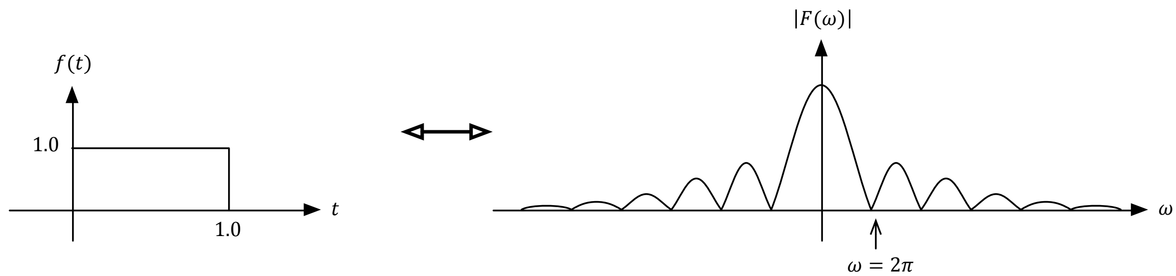

Figure 1 – Pulse Function and its Fourier Transform

With the exception of $\omega = 0$, we see that $|F(\omega)| = 0$ for integer multiples of 2π.

For a selected sample time of $T_s = 0.2$ s and a selected sample set size of $N = 15$, the corresponding periodic $f_D(kT_s)$ is shown in Figure 2, with the points of the fundamental cycle highlighted for emphasis.

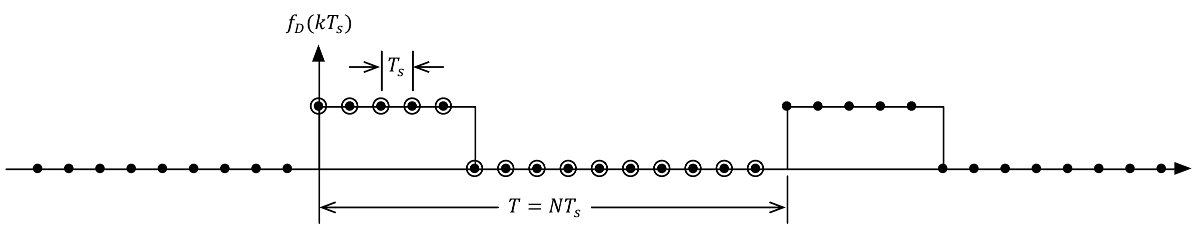

Figure 2 – Periodic Discrete Pulse Function, *N* = 15

The magnitude envelope of the corresponding DFT of $f_D(kT_s)$ is shown in Figure 3, with the points of the fundamental cycle highlighted for emphasis.

It should be understood that the DFT computes the points from $\omega = 0$ to $\omega = (N-1)\Omega_s$ because the values of $F_D(n\Omega_s)$ are from $n = 0$ to $n = N - 1$. Because the Fourier Transform of a real-valued $f(t)$ is symmetric about $\omega = 0$, the points from $\omega = \omega_f$ to $\omega = (N-1)\Omega_s$ can be regarded as being the negative frequencies of the Fourier Transform envelope.

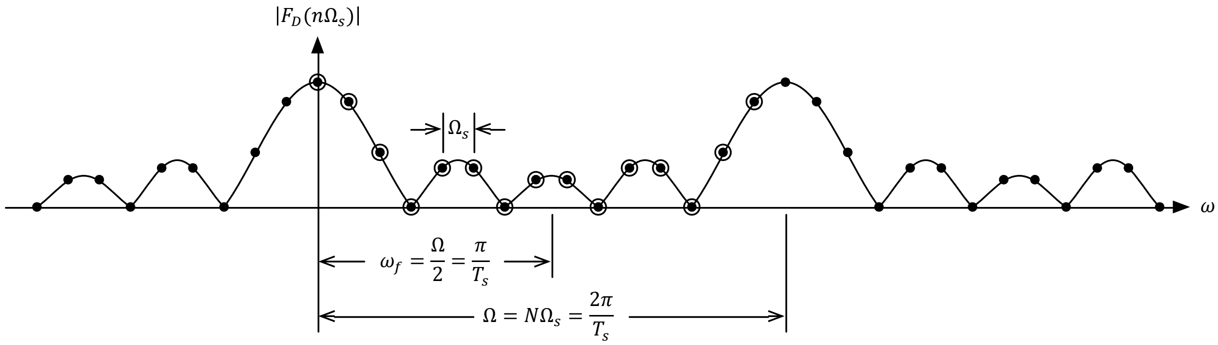

Figure 3 – Discrete Fourier Transform of Pulse Function, *N* = 15

It is important to observe that the sampling period T_s sets the periodic frequency Ω:

$$\Omega = \frac{2\pi}{T_s}$$

whereas the cycle period T sets the frequency spacing Ω_s:

$$\Omega_s = \frac{2\pi}{T} = \frac{2\pi}{NT_s}$$

The value of N for a given T_s essentially establishes the fidelity of $F_D(n\Omega_s)$. By increasing N, we increase the cycle period T (because $T = NT_s$), which thereby reduces the frequency spacing Ω_s, thereby giving us more points within the frequency period Ω established by the given T_s value.

The significance of this observation is that it is the value of T_s, and not N, that governs how much of the original $F(\omega)$ envelope we see in the spectral window defined by the frequency period Ω. Changing N simply affects the fidelity of $F_D(n\Omega_s)$; increasing N simply adds more points to $F_D(n\Omega_s)$ within the same spectral window defined by Ω. To increase how much of the original $F(\omega)$ envelope we see in $F_D(n\Omega_s)$, we need to increase Ω, which means we need to decrease T_s (i.e., we need to sample $f(t)$ more often). Usually this implies an increase in N because faster sampling gives us more data for a given time window, but it is strictly T_s that governs Ω.

In practice, the T_s and Ω_s values are not needed to perform the actual DFT evaluation operations since they only serve as scale factors for interpreting the results. If we note that

$$\Omega_s T_s = \frac{2\pi}{N}$$

we can then express the DFT equations simply in terms of the indexing values:

$$f_D(k) = \frac{1}{N}\sum_{n=0}^{N-1} F_D(n)e^{j\frac{2\pi}{N}kn} \quad ,k = 0,1,2,\dots,N-1$$

$$F_D(n) = \sum_{k=0}^{N-1} f_D(k)e^{-j\frac{2\pi}{N}nk} \quad ,n = 0,1,2,\dots,N-1$$

where $f_D(k)$ implies $f_D(kT_s)$, and $F_D(n)$ implies $F_D(n\Omega_s)$.

As with $f_D(kT_s)$, $f_D(k)$ is periodic:

$$f(k) = f(k+N)$$

because its kernel term can be expanded as

$$e^{j\frac{2\pi}{N}kn} = e^{j(\frac{2\pi}{N}kn+n2\pi)} = e^{j\frac{2\pi}{N}(k+N)n}$$

Likewise, as with $F_D(n\Omega_s)$, $F_D(n)$ is periodic:

$$F_D(n) = F_D(n+N)$$

because its kernel term can be expanded as

$$e^{-j\frac{2\pi}{N}nk} = e^{-j(\frac{2\pi}{N}nk+k2\pi)} = e^{-j\frac{2\pi}{N}(n+N)k}$$

Since the T_s and Ω_s terms are no longer included as part of the DFT evaluation operations, both $f_D(k)$ and $F_D(n)$ are periodic by N.

At this point, it is worth noting that the complexity of the DFT procedure, either forward or inverse, is $O(n^2)$, and therefore it is not used directly in practice unless N is sufficiently small. The next section presents a much more efficient algorithm for computing the DFT.

9 The Fast Fourier Transform

The DFT equations, while concise, are actually computationally expensive when implemented directly from their mathematical expressions. Fortunately, we can optimize the computations such that the evaluation of the DFT becomes quite efficient.

In this document, we will present the popular Cooley-Tukey algorithm. However, there are many variations and permutations of the DFT evaluation, and research is still very much alive (e.g., using GPUs for massively parallel transformations).

Recall the DFT equations:

$$f_D(k) = \frac{1}{N}\sum_{n=0}^{N-1} F_D(n) e^{j\frac{2\pi}{N}kn} \quad , k = 0,1,2,\dots,N-1$$

$$F_D(n) = \sum_{k=0}^{N-1} f_D(k) e^{-j\frac{2\pi}{N}nk} \quad , n = 0,1,2,\dots,N-1$$

and let us focus on the forward DFT, $F_D(n)$.

If we impose the restriction that N must be an even number, we can separate $F_D(n)$ into both even, $k = 2m$, and odd, $k = 2m + 1$, summations, each of length $N/2$:

$$F_D(n) = \sum_{m=0}^{N/2-1} f_D(2m) e^{-j\frac{2\pi}{N}n(2m)} + \sum_{m=0}^{N/2-1} f_D(2m+1) e^{-j\frac{2\pi}{N}n(2m+1)}$$

Let us factor out the $e^{-j\frac{2\pi}{N}n}$ term from the odd sum (it is not dependent the m index):

$$F_D(n) = \sum_{m=0}^{N/2-1} f_D(2m) e^{-j\frac{2\pi}{N}n(2m)} + e^{-j\frac{2\pi}{N}n} \sum_{m=0}^{N/2-1} f_D(2m+1) e^{-j\frac{2\pi}{N}n(2m)}$$

and then rearrange the $e^{-j\frac{2\pi}{N}n(2m)}$ factors to be in terms of $N/2$ so that

$$F_D(n) = \sum_{m=0}^{N/2-1} f_D(2m) e^{-j\frac{2\pi}{N/2}nm} + e^{-j\frac{2\pi}{N}n} \sum_{m=0}^{N/2-1} f_D(2m+1) e^{-j\frac{2\pi}{N/2}nm}$$

If we define the even and odd summation terms as

$$F_{De}(n) = \sum_{m=0}^{N/2-1} f_D(2m) e^{-j\frac{2\pi}{N/2}nm}$$

$$F_{Do}(n) = \sum_{m=0}^{N/2-1} f_D(2m+1) e^{-j\frac{2\pi}{N/2}nm}$$

then our forward DFT expression becomes

$$F_D(n) = F_{De}(n) + e^{-j\frac{2\pi}{N}n} F_{Do}(n) \quad , n = 0,1,2,\ldots,N-1$$

The factor $e^{-j\frac{2\pi}{N}n}$ is known as a twiddle factor, a name which will become apparent later. The twiddle factor is an n^{th} root of unity, and it is common to define

$$W_N = e^{-j\frac{2\pi}{N}}$$

so that

$$W_N^n = (W_N)^n = e^{-j\frac{2\pi}{N}n}$$

Observe that W_N is periodic in N. Figure 4 illustrates the roots of unity for a period of $N = 5$:

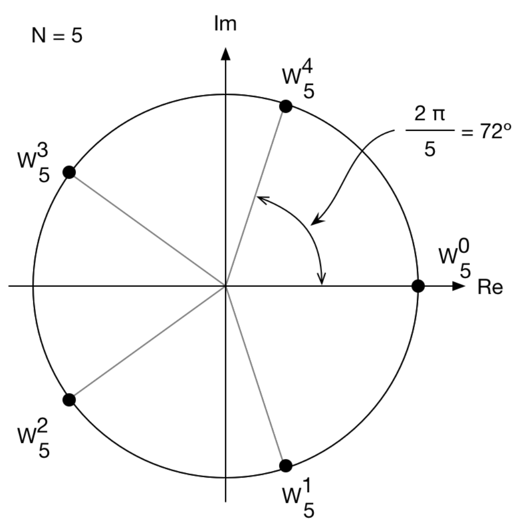

Figure 4 – Roots of Unity, *N* = 5

Table 1 presents some useful identities for W_N:

Table 1 – Some Useful Twiddle Factor Identities

Identity	Derivation
$W_N^0 = 1$	$W_N^0 = e^{-j\frac{2\pi}{N}(0)} = 1$
$W_N^{N/4} = -j$	$W_N^{N/4} = e^{-j\frac{2\pi}{N}\left(\frac{N}{4}\right)} = e^{-j\frac{\pi}{2}} = -j$
$W_N^{N/2} = -1$	$W_N^{N/2} = e^{-j\frac{2\pi}{N}\left(\frac{N}{2}\right)} = e^{-j\pi} = -1$
$W_N^{3N/4} = j$	$W_N^{3N/4} = e^{-j\frac{2\pi}{N}\left(\frac{3N}{4}\right)} = e^{-j\frac{3\pi}{2}} = j$
$W_N^N = 1$	$W_N^N = e^{-j\frac{2\pi}{N}(N)} = e^{-j2\pi} = 1$
$W_N^{k+N} = W_N^k$	$W_N^{k+N} = e^{-j\frac{2\pi}{N}(k+N)} = e^{-j\frac{2\pi}{N}k}e^{-j2\pi} = e^{-j\frac{2\pi}{N}k} = W_N^k$
$W_N^{k+N/2} = -W_N^k$	$W_N^{k+N/2} = e^{-j\frac{2\pi}{N}(k+\frac{N}{2})} = e^{-j\frac{2\pi}{N}k}e^{-j\pi} = -e^{-j\frac{2\pi}{N}k} = -W_N^k$
$W_N^{k+N/4} = -jW_N^k$	$W_N^{k+N/4} = e^{-j\frac{2\pi}{N}(k+\frac{N}{4})} = e^{-j\frac{2\pi}{N}k}e^{-j\frac{\pi}{2}} = -je^{-j\frac{2\pi}{N}k} = -jW_N^k$
$W_N^{k+3N/4} = jW_N^k$	$W_N^{k+3N/4} = e^{-j\frac{2\pi}{N}(k+\frac{3N}{4})} = e^{-j\frac{2\pi}{N}k}e^{-j\frac{3\pi}{2}} = je^{-j\frac{2\pi}{N}k} = jW_N^k$
$W_N^{ak} = W_{N/a}^k$	$W_N^{ak} = e^{-j\frac{2\pi}{N}(ak)} = e^{-j\frac{2\pi}{N/a}k} = W_{N/a}^k$
$W_{aN}^{ak} = W_N^k$	$W_{aN}^{ak} = e^{-j\frac{2\pi}{aN}(ak)} = e^{-j\frac{2\pi}{N}k} = W_N^k$

Using twiddle factors, the even and odd summations can be written as

$$F_{De}(n) = \sum_{m=0}^{N/2-1} f_D(2m) W_{N/2}^{nm}$$

$$F_{Do}(n) = \sum_{m=0}^{N/2-1} f_D(2m+1) W_{N/2}^{nm}$$

and the DFT expression becomes

$$F_D(n) = F_{De}(n) + W_N^n F_{Do}(n) \quad , n = 0,1,2,\dots,N-1$$

Observe that, while $F_D(n)$ is periodic by N, both $F_{De}(n)$ and $F_{Do}(n)$ are periodic by $N/2$:

$$F_{De}\left(n + \frac{N}{2}\right) = F_{De}(n)$$

$$F_{Do}\left(n + \frac{N}{2}\right) = F_{Do}(n)$$

We can then break down $F_D(n)$ into two $N/2$ sequences:

$$F_D(n) = F_{De}(n) + W_N^n F_{Do}(n) \qquad , n = 0,1,2,\dots,\frac{N}{2}$$

$$F_D\left(n + \frac{N}{2}\right) = F_{De}(n) + W_N^{(n+\frac{N}{2})} F_{Do}(n) \quad , n = 0,1,2,\dots,\frac{N}{2}$$

Using the identity $W_N^{n+N/2} = -W_N^n$, we then have

$$F_D(n) = F_{De}(n) + W_N^n F_{Do}(n) \quad , n = 0,1,2,\dots,\frac{N}{2}$$

$$F_D\left(n + \frac{N}{2}\right) = F_{De}(n) - W_N^n F_{Do}(n) \quad , n = 0,1,2,\dots,\frac{N}{2}$$

Notice that we have now halved the number of computations for $F_D(n)$. In addition, note that both $F_{De}(n)$ and $F_{Do}(n)$ are now effectively their own DFT sequences, each of size $N/2$. As long as $N/2$ is also even, we can repeat this decomposition strategy on the subordinate DFT sequences. If $N = 2^r$, where r is a positive integer, we can continue this decomposition transformation until we have reduced the problem into a set of $N/2$ DFTs, each DFT being only 2 points in size.

The algorithm we have just presented is known as the Fast Fourier Transform (FFT) algorithm. Because of the "divide and conquer" approach, we have reduced the complexity of DFT computation from an inefficient $O(n^2)$ to a respectable complexity of $O(n \log n)$.

Furthermore, because of the identity $W_{aN}^{an} = W_N^n$, we can take advantage of the repetition of the twiddle factors throughout the division of the DFT sets, and therefore only N twiddle factors need to be computed. However, because the twiddle factors occur in conjugate pairs, only $N/2$ twiddle factors actually need to be computed.

Let us further examine the structure of the FFT algorithm. As an illustration of the FFT breakdown operations, consider a signal of $N = 8$ points. The FFT "divide and conquer" algorithm transforms the evaluation of one 8-point signal into a set of four 2-point signals. This breakdown is shown in Figure 5.

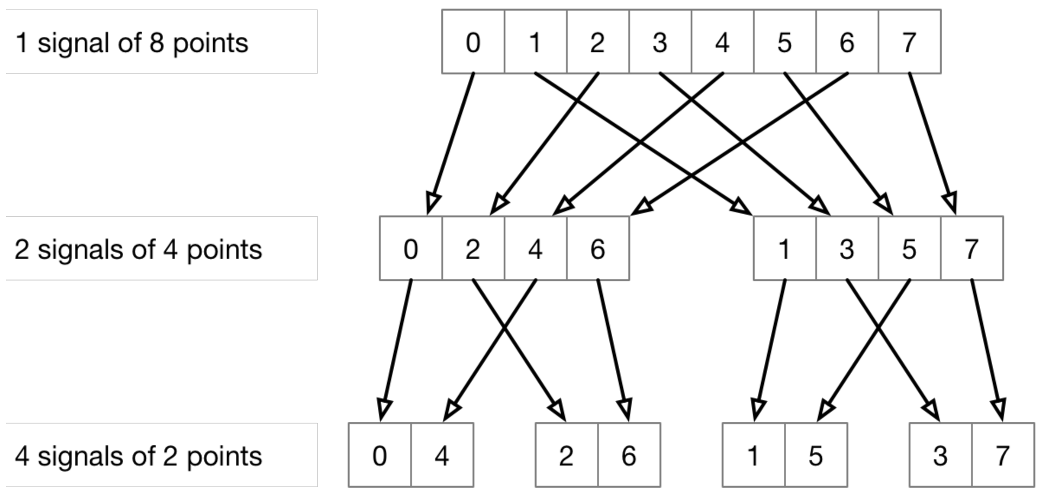

Figure 5 – 8-Point FFT Decomposition

The 2-point DFT is the simplest structure to evaluate. The equations for a 2-point DFT are:

$$\begin{aligned} \mathrm{X}(0) &= \mathrm{x}(0) + W_2^0 x(1) \\ &= \mathrm{x}(0) + x(1) \end{aligned}$$

$$\begin{aligned} \mathrm{X}(1) &= \mathrm{x}(0) - W_2^0 x(1) \\ &= \mathrm{x}(0) - x(1) \end{aligned}$$

The flow diagram for the 2-point DFT equations in Figure 6 shows the familiar FFT butterfly structure.

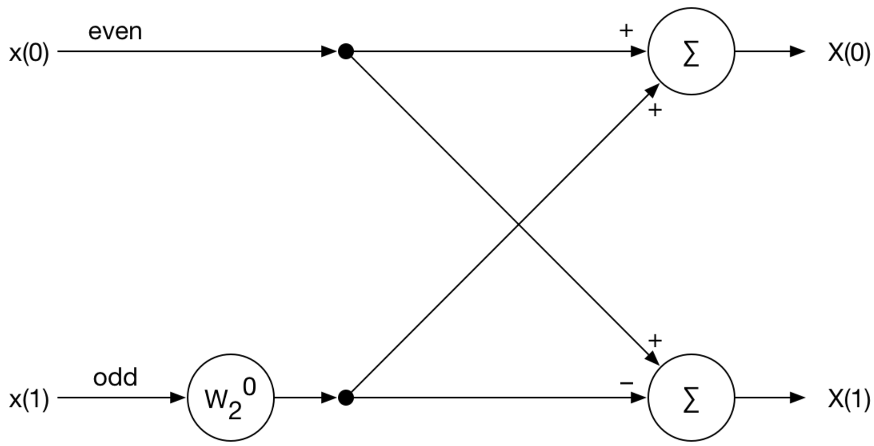

Figure 6 – 2-Point FFT Butterfly Structure

The entire butterfly structure of the "divide and conquer" breakdown for the 8-point problem can be visualized via the trellis diagram of Figure 7.

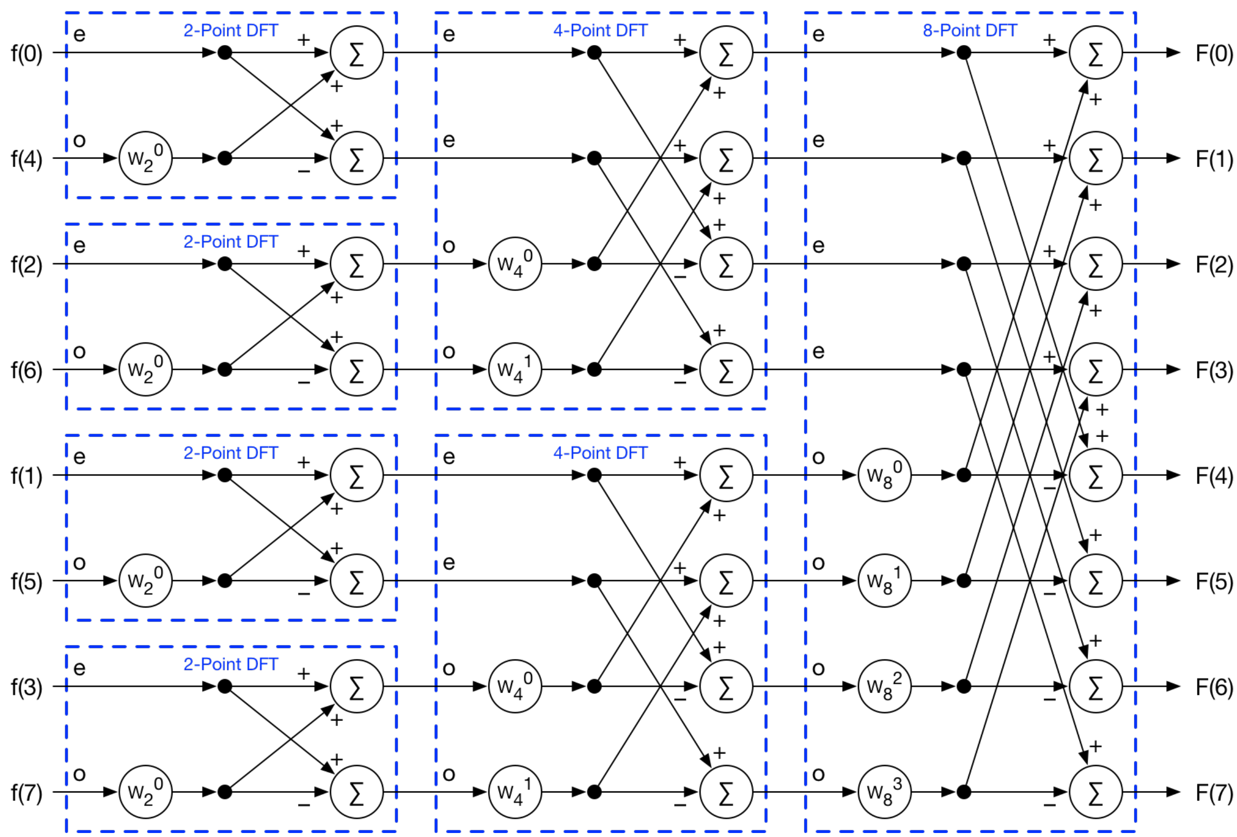

Figure 7 – 8-Point FFT Butterfly Trellis

Observe that the twiddle factors "twiddle" the odd inputs of each subordinate DFT structure in the butterfly trellis. Hence, only the $N/2$ twiddle factors of the lower half of the unit circle are needed. The combinations throughout the butterfly trellis complete the calculations for the upper half of the unit circle.

An appreciation of the butterfly trellis combinations can be obtained by recalling the forward DFT equation, expressed using twiddle factors:

$$F_D(n) = \sum_{k=0}^{N-1} f_D(k) W_N^{nk} \quad , n = 0,1,2,\ldots,N-1$$

This can be re-expressed in matrix notation as follows:

$$\begin{bmatrix} W_N^{(0)(0)} & W_N^{(0)(1)} & W_N^{(0)(2)} & W_N^{(0)(3)} & \cdots & W_N^{(0)(N-1)} \\ W_N^{(1)(0)} & W_N^{(1)(1)} & W_N^{(1)(2)} & W_N^{(1)(3)} & \cdots & W_N^{(1)(N-1)} \\ W_N^{(2)(0)} & W_N^{(2)(1)} & W_N^{(2)(2)} & W_N^{(2)(3)} & \cdots & W_N^{(2)(N-1)} \\ W_N^{(3)(0)} & W_N^{(3)(1)} & W_N^{(3)(2)} & W_N^{(3)(3)} & \cdots & W_N^{(3)(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1)(0)} & W_N^{(N-1)(1)} & W_N^{(N-1)(2)} & W_N^{(N-1)(3)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} f_D(0) \\ f_D(2) \\ f_D(3) \\ f_D(4) \\ \vdots \\ f_D(N-1) \end{bmatrix} = \begin{bmatrix} F_D(0) \\ F_D(1) \\ F_D(2) \\ F_D(3) \\ \vdots \\ F_D(N-1) \end{bmatrix}$$

For the 8-point example, the forward DFT expression is

$$F_D(n) = \sum_{k=0}^{7} f_D(k) W_8^{nk} \quad , n = 0, 1, 2, \ldots, 7$$

and its matrix formulation is

$$\begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \begin{bmatrix} f_D(0) \\ f_D(1) \\ f_D(2) \\ f_D(3) \\ f_D(4) \\ f_D(5) \\ f_D(6) \\ f_D(7) \end{bmatrix} = \begin{bmatrix} F_D(0) \\ F_D(1) \\ F_D(2) \\ F_D(3) \\ F_D(4) \\ F_D(5) \\ F_D(6) \\ F_D(7) \end{bmatrix}$$

Substituting numerical values for the twiddle factors gives us

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.7-0.7j & -j & -0.7-0.7j & -1 & -0.7+0.7j & j & 0.7+0.7j \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.7-0.7j & j & 0.7-0.7j & -1 & 0.7+0.7j & -j & -0.7+0.7j \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.7+0.7j & -j & 0.7+0.7j & -1 & 0.7-0.7j & j & -0.7-0.7j \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.7+0.7j & j & -0.7+0.7j & -1 & -0.7-0.7j & -j & 0.7-0.7j \end{bmatrix} \begin{bmatrix} f_D(0) \\ f_D(1) \\ f_D(2) \\ f_D(3) \\ f_D(4) \\ f_D(5) \\ f_D(6) \\ f_D(7) \end{bmatrix} = \begin{bmatrix} F_D(0) \\ F_D(1) \\ F_D(2) \\ F_D(3) \\ F_D(4) \\ F_D(5) \\ F_D(6) \\ F_D(7) \end{bmatrix}$$

Using matrix multiplication, the expression for $F_D(0)$ evaluates to

$$F_D(0) = f_D(0) + f_D(1) + f_D(2) + f_D(3) + f_D(4) + f_D(5) + f_D(6) + f_D(7)$$

Figure 8 shows the path of the $F_D(0)$ evaluation through the 8-point butterfly trellis.

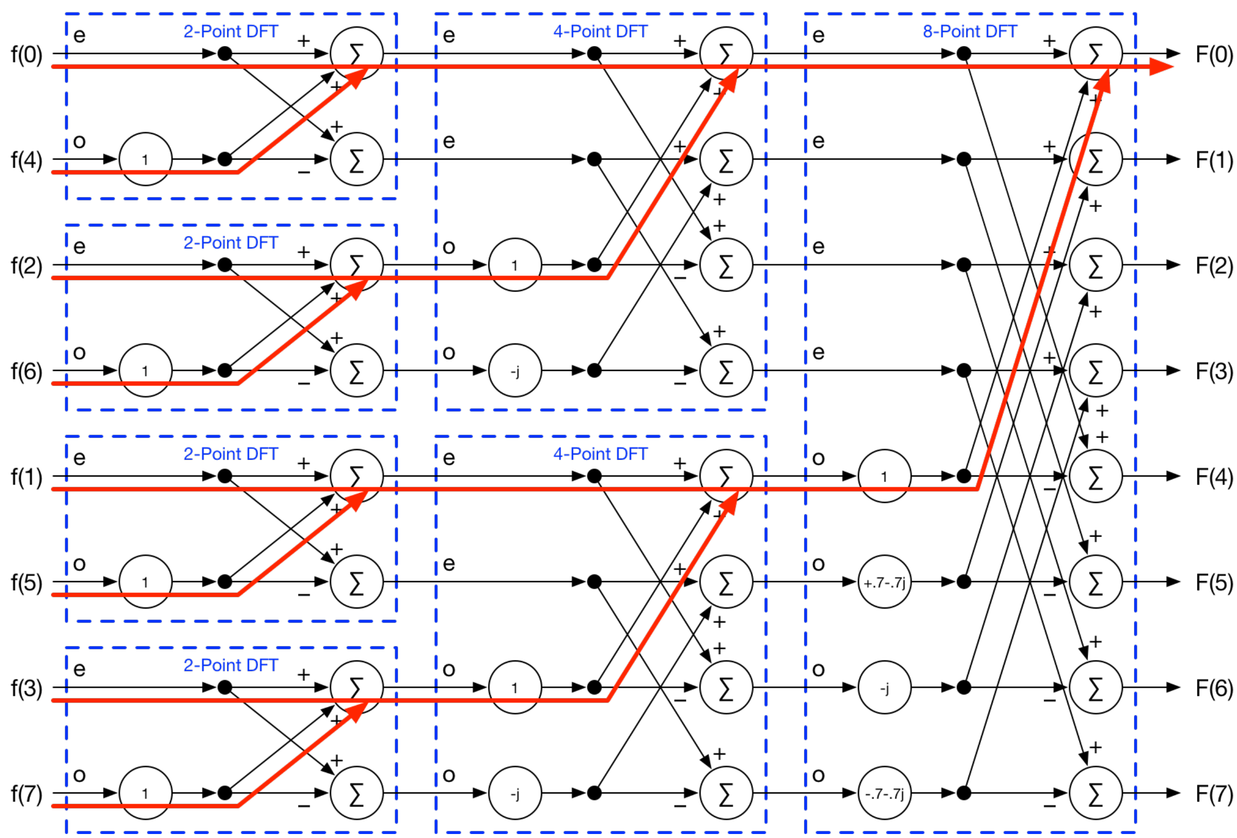

Figure 8 – 8-Point FFT Butterfly Trellis, Path for $F_D(0)$

Similarly, the expression for $F_D(3)$ evaluates to

$$F_D(3) = f_D(0) + (-0.7 - 0.7j)f_D(1) + (j)f_D(2) + (0.7 - 0.7j)f_D(3) + (-1)f_D(4) \\ +(0.7 + 0.7j)f_D(5) + (-j)f_D(6) + (-0.7 + 0.7j)f_D(7)$$

Figure 9 shows the path of the $F_D(3)$ evaluation through the 8-point butterfly trellis.

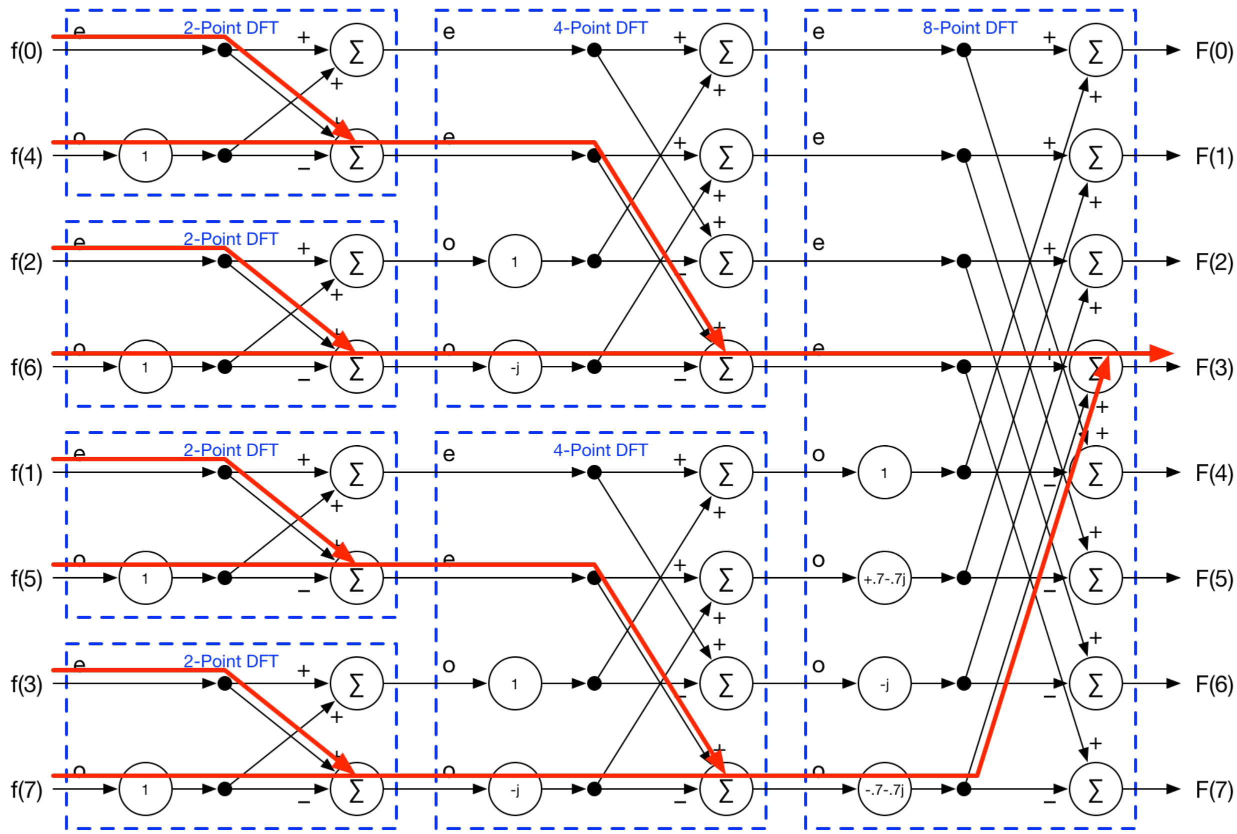

Figure 9 – 8-Point FFT Butterfly Trellis, Path for $F_D(3)$

As was stated before, because of the even-odd structure of the subordinate DFT blocks, only the $N/2$ twiddle factors of the lower half of the unit circle need to be calculated. The combinations throughout the butterfly trellis complete the calculations that involve the twiddle factors of the upper half of the unit circle.

The FFT algorithm can be implemented via in-place calculations using array index mapping, or via recursive function calls. While the recursive approach is more intuitive in terms of the algorithm, the in-place calculation approach is faster and lends itself to vectorization and parallelization. In addition, if the size of the FFT is fixed for a series of FFT runs, the twiddle factors can be computed *a priori* and stored in a table, thereby eliminating the need to perform the transcendental operations at run-time.

10 References

These are the general references I used in writing this document. Their listing order is in the general order as they apply throughout the document.

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