Kalman Filter Equations and Models

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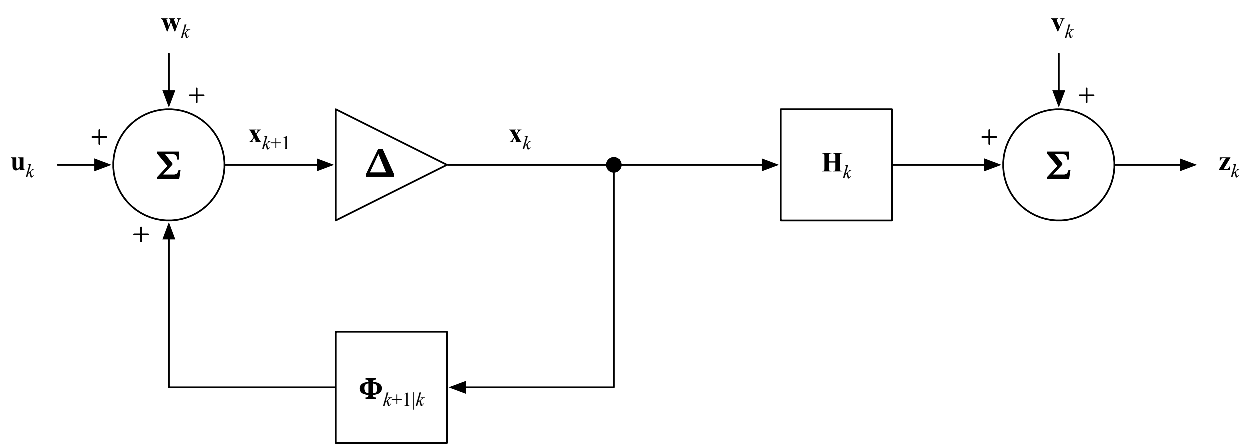
1. Linear System with Linear Measurements - Kalman Filter Estimation

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_{k+1|k}\mathbf{x}_k + \mathbf{u}_k + \mathbf{w}_k$$
$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k$$

where $\mathbf{u}_k$ is a control input, and both $\mathbf{w}_k$ and $\mathbf{v}_k$ are white noise sequences

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)\ ,\ \ \mathbf{Q}_k = E[\mathbf{w}_k\mathbf{w}_k^T]$$
$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)\ ,\ \ \mathbf{R}_k = E[\mathbf{v}_k\mathbf{v}_k^T]$$

A flow diagram of the system model is shown below:



Kalman Filter

The Kalman Filter maintains an estimated state $\hat{\mathbf{x}}_k$ and an estimated state error covariance $\mathbf{P}_k$:

$$\hat{\mathbf{x}}_k = E\{\mathbf{x}_k\}$$
$$\mathbf{P}_k = E\{[\mathbf{x}_k - \hat{\mathbf{x}}_k]\,[\mathbf{x}_k - \hat{\mathbf{x}}_k]^T\}$$

Note that since $\mathbf{P}_k$ is a square covariance matrix,

$$\mathbf{P}_k^T = \mathbf{P}_k$$

Prediction:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{\Phi}_{k|k-1}\hat{\mathbf{x}}_{k-1|k-1} + \mathbf{u}_k$$
$$\mathbf{P}_{k|k-1} = \mathbf{\Phi}_{k|k-1}\,\mathbf{P}_{k-1|k-1}\,\mathbf{\Phi}_{k|k-1}^T + \mathbf{Q}_k$$

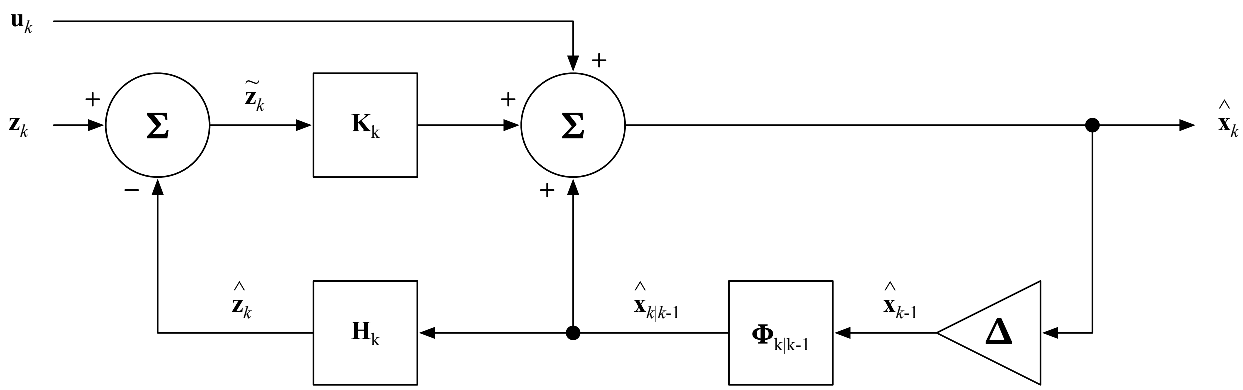
Correction:

$$\hat{\mathbf{z}}_k = \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$
$$\tilde{\mathbf{z}}_k = \mathbf{z}_k - \hat{\mathbf{z}}_k$$

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k$$
$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{z}}_k$$
$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

A flow diagram of the Kalman filter is shown below:



It should be noted that $\mathbf{S}_k$ is the measurement prediction covariance:

$$\mathbf{S}_k = E\{\tilde{\mathbf{z}}_k \tilde{\mathbf{z}}_k^T\}$$

and so

$$\mathbf{S}_k^T = \mathbf{S}_k$$

In addition, because

$$\begin{aligned}
\mathbf{K}_k &= \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1} \\
\mathbf{K}_k \mathbf{S}_k &= \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1} \mathbf{S}_k \\
&= \mathbf{P}_{k|k-1} \mathbf{H}_k^T \\
\left(\mathbf{P}_{k|k-1} \mathbf{H}_k^T\right)^T &= \left(\mathbf{K}_k \mathbf{S}_k\right)^T \\
\mathbf{H}_k \mathbf{P}_{k|k-1} &= \mathbf{S}_k \mathbf{K}_k^T
\end{aligned}$$

the corrected state error covariance $\mathbf{P}_{k|k}$ can be expressed as

$$\begin{aligned}\mathbf{P}_{k|k} &= (\mathbf{I} - \mathbf{K}_k\,\mathbf{H}_k)\,\mathbf{P}_{k|k-1} \\ &= \mathbf{P}_{k|k-1} - \mathbf{K}_k\,\mathbf{H}_k\,\mathbf{P}_{k|k-1} \\ &= \mathbf{P}_{k|k-1} - \mathbf{K}_k\,\mathbf{S}_k\,\mathbf{K}_k^T\end{aligned}$$

The advantage of this form is that it is quadratic and preserves the symmetry of $\mathbf{P}_{k|k}$. In addition, it does not rely on $\mathbf{H}_k$.

It should also be observed that we can denote $\mathbf{P}_{\mathbf{xz},k}$ as the measurement-to-state transformation cross covariance

$$\mathbf{P}_{\mathbf{xz},k} = \mathbf{P}_{k|k-1}\,\mathbf{H}_k^T$$

and denote $\mathbf{P}_{\mathbf{zz},k}$ as the measurement propagation covariance

$$\mathbf{P}_{\mathbf{zz},k} = \mathbf{H}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}_k^T$$

so that

$$\begin{aligned}\mathbf{S}_k &= \mathbf{P}_{\mathbf{zz},k} + \mathbf{R}_k \\ \mathbf{K}_k &= \mathbf{P}_{\mathbf{xz},k}\,\mathbf{S}_k^{-1}\end{aligned}$$

These results are useful in the unscented Kalman Filter formulation.

Lastly, if we combine the covariance prediction and correction equations into one equation, we obtain the discrete Riccati equation for the state estimation error covariance:

$$\mathbf{P}_{k+1|k} = \mathbf{\Phi}_{k|k-1}\mathbf{P}_{k|k-1}\mathbf{\Phi}_{k|k-1}^T - \mathbf{\Phi}_{k|k-1}\mathbf{P}_{k|k-1}\mathbf{H}_k^T\left[\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\right]^{-1}\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{\Phi}_{k|k-1}^T + \mathbf{Q}_k$$

Considerable literature exists on the solution of the Riccati equation, providing a converged value for $\mathbf{P}_\infty$. However, it is evident from the equation that the update of $\mathbf{P}$ is independent of the measurement vector $\mathbf{z}_k$.

2. Linear System with Nonlinear Measurements - Extended Kalman Filter Estimation

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_{k+1|k}\mathbf{x}_k + \mathbf{u}_k + \mathbf{w}_k$$
$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k$$

where $\mathbf{u}_k$ is a control input, and both $\mathbf{w}_k$ and $\mathbf{v}_k$ are white noise sequences

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)\ ,\ \ \mathbf{Q}_k = E[\mathbf{w}_k\mathbf{w}_k^T]$$
$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)\ ,\ \ \mathbf{R}_k = E[\mathbf{v}_k\mathbf{v}_k^T]$$

Extended Kalman Filter

The extended Kalman filter uses linearized matrices to implement the filtering equations.

Prediction:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{\Phi}_{k|k-1}\hat{\mathbf{x}}_{k-1|k-1} + \mathbf{u}_k$$
$$\mathbf{P}_{k|k-1} = \mathbf{\Phi}_{k|k-1}\,\mathbf{P}_{k-1|k-1}\,\mathbf{\Phi}_{k|k-1}^T + \mathbf{Q}_k$$

Correction:

$$\hat{\mathbf{z}}_k = \mathbf{h}(\hat{\mathbf{x}}_{k|k-1})$$
$$\tilde{\mathbf{z}}_k = \mathbf{z}_k - \hat{\mathbf{z}}_k$$

$$\mathbf{H}_k = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}_k}\right|_{\hat{\mathbf{x}}_{k|k-1}}$$

$$\mathbf{S}_k = \mathbf{H}_k\,\mathbf{P}_{k|k-1}\,\mathbf{H}_k^T + \mathbf{R}_k$$
$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\,\mathbf{H}_k^T\,\mathbf{S}_k^{-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k\,\tilde{\mathbf{z}}_k$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\,\mathbf{S}_k\,\mathbf{K}_k^T$$

3. Nonlinear System with Nonlinear Measurements - Extended Kalman Filter Estimation

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k \\ \mathbf{z}_k &= \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k \end{aligned}$$

where $\mathbf{u}_k$ is a control input, and both $\mathbf{w}_k$ and $\mathbf{v}_k$ are white noise sequences

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k) \ , \ \ \mathbf{Q}_k = E[\mathbf{w}_k \mathbf{w}_k^T]$$
$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k) \ , \ \ \mathbf{R}_k = E[\mathbf{v}_k \mathbf{v}_k^T]$$

Extended Kalman Filter

The extended Kalman filter uses linearized matrices to implement the filtering equations.

Prediction:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{f}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_k)$$

$$\mathbf{F}_k = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}_k} \right|_{\hat{\mathbf{x}}_{k|k-1}, \mathbf{u}_k}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \, \mathbf{P}_{k-1|k-1} \, \mathbf{F}_k^T + \mathbf{Q}_k$$

Correction:

$$\hat{\mathbf{z}}_k = \mathbf{h}(\hat{\mathbf{x}}_{k|k-1})$$
$$\tilde{\mathbf{z}}_k = \mathbf{z}_k - \hat{\mathbf{z}}_k$$

$$\mathbf{H}_k = \left. \frac{\partial \mathbf{h}}{\partial \mathbf{x}_k} \right|_{\hat{\mathbf{x}}_{k|k-1}}$$

$$\mathbf{S}_k = \mathbf{H}_k \, \mathbf{P}_{k|k-1} \, \mathbf{H}_k^T + \mathbf{R}_k$$
$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \, \mathbf{H}_k^T \, \mathbf{S}_k^{-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \, \tilde{\mathbf{z}}_k$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k \, \mathbf{S}_k \, \mathbf{K}_k^T$$

4. Linear System with Nonlinear Measurements - Unscented Kalman Filter Estimation

$$\mathbf{x}_{k+1} = \mathbf{\Phi}_{k+1|k}\mathbf{x}_k + \mathbf{u}_k + \mathbf{w}_k$$
$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k$$

where $\mathbf{u}_k$ is a control input, and both $\mathbf{w}_k$ and $\mathbf{v}_k$ are white noise sequences

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)\ , \quad \mathbf{Q}_k = E[\mathbf{w}_k\mathbf{w}_k^T]$$
$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)\ , \quad \mathbf{R}_k = E[\mathbf{v}_k\mathbf{v}_k^T]$$

Unscented Kalman Filter

Prediction:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{\Phi}_{k|k-1}\hat{\mathbf{x}}_{k-1|k-1} + \mathbf{u}_k$$
$$\mathbf{P}_{k|k-1} = \mathbf{\Phi}_{k|k-1}\,\mathbf{P}_{k-1|k-1}\,\mathbf{\Phi}_{k|k-1}^T + \mathbf{Q}_k$$

Generation of sigma points and weights:

$$\mathbf{L}_{k|k-1} = \sqrt{(n+\lambda)\,\mathbf{P}_{k|k-1}}$$

$$\chi_0 = \hat{\mathbf{x}}_{k|k-1}$$
$$\chi_i = \hat{\mathbf{x}}_{k|k-1} + \left[\mathbf{L}_{k|k-1}\right]_i\ , i = 1,\ldots,n$$
$$\chi_{i+n} = \hat{\mathbf{x}}_{k|k-1} - \left[\mathbf{L}_{k|k-1}\right]_i\ , i = 1,\ldots,n$$
$$W_{m,0} = \frac{\lambda}{n+\lambda}$$
$$W_{c,0} = W_{m,0} + (1-\alpha^2+\beta)$$
$$W_{m,i} = W_{c,i} = \frac{1}{2(n+\lambda)}\ , i = 1,\ldots,2n$$

where

$\sqrt{\mathbf{M}}$ is the lower triangular Cholesky decomposition of the symmetric matrix $\mathbf{M}$
$\left[\mathbf{L}\right]_i$ is the i th column of the lower triangular matrix $\mathbf{L}$

W_m is a weighting factor used to predict the measurement vector
W_c is a weighting factor used to predict the correction covariance matrices

$n = \dim \hat{\mathbf{x}}$
λ is a scaling parameter that determines the extent of the spread of the sigma points about the mean
β is a parameter that determines the distribution of the PDF of the state vector; typically set to 2 for Gaussian
α is an adjustment parameter, $[0, 1)$
κ is an adjustment parameter, $[0, \infty)$

$$\lambda = \alpha^2(n + \kappa) - n$$

Some representative values for α, β, and κ are

$$\alpha = 0.5$$
$$\beta = 2$$
$$\kappa = 0$$

Correction:

$$\hat{\mathbf{z}}_k = \sum_{i=0}^{2n} W_{m,i}\, \mathbf{h}(\chi_i)$$
$$\tilde{\mathbf{z}}_k = \mathbf{z}_k - \hat{\mathbf{z}}_k$$

$$\mathbf{P}_{\mathbf{xz},k} = \sum_{i=0}^{2n} W_{c,i} \left[\chi_i - \hat{\mathbf{x}}_{k|k-1}\right] \left[\mathbf{h}(\chi_i) - \hat{\mathbf{z}}_k\right]^T$$

$$\mathbf{P}_{\mathbf{zz},k} = \sum_{i=0}^{2n} W_{c,i} \left[\mathbf{h}(\chi_i) - \hat{\mathbf{z}}_k\right] \left[\mathbf{h}(\chi_i) - \hat{\mathbf{z}}_k\right]^T$$

$$\mathbf{S}_k = \mathbf{P}_{\mathbf{zz},k} + \mathbf{R}_k$$
$$\mathbf{K}_k = \mathbf{P}_{\mathbf{xz},k}\, \mathbf{S}_k^{-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k\, \tilde{\mathbf{z}}_k$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\, \mathbf{S}_k\, \mathbf{K}_k^T$$

It is important to observe that the expression for $\mathbf{P}_{k|k}$ is the form that does not use a linear $\mathbf{H}_k$ matrix, since the unscented transformation does not rely on linearization of $\mathbf{h}(\mathbf{x}_k)$.

5. Constant Velocity Motion with Transformed Range/Bearing Measurement Update

The target is specified in north and east Cartesian coordinates

$$\mathbf{x} = \begin{bmatrix} n \\ v_n \\ e \\ v_e \end{bmatrix}$$

Measurements are north and east values computed from target range and bearing, where bearing is referenced clockwise from north

$$\mathbf{z} = \begin{bmatrix} z_n \\ z_e \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

We model target motion using constant velocity motion with random acceleration adjustments as the process noise

$$\begin{bmatrix} n_k \\ v_{n,k} \\ e_k \\ v_{e,k} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{k-1} \\ v_{n,k-1} \\ e_{k-1} \\ v_{e,k-1} \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix} \begin{bmatrix} w_{an,k} \\ w_{ae,k} \end{bmatrix}$$

Then $\mathbf{Q}_k$ becomes

$$\mathbf{Q}_k = \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix} \begin{bmatrix} \sigma_{an,k}^2 & 0 \\ 0 & \sigma_{ae,k}^2 \end{bmatrix} \begin{bmatrix} \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^2}{2} & T \end{bmatrix} = \begin{bmatrix} \sigma_{an,k}^2\frac{T^4}{4} & \sigma_{an,k}^2\frac{T^3}{2} & 0 & 0 \\ \sigma_{an,k}^2\frac{T^3}{2} & \sigma_{an,k}^2 T^2 & 0 & 0 \\ 0 & 0 & \sigma_{ae,k}^2\frac{T^4}{4} & \sigma_{ae,k}^2\frac{T^3}{2} \\ 0 & 0 & \sigma_{ae,k}^2\frac{T^3}{2} & \sigma_{ae,k}^2 T^2 \end{bmatrix}$$

where $\sigma^2_{an,k}$ and $\sigma^2_{ae,k}$ are the north and east process acceleration noise variances, respectively.

The measurements are obtained with random measurement transformation errors

$$\begin{bmatrix} z_{n,k} \\ z_{e,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} n_k \\ v_{n,k} \\ e_k \\ v_{e,k} \end{bmatrix} + \begin{bmatrix} \gamma_{n,k} \\ \gamma_{e,k} \end{bmatrix}$$

The measurement transformation covariances for $\gamma_{n,k}$ and $\gamma_{e,k}$ are derived from the range and bearing variances $\sigma^2_{r,k}$ and $\sigma^2_{\theta,k}$

$$\mathbf{R}_{\gamma,k} = \mathbf{M}_{r,\theta,k}\ \mathbf{R}_{r,\theta,k}\ \mathbf{M}^T_{r,\theta,k}$$

where

$$\mathbf{M}_{r,\theta,k} = \begin{bmatrix} \cos\theta_k & -r_k \sin\theta_k \\ \sin\theta_k & r_k \cos\theta_k \end{bmatrix}$$

and

$$\mathbf{R}_{r,\theta,k} = \begin{bmatrix} \sigma^2_{r,k} & 0 \\ 0 & \sigma^2_{\theta,k} \end{bmatrix}$$

It should be noted that we can also choose to model target motion using constant velocity motion with random velocity adjustments as the process noise

$$\begin{bmatrix} n_k \\ v_{n,k} \\ e_k \\ v_{e,k} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{k-1} \\ v_{n,k-1} \\ e_{k-1} \\ v_{e,k-1} \end{bmatrix} + \begin{bmatrix} T & 0 \\ 1 & 0 \\ 0 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_{vn,k} \\ w_{ve,k} \end{bmatrix}$$

where

$$T = t_k - t_{k-1}$$

so that $\mathbf{Q}_k$ becomes

$$\mathbf{Q}_k = \begin{bmatrix} T & 0 \\ 1 & 0 \\ 0 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{vn,k}^2 & 0 \\ 0 & \sigma_{ve,k}^2 \end{bmatrix} \begin{bmatrix} T & 1 & 0 & 0 \\ 0 & 0 & T & 1 \end{bmatrix} = \begin{bmatrix} \sigma_{vn,k}^2 T^2 & \sigma_{vn,k}^2 T & 0 & 0 \\ \sigma_{vn,k}^2 T & \sigma_{vn,k}^2 & 0 & 0 \\ 0 & 0 & \sigma_{ve,k}^2 T^2 & \sigma_{ve,k}^2 T \\ 0 & 0 & \sigma_{ve,k}^2 T & \sigma_{ve,k}^2 \end{bmatrix}$$

where $\sigma_{vn,k}^2$ and $\sigma_{ve,k}^2$ are the north and east process velocity noise variances, respectively.

6. Constant Velocity Motion with Direct Range/Bearing Measurement Update

The target is specified in north and east Cartesian coordinates

$$\mathbf{x} = \begin{bmatrix} n \\ v_n \\ e \\ v_e \end{bmatrix}$$

Measurements are target range and bearing, where bearing is referenced clockwise from north

$$\mathbf{z} = \begin{bmatrix} r \\ \theta \end{bmatrix}$$

We model target motion using constant velocity motion with random acceleration adjustments as the process noise

$$\begin{bmatrix} n_k \\ v_{n,k} \\ e_k \\ v_{e,k} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{k-1} \\ v_{n,k-1} \\ e_{k-1} \\ v_{e,k-1} \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix} \begin{bmatrix} w_{an,k} \\ w_{ae,k} \end{bmatrix}$$

Then $\mathbf{Q}_k$ becomes

$$\mathbf{Q}_k = \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix} \begin{bmatrix} \sigma_{an,k}^2 & 0 \\ 0 & \sigma_{ae,k}^2 \end{bmatrix} \begin{bmatrix} \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^2}{2} & T \end{bmatrix} = \begin{bmatrix} \sigma_{an,k}^2 \frac{T^4}{4} & \sigma_{an,k}^2 \frac{T^3}{2} & 0 & 0 \\ \sigma_{an,k}^2 \frac{T^3}{2} & \sigma_{an,k}^2 T^2 & 0 & 0 \\ 0 & 0 & \sigma_{ae,k}^2 \frac{T^4}{4} & \sigma_{ae,k}^2 \frac{T^3}{2} \\ 0 & 0 & \sigma_{ae,k}^2 \frac{T^3}{2} & \sigma_{ae,k}^2 T^2 \end{bmatrix}$$

where $\sigma_{an,k}^2$ and $\sigma_{ae,k}^2$ are the north and east process acceleration noise variances, respectively.

The measurements are obtained with random measurement errors

$$\begin{bmatrix} r_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} \sqrt{n_k^2 + e_k^2} \\ \arctan \dfrac{e_k}{n_k} \end{bmatrix} + \begin{bmatrix} v_{r,k} \\ v_{\theta,k} \end{bmatrix}$$

where the arctan() function is a four-quadrant arctangent function, e.g., `atan2(y,x)`.

$$\mathbf{H}_k = \begin{bmatrix} \dfrac{\partial r}{\partial n} & \dfrac{\partial r}{\partial v_n} & \dfrac{\partial r}{\partial e} & \dfrac{\partial r}{\partial v_e} \\ \dfrac{\partial \theta}{\partial n} & \dfrac{\partial \theta}{\partial v_n} & \dfrac{\partial \theta}{\partial e} & \dfrac{\partial \theta}{\partial v_e} \end{bmatrix}_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}$$

$$= \begin{bmatrix} \dfrac{n}{\sqrt{n^2+e^2}} & 0 & \dfrac{e}{\sqrt{n^2+e^2}} & 0 \\ -\dfrac{e}{n^2+e^2} & 0 & \dfrac{n}{n^2+e^2} & 0 \end{bmatrix}_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}$$

$$\mathbf{R}_k = \begin{bmatrix} \sigma_{r,k}^2 & 0 \\ 0 & \sigma_{\theta,k}^2 \end{bmatrix}$$

7. Constant Velocity Motion with Range and Direction Cosine Measurement Update

The target is specified in north and east Cartesian coordinates

$$\mathbf{x} = \begin{bmatrix} n \\ v_n \\ e \\ v_e \end{bmatrix}$$

Measurements are target range and bearing direction cosines, where bearing is referenced clockwise from north

$$\mathbf{z} = \begin{bmatrix} r \\ \cos\theta \\ \sin\theta \end{bmatrix}$$

We model target motion using constant velocity motion with random acceleration adjustments as the process noise

$$\begin{bmatrix} n_k \\ v_{n,k} \\ e_k \\ v_{e,k} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{k-1} \\ v_{n,k-1} \\ e_{k-1} \\ v_{e,k-1} \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix} \begin{bmatrix} w_{an,k} \\ w_{ae,k} \end{bmatrix}$$

Then $\mathbf{Q}_k$ becomes

$$\mathbf{Q}_k = \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix} \begin{bmatrix} \sigma_{an,k}^2 & 0 \\ 0 & \sigma_{ae,k}^2 \end{bmatrix} \begin{bmatrix} \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^2}{2} & T \end{bmatrix} = \begin{bmatrix} \sigma_{an,k}^2 \frac{T^4}{4} & \sigma_{an,k}^2 \frac{T^3}{2} & 0 & 0 \\ \sigma_{an,k}^2 \frac{T^3}{2} & \sigma_{an,k}^2 T^2 & 0 & 0 \\ 0 & 0 & \sigma_{ae,k}^2 \frac{T^4}{4} & \sigma_{ae,k}^2 \frac{T^3}{2} \\ 0 & 0 & \sigma_{ae,k}^2 \frac{T^3}{2} & \sigma_{ae,k}^2 T^2 \end{bmatrix}$$

where $\sigma^2_{an,k}$ and $\sigma^2_{ae,k}$ are the north and east process acceleration noise variances, respectively.

The measurements are obtained with random measurement errors

$$\begin{bmatrix} r_k \\ \Lambda n_k \\ \Lambda e_k \end{bmatrix} = \begin{bmatrix} r_k \\ \cos\theta_k \\ \sin\theta_k \end{bmatrix} = \begin{bmatrix} \sqrt{n_k^2 + e_k^2} \\ \dfrac{n_k}{\sqrt{n_k^2 + e_k^2}} \\ \dfrac{e_k}{\sqrt{n_k^2 + e_k^2}} \end{bmatrix} + \begin{bmatrix} v_{r,k} \\ v_{\Lambda n,k} \\ v_{\Lambda e,k} \end{bmatrix}$$

$$\mathbf{H}_k = \left.\begin{bmatrix} \dfrac{\partial r}{\partial n} & \dfrac{\partial r}{\partial v_n} & \dfrac{\partial r}{\partial e} & \dfrac{\partial r}{\partial v_e} \\ \dfrac{\partial \Lambda n}{\partial n} & \dfrac{\partial \Lambda n}{\partial v_n} & \dfrac{\partial \Lambda n}{\partial e} & \dfrac{\partial \Lambda n}{\partial v_e} \\ \dfrac{\partial \Lambda e}{\partial n} & \dfrac{\partial \Lambda e}{\partial v_n} & \dfrac{\partial \Lambda e}{\partial e} & \dfrac{\partial \Lambda e}{\partial v_e} \end{bmatrix}\right|_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}$$

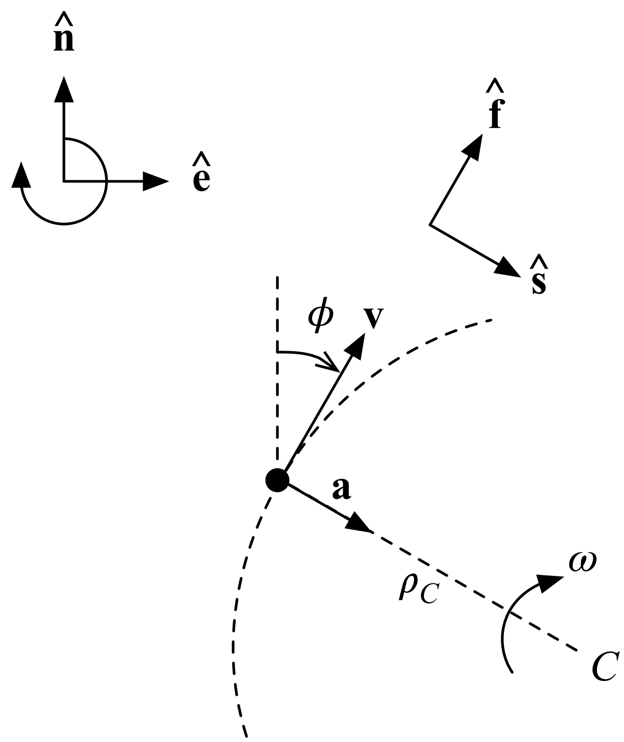
$$= \left.\begin{bmatrix} \dfrac{n}{\sqrt{n^2+e^2}} & 0 & \dfrac{e}{\sqrt{n^2+e^2}} & 0 \\ \dfrac{e^2}{\left(\sqrt{n^2+e^2}\right)^3} & 0 & -\dfrac{ne}{\left(\sqrt{n^2+e^2}\right)^3} & 0 \\ -\dfrac{ne}{\left(\sqrt{n^2+e^2}\right)^3} & 0 & \dfrac{n^2}{\left(\sqrt{n^2+e^2}\right)^3} & 0 \end{bmatrix}\right|_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}$$

$$\mathbf{R}_k = \begin{bmatrix} \sigma^2_{r,k} & 0 & 0 \\ 0 & \sigma^2_{\Lambda n,k} & 0 \\ 0 & 0 & \sigma^2_{\Lambda e,k} \end{bmatrix}$$

where $\sigma_{r,k}^2$, $\sigma_{\Lambda n,k}^2$, and $\sigma_{\Lambda e,k}^2$ are the range and the north and east direction cosine measurement noise variances, respectively.

8. Constant Turn Motion with Direct Range/Bearing Measurement Update

Consider the constant angular motion depicted in the below figure, where $\mathbf{a}$ is the acceleration vector, $\mathbf{v}$ is the velocity vector, ϕ is the heading (referenced clockwise from north), C is the center of turn rotation, ω is the turn rate, ρ_C is the radius of curvature, $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}$ are the north and east unit vectors, respectively, and $\hat{\mathbf{f}}$ and $\hat{\mathbf{s}}$ are the forward and lateral (i.e., sideways) unit vectors, respectively.



From basic engineering mechanics, the velocity, $\mathbf{v}$, for constant angular motion is

$$\begin{aligned}\mathbf{v} &= v\hat{\mathbf{f}} \\ &= \rho_C \omega \hat{\mathbf{f}}\end{aligned}$$

and the acceleration, $\mathbf{a}$, for constant angular motion is

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= v\omega\hat{\mathbf{s}}\end{aligned}$$

where v is the forward velocity

$$\begin{aligned} v &= \|\mathbf{v}\| \\ &= \sqrt{v_n^2 + v_e^2} \end{aligned}$$

For constant turn motion, the heading ϕ evolves at a constant rate ω

$$\phi(t) = \phi_0 + \omega(t - t_0)$$

where ϕ_0 is the heading at time t_0. For notation convenience, we define

$$T \equiv t - t_0$$

so that

$$\phi = \phi_0 + \omega T$$

The rotation of axes formulae relating $\hat{\mathbf{f}}$ and $\hat{\mathbf{s}}$ to $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}$ are

$$\begin{aligned} \hat{\mathbf{f}} &= (\cos\phi)\hat{\mathbf{n}} + (\sin\phi)\hat{\mathbf{e}} \\ \hat{\mathbf{s}} &= (-\sin\phi)\hat{\mathbf{n}} + (\cos\phi)\hat{\mathbf{e}} \end{aligned}$$

Then

$$\begin{aligned} \mathbf{a} &= v\omega\hat{\mathbf{s}} \\ &= (-v\omega\sin\phi)\hat{\mathbf{n}} + (v\omega\cos\phi)\hat{\mathbf{e}} \\ &= a_n\hat{\mathbf{n}} + a_e\hat{\mathbf{e}} \end{aligned}$$

where

$$\begin{aligned} a_n &= -v\omega\sin\phi \\ a_e &= v\omega\cos\phi \end{aligned}$$

But

$$\begin{aligned} \cos\phi &= \cos(\phi_0 + \omega T) \\ &= \cos\phi_0\cos(\omega T) - \sin\phi_0\sin(\omega T) \end{aligned}$$

and

$$\begin{aligned} \sin\phi &= \sin(\phi_0 + \omega T) \\ &= \sin\phi_0\cos(\omega T) + \cos\phi_0\sin(\omega T) \end{aligned}$$

At $t = t_0$ (i.e., $T = 0$),

$$\phi = \phi_0$$

and so

$$\begin{aligned}\mathbf{v} &= v\hat{\mathbf{f}} \\ &= (v\cos\phi_0)\hat{\mathbf{n}} + (v\sin\phi_0)\hat{\mathbf{e}} \\ &= v_{n0}\hat{\mathbf{n}} + v_{e0}\hat{\mathbf{e}}\end{aligned}$$

where

$$\begin{aligned}v_{n0} &= v\cos\phi_0 \\ v_{e0} &= v\sin\phi_0\end{aligned}$$

Thus

$$\begin{aligned}a_n &= -v\omega\left(\sin\phi_0\cos(\omega T) + \cos\phi_0\sin(\omega T)\right) \\ &= -v_{n0}\omega\sin(\omega T) - v_{e0}\omega\cos(\omega T)\end{aligned}$$

$$\begin{aligned}a_e &= v\omega\left(\cos\phi_0\cos(\omega T) - \sin\phi_0\sin(\omega T)\right) \\ &= v_{n0}\omega\cos(\omega T) - v_{e0}\omega\sin(\omega T)\end{aligned}$$

Then, the expressions for north and east velocities are

$$\begin{aligned}v_n &= v_{n0} + \int_0^T a_n(\tau)d\tau \\ &= v_{n0} + v_{n0}\cos(\omega\tau)\Big|_0^T - v_{e0}\sin(\omega\tau)\Big|_0^T \\ &= v_{n0} + v_{n0}(\cos(\omega T) - 1) - v_{e0}(\sin(\omega T) - 0) \\ &= v_{n0}\cos(\omega T) - v_{e0}\sin(\omega T)\end{aligned}$$

and

$$\begin{aligned}v_e &= v_{e0} + \int_0^T a_e(\tau)d\tau \\ &= v_{e0} + v_{n0}\sin(\omega\tau)\Big|_0^T + v_{e0}\cos(\omega\tau)\Big|_0^T \\ &= v_{e0} + v_{n0}(\sin(\omega T) - 0) + v_{e0}(\cos(\omega T) - 1) \\ &= v_{n0}\sin(\omega T) + v_{e0}\cos(\omega T)\end{aligned}$$

Knowing the velocity expressions, we determine the position expressions to be

$$\begin{aligned}
n &= n_0 + \int_0^T v_n(\tau) d\tau \\
&= n_0 + \frac{v_{n0}}{\omega} \sin(\omega \tau) \Big|_0^T + \frac{v_{e0}}{\omega} \cos(\omega \tau) \Big|_0^T \\
&= n_0 + \frac{v_{n0}}{\omega} (\sin(\omega T) - 0) + \frac{v_{e0}}{\omega} (\cos(\omega T) - 1) \\
&= n_0 + v_{n0} \frac{\sin(\omega T)}{\omega} + v_{e0} \frac{\cos(\omega T) - 1}{\omega} \\
&= n_0 + v_{n0} \frac{\sin(\omega T)}{\omega} - v_{e0} \frac{1 - \cos(\omega T)}{\omega}
\end{aligned}$$

and

$$\begin{aligned}
e &= e_0 + \int_0^T v_e(\tau) d\tau \\
&= e_0 - \frac{v_{n0}}{\omega} \cos(\omega \tau) \Big|_0^T + \frac{v_{e0}}{\omega} \sin(\omega \tau) \Big|_0^T \\
&= e_0 - \frac{v_{n0}}{\omega} (\cos(\omega T) - 1) + \frac{v_{e0}}{\omega} (\sin(\omega T) - 0) \\
&= e_0 - v_{n0} \frac{\cos(\omega T) - 1}{\omega} + v_{e0} \frac{\sin(\omega T)}{\omega} \\
&= e_0 + v_{n0} \frac{1 - \cos(\omega T)}{\omega} + v_{e0} \frac{\sin(\omega T)}{\omega}
\end{aligned}$$

Collecting equations, we have the discrete-time model for constant turn rate motion with random acceleration adjustments as the process noise:

$$\begin{aligned}
n_{k+1} &= n_k + \frac{\sin(\omega T)}{\omega} v_{n,k} - \frac{1 - \cos(\omega T)}{\omega} v_{e,k} + \frac{T^2}{2} w_{an,k} \\
v_{n,k+1} &= \cos(\omega T) v_{n,k} - \sin(\omega T) v_{e,k} + T w_{an,k} \\
e_{k+1} &= e_k + \frac{1 - \cos(\omega T)}{\omega} v_{n,k} + \frac{\sin(\omega T)}{\omega} v_{e,k} + \frac{T^2}{2} w_{ae,k} \\
v_{e,k+1} &= \sin(\omega T) v_{n,k} + \cos(\omega T) v_{e,k} + T w_{ae,k} \\
\omega_{k+1} &= \omega_k + T w_{\alpha,k}
\end{aligned}$$

where

$$T = t_{k+1} - t_k$$

and where $w_{an,k}$ and $w_{ae,k}$ are the north and east acceleration disturbances, and $w_{\alpha,k}$ is the angular acceleration disturbance.

In matrix form, the target state vector is

$$\mathbf{x} = \begin{bmatrix} n \\ v_n \\ e \\ v_e \\ \omega \end{bmatrix}$$

and the state model equation is

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{u}_k + \boldsymbol{\Gamma}_k \mathbf{w}_k$$

where

$$\mathbf{f}(\mathbf{x}_k) = \begin{bmatrix} 1 & \frac{\sin(\omega T)}{\omega} & 0 & -\frac{1-\cos(\omega T)}{\omega} & 0 \\ 0 & \cos(\omega T) & 0 & -\sin(\omega T) & 0 \\ 0 & \frac{1-\cos(\omega T)}{\omega} & 1 & \frac{\sin(\omega T)}{\omega} & 0 \\ 0 & \sin(\omega T) & 0 & \cos(\omega T) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_k \\ v_{n,k} \\ e_k \\ v_{e,k} \\ \omega_k \end{bmatrix}$$

$$\boldsymbol{\Gamma}_k = \begin{bmatrix} \frac{T^2}{2} & 0 & 0 \\ T & 0 & 0 \\ 0 & \frac{T^2}{2} & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix}$$

$$\mathbf{w}_k = \begin{bmatrix} w_{an,k} \\ w_{ae,k} \\ w_{\alpha,k} \end{bmatrix}$$

Note that

$$\lim_{\omega \to 0} \frac{\sin(\omega T)}{\omega} \overset{*}{=} \lim_{\omega \to 0} \frac{T\cos(\omega T)}{1} = T$$

and that

$$\lim_{\omega \to 0} \frac{1 - \cos(\omega T)}{\omega} \overset{*}{=} \lim_{\omega \to 0} \frac{T\sin(\omega T)}{1} = 0$$

Hence, for $\omega = 0$, the constant turn rate motion model reduces to the constant velocity model:

$$\mathbf{f}(\mathbf{x}_k)\Big|_{\omega=0} = \begin{bmatrix} 1 & T & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & T & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_k \\ v_{n,k} \\ e_k \\ v_{e,k} \\ \omega_k \end{bmatrix}$$

The measurements are obtained with random measurement errors

$$\begin{bmatrix} r_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} \sqrt{n_k^2 + e_k^2} \\ \arctan \dfrac{e_k}{n_k} \end{bmatrix} + \begin{bmatrix} v_{r,k} \\ v_{\theta,k} \end{bmatrix}$$

where the arctan() function is a four-quadrant arctangent function, e.g., `atan2(y,x)`.

The rest of the system matrices are the same as that of the Constant Velocity Motion model with Direct Range/Bearing Measurement Update.

The linearized form of the state transition function is

$$
\mathbf{F}_k = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}_k}\right|_{\hat{\mathbf{x}}_{k|k-1}} = \begin{bmatrix}
\frac{\partial n}{\partial n} & \frac{\partial n}{\partial v_n} & \frac{\partial n}{\partial e} & \frac{\partial n}{\partial v_e} & \frac{\partial n}{\partial \omega} \\
\frac{\partial v_n}{\partial n} & \frac{\partial v_n}{\partial v_n} & \frac{\partial v_n}{\partial e} & \frac{\partial v_n}{\partial v_e} & \frac{\partial v_n}{\partial \omega} \\
\frac{\partial e}{\partial n} & \frac{\partial e}{\partial v_n} & \frac{\partial e}{\partial e} & \frac{\partial e}{\partial v_e} & \frac{\partial e}{\partial \omega} \\
\frac{\partial v_e}{\partial n} & \frac{\partial v_e}{\partial v_n} & \frac{\partial v_e}{\partial e} & \frac{\partial v_e}{\partial v_e} & \frac{\partial v_e}{\partial \omega} \\
\frac{\partial \omega}{\partial n} & \frac{\partial \omega}{\partial v_n} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial v_e} & \frac{\partial \omega}{\partial \omega}
\end{bmatrix}_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}
$$

The upper 4×4 block terms are simply the n, v_n, e, v_e coefficients of the state transition function. The angular partial derivatives are as follows:

$$
\begin{aligned}
\frac{\partial n}{\partial \omega} &= v_n \left[\frac{T\cos(\omega T)}{\omega} - \frac{\sin(\omega T)}{\omega^2}\right] - v_e \left[\frac{T\sin(\omega T)}{\omega} - \frac{1-\cos(\omega T)}{\omega^2}\right] \\
\frac{\partial v_n}{\partial \omega} &= -\, v_n T\sin(\omega T) - v_e T\cos(\omega T) \\
\frac{\partial e}{\partial \omega} &= v_n \left[\frac{T\sin(\omega T)}{\omega} - \frac{1-\cos(\omega T)}{\omega^2}\right] + v_e \left[\frac{T\cos(\omega T)}{\omega} - \frac{\sin(\omega T)}{\omega^2}\right] \\
\frac{\partial v_e}{\partial \omega} &= v_n T\cos(\omega T) - v_e T\sin(\omega T)
\end{aligned}
$$

$$
\frac{\partial \omega}{\partial n} = \frac{\partial \omega}{\partial v_n} = \frac{\partial \omega}{\partial e} = \frac{\partial \omega}{\partial v_e} = 0
$$

$$
\frac{\partial \omega}{\partial \omega} = 1
$$

Hence

$$
\mathbf{F}_k = \begin{bmatrix}
1 & \dfrac{\sin(\omega T)}{\omega} & 0 & -\dfrac{1-\cos(\omega T)}{\omega} & \begin{array}{r} v_n \left[\dfrac{T\cos(\omega T)}{\omega} - \dfrac{\sin(\omega T)}{\omega^2} \right] \\ - v_e \left[\dfrac{T\sin(\omega T)}{\omega} - \dfrac{1-\cos(\omega T)}{\omega^2} \right] \end{array} \\
0 & \cos(\omega T) & 0 & -\sin(\omega T) & -v_n T\sin(\omega T) - v_e T\cos(\omega T) \\
0 & \dfrac{1-\cos(\omega T)}{\omega} & 1 & \dfrac{\sin(\omega T)}{\omega} & \begin{array}{r} v_n \left[\dfrac{T\sin(\omega T)}{\omega} - \dfrac{1-\cos(\omega T)}{\omega^2} \right] \\ + v_e \left[\dfrac{T\cos(\omega T)}{\omega} - \dfrac{\sin(\omega T)}{\omega^2} \right] \end{array} \\
0 & \sin(\omega T) & 0 & \cos(\omega T) & v_n T\cos(\omega T) - v_e T\sin(\omega T) \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}
$$

Note that

$$
\begin{aligned}
&\lim_{\omega\to 0} \left[\frac{T\cos(\omega T)}{\omega} - \frac{\sin(\omega T)}{\omega^2} \right] \\
&\quad = \lim_{\omega\to 0} \left[\frac{\omega T\cos(\omega T) - \sin(\omega T)}{\omega^2} \right] \\
&\quad \overset{*}{=} \lim_{\omega\to 0} \left[\frac{T\cos(\omega T) - \omega T^2\sin(\omega T) - T\cos(\omega T)}{2\omega} \right] \\
&\quad = \lim_{\omega\to 0} \left[\frac{-T^2\sin(\omega T)}{2} \right] \\
&\quad = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{\omega \to 0} &\left[\frac{T \sin(\omega T)}{\omega} - \frac{1 - \cos(\omega T)}{\omega^2} \right] \\
&= \lim_{\omega \to 0} \left[\frac{\omega T \sin(\omega T) - 1 + \cos(\omega T)}{\omega^2} \right] \\
&\overset{*}{=} \lim_{\omega \to 0} \left[\frac{T \sin(\omega T) + \omega T^2 \cos(\omega T) - T \sin(\omega T)}{2\omega} \right] \\
&= \lim_{\omega \to 0} \left[\frac{T^2 \cos(\omega T)}{2} \right] \\
&= \frac{T^2}{2}
\end{aligned}
$$

Then, for $\omega = 0$, $\mathbf{F}_k$ becomes

$$
\mathbf{F}_k \Big|_{\omega=0} = \begin{bmatrix} 1 & T & 0 & 0 & -v_e \dfrac{T^2}{2} \\ 0 & 1 & 0 & 0 & -v_e T \\ 0 & 0 & 1 & T & v_n \dfrac{T^2}{2} \\ 0 & 0 & 0 & 1 & v_n T \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}
$$

The extended Kalman prediction operations become

$$
\hat{\mathbf{x}}_{k|k-1} = \mathbf{f}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_k)
$$

$$
\begin{aligned}
\mathbf{F}_k &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}_k} \bigg|_{\hat{\mathbf{x}}_{k|k-1}, \mathbf{u}_k} \\
\mathbf{P}_{k|k-1} &= \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{\Gamma}_k \mathbf{Q}_k \mathbf{\Gamma}_k^T
\end{aligned}
$$

where

$$
\mathbf{Q}_k = \begin{bmatrix} \sigma_{an,k}^2 & 0 & 0 \\ 0 & \sigma_{ae,k}^2 & 0 \\ 0 & 0 & \sigma_{\alpha,k}^2 \end{bmatrix}
$$

and where $\sigma^2_{an,k}$ and $\sigma^2_{ae,k}$ are the north and east acceleration disturbance variances, and $\sigma^2_{\alpha,k}$ is the angular acceleration disturbance variance.