We want to solve:

$$ax^2 + bx + c = 0\text{, where } a \neq 0$$

Completing the square gives

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a}\left(\frac{4a}{4a}\right) + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What if we don't divide by "a" first?

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

Observe that

$$(Px + Q)^2 = P^2x^2 + 2PQx + Q^2$$

We want to set

$$\begin{aligned} a &= P^2 \\ b &= 2PQ \end{aligned}$$

And so

$$\begin{aligned} P &= \sqrt{a} \\ Q &= \frac{b}{2P} \\ &= \frac{b}{2\sqrt{a}} \end{aligned}$$

which makes $(Px + Q)^2 = P^2x^2 + 2PQx + Q^2$ become

$$\begin{aligned} \left(\left(\sqrt{a}\right)x + \frac{b}{2\sqrt{a}}\right)^2 &= \left(\sqrt{a}\right)^2 x^2 + bx + \left(\frac{b}{2\sqrt{a}}\right)^2 \\ &= ax^2 + bx + \left(\frac{b}{2\sqrt{a}}\right)^2 \end{aligned}$$

Now we can complete the square as before

$$\left(\sqrt{a}\right)^2 x^2 + bx = -c$$

$$\left(\sqrt{a}\right)^2 x^2 + bx + \left(\frac{b}{2\sqrt{a}}\right)^2 = -c + \left(\frac{b}{2\sqrt{a}}\right)^2$$

$$\left(\left(\sqrt{a}\right)x + \frac{b}{2\sqrt{a}}\right)^2 = -c + \left(\frac{b}{2\sqrt{a}}\right)^2$$

$$\left(\left(\sqrt{a}\right)x+\frac{b}{2\sqrt{a}}\right)^2=-c+\frac{b^2}{4a}$$

$$\left(\left(\sqrt{a}\right)x+\frac{b}{2\sqrt{a}}\right)^2=-c\left(\frac{4a}{4a}\right)+\frac{b^2}{4a}$$

$$\left(\left(\sqrt{a}\right)x+\frac{b}{2\sqrt{a}}\right)^2=-\frac{4ac}{4a}+\frac{b^2}{4a}$$

$$\left(\left(\sqrt{a}\right)x+\frac{b}{2\sqrt{a}}\right)^2=\frac{b^2-4ac}{4a}$$

$$\left(\sqrt{a}\right)x+\frac{b}{2\sqrt{a}}=\pm\sqrt{\frac{b^2-4ac}{4a}}$$

$$\left(\sqrt{a}\right)x+\frac{b}{2\sqrt{a}}=\pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}$$

$$\frac{1}{\sqrt{a}}\left(\left(\sqrt{a}\right)x+\frac{b}{2\sqrt{a}}\right)=\left(\pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}\right)\frac{1}{\sqrt{a}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Same result, just more steps to get there.

Properties of quadratic roots:

$$ax^2 + bx + c = 0$$

has roots x_1 and x_2 such that

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Sum of roots:

$$\begin{aligned}
x_1 + x_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{\left(-b - \sqrt{b^2 - 4ac}\right) + \left(-b + \sqrt{b^2 - 4ac}\right)}{2a} \\
&= \frac{-2b}{2a} \\
&= -\frac{b}{a}
\end{aligned}$$

Product of roots:

$$\begin{aligned}
x_1 x_2 &= \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \\
&= \left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) \\
&= \left(\frac{b^2}{4a^2}\right) - \left(\frac{b^2 - 4ac}{4a^2}\right) \\
&= \frac{b^2 - b^2 + 4ac}{4a^2} \\
&= \frac{4ac}{4a^2} \\
&= \frac{c}{a}
\end{aligned}$$

Another way (and simpler way) of deriving these properties is as follows:

Given

$$
\begin{aligned}
a x^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0
\end{aligned}
$$

has roots x_1 and x_2 such that

$$(x - x_1)(x - x_2) = 0$$

then

$$
\begin{aligned}
(x - x_1)(x - x_2) &= x^2 - (x_1 + x_2)x + x_1x_2 \\
&= x^2 + \frac{b}{a}x + \frac{c}{a}
\end{aligned}
$$

And so

$$x_1 + x_2 = -\frac{b}{a}$$

$$x_1x_2 = \frac{c}{a}$$

Po-Shen Loh method:

We know that

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

has roots x_1 and x_2 such that

$$(x - x_1)(x - x_2) = 0$$

Let x_1 be the "low" root and x_2 be the "high" root. In other words, $x_1 \leq x_2$. Then there exists a midpoint, x_m, halfway between x_1 and x_2 such that $x_1 \leq x_m \leq x_2$, and such that

$$x_m - x_1 = x_2 - x_m$$

Hence,

$$x_m = \frac{x_1 + x_2}{2}$$

Let d denote the halfway distance, such that

$$x_m - x_1 = x_2 - x_m = d$$

We then see that

$$x_1 = x_m - d$$

$$x_2 = x_m + d$$

We know that

$$x_1 + x_2 = -\frac{b}{a}$$

and

$$x_1 x_2 = \frac{c}{a}$$

Therefore

$$
\begin{aligned}
(x_m - d) + (x_m + d) &= -\frac{b}{a} \\
2x_m &= -\frac{b}{a} \\
x_m &= -\frac{b}{2a}
\end{aligned}
$$

and

$$
\begin{aligned}
(x_m - d)(x_m + d) &= \frac{c}{a} \\
x_m^2 - d^2 &= \frac{c}{a} \\
d^2 &= x_m^2 - \frac{c}{a} \\
d &= \sqrt{x_m^2 - \frac{c}{a}}
\end{aligned}
$$

Note that, because d is a distance, we only need the "+" side of the square root. Now that we know d, we can find x_1 and x_2 from

$$
\begin{aligned}
x_1 &= x_m - d \\
x_2 &= x_m + d
\end{aligned}
$$

So, the Po-Shen Loh method consists of the following steps:

$$
x_m = -\frac{b}{2a}
$$

$$
d = \sqrt{x_m^2 - \frac{c}{a}}
$$

$$
\begin{aligned}
x_1 &= x_m - d \\
x_2 &= x_m + d
\end{aligned}
$$

Note that

$$
\begin{aligned}
x_1 &= x_m - d \\
&= x_m - \sqrt{x_m^2 - \frac{c}{a}} \\
&= -\frac{b}{2a} - \sqrt{\left(-\frac{b}{2a}\right)^2 - \frac{c}{a}} \\
&= -\frac{b}{2a} - \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\
&= -\frac{b}{2a} - \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}} \\
&= -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-b - \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

and likewise

$$
\begin{aligned}
x_2 &= x_m + d \\
&= \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

which confirms that the Poh-Shen Loh method is indeed the quadratic formula.