1 JTIX in Wachter (2013)

1.1 Summary

Proposition 1. *JTIX, a model free measure of jump risk, for a ϕ-levered consumption claim in the Wachter (2013) is an affine function of the jump intensity:*

$$JTIX_{t,\tau} = 2\Psi(a_{0,\lambda}(\tau) + a_{1,\lambda}(\tau)\lambda_t)$$

where

$$\begin{aligned}
a_{\lambda,0}(\tau) &= \left[\overline{\lambda}\frac{\kappa}{\kappa - b\sigma_\lambda}\tau - \overline{\lambda}\frac{\kappa}{\kappa - b\sigma_\lambda^2}\left(1 - e^{-(\kappa - b\sigma_\lambda^2)\tau}\right)\right] \\
a_{\lambda,1}(\tau) &= \frac{1}{\kappa - b\sigma_\lambda^2}\left(1 - e^{-(\kappa - b\sigma_\lambda^2)\tau}\right) \\
\Psi &= \left[1 + \phi(\mu_Z - \gamma\sigma_Z^2) + \frac{1}{2}\phi^2(\sigma_Z^2 + (\mu_Z - \gamma\sigma_Z^2)^2) - e^{\phi(\mu_Z - \gamma\sigma_Z^2) + \frac{1}{2}\phi^2\sigma_Z^2}\right] e^{-\frac{1}{2}(2\mu_Z\gamma - \gamma^2\sigma_Z^2)}
\end{aligned}$$

This proposition implies that JTIX in the Wachter (2013) model inherits a similar slow moving behavior of the disaster intensity.

Proposition 2. *$JTIX_{t,\tau}$ has a distribution $JTIX_{t,\tau} \sim 2\Psi(a_{\lambda,0}(\tau) + a_{\lambda,1}(\tau)\xi)$ where ξ has a Gamma distribution $\xi \sim \Gamma(\alpha, \beta)$ with location parameter α and scale parameters $1/\beta$. α and β equal to*

$$\alpha = \frac{2\kappa}{\sigma_\lambda^2}\overline{\lambda}, \;\; \beta = \frac{2\kappa}{\sigma_\lambda^2}$$

This proposition gives a way to simulate the distribution of JTIX within a Wachter (2013) model. In Figure 1, we show the empirical distributions for daily log-JTIX based on SPX options for 30, 90 and 180 days maturities. We compare them with theoretical distributions within the Wachter (2013) model based on propositions 1 and 2

Proposition 1 provideds us with a straighforward way to invert the JTIX measure to get the disaster intensity within the Wachter (2013). We present this inversion in Figure 2 based on 30, 90 and 180 day.

Finally, we compare the disaster intensity obbtained bby inverting JTIX with the from inverting the CAPE ratio (following section II.C from Wachter, 2013) in Figure

1.2 Overview of Wachter (2013) model

Consumption Process Consumption follows a jump-diffusion process

$$\frac{dC_t}{C_t^-} = \mu_c dt + \sigma_c dB_{c,t} + (e^{Z_t} - 1)dN_t, \tag{1}$$

where dN_t is a Poisson counting process with time varying intensity λ_t the process for which we describe below, the distribution of jump size $Z_t \sim \nu(\cdot)$ is time-invariant. Rewrite the process (1) in logs

$$d\log C_t = \left(\mu_c - \frac{1}{2}\sigma_c^2\right)dt + \sigma_c dB_{c,t} + Z_t dN_t. \tag{2}$$

Time varying intensity follows

$$d\lambda_t = \kappa(\overline{\lambda} - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} \tag{3}$$

where the square root ensures that $\lambda_t > 0$.

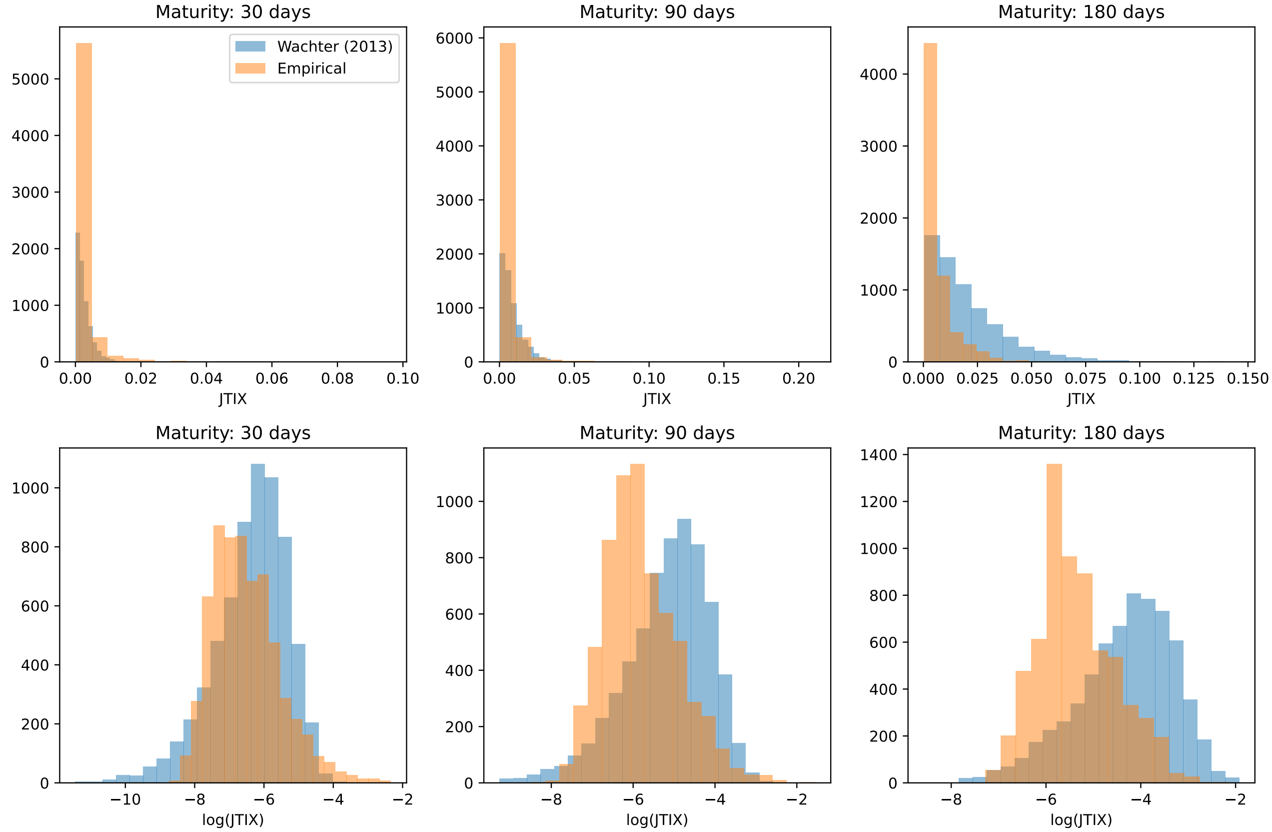

Figure 1: Comparing JTIX in the data and in Wachter 2013

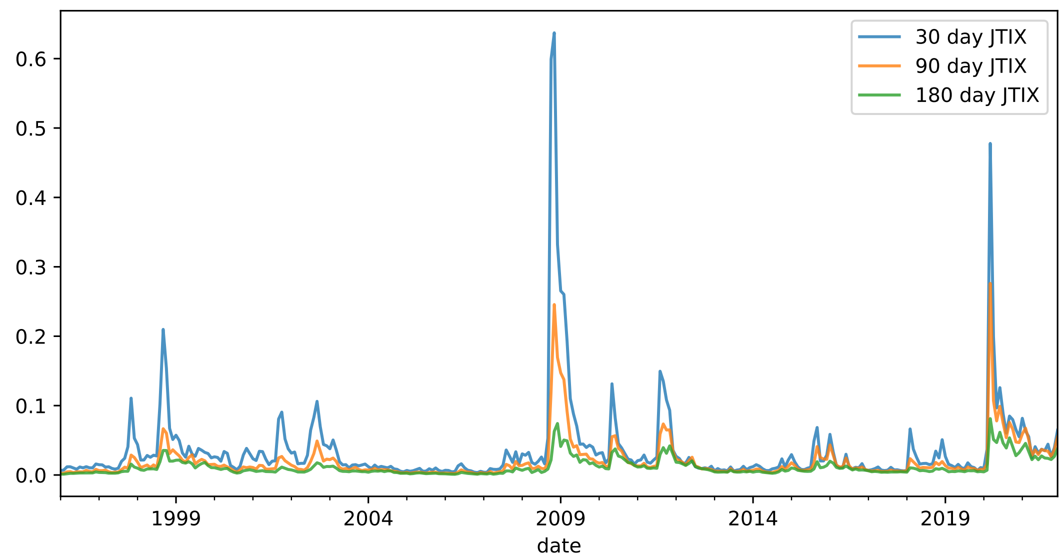

Figure 2: Comparing JTIX in the data and in Wachter 2013

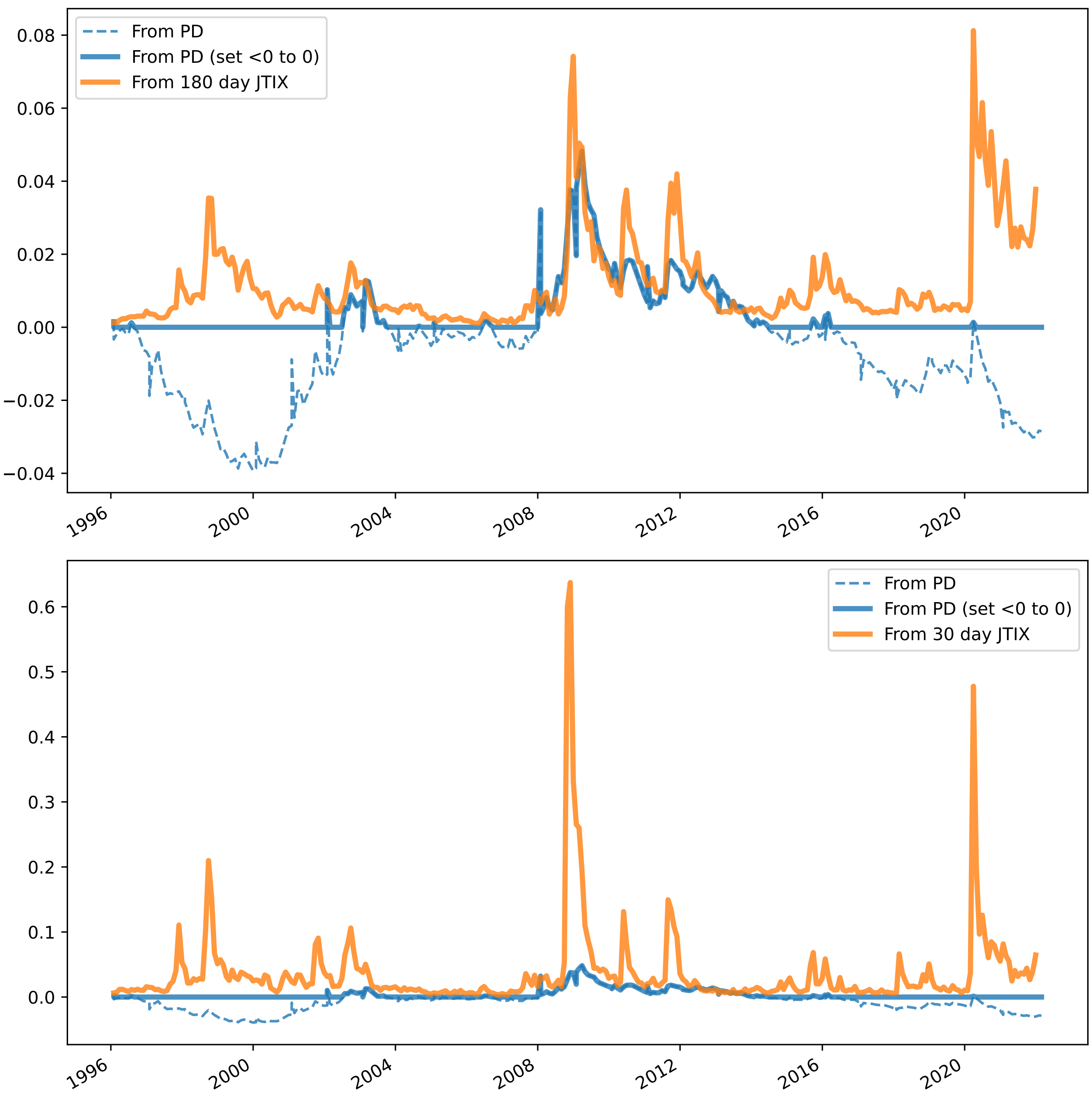

Figure 3: Comparing JTIX in the data and in Wachter 2013

Preferences Wachter (2013) assumes recursive utility with EIS = 1 that allows for a semi closed form solution for the price dividend ratio. In particular, a representative agent is assumed to have a Duffie-Epstein recursive utility

$$V_t = \int_t^\infty f(C_s, V_s)ds$$

where

$$f(C,V) = \beta(1-\gamma)V\left(\log C - \frac{1}{1-\gamma}\log((1-\gamma)V)\right).$$

The state prices under EZ are

$$\pi_t = \exp\left\{\int_0^t f_V(C_s, V_s)ds\right\} f_C(C_t, V_t), \tag{4}$$

and Wachter (2013) shows that it is equal to

$$\pi_t = \exp\left\{-\int_0^t (\beta b\lambda_s - \eta)ds\right\} \beta^\gamma C_t^{-\gamma} e^{a+b\lambda_t} \tag{5}$$

where b is defined in (8) and

$$\begin{aligned} a &= \frac{1-\gamma}{\beta}\left(\mu_c - \frac{1}{2}\gamma\sigma_c^2\right) + (1-\gamma)\log\beta + b\frac{\kappa\bar{\lambda}}{\beta} \\ \eta &= \beta(1-\gamma)\log\beta - \beta a - \beta \end{aligned} \tag{6}$$

Note that in equation (5) the two sources of risk are (1) shocks to consumption as exemplified by $C_t^{-\gamma}$ and (2) shocks to disaster intensity as exemplified by $e^{a+b\lambda_t}$.

Risky assets An asset that is a claim to levered consumption pays a dividend

$$D_t dt = C_t^\phi dt$$

during interval $[t, t+dt]$. The price dividend for such asset is defined as

$$\frac{P_t}{D_t} = E_t\left[\int_t^\infty \frac{\pi_s}{\pi_t}\frac{D_s}{D_t}\right]$$

Wachter (2013) shows that the price dividend ratio in this model is a function of λ_t only and can be written as

$$G(\lambda_t) \equiv \frac{P_t}{D_t} = \int_t^\infty \exp\{a_\phi(\tau) + b_\phi(\tau)\lambda_t\}d\tau$$

where

$$\begin{aligned} a_\phi(\tau) &= \left((\phi-1)\left(\mu_c + \frac{1}{2}(\phi-\gamma)\sigma_c^2\right) - \beta - \frac{\kappa\bar{\lambda}}{\sigma_\lambda^2}(\zeta_\phi + b\sigma_\lambda^2 - \kappa)\right)\tau - \frac{2\kappa\bar{\lambda}}{\sigma_\lambda^2}\log\left(\frac{(\zeta_\phi + b\sigma_\lambda^2 - \kappa)(e^{-\zeta_\phi\tau} - 1) + 2\zeta_\phi}{2\zeta_\phi}\right) \\ b_\phi(\tau) &= \frac{2\mathbb{E}_\nu\left[e^{(1-\gamma)Z} - e^{(\phi-\gamma)Z}\right](1 - e^{-\zeta_\phi\tau})}{(\zeta_\phi + b\sigma_\lambda^2 - \kappa)(1 - e^{-\zeta_\phi\tau}) - 2\zeta_\phi} \end{aligned} \tag{7}$$

and constants b and ζ_ϕ are

$$\begin{aligned} \zeta_\phi &= \sqrt{(b\sigma_\lambda^2 - \kappa)^2 + 2\mathbb{E}_\nu\left[e^{(1-\gamma)Z} - e^{(\phi-\gamma)Z}\right]\sigma_\lambda^2} \\ b &= \frac{\kappa+\beta}{\sigma_\lambda^2} - \sqrt{\left(\frac{\kappa+\beta}{\sigma_\lambda^2}\right)^2 - 2\frac{\mathbb{E}\left[e^{(1-\gamma)Z}\right] - 1}{\sigma_\lambda^2}} \end{aligned} \tag{8}$$

Linearization In order to solve for option prices and, in particular, map the set-up of the model into the affine structure of Duffie, Pan and Singleton (2000), Seo and Wachter (2019) linearize the log price dividend ratio around some λ^* as

$$\log(\hat{G}(\lambda)) = \log(G(\lambda^*)) + \frac{G'(\lambda^*)}{G(\lambda^*)}(\lambda - \lambda^*)$$

$$\hat{G}(\lambda) = G(\lambda^*) \exp\left\{\frac{G'(\lambda^*)}{G(\lambda^*)}(\lambda - \lambda^*)\right\}$$

Denote

$$b^*_\phi \equiv \frac{G'(\lambda^*)}{G(\lambda^*)} = \frac{1}{G(\lambda^*)} \int_0^\infty b_\phi(\tau) \exp\{a_\phi(\tau) + b_\phi(\tau)\lambda^*\} d\tau$$

so that

$$\hat{G}(\lambda) = G(\lambda^*) \exp\{b^*_\phi(\lambda - \lambda^*)\}. \tag{9}$$

1.3 Calculating JTIX in Wachter (2013)

A model independent measure jump measure $JTIX_{t,\tau}$ equals to

$$JTIX_{t,\tau} = 2E^Q_\nu\left[1 + \Delta Z + \frac{1}{2}(\Delta Z)^2 - e^{\Delta Z}\right] E^Q \int_t^{t+\tau} \lambda_{t+s} ds$$

where $E^Q_\nu[\cdot]$ is an expectation with respect to the distribution of the jump ΔZ of a levered consumption claim. The value of a levered consumption claim S_t in Wachter (2013) model is $S_t = G(\lambda_t)C_t^\phi$ implying that conditional on a jump in consumption the jump in $\log(S_t)$ equals to $\phi\Delta Z_t$ where ΔZ_t is the size of the jump in log consumption $\log(C_t)$. To calculate this expression within Wachter (2013) model, we split it into two parts.

Jump distribution The first part can be calculated as

$$\begin{aligned}
&E^Q_\nu\left[1 + \phi Z + \frac{1}{2}(\phi Z)^2 - e^{\phi Z}\right] \\
&= E_\nu\left[\frac{\pi_t}{\pi_{t-}}\left(1 + \phi Z + \frac{1}{2}(\phi Z)^2 - e^{\phi Z}\right)\right] \\
&= E_\nu\left[\frac{\exp\left\{-\int_0^t(\beta b\lambda_s - \eta)ds\right\}\beta^\gamma C_t^{-\gamma}e^{a+b\lambda_t}}{\exp\left\{-\int_0^{t-}(\beta b\lambda_s - \eta)ds\right\}\beta^\gamma C_{t-}^{-\gamma}e^{a+b\lambda_{t-}}}\left(1 + \phi Z + \frac{1}{2}(\phi Z)^2 - e^{\phi Z}\right)\right] \\
&= E_\nu\left[\frac{C_t^{-\gamma}}{C_{t-}^{-\gamma}}\left(1 + \phi Z + \frac{1}{2}(\phi Z)^2 - e^{\phi Z}\right)\right] \\
&= E_\nu\left[\left(e^Z\right)^{-\gamma}\left(1 + \phi Z + \frac{1}{2}(\phi Z)^2 - e^{\phi Z}\right)\right]
\end{aligned}$$

This expression can be evaluated under the assumption that $Z \sim \mathcal{N}(\mu_z, \sigma_Z)$ as follows

$$\begin{aligned}
&E^Q_\nu\left[1 + \phi Z + \frac{1}{2}(\phi Z)^2 - e^{\phi Z}\right] \\
&= E_{\tilde{\nu}}\left[\left(1 + \phi Z + \frac{1}{2}(\phi Z)^2 - e^{\phi Z}\right)\right] e^{-\frac{1}{2}(2\mu_Z\gamma - \gamma^2\sigma_Z^2)}
\end{aligned}$$

where, under $\tilde{\nu}$, $Z \sim \mathcal{N}(\mu_Z - \gamma\sigma_Z^2, \sigma_Z)$ which can be shown by explicitly writing normal density and collecting Z variables. Therefore

$$\Psi = \left[1 + \phi(\mu_Z - \gamma\sigma_Z^2) + \frac{1}{2}\phi^2(\sigma_Z^2 + (\mu_Z - \gamma\sigma_Z^2)^2) - e^{\phi(\mu_Z - \gamma\sigma_Z^2) + \frac{1}{2}\phi^2\sigma_Z^2}\right] e^{-\frac{1}{2}(2\mu_Z\gamma - \gamma^2\sigma_Z^2)}$$

Intensity Following Proposition 6F in Duffie (2004), the risk neutral expectation is

$$E^Q\left[\lambda_{t+s}\right] = E\left[\exp\left\{\int_t^{t+s} r_s ds\right\}\frac{\pi_{t+s}}{\pi_t}\lambda_{t+s}\right] \tag{10}$$

where r_s is the short rate process

$$r_s = \beta + \mu - \gamma\sigma^2 + \lambda_t E_\nu\left[e^{-\gamma Z}(e^Z - 1)\right].$$

Transform method To calculate expectation in (10), we can utilize the transform method from Duffie, Pan and SIngleton (2000). To do this, first, define the state variable

$$X_t = \begin{pmatrix} \log C_t - \log C_0 \\ \lambda_t \end{pmatrix},$$

with law of motion

$$dX_t = \mu(X_t)dt + \sigma(X_t)\begin{pmatrix} dB_t \\ dB_{\lambda,t} \end{pmatrix} + \begin{pmatrix} Z_t \\ 0 \end{pmatrix} dN_t.$$

Next define $(K_0, K_1, H_0, H_1, l_0, l_1, \rho_0, \rho_1)$, parameters (u, v) and function $\theta(c)$:

- Drift of X_s:

$$\mu(X_t) = K_0 + K_1 X_t = \begin{pmatrix} \mu - \frac{1}{2}\sigma_c^2 \\ \kappa\bar{\lambda} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\kappa \end{pmatrix}\begin{pmatrix} \log C_s/C_0 \\ \lambda_s \end{pmatrix}$$

- Element (i, j) of the covariance matrix of X_t:

$$\sigma(X_s) = \begin{pmatrix} \sigma_c & 0 \\ 0 & \sigma_\lambda\sqrt{\lambda_s} \end{pmatrix} \Rightarrow \sigma(X_s)\sigma(X_s)^T = \begin{pmatrix} \sigma_c^2 & 0 \\ 0 & \sigma_\lambda^2\lambda_s \end{pmatrix}$$

  implying

$$\begin{aligned}
(\sigma(X_t)\sigma(X_t)^T)_{11} &= (H_0)_{11} + (H_1)_{11}X_s = \sigma_c^2 + (0,\ 0)\cdot X_s \\
(\sigma(X_t)\sigma(X_t)^T)_{12} &= (H_0)_{12} + (H_1)_{12}X_s = 0 + (0,\ 0)\cdot X_s \\
(\sigma(X_t)\sigma(X_t)^T)_{21} &= (H_0)_{21} + (H_1)_{21}X_s = 0 + (0,\ 0)\cdot X_s \\
(\sigma(X_t)\sigma(X_t)^T)_{22} &= (H_0)_{22} + (H_1)_{22}X_s = 0 + (0,\ \sigma_\lambda^2)\cdot X_s
\end{aligned}$$

- Jump intensity $\lambda(X_s)$

$$\lambda(X_s) = l_0 + l_1 \cdot X_s = 0 + (0,\ 1)X_s$$

- "Discount rate" $R(X_s)$:

$$R(X_s) = \rho_0 + \rho_1 \cdot X_s = -(\mu - \gamma\sigma^2 + \beta(1-\gamma)\log(\beta) - \beta a) + (0,\ b\beta - E\left[e^{-\gamma Z}(e^Z - 1)\right])X_s$$

- Variable multiplying the state in the exponent $u = (-\gamma, b)$
- Variable multiplying the state $v = (0, 1)$

- Function capturing the properties of the jump size

$$\theta(c) = \int_{R^2} e^{c_1 Z_1 + c_2 Z_2} d\nu(Z) = E_\nu[e^{c_1 Z_t}].$$

Under these definitions, the expectation in (10) can be written as

$$E^Q\left[\lambda_{t+\tau}\right] = E\left[\exp\left\{\int_t^{t+\tau}(\rho_0 + \rho \cdot X_s)ds\right\}(v \cdot X_{t+\tau})e^{u \cdot X_{t+\tau}}\right]e^{-b\lambda_t} \tag{11}$$

Duffie, Pan and Singleton (2000) show expectation in (11) can be calculated as

$$\exp\{\alpha(t) + \beta(t) \cdot X_t\}(A(t) + B(t) \cdot X_t)$$

where functions $\alpha(t), \beta(t), A(t)$ and $B(t)$ satisfy the following differential equations

$$\begin{cases} \dot{\beta}(t) = \rho_1 + K_1^T \beta(t) - \dfrac{1}{2}\begin{pmatrix} \sum_{ij}(\beta(t))_i (H_1)_{ij1}(\beta(t))_j \\ \sum_{ij}(\beta(t))_i (H_1)_{ij2}(\beta(t))_j \end{pmatrix} - l_1(\theta(\beta(t)) - 1) & \text{subject to } \beta(t+\tau) = u \\ \dot{\alpha}(t) = \rho_0 + K_0 \cdot \beta(t) - \dfrac{1}{2}\beta(t)^T H_0 \beta(t) - l_0(\theta(\beta(t)) - 1) & \text{subject to } \alpha(t+\tau) = 0 \\ \dot{B}(t) = -K_1^T B(t) - \begin{pmatrix} \sum_{ij}(\beta(t))_i (H_1)_{ij1}(B(t))_j \\ \sum_{ij}(\beta(t))_i (H_1)_{ij2}(B(t))_j \end{pmatrix} - l_1 \nabla\theta(\beta(t)) \cdot B(t) & \text{subject to } B(t+\tau) = v \\ \dot{A}(t) = -K_0 \cdot B(t) - \beta(t)^T H_0 B(t) - l_0 \nabla\theta(\beta(t)) B(t) & \text{subject to } A(t+\tau) = 0 \end{cases}$$

It follows from this system that $\dot{\beta}(s)_1 = 0\ \forall\ s$. Additionally, but substituting the constants defined in Wachter (2013), it can be shown that $\dot{\beta}(t+\tau)_2 = 0$ and, therefore, $\dot{\beta}(s)_2 = 0\ \forall\ s$. These two conditions imply that $\beta(t) = u = (-\gamma, b)$. Using this solutions for $\beta(s)$ we can also show that $\dot{\alpha}(s) = 0\ s\forall\ s$ and, therefore, $\alpha(s) = 0\ \forall\ s$. It is also the case that $\dot{B}(s)_1 = 0\ s\forall\ s$. As a result, the system symplifies to

$$\begin{cases} \dot{B}(t)_2 = (\kappa - \sigma_\lambda^2 b)B(t)_2 & \text{subject to } B(t+\tau)_2 = 1 \\ \dot{A}(t) = -\kappa\overline{\lambda}B(t)_2 & \text{subject to } A(t+\tau) = 0 \end{cases}$$

This system can bbe solved in closed form

$$\begin{cases} B(t)_2 = e^{-(\kappa - \sigma_\lambda^2 b)\tau} \\ A(t) = \dfrac{\kappa}{\kappa - \sigma_\lambda^2 b}\left(1 - e^{-(\kappa - \sigma_\lambda^2 b)\tau}\right) \end{cases}$$

Combining these solution we can write the expectation as

$$E^Q[\lambda_{t+\tau}] = e^{b\lambda_t}\left(\frac{\kappa}{\kappa - \sigma_\lambda^2 b}\left(1 - e^{-(\kappa - \sigma_\lambda^2 b)\tau}\right) + e^{-(\kappa - \sigma_\lambda^2 b)\tau}\lambda_t\right)e^{-b\lambda_t}.$$

Finally, we can integrate this expression to get

$$\int_0^\tau E_0^Q[\lambda_{t+s}]ds = \underbrace{\left[\overline{\lambda}\frac{\kappa}{\kappa - b\sigma_\lambda}\tau - \overline{\lambda}\frac{\kappa}{\kappa - b\sigma_\lambda^2}\left(1 - e^{-(\kappa - b\sigma_\lambda^2)\tau}\right)\right]}_{a_{\lambda,0}(\tau)} + \underbrace{\frac{1}{\kappa - b\sigma_\lambda^2}\left(1 - e^{-(\kappa - b\sigma_\lambda^2)\tau}\right)}_{a_{\lambda,1}(\tau)}\lambda_t$$

Limiting Distribution of JTIX To calculate the limiting distribution of the measure we proceed in two steps. First, we note that λ_t has a stationary Gamma distribution $\propto \lambda^{\alpha-1}e^{-\beta\lambda}$. This can be verified by writing the Kolmogorov Forward Equation

$$\frac{\partial p(\lambda,t)}{\partial t} = -\frac{\partial}{\partial \lambda}[\kappa(\overline{\lambda}-\lambda)p(\lambda,t)] + \frac{\partial^2}{\partial \lambda^2}\left[\frac{\sigma_\lambda^2\lambda}{2}p(\lambda,t)\right],$$

setting $\partial p/\partial t = 0$ and substituting $p(\lambda) = c\lambda^{\alpha-1}e^{-\beta\lambda}$. By setting the coefficients multiplying λ and λ^2 to zero we get

$$\alpha = \frac{2\kappa}{\sigma_\lambda^2}\overline{\lambda},\ \ \beta = \frac{2\kappa}{\sigma_\lambda^2}.$$

Second, we draw random variables $\lambda^{(i)}$'s from the Gamma distribution and transform then to get

$$\log(JTIX)^{(i)} = \log(\Psi(a_{\lambda,0}(\tau) + a_{\lambda,1}(\tau)\lambda^{(i)})).$$

We then compare the distribution of JTIX in Wachter (2013) and in the data for different maturities.