1 Fitting volatility curve with SVI

1.1 General Problem

SVI parametric formulation (e.g. Zeliade, 2009) of the volatility Curve is

$$\sigma^2_{BS}(k) = a + b\left(\rho(k-m) + \sqrt{(k-m)^2+\sigma^2}\right)$$

where $k = \log(Strike/Forward)$ and (m, σ, ρ, a, b) – parameters. This functional form assumes $a \in \mathbb{R}, b \geq 0, |\rho| < 1, m \in \mathbb{R}, \sigma > 0$ and constraint

$$a + b\sigma\sqrt{1-\rho^2} \geq 0 \Rightarrow a \geq -b\sigma\sqrt{1-\rho^2}$$

that insures that this function lies above 0 everywhere. Absence of static arbitrage requires

$$b \leq \frac{4}{(1+|\rho|)T}$$

Zeliade notes that for large maturities, almost affine smiles are not uncommon. This corresponds to the case when $\sigma \to 0$ or when $\sigma \to \infty, a \to -\infty$. To rule out the first limiting case, Zeliade (2009) restricts $\sigma \geq \sigma_{min} > 0$. To rule this the second, Zeliade (2009) assumes that $a \geq 0$. However, I found that it is hard to fit the smile when $a \geq 0$, since implied variances for strikes close to current price are too close to zero, so that we may want to set parameter $a < 0$ which is in general doesn't violate anything.

1.2 Zeliade (2009) method of reducing dimensionality

The approach of Zeliade (2009) then minimizes sum of squared residuals over parameters $\theta = (m, \sigma, \rho, a, b)$

$$\min_\theta \sum_i \left[a + b\left(\rho(k_i - m) + \sqrt{(k_i-m)^2+\sigma^2}\right) - \sigma_i^2\right]^2$$

where k_i – observed $\log(Strike_i/Forward_i)$ and σ_i is the observed implied variance. If we follow Zeliade (2009) assumption that $a \geq 0$, then we can easily transform this problem into a linear one for fixed (m, σ):

1. Divide the parameter θ into two parts $\theta_1 = (m, \sigma)$ and $\theta_2 = (\rho, a, b)$. Fix (m, σ) and substitute $y_i = \frac{k_i - m}{\sigma}$ so that the minimization objective becomes

$$\sum_i \left[a + b\left(\rho\sigma y_i + \sigma\sqrt{y_i^2+1}\right) - \sigma_i^2\right]^2$$

2. Zeliade (2009) works with total variance Tv rather than on variance σ^2 (unclear to me why exactly). Denote total variance $\tilde{v} = Tv$, so that the SVI becomes

$$v(k) = aT + bT\left(\rho(k-m) + \sqrt{(k-m)^2+\sigma^2}\right)$$

and the minimization objective becomes

$$\sum_i \left[aT + b\rho\sigma T y_i + b\sigma T\sqrt{y_i^2+1} - \tilde{v}_i^2\right]^2$$

Replace variables $\tilde{a} := aT, d := b\rho\sigma T, c := b\sigma T$. Now the problem is just a linear least squares regression for the new variables

$$\sum_i \left[\tilde{a} + dy_i + c\sqrt{y_i^2+1} - \tilde{v}_i^2\right]^2$$

3. Now we need to deal with constraints.

$$\rho \in [-1,1] \Rightarrow |d| \leq c$$

$$b \geq 0 \Rightarrow c \geq 0$$

$$b \leq \frac{4}{(1+|\rho|)T} \Rightarrow c \leq \frac{4\sigma}{1+|\rho|} \Rightarrow c + c|\rho| \leq 4\sigma \Rightarrow c + |d| \leq 4\sigma \Rightarrow |d| \leq 4\sigma - c$$

$$c \leq \frac{4\sigma}{1+|\rho|} \Rightarrow c \leq 4\sigma$$

$$0 \leq a \leq \max_i v_i \Rightarrow 0 \leq \tilde{a} \leq \max_i \tilde{v}_i$$

Thus, we can described the parameter space as

$$\mathcal{D} = \begin{cases} 0 \leq c \leq 4\sigma \\ |d| \leq c, |d| \leq 4\sigma - c \\ 0 \leq \tilde{a} \leq \max_i \tilde{v}_i \end{cases}$$

1.3 Simplification of Berger, Dew-Becker and Giglio

Berger, Dew-Becker and Giglio (?) assumes that $\rho = 0$. ρ controls the assymetry of asymptotes of a hyperbola and thus asymmetry of the slopes of wings of the volatility smile. They say that this including this ρ has a minimal effect on the fit. In this case the smile positivity condition simplifies to $a \geq -b\sigma \Rightarrow \tilde{a} \geq -c$. In this case, the optimization simplifies the following procedure

1. For fixed (m, σ) the objective becomes

$$\min_{\tilde{a},d} \sum_i \left[\tilde{a} + c\sqrt{y_i^2+1} - \tilde{v}_i^2\right]^2$$

subject to

$$\mathcal{D} = \begin{cases} 0 \leq c \leq 4\sigma \\ -c \leq \tilde{a} \leq \max_i \tilde{v}_i \end{cases}$$

$\mathcal{D}$ defines a parallelogram in the parameter space and minimization objective is a convex function.

2. Define

$$X = \begin{pmatrix} 1 & \sqrt{y_1+1} \\ \vdots & \vdots \\ 1 & \sqrt{y_n+1} \end{pmatrix}, \tilde{v} = \begin{pmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{pmatrix}$$

   - Estimate linear regression $\beta := (\tilde{a}\ c)' = (X'X)^{-1}X'\tilde{v}$. If $\beta \in \mathcal{D}$ then we found the minimum. If $\beta \notin \mathcal{D}$ proceed further
   - Estimate regression along the side of domain $\mathcal{D}$. Under a linear constraint on parameters $R\beta = b$, $\beta = \arg\min(X\beta - \tilde{v})'(X\beta - \tilde{v})$ is given by

$$\beta = (X'X)^{-1}(X'\tilde{v} + R'\lambda) \text{ where } \lambda = \left[R(X'X)^{-1}R'\right]^{-1}\left[b - R(X'X)^{-1}X'\tilde{v}\right]$$

     Linear constraints for sides of $\mathcal{D}$ are

$$\begin{aligned} (c = 0) &: R = (0\ 1), b = 0 \\ (c = 4\sigma) &: R = (0\ 1), b = 4\sigma \\ (\tilde{a} = -c) &: R = (1\ 1), b = 0 \\ (\tilde{a} = \max_i \tilde{v}_i) &: R = (1\ 0), b = \max_i \tilde{v}_i \end{aligned}$$

     For each of the constraints we need to check that the solution satisfies all other inequalities. If it doesn't, it can't be a solution candidate
   - Estimate objective in 4 vertices

$$\begin{aligned} &\tilde{a} = 0, c = 0 \\ &\tilde{a} = -4\sigma, c = 4\sigma \\ &\tilde{a} = \max_i \tilde{v}_i, c = 0 \\ &\tilde{a} = \max_i \tilde{v}_i, c = 4\sigma \end{aligned}$$

   - Pick the solution along the sides and vertices that has the lowest objective.

Relaxing the constraint from $\tilde{a} \geq 0$ to $\tilde{a} \geq -c$ seems to improve the fit

1.4 Proceeding without assuming $\rho = 0$ and $a \geq 0$

If we don't assume that $\rho = 0$ or $a \geq 0$ and leave constraint $a > -b\sigma\sqrt{1-\rho^2}$ the problem complicates since we can't transform everything into a linear problem. We can proceed in the following way

1. Drop the constraint on $\tilde{a}$ altogether. In this case we can simplify the problem into linear one

2 Appendix

2.1 Regression with linear constraints on parameters

The problem is

$$\min_{\beta} \frac{1}{2}(X\beta - \tilde{v})'(X\beta - \tilde{v}) \text{ subject to } R\beta = b$$

Set up lagrangian

$$\mathcal{L} = \frac{1}{2}(X\beta - \tilde{v})'(X\beta - \tilde{v}) - \lambda'(R\beta - b)$$

First order condition

$$\frac{\partial \mathcal{L}}{\partial \beta} = (X\beta - \tilde{v})'X - \lambda' R = 0 \Rightarrow \beta' X' X - \tilde{v}' X - \lambda' R = 0 \Rightarrow \beta = (X'X)^{-1}(X'\tilde{v} + R'\lambda)$$

Plug this into constraint to get

$$R(X'X)^{-1}(X'\tilde{v} + R'\lambda) = b \Rightarrow R(X'X)^{-1}X'\tilde{v} + R(X'X)^{-1}R'\lambda = b \Rightarrow$$

$$\lambda = \left[R(X'X)^{-1}R'\right]^{-1}\left[b - R(X'X)^{-1}X'\tilde{v}\right]$$

If we plug λ back into the expression for β we can get the final answer.