Notes on Dynamic Portfolio Choice (based on Back's textbook)

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Contents

1 Stochastic Discount Factor Process

Some preliminaries

Definition 1 (Stochastic Discount Factor Process)**.** *For a discrete time dynamic problem, the sequence of random variables $M_1, M_2, \dots$ is called a stochastic discount factor process if*

1. *M_t depends only on date$-t$ information (often this is called $\mathcal{F}_t$-measurable where $\mathcal{F}_t$ – filtration at time t)*

2. *For all i and t*

$$M_t P_{it} = \mathbb{E}_t[M_{t+1}(P_{i,t+1} + D_{it})]$$

Definition 2 (Self-Financing Wealth Process)**.** *Given a portfolio process w such that $w'\iota = 1$ the process defined by*

$$W_{t+1} = W_t(w' R_{t+1})$$

starting from $W_0 \geq 0$ is called a self-financing wealth process

Self-Financing Wealth Process is a particular case of the intertemporal budget constraint

$$W_{t+1} = Y_{t+1} + (W_t - C_t)(w' R_{t+1})$$

where $Y_{t+1} = C_t = 0$

Definition 3 (Martingale Property)**.** *A stochastic process $\{X_t\}$ is said to be a martingale if for all t*

$$X_t = \mathbb{E}_t[X_{t+1}]$$

This is an important concept in finance since it defines a notion of fair game. If a gamble is a fair game you don't expect your wealth to go up or down. This is not directly applicable to asset markets since it goes up on average. However, under a *proper normalization* wealth is going to be a martingale. Thus for any self-financing wealth process we have

$$M_t W_t = \mathbb{E}_t[M_{t+1} W_{t+1}]$$

This follows from the definition of stochastic discount factor process in return form

$$M_t = \mathbb{E}_t[M_{t+1} R_{i,t+1}]$$

stack these equations for all i to get

$$M_t \iota = \mathbb{E}_t[M_{t+1} R_{t+1}]$$

and multiply both sides by w' and by W_t

$$M_t W_t (w'\iota) = \mathbb{E}_t[M_{t+1} W_t (w' R_{t+1})] \implies M_t W_t = \mathbb{E}_t[M_{t+1} W_{t+1}]$$

Valuation over Finite Horizons Using the definition of self-financing wealth process and iterating it forward

$$W_t = \mathbb{E}_t\left[\frac{M_{t+1}}{M_t} W_{t+1}\right] = \mathbb{E}_t\left[\frac{M_{t+1}}{M_t}\mathbb{E}_{t+1}\left[\frac{M_{t+2}}{M_{t+2}} W_{t+2}\right]\right] = \mathbb{E}_t\left[\frac{M_{t+2}}{M_t} W_{t+2}\right] = \cdots = \mathbb{E}_t\left[\frac{M_T}{M_t} W_T\right]$$

for all $T > t$. This shows that the cost of getting W_T at time T is given by W_t.

For non-self financing wealth processes this is a little bit more complicated and needs to be proven by induction

$$W_t + \sum_{s=t+1}^{T} \mathbb{E}_t\left[\frac{M_s}{M_t} Y_s\right] = \sum_{s=t+1}^{T} \mathbb{E}_t\left[\frac{M_s}{M_t} C_s\right] + \mathbb{E}_t\left[\frac{M_T}{M_t} W_T\right]$$

First note that it holds for $t = T$ as we left with only $W_t = W_T$. Next, suppose that it is true for $t = \tau+1, \tau+2, \ldots, T$ and let's prove that it is true for $t = \tau$. Multiply the intertemporal budget constraint by M_{t+1} and take expectations at time t:

$$W_{t+1} = Y_{t+1} + (W_t - C_t)(w' R_{t+1}) \implies \mathbb{E}_t[M_{t+1} W_{t+1}] = \mathbb{E}_t[M_{t+1} Y_{t+1}] + (W_t - C_t) w' \mathbb{E}_t[M_{t+1} R_{t+1}]$$

Use the definition of SDF process to substitute $\mathbb{E}_t[M_{t+1} R_{t+1}] = M_t \iota$

$$\mathbb{E}_t[M_{t+1} W_{t+1}] = \mathbb{E}_t[M_{t+1} Y_{t+1}] + (W_t - C_t) w' M_t \iota$$

$$\mathbb{E}_t[M_{t+1} W_{t+1}] = \mathbb{E}_t[M_{t+1} Y_{t+1}] + (W_t - C_t) M_t$$

Use the conjectured solution at time $\tau+1$ (premultiply it by $M_{\tau+1}$)

$$M_{\tau+1} W_{\tau+1} + \sum_{s=\tau+2}^{T} \mathbb{E}_{\tau+1}\left[M_s Y_s\right] = \sum_{s=\tau+2}^{T} \mathbb{E}_{\tau+1}\left[M_s C_s\right] + \mathbb{E}_{\tau+1}\left[M_T W_T\right]$$

$$M_{\tau+1} W_{\tau+1} = -\sum_{s=\tau+2}^{T} \mathbb{E}_{\tau+1}\left[M_s Y_s\right] + \sum_{s=\tau+2}^{T} \mathbb{E}_{\tau+1}\left[M_s C_s\right] + \mathbb{E}_{\tau+1}\left[M_T W_T\right]$$

Plug this into the modified intertemporal budget constraint

$$\mathbb{E}_\tau[M_{\tau+1} W_{\tau+1}] = \mathbb{E}_\tau[M_{\tau+1} Y_{\tau+1}] + (W_\tau - C_\tau) M_\tau$$

$$\mathbb{E}_\tau\left[-\sum_{s=\tau+2}^{T} \mathbb{E}_{\tau+1}\left[M_s Y_s\right] + \sum_{s=\tau+2}^{T} \mathbb{E}_{\tau+1}\left[M_s C_s\right] + \mathbb{E}_{\tau+1}\left[M_T W_T\right]\right] = \mathbb{E}_\tau[M_{\tau+1} Y_{\tau+1}] + (W_\tau - C_\tau) M_\tau$$

$$-\sum_{s=\tau+2}^{T} \mathbb{E}_\tau\left[M_s Y_s\right] + \sum_{s=\tau+2}^{T} \mathbb{E}_\tau\left[M_s C_s\right] + \mathbb{E}_\tau\left[M_T W_T\right] = \mathbb{E}_\tau[M_{\tau+1} Y_{\tau+1}] + (W_\tau - C_\tau) M_\tau$$

Rearrange this to obtain the result.

2 Dynamic Programming for Portfolio Choice

Define value function to be a function of two variables $V_t(W, X)$ where W – wealth and X is a vector of relevant state-variables that can for example influence returns. Dynamic programming only works in Markovian settings, hence, the state vector X is a Markov process meaning that the distribution of X_{t+1} conditional on all information at date t is the same as conditional only on X_t. We assume that distribution of Y_{t+1} – labor income and R_{t+1} – vector of returns on assets conditional on all information at date t is the same as conditional on X_t.

If the agents maximizes the utility of terminal wealth $\mathbb{E}_t[u(W_T)]$, then his Bellman equation takes form

$$V_t(W, X) = \max_w \mathbb{E}[V_{t+1}(X_{t+1}, Y_{t+1} + Ww'R_{t+1})|X_t = X]$$

where the fact that the value function is indexed by time means that it can changes as there are fewer periods left until the terminal period.

If there is an intermediate utility flow $\delta^t u(C_t)$ then Bellman equation becomes

$$V_t(W, X) = \max_{w,c} \left\{\delta^t u(c) + \mathbb{E}[V_{t+1}(X_{t+1}, Y_{t+1} + Ww'R_{t+1})|X_t = X]\right\}$$

It is useful to normalize the value function by $V_t \to \delta^{-t} V_t$ to get a more familiar representation

$$V_t(W, X) = \max_{w,c} \{u(c) + \delta\mathbb{E}[V_{t+1}(X_{t+1}, Y_{t+1} + Ww'R_{t+1})|X_t = X]\}$$

Inifinite Horizon If the horizon is infinite Bellman equation simplifies to

$$V(W, X) = \max_{w,c} \{u(c) + \delta\mathbb{E}[V(X_{t+1}, Y_{t+1} + Ww'R_{t+1})|X_t = X]\}$$

where $V(\cdot, \cdot)$ doesn't depend on time anymore.

3 CRRA utility

When per period utility function is CRRA, value function is CRRA in wealth with the same coefficient of relative risk aversion. In particular, value function $V_t(W, X)$ takes the following form

$$V_t(W, X) = f_t(X)W^{1-\gamma}$$

where $f_t(\cdot)$ doesn't depend on wealth. To see this in finite horizon case suppose use induction. For $t = T$ we have that the value function is equal to utility function

$$V_T(W, X) = \frac{C_T^{1-\gamma}}{1-\gamma} = \frac{W_T^{1-\gamma}}{1-\gamma} \implies f_T(X) = \frac{1}{1-\gamma}$$

Now suppose that the value function has the conjectured form for $t+1, t+2, \dots, T$. Then the Bellman equation at period t is given by

$$\begin{aligned}
V_t(W, X) &= \max_{c,w} \{u(c) + \delta E\left[V_{t+1}(X_{t+1}, (W-c)w'R_{t+1})|X_t = X\right]\} \\
&= \max_{c,w} \left\{u(c) + \delta E\left[f_{t+1}(X_{t+1})((W-c)w'R_{t+1})^{1-\gamma}|X_t = X\right]\right\} \\
&= [\text{Denote } z = c/W - \text{consumption propensity}] \\
&= \max_{z,w} \left\{\frac{(zW)^{1-\gamma}}{1-\gamma} + \delta E\left[f_{t+1}(X_{t+1})(W(1-z)w'R_{t+1})^{1-\gamma}|X_t = X\right]\right\} \\
&= W^{1-\gamma} \underbrace{\max_{z,w} \left\{\frac{z^{1-\gamma}}{1-\gamma} + \delta E\left[f_{t+1}(X_{t+1})((1-z)w'R_{t+1})^{1-\gamma}|X_t = X\right]\right\}}_{f_t(X)}
\end{aligned}$$

which verifies the conjecture for the finite horizon problem. We can show using the same argument that for log utility value function has the form

$$V_t(W,X) = \alpha_t \log(W) + g_t(X)$$

When the horizon is infinite, we can show by the same argument that CRRA over wealth is one of the solutions to the Bellman equation. In principle, there can be multiple solution to this Bellman equation.

Therefore, CRRA over wealth is indeed the unique solution to the Bellman equation and we have the following expressions

$$\begin{aligned} V(W,X) &= f(X)W^{1-\gamma} && \text{when } \gamma \neq 1 \\ V(W,X) &= \alpha \log(W) + g(X) && \text{when } \gamma = 1 \end{aligned}$$

4 CAPM, ICAPM and CCAPM

Here we derive approximate CCAPM and ICAPM and these approximations are exact in continuous time. CCAPM says that the risk premium is determined by covariances with consummption. ICAPM says that risk premium is determined by covariance with returns on wealth portfolio as well as on future investment opportunities – state variables. The two models are not inconsistent: since optimal consumption depend on wealth and state variables then covariance with wealth and state variables can be collapsed to covariance with consumption.

4.1 Conditional CAPM

Conditional CAPM is the following statement

$$\mathbb{E}_t[R_{i,t+1} - R_{z,t+1}] = \frac{cov_t(R_{i,t+1}, R_{m,t+1})}{var_t(R_{m,t+1})}\mathbb{E}_t[R_{m,t+1} - R_{z,t+1}]$$

where $R_{z,t+1}$ is a zero beta return which in the presence of risk free asset is just the risk free rate. Here we since the zero beta rate is known in advance it doesn't affect the covariance so we can omit it. Conditional CAPM is useful to understand market timing strategies: increasing investment in the asset when the risk premium is high. Contrast conditional CAPM with the traditional CAPM

$$\mathbb{E}[R_i - R_z] = \frac{cov(R_i - R_z, R_m - R_z)}{var(R_m)}\mathbb{E}[R_m - R_z]$$

First note that we have the zero beta rate inside the covariance since it is allowed to vary over the sample.

If the excess returns $R_i - R_z$ and $R_m - R_z$ are iid across time conditional and unconditional CAPM are the same since conditioning on time t infromation doesn't change any moments. Otherwise, the two models are different. To see this denote the conditional CAPM beta for asset i at time t as β_{it}. Then take the unconditional expectation of the conditional CAPM

$$\begin{aligned} \mathbb{E}[R_{i,t+1} - R_{z,t+1}] &= \mathbb{E}[\beta_{it}\mathbb{E}_t[R_{m,t+1} - R_{z,t+1}]] \\ &= \mathbb{E}[\beta_{it}]\mathbb{E}[R_{m,t+1} - R_{z,t+1}] + cov\left(\beta_{it}, \mathbb{E}_t[R_{m,t+1} - R_{z,t+1}]\right) \end{aligned}$$

To see how the covariance term is related to market timing consider the following example of a market timing strategy

Example 4.1 (Condional CAPM and Market Timing). *Suppose that the market return is given by* $R_{m,t+1} = \mu_t + \varepsilon_{t+1}$*. Suppose that* $\mu_t = R_f - \Delta$ *or* $\mu_t = R_f + \Delta$ *with equal probability. Consider a strategy that invests 100% in the risk free asset when* $\mu_t = R_f - \Delta$ *and 100% in the market when* $\mu_t = R_f + \Delta$*. Denote the return of this strategy as* R_t*.*

Consider the elements of unconditional CAPM. The unconditional expected market return is $0.5(R_f + \Delta) + 0.5(R_f - \Delta) = R_f$*. The unconditional CAPM then says that all unconditional returns should equal to the risk free rate. However, the unconditional expected return of this strategy is*

$$\mathbb{E}[R_t] = \underbrace{0.5R_f}_{when\ \mu_t = R_f - \Delta} + \underbrace{0.5(R_f + \Delta)}_{when\ \mu_t = R_f - \Delta} = R_f + \frac{1}{2}\Delta > R_f$$

which is higher than the unconditional expected return predicted by unconditional CAPM. The discrepancy comes from the covariance term: when expected market return is low the beta of this strategy is 0 *and when expected market return is high the beta of this strategy is* 1*.*

4.2 Consumption CAPM

Consumption CAPM states the risk premium of the asset is determined by covariances with aggregate consumption growth. Here to derive it we use a Taylor approximation that is exact in continuous time. The first step is to use the SDF formula for the expected return on any asset:

$$\begin{aligned} 1 &= \mathbb{E}_t[Z_{t+1}R_{i,t+1}] \\ 1 &= \mathbb{E}_t[Z_{t+1}]\mathbb{E}_t[R_{i,t+1}] + cov_t(Z_{t+1}, R_{i,t+1}) \\ \mathbb{E}_t[R_{i,t+1}] &= \frac{1}{\mathbb{E}_t Z_{t+1}} - \frac{1}{\mathbb{E}_t Z_{t+1}} cov_t(Z_{t+1}, R_{i,t+1}) \end{aligned}$$

where Z_{t+1} – one period SDF. Use investor's MRS as one period SDF: $Z_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$

$$\begin{aligned} \mathbb{E}_t[R_{i,t+1}] &= \frac{u'(C_t)}{\beta \mathbb{E}_t u'(C_{t+1})} - \frac{u'(C_t)}{\beta \mathbb{E}_t u'(C_{t+1})} cov_t\left(\beta \frac{u'(C_{t+1})}{u(c_t)}, R_{i,t+1}\right) \\ &= \frac{u'(C_t)}{\beta \mathbb{E}_t u'(C_{t+1})} - \frac{1}{\mathbb{E}_t u'(C_{t+1})} cov_t\left(u'(C_{t+1}), R_{i,t+1}\right) \end{aligned}$$

In principle, we can assume the existence of a representative investors and this equation will be valid for aggregate consumption. We can also take another route that will require approximation that becomes exact in continuous time. Suppose that time periods are small so that we can approximate

$$\begin{aligned} u'(C_{t+1}) &\approx u'(C_t) + u''(C_t)(C_{t+1} - C_t) \\ &= u'(C_t) + u''(C_t)\Delta C_{t+1} \end{aligned}$$

reasonably well. Plug this back into the expression for expected return on asset i

$$\begin{aligned} \mathbb{E}_t[R_{i,t+1}] &= \frac{u'(C_t)}{\beta \mathbb{E}_t u'(C_{t+1})} - \frac{1}{\mathbb{E}_t u'(C_{t+1})} cov_t\left(u'(C_t) + u''(C_t)\Delta C_{t+1}, R_{i,t+1}\right) \\ &= \frac{u'(C_t)}{\beta \mathbb{E}_t u'(C_{t+1})} - \frac{u''(C_t)}{\mathbb{E}_t u'(C_{t+1})} cov_t\left(\Delta C_{t+1}, R_{i,t+1}\right) \\ &= \frac{u'(C_t)}{\beta \mathbb{E}_t u'(C_{t+1})} - C_t \frac{u''(C_t)}{\mathbb{E}_t u'(C_{t+1})} cov_t\left(\frac{\Delta C_{t+1}}{C_t}, R_{i,t+1}\right) \end{aligned}$$

In continuous time the following approximation is exact

$$C_t \frac{u''(C_t)}{\mathbb{E}_t u'(C_{t+1})} \approx \frac{C_t u'(C_t)}{u''(C_t)}$$

hence, we have

$$\mathbb{E}_t[R_{i,t+1}] - \underbrace{\frac{u'(C_t)}{\beta \mathbb{E}_t u'(C_{t+1})}}_{\text{zero-beta return}} = \underbrace{-\frac{C_t u''(C_t)}{u'(C_t)}}_{\text{Relative-Risk Aversion}} cov_t\left(\frac{\Delta C_{t+1}}{C_t}, R_{i,t+1}\right)$$

Consumption CAPM says that risk-premium (expected asset return over the zero beta return) depends on the covariance of return with aggregate consumption growth and the price of this risk in the representative investor's relative risk aversion.

We can also proceed without assuming a representative investor. Suppose that each investor has utility

$u_h(C_{ht})$ and sum the previous equation across agents

$$\begin{aligned}
\mathbb{E}_t[R_{i,t+1}] &= \frac{u_h'(C_{ht})}{\beta\mathbb{E}_t u_h'(C_{h,t+1})} - \frac{u_h''(C_{ht})}{\mathbb{E}_t u_h'(C_{h,t+1})} cov_t\left(\Delta C_{h,t+1}, R_{i,t+1}\right) \\
\frac{\mathbb{E}_t u_h'(C_{h,t+1})}{u_h''(C_{ht})}\mathbb{E}_t[R_{i,t+1}] &= \frac{u_h'(C_{ht})}{\beta u_h''(C_{ht})} - cov_t\left(\Delta C_{h,t+1}, R_{i,t+1}\right) \\
\sum_h \frac{\mathbb{E}_t u_h'(C_{h,t+1})}{u_h''(C_{ht})}\mathbb{E}_t[R_{i,t+1}] &= \sum_h \frac{u_h'(C_{ht})}{\beta u_h''(C_{ht})} - cov_t\left(\Delta C_{t+1}, R_{i,t+1}\right) \\
\mathbb{E}_t[R_{i,t+1}] &= \underbrace{\left[\sum_h \frac{\mathbb{E}_t u_h'(C_{h,t+1})}{u_h''(C_{ht})}\right]^{-1} \sum_h \frac{u_h'(C_{ht})}{\beta u_h''(C_{ht})}}_{R_{z,t+1}} - \left[\sum_h \frac{\mathbb{E}_t u_h'(C_{h,t+1})}{u_h''(C_{ht})}\right]^{-1} cov_t\left(\Delta C_{t+1}, R_{i,t+1}\right) \\
\mathbb{E}_t[R_{i,t+1}] &= R_{z,t+1} + \underbrace{\left[C_t / \sum_h -\frac{\mathbb{E}_t u_h'(C_{h,t+1})}{u_h''(C_{ht})}\right]}_{\Gamma} cov_t\left(\frac{\Delta C_{t+1}}{C_t}, R_{i,t+1}\right)
\end{aligned}$$

Price of risk (the ugly thing that multiplies the covariance term) is consumption multiplied by the *aggregate* absolute risk aversion A^{agg} that is defined as follows

$$\frac{1}{A^{agg}} = \sum_h \frac{1}{A_h} \text{ where } A_h = -\frac{u_h''(C_{ht})}{\mathbb{E}_t u_h'(C_{h,t+1})} \approx -\frac{u_h''(C_{ht})}{u_h'(C_{h,t+1})}$$

so that

$$\Gamma = C_t A^{agg}$$

Since relative risk aversion is absolute risk aversion multiplied by consumption, we can think about Γ as aggregate relative risk aversion.

4.3 Intertemporal CAPM

Intertemporal CAPM provides a theoretical foundation for macroeconomic variables to be priced risk factors. If such factors affect investment opportuinities ICAPM says that they are priced risk factors and command a risk premium. Since we can replace any factors with their projections on returns or excess returns we can extend this approach to returns. If we find that some returns or excess returns are priced risk factors, then it is at least theoretically possible that they work because they are projections of macroeconomic variables.

To understand this logic it is useful to work with the dynamic programming approach interoduced earlier as the macroeconomic variables will affect state X_t that affects the value function. We start with the same formula for the risk premium on an asset

$$\mathbb{E}_t[R_{i,t+1}] = \frac{u'(C_t)}{\beta\mathbb{E}_t u'(C_{t+1})} - \frac{1}{\mathbb{E}_t u'(C_{t+1})} cov_t\left(u'(C_{t+1}), R_{i,t+1}\right)$$

and going to work with the following infinite horizon Bellman equation

$$V(W, X) = \max_{w,c} \{u(c) + \delta\mathbb{E}[V(X_{t+1}, Y_{t+1} + W w' R_{t+1}) | X_t = X]\}$$

Instead of approximating marginal utility around the time-t consumption we are going to use the envelope theorem

$$V_W(X, W) = u'(c)$$

to substitute marginal utility out of this equation. We get

$$\mathbb{E}_t[R_{i,t+1}] = \frac{V_W(X_t, W_t)}{\beta\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} - \frac{1}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(V_W(X_{t+1}, W_{t+1}), R_{i,t+1}\right)$$

Next, we do a Taylor expansion of marginal value of wealth $V_W(X_{t+1}, W_{t+1})$ around X_t – time-t state variables (recall that X is a vector) and $W_t - C_t$ – funds at time$-t$ allocated to investments. The approximation is the following

$$\begin{aligned} V_W(X_{t+1}, W_{t+1}) &= V_W(X_t, W_t - C_t) + \sum_{i=1}^{K} V_{WX_i}(X_t, W_t - C_t)\Delta X_{i,t+1} + V_{WW}(X_t, W_t - C_t)(W_{t+1} - (W_t - C_t)) \\ &= V_W(X_t, W_t - C_t) + \sum_{i=1}^{K} V_{WX_i}(X_t, W_t - C_t)\Delta X_{i,t+1} + V_{WW}(X_t, W_t - C_t)(\Delta W_{t+1} + C_t) \end{aligned}$$

Plug the approximation of $V_W(X_{t+1}, W_{t+1})$ into the expression for risk-premium from above

$$\begin{aligned} \mathbb{E}_t[R_{i,t+1}] =& \frac{V_W(X_t, W_t)}{\beta \mathbb{E}_t V_W(X_{t+1}, W_{t+1})} - \frac{1}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(V_W(X_{t+1}, W_{t+1}), R_{i,t+1}\right) \\ \approx& \frac{V_W(X_t, W_t)}{\beta \mathbb{E}_t V_W(X_{t+1}, W_{t+1})} \\ &- \frac{1}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(\sum_{i=1}^{K} V_{WX_i}(X_t, W_t - C_t)\Delta X_{i,t+1} + V_{WW}(X_t, W_t - C_t)(\Delta W_{t+1} + C_t), R_{i,t+1}\right) \\ =& \frac{V_W(X_t, W_t)}{\beta \mathbb{E}_t V_W(X_{t+1}, W_{t+1})} \\ &- \sum_{j=1}^{K} \frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov\left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1}\right) \\ &- \frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(\frac{\Delta W_{t+1} + C_t}{W_t - C_t}, R_{i,t+1}\right) \end{aligned}$$

Recall that the intertemporal budget constraint is

$$\begin{aligned} W_{t+1} &= Y_{t+1} + (W_t - C_t) w' R_{t+1} \\ \implies \frac{\Delta W_{t+1} + C_t}{W_t - C_t} &= \frac{Y_{t+1} + (W_t - C_t) w' R_{t+1} - W_t + C_t}{W_t - C_t} \\ &= w' R_{t+1} - 1 + \frac{Y_{t+1}}{W_t - C_t} \end{aligned}$$

Hence, the fully expanded expression for ICAPM is

$$\begin{aligned} \mathbb{E}_t[R_{i,t+1}] \approx& \frac{V_W(X_t, W_t)}{\beta \mathbb{E}_t V_W(X_{t+1}, W_{t+1})} \\ &- \sum_{j=1}^{K} \frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov\left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1}\right) \\ &- \frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(w' R_{t+1}, R_{i,t+1}\right) \\ &- \frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(\frac{Y_{t+1}}{W_t - C_t}, R_{i,t+1}\right) \end{aligned}$$

Interpretation There are many terms to deal with

1. First, consider the price of risk for the covariance $cov_t(w' R_{t+1}, R_{i,t+1})$. If we use approximation $\mathbb{E}_t[V_W(X_{t+1}, W_{t+1})] \approx V_W(X_t, W_t - C_t)$ then

$$-\frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} \approx -\frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{V_W(X_t, W_t - C_t)}$$

   is the relative risk aversion of a representative investor's value function. Recall that when we deal with CRRA utility function, value function inherit the same risk aversion coefficient. Hence, with CRRA utility this term is approximately γ.

2. Next, consider the price of risk for covariances with the state variable $cov_t(\Delta X_{j,t+1}/X_{jt}, R_{i,t+1})$. Using the same approximation we can write it as

$$-\frac{X_{j,t}V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} \approx -\frac{X_{j,t}V_{WX_j}(X_t, W_t - C_t)}{V_W(X_t, W_t - C_t))} = -\left.\frac{\partial \log V_W(X, W)}{\partial \log X}\right|_{X=X_t, W=W_t-C_t}$$

Hence, it is the elasticity of the marginal value of wealth with respect to the state variable X_j. What does it mean? To get a better understanding let's focus on a particular case of CRRA per-period utility function and no labor income. As we know the value function inherits the same coefficient of relative risk aversion

$$\begin{aligned} V(W, X) &= f(X)W^{1-\gamma} && \text{when } \gamma \neq 1 \\ V(W, X) &= \alpha \log(W) + g(X) && \text{when } \gamma = 1 \end{aligned}$$

Marginal value of wealth is

$$\begin{aligned} V_W(W, X) &= (1-\gamma)f(X)W^{-\gamma} && \text{when } \gamma \neq 1 \\ V_W(W, X) &= \alpha W^{-1} && \text{when } \gamma = 1 \end{aligned}$$

and its derivative w.r.t. state variable X_j

$$\begin{aligned} V_{WX_j}(W, X) &= (1-\gamma)W^{-\gamma}\frac{\partial f(X)}{X_j} && \text{when } \gamma \neq 1 \\ V_{WX_j}(W, X) &= 0 && \text{when } \gamma = 1 \end{aligned}$$

and derivative of value function w.r.t state X_j is

$$\begin{aligned} V_{X_j}(W, X) &= W^{1-\gamma}\frac{\partial f(X)}{X_j} && \text{when } \gamma \neq 1 \\ V_{X_j}(W, X) &= \frac{\partial g(X)}{\partial X_j} && \text{when } \gamma = 1 \end{aligned}$$

Hence, the ratio of the marginal value of wealth and the value function w.r.t. state X_j is

$$\begin{aligned} \frac{V_{WX_j}(W, X)}{V_{X_j}(W, X)} &= \frac{1-\gamma}{W} && \text{when } \gamma \neq 1 \\ \frac{V_{WX_j}(W, X)}{V_{X_j}(W, X)} &= 0 && \text{when } \gamma = 1 \end{aligned}$$

We have the following observations

- When $\gamma = 1$ marginal value of wealth doesn't respond to changes in state variables. This is the case of the absence of intertemporal hedging demand: changes in investment opportunities don't affect portfolio allocation of a log-investor. Therfore, for such investor the price of return-state-covariance risk is zero.
- When $\gamma < 1$, marginal value of wealth and value function move in the same direction with state variables. Suppose that X_j is state variables increases of which mean "good times" in a sense of good investment opportunities. $X_j \uparrow$ increases the value function of any investor. However, when $\gamma < 1$, i.e. the investor is very aggressive this also corresponds to increase in marginal value of wealth. When there are better investment opportunities an aggressive investor wants more wealth to invest more during such times. In the opposite case, when $X_j \downarrow$ investment opportunities deteriorate meaning that the value goes down, but marginal value of wealth goes down because aggressive investor doesn't want to invest in times of bad investment opportuinities. This is the case of **negative intertemporal hedging demand**. Hence, for such aggressive investor the price of return-state-covariance risk is negative, meaning that this investor values assets that go up when future investment opportunities improve and (weirdly) values assets that go down when future investment opportunities deteriorate.

- When $\gamma > 1$ marginal value of wealth and value of wealth move in the opposite direction with states. In line with the previous argument, when the future opportunities deteriorate and the value function goes down, the marginal value of wealth goes up. Therefore, for such investor the price of return-state-covariance risk is positive meaning that this investor values assets that perform well when future investment opportuinities deteriorate. This is the case of **positive intertemporal hedging demand**.

3. Lastly, we have covariance $cov_t(Y_{t+1}, R_{i,t+1})$ which is straightforward: the price of this return-labor-income-covariance-risk is positive since $V_W W \leq 0$. Therefore, the investor values assets that performs well when labor income goes down.