Real and Financial Options: A Production Based Approach to Option Pricing

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Abstract

This paper builds a bridge between the real decision of firms and prices of equity options. Traditional option pricing literature proposed a variety of reduced form models to fit empirical patterns in option prices. At the same time, the cross sectional effects of firm fundamentals on equity options have not been thoroughly explored. I document a heterogeneous effect of firm fundamentals such as market-to-book on the relative prices of options that varies with the aggregate state of the economy. When the economy is booming, high market-to-book is associated with larger implied volatility skew and is predictive of negatively skewed return distribution. I develop a stylized continuous time production based asset pricing model with real options consistent with this evidence. To match the empirical evidence not only qualitatively but quantitatively as well, I solve a rich dynamic structural model and show that it can fit the empirical moments under countercyclical volatility and a high degree of capital adjustment costs asymmetry. Building further on the continuous time model and the endogenous variance risk premium it generates, I show that it can rationalize recently proposed delta-hedged option strategies based on profitability and book-to-market.

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1 Introduction

Equity options are an important financial instrument that gained in popularity among institutional and individual investors. At the same time, the connection between firm fundamentals and equity option prices has not been thoroughly studied. Traditional option pricing literature expanded on the pioneering work of Black and Scholes (1973) and Merton (1973) to propose a variety of reduced-form models that relaxed the restrictive assumption such as log-normality that the early work in the field was based on. At their core, these models feature sophisticated dynamics for the underlying securities – indices, equities, exchange rates – aimed at capturing the salient features such as stochastic volatility and jumps. A variety of analytical, numerical, and simulation techniques are then used to solve for option prices and the parameters that guide the reduced form dynamics are estimated to match the observed option prices. These mainstream models are widely successful in explaining phenomena such as implied volatility skew – the tendency of out-of-the-money put options to appear more expensive than at-the-money put options – making them immensely useful to practitioners. This success can in part be attributed to the flexibility of the reduced form approach: parameters of a given model can be calibrated to a particular underlying asset. At the same time, these models do not allow to draw a direct connection neither to the fundamentals of the firms whose stock prices underlie the options nor to the state of the economy that these firms operate in.

The role of firm fundamentals takes one of the central places among the numerous explanations for cross-sectional anomalies, such as differences in average returns across growth and value firms, profitable and unprofitable firms, and firms with low and high asset growth. The purpose of these models is explaining deviations from the CAPM, thus requiring to accurately model not only firms' average returns but also variances and covariances with the market. The predictions of these models, however, extend well beyond the first two moments of return distribution. For instance, when productivity shocks have both direct and compositional effects, like in a class of production based asset pricing models that draw a distinction between assets-in-place and growth options, it is possible to obtain rich predictions for the dynamics of return distribution and how it is influenced by fundamentals and the economy.

The final link that brings together firm fundamentals and option prices is the tight relationship between return distributions and option prices. The famous result of Breeden and Litzenberger (1978) shows that options complete the state space with respect to the underlying security value. One can form a portfolio from call and put options to achieve a unit payoff if the security value falls within an interval ΔM and zero otherwise. If the market participants believe that the asset value is likely to end up in a certain interval, for example, well below the current value, the value of the corresponding portfolio will increase as well. This will in turn drive up the value of options from which the portfolio is formed and, in this example, raise the value of out-of-the-money put options. Therefore, the effect of firm fundamentals on option prices can be equivalently formulated as the effect of firm fundamentals on return distribution.

This paper leverages the relationship between option prices, return distributions, and theoretical asset pricing models targeting cross-sectional anomalies to develop a connection between two strands of finance that historically evolved without much interaction. In particular, it builds a bridge between the real decisions of firms and return distributions and, as a result, equity option prices. The starting point is to document that variation in firm fundamentals is indeed associated with differences in option values and the distribution of returns. Focusing on Market-to-Book and Leverage on the firm fundamental side and implied volatility skew – a measure that is tightly linked to (negative) return skewness – on the option side, this paper establishes several facts about the interaction of these variables with each other and with the aggregate state of the economy. For firms with low leverage in a low valuation environment, market-to-book has little to no effect on implied volatility skew. A higher market-to-book is associated with a higher implied volatility skew for low leverage firms when valuations are high. In contrast, for firms with high leverage, market-to-book has a negative effect on skew when economic conditions are bad and a positive effect when economic conditions are good.

The positive relationship between Market-to-Book and implied volatility skew might seem counterintuitive at first. Given the correspondence between option prices and return distributions this suggests that firms with higher market-to-book have a more negatively skewed return distribution – an empirical regularity also documented in this paper. Firms with a high Market-to-Book, commonly referred to as growth firms, are typically associated with having a higher upside potential as they possess many real options: expanding, attracting customers, and launching new products. It is, therefore, natural to assume that such firms would have a positively skewed return distribution contrary to this finding. It turns out that it is crucial to take into account how the value of a firm that possesses real options is determined to understand this empirical result. When aggregate conditions are good, growth firms derive a larger share of their value from real options, and the value of real options is more sensitive to economic conditions. At the same time, the probability of exercising their real options is high, and so are their valuations. This leads to an asymmetry in future returns, where the upside potential is already priced in, but the downside risk remains high due to a higher sensitivity to worsening economic conditions.

To address the heterogeneity in the aggregate state theoretically, this paper modifies a standard continuous time investment model with irreversibility of investment (e.g. Gomes and Schmid, 2010; Back, 2017) by introducing a time-varying aggregate state that determines the price of cash flow risk. Irreversible investments create a wedge between the value of capital the firm currently uses for production – assets-in-place – and the option value of expanding its productive capacity when the economic conditions or firm productivity improve – real or growth options. A firm with no real options or a low probability of exercising them, for example, when economic conditions are bad, or productivity is low, will derive most of its value from assets-in-place. Such a firm will have a low market-to-book ratio. On the other hand, if a firm is very likely to exercise its real options, it will derive a significant portion of its value from real options and will have a high market-to-book.

The value of real options is more sensitive to economic conditions, modeled as the price of risk, than the value of assets-in-place. The sensitivity is higher when economic conditions are good, as the discount rate is lower. This makes a firm that derives a large portion of its value from real options more sensitive to deterioration in economic conditions. This, in turn, manifests itself in negatively skewed risk-neutral and physical distributions of returns for such firms, in line with empirical findings. A straightforward implication of a negatively skewed return distribution is a higher implied volatility skew for growth firms relative to value firms when the price of risk is low.

While the continuous time model helps to illustrate the mechanism, it lacks features that would make it possible to match the empirical moments. Complete irreversibility of investment is unrealistic, and under such assumption a firm would not be able to downscale in response to adverse shocks. The differential impact of market-to-book on skew for low and high leverage firms also requires augmenting the model with debt. To incorporate partial irreversibility and leverage I turn to a discrete-time structural corporate finance model with asymmetric adjustment costs and defaultable debt in the spirit of Begenau and Salomao (2019). I show that under a high degree of irreversibility, when it is much more costly to disinvest than to invest, and strong countercyclicality of the price of risk, model's solution and simulated moments related to the effect of market-to-book on skew can closely match those observed in the data.

The paper concludes by deriving implications of the continuous time model for the average profits of delta-hedged option strategies. In traditional option pricing literature, the performance of such strategies is driven purely by variance risk premium (VRP). While the model presented in the paper does not feature VRP explicitly, the return volatility is correlated negatively with state prices. Notably, the strength of this correlation is determined by firm fundamentals leading to differential exposure of firms to movements in volatility, which can be seen as a fundamentally driven variance risk premium. This qualitative difference is consistent with the performance of recently proposed highly profitable equity option strategies based on firm fundamentals' sorts.

Outline of the Paper Section 2 positions this paper within the existing theoretical and empirical literature on cross-sectional option and equity pricing. Section 3 discusses the empirical strategy of relating firm fundamentals to option prices. Section 4 presents a stylized continuous time model with real options and aggregate risk and discusses the channel through which market-to-book is related to return distributions and option prices. Section 5 presents a rich discrete-time structural model and assesses its fit to the data. Section 6 derives implications of the continuous time model for delta-hedged option strategies.

2 Literature Review

The traditional option pricing or financial engineering literature started with Black and Scholes (1973) and Merton (1973) that showed that when the underlying spot price follows a geometric Brownian motion so that the holding period returns are log-normal, the value of a European option can be calculated in a semi-closed form. This equation is widely known as Black-Scholes-Merton (BSM) equation and is used to this day to compare values of options of different moneyness and maturities. The critical assumption of constant volatility underlying the BSM equation has been challenged by observing the distribution of underlying returns and option prices. In particular, it has been widely documented that equity and index option prices exhibit a phenomenon known as implied volatility skew: out-of-the-money put prices appear to be higher than at-the-money put prices when evaluated against the BSM benchmark. Subsequently, there has been a proliferation of papers that modified the process for the underlying asset to match option prices quantitatively. Heston (1993) showed that stochastic volatility negatively correlated with returns can generate implied volatility skew. Merton (1976) and Bates (1996) showed how discontinuities in the underlying process can generate realistic option prices. Despite the richness of approaches, the calibration of these models relies on matching a set of option prices, also called an implied volatility surface, without considering the fundamental determinants that underlie the choice of parameters. This paper starts with firm fundamentals to derive implications for the distribution of stock returns and option prices as a direct consequence.

The model presented in this paper belongs to a class of production-based asset pricing models that considers firms' real decisions, such as investments, to draw implications about the cross-section of equity returns. Kogan (2001), Zhang (2005), Kogan and Papanikolaou (2010) develop models with (partially) irreversible investments. (Partial) irreversibility generates real options. Firms must decide when to expand optimally, as scaling down is costly or, in extreme cases, impossible. Real options generate variation in market-to-book as the firm can derive more or less value from real options depending on their productivity, aggregate valuations, current scale of operations or other variables. Gomes and Schmid (2010) show that leverage and market-to-book are closely connected if the firm needs to borrow to exercise its real options. Kuehn and Schmid (2014) shows that production-based models can generate realistic credit spreads. This paper complements this literature by explaining novel empirical findings in the cross-sectional of equity options from the lens of production-based asset pricing models.

The literature on consumption-based asset pricing made a step toward understanding option prices. Rare disaster models with a time-varying probability of a disaster as in Gabaix (2012), Wachter (2013), Seo and Wachter (2019) generate realistic skew for *index* options. Conversely, this paper aims to explain the cross-section of *equity* options.

One of the insights of Merton (1973) is that the option can be dynamically replicated under BSM assumptions with a position in the underlying asset and a riskless bond. Therefore, a delta-hedged position produces exactly zero excess return under BSM assumptions. However, in the presence of other forms of risk, like stochastic volatility, the excess profits of delta-hedged strategies will deviate from zero. Coval and Shumway (2001) empirically studies market-neutral index option straddles – a long position in both a call and a put. They find that, on average, such a strategy produces negative returns as it protects the investor from increases in volatility, indicating a presence of variance risk premium. Bakshi and Kapadia (2003) theoretically show that a delta-hedged option strategy's average profits in a stochastic volatility environment are exposed only to the variance risk premium. This paper shows theoretically that companies with different fundamentals have different profits from delta-hedged strategies – a form of fundamentals-driven variance risk premium. In particular, this theoretical insight sheds light on recently proposed option strategies based on fundamentals sorted portfolios (Zhan et al., 2022).

Studying the cross-section of equity option returns is not new to the literature. Goyal and Saretto (2009) shows that option straddles formed on firms with a large difference between historical and implied volatility produce high returns relative to straddles based on firms with a small difference. Cao and Han (2013) looks at the delta-hedged return of equity options formed based on idiosyncratic volatility. Zhan et al. (2022) forms delta-hedged option positions based on fundamental sorts. Christoffersen et al. (2018) documents a strong factor structure in implied volatility levels and skew. The contribution of this paper is to derive theoretical results about the performance of delta-hedged positions in a production-based model and provide a resolution to highly profitable trading strategies proposed by Zhan et al. (2022).

3 Firm Fundamentals and Option Prices in the Data

This section studies the relationship between firm fundamentals, implied volatilities, and return distributions empirically. I start by describing the data and variables used in the empirical analysis. Importantly, I formalize implied volatility skew and illustrate the differences in skew for two companies, value and growth, in different economic regimes to motivate the susequent analysis. I then present the empirical specification that allows to identify the heterogeneous effect of firm fundamentals such as market-to-book on implied volatility skew and forward looking return skewness.

3.1 Data and Variable definitions

Overview of Option Data Option prices and implied volatilities are obtained from OptionMetrics (OM) – standard dataset for empirical option research. OM contains equity option prices since January, 1996 and the sample used in this paper ends in December, 2021. To relate firm fundamentals such as market-to-book to implied volatilities we need to first understand how to reduce the immense dimensionality of option data. For every firm i at date t there is a set of options with different strikes K and maturities T. In addition to many degrees of freedom, the unprocessed data makes it hard to compare the prices of different options across firms and economic regimes. Below, I describe how I aggregate the set of strikes and maturities, also called a volatility surface, into a single implied volatility skew variable defined at the same frequency as firm fundamentals. Another issue that arises when working with option data is illiquidity. Consider a situation when the price of a particular security is well above the price of a call option. The price of this option has an almost unit sensitivity to the price of the underlying. As a result, there are few reasons to take a position in an option. This results in few contracts being traded making the prices non-informative and noisy. A standard practice, that I follow in this paper, is to heavily filter the option data to minimize the effects of noise of the estimates. Table B.1 presents the full set of filters used for the remainder of the paper. These filters exclude options with zero open interest and trading volume and excludes options whose prices violate simple arbitrage bounds.

Constructing Implied Volatility Skew Firm fundamentals are defined at firm-time frequency. Therefore, we need to aggregate high dimensional option prices to the same level. The simplest way to do it is to choose a particular maturity and moneyness – the ratio of strike to the underlying spot price. However, there are a few problems with such approach. First, not all maturities are available at any given point in time. The option that had a 30 day maturity 5 days ago, has a maturity of 25 days today, and it is unlikely that there exists an option with 30 days maturity today as options have a fixed expiration calendar. Fixing moneyness at a certain level leads to the same problem. Once the contracts are specified, variation in the price of the underlying (spot price) changes the strike relative to spot price (moneyness) over time.

The second problem is that the same moneyness may have different interpretations in different economic regimes and even across different underlying assets. Moneyness of 80\$, meaning that the strike price of 20% below the spot price, is very different when the volatility is 10% compared to when volatility is 30%. Therefore, a good measure of moneyness should take into account the differences in volatility. In practice, option traders use option delta – the sensitivity of the value of the option to changes in the underlying spot price. A similar sensitivity to the underlying asset and thus to the underlying fundamental shocks allows to compare prices across options across time and firms.

To address both of these issues, I follow Carr and Wu (2020) who use a different measure of moneyness closely related to option delta and interpolate implied volatilities to a fixed set of moneyness and maturities via a Gaussian kernel. For a particular option contract j written on the underlying equity i at date t, moneyness is defined as

$$x_{j,t} = \frac{\log\left(K_j/S_{i,t}\right) + \frac{1}{2}I_{j,t}^2\tau_{j,t}}{I_{j,t}\sqrt{\tau_{j,t}}}, \tag{1}$$

where K_j is the strike price, $S_{i,t}$ is the underlying or spot price, $I_{j,t}$ is the implied volatility obtained from inverting the BSM equation of option contract j at time t and $\tau_{j,t}$ is the remaining time to maturity for contract j at time t. Under BSM assumptions the return volatility until the maturity is $I\sqrt{\tau}$. Therefore, moneyness measure x indicates how many standard deviations the log strike is away from the average log return under the risk-neutral measure.

To interpolate implied volatilities to a standard set of moneyness and maturities, I use Gaussian kernel weights. For a given target moneyness x and maturity τ, the weight on an option contract j at time t is defined as

$$w_{j,t}(x,\tau) = (1 - |\delta_{j,t}|)\, \mathbb{I}_{|\delta_{j,t}|<0.8} \exp\left(-\frac{(x_{j,t}-x)^2}{2h_x^2}\right) \exp\left(-\frac{(\ln\tau - \ln\tau_{j,t})^2}{2h_\tau^2}\right), \tag{2}$$

where $\delta_{j,t}$ is option delta, $\mathbb{I}_A$ is an indicator of event A and h_x and h_τ are standard Gaussian bandwidths. In this interpolation scheme, the farther away a particular option j is from the target moneyness and maturity the lower weight its implied volatility will get. In addition, out-of-the-money options with $|\delta| < 0.5$ that are usually more actively traded get a larger weight. The interpolated implied volatility at target x and τ is then

$$I(x,\tau;\mathcal{J}(i,t)) = \sum_{j\in\mathcal{J}(i,t)} \frac{w_{j,t}(x,\tau)}{\sum_{j\in\mathcal{J}(i,t)} w_{j,t}(x,\tau)} I_{j,t}, \tag{3}$$

where $\mathcal{J}(i,t)$ is the set of available option contracts for firm i at time t. To simplify the notation I will use the following definition for the rest of the paper

$$I_{i,t}(x,\tau) \equiv I(x,\tau;\mathcal{J}(i,t)). \tag{4}$$

For a particular firm i, date t and maturity τ, I define implied volatility skew as

$$Skew_{i,t}(\tau) \equiv I_{i,t}(-2.0,\tau) - I_{i,t}(0.0,\tau), \tag{5}$$

the difference in implied volatility for out-of-the-money put options with $x = -2.0$ and at-the-money options with $x = 0.0$. The results are not sensitive to a particular choice of relative moneyness.

Fundamentals and Return Data I use a standard approach to construct fundamental variables. For balance sheet information I use quarterly reports and require a lag of at least 3 months after reporting quarter end to ensure that the information was made public. I then calculate Book-to-Market following Fama and French (1993) and take it reciprocal to obtain market-to-book MB. Unlike in the literature on the value anomaly the focus of this paper is on growth firms making market-to-book a more natural characteristic that connects more explicitly with firm values derived in the models studied in the next two sections. Operating Profitability OP is defined as Revenue less Costs of Goods Sold normalized by assets (Novy-Marx, 2013), Asset growth AG defined as quarter on quarter assets growth rate, and Leverage Lev defined as long-term debt normalized by assets. While Leverage is not a standard variable used in empirical asset pricing, it is important to include it when studying option prices, as leverage can significantly change the distribution of equity returns and is significantly correlated with market-to-book.

To make the estimation stable across different economic regimes, I normalize fundamental variables cross-sectionally following Gu et al. (2020). Each month I rank companies based on their fundamental characteristic (e.g. market-to-book) and map this rank into an interval from -1 to 1. An additional benefit of this approach is an ease of interpretation: varying market-to-book from -0.5 to 0.5 means comparing 25th and 75th percentile firms based on a particular fundamental variable. As is standard in the literature, I exclude utility, financial and unclassified firms based on Fama-French industry classification.

Realized Skewness Anticipating the mechanism of the model, in addition to estimating the effect of firm fundamentals on implied volatility skew, I will also estimate their effect on the return distribution. In particular, I will predict realized skewness of daily returns from time t to $t+h$ with firm fundamental information available at date t. Realized skewness is defined as

$$RS_{i,t}(h) = \frac{\frac{1}{h}\sum_{k=1}^{h}\left(r_{t+k} - \bar{r}_{i,t}(h)\right)^3}{\left(\frac{1}{h}\sum_{k=1}^{h}\left(r_{t+k} - \bar{r}_{i,t}(h)\right)^2\right)^{3/2}}, \text{ where } \bar{r}_{i,t}(h) \equiv \frac{1}{h}\sum_{k=1}^{h} r_{t+k}. \tag{6}$$

Note that the third moment in the numerator is normalized by the scaled second moment in the denominator.

3.2 Illustration of Growth and Value Firms

To shed more light on the empirical strategy, I start by presenting an example of the effect I aim to systematically identify in the section. The top two panels of Figure 1 show the implied volatility as a function of moneyness defined in equation (1) for two companies. First is NVidia, a graphics chips manufacturer, that rose in its prominence with popularity of BitCoin and Machine Learning. NVidia is on the forefront of innovation and is an example of a growth company. The second one is Kellogg – a food manufacturing company that focuses on breakfast cereals and is a value company. The top left panel plots IVs in the midsts of the Great Financial Crisis and the top right panel plots IVs during a market boom. We first see that NVidia IV is always higher which is to be expected due to more volatile nature of its cash flows and growth prospects. It is also the case that for both companies, IVs are larger during an economic recession.

The bottom panels of Figure 1 show the difference of implied volatility at a particular moneyness x and ATM implied volatility $I_{i,t}(x,\tau) - I_{i,t}(0,\tau)$ where $I_{i,t}(x,\tau)$ is defined in equation (4). In simpler terms, I used implied volatilities from the top panels and subtracted their value at $x = 0$. This immediately implies that the value at $x = -2$ on the bottom two panels of Figure 1 is the measure of implied volatility skew defined in equation (5). Here we can clearly see that when the economy is in a recession (bottom-left panel), IV skews for both NVidia (growth) and Kellogg (value) are very similar. On the other hand, when the economy is booming (bottom-right panel), IV skew of Nvidia (growth) is much larger than that of Kellogg (value).The goal of the empirical strategy that I formalize below is to systematically estimate the effect of market-to-book on implied volatility skew across different economic regimes. At the same time, it will important to control for other firm characteristics that may be correlated with valuation ratios and can influences return distributions and implied volatility as a result

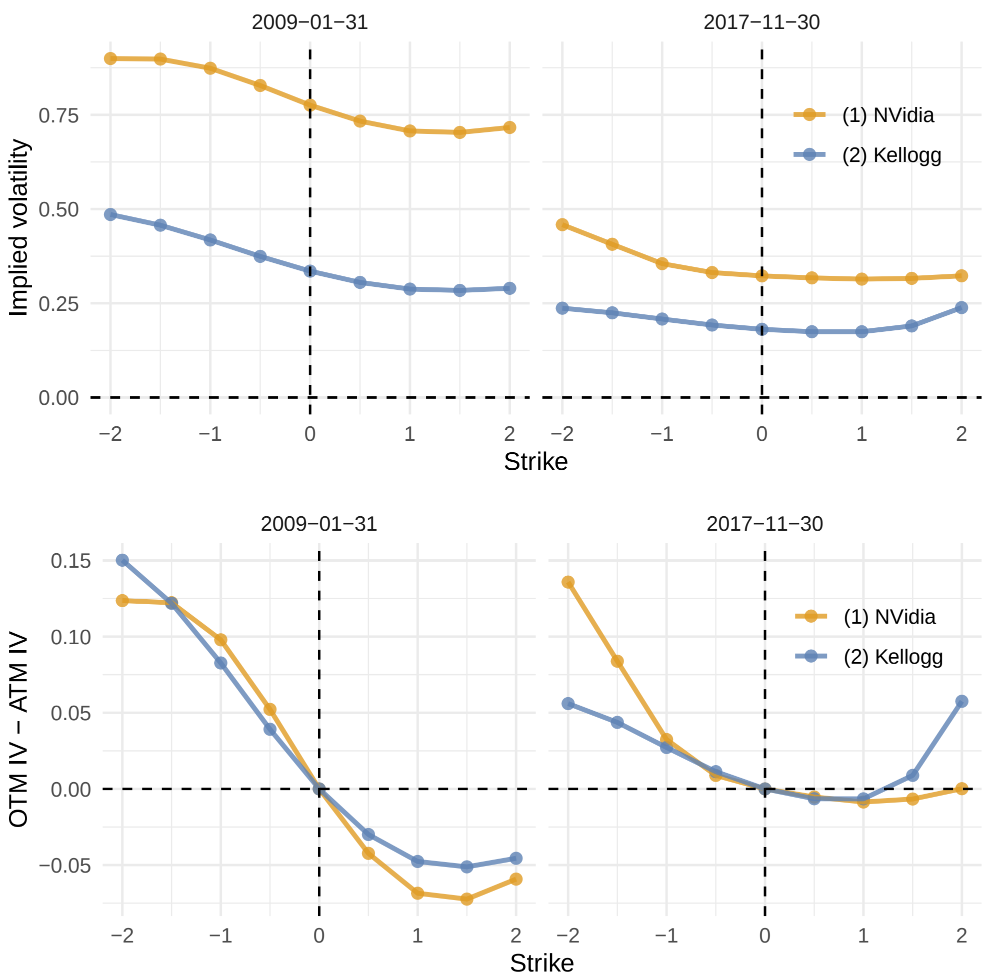


Figure 1: Example of growth and value implied volatilities in different economic regimes

Notes: The top panels present implied volatilities on a standardized grid $I_{i,t}(x,\tau)$ as defined in equation (4) as a function of normalized moneyness x. The bottom panels present implied volatilities relative to their values for $x = 0$, i.e. $I_{i,t}(x,\tau) - I_{i,t}(0,\tau)$. The values on the far left for $x = -2$ are exactly equal to $Skew_{i,t}(\tau)$ defined in equation (5).

Summary statistics I present the summary statistics in Tables 1, 2, and 3. Table 1 compares the number of observations (firm-months) in the sample used in this paper (Skew sample, henceforth) to the standard Compustat-CRSP sample (CC sample, henceforth). When we consider the time periods from 1996 (when OptionMetrics start) to 2021 in the first row, Skew sample has just

Sample	Skew (000)	CRSP-Comp (000)	Ratio (%)
Full	141	1312	10.77
Post 2003	89	560	15.93

Table 1: Comparing Compustat-CRSP and Skew Samples, Number of Observations

Notes: This table compares the number of observations across two samples (1) sample of firms with option prices available (Skew sample) and (2) CRSP-Compustat sample.

under 11% of the observations of the CC sample. There are two main reasons for such a small observation count. First is that while the the number of firms with traded options increases over time, the share remains small. Second is the set of restrictive filters employed to discard thinly traded options with non-informative and noisy prices. The early sample of OptionMetrics has a worse coverage. Therefore, when we compare 2004 through 2021 in the second row, the size of the Skew sample is 16% of the size of CC sample.

Table 2 compares the average characteristics of firms in the Skew sample to those in the CC sample. In the first two rows we see that the firms in the Skew sample are significantly larger when measured both in terms of market value and assets. This is less pronounced in the later part of the sample. While only 11% of firm-month observationshave option data available, when measured by value the data covers an overwhelming majority of observations. The following two rows show that firms in the Skew sample have higher valuations when measured against either book equity or book assets. Firms in the skew sample also have higher profitability, slightly higher leverage and slightly lower investments. The full summary statistics including average, standard deviations and quantiles across these two samples are presented in Table B.2. The full table reveals that the differences across the two samples highlighted above are present across all quantiles of the distributions and are not driven by outliers that would significantly influence the averages. I compare the time series of average characteristics across Skew and CC samples in Figure B.2.

Panel C of Table 3 presentes summary statistics for the main variable of interest $Skew$ defined in equation (5) along with ATM implied IV $I_{i,t}(0.0, \tau)$ for comparison. The skew is positive and large on average and is volatile. I present the time series of implied volatility skew quantiles in Figure B.3. Skew both varies over time and has apparent cyclical patterns: it tends to increase in economic downturns and decrease in economic expansions.

Figure 2 presents the time series of price-dividend ratio (PD) and CAPE that will be used as a measure of the aggregate state in the empirical analysis below. The short period for which equity options are available feature a substantial variation in the price-dividend ratio: very high valuations during the dot-com bubble, a substantial decline afterwards, large decrease in valuations during the Financial Crisis of 2008 and an increase post COVID pandemic. The initial period of high valuations may raise worries that the results presented below will be driven by the dot-com bubble. However, I show that even when this period is excluded the estimates stay quantitatively similar.

Characteristic	Full	Post 2003
Market Equity (ratio)	9.487	5.255
Assets (ratio)	5.134	3.057
Market-to-Book (diff)	2.011	1.774
Market-to-Book (A) (diff)	0.676	0.547
Profitability (diff)	0.02	0.016
Leverage (diff)	0.051	0.061
Investments (diff)	-0.02	-0.023

Table 2: Comparing characteristics across Compustat-CRSP and OptionMetrics samples

Notes: This table compares the sample of firms with option prices available – Skew sample – and CRSP-Compustat sample. Panel A compare the number of observations across two subsamples: full sample that starts in January 1996 and a subsample that starts in January 2003. Both samples end in December 2022. Panel B compare average firm characteristic (such as market equity) in the Skew and CC samples. The comparison is either (1) **ratio** of average characteristic in the Skew sample to average characteristic in the CC samples, or (2) **diff**erence between average characteristic in the Skew and CC samples as indicated in the second column.

Panel C: Skew summary statistics

	Mean	st.dev.	Quantile				
			5	25	50	75	95
Skew (%)	9.3	8.53	0.77	5.17	7.89	12.17	23.67
ATM IV (%)	44.75	25.37	18.2	27.52	38.25	54.14	94.03

Table 3: Skew summary statistics

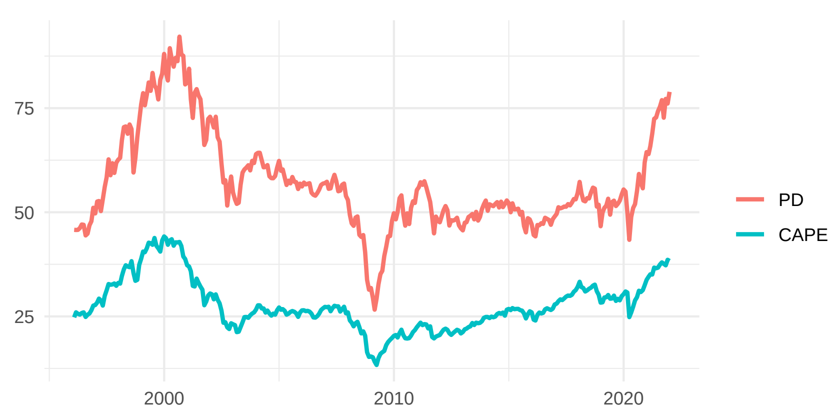


Figure 2: Time series of aggregate variables

Empirical strategy The empirical strategy relates firm fundamentals to $Skew_{i,t}(\tau)$ defined in equation (5). The hypothesis states that this relationship is time varying and in particular depends on whether the economy is booming or is in a recession. I use aggregate price dividend ratio PD_t as a proxy for aggregate conditions: when PD is high the economy is in a boom and when it is low the economy is in a bust. In a simple linear empirical model

$$Skew_{i,t}(\tau) = \alpha_t + B(PD_t) \cdot MB_{i,t} + \varepsilon_{i,t} \tag{7}$$

for a fixed τ, some function $B(PD)$ and time fixed effect α_t that captures aggregate variation in skew across firms due to a strong factor structure for IV skew documented in Christoffersen et al. (2018). I further parametrize $B(PD)$ to be linear in PD yielding

$$Skew_{i,t}(\tau) = \alpha_t + (\beta + \gamma \cdot PD_t) \cdot MB_{i,t} + \varepsilon_{i,t}. \tag{8}$$

This specification is the baseline empirical specification. I plot the estimate of the coefficient on market-to-book $(\hat{\beta} + \hat{\gamma} \cdot PD_t)$ along with 95% confidence interval as a function of price-dividend ratio PD in Figure 3 in red. The estimates for coefficients $\hat{\beta}$ and $\hat{\gamma}$ are presented in column (1) of 4. Market-to-book indeed has a heterogeneous impact on skew. When the economy is booming (right side of the figure, high P/D) firms with higher market-to-book have higher skew while when the economy is in a downturn (left side of the figure, low P/D) such firms have lower skew.

There are other variables that may determine the implied volatility skew, for example, leverage is correlated with market-to-book and also affects the return distribution. Other fundamental characteristics may also be correlated with both market-to-book and IV skew leading to an omitted variable bias. Additionally, it might be the case that skew is a property of a particular stock and has nothing to do with market-to-book. To proxy for general tendency of a stock to have a skewed distribution, I include past realized return skewness as a control along with realized return volatility. Then the saturated version of equation (8) takes the following form

$$Skew_{i,t}(\tau) = \alpha_t + \left(\beta^{MB} + \gamma^{MB} \cdot PD_t\right) \cdot MB_{i,t} + \sum_k (\beta^k + \gamma^k \cdot PD_t) \cdot X^k_{i,t} + B' Z_{i,t} + \varepsilon_{i,t} \tag{9}$$

where $k \in \{Lev, OP, AG\}$ and $Z_{i,t}$ includes past realized volatility and skewness. The errors are double clustered at the firm and time levels. The estimates for coefficients $\hat{\beta}^{MB}$, $\hat{\gamma}^{MB}$, $\hat{\beta}^{Lev}$, and $\hat{\beta}^{Lev}$ are presented in column (2) of 4. Other coefficients are reported in Table B.3. I show the estimate of the coefficient on market-to-book $(\hat{\beta}^{MB} + \hat{\gamma}^{MB} \cdot PD_t)$ as a function PD in Figure 3 in blue. The estimates are very similar both quantitatively and qualitatively with wider confidence band. The estimated effect of market-to-book on skew is statistically similar for low aggregate states but is larger for hight aggregate states. The magnitude of the effect is non-trivial: a move from the lowest to highest market-to-book is associated with a 3% increase in implied volatility skew in high aggregate states, an over 42% of the 25th-75th interquantile range of the skew distribution.

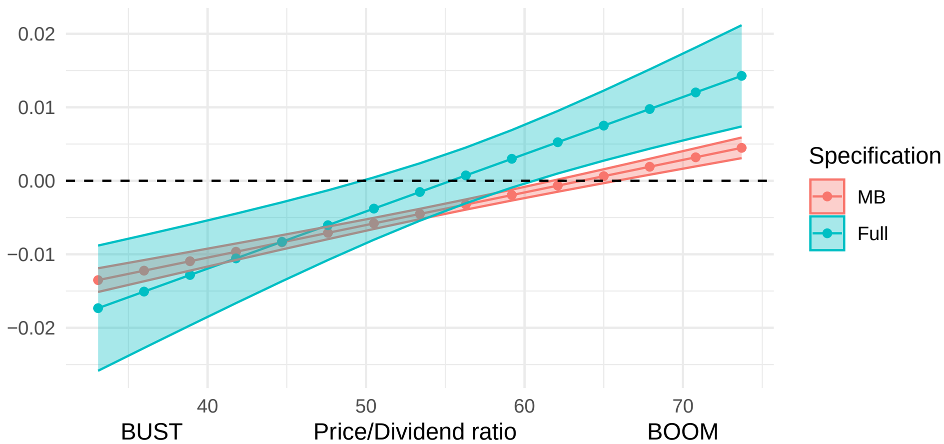


Figure 3: Heterogeneous effect of market-to-book on *Skew*

	Dependent variable: Skew			
	(1) MB	(2) Full	(3) Lev Q1	(4) Lev Q4
MB: β^{MB}	-0.028^{***}	-0.043^{***}	-0.027^{*}	-0.069^{***}
	(0.002)	(0.010)	(0.015)	(0.021)
$MB \times PD$: γ^{MB}	0.044^{***}	0.078^{***}	0.066^{**}	0.115^{***}
	(0.003)	(0.017)	(0.026)	(0.035)
Lev: β^{Lev}		0.032^{***}		
		(0.009)		
$Lev \times PD$: γ^{Lev}		-0.042^{***}		
		(0.015)		
Time FE		X	X	X
Other Fundamentals		X	X	X
Observations	174,283	145,722	49,319	28,609
R^2	0.002	0.220	0.203	0.200
Adjusted R^2	0.002	0.218	0.198	0.191
Residual Std. Error	0.085	0.074	0.070	0.098
Note:				*p<0.1; **p<0.05; ***p<0.01

Table 4: Regression of Implied Volatility Skew on Firm Fundamentals

Notes: The table shows regression estimates for the heterogeneous effect of firm fundamentals on implied volatility skew. The dependent variable *Skew* is defined as implied volatility for standardized moneyness equals to -2 less at the money implied volatility as in equation 5. The right-hand-side firm fundamental variables: market-to-book MB, leverage Lev, asset growth AG, operating profitability OP are cross-sectionally ranked and normalized between -1 and 1. For specifications that include other fundamentals, Table B.3 shows the respective coefficients. Other controls include 24 realized return volatility and skewness. Errors are double clustered at firm and month levels.

Differences in leverage While the estimates of equation (9) reported in Figure 3 control for leverage, they mask an important source of heterogeneity. To uncover it, I estimate equation (9) separately for firms in the lowest and highest cross-sectional quartiles of leverage. Specifically, each month, I rank firms by their leverage and consider the firms with leverage below the 25% percentile and above the 75% percentile the of leverage distribution. I then estimate equation (9) without controlling for leverage.

I report the estimates in Figure 4 and columns 3 and 4 of Table 4. The effect of market-to-book on IV skew is much flatter for firms in the first quartile of the leverage distribution. The effect of market-to-book is positive when the aggregate state is high and is statistically indistinguishable from zero when the aggregate state is low. For firms in the fourth leverage quartile, on the other hand, higher market-to-book is associated with a lower implied volatility skew when the aggregate state is low. When compared to Figure (9), we notice that the negative effect of market-to-book on IV skew for low aggregate states is driven by firms with high leverage. However, when the aggregate state is high, the estimated relationship between market-to-book and implied volatility skew is similar for firms in the first and fourth quartiles of leverage distribution.

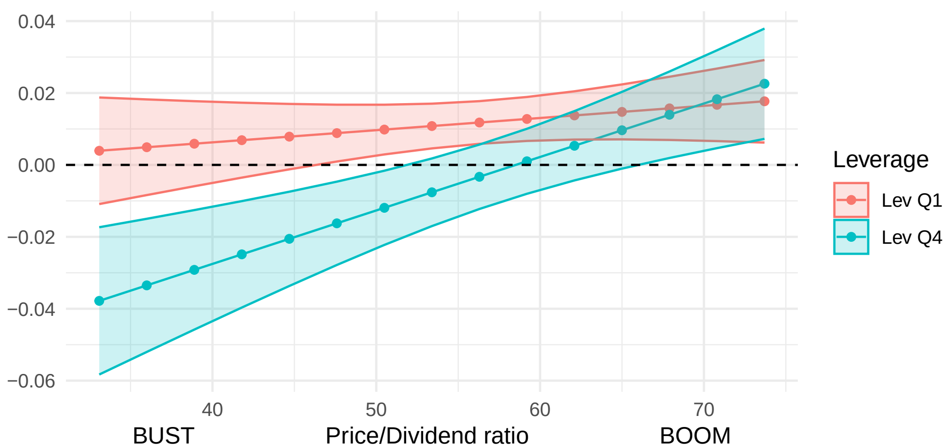


Figure 4: Heterogeneous effect of market-to-book on *Skew* for difference leverage quartiles

The heterogeneity of the estimated effect in firms' leverage will be important to match when I consider the discrete-time structural model in Section 5. It is important to note that the firms in the first quartile of leverage distribution have very little leverage. As I present in Figure B.4, the average leverage in 1st quartile fluctuates between 1% and 10% while spending the majority of time below 5%. In line with this, I will consider two versions of the model: (1) a model with equity only financing and (2) a model with both equity and debt financing. These models will target the respective estimates in Figure 4.

I show the robustenss of the heterogeneous effect of market-to-book as the function of the aggregate state in Figure B.5. In particular, I show that the effect stays quantitatively similar when other variables are used as the measure of the aggregate state: CAPE and log Price-Dividend in panels (a) and (b). The time series of both price-dividend ratio and CAPE highlights that a prolonged period of very high price dividend ratio is a phenomenon that is unique to the dot-com bubble. One concern is that this period may drive the results on the right part of Figure 4. I confirm that this is not the case by showing that the effect presented in Panel (c) of Figure B.5 stays quantitatively similar when estimated on the sample that starts in January 2004.

Return skewness Implied volatility skew goes hand in hand with negative skewness of return distribution, a particular case of a more general result from Breeden and Litzenberger (1978)[1]. The caveat is that Breeden and Litzenberger (1978) result describes the risk-neutral as opposed to physical distribution of underlying holding period returns. In the models presented in subsequent section the return distribution will be negatively skewed under both physical and risk-neutral measures. To make sure that the negative skewness implied by equity options is not driven solely by the risk adjustment in the risk-neutral measure, we need to establish that market-to-book is predictive of physical or realized return skewness.

To test this hypothesis, I will estimate a similar model to equation (9) with realized skewness defined in equation (6) on the left-hand-side. In particular, I estimate

$$-RS_{i,t}(h) = \alpha_t + \left(\beta^{MB} + \gamma^{MB} \cdot PD_t\right) \cdot MB_{i,t} + \sum_k (\beta^k + \gamma^k \cdot PD_t) \cdot X^k_{i,t} + B' Z_{i,t} + \varepsilon_{i,t} \tag{10}$$

where h is 3 months or 63 trading days. Note that to make the comparison with the previous estimates simpler, I flip the sign of the dependent variable as lower (more negative) return skewness is expected to correspond to higher implied volatility skew. I present the estimated coefficient on market-to-book in Figure 5 for the 1st and 4th quartile of leverage distribution. When a firm has low leverage, a higher market-to-book is associated with more negative skewness of returns going forward in high but not low aggregate states. This is in line with a higher implied volatility skew being a sign of expectations of a more negatively skewed return distribution and, therefore, is in line with the empirical estimates in Figure 4 for firms with low leverage. When a firm, in contrast, has high leverage, there is no relationship between market-to-book and realized return skewness.

[1] A simple of example of how negatively skewed return distribution results in a positive implied volatility skew is presented in the Appendix A.1

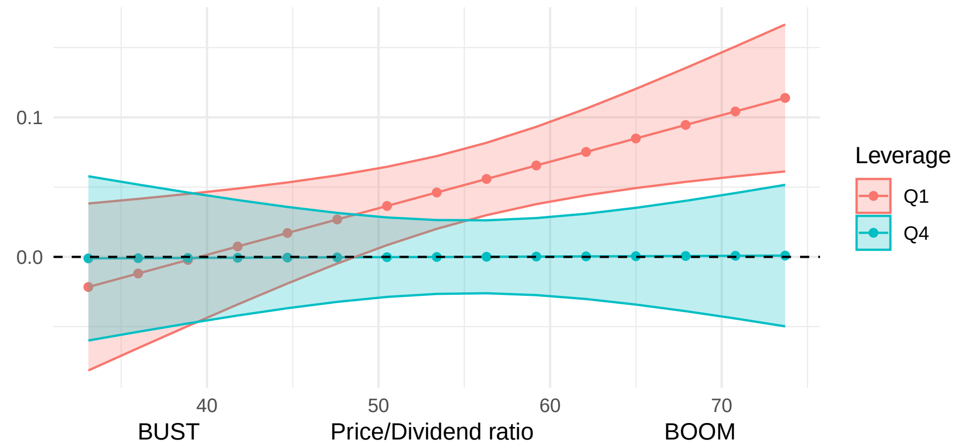


Figure 5: Heterogeneous effect of market-to-book on realized return skewness

4 Continuous Time Model with Real Options

This section presents the continuous time model that will highlight the mechanism and intuition behind the empirical estimates for the effect of market-to-book on implied volatility skew for low leverage firms described in the previous section. I will start with a simple model with real option similar to Gomes and Schmid (2010) and the textbook exposition in Back (2017). In these models, a firm faces stochastic productivity shocks and possess a perpetual option to expand its capacity that is irreversible once exercised. The firm would optimally exercise its option when its productivity increases significantly and cross a particular threshold. These models do not include an aggregate state and, therefore, cannot speak to a heterogeneous effect of firm fundamentals on return distributions and implied volatility skew identified in the previous section. Therefore, I extend the model to include an aggregate variable that will, first, affect the price of cash flow risk and, second, be correlated with the firm productivity state. In this extended model, the firm will optimally exercise its real option when its productivity crosses a threshold that depends on the aggregate state.

Set-Up First consider a mature firm with no option to expand its capacity that operates with a fixed level of capital K generating a stochastic stream of cash flows $x_t K^\alpha$ where x_t is the productivity that follows a geometric brownian motion

$$\frac{dx_t}{x_t} = \mu_x dt + \sigma_x dB_{x,t}. \tag{11}$$

Instead of introducing a representative firm owner with exogenous consumption process as in consumption based asset pricing models, I use an exogenously specified process for the stochastic discount factor π_t:

$$\frac{d\pi_t}{\pi_t} = -rdt - b\sqrt{\lambda_t}\sigma_x dB_{x,t}, \tag{12}$$

where $b\sqrt{\lambda_t}$ is the price of risk of cash flow shocks. Time varying price of risk is the main ingredient to the heterogeneous effect of market-to-book on implied volatility skew. I assume that λ_t follows a square root process that is used extensively in interest rate modelling starting from Cox et al. (1985):

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}, \tag{13}$$

where the correlation of cash flow shock and price of cash flow risk shock is $dB_x dB_\lambda = \rho dt$ and $\rho < 0$ so that on average negative cash flows shocks are accompanied by an increase in price of risk. This specification is quantitatively similar to the time varying rare disaster model in Wachter (2013). This specification generates a steady state distribubtion illustrated in Figure 6, where the economy spends most of its time in the low price of risk λ region with low risk premium and low volatility. At the same time, there is a long tail where both risk premium and volatility is high. When a firm doesn't have an option to expand its productive capacity, the value of the firm with capital K is equal to

$$V^M(x_t, \lambda_t, K) = E_t \left[\int_t^\infty \frac{\pi_\tau}{\pi_t} x_\tau K^\alpha d\tau \right], \tag{14}$$

that can be rewritten as a recursive Bellman equation

$$V^M(x_t, \lambda_t, K) = x_t K^\alpha dt + E\left[\frac{\pi_{t+dt}}{\pi_t} V^M(x_t + dx_t, \lambda_t + d\lambda_t, K) \right]. \tag{15}$$

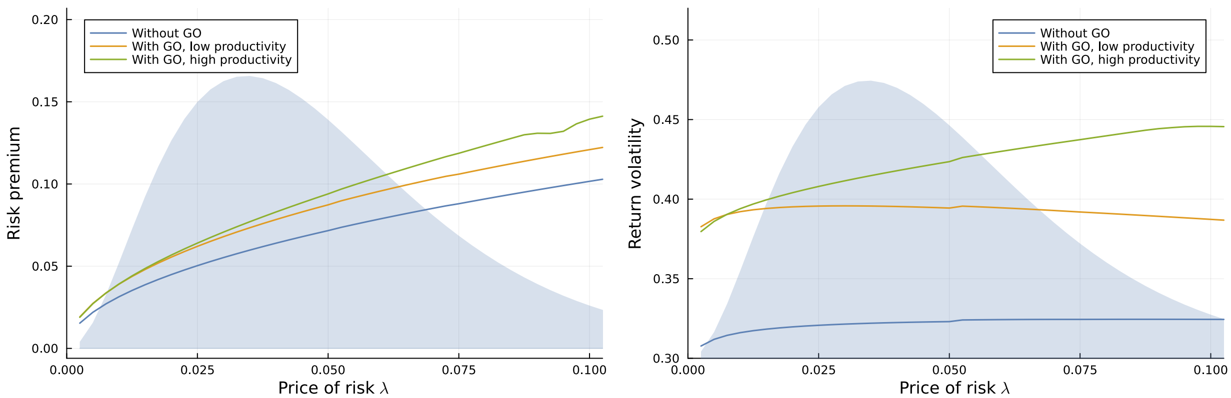
Note that a mature firm doesn't have any control variables.

A young firm, in contrast, has an option to expand its productive capacity from K_0 to K_1 by raising $K_1 - K_0$ from its shareholders. Thus, at any point in time, the firm is choosing between expanding (maturing) and maintaining the same capacity (staying young). Formally, the value of the young firm $V^Y(x_t, \lambda_t, K_0)$ solves the following recursive equation

$$\begin{aligned} V^Y(x_t, \lambda_t, K_0) = \max \big\{ & V^M(x_t, \lambda_t, K_1) - (K_1 - K_0)\,, \\ & x_t K_0^\alpha dt + E\left[\frac{\pi_{t+dt}}{\pi_t} V^Y(x_t + dx_t, \lambda_t + d\lambda_t, K_0) \right] \Big\} \end{aligned} \tag{16}$$

While the simpler version of this model with constant price of risk does admit a closed form solution as in Gomes and Schmid (2010), in the presence of time-varying price of risk I resort to solving the model numerically. The solution approach uses a finite-differences method which consists of the following steps. First, discretize both x and λ on a fine grid. Second, "discretize" the Hamilton-Jacobi-Bellman operator that corresponds to non-exercise decision in equation 16 to form a "transition" matrix. Finally, we can iterate the model with the discretized transition matrix similar to a standard value function iteration while accounting for real option exercise decision. The continuous time nature of the model will manifests itself in a sparse transition matrix: over a very small time period the state can transition only to neighboring states. To achieve better convergence properties the baseline finite-differences method requires several tweaks that I outline in full details in Appendix A.2 following a detailed exposition in Achdou et al. (2022).

Calibration I choose the following parameters $\alpha = 0.65, r = 0.08, K_0 = 2.0, K_1 = 5.0, \bar{\lambda} = 0.05, b = 10, \sigma_\lambda = 0.05, \mu_x = 0.03, \sigma_x = 0.1, \rho = -0.5, \kappa = 0.08, \phi = 3.0$. The asset pricing moments implied by this calibration as well as the steady state distribution of the price of risk λ are presented in Figure 6. The plotted region represents the 95% region for steady state distribution of λ.



(a) Risk premium (b) Volatility

Figure 6: Asset pricing moments for baseline calibration of continuous time model

Model solution The only choice the firm has is whether to exercise its growth option. Optimally, the firm decides to expand its capacity when its productivity hits a price specific value $\overline{x}(\lambda)$. I illustrate this function in Figure 7. Note that the exercise boundary has a concave shape in the price of risk λ. Under a shock that increases the price of risk λ the firm will be pushed farther away from the exercised boundary when the initial price of risk is low relative to when it is initially high. As a result, the value of the firm will experience a sharper decline when the initial price of risk is low.

I show the solution of the model, the value of a young firm V^Y as a function of productivity x for a particular price of risk λ in Figure 8. The value of a young firm *without* real options (green dotted line) is below the value of a young firm *with* real options (solid blue line). The gap between two lines represents the value of real options that the firm possesses. When productivity is low, this gap in values tends to zero, as the firm is far from the exercise boundary. In option terminology, the real option is far out-of-the-money. As productivity x approaches zero, the value of a young firm approaches the value of the firm without growth options. When productivity increases, the gap between the two lines increases representing an increase in the option value. Finally, when productivity is sufficiently large the firm exercises its real option and expands. At this point, $x(\lambda)$, the value of a young firm smooth pastes into the value of exercising the real option (dashed orange line) which is the first term in equation (16). Importantly, we see that when productivity x is large,

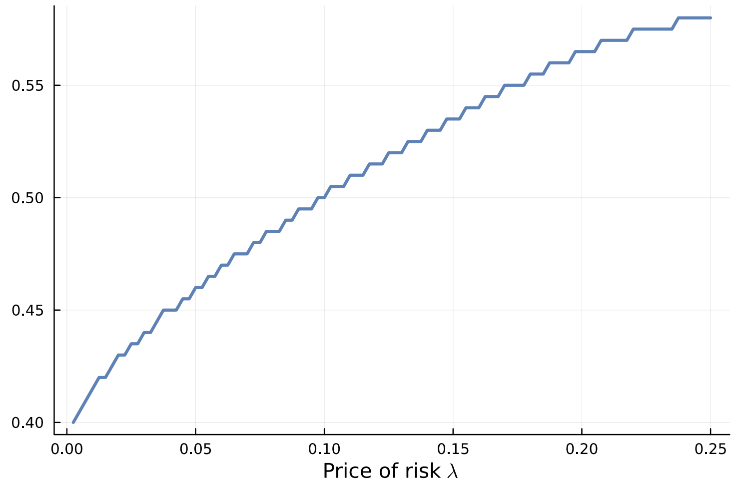


Figure 7: Real option exercise boundary $\overline{x}(\lambda)$

the sensitivity of the value of a young firm to x increases. In option terminology, the firms has a higher delta with respect to productivity.

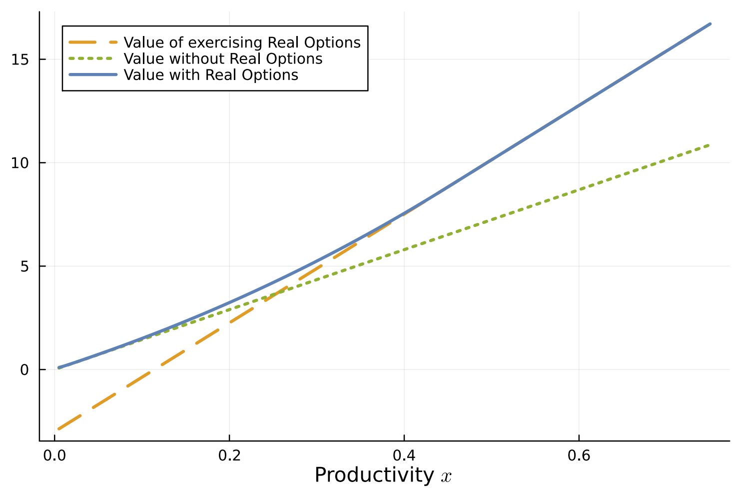


Figure 8: Young firm value

Notes: The figure compares the value of young firm with real options V^Y (solid blue line) to the value firm without real options $V^M\,(K_0/K_1)^\alpha$ (dotted green line) and the value of exercising the real option $V^M - (K_1 - K_0)$ (dashed orange line). Note, that to obtain the value of firm with out real options, we can simply rescale the value of mature firm as its value is homogenuous of degree 1 w.r.t. effective capital K^α. Price of risk is equal to $\lambda = 0.039$.

Equity dilution The recursive expression for the value of the firm presented in equation (16) makes it amenable to standard numerical techniques to solve for the value of the young firm V^Y.

Deriving return distribution requires an additional step. Suppose that a young firm has 1 share outstanding at time t. When the firm exercises its real option at some time $\tilde{t} \geq t$ and expands, it raises equity. I assume that the firm issues η new shares to fund the cost $K_1 - K_0$ to expand its capacity. If an investor owned 1 share in this company at time 0, giving him an ownership share of 100%, after the firm exercised its option and issued new equity this investor will be only entitled to a share of $\frac{1}{1+\eta} \cdot 100\%$ of the firm at some future time $T > 0$. To derive the amount of shares issued η, I assume that the amount raised at post-money valuation exactly covers the expansion costs. Formally, this means

$$\eta(\lambda_{\tilde{t}})(V^M(x(\lambda_{\tilde{t}}), \lambda_{\tilde{t}}, K_1) - (K_1 - K_0)) = K_1 - K_0, \tag{17}$$

where $\eta(\lambda_{\tilde{t}})$ is a function of price of risk $\lambda_{\tilde{t}}$, as the exercise boundary $x(\lambda_{\tilde{t}})$ is a function of $\lambda_{\tilde{t}}$. Expressing $\eta(\lambda_{\tilde{t}})$ we get

$$\eta(\lambda_{\tilde{t}}) = \frac{K_1 - K_0}{V^M(x(\lambda_{\tilde{t}}), \lambda_{\tilde{t}}, K_1) - (K_1 - K_0)}. \tag{18}$$

We can formally define the stochastic and potentially infinite option exercise time $\tilde{t}$ as

$$\tilde{t} = \inf_t \{t : x_t \geq \overline{x}(\lambda_t)\}, \tag{19}$$

i.e the first time productivity x_t crossed the exercise boundary $x(\lambda_t)$. The dilution over the holding period from $t = 0$ to $t = T$ is

$$\eta_{0,T} = \begin{cases} 0 \text{ if } \tilde{t} > T \\ \eta(\lambda_{\tilde{t}}) \text{ if } \tilde{t} \leq T \end{cases} \tag{20}$$

This condition emphasizes that between time t and T a share issuance may or may not occur and if it does the dilution coefficient will depend on the aggregate state λ at the time of exericing the option.

Path dependence in returns significantly complicates the derivation of return distribution and requires expanding the state-space with an additional dimension to account for *dilution adjusted flow* of companies into maturity. To understand the approach, note that in addition to productivity and aggregate states (x, λ) the diluted firm value at time T, $\frac{V_T}{1+\eta_{t,T}}$ depends only on the aggregate state the prevailed when the exercise occured $\lambda_{\tilde{t}}$. Suppose that at some time $\tau < T$, the mass of firms that have not yet exercised their option is given by $g_Y(x, \lambda)$ while the distribution of firm that exercised their option when the aggregate state was $\tilde{\lambda}$ is given by $g_\eta(x, \lambda, \tilde{\lambda})$. How do these distributions change in the next instant of time ignoring the effect of diffusion in x and λ? The firms that are at the exerice boundary with $x = \bar{x}(\lambda)$ will exercise their options meaning that between times τ and $\tau + d\tau$, distribution g_Y will change as

$$g_Y(x, \lambda) \to g_Y(x, \lambda)\mathbb{I}\{x \neq \bar{x}(\lambda)\} + 0 \cdot \mathbb{I}\{x = \bar{x}(\lambda)\}$$

while distribution g_η will change as

$$g_\eta(x,\lambda,\tilde{\lambda}) \to \left[g_\eta(x,\lambda,\tilde{\lambda}) + g_Y(x,\lambda)\right]\mathbb{I}\{x = \bar{x}(\lambda), \tilde{\lambda} = \lambda\} + g_\eta(x,\lambda,\tilde{\lambda})\left(1 - \mathbb{I}\{x = \bar{x}(\lambda), \tilde{\lambda} = \lambda\}\right)$$

where $g_Y(x,\lambda)\mathbb{I}\{x = \bar{x}(\lambda), \tilde{\lambda} = \lambda\}$ accounts for the change in the distribution due to option exercise. I provide more details on the implementation of this idea in Appendix A.3. With these distributions, expectation of an arbitrary function of the diluted firm value V can be obtained as

$$E[f(V)] = \int f(V^Y(x,\lambda))g_Y(x,\lambda) + \int f\left(\frac{V^M(x,\lambda)}{1+\eta(\tilde{\lambda})}\right) g_\eta(x,\lambda,\tilde{\lambda}) \tag{21}$$

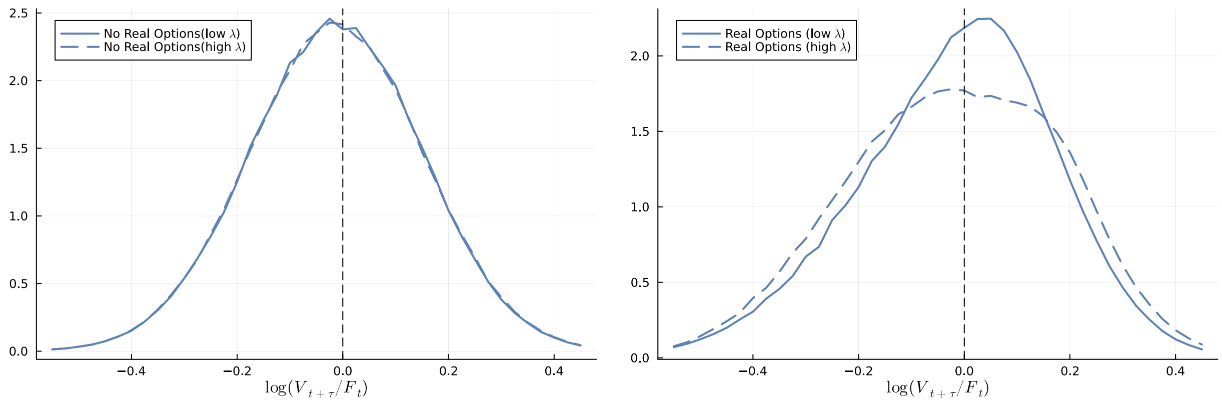
Return distribution I present the distribution of firm value $V_{t+\tau}$ relative to the forward price

$$F_{t,t+\tau} \equiv E_t^Q\left[V_T\right] \tag{22}$$

for firms with and without real options in Figure 9 for different starting price of risk λ. First, consider a firm without real option presented in Panel (a). Note that return distribution is not exactly symmetric due to the presence of price of risk shocks. More importantly, changing price of risk λ doesn't change the return distribution of a firm with no real option in a meaningful way. The sensitivity to λ is different when we consider a firm with real options presented in Panel (b). When the price of risk is high (dashed line), the distribution is relatively symmetric and resembles the distribution of a firm with no real options albeit wider as the firm is sensitive not only to changes in the value of assets-in-place, but also to changes in the value of real options. However, when the price of risk is low (solid line) corresponding to an aggregate state with high valuations, the distribution changes its form markedly. Instead of being symmetric, it is negatively skewed in line with the going hypothesis and with empirical evidence for low leverage firms presented in the previous section. I show the value distribution moments associated with these distributions formally in Table 5 where we can calearly see that while for the firm without growth options a smaller price of risk decrease skewness only modestly, the effect for a firm with growth options is much more prounounced.

Implied volatility To derive implications for implied volatilities, I use the distributions presented in Figure 9 to derive values of options at different strikes and invert the Black-Scholes formula to obtain implied volatilities. I present implied volatilities for firms with and without real options in Figure 10. A direct consequence of Breeden and Litzenberger (1978) result is that there is a tight connection between return distributions and option prices. Implied volatility, in turn, is simply a normalization that allows to compare option prices at different strikes and maturities. Therefore, analyzing implied volatilities on Figure 10 is very similar to analyzing return distributions in Figure 9.

The first observation in Figure 10 is that firms without real options (blue lines) have very flat



(a) No Growth Options

(b) With Growth Options

Figure 9: Distribution of firm value for firms with and without real options

Notes: Both panels show the probability density function for the log diluted firm value relative to forward price $\log(V_T/F_t)$ where $T-t$ is taken to be three months. The PDF is obtained by applying equation (21) with $f(x) = \mathbb{I}\{x \leq \vartheta\}$ and then numerically differentiating. Panel (a) show the distribution for a firm without growth options for low (solid blue line) and high (dashed blue line) price of risk. Panel (b) shows the same for a firm **with** growth options. The low price of risk is $\lambda = 0.039$ and high price of risk is $\lambda = 0.078$. For all distributions, the productivity state is $x = 0.234$

		Variance	Skewness	Normalized Skewness
No GO	low λ	0.0270	-0.0002	-0.0504
No GO	high λ	0.0263	-0.0002	-0.0384
With GO	low λ	0.0342	-0.0022	-0.3480
With GO	high λ	0.0406	-0.0016	-0.2020

Table 5: Value distribution moments for firms with and without growth options

Notes: The table shows moments of the distribution of log diluted firm value relative to forward price $\log(V_T/F_t)$. Variance and Skewness in the first two columns are defined in the standard way as $E^Q[(x-E^Q[x])^2]$ and $E^Q[(x-E^Q[x])^3]$ while the Normalized Skewness is defined as Skewness divided by Variance$^{3/2}$ corresponding to the empirical measure os skewness defined in equation (6) and used in the empirical analysis in Section 3. These moments correspond to distributions presented in Figure 9 implied volatilities. When price of risk is higher (dashed blue line), implied volatility increases as the return volatility increases, but remains flat as the function of terminal price. However, firms with real options (orange and green lines) have more interesting implied volatility shapes. When the price of risk is low (orange) corresponding to high valuations, there is a significant skew: implied volatility for low strikes is significantly above the at-the-money implied volatility (corresponding to zero log return located at the vertical dashed line). But it is not the case for a high price of risk (green line) that corresponds to low valuations.

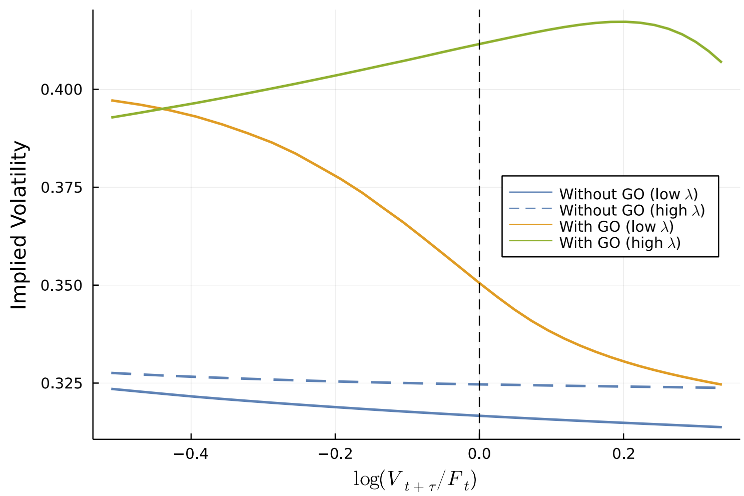


Figure 10: Implied volatility for firms with and without real options

Notes: This figure presents implied volatilities associated with distributions presented in Figure 9. To obtain option prices I use function $f(V) = \max\{V \leq F \cdot \exp(\vartheta), 0\}$ and vary ϑ move along the x-axis. Once option prices are calculated I numerically invert the Black-Scholes-Merton equation.

The model developed in this section proposes a mechanism for the positive effect of market-to-book or productivity on implied volatility skew when the aggregate state is high. Firms with high productivity derive a large portion of their value from real options and this value is particularly sensitive to the aggregate state when it is initially high. As the result, the value of such firms will be adversly affected by deteorating aggregate conditions while the value of the real option is already priced in. This generates a negatively skewed return distribution that manifests itself in a positive implied volatility skew.

5 Discrete Time Model

The continuous time model presented in the previous section provides a stylized explanation for higher implied volatility skew and negatively skewed distribution of firm value for growth companies when the economy is booming. In that model, the difference between growth and value companies was driven by a one-shot opportunity to expand productive capacity, this decision was, moreover, indivisible and irreversible. This section presents a richer model that features two main components not present in the continuous time model. This mode builds on the large literature of dynamic corporate finance models that were successful in matching asset pricing moments for both debt and equity. First, instead of having an ability to increase capital to a predetermined level and not being able to reverse the decision, I follow a standard approach to modelling real options with a partial irreversibility of investment in a form of asymmetric adjustment costs. Second, in order to speak to the effect of leverage on implied volatilities, I introduce defaultable debt. In contrast to the stylized model in the previous section that was used to illustrate the effect of market-to-book on implied volatility skew, the richer model in this section serves a goal of quantitatively matching the empirically estimated effect from Section 3.

5.1 Firm's problem

The model is set in discrete time. A firm starts each period t with capital K_t, face value of borrowings B_t and two productivity states. x_t is an aggregate productivity state that will affect the SDF discussed below and y_t is an idiosyncratic productivity state. The firm makes two types of decisions. First is whether to continue operating. If the firm decides to stop operating and exits the market due to a low realization of the shock and/or large leverage, equity holders get a continuation value of zero. If the company decides to continue operating, the firm chooses how much to (dis)invest and to borrow to maximize the shareholders' continuation value.

When the firm made its investment K_{t+1} and borrowing B_{t+1} decisions, it will be left with an amount

$$d(S_t, K_{t+1}, B_{t+1}) = (1-\tau)\left[\underbrace{e^{\beta_x x + y} K_t^{\alpha}}_{\text{Operating cash flows}} + \underbrace{\frac{B_{t+1}}{1+R} - B_t}_{\text{Net borrowings}} - \underbrace{\widetilde{\Phi}(K_{t+1}, K_t)}_{\text{Investment costs}}\right] \tag{23}$$

available for distribution to shareholders, where $e^{\beta_x x_t + y_t}$ is firm's productivity that depends on both aggregate x and idiosyncratic states y. τ is the tax rate. $\frac{B_{t+1}}{1+R}$ are the proceeds from promising to repay B_{t+1} in period $t+1$ which is discuss in more details below. $\widetilde{\Phi}$ is investment cost function that depends on both the current period and next period capital through investments and can be written as $\widetilde{\Phi}(K_{t+1}, K_t) \equiv \Phi(I_t(K_{t+1}, K_t), K_t)$ where $I_t \equiv K_{t+1} - (1-\delta)K_t$. S_t is a short notation for firm's state in period t: $S_t \equiv (K_t, B_t, x_t, y_t)$. Following Zhang (2005), adjustment costs are asymmetric

$$\widetilde{\Phi}(I_t, K_t) \equiv I_t + \mathbb{I}_{\{I_t>0\}} \cdot \phi^{+} \left(\frac{I_t}{K_t}\right)^2 K_t + \mathbb{I}_{\{I_t<0\}} \cdot \phi^{-} \left(\frac{I_t}{K_t}\right)^2 K_t \tag{24}$$

with a larger adjustment cost $\phi^- > \phi^+$ when the firm disinvests. This is the critical element of generating real options that was taken to the extreme in the continuous time model presented in Section 4 where the firm didn't have means to disinvest.

The full problem of the firm is

$$V_t(S_t) = \max\left\{0, \max_{K_{t+1},B_{t+1}} d(S_t, K_{t+1}, B_{t+1}) + E\left[M(x_{t+1}|x_t)V(S_{t+1})|\, S_t\right]\right\}. \tag{25}$$

where the max lets the firm to cease operating and default if the continuation value is low enough.

Debt pricing When the firm promises to repay B_{t+1} in the next period, there is a possibility that the firm will exit and will not make whole on its promise. The proceeds from borrowing $\frac{B_{t+1}}{1+R}$, therefore, should reflect that possibility and the interest rate on firm's borrowings will deviate from the risk free interest rate. I follow Begenau and Salomao (2019) to price firm's debt. In particular, when the firm promises to repay B_{t+1} the proceeds at time t are

$$\underbrace{\frac{B_{t+1}}{1+R}}_{\text{Proceeds at } t} \equiv \underbrace{E\left[M_{t+1}\mathbb{I}_{V_{t+1}>0}B_{t+1}\right]}_{\text{Not default}} + \underbrace{E\left[M_{t+1}\mathbb{I}_{V_{t+1}=0}\min\{\theta(1-\delta)K_{t+1}, \overline{RC}\cdot B_{t+1}\}\right]}_{\text{Default}}, \tag{26}$$

where the first term corresponds to the company making its debtholders whole and the second term corresponds to default. In the case of default, debtholders can recover a share θ of non-depreciated capital or some maximum recuperation value $\overline{RC}$ of the face value of the borrowings whichever is smaller. The latter component is introduced by Begenau and Salomao (2019) to prevent debtholders from recovering capital in excess of the face value of their debt.

Stochastic discount factor The stochastic discount factor (SDF) is based on an Epstein and Zin (1989) utility investor with an exogenous consumption stream $C(x) = e^x$. Specifically, following Campbell (2018), I define

$$M(x_{t+1}|x_t) = \beta\left(\frac{C(x_{t+1})}{C(x_t)}\right)^{-1/\psi}\left(\frac{U(x_{t+1})}{E_t\left[U(x_{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{-(\gamma-1/\psi)}, \tag{27}$$

where β is the time preference parameter and $U(x)$ is the representative owner's lifetime utility that solves

$$U(x_t) = \left\{(1-\beta)C(x_t)^{1-1/\psi} + \beta\left(E_t\left[U(x_{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right)^{1-1/\psi}\right\}^{\frac{1}{1-1/\psi}}. \tag{28}$$

The presence of debt in the model makes it important to use Epstein-Zin based SDF instead of CRRA based. As explained in Campbell (2018), under CRRA utility high value of risk aversion would explain the equity risk premium at the expense of plausibility of the risk free rate. While many dynamic structural models model SDF in a reduced form without specifying the utility function as in Zhang (2005), they are prone to the same issue. For example, the parametrization in Zhang (2005) results in a risk-free rate that increases sharply when the aggregate state declines.

Aggregate Productivity		Idiosyncratic Productivity		Behavioral		Production	
μ_x	0.1	μ_y	0.1	β	0.965	δ	0.12
ρ_x	0.95	ρ_y	0.7	γ	50	ϕ^+	2.0
σ_x	0.015	σ_y	0.346	ψ	1.0	ϕ^-	15.0
b	0.0					τ	0.3
β_x	4.0						

Table 6: Discrete Model Parameters

Using the baseline calibration in Table I and expressions for the risk-free rate Zhang (2005), we can derive that the risk free rate increases to 10.6% when the aggregate state is 1 standard deviation below the steady state value and to 25% when it is two standard deviations below. Such a high risk-free rate has potentially severe implications for models with debt: in low aggregate states levered firms will incur prohibitively high costs of servicing and rolling over their debt. In contrast, a model based on Epstein-Zin prefences generates a much flatter risk-free rate when the aggregate state varies.

Stochastic states Both aggregate and idiosyncratic states follow AR(1) processes

$$
\begin{aligned}
x_{t+1} &= (1-\rho)\bar{x} + \rho_x x_t + v(x_t)\varepsilon_{x,t+1} \\
y_{t+1} &= (1-\rho)\bar{y} + \rho_y y_t + \sigma_y \varepsilon_{y,t+1}
\end{aligned}
\tag{29}
$$

where ε_x and ε_y are standard normal random variables and

$$
v(x) \equiv \sigma_x \cdot \frac{2}{1+e^{bx}}. \tag{30}
$$

The second term in equation (30) makes volatility of x_t state dependent. In particular, the larger the value of b is, the more countercyclical volatility of the aggregate state will be.

Mapping the model to the data Solution of the discrete time model is standard. I first discretize the aggregate state space x_t. I iterate equation (28) until convergence to obtain agent's utility. I then apply equation (27) to obtain the SDF. After the SDF is obtained for all pairs possible transition pairs (x_t, x_{t+1}) on the discretized grid on the aggregate state grid, I discretize the remaining firm state variables (K_t, B_t, y_t) and iterate the Bellman equation (25) until convergence. The baseline parameters used for solving the simulating the model are presented in Table 6 where the highlighted entries for the countercyclicality of volatility b and asymmetry of adjustment costs ϕ^- will be targeted in the calibration below when I match the empirical estimates for the effect of market-to-book on implied volatility skew.

After I solve the model, the next step is to bring the solution closer to the data. The first step is to simulate the model. I simulate 250 economies over 100 quarters to obtain unconditional distribution of each state $p(S)$ for $S = (K, B, x, y)$. For each distinct state S I solve for option prices given the risk neutral transition equation and invert the Black-Scholes-Merton equation to obtain implied volatilities. I calculate implied volatility skew as the difference between $I(-2.0, \tau)$ and $I(2.0, \tau)$. To mirror the specification estimated in the data, I form right hand side variables similar to the main specification in equation (9) as

$$MB = \frac{V + PB}{K},\ Lev = \frac{B}{K},\ Prof = y,\ Inv = K_{t+1} - (1-\delta)K_t. \tag{31}$$

where following Kuehn and Schmid (2014), I define market-to-book as the ratio between the market value of equity V and debt PB and capital K where $P \equiv 1/(1+R)$ is the market price of debt from (26) that takes into account the possibility of default.

Similarly to the data, I normalize the variables cross-sectionally to be uniform from -1 to 1 and estimate the following regressions across states S:

$$Skew(S) = (x \text{ FE}) + (\beta^{MB} + \gamma^{MB} \cdot x) \cdot MB(S) + \sum_k (\beta^k + \gamma^k \cdot x) \cdot X(S)^k + \varepsilon(S) \tag{32}$$

where $k \in \{Lev, Prof, Inv\}$ where I use x as the aggregate state variable similarly to price-dividend ratio in the data. I weight the observations by the frequency of state $p(S)$.

5.2 Simulation results

As I outlined in section 3, I will estimate model with and without leverage to fit the respective estimates from Figure 4.

Model with financial leverage For firms with high leverage, the mechanism behind the effect of market-to-book on implied volatility skew is relatively straightforward in the model. For the same aggregate state, firms with higher market-to-book will be the more productive firms. When the aggregate state is low and the firm has high leverage, its return distribution will be negatively skewed as the firm is closer to default and exit. An increase in profitability pushes the firm away from default leading to a less negatively skewed returns distribution. This, in turn, lowers implied volatility skew. Therefore, even with moderate adjustment costs asymmetry and no countercycliclical volatility, a model with financial leverage will generate estimates similar to the empirical results for firms with high leverage in Figure 4.

To directly compare the regression estimates across empirics and simulation, we need to transform the aggregate state, PD in the data and aggregate productivity x in the simulation, onto the same scale. I map the full range of PD ratio in the sample, linearly aggregate states in the model.[2] The empirical estimates presented mapped to simulation estimates for the model with debt are

[2]Specifically, I associate a particular value of PD with aggregate state with index $3 + 14 \times (PD - \min PD)/(\max PD - \min PD)$ from the model. Meaning that I associate the lowest PD ratio in my sample with the 3rd aggregate state and the largest PD ratio with the 17th (3rd from the end) aggregate state.

presented on the right panel of Figure 11. The model with debt has no issuance in generating a large negative effect of market-to-book on implied volatility skew when the aggregate state is low.

Model without leverage In contrast, it is more challenging to fit the empirical estimates for firms with low leverage in a model with no debt financing. In particular, a large and significant positive effect of market-to-book on implied volatility skew in high aggregate states of the world requires both a substantial adjustment costs asymmetry ϕ^+/ϕ^- and highly countercycliclical volatility – high b. To illustrate the influence of these parameters, I solve the model for different parameters, simulate and re-estimate regressions in equation (32). Instead of comparing the full fit relative to the empirical estimates as in the left Panel of Figure 11, I focus on the right-most estimate for the largest considered aggregate state and compare it with the corresponding empirical estimate. In Figure 12, the dashed line shows the empirical estimate – the fitted effect of market-to-book on implied volatility skew for the largest price-dividend ratio. The corresponding fitted simulation effects are presented as a function of countercyclicality of volatility b for different values of ϕ^+ (different panels) and ϕ^- (different line colors). We see that a higher b, as we move to the right, raises the simulated effect and brings it closer to the empirically estimated effect plotted as the horizontal dashed line. Similarly, for a fixed ϕ^+ a higher asymmetry of adjustment costs, i.e. a higher ϕ^-, generally raises the simulated effect.

As a result, both asymmetry in adjustment costs and countercyclical volatility are the necessary ingridients for generating a positive effect of market-to-book on IV skew similar to the stylized model presented in section 4. At the same time, the discrete time model presented in this section is able to match the empirical estimates also quantitatively. The parameters that generate the best fit from the analysis in Figure 12 are $\phi^+ = 10$, $\phi^-/\phi^+ = 12.5$ and $b = 10$. The asymmetry of adjustment costs is large, but is below its value in Zhang (2005). Countercyclicality $b = 10$ implies that the the maximum achievable Sharpe ratio defined as

$$\max SR(x_t) = \frac{\sigma_t(M(x_{t+1}|x_t))}{E_t[M(x_{t+1}|x_t)]}$$

is just under 0.5 for the average aggregate state of the world and declines to 0.25 for a highest aggregate state which seem empirically plausible.

Differences with Zhang (2005) The mechanism considered in the continuous time model in section 4 and in the discrete time model for equity only financing appears to contradict the one in Zhang (2005). In the models presented in this paper, firms with a high market-to-book appear to be riskier in high aggregate state of the world based on both implied volatility skew and return skewness. Zhang (2005), in contrast, argues that value firms are the ones that are riskier especially in low aggregate states.

The model in Zhang (2005) rationalizes the value premium with a production based model with asymmetric adjustment costs. The value firms, the less productive firms in the cross-section, are

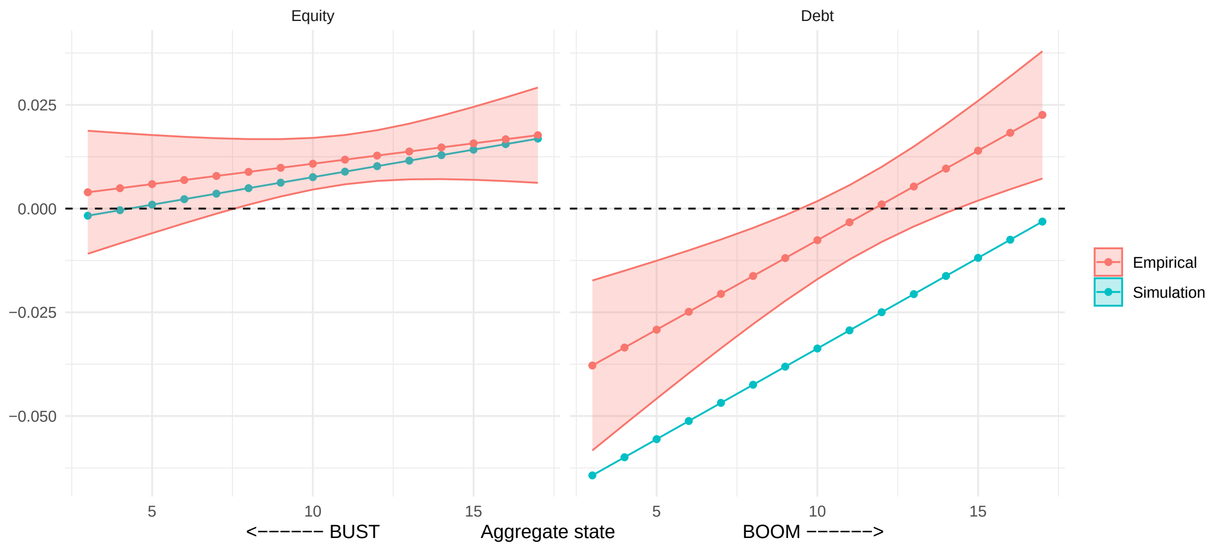


Figure 11: Comparing empirical and discrete model estimates

Notes: The figure presents the simulated $\hat{\beta}^{MB} + \hat{\gamma}^{MB} \cdot x$ (blue) and empirical $\hat{\beta}^{MB} + \hat{\gamma}^{MB} \cdot PD$ (red) estimates for the effect of market-to-book on implied volatility skew. PD and x are linearly matched such tha the lowest PD in the post-1996 sample is equal to the 3rd aggregate state and the highest PD is equal to the 3rd aggregate state from the end. 3rd and 3rd from the end aggregate states correspond to $approx 2.5\%$ percentiles of the unconditional distribution of x. The left panel compares the empirical estimates for the firms in the lowest leverage quartile to the model without debt. The right panel compares the empirical estimates for the firms in the highest leverage quartile to the model with defaulatble debt. In both empirical and simulation regressions, firm fundamentals are cross-sectionally ranked.

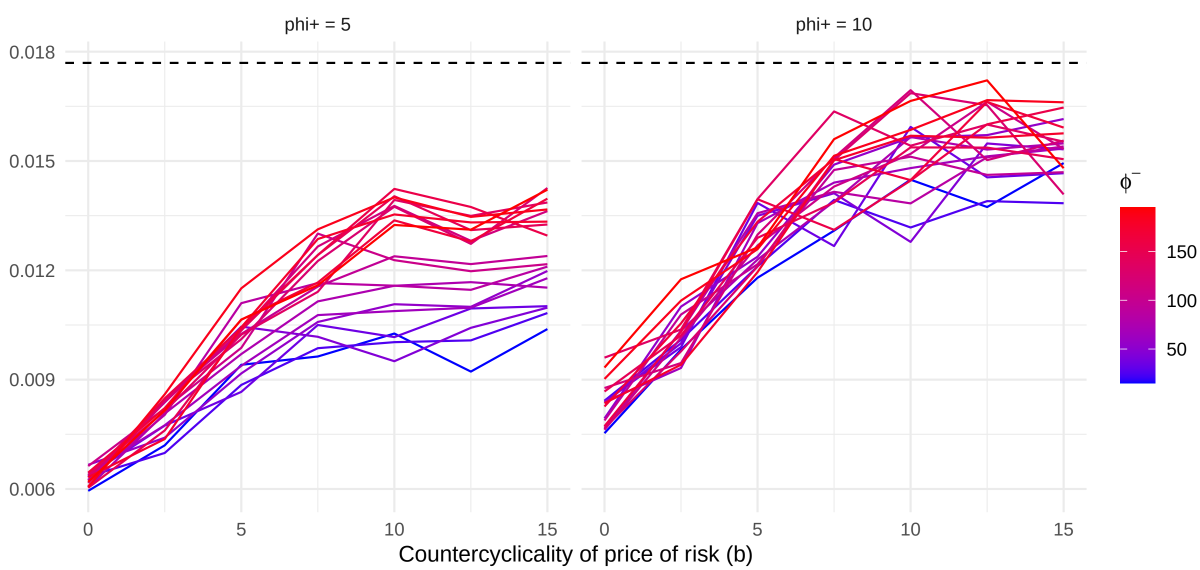


Figure 12: Effect of market-to-book on IV skew for high aggregate states

at the risk of needing to scale down and thus incur substantial adjustment costs as the costs of disinvestment are much higher than the costs of investment. This effect is intensified when the aggregate state is low. This makes value firms more risky especially when the aggregate state is low generating a value premium.

While much of the discussion in Zhang (2005) is centered around adjustment costs, the fixed costs of operations – operational leverage – is the crucial ingridient in the model. If we denote stochastic cash-flows as cf_t and fixed costs as f we can decompose the growth rate of cash flows available for distribution to equityholders as

$$\frac{\Delta(cf_t - f)}{cf_t - f} = \frac{\Delta cf_t}{cf_t} \frac{cf_t}{cf_t - c_f}. \tag{33}$$

Under fixed volatility of $\Delta cf_t / cf_t$, the difference in volatility of cash flows available for distribution is driven by the second term in the cross-section. This term indicates how close a firm is to its "subsistance" level. The lower the cash flows cf_t are, the larger the second term is implying that firms with lower productivity have riskier cash flows. In line with this argument, Figure 2 in Kogan and Papanikolaou (2012) shows that a firm with operating leverage and symmetric adjustment costs has a U-shaped risk exposure as a function of productivity – the same shape that generates the main results in Zhang (2005). Kogan and Papanikolaou (2012) then show that adding asymmetric adjustment costs only slightly elevates the risk exposure when the productivity is low. In this way, the parametrization in Zhang (2005) is more similar to a model with debt. And, in fact, under the baseline calibration it produces a similar effect to the one presented in the right panel of Figure 11.

6 Delta Hedged Option Trading Strategies

The continuous time model in Section 4 feature a stochastic aggregate state λ that determines the price of cash flow risk. The changes in price of risk make the volatility in the model time varying. At the same time, we have observed that return volatility of firms with different fundamentals react differently to changes in the aggregate states as exemplified in Table 5. This heterogeneity points to an existence of fundamentally driven exposure to the variance risk premium and to a potential mechanism how firm fundamentals may influence trading strategies that aim at exploiting it. This paper is not the first one to suggest variance risk premium strategies in equity options based on firm fundamentals. Most recently, Zhan et al. (2022) sorts companies based on firm fundamentals such as profitability, then takes long positions in covered calls for the top quintile and short positions in covered calls for the bottom quintile. These strategies have large Sharpe ratios that authors claim stay significant even after accounting for transaction costs. While Zhan et al. (2022) points out that these strategies are highly profitable and provides evidence that they are robust to different economic regimes, they do not offer any insights as to why.

Studying the performance of naked option positions is inherently challenging due to a significant asymmetry of returns. For example, a long position in an at-the-money call option has a roughly 50% probability of losing all the value if the option expires out-of-the-money generating a -100% return. On the other hand, a delta-hedged option position is more amenable for both theoretical and empirical analysis. Under BSM assumptions a continuously delta-hedged option position generates a zero return for an investor almost surely. In fact, the possibility to delta-hedge continuously underlies the derivation of BSM equation. Even when only discrete delta-hedging is allowed, the return distribution of a discretely hedged option position is normal with variance decreasing with the square root of the hedging frequency (e.g. Bertsimas et al., 2000, 2001).

Deviations from BSM assumptions like stochastic volatility and their implications for the returns of delta-hedged option positions have also been studied. An important theoretical insight from Bakshi and Kapadia (2003) states that profits of a delta hedged option position are related to variance risk premium. In particular, define the profit as

$$Profit_{t,t+T} = \underbrace{C_{t+T} - C_t}_{\Delta\text{call value}} - \underbrace{\int_t^{t+T} \Delta_u dF_u}_{\text{delta-hedge}} - \underbrace{\int_t^{t+T} r\left(C_u - \Delta_u F_u\right) du}_{\text{financing costs}}. \tag{34}$$

The first term is the change in the value of a long position in a call option. The second term are the profits from the delta-hedge position in the futures contract on the underlying with maturity at $t+T$: at every instant u, a trader takes a position of size $-\Delta_u$ where Δ_u is the delta of a call option at instant u to make the overall position delta neutral. In the next instant, the trader makes profits on the position in the underlying futures $-\Delta_u(F_{u+du} - F_u) = -\Delta_u dF_u$. The last term is the financing of position. If it is not costless to enter the position ($C_u \neq \Delta_u F_u$), a trader needs to borrow/lend the difference at the risk free rate r.

Bakshi and Kapadia (2003) prove that in an environment with stochastic volatility σ_t, the expected profits of the position described in equation (34) are

$$E[Profit] = \int_t^{t+T} E_t\left[\frac{\partial C}{\partial \sigma}\theta(\sigma)\right] du, \tag{35}$$

where $\theta(\sigma)$ is the variance risk premium

$$\theta(\sigma) = cov_t\left(-\frac{d\pi_t}{\pi_t}, d\sigma_t\right), \tag{36}$$

C is the price of a call option, σ_t is stochastic volatility, $\partial C/\partial\sigma$ is the sensitivity of a call option to changes in volatility also known as option *Vega* and π_t is the stochastic discount factor process. The model of a firm with real options presented above doesn't feature stochastic volatility explicitly. However, the presence of time varying price of risk makes volatility state dependent as is exemplified in Panel (b) of Figure 6. Therefore, it is possible to derive a result similar to Bakshi and Kapadia (2003) for the model with real options where the profits of a strategy will be driven by the price of risk λ.

In order to derive the expected profits of a delta-hedged option strategy, it is useful to set some notation to highlight the differences between the standard approach to option pricing taken in the financial engineering literature and the model presented in this paper. The model features 2 state variables x and λ and the value of a call option can be expressed as a function of (x, λ). Denote this function $\widetilde{C}(x, \lambda)$. Similarly, the futures price is also a function of same state variables $F(x, \lambda)$. Alternatively, following financial engineering approach, the value of a call option can be expressed as a function of a futures price $F(x, \lambda)$ and the price of risk λ such that

$$C(F(x, \lambda), \lambda) = \widetilde{C}(x, \lambda). \tag{37}$$

While $\widetilde{C}$ allows to directly calculate the value of an option within the model, C is useful for understanding quantities such as option delta, i.e. $\partial C/\partial F$. Using this notation, I prove the following proposition:

Proposition 1. *(Real options delta-hedged profits) Expected profits from a continuously delta hedged long option position are*

$$E[Profit] = \int_t^{t+\tau} \sigma_x \sigma_\lambda b \rho E_t \left[\frac{\partial \widetilde{C}}{\partial \lambda} \lambda_u \right] du, \tag{38}$$

where

$$\frac{\partial \widetilde{C}}{\partial \lambda} \equiv \frac{\partial C}{\partial \lambda} - \frac{\partial F/\partial \lambda}{\partial F/\partial x} \cdot \frac{\partial C}{\partial F}. \tag{39}$$

Proof. Appendix A.4 □

First note that the general form of expected profits is similar to the result in Bakshi and Kapadia presented in equation (35). Expected profits are driven by the sensitivity of the call price to the additional source of risk – price of risk λ. If cash flow shocks are not correlated with price of risk shocks ($\rho = 0$), then expected profits are zero. The difference comes from the way the sensitivity $\partial\widetilde{C}/\partial\lambda$ is calculated. In a standard option pricing set-up including BK, the initial spot price is exogenously given. In contrast, in the real options model, the initial value of a firm is determined by both productivity x and price of risk λ. The first term in the expression, $\partial C/\partial\lambda$, is similar to BK. However, in an environment where the second state affects the valuation of the firm itself like in a real option model with a stochastic price of risk, we need to correct the sensitivity to remove the effect of λ that comes from the effect on the futures price. This term appears since both the underlying futures price and the value of the option are exposed to price of risk shocks. Therefore, we need to consider the whole portfolio consisting of a long position in a call and a short position in the underlying.

The second expression $\frac{\partial F/\partial \lambda}{\partial F/\partial x}\frac{\partial C}{\partial F}$ is easy to sign. Higher price of risk λ decreases the futures price F and higher productivity x increases it. Delta of a call option $\partial C/\partial F$ is positive. Therefore, the second term has a negative sign. The first term $\partial C/\partial\lambda$, however, can be both positive and negative. On the one hand, higher price of risk λ given the same futures price F decreases the value of an option as it carries more systematic risk. On the other hand, higher λ also raises the volatility, thus, increasing the value of an option. As a result, the senstivity has two competing effects.

6.1 Theoretical performance

I use the numerical solution to the model with real options to derive implications for expected profits of a delta hedged option strategy described in Proposition 1. While the expected profits depend on the expectation of the full path $(\partial\widetilde{C}/\partial\lambda)\lambda$ over the holding period $[t, t+T]$, we can get a sense of its sign, by considering the sign at the start of the holding period t. Note that as $\sigma_x, \sigma_\lambda, b > 0$, the initial sign is determined by $\rho\, \partial\widetilde{C}/\partial\lambda\Big|_t$. In the numerical solution, I assumed $\rho < 0$ so that decreases in productivity on average coincide with increases in the price of risk. In that case, the sign will be determined by $-\,\partial\widetilde{C}/\partial\lambda\Big|_t$.

I present the initial partial sensitivity $\partial\widetilde{C}/\partial\lambda\Big|_t$ as a function of price of risk λ for different levels of productivity x in Figure 13. First, consider a firm with no real options (thick black line). For such firm, the partial sensitivity is positive implying that the delta hedged position is likely to have negative returns. When a firm has real options, but the productivity is low (dark blue lines), it is not very different from a firm with no real options as its real option is very far out of the money. However, as productivity increases, the real option moves in the money and significantly changes the sensitivity. At first, the partial sensitivity increases leading to even lower profits. Eventually, when productivity is high enough (yellow lines), the partial sensitivity turns negative for low prices of risk λ. As a result, when price of risk is low and productivity is high, the expected profits of a delta hedged strategies turn positive.

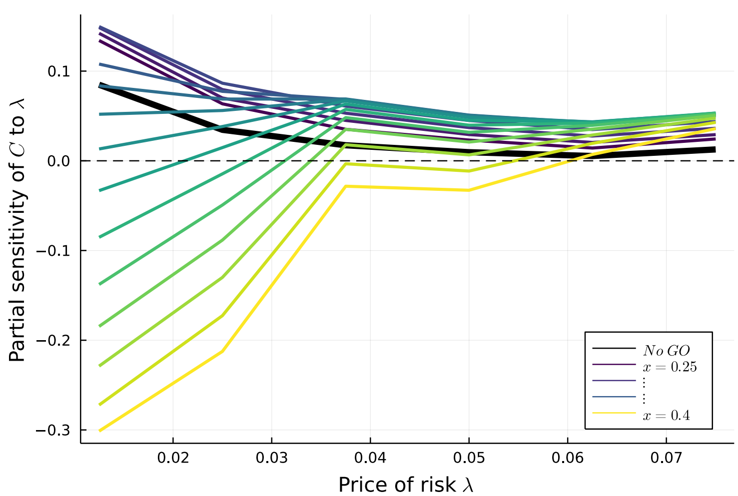


Figure 13: Partial sensitivity of the value of the option to price of risk

To understand the result, it is useful to consider the two offsetting effects that determine the sign of $\partial C/\partial\lambda$. As is mentioned above there is a direct impact from volatility that raises the value of the option. Additionally, a higher price of risk simply lower the valuations of all assets with positive exposure to cash flow shocks such as the call option with a positive delta. When price of risk λ is low, return volatility of a more productive firm is more sensitive to changes in the price of risk than return volatility of a less productive firm as can be observed in Panel (b) of Figure 6. As a result, for low λ and high x, the effect coming through volatility dominates the direct effect that lowers the value of the option.

6.2 Empirical performance of delta hedged trading strategies

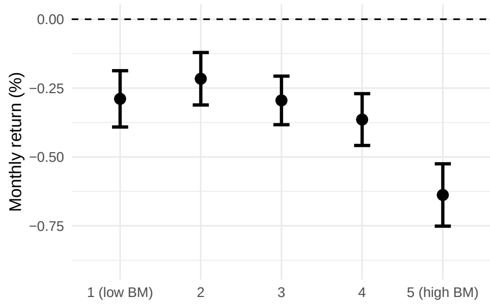
Construction I use Compustat, CRSP, and OptionMetrics as described in Section 3 to construct delta hedged strategies based on firm fundamentals. Since the continuous time model doesn't feature debt, I will only consider companies with leverage below the median value for each month. At the end of each month t, I sort companies by a fundamental characteristic (either profitability or market-to-book). Over the next month, I discretely delta-hedge the option position at the closing price at daily frequency to arrive at the discretized version of equation (34)

$$\widehat{Profit}_{t+1} = \underbrace{C_{t+1} - C_t}_{\Delta\text{call value}} + \underbrace{\sum_{h\in\mathcal{H}(t+1)} \Delta_{h-1}(S_h - S_{h-1})}_{\text{delta-hedge}} - \underbrace{\sum_{h\in\mathcal{H}(t+1)} r\left(C_{h-1} - \Delta_{h-1}S_{h-1}\right)}_{\text{financing costs}} \tag{40}$$

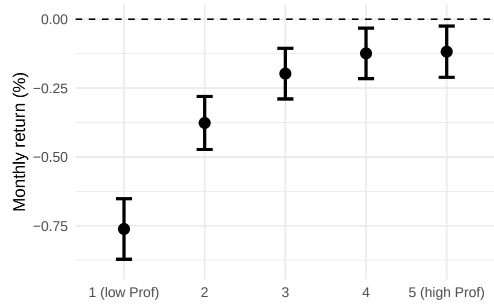
where $\mathcal{H}(t+1)$ is the set of days in month $t+1$, S_h is the spot price at day h, C_h – call price at day h and Δ_{h-1} is the option delta at day h. I construct such P&L for each company with option data available so that $Profit_{i,t+1}$ is firm i specific. While Proposition 1 is not specific to a call or a put position, I use a portfolio of 1 call and 1 put also known as a straddle, to evaluate delta-hedged returns. I choose options with maturity closest to 2-months period and strike closest to at-the-money, i.e. with initial delta closest to 0.5.

I then calculate returns on equal weighted portfolios of straddles for 5 bins based on fundamental characteristics sorts at the end of month t. To normalize the profits of the strategies, I follow the practitioner's approach of normalizing by the spot price at the time of entering the trade so that returns are normalized by a unit of exposure. I present profits of a delta hedged strategy for each bin in Figure 14. Panel (a) presents sorts on book-to-market and Panel (b) sorts on operating profitability.

Average returns are significantly different across bins both for market-to-book and profitability sorts. The average returns on the highest book-to-market bin is negative and significantly lower than on the lowest book-to-market bins. Delta hedged straddles formed on value companies lose significantly more money than straddles formed on the growth companies. In the continuous time model from Section 4, I showed that firms with real options that are more in-the-money, and thus the firms with lower book-to-market as their real options are a larger share of their value, have higher delta-hedged option returns relative to firms with real options that are out-of-the-money –



(a) Book-to-Market sorts



(b) Profitability sorts

Figure 14: Performance of delta hedged strategies based of fundamental sorts

firms with higher book-to-market. The results in Panel (a) confirm this theoretical prediction.

Similarly, in Panel (b) we can observe that straddles formed on high profitability companies lose significantly less money than straddles formed on low profitability firms. In the continuous time model, we observed exactly that. Firms with higher profitability are the ones that are closer to exercising their real options and have higher profits from a delta-hedged option strategy, while low profitability firms have lower returns from the same strategy.

The strategy based on operating profitability sorts is not new and is similar to one of the strategies proposed in Zhan et al. (2022). Instead of a daily delta hedged strategy Zhan et al. (2022) proposed a covered call strategy where they take a delta neutral call position by taking a long position in a call and selling Δ of the underlying without rebalancing the position in the underlying over the holding period of the trade. They report an annualized Sharpe ratio of 2.67 that is higher, but not significantly so, than the Sharpe ratio of 2.30 that I report in Panel (b) of Table 7. At the same time, Panel (a) of Table 7 shows that sorts on book-to-market produce a less impressive Sharpe ratio of 1.04.

Panel (a): sorted on Book-to-Market (BM)

	Low BM	2	3	4	high BM	High - Low
mean (%)	-0.289	-0.216	-0.295	-0.364	-0.638	-0.349
stdev (%)	1.80	1.68	1.56	1.66	2.00	1.16
Sharpe (ann.)	-0.556	-0.446	-0.657	-0.761	-1.11	-1.04

Panel (b): Sorted on Operating Profitability (OP)

	Low OP	2	3	4	High OP	High - Low
mean (%)	-0.762	-0.377	-0.198	-0.124	-0.118	0.644
stdev (%)	1.94	1.69	1.62	1.62	1.64	0.971
Sharpe (ann.)	-1.36	-0.770	-0.422	-0.265	-0.249	2.30

Table 7: Average returns on straddle portfolios formed based on firm fundamentals

7 Conclusion

This paper studies the connection between firm fundamentals and value of equity options both empirically and theoretically. I started by documenting the heterogeneous relationship between a firm's market-to-book and implied volatility skew. If a firm has little leverage, market-to-book is associated with a higher implied volatility skew when the economy is booming. On the other hand, if the economy is in a downturn, the relationship is close to zero. I documented that higher market-to-book in high aggregate states predicts a more negatively skewed return in line with the tight relationship between option prices and return distributions. To rationalize this phenomenon, I developed a real option model where a firm has an opportunity to increase its scale. In the optimum, the firm exercises its real option when its productivity is high and the economy is doing well modeled as a low price of risk. When the firm is close to the exercise boundary, its value becomes more sensitive to the aggregate economic conditions and shocks to its productivity due to a larger "delta" of its real option value to productivity and price of risk shocks. At the same time, the contribution of the real option to the value of the firm is large resulting in a high valuation. Combined these two effects generate a negatively skewed return distribution for the firm thus leading to implied volatility skew.

I then analyzed a fully dynamic model, where in contrast to the stylized model the firm can both invest and disinvest and the real options are modeled with asymmetric investment costs. I showed that the same components that produced a heterogeneous effect of market-to-book on implied volatility skew in the stylized model are necessary to obtain a similar effect in the full dynamic model. Moreover, the model can match the empirical regressions quantitatively. I use the full dynamic model to study firms with high leverage. In the data, larger market-to-book significantly reduces implied volatility skew when the economy is in a downturn. I match this pattern in a model with financial leverage which increases the sensitivity of a firm's value to aggregate shocks particularly when the aggregate state is low.

Finally, I use the stylized model set in continuous time to study the performance of delta hedged equity option strategies based on fundamental sorts and in particular, profitability and market-to-book sorts. The existing literature showed that the performance of such strategies will depend on variance risk premium. While my model doesn't feature it explicitly, variation in the price of risk generates variation in return volatility thus generating implicit variance risk premium. The differential exposure of firms with high and low market-to-book to this implicit VRP drives variation in the average profits of delta hedged option strategies.

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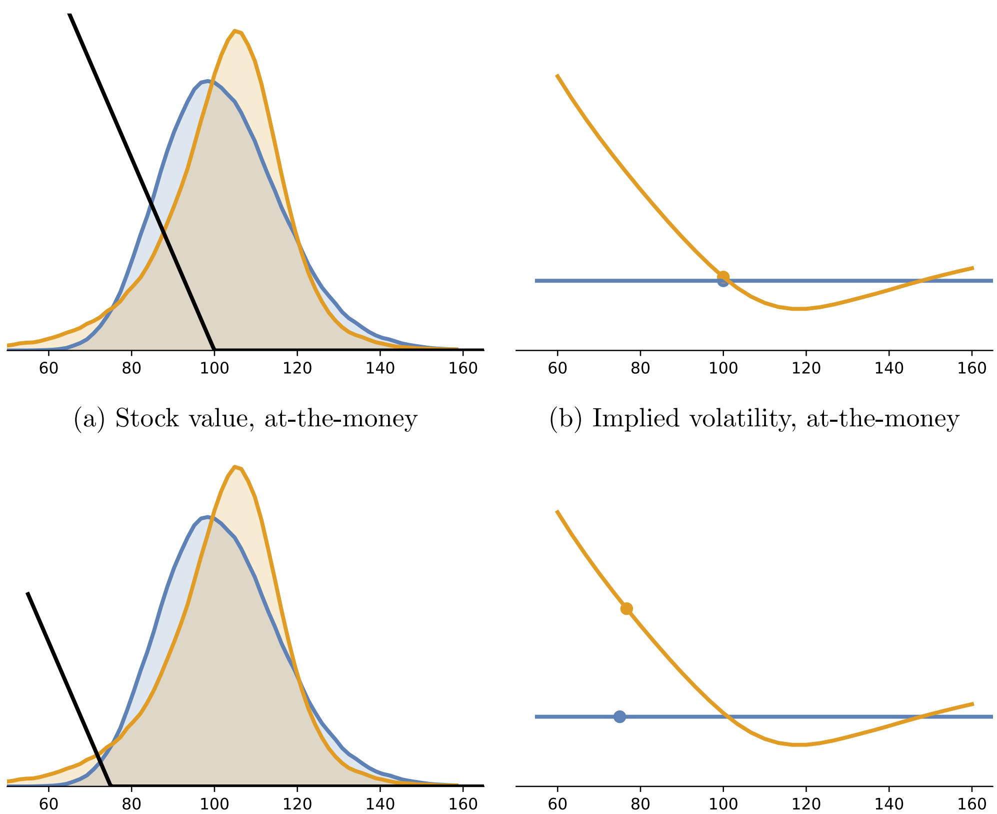
A Appendix: Derivations and Proofs

A.1 Return Distributions and Implied Volatilities

Figure A.1 presents the connection between stock distribution and implied volatility as an example of more general result in Breeden and Litzenberger (1978). Panels (a) and (c) show the distribution of the terminal value of a stock. The blue distribution is lognormal in accordance with the model in Black and Scholes (1973) and Merton (1973). For a lognormal distribution, regardless of the strike of an option, payout function of which shown as the black line, the implied volatility obtained by inverting the option prices with the Black-Scholes-Merton equation is constant and is equal to the true volatility of the lognormal distribution. This is presented as the blue lines in panels (b) and (d).

In contrast, the orange distribution is generated under a model with stochastic volatility, where the level of volatility increases when the spot price declines. This results in a negatively skewed distribution for the terminal distribution of the stock. As a result, the implied volatility shown in panels (b) and (d) is not constant as the function of the strikes. In particular, for out-of-the-money put options with strikes below the current spot price of 100, implied volatility is higher than under the log-normal benchmark despite being the same when the strike is at the current spot price.



(a) Stock value, at-the-money (b) Implied volatility, at-the-money

(c) Stock value, out-of-the-money (d) Implied volatility, out-of-the-money

Figure A.1: Distributions and implied volatility

A.2 Finite Differences Method for Solving the Continuous Time Model

Before I derive the value for the mature firm, I consider the adjustment to the dynamics of state variables x and λ due to systematic risk, i.e. I derive the risk-neutral dynamics of the state that will simply further derivations. The value of stochastic V_{t+dt} at time t can is

$$\mathcal{P}(V) = E_t\left[\left(1+\frac{d\pi}{\pi}\right)(V+dV)\right]$$

Expanding we get

$$(1-rdt)\,V + E_t\left[\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial V}{\partial \lambda}d\lambda + \frac{1}{2}\frac{\partial^2 V}{\partial \lambda^2}(d\lambda)^2 + \frac{\partial^2 V}{\partial x\partial \lambda}(dx)(d\lambda) + \frac{\partial V}{\partial x}\frac{d\pi}{\pi}dx + \frac{\partial V}{\partial \lambda}\frac{d\pi}{\pi}d\lambda\right]$$

Collecting terms with $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial \lambda}$ we get

$$(1-rdt)\,V + E_t\left[\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}\left(dx + \frac{d\pi}{\pi}dx\right) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial V}{\partial \lambda}\left(d\lambda + \frac{d\pi}{\pi}d\lambda\right) + \frac{1}{2}\frac{\partial^2 V}{\partial \lambda^2}(d\lambda)^2 + \frac{\partial^2 V}{\partial x\partial \lambda}(dx)(d\lambda)\right].$$

The terms that multiply $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial \lambda}$ are

$$\begin{aligned}
dx + \frac{d\pi}{\pi}dx &= x\mu_x dt + x\sigma_x dB_x - xb\sqrt{\lambda}\sigma_x^2 dt = x(\mu_x - b\sqrt{\lambda}\sigma_x^2)dt + x\sigma_x dB_x \\
d\lambda + \frac{d\pi}{\pi}d\lambda &= \kappa(\bar{\lambda}-\lambda)dt + \sigma_\lambda\sqrt{\lambda}dB_\lambda - b\sigma_x\sigma_\lambda\lambda\rho dt = (\kappa + b\sigma_x\sigma_\lambda\rho)\left(\frac{\kappa}{\kappa + b\sigma_x\sigma_\lambda\rho} - \lambda\right)dt + \sigma_\lambda\sqrt{\lambda}dB_\lambda
\end{aligned}$$

Therefore, we can write the risk-neutral dynamics for x and λ as

$$\begin{aligned}
dx &= x\mu_x^Q + x\sigma_x dB_x^{\mathbb{Q}} \\
d\lambda &= \kappa^Q(\bar{\lambda}^Q - \lambda)dt + \sigma_\lambda\sqrt{\lambda}dB_\lambda^{\mathbb{Q}}
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
\mu_x^Q &\equiv \mu_x - b\sqrt{\lambda}\sigma_x^2 \\
\kappa^Q &\equiv \kappa + b\sigma_x\sigma_\lambda\rho \\
\bar{\lambda}^Q &\equiv \frac{\kappa}{\kappa + b\sigma_x\sigma_\lambda\rho}\bar{\lambda}
\end{aligned} \tag{A.2}$$

I will use the risk-neutral dynamics to value the mature and young firms going forward.

A.2.1 Value of the mature firm

To derive the value of the mature firm, I first expand the Bellman equation for the value of the mature firm from equation (15) allowing for time dependency of the value function. I price the firm under the risk-neutral measure from equations (A.1) and (A.2):

$$\begin{aligned}
V =& xK^\alpha dt + E^Q\left[(1-rdt)(V+dV)\right] \\
=& xK^\alpha dt + E^Q\left[V - rVdt + \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial V}{\partial \lambda}d\lambda + \frac{1}{2}\frac{\partial^2 V}{\partial \lambda^2}(d\lambda)^2 + \frac{\partial^2 V}{\partial x\partial \lambda}(dx)(d\lambda)\right].
\end{aligned}$$

Cancelling terms and rearranging we get

$$-\frac{\partial V}{\partial t}dt + rVdt = xK^{\alpha}dt + E^Q\left[\frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial V}{\partial \lambda}d\lambda + \frac{1}{2}\frac{\partial^2 V}{\partial \lambda^2}(d\lambda)^2 + \frac{\partial^2 V}{\partial x \partial \lambda}(dx)(d\lambda)\right].$$

Expanding the processes for x and λ and cancelling dt we get

$$\begin{aligned}
-\frac{\partial V}{\partial t}dt + rVdt &= xK^{\alpha}dt + \frac{\partial V}{\partial x}x\mu_x^Q dt + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}x^2\sigma_x^2 dt + \frac{\partial V}{\partial \lambda}\kappa^Q(\bar{\lambda}^Q - \lambda)dt + \frac{1}{2}\frac{\partial^2 V}{\partial \lambda^2}\sigma_\lambda^2\lambda dt + \frac{\partial^2 V}{\partial x \partial \lambda}\sigma_x x \sigma_\lambda \rho\sqrt{\lambda}dt,\\
-\frac{\partial V}{\partial t} + rV &= xK^{\alpha} + \frac{\partial V}{\partial x}x\mu_x^Q + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}x^2\sigma_x^2 + \frac{\partial V}{\partial \lambda}\kappa^Q(\bar{\lambda}^Q - \lambda) + \frac{1}{2}\frac{\partial^2 V}{\partial \lambda^2}\sigma_\lambda^2\lambda + \frac{\partial^2 V}{\partial x \partial \lambda}\sigma_x x \sigma_\lambda \rho\sqrt{\lambda}.
\end{aligned}$$

Conjecture a solution $V(x, \lambda) = xv(\lambda)$:

$$-x\frac{\partial v}{\partial t} + rxv = xK^{\alpha} + vx\mu_x^Q + xv'\kappa^Q(\bar{\lambda}^Q - \lambda) + \frac{1}{2}xv''\sigma_\lambda^2\lambda + v'\sigma_x x\sigma_\lambda\rho\sqrt{\lambda}.$$

where we can cancel the x thus verifying the conjecture

$$-\frac{\partial v}{\partial t} + rv = K^{\alpha} + v\mu_x^Q + v'\kappa^Q(\bar{\lambda}^Q - \lambda) + \frac{1}{2}v''\sigma_\lambda^2\lambda + v'\sigma_x\sigma_\lambda\rho\sqrt{\lambda}. \tag{A.3}$$

To solve the equation for the value of the firm that is determined by setting $\partial v/\partial t = 0$ in equation (A.3), I discretize price of risk λ on an equidistant grid $\{\lambda_1, \ldots, \lambda_J\}$ with step $\Delta\lambda$. The value is $\{v_1, \ldots, v_J\}$, accordingly. I then discretize the derivatives in each point on the grid as

$$\begin{aligned}
(v')_j &\approx \lambda_j^+\frac{v_{j+1} - v_j}{\Delta\lambda} + (1 - \lambda_j^+)\frac{v_j - v_{j-1}}{\Delta\lambda}\\
(v'')_j &\approx \frac{v_{j+1} - 2v_j + v_{j-1}}{(\Delta\lambda)^2}
\end{aligned}$$

where $\lambda_j^+ \equiv \mathbb{I}_{\{\lambda \leq \bar{\lambda}^Q\}}$ is the indicator for the upwind discretization scheme. Under upwind scheme, the derivative is taken in the direction of the drift of a state variable allowing for better stability of the solution (Achdou et al., 2022). Substituting these into equation A.3 we get a system of equation $j = 1, \ldots, J$ with one equation for each point on the λ-grid

$$\begin{aligned}
-\left(\frac{\partial v}{\partial t}\right)_j + rv_j = &K^{\alpha} + v_j\mu_x^Q + \left[\lambda_j^+\frac{v_{j+1} - v_j}{\Delta\lambda} + (1 - \lambda_j^+)\frac{v_j - v_{j-1}}{\Delta\lambda}\right]\left[\kappa^Q(\bar{\lambda}^Q - \lambda_j) + \sigma_x\sigma_\lambda\rho\sqrt{\lambda}\right]\\
&+ \frac{1}{2}\frac{v_{j+1} - 2v_j + v_{j-1}}{(\Delta\lambda)^2}\sigma_\lambda^2\lambda_j.
\end{aligned}$$

Collect terms for the same indices

$$\begin{aligned}
-\left(\frac{\partial v}{\partial t}\right)_j + rv_j = K^{\alpha} &+ v_{j-1}\left[-\frac{1 - \lambda_j^+}{\Delta\lambda}\left[\kappa^Q(\bar{\lambda}^Q - \lambda_j) + \sigma_x\sigma_\lambda\rho\sqrt{\lambda_j}\right] + \frac{\sigma_\lambda^2\lambda_j}{2(\Delta\lambda)^2}\right]\\
&+ v_j\left[\mu_x^Q + \left(\frac{-\lambda_j^+ + (1 - \lambda_j^+)}{\Delta\lambda}\right)\left[\kappa^Q(\bar{\lambda}^Q - \lambda_j) + \sigma_x\sigma_\lambda\rho\sqrt{\lambda_j}\right] - \frac{1}{(\Delta\lambda)^2}\sigma_\lambda^2\lambda_j\right]\\
&+ v_{j+1}\left[\frac{\lambda_j^+}{\Delta\lambda}\left[\kappa^Q(\bar{\lambda}^Q - \lambda_j) + \sigma_x\sigma_\lambda\rho\sqrt{\lambda_j}\right] + \frac{1}{2(\Delta\lambda)^2}\sigma_\lambda^2\lambda_j\right]
\end{aligned}$$

Replacing terms we can write the system more compactly as

$$-\left(\frac{\partial v}{\partial t}\right)_j + rv_j = K^{\alpha} + a_jv_{j-1} + b_jv_j + c_jv_{j+1}$$

Finally, we need to specify the timing. Since v's are next "period" values, we will designate them an index $v^{(t+1)}$ so that the equation becomes

$$-\frac{v_j^{(t+1)} - v_j^{(t)}}{\Delta t} + r v_j^{(t+1)} = K^\alpha + a_j v_{j-1}^{(t+1)} + b_j v_j^{(t+1)} + c_j v_{j+1}^{(t+1)}. \tag{A.4}$$

Stacking all equations together we can write the system that we will iterate at until convergence as

$$-\frac{1}{\Delta t}(v^{(t+1)} - v^{(t)}) + r v^{(t+1)} = K^\alpha + \mathbf{A}^Q v^{(t+1)} \tag{A.5}$$

or in the form amenable for iteration as

$$v^{(t)} = (1 - r\Delta t) v^{(t+1)} + K^\alpha \Delta t + \Delta t \mathbf{A}^Q v^{(t+1)} \tag{A.6}$$

where $\mathbf{A}^Q$ is the risk-neutral transition matrix

$$\mathbf{A}^Q \equiv \begin{pmatrix} a_1 + b_1 & c_1 & 0 & 0 & \dots & 0 \\ a_2 & b_2 & c_2 & 0 & \dots & 0 \\ 0 & a_3 & b_3 & c_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & 0 \\ 0 & 0 & 0 & a_{J-1} & b_{J-1} & c_{J-1} \\ 0 & 0 & 0 & 0 & a_J & b_J + c_J \end{pmatrix} \tag{A.7}$$

Note that in the first and last element on the diagonal we added the boundaries that go outside the state space: a_1 that should multiply v_0 and c_J that should multiply v_{J+1}. This imposes a reflective barrier boundary condition. Additionally, note that duer to upwind scheme the only terms that will be used for calculating a_1 and c_J are the diffusive terms and not the terms related to the drift.

A.2.2 Value of a young firm

Unlike for mature firm, we can't conjecture a simpler solution $xv(\lambda)$. To derive the transition matrix I, therefore, discretize the transition part, defined by operator $\mathcal{B}^Q(\cdot)$

$$\mathcal{B}^Q(V) \equiv \frac{\partial V}{\partial x} x \mu_x^Q + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} x^2 \sigma_x^2 + \frac{\partial V}{\partial \lambda} \kappa^Q (\bar{\lambda}^Q - \lambda) + \frac{1}{2}\frac{\partial^2 V}{\partial \lambda^2} \sigma_\lambda^2 \lambda + \frac{\partial^2 V}{\partial x \partial \lambda} \sigma_x x \sigma_\lambda \rho \sqrt{\lambda} \tag{A.8}$$

directly by additionally discretizing productivity x on an equidistant grid $\{x_1, \dots, x_I\}$ with step Δx and defining the value of a young firm for state (x_i, λ_j) as $V_{i,j}$. Similarly to the mature firm I use upwind scheme for both x and λ. To simplify notation denote the indicators for using the forward derivative as $x_j^+ \equiv \mathbb{I}_{\{\mu^Q_{x,j}\geq 0\}}$ and $\lambda_j^+ \equiv \mathbb{I}_{\{\lambda_j \leq \bar{\lambda}^Q\}}$. With this notation

$$\begin{aligned}
\left(\frac{\partial V}{\partial x}\right)_{i,j} &\approx x_j^+ \frac{V_{i+1,j}-V_{i,j}}{\Delta x} + (1-x_j^+)\frac{V_{i,j}-V_{i-1,j}}{\Delta x} \\
\left(\frac{\partial^2 V}{\partial x^2}\right)_{i,j} &\approx \frac{V_{i+1,j}-2V_{i,j}+V_{i-1,j}}{(\Delta x)^2} \\
\left(\frac{\partial V}{\partial \lambda}\right)_{i,j} &\approx \lambda_j^+ \frac{V_{i,j+1}-V_{i,j}}{\Delta \lambda} + (1-\lambda_j^+)\frac{V_{i,j}-V_{i,j-1}}{\Delta \lambda} \\
\left(\frac{\partial^2 V}{\partial \lambda^2}\right)_{i,j} &\approx \frac{V_{i,j+1}-2V_{i,j}+V_{i,j-1}}{(\Delta \lambda)^2}
\end{aligned}$$

The cross term features upwind scheme for both x and λ

$$\begin{aligned}
\left(\frac{\partial^2 V}{\partial x \partial \lambda}\right)_{i,j} \approx\ & x_j^+\lambda_j^+ \frac{1}{\Delta\lambda}\left(\frac{V_{i+1,j+1}-V_{i,j+1}}{\Delta x} - \frac{V_{i+1,j}-V_{i,j}}{\Delta x}\right) \\
&+ (1-x_j^+)\lambda_j^+ \frac{1}{\Delta\lambda}\left(\frac{V_{i,j+1}-V_{i-1,j+1}}{\Delta x} - \frac{V_{i,j}-V_{i-1,j}}{\Delta x}\right) \\
&+ x_j^+(1-\lambda_j^+) \frac{1}{\Delta\lambda}\left(\frac{V_{i+1,j}-V_{i,j}}{\Delta x} - \frac{V_{i+1,j-1}-V_{i,j-1}}{\Delta x}\right) \\
&+ (1-x_j^+)(1-\lambda_j^+) \frac{1}{\Delta\lambda}\left(\frac{V_{i,j}-V_{i-1,j}}{\Delta x} - \frac{V_{i,j-1}-V_{i-1,j-1}}{\Delta x}\right)
\end{aligned}$$

I then substitute the discretized derivatives into the transition dynamics in equation (A.8) and collect the terms

$$V_{i-1,j-1}:\ \sqrt{\lambda_j} x_i \sigma_x \sigma_\lambda \rho \frac{(1-x_j^+)(1-\lambda_j^+)}{\Delta x \Delta \lambda}$$

$$V_{i-1,j}:\ -\frac{1-x_j^+}{\Delta x} x_i \mu^Q_{x,j} + \frac{x_i^2\sigma_x^2}{2(\Delta x)^2} + x_i\sqrt{\lambda_j}\sigma_x\sigma_\lambda\rho\left[\frac{(1-x_j^+)\lambda_j^+}{\Delta x\Delta\lambda} - \frac{(1-x_j^+)(1-\lambda_j^+)}{\Delta x\Delta\lambda}\right]$$

$$V_{i,j-1}:\ -\frac{1-\lambda_j^+}{\Delta\lambda}\kappa^Q(\bar{\lambda}^Q-\lambda_j) + \frac{\sigma_\lambda^2\lambda_j}{2(\Delta\lambda)^2} + x_i\sqrt{\lambda_j}\sigma_x\sigma_\lambda\rho\left[\frac{x_j^+(1-\lambda_j^+)}{\Delta\lambda\Delta x} - \frac{(1-x_j^+)(1-\lambda_j^+)}{\Delta\lambda\Delta x}\right]$$

$$\begin{aligned}
V_{i,j}:\ & x_i\mu^Q_{x,j}\left(-\frac{x_j^+}{\Delta x} + \frac{1-x_j^+}{\Delta x}\right) - \frac{x_i^2\sigma_x^2}{(\Delta x)^2} + \kappa^Q(\bar{\lambda}^Q-\lambda_j)\left(-\frac{\lambda_j^+}{\Delta\lambda} + \frac{1-\lambda_j^+}{\Delta\lambda}\right) - \frac{\lambda_j\sigma_\lambda^2}{(\Delta\lambda)^2} \\
&+ \frac{x_i\sqrt{\lambda_j}\sigma_x\sigma_\lambda\rho}{\Delta\lambda\Delta x}\left(x_j^+\lambda_j^+ - (1-x_j^+)\lambda_j^+ - x_j^+(1-\lambda_j^+) + (1-x_j^+)(1-\lambda_j^+)\right)
\end{aligned}$$

$$V_{i,j+1}:\ \kappa^Q(\bar{\lambda}^Q-\lambda_j)\frac{\lambda_j^+}{\Delta\lambda} + \frac{\sigma_\lambda\lambda_j^2}{2(\Delta\lambda)^2} + x_i\sqrt{\lambda_j}\sigma_x\sigma_\lambda\rho\left[-\frac{x_j^+\lambda_j^+}{\Delta\lambda\Delta x} + \frac{(1-x_j^+)\lambda_j^+}{\Delta\lambda\Delta x}\right]$$

$$V_{i+1,j}:\ x_i\mu^Q_{x,j}\frac{x_j^+}{\Delta x} + \frac{x_i^2\sigma_x^2}{2(\Delta x)^2} + x_i\sqrt{\lambda_j}\sigma_x\sigma_\lambda\rho\left[-\frac{x_j^+\lambda_j^+}{\Delta\lambda\Delta x} + \frac{x_j^+(1-\lambda_j^+)}{\Delta\lambda\Delta x}\right]$$

$$V_{i+1,j+1}:\ x_i\sqrt{\lambda_j}\sigma_x\sigma_\lambda\rho\frac{x_j^+\lambda_j^+}{\Delta\lambda\Delta x}$$

$$V_{i+1,j-1}:\ -x_i\sqrt{\lambda_j}\sigma_x\sigma_\lambda\rho\frac{x_j^+(1-\lambda_j^+)}{\Delta\lambda\Delta x}$$

$$V_{i-1,j+1}: \ -x_i\sqrt{\lambda_j}\sigma_x\sigma_\lambda\rho\frac{(1-x_j^+)\lambda_j^+}{\Delta\lambda\Delta x}$$

With these terms we can rewrite the transition part of the Bellman equation (A.8) as

$$a_{i,j}V_{i-1,j}+b_{i,j}V_{i,j}+c_{i,j}V_{i+1,j}+f_{i,j}V_{i-1,j-1}+g_{i,j}V_{i,j-1}+h_{i,j}V_{i+1,j-1}+m_{i,j}V_{i-1,j+1}+n_{ij}V_{i,j+1}+p_{ij}V_{i+1,j+1},$$

that can be combined in the matrix form as

$$\mathbf{B}^Q V$$

where the transition matrix $\mathbf{B}^Q$ is

$$\mathbf{B}^Q \equiv \begin{pmatrix} \mathbf{K}_1+\mathbf{L}_1 & \mathbf{M}_1 & 0 & 0 & & \\ \mathbf{K}_2 & \mathbf{L}_2 & \mathbf{M}_2 & 0 & & \\ 0 & \mathbf{K}_3 & \mathbf{L}_3 & \mathbf{M}_3 & 0 & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & \mathbf{K}_{J-1} & \mathbf{L}_{J-1} & \mathbf{M}_{J-1} \\ & & & & \mathbf{K}_J & \mathbf{L}_J+\mathbf{M}_J \end{pmatrix}$$

and matrices $\mathbf{K}_j$, $\mathbf{L}_j$ and $\mathbf{M}_j$ are defined as

$$\mathbf{K}_j \equiv \begin{pmatrix} f_{1,j}+g_{1,j} & h_{1,j} & 0 & 0 & \dots & 0 \\ f_{2,j} & g_{2,j} & h_{2,j} & 0 & \dots & 0 \\ 0 & f_{3,j} & g_{3,j} & h_{3,j} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & 0 \\ 0 & 0 & 0 & f_{I-1,j} & g_{I-1,j} & h_{I-1,j} \\ 0 & 0 & 0 & 0 & f_{I,j} & g_{I,j}+h_{I,j} \end{pmatrix},$$

$$\mathbf{L}_j \equiv \begin{pmatrix} a_{1,j}+b_{1,j} & c_{1,j} & 0 & 0 & \dots & 0 \\ a_{2,j} & b_{2,j} & c_{2,j} & 0 & \dots & 0 \\ 0 & a_{3,j} & b_{3,j} & c_{3,j} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & 0 \\ 0 & 0 & 0 & a_{I-1,j} & b_{I-1,j} & c_{I-1,j} \\ 0 & 0 & 0 & 0 & a_{I,j} & b_{I,j}+c_{I,j} \end{pmatrix}, \text{ and}$$

$$\mathbf{L}_j \equiv \begin{pmatrix} m_{1,j}+n_{1,j} & p_{1,j} & 0 & 0 & \dots & 0 \\ m_{2,j} & n_{2,j} & p_{2,j} & 0 & \dots & 0 \\ 0 & m_{3,j} & n_{3,j} & p_{3,j} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & 0 \\ 0 & 0 & 0 & m_{I-1,j} & n_{I-1,j} & p_{I-1,j} \\ 0 & 0 & 0 & 0 & m_{I,j} & n_{I,j}+p_{I,j} \end{pmatrix}.$$

The non-discretized version of a young firm's problem is

$$V = \max\{U - (K_1 - K_0), xK_0^\alpha dt + (1 - rdt)V + \frac{\partial V}{\partial t}dt + \mathcal{B}^Q(V)dt\}$$

where U and V the values of mature and young firms, respectively, and are functions of x and λ. This expression can be discretized as

$$V^{(t)} = \max\{U - (K_1 - K_0), xK_0^\alpha + (1 - r\Delta t)V^{(t+1)} + \mathbf{B}^Q V^{(t+1)}\Delta t\}, \tag{A.9}$$

where U and V are now the discretized values over the grids for x and λ to match matrix $\mathbf{B}^Q$.

$$U \equiv (U_{1,1}, \dots, U_{I,1}, U_{1,2}, \dots, U_{I,2}, \dots)^\top$$
$$V \equiv (V_{1,1}, \dots, V_{I,1}, V_{1,2}, \dots, V_{I,2}, \dots)^\top$$

Equation A.9 is iterated until convergence (until the difference between $V^{(t+1)}$ and $V^{(t)}$ becomes small according to, e.g., sup-norm). At each iteration, the algorithm will compare the values of maturing or staying young thus calculating the discretized exercise boundary $\bar{x}(\lambda_j)$.

A.3 Deriving terminal value distribution

The connvenient property of the discretization approach presented in the previous section is that the transition dynamics can be easily calculated once $\mathbf{A}^Q$ and $\mathbf{B}^Q$ are calculated. In particular, the transitional matrix dynamics will be given by adding an identity matrix to the transpose of the corresponding matrix. Suppose that at time t the distribution of states is $\xi^{(t)}$ meaning that all entries of $\xi^{(t)}$ are non-negative and sum to one. In the next discretized instant $t + \Delta t$ the distribution of states will be

$$\xi^{(t+\Delta t)} = \left(I + A^Q \Delta t\right)^\top \xi^{(t)}. \tag{A.10}$$

To calculate value of an option we need calculate the distribution of states at some future time $t + \tau$ starting from a particular initial (x, λ) at t. To achieve this, we start with $\xi^{(t)}$ that has only one non-zero entry equal to one at the position of the initial state. We then iterate equation (A.10) until time reaches $t + \tau$

When the firm doesn't have a choice to expand solving for the terminal distribution of states is straightforward. However, as outlined in the main text, real options generate additional complexity.

Before proceeding further, define the following variables. Let ξ denote the distribution over young firms and χ the dilution adjusted distribution of mature firms. This vector shows how much dilution the given initial stake in a firm will give the investor in each particular state. At initial point when the firm is young and is away from the exercise boundary, χ is a vector of zeros and ξ is zero everywhere except one place that indicates the initial state (x, λ). Define vector $\mathbf{I}_E$ that is equal to one in for states where the firm exercises its real option, $x_i \geq \bar{x}(\lambda_j)$, and equal to zero otherwise. Define vector η as the dilution coefficient

$$\eta \equiv \frac{U - (K_1 - K_0)}{U}.$$

Suppose that at some point in time the distribution of the states of young firms is ξ while the distribution of states of mature firms χ is zero everywhere. If after transition to the next time period, there are some firms that are pushed over the boundary of exercising their real options, the distribution of states under which they transitioned will be determined by

$$\left(I + B^Q \Delta t\right)^\top \xi^{(t)} \star \mathbf{I}_E$$

where $\star$ denotes element-wise multiplication. Accordingly, the sum of the elements in this vector is equal to the total mass of firms that transitions to maturity.

If we are holding an option we are not interested in the total mass of firms that transitioned to maturity, rather by the dilution adjusted mass of firms. The distribution of dilution adjusted transition to maturity is

$$\left(I + B^Q \Delta t\right)^\top \xi^{(t)} \star \mathbf{I}_E \star \eta.$$

This vector determins the dilution adjusted inflow into the dilution adjusted distribution of mature firms. In addition, mature firms also have transition dynamics determined by $\mathbf{A}^Q$. As a result, the dynamics of the distribution of states will be determined by the

$$\begin{cases} \xi^{(t+\Delta t)} = \underbrace{(\mathbf{I} + \mathbf{B}^Q \Delta t)^\top \xi^{(t)}}_{\text{Young transition}} - \underbrace{\left(\mathbf{I} + \mathbf{B}^Q \Delta t\right)^\top \xi^{(t)} \star \mathbf{I}_E}_{\text{Outflow to maturity}} \\ \chi^{(t+\Delta t)} = \underbrace{(\mathbf{I} + \mathbf{A}^Q \Delta t)^\top \chi^{(t)}}_{\text{Mature transition}} + \underbrace{\left(\mathbf{I} + \mathbf{B}^Q \Delta t\right)^\top \xi^{(t)} \star \mathbf{I}_E \star \eta}_{\text{Diluted inflow to maturity}} \end{cases} \tag{A.11}$$

After we iterated equations (A.11) until $t + \tau$ the distribution of the terminal value will be determined by the combined distribution $\xi^{(t+\tau)}$ over V and distribution $\chi^{(t+\tau)}$ over U. The value of a forward contract is

$$F = \left(\xi^{(t+\tau)}\right)^\top V + \left(\chi^{(t+\tau)}\right)^\top U \tag{A.12}$$

and the value of a call option with strike K is

$$C = e^{-r\tau}\left[\left(\xi^{(t+\tau)}\right)^{\top}(V-K)^{+} + \left(\chi^{(t+\tau)}\right)^{\top}(U-K)^{+}\right] \tag{A.13}$$

where $(\cdot)^{+}$ denotes the positive part of the expression in parentheses.

A.4 Profits of a Delta Hedged Strategy

This section proves proposition 1. Expected profits from a continuously delta hedged long option position

$$Profit_{t,t+T} = \underbrace{C_{t+T} - C_t}_{\Delta\text{call value}} - \underbrace{\int_t^{t+T} \Delta_u dF_u}_{\text{delta-hedge}} - \underbrace{\int_t^{t+T} r\left(C_u - \Delta_u F_u\right) du}_{\text{financing costs}} \tag{A.14}$$

are

$$E[Profit] = \int_t^{t+\tau} \sigma_x\sigma_\lambda b\rho E_t\left[\frac{\partial\widetilde{C}}{\partial\lambda}\lambda_u\right] du, \tag{A.15}$$

where

$$\frac{\partial\widetilde{C}}{\partial\lambda} \equiv \frac{\partial C}{\partial\lambda} - \frac{\partial F/\partial\lambda}{\partial F/\partial x}\cdot\frac{\partial C}{\partial F}. \tag{A.16}$$

Proof. First, consider an arbitrary asset V within the same economy with state variables (x,λ). This asset has (possibly stochastic) dividend flow $D_t dt$ per interval of time dt. By the fundamental theorem of asset pricing, the discounted expected return is

$$1 = E_t\left[\frac{\pi_t + d\pi_t}{\pi_t}\frac{V_t + dV_t + D_t dt}{V_t}\right]. \tag{A.17}$$

Multiply both sides by $\pi_t V_t$ and expand dV_t using Ito's lemma

$$\begin{aligned}\pi_t V_t =& E_t\left[(\pi_t + d\pi)\left(V + \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial V}{\partial\lambda}d\lambda + \frac{1}{2}\frac{\partial^2 V}{\partial\lambda^2}(d\lambda)^2 + \frac{\partial^2 V}{\partial x\partial\lambda}(dx)(d\lambda) + D_t dt\right)\right]\\ =& E_t\left[\pi_t V_t + \pi\frac{\partial V}{\partial t}dt + \pi\frac{\partial V}{\partial x}dx + \pi\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \pi\frac{\partial V}{\partial\lambda}d\lambda + \pi\frac{1}{2}\frac{\partial^2 V}{\partial\lambda^2}(d\lambda)^2 + \pi\frac{\partial^2 V}{\partial x\partial\lambda}(dx)(d\lambda) + \pi D_t dt\right]\\ &+ E_t\left[V d\pi + \frac{\partial V}{\partial x}dx d\pi + \frac{\partial V}{\partial\lambda}d\lambda d\pi\right].\end{aligned} \tag{A.18}$$

Cancel $\pi_t V_t$ and divide through by π_t to obtain

$$\begin{aligned}0 = E_t&\left[\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial V}{\partial\lambda}d\lambda + \frac{1}{2}\frac{\partial^2 V}{\partial\lambda^2}(d\lambda)^2 + \frac{\partial^2 V}{\partial x\partial\lambda}(dx)(d\lambda) + D_t dt\right]\\ &+E_t\left[V\frac{d\pi}{\pi} + \frac{\partial V}{\partial x}dx\frac{d\pi}{\pi} + \frac{\partial V}{\partial\lambda}d\lambda\frac{d\pi}{\pi}\right].\end{aligned} \tag{A.19}$$

Rearrange to express the second derivatives

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + &\frac{1}{2}\frac{\partial^2 V}{\partial\lambda^2}(d\lambda)^2 + \frac{\partial^2 V}{\partial x\partial\lambda}(dx)(d\lambda)\\ &= -E_t\left[\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial\lambda}d\lambda + D_t dt + V\frac{d\pi}{\pi} + \frac{\partial V}{\partial x}dx\frac{d\pi}{\pi} + \frac{\partial V}{\partial\lambda}d\lambda\frac{d\pi}{\pi}\right].\end{aligned} \tag{A.20}$$

I will use this equation to substitute into the law of motion for the option.

Consider the first part of the profits in equation (A.14)

$$C_{t+\tau} - C_t = \int_t^{t+\tau} \frac{\partial C}{\partial t} du + \frac{\partial C}{\partial x} dx + \frac{\partial C}{\partial \lambda} d\lambda + \int_t^{t+\tau} \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(dx)^2 + \frac{1}{2}\frac{\partial^2 C}{\partial \lambda^2}(d\lambda)^2 + \frac{\partial^2 C}{\partial x \partial \lambda}(dx)(d\lambda). \quad \text{(A.21)}$$

Use equation (A.20) for the call option $V = C$ with a zero dividend flow of $D_t = 0 \ \forall \ t$

$$\begin{aligned} C_{t+\tau} - C_t = & \int_t^{t+\tau} \frac{\partial C}{\partial t} du + \frac{\partial C}{\partial x} dx + \frac{\partial C}{\partial \lambda} d\lambda \\ & - \int_t^{t+\tau} E_u \left[\frac{\partial C}{\partial t} du + \frac{\partial C}{\partial x} dx + \frac{\partial C}{\partial \lambda} d\lambda + C\frac{d\pi}{\pi} + \frac{\partial C}{\partial x} dx \frac{d\pi}{\pi} + \frac{\partial C}{\partial \lambda} d\lambda \frac{d\pi}{\pi} \right]. \end{aligned} \quad \text{(A.22)}$$

Cancel terms and rearrange this expression to obtain

$$\begin{aligned} C_{t+\tau} - C_t = & \int_t^{t+\tau} \frac{\partial C}{\partial x} \left[dx - E_t[dx] - \left(\frac{d\pi}{\pi}\right) dx \right] \\ & + \int_t^{t+\tau} \frac{\partial C}{\partial \lambda} \left[d\lambda - E_t[d\lambda] - \left(\frac{d\pi}{\pi}\right) d\lambda \right] - \int_t^{t+\tau} C_u E_t \left[\frac{d\pi}{\pi} \right]. \end{aligned} \quad \text{(A.23)}$$

I now move to the second part of delta hedged profits in equation (A.14). Expand the forward dynamics as

$$\int_t^{t+\tau} \Delta_u dF_u = \int_t^{t+\tau} \Delta_u \left[\frac{\partial F}{\partial t} du + \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial \lambda} d\lambda + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial \lambda^2}(d\lambda)^2 + \frac{\partial^2 F}{\partial x \partial \lambda}(d\lambda)(dx) \right]. \quad \text{(A.24)}$$

Use equation (A.20) to substitute the second order terms for $V = F$ with zero dividend flow

$$\begin{aligned} \int_t^{t+\tau} \Delta_u dF_u = & \int_t^{t+\tau} \Delta_u \left[\frac{\partial F}{\partial t} du + \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial \lambda} d\lambda \right] \\ & - \int_t^{t+\tau} \Delta_u E_u \left[\frac{\partial F}{\partial t} du + \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial \lambda} d\lambda + F\frac{d\pi}{\pi} + \frac{\partial F}{\partial x}\frac{d\pi}{\pi} dx + \frac{\partial F}{\partial \lambda}\frac{d\pi}{\pi} d\lambda \right] \\ = & \int_t^{t+\tau} \Delta_u \left[\frac{\partial F}{\partial x} \left(dx - E_u[dx] - \frac{d\pi}{\pi} dx \right) \right] \\ & + \int_t^{t+\tau} \Delta_u \left[\frac{\partial F}{\partial \lambda} \left(d\lambda - E_u[d\lambda] - \frac{d\pi}{\pi} d\lambda \right) \right] - \int_t^{t+\tau} \Delta_u F E_u \left[\frac{d\pi}{\pi} \right]. \end{aligned} \quad \text{(A.25)}$$

Now combine all three parts of profits (A.14)

$$\begin{aligned} Profit = & \int_t^{t+\tau} \left(\frac{\partial C}{\partial x} - \Delta_u \frac{\partial F}{\partial x} \right) \left(dx - E_u[dx] - \frac{d\pi}{\pi} dx \right) \\ & + \int_t^{t+\tau} \left(\frac{\partial C}{\partial \lambda} - \Delta_u \frac{\partial F}{\partial \lambda} \right) \left(d\lambda - E_u[d\lambda] - \frac{d\pi}{\pi} d\lambda \right) \\ & - \int_t^{t+\tau} (C_u - \Delta u F_u) E_u \left[\frac{d\pi}{\pi} \right] - \int_t^{t+\tau} r(C_u - \Delta_u F_u) du \end{aligned} \quad \text{(A.26)}$$

where the last row cancels as $E[d\pi/\pi] = -r$. To proceed we need to calculate the delta of the option Δ_u. To do this, introduce a function $\widetilde{C}(F, \lambda)$. This function is defined so that

$$\widetilde{C}(F(x,\lambda),\lambda) = C(x,\lambda). \quad \text{(A.27)}$$

Full differential on this expression is

$$\frac{\partial \widetilde{C}}{\partial F}\left(\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial \lambda}d\lambda\right)=\frac{\partial C}{\partial x}dx+\frac{\partial C}{\partial \lambda}d\lambda. \tag{A.28}$$

Set $d\lambda=0$ to obtain

$$\frac{\partial \widetilde{C}}{\partial F}\frac{\partial F}{\partial x}=\frac{\partial C}{\partial x}, \tag{A.29}$$

implying that delta of the call option is

$$\Delta\equiv\frac{\partial \widetilde{C}}{\partial F}=\frac{\partial C/\partial x}{\partial F/\partial x}. \tag{A.30}$$

We can now substitute the expression to simplify the profits and note that the first term in the expression cancels

$$\begin{aligned}Profit=&\int_t^{t+\tau}\underbrace{\left(\frac{\partial C}{\partial x}-\frac{\partial C/\partial x}{\partial F/\partial x}\frac{\partial F}{\partial x}\right)}_{=0}\left(dx-E_u[dx]-\frac{d\pi}{\pi}dx\right)\\&+\int_t^{t+\tau}\left(\frac{\partial C}{\partial \lambda}-\frac{\partial C/\partial x}{\partial F/\partial x}\frac{\partial F}{\partial \lambda}\right)\left(d\lambda-E_u[d\lambda]-\frac{d\pi}{\pi}d\lambda\right)\\=&\int_t^{t+\tau}\left(\frac{\partial C}{\partial \lambda}-\frac{\partial C/\partial x}{\partial F/\partial x}\frac{\partial F}{\partial \lambda}\right)\left(d\lambda-E_u[d\lambda]-\frac{d\pi}{\pi}d\lambda\right).\end{aligned} \tag{A.31}$$

Expected profits are

$$E_t[Profit]=E_t\left[\int_t^{t+\tau}\left(\frac{\partial C}{\partial \lambda}-\frac{\partial C/\partial x}{\partial F/\partial x}\frac{\partial F}{\partial \lambda}\right)(d\lambda-E_u[d\lambda])\right]-E_t\left[\int_t^{t+\tau}\left(\frac{\partial C}{\partial \lambda}-\frac{\partial C/\partial x}{\partial F/\partial x}\frac{\partial F}{\partial \lambda}\right)\frac{d\pi}{\pi}d\lambda\right] \tag{A.32}$$

Since $d\lambda-E_u[d\lambda]$ is an Ito process with zero drift, the first part of the expected profits is zero resulting in

$$E_t[Profit]=-E_t\left[\int_t^{t+\tau}\left(\frac{\partial C}{\partial \lambda}-\frac{\partial C/\partial x}{\partial F/\partial x}\frac{\partial F}{\partial \lambda}\right)\frac{d\pi}{\pi}d\lambda\right] \tag{A.33}$$

Finally, substitute the law of motion for π and λ to obtain the expression from the proposition

$$E_t[Profit]=-\int_t^{t+\tau}\sigma_x\sigma_\lambda b\rho E_t\left[\left(\frac{\partial C}{\partial \lambda}-\frac{\partial C/\partial x}{\partial F/\partial x}\frac{\partial F}{\partial \lambda}\right)\Bigg|_u\lambda_u\right]=-\int_t^{t+\tau}\sigma_x\sigma_\lambda b\rho E_t\left[\frac{\partial \widetilde{C}}{\partial \lambda}\Bigg|_u\lambda_u\right] \tag{A.34}$$

□

B Appendix: Tables and Figures

Filter	**Reason**
Open interest is greater than zero	Avoid thinly traded options
Bid price is greater than zero	Avoid thinly traded options
Bid-Ask spread is positive	Avoid spurious options
Calls: $K > P$ and Puts: $K < P$	OTM options for liquidity
Calls: $\mathcal{C} < P_{it}$	No arbitrage
Calls: $\mathcal{C} \geq \max\{0, P - PV(K) - PV(D)\}$	No arbitrage
Puts: $\mathcal{P} < K$	No arbitrage
Puts: $\mathcal{P} \geq \max\{0, PV(K) + PV(D) - P\}$	No-arbitrage
Options has standard settlements	Use standard options
Option implied volatility and delta are not missing	Avoid spurious options

Table B.1: Filters applied to option data. From Siriwardane (2015). P is the prices of the underlying, $\mathcal{P}$ and $\mathcal{C}$ are put can call option prices, respectively, $PV(K)$ is the present value of the strike, $PV(D)$ is the present value of dividends.

Characteristic	Compustat-CRSP sample							Skew Sample							
	Mean	StD	Q01	Q25	Q50	Q75	Q99	Mean	St.Dev	Q01	Q25	Q50	Q75	Q99
Panel A: January 1996 – December 2021															
Market Equity	2.905	10.701	0.004	0.061	0.269	1.303	50.684	15.916	25.668	0.138	1.848	5.857	18.266	131.828
Assets	4.038	15.036	0.005	0.077	0.367	1.698	79.076	14.975	26.736	0.068	1.499	5.056	15.869	145.299
Market-to-Book	3.193	4.678	0.273	1.109	1.819	3.327	24.26	4.931	6.53	0.477	1.672	2.912	5.349	35.563
Market-to-Book (A)	1.679	1.914	0.125	0.661	1.089	1.938	9.693	2.277	2.121	0.417	0.999	1.56	2.709	10.801
Profitability	0.009	0.062	-0.259	0.004	0.019	0.038	0.118	0.027	0.045	-0.151	0.016	0.031	0.048	0.118
Leverage	0.218	0.22	0	0.027	0.161	0.34	0.928	0.274	0.215	0	0.108	0.25	0.396	0.921
Investments	0.267	1.129	-0.501	-0.029	0.065	0.217	4.458	0.212	0.79	-0.375	-0.014	0.067	0.209	2.614
Panel B: January 2004 – December 2021															
Market Equity	4.05	13.36	0.007	0.106	0.457	2.063	64.682	16.454	27.375	0.13	1.713	5.469	17.75	135.043
Assets	5.598	18.736	0.007	0.132	0.625	2.615	110.791	15.427	28.375	0.064	1.404	4.852	15.471	148.81
Market-to-Book	3.3	5.042	0.276	1.141	1.856	3.392	25.363	4.903	6.658	0.47	1.654	2.87	5.236	36.605
Market-to-Book (A)	1.655	1.71	0.12	0.681	1.127	1.974	8.905	2.227	1.975	0.422	0.998	1.545	2.667	10.225
Profitability	0.01	0.058	-0.243	0.004	0.019	0.037	0.114	0.025	0.046	-0.159	0.015	0.031	0.047	0.118
Leverage	0.215	0.223	0	0.026	0.156	0.333	0.952	0.278	0.221	0	0.108	0.253	0.404	0.939
Investments	0.2	0.76	-0.477	-0.028	0.056	0.185	3.474	0.19	0.568	-0.371	-0.016	0.064	0.202	2.501

Table B.2: Summary Statistics for Firm Fundamentals across Compustat-CRSP and Skew Samples

	Dependent variable: Skew			
	MB	Full	Lev Q1	Lev Q4
	(1)	(2)	(3)	(4)
MB: β^{MB}	-0.028^{***}	-0.043^{***}	-0.027^{*}	-0.069^{***}
	(0.002)	(0.010)	(0.015)	(0.021)
$MB \times PD$: γ^{MB}	0.044^{***}	0.078^{***}	0.066^{**}	0.115^{***}
	(0.003)	(0.017)	(0.026)	(0.035)
Lev: β^{Lev}		0.032^{***}		
		(0.009)		
$Lev \times PD$: γ^{Lev}		-0.042^{***}		
		(0.015)		
OP: β^{OP}		-0.011	-0.001	-0.018
		(0.010)	(0.012)	(0.024)
$OP \times PD$: γ^{OP}		-0.002	-0.004	-0.009
		(0.017)	(0.020)	(0.040)
AG: β^{AG}		0.011	0.015	0.004
		(0.010)	(0.013)	(0.018)
$AG \times PD$: γ^{AG}		-0.016	-0.023	-0.003
		(0.017)	(0.024)	(0.030)
$RetVar$		0.090	-0.133	-0.149
		(0.099)	(0.109)	(0.251)
$RetVar \times PD$		-0.133	0.163	0.259
		(0.134)	(0.143)	(0.395)
$RetSkewness$		0.014^{**}	0.020^{*}	0.005
		(0.007)	(0.010)	(0.017)
$RetSkewness \times PD$		-0.024^{*}	-0.036^{*}	-0.011
		(0.012)	(0.019)	(0.028)
Constant	0.093^{***}			
	(0.0002)			
Observations	174,283	145,722	49,319	28,609
R^2	0.002	0.220	0.203	0.200
Adjusted R^2	0.002	0.218	0.198	0.191
Residual Std. Error	0.085	0.074	0.070	0.098
Note:				*p<0.1; **p<0.05; ***p<0.01

Table B.3: Regression of Skew on Firm Fundamentals, extended table

Figure B.2: Time series of firm characteristics across Compustat-CRSP and OptionMetrics samples

	To:				
From:	$Q1$	$Q2$	$Q3$	$Q4$	Out of sample
$Q1$	95.4	3.3	1.3	1.1	2.4
$Q2$	3.2	92.8	4.2	1.0	2.2
$Q3$	1.0	4.4	92.9	3.0	2.2
$Q4$	1.1	0.9	3.0	96.2	2.3

Table B.4: Transition Probability between leverage quartiles

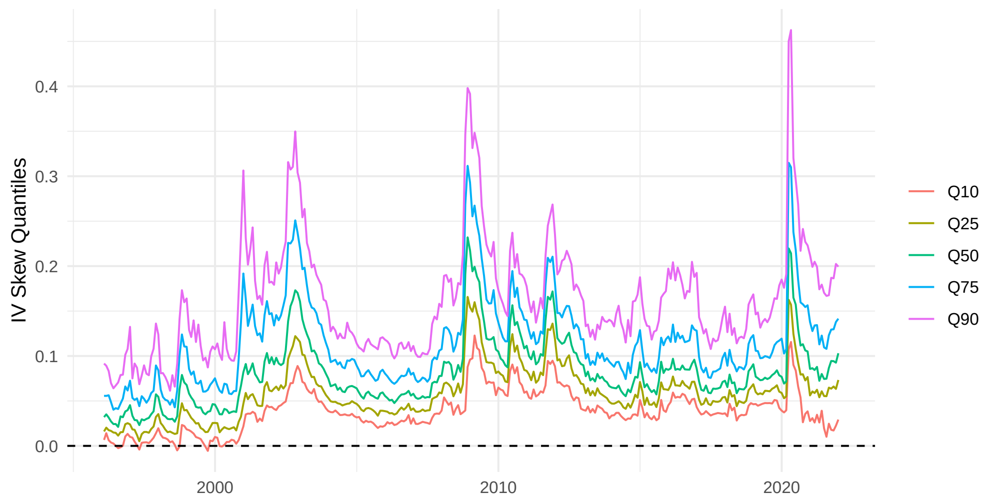


Figure B.3: Time series of skew quantiles

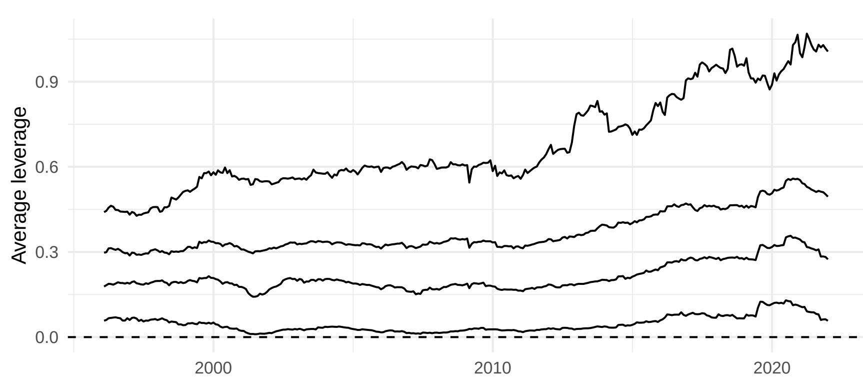
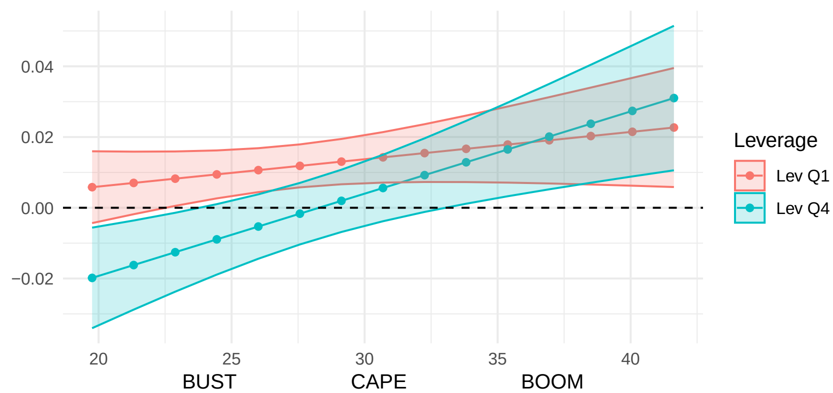
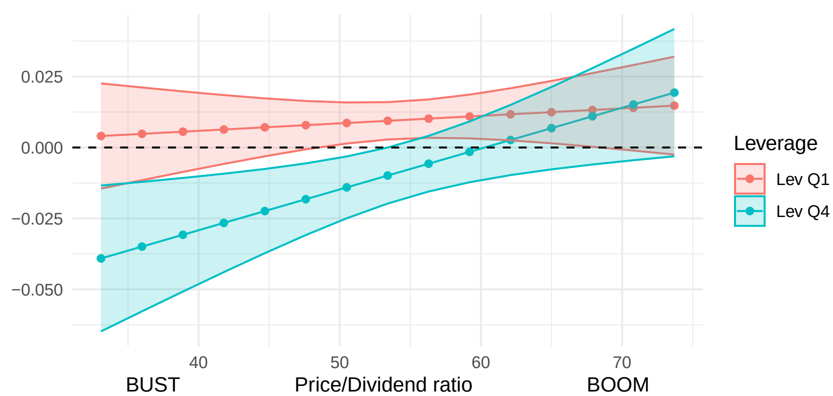


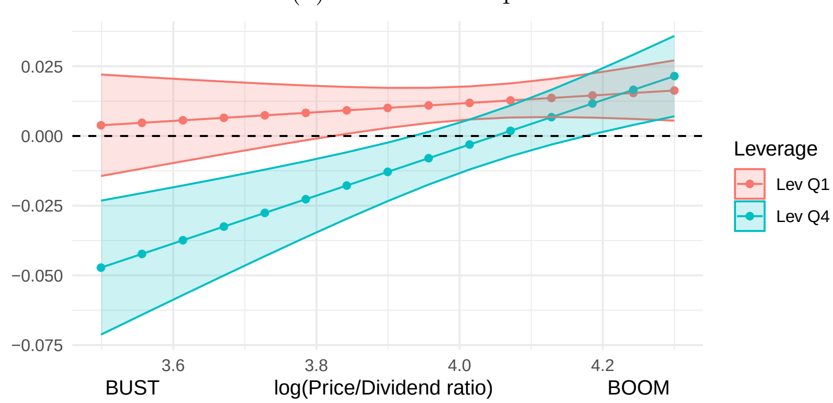
Figure B.4: Average leverage for each leverage quartile



(a) Function of CAPE



(b) Post 2003 sample



(c) Function of log(PD)

Figure B.5: Robustensss for the heterogenous effect of M/B on IV skew