Asset Pricing Notes. Chapter 4: Stochastic Discount Factor

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December 7, 2021

1 Complete Markets

1.1 Preliminaries

Suppose that there is a discrete set of states $s = 1, \ldots, S$ that has strictly positive probability $\pi(s)$. For now assume complete markets

Definition 1 (Complete Markets)**.** *For any state s there is an asset called Arrow-Debreu security that pays* 1 *if state s occurs and 0 otherwise. The price of this asset is denoted $q(s)$.*

We assume that the Law of One Price Holds

Definition 2 (Law of One Price (LOOP))**.** *Two assets with identical payoffs in every state $s = 1, \ldots, S$ have the same price*

Suppose that there is an asset the pays $X(s)$ in state s. Then, by the LOOP its price

$$\begin{aligned}
P(X) &= \sum_{s=1}^{S} q(s)X(s) \\
&= \sum_{s=1}^{S} \pi(s) \underbrace{\frac{q(s)}{\pi(s)}}_{M(s)} X(s) \\
&= \sum_{s=1}^{S} \pi(s)M(s)X(s) \\
&= \mathbb{E}[M(s)X(s)] = \mathbb{E}[MX]
\end{aligned}$$

Note that $M(s)$ is a random variable and its values is determined by state s that occurs. It is called the Stochastic Discount Factor. The above expression shows that price of an asset is the expected ($\mathbb{E}$), discounted ($M(s)$) values of its payoff ($X(s)$). Compared to riskless discounting SDF also incorporates risk. When consumption in a given state s is valuable $q(s)$ is high meaning that $M(s)$ is high. As a result payoff of any asset will be overweighted in that state compared to a riskless case.

Consider a simple example of a riskless asset: its payoff doesn't depend on the state and always gives 1. By the formula above its price is

$$P = \mathbb{E}[MX] = \mathbb{E}[M \cdot 1] = \mathbb{E}[M]$$

The (gross) risk free rate can be obtained by dividing the payoff 1 by the price P

$$1 + R_f = \frac{1}{\mathbb{E}[M]}$$

Hence, the risk free rate is a reciprocal of the expectation of the SDF.

1.2 Risk-Neutral Probabilities

Consider the value of a risk free asset

$$\mathbb{E}[M] = \sum_{s=1}^{S} \pi(s)M(s) \implies 1 = \sum_{s=1}^{S} \underbrace{\pi(s)\frac{M(s)}{\mathbb{E}[M]}}_{\pi^*(s)}$$

$\pi^*(s)$ is a valid probability measure since $\sum_{s=1}^{S} \pi^*(s) = 1$. Compared to actual (physical) probability $\pi(s)$, $\pi^*(s)$ is high when $M(s)$ is high, i.e. in states when the wealth is valuable. Now rearrange the pricing equation to get $\pi^*(s)$ inside of it

$$\begin{aligned}
P(X) &= \sum_{s=1}^{S} \pi(s)M(s)X(s) \\
&= \mathbb{E}[M] \sum_{s=1}^{S} \pi(s)M(s)\frac{1}{\mathbb{E}[M]}X(s) \\
&= \mathbb{E}[M] \sum_{s=1}^{S} \pi^*(s)X(s) \\
&= \frac{1}{1+R_f} \sum_{s=1}^{S} \pi^*(s)X(s)
\end{aligned}$$

Now the value of the asset is expectation of its payoff where expectation is taken with distorted probabilities $\pi^*(s)$ discounted at the risk free rate. These probabilities are called risk-neutral and we denote the expectation w.r.t. to these probabilities as $\mathbb{E}^*$[1]:

$$P(X) = \frac{1}{1+R_f}\mathbb{E}^*[X]$$

1.3 Utility Maximization

In complete markets a dynamic model can be collapsed to a static model where every agent buys state (and path contingent securities) at period zero and they live consuming according to her A-D securities purchased for that state. In two period model with discrete states, the problem is

$$\begin{gathered}
\max u(C_0) + \beta \sum_{s=1}^{S} \pi(s)u(C(s)) \\
\text{subject to } C_0 + \sum_{s=1}^{S} q(s)C(s) = W_0
\end{gathered}$$

Substitute C_0 from the budget constraint

$$\max u\left(W_0 - \sum_{s=1}^{S} q(s)C(s)\right) + \beta \sum_{s=1}^{S} \pi(s)u(C(s))$$

$$FOC_{C(s)} : -q(s)u'(C_0) + \beta\pi(s)u'(C(s)) = 0 \implies \frac{q(s)}{\pi(s)} = \beta\frac{u'(C(s))}{u'(C_0)} \; \forall s = 1, \ldots, S$$

Therefore, in complete markets model SDF is

$$M(s) = \frac{q(s)}{\pi(s)} = \beta\frac{u'(C(s))}{u'(C_0)}$$

[1] Often the actual (physical) probability is denoted as $\mathbb{P}$ and risk neutral probability as $\mathbb{Q}$. Then the expectation with respect to actual probability is $\mathbb{E}^{\mathbb{P}}$ and expectation w.r.t. to risk-neutral probability is $\mathbb{E}^{\mathbb{Q}}$

1.4 Growth Optimal Portfolio

Denote the return on the portfolio held by the log-investor in state s as $R_{GO}(s)$. This return can be written as

$$1 + R_{GO}(s) = \frac{C(s)}{W_0 - C_0}$$

since $W_0 - C_0$ is the total amount left after initial consumption and invested in A-D securities and this portfolio returns $C(s)$ in state s. With log-utility the first order condition implies

$$q(s)/C_0 = \beta\pi(s)/C(s) \implies q(s)C(s) = C_0\beta\pi(s)$$

sum across states

$$\sum_{s=1}^{S} q(s)C(s) = \sum_{s=1}^{S} C_0\beta\pi(s) = \beta C_0$$

From the budget constraint

$$C_0 + \sum_{s=1}^{S} q(s)C(s) = W_0 \implies W_0 - C_0 = \sum_{s=1}^{S} q(s)C(s) = \beta C_0$$

Therefore, the return on the GO portfolio is

$$1 + R_{GO}(s) = \frac{C(s)}{W_0 - C_0} = \frac{C(s)}{\beta C_0}$$

Notice that it is exactly the reciprocal of the SDF of the log-investor

$$M(s) = \beta\frac{C_0}{C(s)} = \frac{1}{1 + R_{GO}(s)}$$

1.5 Perfect Risk Sharing

The assumption of complete markets implies very strong statement about the relative marginal utilities across agents: marginal utilities across agents are perfectly correlated. To see why consider two agents with utilties u_j and u_k and assume for simplicity that they have the same belief about the probability of each state and have the same time preferences. Then, since they should agree on the price of A-D securities

$$\frac{q(s)}{\pi(s)} = \beta\frac{u_j(C_j(s))}{u_j(C_{j0})} = \beta\frac{u_k(C_k(s))}{u_k(C_{k0})}$$

Take this equation for two states s and s^* and divide them to get

$$\frac{u_j(C_j(s))}{u_j(C_j(s^*))} = \frac{u_k(C_k(s))}{u_k(C_k(s^*))} \tag{1}$$

This is the condition of **perfect risk sharing**.

With perfect risk sharing the allocation in the economy is Pareto optimal. Suppose that a planner allocates $\overline{C}(s)$ of conusmption across agents. He solves

$$\max \lambda_j \sum_s \pi(s)u_j(C_j(s)) + \lambda_k \sum_s \pi(s)u_k(C_k(s)) \text{ subject to } C_j(s) + C_k(s) = \overline{C}(s)$$

first order condition is

$$\lambda_j u_j'(C_j(s)) = \lambda_k u_k'(C_k(s)) \implies \frac{u_j'(C_j(s))}{u_k'(C_k(s))} = \frac{\lambda_k}{\lambda_j} \implies \frac{u_j'(C_j(s))}{u_k'(C_k(s))} = \frac{u_j'(C_j(s^*))}{u_k'(C_k(s^*))} \implies \frac{u_j(C_j(s))}{u_j(C_j(s^*))} = \frac{u_k(C_k(s))}{u_k(C_k(s^*))}$$

Hence, the allocation is the same as in the competitive case with complete markets in equation (1).

1.6 Representative Agent

Go back to equation (1). If agent j has higher marginal utility in state s than in state s^* so does agent k

$$u_j'(C_j(s)) \geq u_j'(C_j(s^*)) \implies u_k'(C_k(s)) \geq u_k'(C_k(s^*))$$

In complete markets all agents have the same ordering of states in terms of marginal utilities. Since marginal utility is a monotone function we can *rename* states so that to get the following ordering for all agents

$$C_j(1) \leq C_j(2) \leq \cdots \leq C_j(S) \; \forall j$$

note that this doesn't mean that all agents have the same consumption in each state just that the ordering is the same. Therefore, we can sum this ordering across agents to arrive at the ordering of the aggregate consumption

$$\overline{C}(1) \leq \overline{C}(2) \leq \cdots \leq \overline{C}(S) \tag{2}$$

For the same reason we can order SDFs

$$M(1) \geq M(2) \geq \cdots \geq M(S) \tag{3}$$

Given orderings (2) and (3) we can always come up with a decreasing mapping $g(\cdot) : g' > 0, g'' \leq 0$ such that

$$g(\overline{C}(s)) = M(s) \implies \frac{g(\overline{C}(s))}{g(\overline{C}(s^*))} = \frac{M(s)}{M(s^*)} \; \forall (s, s^*)$$

Note that for individual consumer we have

$$M(s) = \beta \frac{u'(C(s))}{u'(C_0)} \implies \frac{u'(C(s))}{u'(C(s^*))} = \frac{M(s)}{M(s^*)}$$

hence $g(\cdot)$ can be interpreted as the marginal utility of a representative investor and to get the the actual utility function $v(\cdot)$ we need to integrate $g(\cdot)$:

$$v'(\overline{C}(s)) = g(\overline{C}(s))$$

This means that with complete markets the market portfolio (the portfolio that gives consumption) is *efficient* in the sense that we can find a concave utility function that would induce agent to hold the market portfolio

There are several important points

1. $v(\cdot)$ doesn't necessarily preserve the form of individual utility functions
2. In general, reallocation of wealth can alter $v(\cdot)$
3. The demand of the *representative investor* need not coincide with aggregate demand for out-of-equilibrium prices

2 Incomplete Markets

In complete markets we had **existence**, **uniqueness** and **positivity** of the stochastic discount factor that followed from

$$M(s) = \frac{q(s)}{\pi(s)}$$

In incomplete markets we need to make additional assumptions to get the same properties. But even then we face some tradeoffs

1. Under some conditions we can always construct an SDF that lies in the space of tradable assets. However, this SDF is notguaranteed to be positive for every state. Moreover, it is not the only SDF that prices assets. There can be other SDF and the SDF constructed from the tradable assets is the projection of these other SDFs.
2. Under slightly stronger conditions can guarantee existence of positive SDF. However, it is not guaranteed to lie in the space of tradable assets.

2.1 Constructing SDF from Tradable Assets

We work with a set of asset payoffs X with prices P and denote the *payoff space* as Ξ. We need two assumptions to construct an SDF

Assumption 1 (Portfolio Formation)**.** $X_1, X_2 \in \Xi \implies aX_1 + bX_2 \in \Xi \; \forall a, b \in \mathbb{R}$

Assumption 2 (Law of One Price)**.** $P(aX_1 + bX_2) = aP(X_1) + bP(X_2)$

Under this assumptions

Theorem 1. *There exists a unique payoff X^* in the payoff space Ξ with the property $P(X) = \mathbb{E}[X^* X]$ for all $X \in \Xi$.*

Proof. There are S states and N *basis* payoffs $X_1, \ldots, X_N$. Basis payoffs are a vector

$$X_i = (X_i(1) \ldots X_i(S))'$$

that specifies how much X_i gives in each state such that $(X_1, \ldots, X_N)$ are linearly independent. We assume that $N < S$ so that markets are incomplete. Denote

$$\mathbb{X} = (X_1 \ldots X_N)' = \begin{pmatrix} X_1(1) & \cdots & X_N(1) \\ \vdots & \ddots & \vdots \\ X_1(S) & \cdots & X_N(S) \end{pmatrix}' \text{ and } \mathbb{P} = (P(X_1) \ldots P(X_N))'$$

Under assumption 1 we can represent the set of payoffs as linear combinations of basis payoffs

$$\Xi = \{\mathbb{X}'c, c \in \mathbb{R}^N\}$$

We want to find $X^* = \mathbb{X}'c^*$ – linear combination of basis payoffs such that it prices all basis payoffs

$$\begin{pmatrix} P(X_1) \\ \vdots \\ P(X_N) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[X_1' X^*] \\ \vdots \\ \mathbb{E}[X_N' X^*] \end{pmatrix}$$

Then by assumption 2 X^* will price all other assets in the payoff space. Write the condition in matrix form as

$$\mathbb{P} = \mathbb{E}[\mathbb{X}\mathbb{X}'c^*] \implies c^* = (\mathbb{E}[\mathbb{X}\mathbb{X}'])^{-1} \mathbb{P} \implies \mathbb{X}'c^* = X^* = \mathbb{X} (\mathbb{E}[\mathbb{X}\mathbb{X}'])^{-1} \mathbb{P}$$

□

There can be other SDF of the form

$$M = X^* + \varepsilon \text{ where } \mathbb{E}[\mathbb{X}\varepsilon] = \mathbb{E}[\mathbb{X}(M - X^*)] = 0$$

This says that X^* is a projection of every SDF on the space of tradable assets and, hence, can be thought as the portfolio that best mimics the behavior of any SDF.

2.2 Existence of Positive SDF

If we introduce an additional assumption

Assumption 3 (Absence of Arbitrage)**.** *A payoff space Ξ and pricing function $P(X)$ have absence of arbitrage if*

1. $\forall s, X(s) \geq 0 \implies P(X) \geq 0$
2. $\forall s, X(s) \geq 0$ *and* $\exists s : X(s) > 0$ *with positive probability* $\implies P(X) > 0$

we can guarantee the existence of strictly positive SDF. In particular, we have the following theorems

Theorem 2. $P = \mathbb{E}[MX]$ *and* $M(s) > 0 \forall s \implies$ ***absence of arbitrage***

Proof. There are two short parts

1. If $\forall s X(s) \geq 0 \implies P(X) = \sum_s \pi(s)M(s)X(s) \geq 0$
2. If additionally $X(s) > 0$ for some state $\implies P(X) = \sum_s \pi(s)M(s)X(s) > 0$ since we assume $\pi(s) > 0 \forall s$

$\square$

Theorem 3. ***Absence of arbitrage*** $\implies \exists M : M(s) > 0 \forall s$ *and* $P(X) = \mathbb{E}[MX]$

Proof. See the textbook $\square$

3 Properties of the SDF

There is a set of standard results that follow from the fundamental pricing equation $P = E[MX]$.

3.1 No distributional assumptions

Write the fundamental equation as

$$\begin{aligned} P &= E[MX] \\ &= E[M]E[X] + cov(M, X) \\ &= \frac{1}{1+R_f}E[X] + \underbrace{cov(M, X)}_{\text{risk adjustement}} \end{aligned}$$

In a more standard return formulation

$$\begin{aligned} 1 &= E[M_{t+1}(1 + R_{i,t+1})] \\ &= E[M_{t+1}]E[1 + R_{i,t+1}] + cov(M_{t+1}, R_{i,t+1}) \end{aligned}$$

rearrange to get

$$\begin{aligned} E[1 + R_{i,t+1}] &= \frac{1}{E[M_{t+1}]}(1 - cov_t(M_{t+1}, R_{i,t+1})) \\ &= (1 + R_{f,t+1})(1 - cov_t(M_{t+1}, R_{i,t+1})) \end{aligned}$$

with no risk adjustment (covariance term is zero), the expected return of any asset is equal to the risk free return. Note that this doesn't mean that the asset has zero risk as measured by variance. What matters is the systematic - covariance risk. Consider the excess return

$$\begin{aligned} E[R_{i,t+1} - R_{f,t+1}] &= (1 + R_{f,t+1})(1 - cov_t(M_{t+1}, R_{i,t+1})) - R_{f,t+1} \\ &= -(1 + R_{f,t+1})cov_t(M_{t+1}, R_{i,t+1})) \\ &= -(1 + R_{f,t+1})cov_t(M_{t+1}, R_{i,t+1} - R_{f,t+1})) \end{aligned} \tag{4}$$

where the equality follows since $R_{f,t+1}$ is knows at date t.

Factor structure of expected returns We can write the excess return in a factor structure form

$$\begin{aligned} E[R_{i,t+1} - R_{f,t+1}] &= -(1 + R_{f,t+1})cov_t(M_{t+1}, R_{i,t+1} - R_{f,t+1})) \\ &= \frac{cov_t(M_{t+1}, R_{i,t+1} - R_{f,t+1}))}{var_t(M_{t+1})} \cdot (-(1 + R_{f,t+1})var_t(M_{t+1})) \\ &= \beta_{it} \cdot \lambda_t \end{aligned} \tag{5}$$

where β_{it} is the *quantity of risk* and λ_t is the *price of risk*. This is similar to CAPM where the quantity of risk is determined by the beta with the market return and price of risk as the expected return on the market. For some models we can get a richer structure so that excess return is linear in multiple sources of risk. For example, the risk can consist of wealth portfolio, labor income and future investment opporuinities (as is the case with ICAPM). Then the excess return of the asset depends on covariances with these risks.

To see this suppose that the SDF is linear in K common factors $f_{k,t+1}$ with conditional zero mean and are orthogonal to one another

$$M_{t+1} = a_t - \sum_{k=1}^{K} b_{kt} f_{k,t+1}$$

where a_t pins down the risk free rate since $P_{f,t} = E_t M_{t+1} = a_t \implies 1 + R_{f,t+1} = 1/a_t$. Now use the equation (4) to write

$$\begin{aligned}
E[R_{i,t+1} - R_{f,t+1}] &= \frac{1}{a_t} cov_t(\sum_{k=1}^{K} b_{kt} f_{k,t+1}, R_{i,t+1} - R_{f,t+1})) \\
&= \frac{1}{a_t} \sum_{k=1}^{K} b_{kt} \sigma_{ikt} \\
&= \frac{1}{a_t} \sum_{k=1}^{K} \frac{\sigma_{ikt}}{\sigma_{kt}^2} \cdot b_{kt} \sigma_{kt}^2 \\
&= \frac{1}{a_t} \sum_{k=1}^{K} \beta_{ikt} \cdot \lambda_{kt}
\end{aligned}$$

Volatility Bounds Using the same equation we can put a lower bound on the volatility of the SDF. First decompose covariance in equation (4) into the product of standard deviations and correlation

$$\begin{aligned}
E[R_{i,t+1} - R_{f,t+1}] &= -(1 + R_{f,t+1}) \sigma_t(M_{t+1}) \sigma_t(R_{i,t+1}) corr_t(M_{t+1}, R_{i,t+1} - R_{f,t+1})) \\
&= -\frac{\sigma_t(M_{t+1}) \sigma_t(R_{i,t+1} - R_{f,t+1}) corr_t(M_{t+1}, R_{i,t+1} - R_{f,t+1}))}{E_t[M_{t+1}]} \\
&\leq \frac{\sigma_t(M_{t+1}) \sigma_t(R_{i,t+1} - R_{f,t+1})}{E_t[M_{t+1}]}
\end{aligned} \tag{6}$$

where we used that correlation is between -1 and 1. Rearrange to get

$$\frac{E[R_{i,t+1} - R_{f,t+1}]}{\sigma_t(R_{i,t+1} - R_{f,t+1})} \leq \frac{\sigma_t(M_{t+1})}{E_t[M_{t+1}]} \tag{7}$$

Volatility of the SDF is bounded by any assets Sharpe Ratio. The tightest bound is achieved for the asset (or portfolio) with the highest sharpe ratio – **tangent portfolio**.

3.2 Joint Lognormality

We can simplify many derivations if we assume joint lognormality of returns and SDF[2]. Let's see how it helps us with the fundamental equation

$$\begin{aligned}
1 &= E_t[M_{t+1}(1 + R_{i,t+1})] \\
0 &= \log E_t[M_{t+1}(1 + R_{i,t+1})] \\
&= E_t[\log(M_{t+1}(1 + R_{i,t+1}))] + \frac{1}{2} Var_t(\log(M_{t+1}(1 + R_{i,t+1}))) \\
&= E_t[m_{t+1} + r_{i,t+1}] + \frac{1}{2} Var_t(m_{t+1} + r_{i,t+1}) \\
&= E_t[m_{t+1}] + E_t[r_{i,t+1}] + \frac{1}{2} Var_t(m_{t+1}) + \frac{1}{2} Var_t(r_{i,t+1}) + cov_t(m_{t+1}, r_{i,t+1}) \\
&= E_t[m_{t+1}] + E_t[r_{i,t+1}] + \frac{1}{2}\sigma_{mt}^2 + \frac{1}{2}\sigma_{it}^2 + \sigma_{imt}
\end{aligned} \tag{8}$$

[2] Joint lognormality is helpful since the product of two lognormal random varables is a lognormal

For the **riskless asset** this equation becomes

$$\begin{aligned}
0 &= E_t[m_{t+1}] + r_{f,t+1} + \frac{1}{2}\sigma_{mt}^2 \\
r_{f,t+1} &= -E_t[m_{t+1}] - \frac{1}{2}\sigma_{mt}^2
\end{aligned} \tag{9}$$

The **log risk premium with Jensen adjustment**

$$\begin{aligned}
E_t[r_{i,t+1}] - r_{f,t+1} + \frac{1}{2}\sigma_{it}^2 &= -\left(E_t[m_{t+1}] + \frac{1}{2}\sigma_{mt}^2 + \frac{1}{2}\sigma_{it}^2 + \sigma_{imt}\right) - \left(-E_t[m_{t+1}] - \frac{1}{2}\sigma_{mt}^2\right) + \frac{1}{2}\sigma_{it}^2 \\
&= -\sigma_{imt}
\end{aligned} \tag{10}$$

which analogous to equation (4).

We can also bound the volatility of the SDF in the lognormal setting from equation (10)

$$\begin{aligned}
E_t[r_{i,t+1}] - r_{f,t+1} + \frac{1}{2}\sigma_{it}^2 &= -\sigma_{imt} = -\sigma_{it}\sigma_{mt}\rho_{imt} \leq \sigma_{it}\sigma_{mt} \\
\sigma_{mt} &\geq \frac{E_t[r_{i,t+1}] - r_{f,t+1} + \frac{1}{2}\sigma_{it}^2}{\sigma_{it}}
\end{aligned} \tag{11}$$

3.3 Hansen-Jagannathan Bound

In the previous part we derived the bounds on the variance of the stochastic discount factor when we knew the risk free rate. In practice, when we write a model, risk free is also pinned down by the model parameters. Therefore, we might need to have a tool for an assessment of the model without the knowledge of the risk free rate.

Suppose that we have

- $\mathbb{X}$ – $N \times S$ matrix of basis payoffs
- $\mathbb{P}$ – vector of prices of basis payoffs
- $\overline{M}$ – mean of the SDF that is unknown.

Now suppose that we fix some mean of the SDF at $\overline{M}$. We can augment the payoff space with a payoff vector that returns $1/\overline{M}$ in every state of the worls. Denote this payoff space as $\mathbb{X}_{\overline{M}} = [1/\overline{M}, \mathbb{X}]$ and augmented price vector as $\mathbb{P}_{\overline{M}} = [1, \mathbb{P}]$. From the construction of an SDF that lies in the payoff space we know that it will be a linear combination of a constant and the risky payoffs. Since any SDF should have mean of $\overline{M}$ (by construction) we can write it as

$$M^*(\overline{M}) = \overline{M} + \underbrace{(\mathbb{X} - E\mathbb{X})'\beta_{\overline{M}}}_{\text{has zero mean}}$$

The same SDF $M^*(\overline{M})$ constructed from the assets of augmented economy is also a valid SDF of the original economy (prices all assets $\implies$ price a subset of assets). Therefore, we can write

$$\begin{aligned}
\mathbb{P} &= E[M^*(\overline{M})\mathbb{X}] \\
&= E[M^*(\overline{M})]E[\mathbb{X}] + cov(M^*(\overline{M}), \mathbb{X}) \\
&= \overline{M}E[\mathbb{X}] + cov((\mathbb{X} - E\mathbb{X})'\beta_{\overline{M}}, \mathbb{X}) \\
&= \overline{M}E[\mathbb{X}] + cov(\mathbb{X}, \mathbb{X})\beta_{\overline{M}} \\
&= \overline{M}E[\mathbb{X}] + \Sigma\beta_{\overline{M}} \\
\implies \beta_{\overline{M}} &= \Sigma^{-1}\left[\mathbb{P} - \overline{M}E[\mathbb{X}]\right]
\end{aligned}$$

Using $\beta_{\overline{M}}$ we can calculate the variance of SDF as

$$\begin{aligned}
var(M^*(\overline{M})) &= var((\mathbb{X} - E\mathbb{X})'\beta_{\overline{M}}) \\
&= var(\mathbb{X}'\beta_{\overline{M}}) \\
&= \beta_{\overline{M}}'\Sigma\beta_{\overline{M}} \\
&= \left[\mathbb{P} - \overline{M}E[\mathbb{X}]\right]'\Sigma^{-1}\Sigma\Sigma^{-1}\left[\mathbb{P} - \overline{M}E[\mathbb{X}]\right] \\
&= \left[\mathbb{P} - \overline{M}E[\mathbb{X}]\right]'\Sigma^{-1}\left[\mathbb{P} - \overline{M}E[\mathbb{X}]\right]
\end{aligned}$$

The variance of this SDF $M^*(\overline{M})$ gives a lower bound for the variance of any SDF with the same mean. Then

$$\sigma^2(M(\overline{M})) = \sigma^2(M^*(\overline{M})) + \sigma^2(\varepsilon) \geq \sigma^2(M^*(\overline{M}))$$

Equivalently, we can divide by $\overline{M}$ to get

$$\frac{\sigma^2(M(\overline{M}))}{E[M(\overline{M})]} \geq \frac{\sigma^2(M^*(\overline{M}))}{\overline{M}}$$

Geometry of HJ Frontier Notice that from the previous discussion pairs $\{\overline{M}, \sigma^2(M^*(\overline{M}))\}\forall\overline{M}$ trace a frontier of a minimum variance of the SDF for a given mean. In order to see how this frontier looks like we can do the following. First define the benchmark return which is the return on the "SDF portfolio" in the augmented economy

$$1 + R^*(\overline{M}) = \frac{M^*(\overline{M})}{P(M^*(\overline{M}))} = \frac{M^*}{E[M^*M^*]} = \frac{M^*}{E[(M^*)^2]}$$

Then we can write

$$\frac{\sigma(1 + R^*(\overline{M}))}{E[1 + R^*(\overline{M})]} = \frac{\sigma(M^*)}{E[(M^*)^2]} \times \frac{E[(M^*)^2]}{E[M^*]} = \frac{\sigma(M^*)}{\overline{M}}$$

Hansen and Richard (1987) show that the benchmark return is the traded return with the smallest uncentered second moment. Consider the variance of the benchmark return

$$\sigma^2(1 + R^*) = E[(1 + R^*)^2] - (E[1 + R^*])^2 \implies E[(1 + R^*)^2] = \sigma^2(1 + R^*) + (E[1 + R^*])^2$$

Hence, the portfolio with the lowest $\sigma^2(1+R^*)+(E[1+R^*])^2$ and hence the smallest $\sqrt{\sigma^2(1 + R^*) + (E[1 + R^*])^2}$. Notice that the latter is the distance from the origin to the corresponding portfolio on (mean, st.dev) diagram that we use in mean-variance analysis. Hence, we can construct this payoff as is shown on the figure below

Notice that the angles that are highlighted are the same.

Main Conclusion The biggest Sharpe Ratio for a given mean of the SDF $\overline{M}$ is the lower bound on $\frac{\sigma(\overline{M})}{\overline{M}}$

3.4 Entropy bounds

Assumption of lognormality simplifies the derivations but also over-simplifies the world in many cases. Therefore, we need to some techniques to work with more general distributions.

Definition 3 (Entropy)**.** *Entropy of a random variables X is*

$$L(X) = \log EX - E\log X$$

By Jensen's inequality it is non-negative and it is equal to half-variance if X si lognormal. In general entropy depends on higher moments as well

Definition 4 (Cumulant-Generating Function)**.** *Cumulant generating function for random variable z is defined as*

$$\mathbf{c}(\theta, z) = \log E\exp(\theta z)$$

It has the property that

$$\mathbf{c}(\theta, z) = \sum_{n=1}^{\infty} \frac{\kappa_n(z)\theta^n}{n!}$$

where $\kappa_n(z)$'s are cumulants of z: $\kappa_1(z)$ – mean of z, $\kappa_2(z)$ – variance of z, $\kappa_3(z)/\sigma^2$ – skewness, $\kappa_4(z)$ – excess kurtosis and so on. We can write entropy in terms of the cumulant generating function.

$$\begin{aligned}
L(X) &= \log EX - E\log X \\
&= \log E\exp(x) - Ex \\
&= \mathbf{c}(1, \log(X)) - \kappa_1(\log(X)) \\
&= \sum_{n=2}^{\infty} \frac{\kappa_n(\log(X))}{n!}
\end{aligned}$$

Entropy of the SDF in two period model First, consider the case of two periods. Recall from the definition of the risk-neutral probability

$$\pi^*(s) = \pi(s)\frac{M(s)}{\mathbb{E}[M]} = \pi(s)\frac{M(s)}{P_f} \implies M(s) = \frac{\pi^*(s)}{\pi(s)}P_f$$

Entropy of the SDF is then

$$\begin{aligned}
L(M) &= L\left(\frac{\pi^*(s)}{\pi(s)}P_f\right) \\
&= \log E\left[\frac{\pi^*(s)}{\pi(s)}P_f\right] - E\log\left(\frac{\pi^*(s)}{\pi(s)}P_f\right) \\
&= \log P_f E\left[\frac{\pi^*(s)}{\pi(s)}\right] - E\log\left(\frac{\pi^*(s)}{\pi(s)}\right) - E\log P_f \\
&= \log E\left[\frac{\pi^*(s)}{\pi(s)}\right] - E\log\left(\frac{\pi^*(s)}{\pi(s)}\right) \\
&= \log\sum_s \pi(s)\left[\frac{\pi^*(s)}{\pi(s)}\right] - E\log\left(\frac{\pi^*(s)}{\pi(s)}\right) \\
&= \log\sum_s \pi^*(s) - E\log\left(\frac{\pi^*(s)}{\pi(s)}\right) \\
&= -E\log\left(\frac{\pi^*(s)}{\pi(s)}\right)
\end{aligned}$$

Hence, in a two period model entropy of the SDF can be interpreted as the (Kullback-Leibler) distance between physical and risk-neutral probabilities.

Entropy of the SDF in a multiperiod model Consider the conditional entropy of $M(1+R_{i,t+1})$

$$L_t(M_{t+1}(1+R_{i,t+1})) = \log\underbrace{E_t[M_{t+1}(1+R_{i,t+1})]}_{=1} - E_t\log(M_{t+1}(1+R_{i,t+1})) = -E_t m_{t+1} - E_t r_{i,t+1} \geq 0$$

$$\implies E_t r_{i,t+1} \leq -E_t m_{t+1}$$

Now consider the conditional entropy of the SDF itself

$$\begin{aligned}
L_t(M_{t+1}) &= \log E_t M_{t+1} - E_t\log M_{t+1} \\
&= \log E_t M_{t+1} - E_t m_{t+1} \\
&= \log\frac{1}{1+R_{f,t+1}} - E_t m_{t+1} \\
&= -r_{f,t+1} - E_t m_{t+1} \\
&\geq E_t r_{i,t+1} - r_{f,t+1}
\end{aligned}$$

Conditional log-risk premium is the lower bound for the entropy of the SDF. Growth optimal portfolio is the portfolio with the largest expected log-return.

Unconditional Entropy of SDF As there is las of total variance

$$Var(X_{t+1}) = EVar_t(X_{t+1}) + Var(E_t Xt+1)$$

there is a law of total entropy

$$L(X_{t+1}) = EL_t(X_{t+1}) + L(E_t X_{t+1})$$

Use the fact that the conditional expectation of the SDF is the price of the one period risk-free asset and the previous result to get

$$L(M_{t+1}) \geq E[E_t r_{i,t+1} - r_{f,t+1}] + L(P_{f,t+1}) = E[r_{i,t+1} - r_{f,t+1}] + L\left(\frac{1}{1+R_{f,t+1}}\right)$$

Hence, unconditionally entropy of the SDF is bounded by both the unconditional expectation of risk premium on the growth-optimal portfolio and by variation in the risk-free rate.