Real and Financial Options

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Abstract

This paper builds a bridge between the real decision of firms and prices of equity options. Traditional option pricing literature proposed a variety of reduced form models to fit empirical patterns in option prices. At the same time, the cross sectional effects of firm fundamentals on equity options have not been thoroughly explored. I document a heterogeneous effect of firm fundamentals such as book-to-market on the relative prices of options that varies with the aggregate state of the economy. I develop a stylized production based asset pricing model with real options consistent with this evidence. A full structural model with leverage is able to match the observed patterns both qualitatively and quantitatively. Additionally, I show that the real options model can rationalize recently proposed delta-hedged option strategies based on profitability and book-to-market.

1 Introduction

Equity options are an important financial instruments. At the same time, the connection between firm fundamentals and equity option prices has not been thoroughly studied. Traditional financial engineering literature that followed the pioneering work of Black and Scholes (1973) and Merton (1973) evolved separately from financial economics and, in particular, from cross sectional equity pricing both in theory and in empirics. Over the last several decades financial engineering proposed a variety of reduced form models aimed at better explaining the observed option prices and rationalizing phenomena like implied volatility skew – a persistent property in the index and equity option markets.

Financial engineering approach is very successful: industry practitioners use these models to generate trading ideas by observing deviations from model calibrations and to manage risks. This stems from the intrinsic flexibility of the approach: parameters of a given model can be calibrated to a particular underlying, equity or index. A natural question arises: can firm fundamentals guide us in calibrating these parameters? Do underlying differences in firms' business models and capital structures lead to systematic differences in calibrations? There has a been a proliferation of models explaining cross-sectional anomalies such as difference in returns across growth and value firms, profitable and unprofitable firms, firms with low and high asset growth. In addition to implications about average returns, such models have direct implications for the distribution of returns. Studying option prices is parallel to studying the distribution of equity returns (Breeden and Litzenberger, 1978) and thus the characteristics that were found predictive of average returns have a potential to predict the distribution of stock returns both theoretically and empirically.

In this paper, I develop a connection between two strands of finance that historically evolved without much interaction. In particular, I build a bridge between the real decisions of firms and return distribution and, as a result, equity option prices. I start by documenting a heterogeneous effect of firm fundamentals and, in particular, market-to-book ratio on the implied volatility skew – the measure of expensiveness of out-of-the-money puts relative to at-the-money options. When aggregate economic conditions and valuations are low, there is little to no effect of market-to-book on skew. However, when valuations are high firms with higher market-to-book have a higher skew.

The heterogenous effect of fundamentals depending on the aggregate state might seem counterintuitive when viewed through the lens of returns distribution. Growth firms in good aggregate economic conditions have a more negatively skewed distribution relative to value firms – the empirical regularity that I also document in my empirical analysis. At the same time, growth firms should have higher upside potential as they possess many real options: ability to expand, attract customers, launch new products. The crucial ingredient is to think about how the value of these firms is determined. When aggregate conditions are good, growth firms derive a larger share of their value from real options and the value of real options is more sensitive to economic conditions. At the same time, the probability of exercising their real options is high and so are their valuations. This leads to an asymmetry in the future returns, where the upside potential is already priced but the downside risk remains high due to a higher sensitivity to worsening economic conditions.

To address the heterogeneity in the aggregate state theoretically, I modify a standard continuous time model with real options (e.g. Gomes and Schmid, 2010; Back, 2017) by introducing time varying aggregate state that determines the price of cash flow risk. Real options generate a difference between value of capital the firm currently uses for production – assets-in-place – and option value of expanding its productive capacity when the economic conditions improve – real or growth options. A firm that doesn't have any real options or has a very small probability of exercising them, for example, when the economic conditions are bad or productivity is low, will derive most of its value from assets-in-place. Such firm will have a low market-to-book ratio. On the other hand, if a firm is very likely to exercise its real options, it will derive a significant share of its value from real option and will have a high market-to-book.

The value of real option is more sensitive to economic conditions, modeled as price of risk, than the value of assets-in-place. The sensitivity is higher when economic conditions are good, as the discount rate is lower. This makes a firm with that derive a large proportion of its value from real options more sensitive to deterioration in economic conditions, i.e. to an increase in the price of risk. I show that this manifests itself in a negatively skewed risk-neutral and physical distribution of returns for such firms, in line with empirical findings. A straightforward implication of a negatively skewed return distribution is higher implied volatility skew for growth firms relative to value firms when the price of risk is low.

The continuous time model I develop in this paper allows to derive implications for the average profits of delta hedged option strategies. In traditional option pricing literature the performance of such strategies is driven purely by variance risk premium (VRP). While my model doesn't feature VRP explicitly, volatility of firm's returns is correlated negatively with state prices. Importantly, the strength of this correlation is determined by firm fundamentals leading to differential exposure of firms to movements in volatility, that can be seen as fundamentally driven firm level variance risk premium. I show that this qualitative difference is consistent with the performance of recently proposed highly profitable equity option strategies based on firm fundamentals' sorts.

I conclude my analysis by presenting a structural dynamic firm model set in discrete time that features more realistic firm dynamics. The model includes recent developments in production based asset pricing such as partial irreversibility of investments, external financing constraints and leverage similar to Begenau and Salomao (2019). I show that the aggregate state dependent sensitivity of implied volatility skew to market-to-book from the simulated model is similar to empirical sensitivity both quantitatively and qualitatively.

Outline of the Paper I position this paper within the existing theoretical and empirical literature on cross-sectional option and equity pricing in Section 2. I discuss empirical strategy of relating firm fundamentals to option prices in Section 3. I present a stylized continuous time model with real options and aggregate risk and discuss the channel through which market-to-book is related to the cross Section of option prices in Section 4. I derive implications of the continuous time model for delta-hedged option strategies in Section 4.2. I present a rich discrete time structural model and assess its fit to the data in Section 5.

2 Literature Review

The traditional option pricing or financial engineering literature started with Black and Scholes (1973) and Merton (1973). They showed that when the underlying spot price follows a geometric brownian motion so that the holding period returns are log-normal, the value of a European option can be calculated in a semi-closed form. This equation is widely known as Black-Scholes-Merton (BSM) equation and is used to this day to compare values of option of different moneyness and maturities. The critical assumption of constant volatility underlying BSM equation has been challenged by observing both the distribution of underlying returns and option prices themselves. In particular, it has been widely documented that equity and index option prices exhibit a phenomenon known as implied volatility skew: out-of-the-money put prices appear to be higher than at-the-money put prices when evaluated against the BSM benchmark. Subsequently there has been a proliferation of papers that modified the process for the underlying asset to match option prices quantitatively. Heston (1993) showed that stochastic volatility that is negatively correlated with returns can generate implied volatility skew. Merton (1976) and Bates (1996) showed how the presence of discontinuities in the underlying process can generate realistic option prices. Despite the richness of approaches, the calibration of these models relies on matching implied volatility surfaces without considering the fundamental determinants that underlie the choice of parameters. In this paper, I start with firm fundamentals to derive implications for the distribution of stock returns and option prices as a result.

The model presented in this paper belongs to a class of production based asset pricing models that considers firms' real decisions such as investments to draw implications about the cross section of equity returns. Kogan (2001), Zhang (2005), Kogan and Papanikolaou (2010) develop models with (partially) irreversible investments. (Partial) irreversibility generates real options: firms need to decide when to optimally expand as scaling down is costly or, in extreme cases, impossible. Real options generate variation in market-to-book as the firm may can derive more or less of its value from real options. Gomes and Schmid (2010) show that leverage and market-to-book are closely connected if the firm needs to borrow to exercise its real options. Kuehn and Schmid (2014) shows that production based models can generate realistic credit spreads. I complement this literature by explaining novel empirical findings in the cross-sectional of equity options from the lense of production based asset pricing models.

The literature on consumption based asset pricing made a step towards understanding option prices. Rare disaster models with time varying probability of a disaster as in Gabaix (2012), Wachter (2013), Seo and Wachter (2019) generate realistic skew for *index* options. This paper, on the other hand, aims at explaining the cross-section of *equity* options.

One of the insights of Merton (1973) is that under BSM assumptions the option can be dynamically replicated with a position in the underlying asset and a riskless bond. Therefore, a delta-hedged position produces exactly zero excess return under BSM assumptions. However, in the presence of other forms of risk, like stochastic volatility, the excess profits of delta hedged strategies will deviate from zero. Coval and Shumway (2001) empirically studies market neutral index option straddles – a long position in both a call and a put. They find that on average such strategy produces negative return as it protects the investor from increases in volatility indicating a presence of variance risk premium. Bakshi and Kapadia (2003) theoretically show that average profits of a delta hedged option strategies in a stochastic volatility environment are exposed only to variance risk premium. In this paper, I show theoretically that companies with different fundamentals have different profits from delta-hedged strategies – a form of fundamentals driven variance risk premium. In particular, this theoretical insight sheds light on recently proposed option strategies based on fundamentals sorted portfolios (Zhan et al., 2022).

Studying the cross-section of equity option returns is not new to the literature. Goyal and Saretto (2009) shows that option straddles formed on firms with large difference between historical and implied volatility produce high returns relative to straddles based on firms with a small difference. Cao and Han (2013) looks at delta-hedged return of equity options formed based on idiosyncratic volatility. Zhan et al. (2022) forms delta-hedged option positions based on fundamental sorts. Christoffersen et al. (2018) documents a strong factor structure in implied volatility levels and skew. My contribution is deriving theoretical results about the performance of delta hedged positions in a production based model and provide a resolution to highly profitable trading strategies proposed by Zhan et al. (2022).

3 Firm Fundamentals and Option Prices in the Data

In this section, I present the empirical strategy to estimate the relationship between firm fundamentals and implied volatility skew. I first describe the data and variables I use in my empirical analysis. To motivate the empirical strategy, I present an example of two companies, value and growth, across two economic regimes. I then show the main empirical specification and estimation results.

Data Option prices and implied volatilities are obtained from OptionMetrics (OM). OM contains equity option prices since January, 1996. Option data has high dimensionality: for every firm i at date t there is a set of options with different strikes K and maturities T. Below I describe how I aggregate the set of strikes and maturities, also called a volatility surface, into a single variable amenable for empirical analysis. Another issue that arises when working with option data is illiquidity. When there is little optionality left, for example, when the current price of equity is well above or well below the strike, there are few reason to trade options as the payoff is essentially linear. This results in low liquidity and non-informative prices and often to violation of simple arbitrage bounds. Therefore, I use a standard set of filters (e.g. Siriwardane, 2015) presented in Table 2. These filters excludes options with no open interest and volume and excludes options whose prices violate simple arbitrage bounds. Data on firm fundamentals comes from CRSP and Compustat. Aggregate variables such as aggregate price-dividend ratio come from Welch and Goyal (2008) that is regularly updated and is publicly available on Amit Goyal's website.

Option variables Firm fundamentals are available only at firm-time-period level.Therefore, I need to aggregate high dimensional option prices to the same level. The simplest way to do it is to choose a particular maturity and moneyness – the ratio of strike to the underlying spot price. There are a few problems with such approach. First, not all maturities are available at any given point in time. The option that had a 30 day maturity 5 days ago, has a maturity of 25 days today, and it is unlikely that there exists an option with 30 days maturity today as options have fixed expiration calendar. Fixing moneyness at the same level has the same issue.

The second problem is that the same moneyness may have different interpretations in different economic regimes and even across different underlying assets. Moneyness of 0.8 is very different when the volatility is 10% compared to when volatility is 30%. Therefore, a good measure of moneyness will account for differences in volatility. In practice, option traders use option delta – the sensitivity of the value of the option to changes in the underlying spot price. Delta of an option is also the risk-neutral probability of an option ending up in-the-money and, therefore, accounts for underlying volatility.

To address both of these problems, I follow Carr and Wu (2020) who use a different measure of moneyness which is closely related to option delta and uses interpolation via a Gaussian kernel to fix moneyness and maturity. For a particular option contract j written on the underlying equity i at date t, moneyness is defined as

$$x_{j,t} = \frac{\log\left(K_j/S_{i,t}\right) + \frac{1}{2}I_{j,t}^2\tau_{j,t}}{I_{j,t}\sqrt{\tau_{j,t}}}, \tag{1}$$

where K_j is the strike price, $S_{i,t}$ is the underlying or spot price, $I_{j,t}$ is the implied volatility obtained from inverting the BSM equation of option contract j at time t and $\tau_{j,t}$ is the remaining time to maturity for contract j at time t. Under BSM assumptions the return volatility until the maturity is $I\sqrt{\tau}$. Therefore, moneyness measure x indicates how many standard deviations the log strike is away from the average log return under the risk-neutral measure.

To interpolate implied volatilities to a standard set of moneyness and maturities, I use Gaussian kernel weights. For a given target moneyness x and maturity τ, the weight on a particular option contract j at time t is

$$w_{j,t}(x,\tau) = \left(1 - |\delta_{j,t}|\right)\mathbb{I}_{|\delta_{j,t}|<0.8}\exp\left(-\frac{(x_{j,t}-x)^2}{2h_x^2}\right)\exp\left(-\frac{(\ln\tau - \ln\tau_{j,t})^2}{2h_\tau^2}\right), \tag{2}$$

where $\delta_{j,t}$ is option delta, $\mathbb{I}_A$ is an indicator of event A and h_x and h_τ are standard Gaussian bandwidths. The interpolated implied volatility at target x and τ is then

$$I(x,\tau;\mathcal{J}(i,t)) = \sum_{j\in\mathcal{J}(i,t)} \frac{w_{j,t}(x,\tau)}{\sum_{j\in\mathcal{J}(i,t)} w_{j,t}(x,\tau)} I_{j,t}, \tag{3}$$

where $\mathcal{J}(i,t)$ is the set of available option contracts for firm i at time t. To simplify the notation I will use the following definition for the rest of the paper

$$I_{i,t}(x,\tau) \equiv I(x,\tau;\mathcal{J}(i,t)). \tag{4}$$

For a particular firm i, date t and maturity τ, I define implied volatility skew

$$Skew_{i,t}(\tau) \equiv I_{i,t}(-2.0,\tau) - I_{i,t}(0.0,\tau), \tag{5}$$

the difference in implied volatility for out-of-the-money put options with $x = -2.0$ and at-the-money options with $x = 0.0$. The results are not sensitive to a particular choice of relative moneyness.

Fundamental variables I use a standard approach to construct fundamental variables. For balance sheet information I use quarterly reports and require a lag of at least 3 months to make sure the information was made public. I then calculate market-to-book MB following Fama and French (1993), Operating Profitability OP as Revenue less Costs of Goods Sold normalized by assets (Novy-Marx, 2013), Asset growth AG defined as quarter on quarter assets growth rate, and Leverage Lev defined as long-term debt normalized by assets. While Leverage is not a standard variable used in empirical asset pricing, it is important to include it when studying option prices, as leverage can significantly change the distribution of equity returns and is significantly correlated with market-to-book.

To make the estimation stable across different economic regimes, I normalize fundamental variables cross-sectionally following Gu et al. (2020). Each month I rank companies based on their fundamental characteristic (e.g. market-to-book) and map this rank into an interval from -1 to 1. An additional benefit of this approach is an ease of interpretation: varying market-to-book from -0.5 to 0.5 means comparing 25th and 75th percentile firms based on a particular fundamental variable. As is standard in the literature, I exclude utility, financial and unclassified firms based on Fama-French industry classification.

Realized Skewness Anticipating the mechanism of the model, in addition to estimating the effect of firm fundamentals on implied volatility skew, I will also estimate their effect on return distribution. In particular, I will predict realized skewness of daily return from time t to $t+h$ with firm fundamental information available at date t. Realized skewness is defined as

$$RS_{i,t}(h) = \frac{\frac{1}{h}\sum_{k=1}^{h}\left(r_{t+k} - \bar{r}_{i,t}(h)\right)^3}{\left(\frac{1}{h}\sum_{k=1}^{h}\left(r_{t+k} - \bar{r}_{i,t}(h)\right)^2\right)^{3/2}}, \text{ where } \bar{r}_{i,t}(h) \equiv \frac{1}{h}\sum_{k=1}^{h} r_{t+k}. \tag{6}$$

Note that the third moment in the numerator is normalized by scaled second moment in the denominator.

Example of growth and value companies To shed more light on the empirical strategy, in Figure 1 I present an example of the effect that I will further systematically estimate using the cross section of equity options. The top two panels plot, the implied volatility as a function of moneyness defined in equation 1 for two companies. First is NVidia, a graphics chips manufacturer, that rose in its prominence with popularity of BitCoin and Machine Learning. NVidia is on the forefront of innovation and is an example of a growth company. The second one is Kellogg – a food manufacturing company that focuses on breakfast cereals and is a value company. The top left panel plots IVs in the midsts of the Great Financial Crisis and the top right panel plots IVs during a market boom. We first see that NVidia IV is always higher which is to be expected due to more volatile cash flows and growth prospects. It is also the case that for both companies, IVs are larger during an economic recession.

The bottom panels of Figure 1 show the difference of implied volatility at a particular moneyness x and ATM implied volatility $I_{i,t}(x,\tau) - I_{i,t}(0,\tau)$ where $I_{i,t}(x,\tau)$ is defined in equation (4). In simpler terms, I used implied volatilities from the top panels and subtracted their value at $x = 0$. This immediately implies that the value at $x = -2$ on the bottom two panels of Figure 1 is the measure of implied volatility skew defined in equation (5). Here we can clearly see that when the economy is in a recession (bottom-left panel), IV skews for both NVidia (growth) and Kellogg (value) are very similar. On the other hand, when the economy is booming (bottom-right panel), IV skew of Nvidia (growth) is much larger than that of Kellogg (value). The goal of the empirical strategy that I formalize below is to systematically measure the difference in skew between as a function of both market-to-book and the aggregate state.

Empirical strategy The empirical strategy relates firm fundamentals to $Skew_{i,t}(\tau)$ defined in equation (5). The hypothesis states that this relationship is time varying and in particular depends on whether the economy is booming or is in a recession. I use aggregate price dividend ratio PD_t as a proxy for aggregate conditions: when PD is high the economy is in a boom and when it is low the economy is in a bust. In a simple linear empirical model

$$Skew_{i,t}(\tau) = \alpha_t + B(PD_t) \cdot MB_{i,t} + \varepsilon_{i,t} \tag{7}$$

for a fixed τ, some function $B(PD)$ and time fixed effect α_t that captures aggregate variation in skew across firms due to a strong factor structure for IV skew documented in Christoffersen et al. (2018). I further parametrize $B(PD)$ to be linear in PD yielding

$$Skew_{i,t}(\tau) = \alpha_t + (\beta + \gamma \cdot PD_t) \cdot MB_{i,t} + \varepsilon_{i,t}. \tag{8}$$

This specification is the baseline empirical specification. I plot the estimate of the coefficient on market-to-book $(\hat{\beta} + \hat{\gamma} \cdot PD_t)$ along with 95% confidence interval as a function of price-dividend ratio PD in Figure 2 in red. Market-to-book indeed has a heterogenous impact on skew. When the economy is booming shown (right side of the figure) firms with higher market-to-book have higher skew while when the economy is in a downturn (left side of the figure) such firms have lower skew.

There are other variables that may determine the implied volatility skew, for example, leverage is correlated with market-to-book and also affects the return distribution. Other fundamental characteristics may also be correlated with both market-to-book and IV skew leading to an omitted variable bias. Additionally, it might be the case that skew is a property of a particular stock and has nothing to do with market-to-book. To proxy for general tendency of a stock to have a skewed distribution, I include past realized skewness as a control along with realized volatility. Then the saturated version of equation (8) takes the following form

$$Skew_{i,t}(\tau) = \alpha_t + \left(\beta^{MB} + \gamma^{MB} \cdot PD_t\right) \cdot MB_{i,t} + \sum_k (\beta^k + \gamma^k \cdot PD_t) \cdot X^k_{i,t} + B' Z_{i,t} + \varepsilon_{i,t} \tag{9}$$

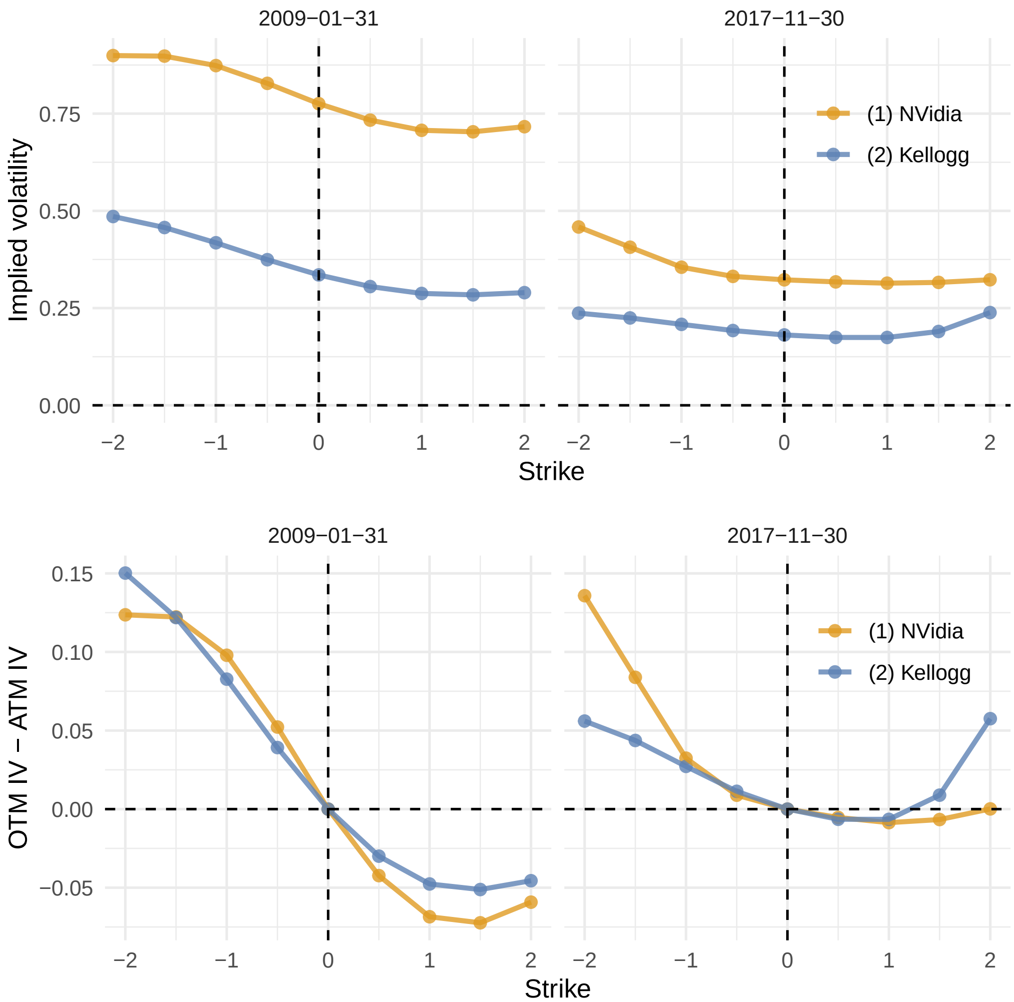


Figure 1: Example of growth and value implied volatilities in different economic regimes

where $k \in \{Lev, OP, AG\}$ and $Z_{i,t}$ includes past realized volatility and skewness. I show the estimate of the coefficient on market-to-book $(\hat{\beta}^{MB} + \hat{\gamma}^{MB} \cdot PD_t)$ as a function PD in Figure 2 in blue. The estimate is very similar both quantitatively and qualitatively with wider confidence band. Now we can't reject the hypothesis that when the economy is in a recession (left part of the figure) market-to-book has an effect on implied volatility skew. Nevertheless, it is still the case that when the economy is booming, firms with higher market-to-book have higher IV skew and the point estimate is significantly higher than in the case of simple model shown in red. The magnitude of the effect is non-trivial. The average difference between 25th and 75th cross-sectional quantile of Skew is 0.095 implying that going from lowest to highest market-to-book firm is associated with an increase of 0.060 – two thirds of the 25-75 interquantile range.

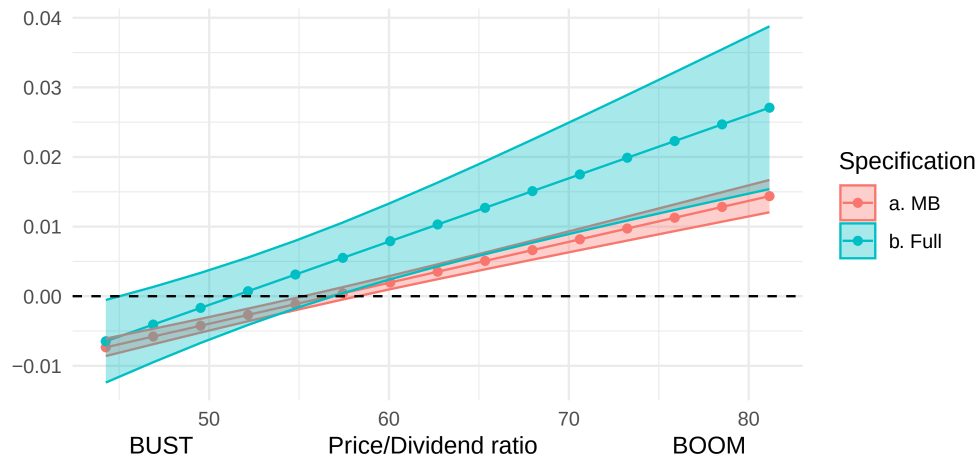


Figure 2: Heterogenous effect of market-to-book on $Skew$

As I showed before, the implied volatility skew goes hand in hand with negative skewness of return distribution, a particular case of a more general result from Breeden and Litzenberger (1978). The caveat is that Breeden and Litzenberger (1978) result describes the risk-neutral as opposed to physical distribution of underlying holding period returns. In the model, outlined in the next section, on the other hand, the mechanism will primarily come from a negative skew in physical distribution. In fact, the difference between risk-neutral physical distribution will be relatively small. Therefore, I need to establish that the same firms that have implied volatility skew, also have a negatively skewed distribution of physical returns going forward.

To test this hypothesis, I will estimate a similar model to equation (9) with realized skewness defined in equation (6). In particular, I estimate

$$RS_{i,t}(h) = \alpha_t + \left(\beta^{MB} + \gamma^{MB} \cdot PD_t\right) \cdot MB_{i,t} + \sum_k (\beta^k + \gamma^k \cdot PD_t) \cdot X_{i,t}^k + B' Z_{i,t} + \varepsilon_{i,t} \tag{10}$$

where h is 3 months or 63 trading days. I present the estimated coefficient on market-to-book in Figure 3 for the baseline and saturated specifications. The red line shows a specification for market-to-book only. The coefficient is significant and negative when the price dividend ratio is high indicating that firms with larger market-to-book have more negatively skewed returns going forward confirming the reasoning from before. When other firm fundamentals are included the relationship stays very similar as can be seen from blue line on Figure 3.

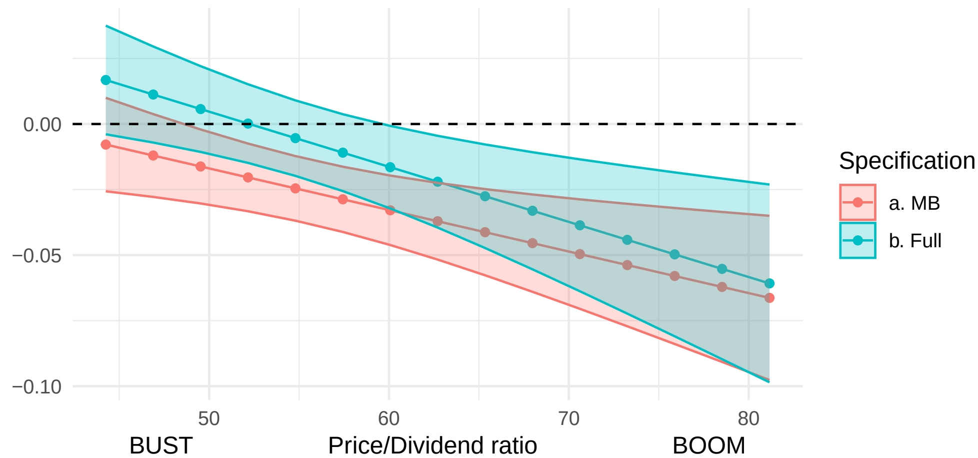


Figure 3: Heterogenous effect of market-to-book on realized return skewness

4 Continuous Time Model with Real Options

In this section, I outline the continuous time model that will highlight the mechanism and intuition behind the empirical results in the previous section. I will start with a simple model with real option similar to Gomes and Schmid (2010) and the textbook exposition in Back (2017). In these models, a firm faces stochastic productivity shocks and possess a perpetual option to expand its capacity. For the model to shed light on the difference between high and low aggregate state I will modify the model to include a time varying price of cash flow risk. With this modification, the firm will optimally exercise its real option when its productivity hits a threshold that depends on the aggregate state.

Set-Up First consider a mature firm that operates with fixed capacity K that produces a stochastic stream of cash flows $x_t K^\alpha$ where x_t is the productivity that follows a geometric brownian motion

$$\frac{dx_t}{x_t} = \mu_x dt + \sigma_x dB_{x,t}. \tag{11}$$

Instead of introducing a representative firm owner with exogenous consumption process as in consumption based asset pricing models, I use an exogenously specified process for the stochastic discount factor π_t:

$$\frac{d\pi_t}{\pi_t} = -rdt - b\sqrt{\lambda_t}\sigma_x dB_{x,t}, \tag{12}$$

where $b\sqrt{\lambda_t}$ is the price of risk of cash flow shocks. Time varying price of risk is the main ingredient to the heterogenous effect of market-to-book on implied volatility skew. I assume that λ_t follows a square root process that is used extensively in interest rate modelling starting from Cox et al. (1985):

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}, \tag{13}$$

where the correlation of cash flow shock and price of cash flow risk shock is $dB_x dB_\lambda = \rho dt$ and $\rho < 0$ so that on average negative cash flows shocks are accompanied by an increase in price of risk. This specification is quantitatively similar to the time varying rare disaster model in Wachter (2013), where the latent disaster intensity follows the same process. When the firm doesn't have an option to expand its productive capacity, the value of the firm with capital K is equal to

$$V^M(x_t, \lambda_t, K) = E_t \left[\int_t^\infty \frac{\pi_\tau}{\pi_t} x_\tau K^\alpha d\tau \right], \tag{14}$$

that can be rewritten as a recursive Bellman equation

$$V^M(x_t, \lambda_t, K) = x_t K^\alpha dt + E\left[\frac{\pi_{t+dt}}{\pi_t} V^M(x_t + dx_t, \lambda_t + d\lambda_t, K) \right]. \tag{15}$$

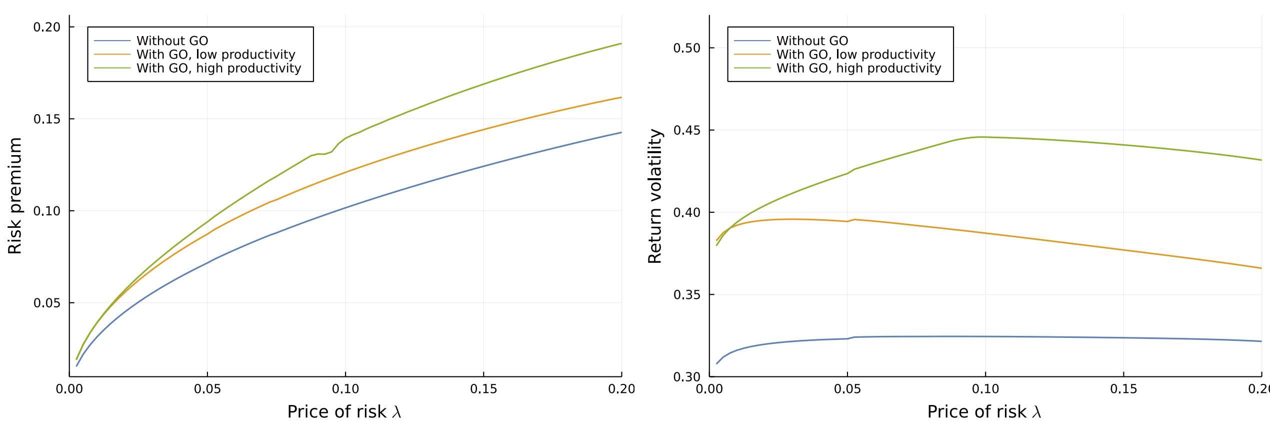
Note that a mature firm doesn't have any control variables.

A young firm, in contrast, has an option to expand its productive capacity from K_0 to K_1 by raising $K_1 - K_0$ from its shareholders. Thus, at any point in time, the firm is choosing between expanding (maturing) and maintaining the same capacity (staying young). Formally, the value of the young firm $V^Y(x_t, \lambda_t, K_0)$ solves the following recursive equation

$$\begin{aligned} V^Y(x_t, \lambda_t, K_0) = \max \Big\{ & V^M(x_t, \lambda_t, K_1) - (K_1 - K_0)\,, \\ & x_t K_0^\alpha dt + E\left[\frac{\pi_{t+dt}}{\pi_t} V^Y(x_t + dx_t, \lambda_t + d\lambda_t, K_0) \right] \Big\} \end{aligned} \tag{16}$$

While the simpler version of this model with constant price of risk does admit a closed form solution as in Gomes and Schmid (2010), in the presence of of time-varying price of risk I resort to solving the model numerically. I discretize transition dynamics of productivity x and price of risk λ via a finite-differences method as in Achdou et al. (2022). I provide more details on this method in appendix A.2.

Calibration I choose the following parameters $\alpha = 0.65, r = 0.08, K_0 = 2.0, K_1 = 5.0, \bar{\lambda} = 0.05, b = 10, \sigma_\lambda = 0.05, \mu_x = 0.03, \sigma_x = 0.1, \rho = -0.5, \kappa = 0.08, \phi = 3.0$. The asset pricing moments implied by this calibration are presented in Figure 4.



(a) Risk premium (b) Volatility

Figure 4: Asset pricing moments for baseline calibration of continuous time model

Model solution The only choice of the firm is whether to exercise its growth option. Optimally, the firm decides to expand its capacity when its productivity hits a price specific value $\overline{x}(\lambda)$. I illustrate this function in Figure 5. Note that the exercise boundary has a concave shape in the price of risk λ. When the price of risk is low and there is a shock that increases it, the firm will be pushed farther away from the exercise boundary relative to when the price of risk is initially high. As a result, the value of the firm will experience a sharper decline.

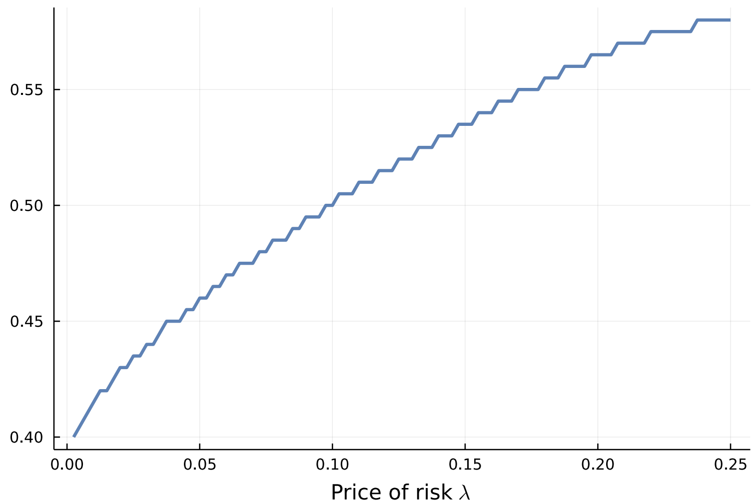


Figure 5: Real option exercise boundary $\overline{x}(\lambda)$

I show the solution of the model, the value of a young firm V^Y as a function of productivity x for a particular price of risk λ in Figure 6. The value of a young firm *without* real options (green dashed line) is below the value of a young firm *with* real options (solid blue line). The gap between two lines represents the value of real options that the firm possesses. When productivity is low, this value tends to zero, as the firm is far from the exercise boundary, and the young firm value smooth pastes into the value of a firm with no real options. In option terminology, the real option is far out of the money. When productivity increases, the gap between the two lines increases representing an increase in option value. Finally, when productivity is sufficiently large the firm exercises its real option and expands. At this point, $x(\lambda)$, the value of a young firm smooth pastes into the value of exercising the real option, the first term in equation (16).

Importantly, we see that when productivity x is large, the sensitivity of the value of a young firm to x increases. In option terminology, the firms has a higher delta with respect to productivity.

Equity dilution The recursive expression for the value of the firm presented in equation (16) makes it amenable to numerical techniques to solve for the value of the young firm V^Y. Deriving return distribution requires an additional step. Suppose that a young firm has 1 share outstanding. When the firm exercises its real option and expands, it raises equity. I assume that the firm issues η new shares to fund the cost $K_1 - K_0$ to expand its capacity. If an investor owned 1 share in this company at time 0, giving him ownership share of 100%, after the firm exercised its option and issued new

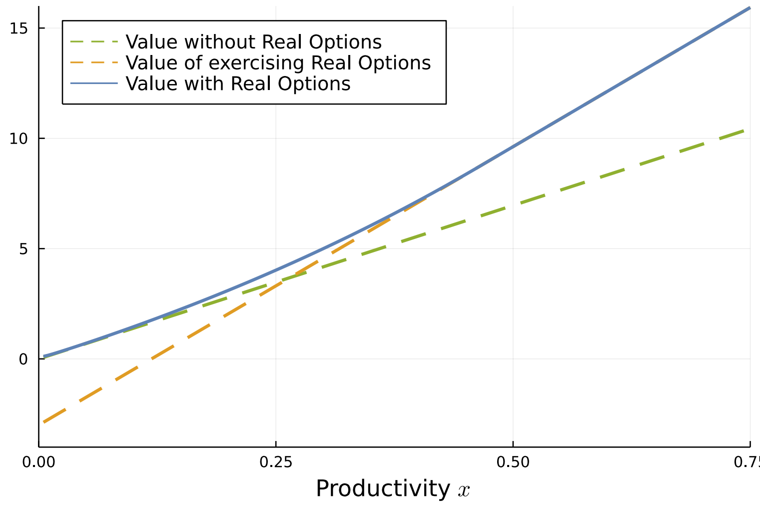


Figure 6: Young firm value

equity this investor will be only entitled to share $\frac{1}{1+\eta} \cdot 100\%$ of the firm at some future time $T > 0$. To derive the amount of shares issued, first observe that the amount raised at post-money valuation should cover the expansion costs

$$\eta(\lambda)(V^M(x(\lambda), \lambda, K_1) - (K_1 - K_0)) = K_1 - K_0, \tag{17}$$

where $\eta(\lambda)$ is a function of price of risk λ, as the exercise boundary $x(\lambda)$ is a function of λ. Expressing $\eta(\lambda)$ we get

$$\eta(\lambda) = \frac{K_1 - K_0}{V^M(x(\lambda), \lambda, K_1) - (K_1 - K_0)}. \tag{18}$$

An important complication of determining dilution is that λ is not the final state at time $T > 0$, (x_T, λ_T), but rather the state at which share issuance occurred. In particular, it is the price of risk λ when the issuance occurred if at all. Formally, the ownership share at time T will depend on $\lambda_{\tilde{t}}$ where $\tilde{t}$ is a stochastic time

$$\tilde{t} = \inf_t \{t : x_t \geq x(\lambda_t)\}, \tag{19}$$

i.e the first time productivity x_t crossed the exercise boundary $x(\lambda_t)$. The dilution over the holding period from $t = 0$ to $t = T$ is

$$\eta_{0\to T} = \begin{cases} 0 \text{ if } \tilde{t} > T \\ \eta(\lambda_{\tilde{t}}) \text{ if } \tilde{t} \leq T \end{cases} \tag{20}$$

Path dependence in returns significantly complicates the derivation of return distribution and requires expanding the state-space with an additional dimension to account for *dilution adjusted flow* of companies into maturity. I explain the additional state variable as well as the approach I take to deriving the return distribution in Appendix A.3.

Return distribution I present return distribution of firms with and without real options in Figure 7 for different starting price of risk λ. First, consider a firm without real option (solid and dashed blue lines). First note that return distribution is not exactly symmetric due to the presence of price of risk shocks. More importantly, changing price of risk λ doesn't change the return distribution of a firm with no real option in a meaningful way. The sensitivity to λ is different when we consider a firm with real options (orange and green lines). When the price of risk is high (green), the distribution is relatively symmetric and resembles the distribution of a firm with no real options (blue) albeit wider as the firm is sensitive not only to changes in the value of assets-in-place, but also to changes in the value of real options. However, when the price of risk is low (orange) corresponding to an aggregate state with high valuations, the distribution changes its form markedly. Instead of being symmetric, it is negatively skewed in line with the going hypothesis and with empirical evidence presented in the previous section.

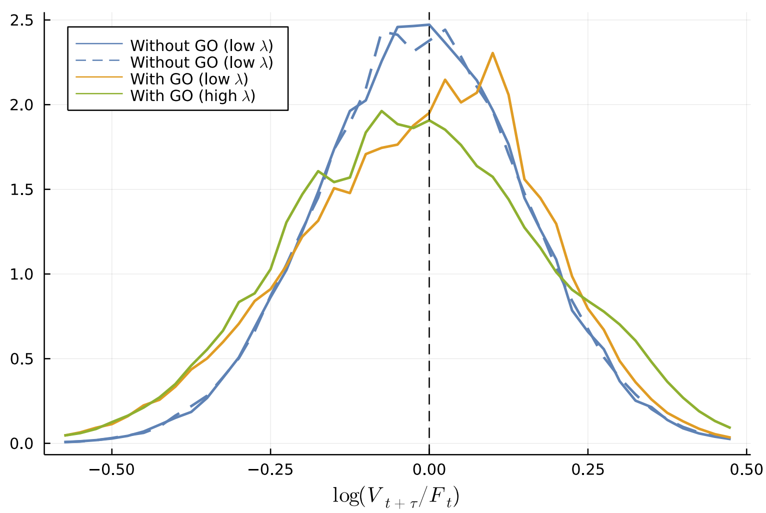


Figure 7: Log return distribution for firms with and without real options

Implied volatility To derive implications for implied volatilities, I use the distributions presented in Figure 7 to derive values of options at different strikes and invert the Black-Scholes formula to obtain implied volatilities. I present implied volatilities for firms with and without real options in Figure 8. A direct consequence of Breeden and Litzenberger (1973) result is that there is a direct connection between return distribution and option prices. Implied volatility, in turn, is simply a normalization that allows to compare option prices at different strikes and maturities. Therefore, analyzing implied volatilities on Figure 8 is very similar to analyzing return distributions in Figure 7.

The first observation in Figure 8 is that firms without real options (blue lines) have very flat implied volatilities. When price of risk is higher (dashed blue line), implied volatility is higher as the return volatility increases, but stays flat. However, firms with real options (orange and green lines) have more interesting implied volatility shapes. When price of risk is low (orange) corresponding to high valuations, there is a significant skew: implied volatility for low strikes is significantly above the at-the-money implied volatility (corresponding to zero log return located at the vertical dashed line). But it is not the case for high prices of risk (green line) that corresponds to low valuations.

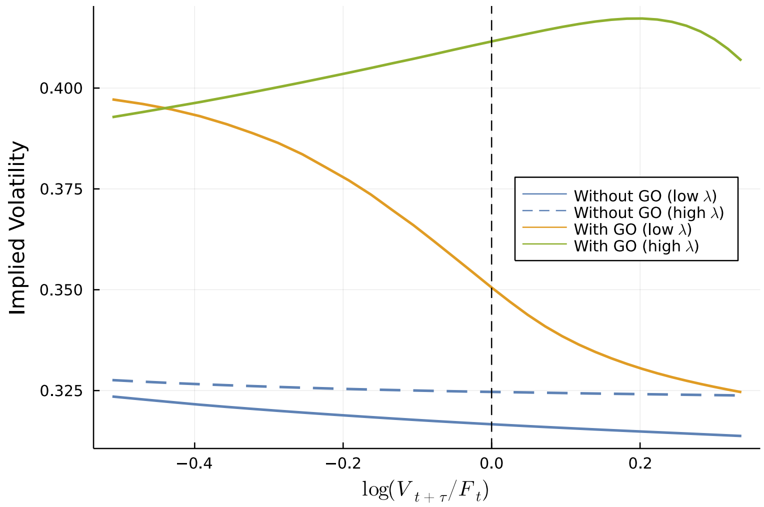


Figure 8: Implied volatility for firms with and without real options

4.1 Delta Hedged Trading Strategies

The literature has proposed several option based strategies based on firm fundamentals. Zhan et al. (2022) sorts companies based on firm fundamentals such as profitability, then takes long positions in covered calls for the top quintile and short positions in covered calls for the bottom quintile. These strategies have large Sharpe ratios that authors claim stay significant even after accounting for transaction costs. While Zhan et al. (2022) points out that these strategies are highly profitable and provide evidence that they are robust to different economic regimes, they do not offer any insights as to why.

Studying the performance of naked option positions is inherently challenging due to a significant asymmetry of returns. For example, a long position in an at-the-money call option has a roughly 50% probability of losing all the value is the option expires out-of-the-money generating a -100% return. On the other hand, a delta-hedged option position is more amenable for both theoretical and empirical analysis. Under BSM assumptions a continuously delta-hedged option position generates a zero return for an investor almost surely. In fact, the possibility to delta-hedge continuously underlies the derivation of BSM equation. Even when only discrete delta-hedging is allowed, the return distribution of a discretely hedged option position is normal with variance decreasing with the square root of the hedging frequency (e.g. Bertsimas et al., 2000, 2001).

Deviations from BSM assumptions like stochastic volatility and their implications for the returns of delta-hedged option positions have also been studied. An important theoretical insight from Bakshi and Kapadia (2003) states that profits of a delta hedged option position are related to variance risk premium. In particular, define the profit as

$$Profit_{t,t+T} = \underbrace{C_{t+T} - C_t}_{\Delta\text{call value}} - \underbrace{\int_t^{t+T} \Delta_u dF_u}_{\text{delta-hedge}} - \underbrace{\int_t^{t+T} r\,(C_u - \Delta_u F_u)\,du}_{\text{financing costs}}\,. \tag{21}$$

The first term is the change in the the value of a long position in a call option. The second term are the profits from the delta-hedge position in the futures contract on the underlying with maturity at

$t+T$: at every instant u, a trader takes a position of size $-\Delta_u$ where Δ_u is the delta of a call option at instant u to make the overall position delta neutral. In the next instant, the trader makes profits on the position in the underlying futures $-\Delta_u(F_{u+du}-F_u) = -\Delta_u dF_u$. The last term is the financing of position. If it is not costless to enter the position ($C_u \neq \Delta_u F_u$), a trader needs to borrow/lend the difference at the risk free rate r.

Bakshi and Kapadia (2003) prove that the expected profits of the position described in equation (21) are

$$E[Profit] = \int_t^{t+T} E_t \left[\frac{\partial C}{\partial \sigma} \theta(\sigma) \right] du, \tag{22}$$

where $\theta(\sigma)$ is the variance risk premium

$$\theta(\sigma) = cov_t \left(-\frac{d\pi_t}{\pi_t}, d\sigma_t \right), \tag{23}$$

C is the price of a call option, σ_t is stochastic volatility, $\partial C/\partial \sigma$ is the sensitivity of a call option to changes in volatility also known as option *Vega* and π_t is the stochastic discount factor process. The model of a firm with real options presented above doesn't feature stochastic volatility explicitly. However, the presence of time varying price of risk makes volatility state dependent. Therefore, it is possible to derive a result similar to Bakshi and Kapadia (2003) for the model with real options where the profits of a strategy will be driven by the price of risk λ.

In order to derive the expected profits of a delta-hedged option positions, it is useful to set some notation to highlight the differences between the standard approach to option pricing taken in the financial engineering literature and the model presented in this paper. The model features 2 state variables x and λ and the value of a call option can be expressed as a function of (x, λ). Denote this function $\widetilde{C}(x, \lambda)$. Similarly, the futures price is also a function of same state variables $F(x, \lambda)$. Alternatively, following financial engineering approach, the value of a call option can be expressed as a function of a futures price $F(x, \lambda)$ and the price of risk λ such that

$$C(F(x, \lambda), \lambda) = \widetilde{C}(x, \lambda). \tag{24}$$

While $\widetilde{C}$ allows to directly calculate the value of an option within the model, C is useful for understanding quantities such as option delta, i.e. $\partial C/\partial F$. Using this notation, I prove the following proposition:

Proposition 1. *(Real options delta-hedged profits) Expected profits from a continuously delta hedged long option position are*

$$E[Profit] = \int_t^{t+\tau} \sigma_x \sigma_\lambda b \rho E_t \left[\frac{\partial \widetilde{C}}{\partial \lambda} \lambda_u \right] du, \quad \frac{\partial \widetilde{C}}{\partial \lambda} \equiv \frac{\partial C}{\partial \lambda} - \frac{\partial F/\partial \lambda}{\partial F/\partial x} \cdot \frac{\partial C}{\partial F} \tag{25}$$

Proof. Appendix A.4 □

First note that the general form of expected profits is similar to the result in Bakshi and Kapadia presented in equation (22). Expected profits are driven by the sensitivity of the call price to the additional source of risk – price of risk λ. If cash flow shocks are not correlated with price of risk shocks ($\rho = 0$), then expected profits are zero. The difference comes from how sensitivity $\partial \widetilde{C}/\partial \lambda$ is calculated. In a standard option pricing set-up including BK, the initial spot price if exogenously given. In contrast, in real options model, the initial value of a company is determined by both productivity x and price of risk λ. The first term in the expression, $\partial C/\partial\lambda$, is similar to BK. However, in an environment where the second state affects the valuation of the stock itself like in a real option model with a stochastic price of risk, we need to correct the sensitivity to remove the effect of λ that comes from the effect on the futures price. This term appears since both the underlying futures price and the value of the option are exposed to price of risk shocks. Therefore, we need to consider the whole portfolio consisting of a long position in a call and a short position in the underlying.

The second expression $\frac{\partial F/\partial\lambda}{\partial F/\partial x}\frac{\partial C}{\partial F}$ is easy to sign. Higher price of risk λ decreases the futures price F and higher productivity x increases it. Delta of a call option $\partial C/\partial F$ is positive. Therefore, the second term has a negative sign. The first term $\partial C/\partial\lambda$, however, can be both positive and negative. On the one hand, higher price of risk λ given the same futures price F decreases the value of an option as it carries more systematic risk. On the other hand, higher λ also raises the volatility, thus, increasing the value of an option. As a result, the senstivity has two competing effects.

Theoretical performance I use the numerical solution to the model with real options to derive implications for expected profits of a delta hedged option strategy described in Proposition 1. While the expected profits depend on the expectation of the full path $(\partial\widetilde{C}/\partial\lambda)\lambda$ over the holding period $[t, t+T]$, we can get a sense of its sign, by considering the sign at the start of the holding period t. Note that as $\sigma_x, \sigma_\lambda, b > 0$, the initial sign is determined by $\rho\, \partial\widetilde{C}/\partial\lambda\Big|_t$. In the numerical solution, I assumed $\rho < 0$ so that decreases in productivity on average coincide with increases in prices of risk. In that case, the sign will be determined by $-\,\partial\widetilde{C}/\partial\lambda\Big|_t$.

I present the initial partial sensitivity $-\,\partial\widetilde{C}/\partial\lambda\Big|_t$ as a function of price of risk λ for different levels of productivity x in Figure 9. First, consider a firm with no real options (thick black line). For such firm, the partial sensitivity is positive implying that the delta hedged position is likely to have negative returns. When a firm has real options, but the productivity is low (dark blue lines), it is not very different from a firm with no real options as its real option is very far out of the money. However, as productivity increases, the real option moves in the money and significantly changes the sensitivity. At first, the partial sensitivity increases leading to even lower profits. Eventually, when productivity is high enough (yellow lines), the partial sensitivity turns negative for low prices of risk λ. As a result, when price of risk is low and productivity is high, the expected profits of a delta hedged strategies turn positive.

To understand the result, it is useful to consider the two offsetting effects that determine the sign of $\partial C/\partial\lambda$. As is mentioned above there is a direct impact from volatility that raises the value of the option. Additionally, higher price of risk simply lower the valuations of all assets with positive exposure to cash flow shocks as is the call option with a positive delta. When price of risk λ is low, return volatility of a more productive firm is more sensitive to changes in the price of risk than return volatility of a less productive firm as can be observed in Panel (b) of Figure 4. As a result, for low λ and high x, the effect coming through volatility dominates the direct effect that lowers the value of the option.

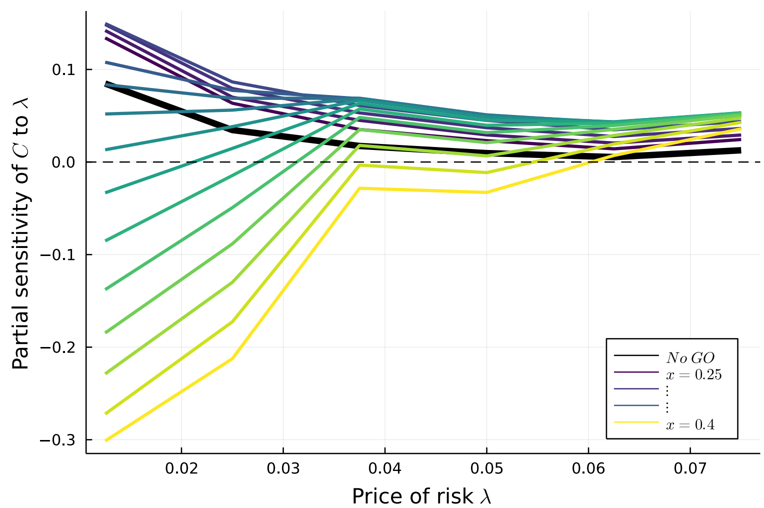


Figure 9: Partial sensitivity of the value of the option to price of risk

4.2 Empirical performance of delta hedged trading strategies

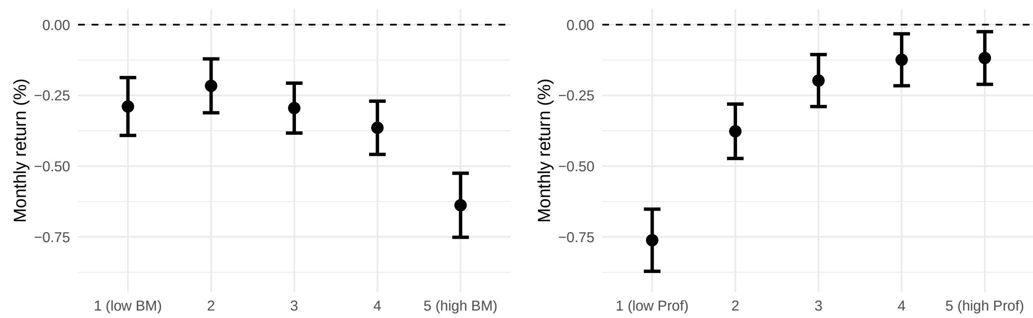
Construction I use the same datasets to construct delta-hedged option strategies based on firm fundamentals. At the end of each month t, I sort companies by a fundamental characteristic (either profitability or market-to-book). Over the next month, I discretely delta-hedge the option position at the closing price to arrive at the discretized version of equation (21)

$$\widehat{Profit}_{t+1} = \underbrace{C_{t+1} - C_t}_{\Delta\text{call value}} + \underbrace{\sum_{h \in \mathcal{H}(t+1)} \Delta_{h-1}(S_h - S_{h-1})}_{\text{delta-hedge}} - \underbrace{\sum_{h \in \mathcal{H}(t+1)} r\left(C_{h-1} - \Delta_{h-1} S_{h-1}\right)}_{\text{financing costs}} \tag{26}$$

where $\mathcal{H}(t+1)$ is the set of days in month $t+1$, S_h is the spot price at day h, C_h – call price at day h and Δ_{h-1} is the option delta at day h. I construct such P&L for each company with option data available so that $Profit_{i,t+1}$ is firm i specific. While Proposition 1 is not specific to a call or a put position, I use a portfolio of 1 call and 1 put also known as a straddle, to evaluate delta-hedged returns. I choose options with maturity closest to 2-months period and strike closest to at-the-money, i.e. with initial delta closest to 0.5.

I then calculate returns on equal weighted portfolios of straddles for 5 bins based on fundamental characteristics sorts at the end of month t. To normalize the profits of the strategies, I follow the practitioner's approach of normalizing by the spot price at the time of entering the trade so that returns are normalized by the unit of exposure. I present profits of a delta hedged strategy for each bin in Figure 10. Panel (a) presents sorts on book-to-market and Panel (b) sorts on operating profitability.

Average returns are significantly different across bins both for market-to-book and profitability sorts. The average returns on the highest book-to-market bin is negative and significantly lower than on the lowest book-to-market bins. Delta hedged straddles formed on value companies lose significantly more money than straddles formed on the growth companies. In the continuous time model in the previous section I showed that firms with real options that are more in-the-money, and thus the firms with lower book-to-market as their real options are a larger share of their value, have higher delta-



(a) Book-to-Market sorts (b) Profitability sorts

Figure 10: Performance of delta hedged strategies based of fundamental sorts

hedged option returns relative to firms with real options that are out-of-the-money – firms with higher book-to-market. The results in Panel (a) confirm this theoretical prediction.

Similarly, in Panel (b) we can observe that straddles formed on high profitability companies lose significantly less money than straddles formed on low profitability firms. In the continuous time model, we observed exactly that. Firms with higher profitability are the ones that are closer to exercising their real options and have higher profits from a delta-hedged option strategy, while low profitability firms have lower returns from the same strategy.

The strategy based on operating profitability sorts is not new and is similar to one of the strategies proposed in Zhan et al. (2022). Instead of a daily delta hedged strategy Zhan et al. (2022) proposed a covered call strategy where they take a delta neutral call position by taking a long position in a call and selling Δ of the underlying without rebalancing the position in the underlying over the holding period of the trade. They report an annualized Sharpe ratio of 2.67 that is higher, but not significantly so, than the Sharpe ratio of 2.30 that I report in Panel (b) of Table 1. At the same time, Panel (a) of Table 1 shows that sorts on book-to-market produce a less impressive Sharpe ratio of 1.04, which is likely the reason that this strategy wasn't reported in Zhan et al. (2022).

Panel (a): sorted on Book-to-Market (BM)

	Low BM	2	3	4	high BM	High - Low
mean (%)	-0.289	-0.216	-0.295	-0.364	-0.638	-0.349
stdev (%)	1.80	1.68	1.56	1.66	2.00	1.16
Sharpe (ann.)	-0.556	-0.446	-0.657	-0.761	-1.11	-1.04

Panel (b): Sorted on Operating Profitability (OP)

	Low OP	2	3	4	High OP	High - Low
mean (%)	-0.762	-0.377	-0.198	-0.124	-0.118	0.644
stdev (%)	1.94	1.69	1.62	1.62	1.64	0.971
Sharpe (ann.)	-1.36	-0.770	-0.422	-0.265	-0.249	2.30

Table 1: Average returns on straddle portfolios formed based on firm fundamentals

5 Discrete Time Model

The continuous time model presented before provides a stylized explanation for higher implied volatility skew of growth companies when the economy is booming modeled as a lower price of risk. In that model, the difference between growth and value companies was driven by a one-shot opportunity to expand productive capacity, and, moreover, this decision was indivisible. In this section, I present a richer model that introduces partial irreversibility of investments, operating leverage and borrowing. This model builds on the large literature of dynamic corporate finance models that were successful in matching asset pricing moments for both debt and equity.

5.1 Firm's problem

The model is set in discrete time. A firm starts each period t with capital K_t, face value of borrowings B_t and two productivity states. x_t is an aggregate productivity state that will affect the SDF discussed below and y_t is an idiosyncratic productivity state. The firm makes two types of decisions. First is whether to continue operating. If the firm decides to stop operating and exits the market due to a low realization of the shock and/or large leverage, equity holders get a continuation value of zero. If the company decides to continue operating, the firm chooses how much to (dis)invest and to borrow to maximize the shareholders' continuation value.

When the firm made its investment and borrowing decisions, it will have the following amount available for distribution to shareholders

$$d(S_t, K_{t+1}, B_{t+1}) = \underbrace{e^{\beta_x x + y} K_t^{\alpha} - c_f}_{\text{Operating cash flows}} + \underbrace{\frac{B_{t+1}}{1+R} - B_t}_{\text{Net borrowings}} - \underbrace{\Phi(K_{t+1}, K_t)}_{\text{Investment costs}} \tag{27}$$

where $e^{\beta_x x_t + y_t}$ is firm's productivity that depends on both aggregate x and idiosyncratic states y, c_f is the fixed costs that acts as the source of operational leverage, $\frac{B_{t+1}}{1+R}$ are the proceeds from promising to repay B_{t+1} in period $t+1$ and Φ is investment cost function that depends on both the current period and next period capital through investments and can be written as $\Phi(K_{t+1,K_t}) \equiv \widetilde{\Phi}(I_t(K_{t+1}, K_t), K_t)$ where $I_t \equiv K_{t+1} - (1-\delta)K_t$. S_t is a short notation for firm's state in period t $S_t = (K_t, B_t, x_t, y_t)$.

The firm faces external financing costs. If the firm wants to distribute/raise d dollars from its shareholders, the shareholders will receive/provide $F(d)$ where

$$F(d) = d + (\chi_d + c_d d)\mathbb{I}_{d<0}. \tag{28}$$

When dividends are non-negative, the shareholders receive the funds available to them, i.e. $F(d) = d$. However, when the firm raises equity and dividends are negative, there are both fixed χ_d and proportional c_d equity issuance costs. Therefore, function $F(d_t)$ determines the period t flow to/from shareholders from/to the firm. The problem of the firm is

$$V_t(S_t) = \max\left\{0, \max_{K_{t+1}, B_{t+1}} F(d(S_t, K_{t+1}, B_{t+1})) + E\left[M(x_{t+1}|x_t)V(S_{t+1})|\, S_t\right]\right\}. \tag{29}$$

Debt pricing When the firm promises to repay B_{t+1} in the next period, there is a possibility that the firm will exit and will not make whole on its promise. The proceeds from borrowing $\frac{B_{t+1}}{1+R}$, therefore, should reflect that possibility and the interest rate on firm's borrowings will deviate from the risk free interest rate. I follow Begenau and Salomao (2019) to price firm's debt. In particular, when the firm promises to repay B_{t+1} the proceeds at time t are

$$\underbrace{\frac{B_{t+1}}{1+R}}_{\text{Proceeds at } t} \equiv \underbrace{E\left[M_{t+1}\mathbb{I}_{V_{t+1}>0}B_{t+1}\right]}_{\text{Not default}} + \underbrace{E\left[M_{t+1}\mathbb{I}_{V_{t+1}=0}\min\{\theta(1-\delta)K_{t+1}, \overline{RC}\cdot B_{t+1}\}\right]}_{\text{Default}}, \tag{30}$$

where the first term corresponds to the company making its debtholders whole and the second term corresponds to default. In the case of default, debtholders can recover a share θ of non-depreciated capital or some maximum recuperation value $\overline{RC}$ of the face value of the borrowings whichever is smaller. The latter component is introduced by Begenau and Salomao (2019) to prevent debtholders from recovering capital in excess of the face value of their debt.

Stochastic discount factor The stochastic discount factor (SDF) is based on an Epstein and Zin (1989) utility investor with an exogenous consumption stream $C(x) = e^x$. Specifically, following Campbell (2018), I define

$$M(x_{t+1}|x_t) = \beta\left(\frac{C(x_{t+1})}{C(x_t)}\right)^{-1/\psi}\left(\frac{U(x_{t+1})}{E_t\left[U(x_{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{-(\gamma-1/\psi)}, \tag{31}$$

where β is the pure discount factor and $U(x)$ is the representative owner's lifetime utility that solves

$$U(x_t) = \left\{(1-\beta)C(x_t)^{1-1/\psi} + \beta\left(E_t\left[U(x_{t+1})^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right)^{1-1/\psi}\right\}^{\frac{1}{1-1/\psi}}. \tag{32}$$

Stochastic states Both aggregate and idiosyncratic states follow AR(1) processes

$$\begin{aligned} x_{t+1} &= (1-\rho)\bar{x} + \rho_x x_t + \sigma_x\varepsilon_{x,t+1} \\ y_{t+1} &= (1-\rho)\bar{y} + \rho_y y_t + \sigma_y\varepsilon_{y,t+1} \end{aligned} \tag{33}$$

where ε_x and ε_y are standard normal random variables.

Mapping model to the data After I solve the model, the next step is to bring the solution closer to the data. The first step is to simulate the model. I simulate 250 economies over 100 quarters to obtain frequencies of each distinct state $S = (K, B, x, y)$. For each distinct state S I solve for option prices and invert the BSM equation to obtain implied volatilities. To mirror the specification estimated in the data, I form right hand side variables similar to the main specification in equation (9) as

$$MB = \frac{V-B}{K},\ Lev = \frac{B}{K},\ Prof = y,\ Inv = K_{t+1} - (1-\delta)K_t. \tag{34}$$

Similarly to the data, I normalize the variables cross-sectionally to be uniform from -1 to 1 and estimate the following regressions

$$Skew_{i,t} = (x\ \text{FE}) + (\beta^{MB} + \gamma^{MB}\cdot x)\cdot MB_{i,t} + \sum_k(\beta^k + \gamma^k\cdot x)\cdot X^k_{i,t} + \varepsilon_{i,t} \tag{35}$$

where $k \in \{Lev, Prof, Inv\}$ where I use x as the aggregate state variable similarly to price-dividend ratio in the data.

Simulation results I compare estimates of the effect of market-to-book on skew in the discrete time model vs empirical estimates in Figure 11.

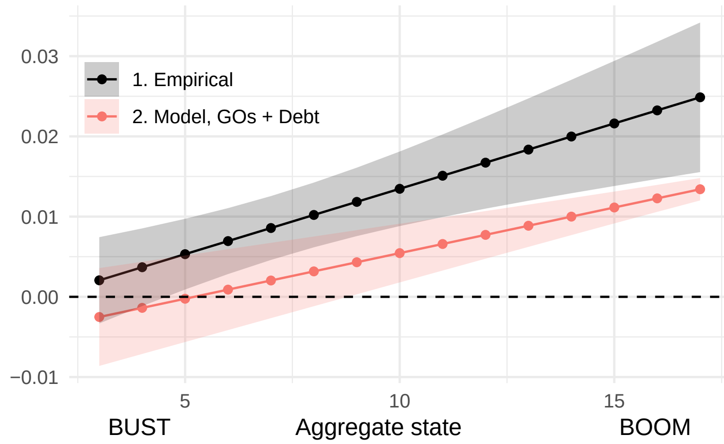


Figure 11: Comparing empirical and discrete model estimates

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A Derivations and Proposition Proofs

A.1 Return Distribution and Implied Volatilities

A.2 Finite Differences Method for Solving the Continuous Time Model

A.3 Kolmogorov Forward Equation for Studying Distributions

A.4 Profits of a Delta Hedged Strategy

B Data Processing

B.1 Option Prices Sample and Filters

Filter	Reason	Example of Source (if possible)
Open interest is greater than zero	Avoid thinly traded options	Bakshi et al. (2003) - BKM
Bid price is greater than zero	Avoid thinly traded options	BKM and sources therein
Bid-Ask spread is positive	Avoid spurious options	Du and Kapadia (2013) - DK
Calls: $K > P_{it}$ and Puts: $K < P_{it}$	OTM options for liquidity	BKM, DK
Calls: $\mathscr{C}_i(t,\tau;K) < P_{it}$	No-arbitrage	
Calls: $\mathscr{C}_i(t,\tau;K) \geq \max(0, P_{it} - PV(K) - PV(D_i(t,\tau)))$	No-arbitrage	
Puts: $\mathscr{P}_i(t,\tau;K) < K$	No-arbitrage	
Puts:$\mathscr{P}_i(t,\tau;K) \geq \max(0, K - P_{it})$	No-arbitrage. Trivially satisfied	
Puts: $\mathscr{P}_i(t,\tau;K) \geq \max(0, PV(K) + PV(D_i(t,\tau)) - P_{it})$	No-arbitrage	
Option has standard settlement	Use standard options	
Option Implied Volatility and Delta Listed	Avoid spurious options	

Notes: This table lists all of the filters that are applied to each firm's option price in order to compute daily measures of $\widehat{\mathbb{D}}_{it}$ as defined in Equation (20). Please refer to Appendix B for more details.

Table 2: Filters applied to option data

B.2 Implied Volatility Interpolation

C Figures and Tables

D Other Content