Asset Pricing Notes. Chapter 2: Static Portfolio Choice

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1 Small Reward for Risk

Before going to the full problem where we consider CARA and CRRA utility functions and use convenient distributions for return to derive closed form solution, it is useful to derive some results for a general case utility functions to see how this approach can take us. We will shortly see that we can show something when the risk is small compared to the wealth of the investor. Here we consider a static portfolio choice with one risk asset and one riskless asset. The objective of the investors is to maximize next period wealth. This is equivalent to having investor consume only in the next period.

Suppose that the investor invests **dollar** amount θ in a risky asset with payoff $\tilde{x} = k\mu + \tilde{y}$ where we will take $k \to 0$ and assume that $E\tilde{y} = 0$. The objective is to maximize

$$\max_\theta V(\theta) = \max_\theta \mathbb{E}u(W_0 + \theta\tilde{x})$$

First order condition w.r.t. θ is

$$\mathbb{E}[\tilde{x}u'(W_0 + \theta^*(k)(\mu k + \tilde{y}))] = 0$$
$$\mathbb{E}[(\mu k + \tilde{y})u'(W_0 + \theta^*(k)(\mu k + \tilde{y}))] = 0$$

where $\theta^*(k)$ means that the optimal decision of how much to invest in the risky asset depends on k. Note that for $k = 0$ the first order condition implies

$$\mathbb{E}[\tilde{y}u'(W_0 + \theta^*(0)\tilde{y})] = 0 \implies cov(\tilde{y}, u'(W_0 + \theta^*(0)\tilde{y})) = 0$$

Since $u'(W_0 + \theta^*(0)\tilde{y})$ is a monotonous decreasing function of $\tilde{y}$ covariation is negative unless $\theta^*(0) = 0$. This is a standard risk aversion result: a risk averse agent always foregoes an investment that gives him zero mean payoff. Differentiate the first order condition w.r.t. k to get

$$E[\mu u'(W_0 + \theta^*(k)(\mu k + \tilde{y})) + \tilde{x}u''(W_0 + \theta^*(k)(\mu k + \tilde{y}))(\theta^{*'}(k)(\mu k + \tilde{y}) + \theta^*(k)\mu)] = 0$$

Evaluate this expression for $k = 0$ and use $\theta^*(k) = 0$

$$E[\mu u'(W_0) + \tilde{y}u''(W_0)\theta^{*'}(0)\tilde{y}] = 0 \implies \theta^{*'}(0) = -\frac{\mu u'(W_0)}{E[\tilde{y}^2]u''(W_0)}$$

Now that we have both $\theta^*(0)$ and $\theta^{*'}(0)$ we can write the Taylor expansion of function $\theta^*(k)$ around a small gamble $k = 0$

$$\theta^*(k) = \theta^*(0) + \theta^{*'}(0)k = 0 - \frac{k\mu u'(W_0)}{E[\tilde{y}^2]u''(W_0)} = \frac{E[\tilde{x}]}{E[\tilde{x}^2](-u''(W_0)/u'(W_0))} = \frac{E[\tilde{x}]}{Var(\tilde{x})A(W_0)}$$

We get mean-variance ratio divided by coefficient of Absolute Risk Aversion. The share of wealth invested in the risky asset is given by

$$s^*(W_0) = \frac{\theta^*(k)}{W_0} = \frac{E[\tilde{x}]}{Var(\tilde{x})}\frac{1}{R(W_0)} \text{ where } R(W_0) = -\frac{W_0 u''(W_0)}{u'(W_0)}$$

$R(W_0)$ – coefficient of relative risk aversion. As we will see in more specific cases below this is a recurring theme of static portfolio choice problems: an investors invests in a risky asset proportionally to its mean payoff and inversely proportional to its variance and some notion of risk aversion.

2 CARA-Normal Case

The simplest example of Preferences-Distribution assumptions that work is CARA (Constant Absolute Risk Aversion) and Normal risks. This case is particularly tractable and will prove to be useful when considering Rational Expectations Equilibrium much later. We will also see that CARA-Normal gives the same portfolio choice as mean variance optimization that lies at the heart of CAPM[1].

The problem is the following

$$\max_\theta E[-\exp(-A(W_0 + \theta\tilde{x}))] \text{ where } \tilde{x} \sim \mathcal{N}(\mu, \sigma^2)$$

Use moment generating function for normal $E(\exp(\mathcal{N}(\mu,\sigma^2))) = \exp(\mu + \frac{1}{2}\sigma^2)$ to write the problem as

$$\max_\theta -\exp(\mathbb{E}[-A(W_0 + \theta\tilde{x}) + \frac{1}{2}Var(-A(W_0 + \theta\tilde{x}))])$$

$$\max_\theta -\exp(-A(W_0 + \theta\mathbb{E}[\tilde{x}]) + \frac{1}{2}A^2\theta^2 Var(\tilde{x}))$$

$$\max_\theta -\exp(-A(\theta\mathbb{E}[\tilde{x}] - \frac{1}{2}A\theta^2 Var(\tilde{x})))$$

$$\max_\theta \theta E[\tilde{x}] - \frac{1}{2}A\theta^2 Var(\tilde{x})$$

First order condition

$$\mu - A\theta^*\sigma^2 = 0 \implies \theta^* = \frac{\mu}{A\sigma^2}$$

Again, amount allocated to the risky asset is proportional to its mean and inversely proportional to its variance and Absolute Risk Aversion. Note the important feature of CARA (which, well, comes from its name). Wealth doesn't show up in the expression above. Hence, the **dollar** amount invested in the risky asset is independent of wealth. However, this is not true for proportional gambles such as return. This means that in general investor with CARA preferences may accept $+20\%/-20\%$ gamble for low levels of wealth and reject for high levels of wealth, i.e. CARA exhibits increasing Relative Risk Aversion. This is problematic if we think about a growing economy with constant proportional risks: since the absolute amount of risk increases the agent becomes more risk averse over proportional gambles over time which results in upward sloping risk premium which is counterfactual. Moreover, in dynamic setting we face another difficulty: compounded normal return are no longer normal and converge in the limit to a lognormal distribution

3 CRRA-Lognormal Case

Problems of CARA-Normal case just discussed make another approach more appealing. Consider the case of lognormal returns (so that $r \equiv \log(1+R) \sim \mathcal{N}(\cdot,\cdot)$) . If returns are iid then the geometric average return

$$[(1+R_1)\cdot\dots\cdot(1+R_T)]^{1/T}$$

and take its log

$$\log[(1+R_1)\cdot\dots\cdot(1+R_T)]^{1/T} \sim \frac{1}{T}\sum_{t=1}^{T}\log(1+R_t)$$

In order to proceed we need a convenient utility functions and it turns out it is Constant Relative Risk Aversion (CRRA) utility function. One period investor solves the following problem

$$\max \mathbb{E}_t\left[\frac{W_{t+1}^{1-\gamma}}{1-\gamma}\right] \text{ where } W_{t+1} \sim \textit{lognormal}$$

[1]Mean variance preferences are not equivalent to CARA though. In particular, for not-normal payoffs they may produce weird things. For example, is mean-variance agent is to choose between a lottery that gives 1 for sure and a lottery that gives 2 and 10 with equal probability he will choose the first ones, even though, the second one dominates

When $\gamma < 1$ we can just drop the denominator and consider the problem $E_t W_{t+1}^{1-\gamma}$. When $\gamma > 1$ we need to be more careful. First, drop $1-\gamma$ and convert the problem into a minimization problem.

$$\min E_t[W_{t+1}(1-\gamma)]$$

Focus on case $\gamma < 1$. Then we can take log of the objective function to get problem

$$\log E_t[W_{t+1}^{1-\gamma}] \tag{1}$$

We are going to use a very useful trick. Suppose that x is lognormal random variable. Then $\log(x)$ is normal. We can then use moment generating function to calculate expectation of

$$E\exp(\log(x)) = \exp(E\log(x) + \frac{1}{2}Var(\log(x)))$$

Meaning that if x is lognormal

$$\boxed{logEx = E\log(x) + \frac{1}{2}Var(\log(x))} \tag{2}$$

Use this to calculate expectation in equation (1).

$$\max \log E_t[W_{t+1}^{1-\gamma}] = \max E_t\left[(1-\gamma)w_{t+1}\right] + \frac{1}{2}(1-\gamma)^2 Var(w_{t+1})$$

Where $w_{t+1} \equiv \log W_{t+1}$ and in general lowercase letters denote log of uppercase variables. Budget constraint in logs

$$W_{t+1} = W_t(1+R_{p,t+1}) \implies w_{t+1} = w_t + r_{p,t+1}$$

plug this into the objective function to get

$$\max E_t\left[(1-\gamma)(w_t + r_{p,t+1})\right] + \frac{1}{2}(1-\gamma)^2 Var(w_t + r_{p,t+1})$$

Next cancel $1-\gamma$. Note that if we worked with a problem where $\gamma > 1$ and had max instead of the min at this point this is where we would've changed it back to max:

$$\max E_t\left[r_{p,t+1}\right] + \frac{1}{2}(1-\gamma)Var(r_{p,t+1}) \tag{3}$$

At the first glance this equation may look problematic at first. If $\gamma > 1$ this equation seems to imply that the investor likes variance. However, we need to be careful. Take the first part of the variance in equation 3

$$E_t\left[r_{p,t+1}\right] + \frac{1}{2}Var(r_{p,t+1}) = E_t\left[\log(1+R_{p,t+1})\right] + \frac{1}{2}Var(\log(1+R_{p,t+1})) = \log E_t(1+R_{p,t+1})$$

Therefore, equation 3 becomes

$$\max \log E_t(1+R_{p,t+1}) - \frac{1}{2}\gamma Var(r_{p,t+1}) \tag{4}$$

Investor trades-off log average returns and variance of log-returns. This is a recurring theme that we are going to face further when dealing with CRRA of Epstein-Zin preferences.

Portfolio Return Approximation Next we need to deal with another recurring problem: how to reconcile multiplicative and additive terms when we have logs? Return on the portfolio that consist of α_t share of a risky asset and $1-\alpha_t$ of riskless asset is given by

$$\alpha_t(1+R_{t+1}) + (1-\alpha_t)(1+R_{f,t+1}) \tag{5}$$

we can't easily take logs of this. Fortunately, there is a useful approximation that is more accurate as the time interval goes to zero and is exact in continuous time

Result 1 (Portfolio Return Approximation)**.** *As the time interval shrinks the following approximation becomes better and it is exact in continuous time*

$$r_{p,t+1} - r_{f,t+1} = \alpha_t(r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_t^2 \text{ where } \sigma_t = Var_t(r_{t+1}) \tag{6}$$

Proof. Consider the log excess return on the portfolio

$$\begin{aligned} r_{p,t+1} - r_{f,t+1} &= \log(1 + R_{p,t+1}) - \log(1 + R_{f,t+1}) \\ &= \log(\alpha_t(1 + R_{t+1}) + (1 - \alpha_t)(1 + R_{f,t+1})) - \log(1 + R_{f,t+1}) \\ &= \log\left(1 + \alpha_t \frac{1 + R_{t+1}}{1 + R_{f,t+1}}\right) \\ &= \log\left(1 + \alpha_t \left(\frac{1 + R_{t+1}}{1 + R_{f,t+1}} - 1\right)\right) \\ &= \log\left(1 + \alpha_t \left(\exp(r_{t+1} - r_{f,t+1}) - 1\right)\right) \end{aligned}$$

Taylor expansion of function $f(x) = \log(1 + \alpha(\exp(x) - 1))$ around $x = 0$ gives

$$\begin{aligned} \log(1 + \alpha(\exp(x) - 1)) &= 0 + \left.\frac{\alpha \exp(x)}{1 + \alpha(\exp(x) - 1)}\right|_{x=0} x + \left.\frac{\alpha \exp(x)(1 + \alpha(\exp(x) - 1)) - \alpha^2 \exp(2x)}{(1 + \alpha(\exp(x) - 1))^2}\right|_{x=0} \frac{x^2}{2} + \bar{o}(x^2) \\ &= \alpha x + \frac{1}{2}(\alpha - \alpha^2)x^2 + \bar{o}(x^2) = \alpha x + \frac{1}{2}\alpha(1 - \alpha)x^2 + \bar{o}(x^2) \end{aligned}$$

Applying this to $r_{p,t+1} - r_{f,t+1}$ we get

$$r_{p,t+1} - r_{f,t+1} \approx \alpha(r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha(1 - \alpha) \underbrace{(r_{t+1} - r_{f,t+1})^2}_{\to \sigma^2 \text{ as time interval } \to dt}$$

□

A way to remember this is that when $\alpha_t = 0$ or 1 we should have

$$r_{p,t+1} - r_{f,t+1} = r_{t+1} - r_{f,t+1} \text{ or } r_{t+1} - r_{f,t+1} = 0$$

Under this approximation we can see that $Var(r_{p,t+1}) = \alpha_t^2 \sigma_t^2$. Plug the expression for $r_{p,t+1}$ from (6) into equation (3) to get

$$\max_{\alpha_t} E_t \left[r_{f,t+1} + \alpha_t(r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t(1 - \alpha_t)\sigma_t^2\right] + \frac{1}{2}(1 - \gamma)\alpha_t^2\sigma_t^2$$

$$\max_{\alpha_t} r_{f,t+1} + \alpha_t(E_t r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t\sigma_t^2 - \frac{1}{2}\gamma\alpha_t^2\sigma_t^2$$

First order condition

$$E_t r_{t+1} - r_{f,t+1} + \frac{1}{2}\sigma_t^2 = \gamma\alpha_t\sigma_t^2$$

$$\alpha_t = \frac{E_t r_{t+1} - r_{f,t+1} + \frac{1}{2}\sigma_t^2}{\gamma\sigma_t^2} \tag{7}$$

The solution to portfolio choice problem is again proportional to expected return and inversely proportional to its variance and risk aversion.

Growth-Optimal Portfolio Notice another important thing about the problem in equation (3). When $\gamma = 1$ and we have log-utility investor the variance term cancels and the investors ends up solving

$$\max \mathbb{E}_t[r_{p,t+1}] \tag{8}$$

Such portfolio is called growth optimal portfolio. Growth optimal portfolio has the property that when investment horizon increases it outperforms every other portfolio with increasing probability. To see this note that from the maximization problem

$$r_{t+1}^{GO} - r_{p,t+1} \sim \mathcal{N}(\Delta, \sigma^2) \text{ where } \Delta > 0$$

Under the assumption of iid return as horizon increases this different becomes

$$r^{GO}_{t+k} - r_{p,t+k} \sim \mathcal{N}(k\Delta, k\sigma^2) \text{ where } \Delta > 0$$

Recall that the probability that for normal

$$\mathbb{P}(r^{GO}_{t+k} - r_{p,t+k} < 0) = \Phi\left(\frac{0 - k\Delta}{\sqrt{k}\sigma}\right) = \Phi\left(-\sqrt{k}\frac{\Delta}{\sigma}\right) \to_{k\to\infty} 0$$

this goes to zero as the investment horizon increases.

It will be erroneous, however, to claim that every investor should, therefore, pick a growth optimal portfolio. As we will in later chapters covariance with something that the investors cares about (e.g. consumption) is the right measure that determines portfolios allocation.