$$T = \frac{K^{-1} S r^2}{6} - \frac{a}{r} + b$$

~~ATTn~~

$$\Delta T = -a \frac{1-\eta^2}{d\,\eta} + \frac{K^{-1} S}{6} \frac{d^2}{1-\eta}$$

$\cancel{r},\ a \to a\, d\, \Delta T$

$s \to s \frac{\Delta T\, \nu}{d^2}$

$$\Rightarrow 1 = -a \frac{(1-\eta)^2}{\eta} + \frac{K^{-1} S\, d^3}{6\, d^2} \left(\frac{1}{1-\eta} \right)$$

$$= \boxed{-a \frac{(1-\eta)^2}{\eta} + S\, P_t \frac{1}{1-\eta} = 1}$$

$$\frac{\partial T}{\partial r} = \frac{S r}{3K} + \frac{a}{r^2}$$

$a =$

$$-K \frac{\partial T}{\partial r} = -K \left(\frac{S r}{3K} + \frac{a}{r^2} \right)$$

$$-K \frac{\partial T}{\partial r}\bigg|_{r_0} = -K \left(\frac{S r_0}{3K} + \frac{a}{r_0^2} \right)$$

$$T = T_s(r) + \Theta(\vec{r}) \quad , \quad C = C_s(r) + \chi(\vec{r})$$

$$\rho = \rho_0 + \frac{\partial \rho}{\partial T}\delta T + \frac{\partial \rho}{\partial C}\delta C = \rho_0\left[1 - \alpha\,\delta T - \alpha'\delta C\right]$$

$$\alpha = -\frac{1}{\rho_0}\frac{\partial \rho}{\partial T} \gtrless 0$$

$$\alpha' = -\frac{1}{\rho_0}\frac{\partial \rho}{\partial C} \gtrless 0$$

$$R_T^* = +\rho_0 \gamma \alpha$$

$$R_C^* = +\rho_0 \gamma \alpha'$$

$$\rho\vec{g} = -\rho\gamma r\,\hat{r}$$

$$= -\rho_0 \gamma r\left[1 - \alpha\,\delta T - \alpha'\delta C\right]\hat{r}$$

$$\rightarrow \left[R_T^* r\,\delta T + R_C^* r\,\delta C + \cancel{\rho_0 \gamma r}\right]\hat{r}$$

$$\rightarrow (R_T^*\,\delta T\, r + R_C^*\,\delta C\, r)\,\hat{r}$$

$$\Rightarrow \underline{R_T^*\,\Theta\,\vec{r} + R_C^*\,\chi\,\vec{r}}$$

Now, all of this only works if Θ and χ are small perturbations above a static profile T_s and C_s. So, the static profiles must already be in accordance with the expected convecting regime.

Temperature: $R_t = \frac{\alpha \gamma d^4}{\nu^2} T^*$, $P_t = \nu/\kappa$

Profiles:	conduction (0)	Internal Heating (1)
$T_s(r)/T^*$	$\left(\frac{\eta}{r} - (1-\eta)\right)/\left[(1-\eta)^2 P_t\right]$	$\left(\frac{r^2}{2} - 1\right)/P_t$
T^*	$\beta d \nu/\kappa$	$\beta d^2 \nu/\kappa$
β	$\Delta T/d$	$\beta = \Delta T/(r_0^2 - r_i^2)$

Composition: $R_c = \frac{\alpha' \gamma d^4}{\nu^2} C^*$, $P_c = \nu/D$

Profiles:	Homog. (0)	Diffusive (2)	Internal sources (1)
$C_s(r)/C^*$	1	$\left(\frac{\eta}{r(1-\eta)^2} - \frac{1}{1-\eta}\right)/P_c$	$\left(\frac{r^2}{2} - 1\right)/P_c$
C^*	$\beta' \cancel{C_0}$	$\beta' d \nu/D$	$\beta' d^2 \nu/D$
β'	C_0	$\Delta C/d$	$\Delta C/(r_0^2 - r_1^2)$

Boundary conditions: Thermal case.

We assume a temperature of the form:

$$T(\vec{r},t) = T_s(r) + \Theta(\vec{r},t) \quad , \quad \Theta \ll T_s$$

In the absence of flow the temperature eq. can be ~~written~~, for the static profile:

$$\cancel{\frac{\partial T_s}{\partial t}}^{0} = \frac{1}{P_t}\nabla^2 T_s + S$$

$$\Rightarrow \nabla^2 T_s = S \cdot P_t$$

$\boxed{S=0}$

$$\nabla^2 T_s = 0 \Rightarrow \frac{1}{r^2}\partial_r\left(r^2 \partial_r T_s\right) = 0$$

$$\Rightarrow \partial_r\left(r^2 \partial_r T_s\right) = 0 \Rightarrow$$

$$\Rightarrow \left[r^2 \partial_r T_s\right]_{r_i}^{r} = cst = \beta$$

$$\Rightarrow r^2 \partial_r T_s = \beta + r_i^2 T_s'(r_i)$$

$$\Rightarrow T_s'(r) = \frac{\beta}{r^2} + \frac{r_i^2}{r^2} T_s'(r_i)$$

$$\Rightarrow T_s'(r) = \frac{T_s'(r_0)\, r_0^2}{r^2}$$

$$\left[T_s(r)\right]_{r_i}^{r_0} = T_s'(r_0)\, r_0^2 \left[-\frac{1}{r}\right]_{r_i}^{r}$$

$$r_i = r_0 - 1 = \frac{1}{1-m} - \frac{1-m}{1-m} = \frac{m}{1-m}$$

$$\frac{r_0}{r_0} = \frac{r_i+1}{r_0}$$

$$\Rightarrow 1 = m \pm \frac{1}{r_0}$$

$$r_0 = \frac{1}{1-m}$$

$$\beta = T_s'(r_0)\, r_0^2 - T_s'(r_i)\, r_i^2 = T_s'(r_0)\frac{1}{(1-m)^2} - T_s'(r_i)\left(\frac{m}{1-m}\right)^2 = \frac{T_s'(r_0) - m^2 T_s'(r_i)}{(1-m)^2}$$

$\Rightarrow$

$$T_S(r) = T_S(r_i) - T_S'(r_0)\, r_0^2 \left(\frac{1}{r} - \frac{1}{r_i} \right)$$

$$= T_S(r_i) - T_S'(r_0) \left(\frac{1}{1-\eta} \right)^2 \left(\frac{1}{2} - \frac{1-\eta}{\eta} \right)$$

$$= T_S(r_i) + T_S'(r_0) \left(\frac{1}{\eta(1-\eta)} \right) - \frac{T_S'(r_0)}{(1-\eta)^2}\, \frac{1}{r}$$

$$T_S(r_0) = T_S(r_i) + T_S'(r_0) \frac{1}{\eta(1-\eta)} - \frac{T_S'(r_0)}{1-\eta}$$

$$= T_S(r_i) + \frac{T_S'(r_0)}{1-\eta} \left(\frac{1-\eta}{\eta} \right)$$

$$= T_S(r_i) + \frac{T_S'(r_0)}{\eta}$$

$$T_S'(r_0) = \eta \Delta T_S$$

$$T_S(r) = T_S(r_i) - T_S'(r_0) \left(\frac{1}{1-\eta} \right)^2 \left(\frac{1}{r} - \frac{1-\eta}{\eta} \right)$$

or

$$T_S(r) = T_S(r_i) - \Delta T_S\, \eta \left(\frac{1}{1-\eta} \right)^2 \left(\frac{1}{r} - \frac{1-\eta}{\eta} \right)$$

So! If there are no internal sources, I can specify either the top and bottom temperatures or a temperature and a flux! The flux at the other boundary will immediately follow and differ from the first, only through a geometric factor.

The case with $S \neq 0$

$$\partial_r \left(r^2 \partial_r T_S \right) = K^{-1} r^2 S$$

$$\Rightarrow T_S = b - \frac{a}{r} + \frac{K^{-1}}{6} r^2 S \quad , \ a, b \text{ to be determined}$$

It has to match:

$$\left(T_S(r_i) = T_i \text{ or } T_S'(r_i) = T_i' \right) \text{ and } \left(T_S(r_o) = T_o \text{ or } T_S'(r_o) = T_o' \right)$$

Fixed T at boundaries:

$$T_S(r_i) = b - \frac{a}{r_i} + \frac{K}{6} r_i^2 S = T_i \Rightarrow b = T_i + \frac{a}{r_i} - \frac{K}{6} r_i^2 S$$

$$T_S(r_o) = b - \frac{a}{r_o} + \frac{K}{6} r_o^2 S = T_o \Rightarrow b = T_o + \frac{a}{r_o} - \frac{K}{6} r_o^2 S$$

$$\Rightarrow \underbrace{T_i - T_o}_{\Delta T} + a \underbrace{\left(\frac{1}{r_i} + \frac{1}{r_o} \right)}_{\frac{r_i + r_o}{r_i r_o}} - \frac{K}{6} S \left(r_i^2 - r_o^2 \right) = 0$$

$$\Rightarrow a = \left[\frac{K}{6} S (r_i - r_o)(r_i + r_o) - (T_i - T_o) \right] \times \frac{r_i r_o}{r_i + r_o} \quad ,$$

$$\left| \begin{array}{l} d = r_o - r_i \\ m = \dfrac{r_i}{r_o} \end{array} \right.$$

$$\Rightarrow a = \frac{K}{6} S\, d\, r_i r_o - \frac{\Delta T\, r_i r_o}{r_i + r_o}$$

$$= \frac{KS}{6} \frac{d^3 m}{(1-m)^2} - \frac{\Delta T\, d^2 m (1-m)}{(1-m)^2 d (1+m)} = \frac{KS}{6} \frac{d^3 m}{(1-m)^2} - \frac{\Delta T\, d m}{(1-m^2)}$$

$$b = T_o + \frac{K S d^2 m}{6(1-m)} - \frac{\Delta T\, m}{1+m} - \frac{K}{6} \frac{S d^2}{(1-m)^2}$$

$$= T_o - \frac{\Delta T\, m}{1+m} + \frac{K S d^2}{6(1-m)} \left(m - \frac{1}{1-m} \right)$$

$$\Rightarrow r_o = d + \frac{r_i r_o}{r_o}$$

$$\Rightarrow r_o = d + m r_o$$

$$\Rightarrow r_o (1-m) = d$$

$$\Rightarrow r_o = \frac{d}{(1-m)}$$

$$r_i = r_o - \frac{d(1-m)}{(1-m)} = \frac{d m}{(1-m)}$$

$$T_s(r) = T_0 - \Delta T \frac{\eta}{1-\eta} + \frac{KSd^2}{6(1-\eta)} + \dots$$

So, in the general case, one can impose:

(1) – Temp. at top and bottom;

(2) – Temp. at top; Q at bottom;

(3) – Q at top; T at bottom;

(4) – Q at Top; Q at bottom $s \neq 0$.

With $s = 0$, (4) does not make sense as $Q_\sigma = f(Q_i)$.

Adimensionalization

$$R_T = \frac{\gamma \alpha_T d^4 \Delta T}{\nu^2}$$

$\vec{u} \to \frac{\nu}{d}\vec{u}$; $\vec{\nabla} \to \frac{1}{d}\vec{\nabla}$

$\vec{r} \to d\,\vec{r}$

$t \to \frac{d^2}{\nu} t$

$\vec{B} \to \frac{\nu}{d}\sqrt{\mu C_0}\,\vec{B}$

$\Delta T \to \Theta T^*$; $\Delta C \to X C^*$

$$\frac{\partial T}{\partial t} = -\vec{u}\cdot\vec{\nabla}T + K\nabla^2 T + S; \quad T = T_s^* + \Delta T \; ; \; \partial_t T_s^* = 0$$

$$\Rightarrow \frac{\nu}{d^2}\frac{\partial \Delta T}{\partial t} = -u_r \partial_r T_s^* + \vec{u}\cdot\vec{\nabla}\Delta T + K\nabla^2(T_s^* + \Delta T) + S\cancel{\frac{T^*}{d^2}}$$

$$\Rightarrow \frac{\cancel{T^*}\nu}{\cancel{d^2}}\frac{\partial\Theta}{\partial t} = -\frac{\nu\cancel{T^*}}{\cancel{d^2}} u\,\partial_r T_s - \frac{\nu\cancel{T^*}}{\cancel{d^2}}\vec{u}\cdot\vec{\nabla}\Theta + \frac{K\cancel{T^*}}{\cancel{d^2}}\nabla^2(T_s + \Theta) + \cancel{\frac{S T^* \nu}{d^2}}$$

$$\Rightarrow \frac{\partial\Theta}{\partial t} = -u\,\partial_r T_s - \vec{u}\cdot\vec{\nabla}\Theta + \frac{1}{P_T}\nabla^2(T_s + \Theta) + S$$

$\Theta = \Theta_m(\theta,\phi,t) R_m(r)$; $T_s = T_{sm} R_m(r)$; $\frac{1}{P_T}\nabla^2 T_s = -S$

$$\sum_m R_m \partial_t \Theta_m = \sum_m \left\{ -u_r T_{sm} \partial_r R_m - \vec{u}\cdot\vec{\nabla}_H \Theta - u_r \Theta_m \partial_r R_m + \frac{1}{P_T}\left[\nabla_H^2 \Theta + \nabla_r^2 R_m \Theta_m\right]\right\}$$

CN: Part

$$\sum_m \left\{ R_m \Theta_m^{i+1} - \frac{\delta t}{2P_T}\left[R_m \nabla_H^2 \Theta_m^{i+1} + \nabla_r^2 R_m \Theta_m^{i+1}\right]\right\} = RHS^{i,\,i-1}$$

The best adimensionalyzation for the temperature is to have $T_i = T_0 + \Delta T$.

So $\Delta T = -a\left(\frac{1}{r_i} + \frac{1}{r_0}\right) + \frac{\cancel{\kappa} S}{6\,K}(r_i^2 - r_0^2)$

$$= -a\,\frac{1-\eta}{d}\left(\frac{1}{\eta} + \frac{\eta}{\eta}\right) + \frac{\cancel{\kappa}\,S}{6\,K}\left[\frac{d^2\eta^2}{(1-\eta)^2} - \frac{d^2}{(1-\eta)^2}\right]$$

$$\Rightarrow \Delta T = -a\,\frac{1-\eta^2}{d\,\eta} + \frac{\cancel{\kappa} S}{K6}\,\frac{d^2}{(1-\eta)^2}$$

Now, the only input parameter that relates to a is R_E so we can replace it in the above equation to give:

$$\frac{\nu^2 R_t}{\gamma\,\alpha_T\,d^4} + \frac{S}{6K}\,\frac{d^2}{(1-\eta)^2} = a\,\frac{1-\eta^2}{d\,\eta}$$

The heat flux across the outer boundary is:

$$q_0 = -K\left.\frac{\partial T}{\partial r}\right|_{r_0} = -\frac{a\,K}{r_0^2} - \frac{r_0 S}{3}$$

A natural adimensionalyzation is to make it as the reference.

Then a can be written as a function of q_0 as:

$$a = -\frac{r_0^3 S}{K3} - \frac{q_0\,r_0^2}{K}$$

with $a \to a\,d\,\Delta T$

$S \to \frac{S\,\Delta T\,\nu}{d^2}$

$r_0 = \frac{d}{1-\eta}$

$q_0 \to \frac{\Delta T}{d} K\,q_0$

$$\Rightarrow a\,\cancel{d}\,\cancel{\Delta T} = -\frac{d^{\cancel{3}}\,S\,\cancel{\Delta T}\,\nu}{(1-\eta)^3\,K\,3\,\cancel{d^2}} - \frac{\cancel{\Delta T}\,\cancel{K}\,q_0}{\cancel{d}\,\cancel{K}}\,\frac{d^2}{(1-\eta)^2}$$

$$a = -\frac{S}{(1-\eta)^3}\,P_r - \frac{q_0}{(1-\eta)^2}$$

At the boundaries, either the temperature is fixed or the heat flux is fixed:

- For fixed value, we assume the temperature deviation for the static profile is 0 at the given boundary.

We can say:

$$\sum_m R_m(\lambda_K)\,\Theta_m^{i+1} = 0 \quad , \quad K = i, o$$

Alternatively, one may want to specify a temperature anomaly. Then

$$\sum_m R_m(\lambda_K)\,\Theta_m^{i+1} = \sum_m \Theta_m^{i} R_m(\lambda_K)$$

- For fixed flux boundaries we have:

$$\sum_m \partial_\lambda R_m(\lambda_K)\,\Theta_m^{i+1} = \sum_m \partial_\lambda R_m(\lambda_K)\,\Theta_m^{i} = Q(\lambda_K)$$

Now, $Q(\lambda_K)$ is related to the static profile by

$$Q(\lambda_K) = \frac{\partial T_S}{\partial \lambda}(\lambda_K) =$$

Boundary condition for the flow.

②

$$\bar\nabla\times(\bar\nabla\times\bar r r S) = r\bar\nabla\times\left(\frac{1}{\sin\theta} r\frac{\partial S}{\partial\phi}\hat\theta - r\frac{\partial S}{\partial\theta}\hat\phi\right)$$

$$= \hat r\,\frac{1}{\cancel{r}\sin\theta}\left[\partial_\theta(\sin\theta\,\partial_\theta S) - \frac{1}{\sin\theta}\partial_\phi^2 S\right]\hat r \; +$$

$$+\frac{\hat\theta}{\cancel{r}}\partial_r\left(\cancel{r}\frac{\partial S}{\partial\theta}\right) + \frac{\hat\phi}{r}\partial_r\left(\cancel{r}\frac{1}{\sin\theta}\frac{\partial S}{\partial\phi}\right) = \nabla_H^2 S\,\hat r + \frac{\partial_r}{\cancel{r}}(r\,\bar\nabla_H S)$$

$$(1) \Rightarrow \begin{cases} \dfrac{1}{\cancel{r}\sin\theta}\dfrac{\partial T}{\partial\phi} + \dfrac{1}{\cancel{r}}\partial_r\left(r\dfrac{\partial S}{\partial\theta}\right) = 0 \\[2ex] \dfrac{\partial T}{\partial\theta} - \partial_r\left(r\dfrac{1}{\sin\theta}\dfrac{\partial S}{\partial\phi}\right) = 0 \end{cases}$$

$$\frac{\partial_r}{r}(r\bar\nabla_H S) + \bar\nabla_H(T\bar r) = 0$$

$$\Rightarrow \begin{cases} \dfrac{1}{\sin\theta}\dfrac{\partial T}{\partial\phi} = -\dfrac{\partial}{\partial\theta}\left(\cancel{r}\,\partial_r S\right) \\[2ex] \dfrac{\partial T}{\partial\theta} = \dfrac{1}{\sin\theta}\partial_\phi\left(\partial_r(rS)\right) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{1}{\sin\theta}\dfrac{\partial Y_\ell^m}{\partial\phi}R_n t_\ell^m = -\partial_\theta Y_\ell^m\,\partial_r(rR_n)\,s_\ell^m \\[2ex] \dfrac{1}{\cancel{\sin}}\dfrac{\partial Y_\ell^m}{\partial\theta}R_n t_\ell^m = \dfrac{1}{\sin\theta}\partial_\phi Y_\ell^m\,\partial_r(rR_n)\,s_\ell^m \end{cases}$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{\partial Y_\ell^m t_\ell^m}{\partial\phi}\Big/\partial_\theta Y_\ell^m = -\partial_r(rR_n)\,s_\ell^m\big/R_n = 0 \Rightarrow \underline{t_\ell^m = s_\ell^m = 0}$$

No penetration

$$\left[\nabla \times (\nabla \times \vec{r} S) + \bar{\nabla} \times \vec{r} T\right]_r = 0$$

$$\Rightarrow -\nabla_H^2 S = 0 \Big|_{r=r_i, r_o}$$

$$\Rightarrow \nabla_H^2 S = 0 \Big|_{r=r_i, r_o}$$

$$\Rightarrow \Rightarrow \sum_{l, m} -\frac{l(l+1)}{r} S_l^m(r) \Big|_{r=r_{i/o}} = 0 \Rightarrow S_l^m(r)\Big|_{r=r_{i/o}} = 0$$

No slip

$$\left[\nabla \times (\nabla \times \vec{r} S) + \bar{\nabla} \times \vec{r} T\right]_H = \vec{0}$$

$$\Rightarrow \begin{cases} \frac{1}{\sin\theta}\frac{\partial T}{\partial \phi} + [\nabla \times (\nabla \times \vec{r} S)]_\theta = 0 \\ -\frac{\partial T}{\partial \theta} + [\bar{\nabla} \times (\bar{\nabla} \times \vec{r} S)]_\phi = 0 \end{cases} \qquad (1)$$

$$\bar{\nabla} \times (\vec{r} T) = \frac{\partial T}{\partial \phi}\frac{1}{\sin\theta}\hat{\theta} - \frac{\partial T}{\partial \theta}\hat{\phi}$$

Free slip or stress free ?

$$\frac{\partial \vec{u}}{\partial r} = 0 \Big|_{r=r_i, r_o}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(\vec{\nabla}\times\left(\vec{\nabla}\times \vec{r} S\right)\right) = 0 \;;\; \partial_r \vec{\nabla}\times\left(\vec{r}\, T\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial S}{\partial r} = 0 \;,\; \frac{\partial T}{\partial r} = 0}$$