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The Glasgow dynamo code ~~makes~~ uses ~~of~~ the following adimensional parameters:

$\eta = \frac{r_i}{r_o}$

$d = r_o - r_i$
$= r_o(1-\eta)$

$P_r = \frac{\nu}{\kappa}$

$T_a = \left(\frac{2\Omega d^2}{\nu}\right)^2$

$R_a = \frac{\alpha \gamma \beta d^6}{\nu \kappa} \quad (R_a = R_a(\eta))$

$P_m = \frac{\nu}{\lambda}$

$C_o = \sqrt{T_a} = \frac{2\Omega d^2}{\nu} = \frac{2}{E}$

$\alpha = -\frac{\partial \rho}{\partial T}\frac{1}{\rho}$

$\beta : T_s = T_0 - \beta d^2 r^2/2$

$\gamma : \vec{g} = -\gamma \vec{r}$

$\nu \to$ viscosity

$\kappa \to$ thermal diff.

$\mu \to$ mag. permeability

$\lambda \to$ mag. diff.

$E \to$ Ekman number

In the aspect ratio dynamos we have

$\eta \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$

$P_r = 0.75$

$\left.\begin{array}{l} T_a = 4.0 \times 10^8 \\ C_o = 2 \times 10^4 \end{array}\right\} \to E = 1 \times 10^{-4}$

$R_a = \{6.1 \times 10^4, 1.8e5, 4.1e5, 8.2e5, 1.5e6, 2.5e6, 4e6, 6e6\}$
corresponding in reverse order to η.

$P_m = 1.5$

In this code, time is measured in multiples of $\frac{d^2}{\nu}$, that is:

$$t = \frac{t_G^* \tau_\Theta}{2\Omega} = \frac{t_G^* d^2}{\nu} = t_G^* \frac{1}{E\Omega}$$

Note that this is a different time scaling than the one in the Leeds dynamo code where:

$$t = t_L^* \frac{d^2}{\eta} = t_L^* \frac{1}{2\Omega R_o} .$$

This means that:

$$t_G^* = t_L^* \frac{\nu}{\eta} = t_L^* P_m .$$

For the case of $P_m = 1.5$, one time unit in the Leeds code represent 1.5 time units in the Glasgow code.

The volume ocupied by the shell is

$$\eta = \frac{r_i}{r_o}$$

$$V = \frac{4}{3}\pi(r_o^3 - r_i^3)$$

$$= \frac{4}{3}\pi r_o^3 (1-\eta^3) \quad , \quad r_o = \frac{d}{(1-\eta)^3} \quad r_i = \eta r_o \quad \cancel{r_o = r_i + d = d(\eta+1)}$$

$$= \frac{4}{3}\pi d^3 \frac{(1-\eta^3)}{(1-\eta)^3} \quad , d=1 \text{ in adimensional units}$$

$$= \frac{4}{3}\pi \frac{(1-\eta^3)}{(1-\eta)^3}$$

$$r_o\left(\frac{r_o - r_i}{r_o}\right) = d \Rightarrow$$

$$r_o = \frac{d}{1-\eta}$$

η	$V(\eta)$	$\approx$
0.1	$\frac{4}{3}\pi \frac{(1-1\times10^{-3})}{(0.9)^3} =$	5.74
0.2		8.12
0.3		11.88
0.4		18.15
0.5		29.32
0.6		51.31
0.7		101.93
0.8		255.52

Energy densities

real $\qquad$ adimensional

$$E_b = \frac{1}{2\mu_0}\int B^{*2}\, d\Omega^* \quad , \quad B^* = B\,\frac{\nu(\mu_0\rho)^{1/2}}{d}$$

$$= \frac{\nu^2\rho}{2d^2}\int B^2\, d\Omega^*$$

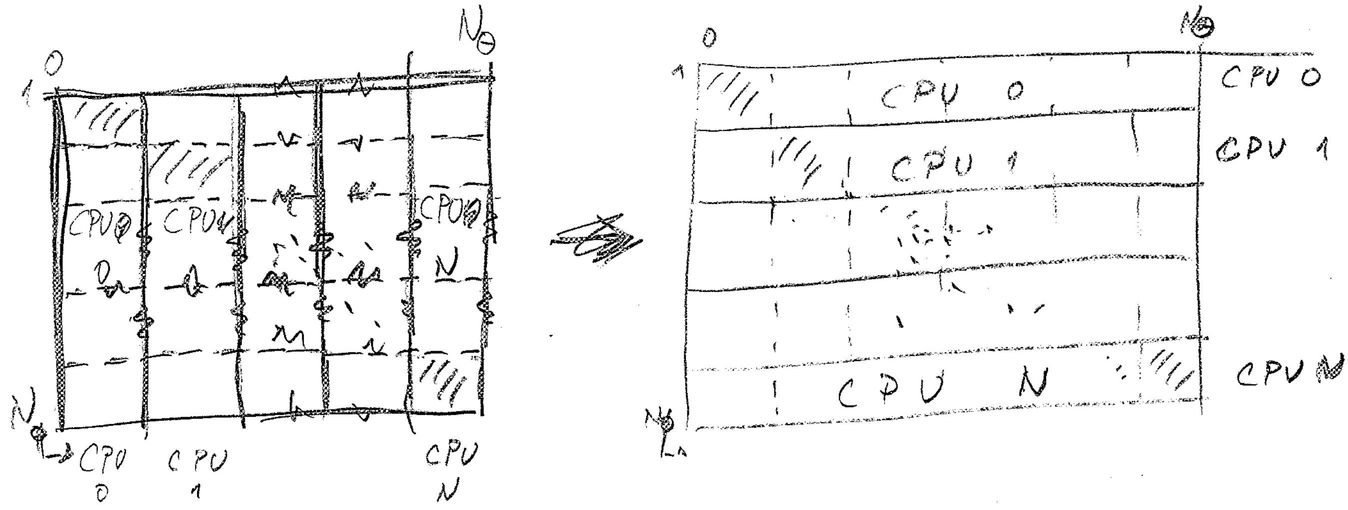
$$E_k = \frac{1}{2}\rho\int v^{*2}\, d\Omega^* \qquad v^* = v\,\frac{\nu}{d}$$

$$= \frac{1}{2}\rho\,\frac{\nu^2}{d^2}\int v^2\, d\Omega^*$$

Scaled by the same factor !!!

Parallelization

$t\_\theta 2\phi$ ⟶



⟵ $t\phi 2\theta$

Send

0 → 1, 2, ..., MPI_SIZE-1

1 → 2, 3, ..., MPI_SIZE-1, 0

2 → 3, 4, ..., MPI_SIZE-1, 0, 1

⋮

MPI_SIZE-1 → 0, 1, ..., MPI_SIZE-2

Receive

0 ← MPI_SIZE-1, MPI_SIZE-2, ..., 1

1 ← 0, MPI_SIZE-1, ..., 2

Eg. $N\phi\_s = 18$, MPI_size = 4

ceil(18/4) = 5 = blk_max_size
= blk_max_size(0)

blk_size = 5

MPI_SIZE = 4

Rank	blk_start(rank)	
0	1	
1	6	= 1+5
2	11	= 6+5
3	16	= 11+5

Rank

CPU0			CPU1			CPU2			CPU3			
1	...	5	6	...	10	11	...	15	16	17	18	φ/m
0	...	4	5	...	9	10	...	14	15	...		θ/l

Discretising the ~~temperature~~ eq. ①

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \bar{u}\cdot\bar{\nabla} T \quad , \quad T = \Theta + T_0 - \beta \frac{d^2 r^2}{2}$$

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T - \bar{u}\cdot\bar{\nabla} T$$

$$= \kappa \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\kappa}{r^2}\nabla_H^2 T - \vec{u}\cdot\bar{\nabla} T$$

$$= \kappa\left(\frac{2}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) + \frac{\kappa}{r^2}\hat{\nabla}_H^2 T - \bar{u}\cdot\bar{\nabla} T$$

$$\frac{\partial T}{\partial t} = \frac{\partial \Theta}{\partial t} = \kappa\left(\frac{2}{r}\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial r^2}\right) - \kappa\left(\frac{2}{r}\frac{\beta d^2 \cdot 2r}{2} + \beta d^2\right) + \frac{\kappa}{r^2}\nabla_H^2 \Theta +$$

$$+ \bar{u}\cdot\bar{\nabla}\Theta - \vec{u}\cdot\bar{\nabla}\left(\beta \frac{d^2 r^2}{2}\right)$$

$$= \kappa\left(\frac{2}{r}\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial r^2} - 3\beta d^2\right) + \frac{\kappa}{r^2}\nabla_H^2 \Theta + \bar{u}\cdot\bar{\nabla}\Theta - \vec{u}\cdot\vec{r}\,\beta d^2$$

$t \to t\, d^2/\nu$

$\Theta \to \Theta \Delta T$

$r \to r d$

$u \to u U$

$U = \nu/d$

$$= \frac{\kappa \Delta T}{d^2}\left(\frac{2}{r}\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial r^2} - \frac{3\beta d^2}{\Delta T} + \frac{\nabla_H^2}{r^2}\Theta\right) + \frac{\nu \Delta T}{d\, d^2} - \frac{\beta \nu d^3}{d}\vec{u}\cdot\hat{r}$$

$$\frac{\nu \Delta T}{d^2}\frac{\partial \Theta}{\partial t} = \frac{\kappa}{\nu}\left(\quad\right) + \boxed{\frac{U}{\nu}d}^{\,1}\,\bar{u}\cdot\bar{\nabla}\Theta - \bar{u}\cdot\hat{r}\,\frac{\beta \nu d^3}{\nu \Delta T}$$

$$\frac{\partial \Theta}{\partial t} = \frac{1}{P_r}\left(\nabla^2\Theta - \frac{3\beta d^2}{\Delta T}\right) + \bar{u}\cdot\bar{\nabla}\Theta - \bar{u}\cdot\hat{r}\boxed{\frac{\beta d^2}{\Delta T}}$$

$$\frac{\partial \Theta}{\partial t} = \frac{1}{P_r}\nabla^2\Theta - 3\frac{\beta d^2}{\Delta T}\frac{1}{P_r} + \bar{u}\cdot\bar{\nabla}\Theta - \frac{\beta d^2}{\Delta T}\bar{u}\cdot\hat{r}$$

2

$$\frac{\partial \theta}{\partial t} = \frac{1}{P_r}\nabla^2\theta \mp \bar{u}\cdot\bar{\nabla}\theta + \frac{\beta d^2}{\Delta T}\left(\bar{u}\cdot\hat{r} + \frac{3}{R}\right)$$

$$\Delta T = \nu^2/\gamma\alpha d^4 \longrightarrow \frac{\beta d^2}{\Delta T} = \frac{\cancel{\nu}\alpha\beta d^6}{\nu^2} = \frac{\nu}{\kappa}\,\frac{\gamma\alpha\beta d^6}{\nu\lambda P_r} = \frac{R_a}{P_r}$$

Discretization in time

$$\frac{\partial \theta}{\partial t} = \frac{\theta^{i+1}-\theta^{i}}{\tau^{i}} = \frac{1}{P_r}\,\frac{\nabla^2\theta^{i}+\nabla^2\theta^{i+1}}{2} + \frac{R_a}{P_r}u_r^{i} - \bar{u}^{i}\cdot\nabla\theta^{i}$$

Approximated by 2nd of A-B

$$\left(\theta^{i+1} - \frac{\tau_i}{2P_r}\nabla^2\theta^{i+1}\right) = \theta^{i} + \frac{\tau_i}{2P_r}\nabla^2\theta^{i} + \boxed{\tau_i\frac{R_a}{P_r}u_r^{i} - \bar{u}^{i}\nabla\theta^{i}\tau_i}$$

$$\theta^{i+1} = \left(\mathbb{1} - \frac{\tau_i}{2P_r}\nabla^2\right)^{-1}\left(\theta_i + \frac{\tau_i}{2P_r}\nabla^2\theta^{i} + \underbrace{\tau_i\frac{R_a}{P_r}u_r^{i} - \bar{u}^{i}\cdot\bar{\nabla}\theta^{i}\tau_i}_{A}\right)$$

(A) $\left(R_a u_r^{i} - u^{i}\cdot\nabla\theta^{i}\right)h_1 - \left(R_a u_r^{i-1} - u^{i-1}\cdot\nabla\theta^{i-1}\right)h_2$

$$\begin{cases} h_1 = \dfrac{h_i}{h_{i-1}}\,\dfrac{1}{2}\left(2h_{i-1} + h_i\right) \\[2ex] h_2 = \dfrac{1}{2}\,\dfrac{h_i}{h_{i-1}}\,h_i \end{cases}$$

2 Step Adams-Bashford
with variable time step

$$y_{n+1} = \int_{t_n}^{t_{n+1}} \left(f_n \frac{t - t_{n-1}}{t_n - t_{n-1}} + f_{n-1} \frac{t - t_n}{t_{n-1} - t_n} \right) dt =$$

$$= -f_n \frac{t_{n-1} h_n}{h_{n-1}} + f_{n-1}\, t_n \frac{h_n}{h_{n-1}} + \frac{f_n}{h_{n-1}} \frac{1}{2} \left[t^2 \right]_{t_n}^{t_{n+1}} - \frac{f_{n-1}}{h_{n-1}} \frac{1}{2} \left[t^2 \right]_{t_n}^{t_{n+1}}$$

$$\left[t^2 \right]_{t_n}^{t_{n+1}} = t_{n+1}^2 - t_n^2 = (t_{n+1} - t_n)(t_{n+1} + t_n)$$

$$= h_n (t_{n+1} + t_n)$$

$$= -f_n\, t_{n-1} \frac{h_n}{h_{n-1}} + f_n \frac{1}{2} \frac{h_n}{h_{n-1}} (t_{n+1} + t_n) + f_{n-1} \left[t_n \frac{h_n}{h_{n-1}} - \frac{1}{2} \frac{h_n}{h_{n-1}} (t_{n+1} + t_n) \right]$$

$$= f_n \frac{h_n}{h_{n-1}} \frac{1}{2} \Big(\underbrace{t_{n+1} + \underbrace{t_n - 2t_{n-1}}_{-t_{n-1} + h_{n-1}}}_{h_n + h_{n-1}} \Big) + f_{n-1} \frac{h_n}{h_{n-1}} \frac{1}{2} \Big(\underbrace{\cancel{2} t_n - t_{n+1}}_{-h_n} \cancel{- t_n} \Big)$$

$$= \underbrace{f_n \underbrace{\frac{h_n}{h_{n-1}}}_{\alpha} \frac{1}{2} \left(2h_{n-1} + h_n \right)}_{h_1} - f_{n-1} \underbrace{\underbrace{\frac{1}{2} \frac{h_n}{h_{n-1}}}_{\alpha}}_{h_2} h_n$$

$$A^{-1} = \left\{ -\frac{\partial^2 R}{\partial r^2} \frac{\sigma_i}{2} - \frac{\left(2 - l(l+1)\right)}{r^2} R \frac{\sigma_i}{2} + R \right\}^{-1}$$

$$u^{i+1} = u^i + \frac{\sigma_i}{2} \nabla^2 u^i + \left(-R_a T^i + \Omega \times u^i + (\nabla \times \bar{B}^i) \times \bar{B}^i\right) \sigma_i \times 1.5 -$$

$$-\left(R_a T^{i-1} + \bar{\Omega} \times u^{i-1} + (\nabla \times \bar{B}^{i-1}) \times \bar{B}^{i-1}\right) \sigma_{i-1}\, 0.5$$

$$u^1 = u^0 +$$

$$u^{i+1} = \cancel{A}$$

$$\frac{\partial \vec{u}}{\partial t} = \nu \nabla^2 \vec{u} + \vec{F}$$

$$\vec{u}_p = \vec{\nabla} \times (\vec{\nabla} \times p \vec{r})$$

$$\vec{u}_T = \vec{\nabla} \times (t\, r^2 \hat{r})$$

$$t = \sum t^m_{nl}(t) \; ; \; p = \sum p^m_{nl}(t)\, R_n Y^m_l$$

$$\begin{cases} \dfrac{\partial \vec{u}_p}{\partial t} = \nu \nabla^2 \vec{u}_p + \vec{F}_p \\[2ex] \dfrac{\partial \vec{u}_T}{\partial t} = \nu \nabla^2 \vec{u}_T + \vec{F}_T \end{cases}$$

$$\Rightarrow \begin{cases} \vec{\nabla} \times (\vec{\nabla} \times (\dot{p}\, \vec{r})) = +\vec{\nabla} \times (\vec{\nabla} \times (\nu \nabla^2 (p \vec{r}))) + \vec{F}_p \\[2ex] \vec{\nabla} \times (\dot{t}\, r^2 \hat{r}) = -\vec{\nabla} \times (\nu \nabla^2 t\, r^2 \hat{r}) + \vec{F}_T \end{cases}$$

$$\sum \dot{p}^{m}_{ml}\, \vec\nabla \times \overbrace{\left[\vec\nabla \times (R_m Y^{m}_{l} \hat{r})\right]}^{\vec{A}^{m}_{ml}} = \sum p^{m}_{ml}\, \nu\, \vec\nabla \times \left(\vec\nabla \times \left(\nabla^2 (\hat{r} R_m Y^{m}_{l})\right)\right) + \vec{F}_P$$

$$\sum \dot{t}^{lm}_{ml}\, \vec\nabla \times (R_m Y^{m}_{l} r^2 \hat{r}) = -\sum t^{m}_{ml}\, \nu\, \vec\nabla \times \left(\nabla^2 (r^2 \hat{r} R_m Y^{m}_{l})\right) + \vec{F}_T$$

$[\nabla \times \nabla \times]_r$

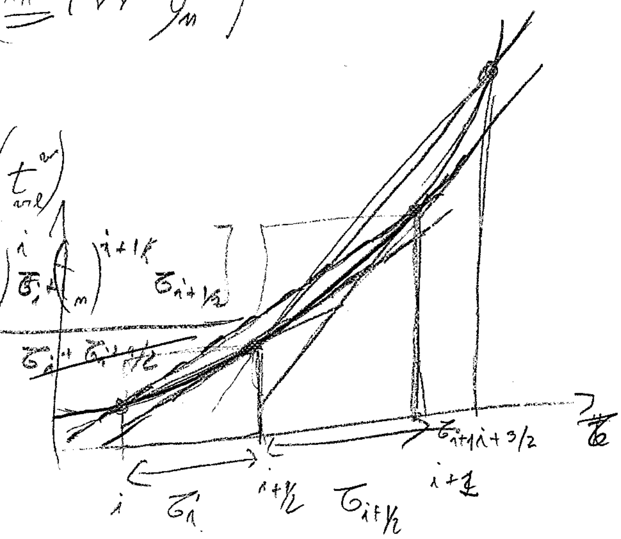
$$\Rightarrow \sum \dot{p}^{m}_{ml} \left[\vec\nabla \times \vec\nabla \times \left(-\nabla^2 (R_m Y^{m}_{l} \hat{r})\right)\right]_r = \sum \nu\, p^{m}_{ml} \dots + \left[\vec\nabla \times \vec\nabla \times \vec{F}_P\right]_r$$

$[\nabla \times]_r$

$$+\sum \dot{t}^{m}_{ml} \frac{l(l+1)}{r^2} R_m Y^{m}_{l} r^2 = -\sum \nu\, t^{m}_{ml} \left[\vec\nabla \times \left(\vec\nabla \times \left(\nabla^2 (r^2 R_m Y^{m}_{l} \hat{r})\right)\right)\right]_r + \left[\vec\nabla \times \vec{F}_T\right]_r$$

$$\dot{t}^{m}_{ml}\, l(l+1) \cancel{r^2} R_m \cancel{Y^{m}_{l}} = -\left\{\sum \nu\, t^{m}_{ml} \left[\nabla \times \nabla \times \left(\nabla^2 (r^2 R_m Y^{m}_{l} \hat{r})\right)\right]_r + \left[\vec\nabla \times \vec{F}_T\right]_r\right\}_{ml}$$

$$\frac{\int \cancel{p\,ls}\, Y^{m}_{l}\, d\Sigma}{l(l+1)} \qquad \left(\underline{RHS} + \nu \nabla^2 t^{lm}_{m}\right)$$



$$\frac{(t^{m}_{ml})^{i+1} - (t^{m}_{ml})^{i}}{\cancel{\sigma_i + \sigma_{i+1/2}}} R = \frac{\sigma_i RHS^{i} + RHS^{i+1} \sigma_{i+1/2}}{\cancel{\sigma_i + \sigma_{i+1/2}}} + \nu \nabla^2 \frac{\left[(t^{m}_{ml})^{i} \sigma_i + (t^{m}_{m})^{i+1} \sigma_{i+1/2}\right]}{\cancel{\sigma_i + \sigma_{i+1/2}}}$$

$$(t^{m}_{ml})^{i+1} R - \left(RHS^{i+1} + \nu \nabla^2 (t^{m}_{ml})^{i+1}\right) \sigma_{i+1/2} =$$

$$(t^{lm}_{m})^{i} R + \left(RHS^{i} + \nu \nabla^2 (t^{lm}_{ml})^{i}\right) \sigma_i$$

$$(t^{m}_{ml})^{i+1} \simeq$$

$$\frac{(t^{lm}_{m})^{i+3/2} - (t^{m}_{ml})^{i+1/2}}{\sigma_{i+1/2} + \sigma_{i+1}} = \frac{\sigma_{i+1/2} RHS^{i+1/2} + \sigma_{i+1} RHS^{i+3/2}}{\sigma_{i+1/2} + \sigma_{i+1}} + \dots$$

(A)

$$\frac{\partial T}{\partial \tau_a} = +\eta \nabla^2 T \quad , \quad T = t(\tau)\, R(r)\, Y(\theta\phi)$$

$$\frac{\partial t}{\partial \tau} R \cancel{Y} = \eta (\nabla_r^2 R) Y t - \eta \frac{l(l+1)}{r^2} t R \cancel{Y}$$

$$\frac{t^{i+1} - t^i}{\tau^i} R = \eta \left(\nabla_r^2 R - \frac{l(l+1)}{r^2} R \right) \left(\frac{t^{i+1} + t^i}{2} \right)$$

$$\left[R - \frac{\eta}{2} \tau_i \left(\nabla_r^2 R - \frac{l(l+1)}{r^2} R \right) \right] t^{i+1} =$$

$$= \left[R + \frac{\eta \tau_i}{2} \left(\nabla_r^2 R - \frac{l(l+1)}{r^2} R \right) \right] t^i$$

$$t_r^{i+1} = \left(R - \frac{\eta \tau_i}{2} \left(\nabla_r^2 R - \frac{l(l+1)}{r^2} R \right) \right)^{-1} \left(R + \frac{\eta \tau_i}{2} \left(\nabla_r^2 R - \frac{l(l+1)}{r^2} R \right) \right) t^i$$

$$\frac{\partial T}{\partial \tau_a} = \vec{u} \cdot \vec{\nabla} T + \eta \nabla^2 T$$

$$\frac{\partial T}{\partial \tau} = t \vec{u} \cdot \vec{\nabla}(RY) + \eta \left(\nabla_r^2 R - \frac{l(l+1)}{r^2} R \right) Y t$$

$$= t \boxed{u_r (\nabla_r R) Y} + t\, \vec{u}_H \cdot \left(\vec{\nabla}_H Y\right) R + \eta \left(\nabla_r^2 R - \frac{l(l+1)}{r^2} R \right) Y t \quad \hat{\Omega}$$

$$t^i \left\{ \left[u_r^i (\nabla_r R) Y + \vec{u}_H^i \cdot \vec{\nabla}_H Y R + \eta \left(\nabla_r^2 R - \frac{l(l+1)}{r^2} R \right) Y \right] \frac{\tau_i}{2} + RY \right\} =$$

$$= t_r^{i+1} \left\{ RY - \left[u_r^{i+1} (\nabla_r R) Y + \quad \cdots \quad \right] \frac{\tau_i}{2} \right\}$$

$$\hat{F}^i$$

$$\cancel{\left(t^i \{ \vec{u}_z^i (\nabla_z R) Y + \vec{u}_H^i \cdot \nabla_H Y R \} + t^i \hat{\Omega}^i\right.}$$

$$\cancel{\Rightarrow t^{i+1} \hat{F}^{i+1} + t^{i+1} \hat{\Omega}^{i+1}}$$

$$\cancel{\Rightarrow t^{i+1} \hat{F}^{i+1} + t^{i+1} \hat{\Omega}^{i+1} - \hat{F}^i = t^i \hat{\Omega}^i}$$

$$\cancel{\Rightarrow t^{i+1} \left\{ (\hat{\Omega}^i)^{-1} \left(\hat{F}^{i+1} + \hat{\Omega}^{i+1} \right) \right\} - (\Omega^i)^{-1} \hat{F}^i = t^i}$$

$$u^{i+1} = (\Omega)^{-1} \left\{ \Omega^i u^i + \hat{F}^i \sigma_i \times \frac{3}{2} - F^{i-1} \sigma_i \frac{1}{2} \right\}$$

$$\Omega^{i+1} u^{i+1} = \Omega^i u^i + F^i \sigma \frac{3}{2} - F^{i-1} \sigma \frac{1}{2}$$

$$\left(u_n^{i+1} - \frac{\nu\sigma}{2} \nabla^2 u^{i+1} \right) = u^i + \frac{\nu\sigma}{2} \nabla^2 u^i + F^i \sigma + \frac{F^i - F^{i-1}}{2} \sigma$$

$$\Rightarrow \frac{u^{i+1} - u^i}{\sigma} = \frac{\nu}{2} \nabla^2 (u^{i+1} + u^i) + F^i + \frac{F^i - F^{i-1}}{2}$$