Persistence of the Conley Index in Combinatorial Dynamical Systems

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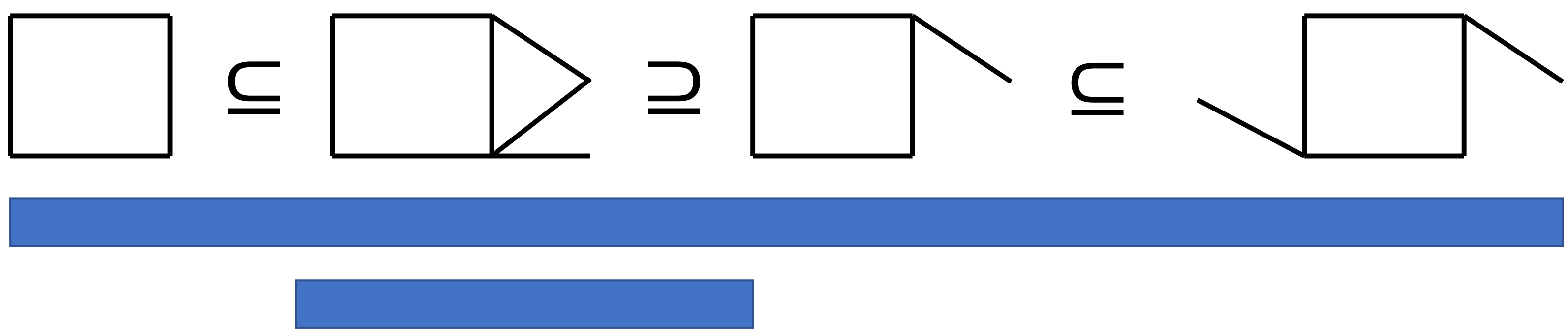
The Ohio State University

Student Presentation

Jagiellonian University in Krakow

Overview

Persistence: capture changing homology of spaces



But what about dynamical systems?

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Motivating Example: Hopf Bifurcation

$$x' = -y + x(\lambda - x^2 - y^2)$$

$$y' = x + y(\lambda - x^2 - y^2)$$

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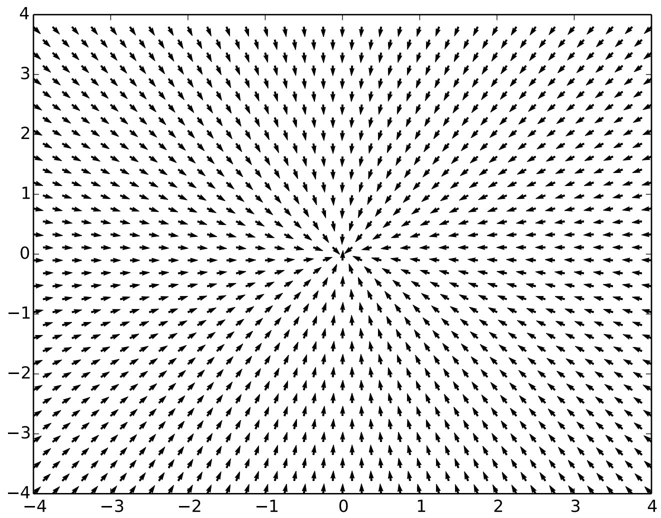
Motivating Example: Hopf Bifurcation



$$\lambda \ll 0$$

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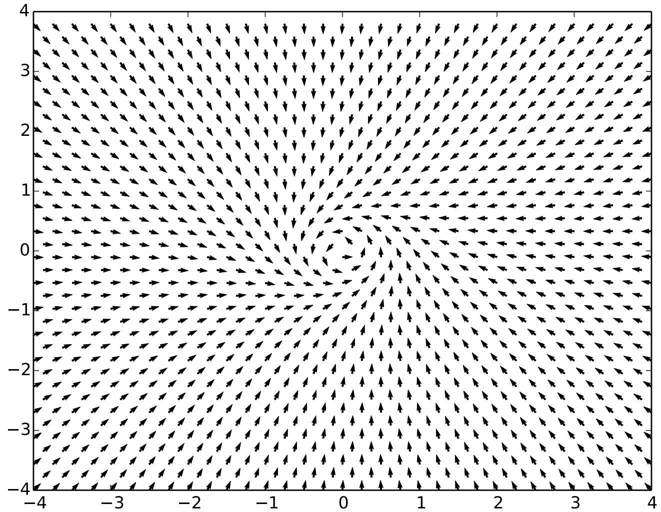

$$\lambda = 0$$

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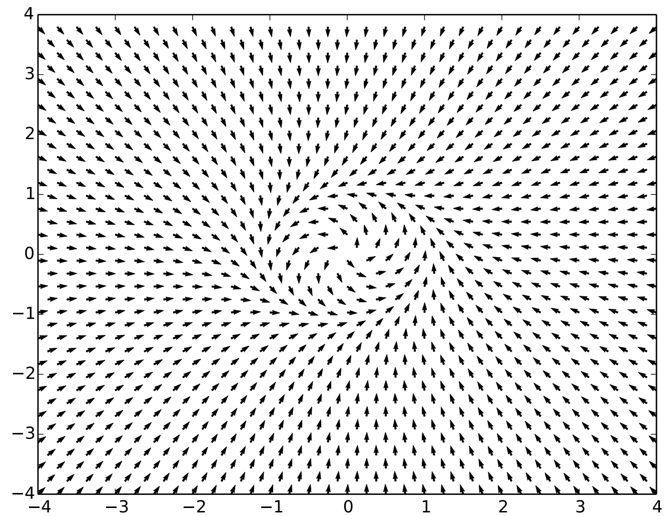
Motivating Example: Hopf Bifurcation



$$\lambda = 1$$

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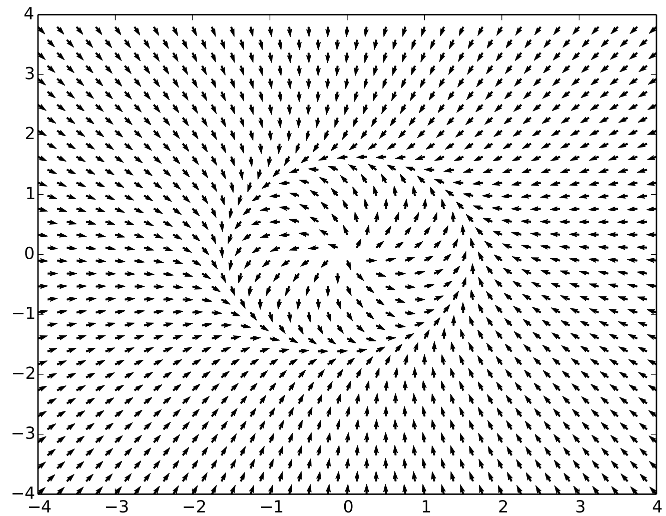
Motivating Example: Hopf Bifurcation



$$\lambda = 2.5$$

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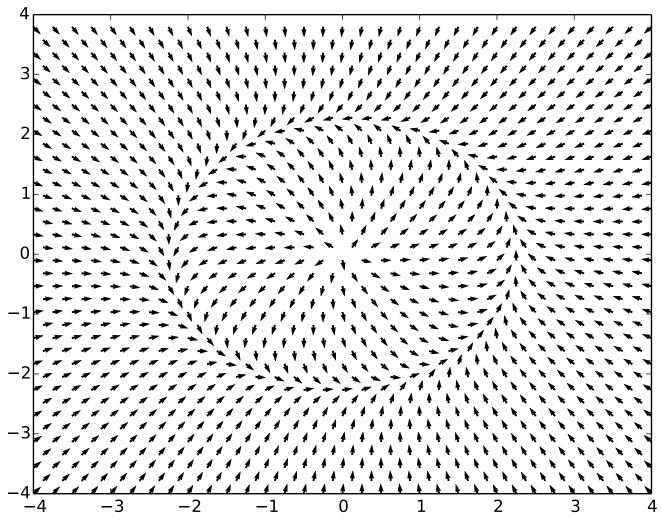

$$\lambda = 5$$

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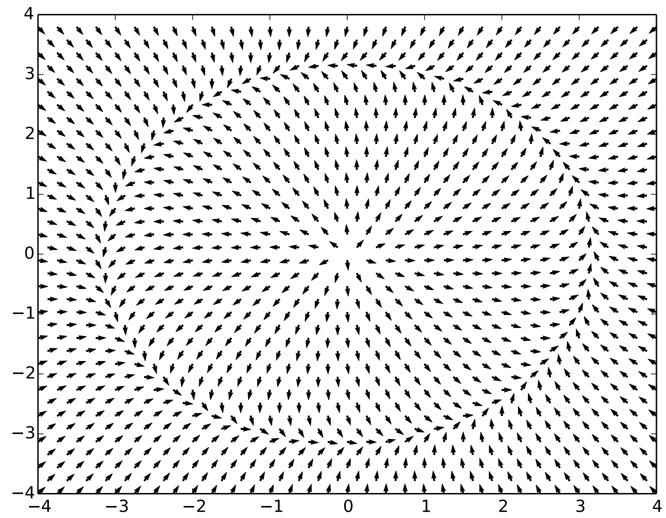
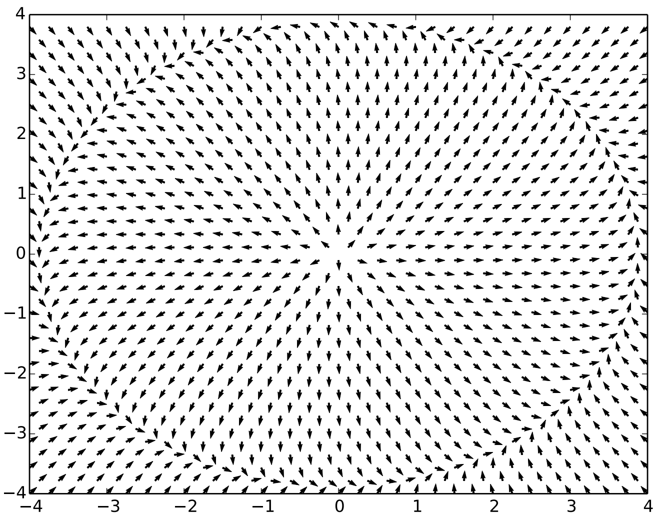
Motivating Example: Hopf Bifurcation



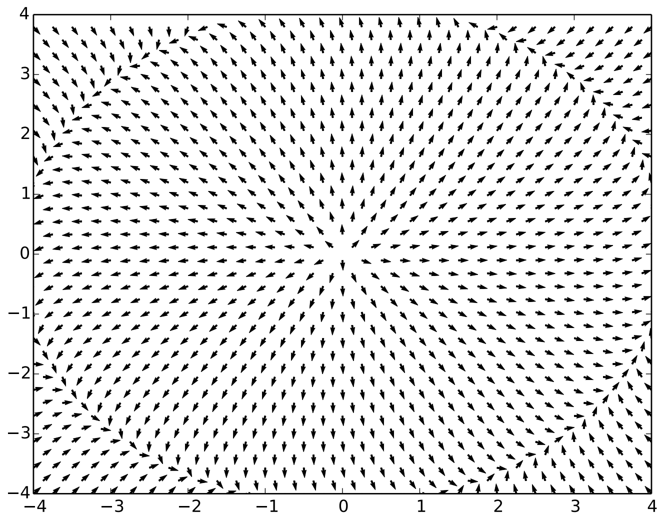
$\lambda = 10$

Motivating Example: Hopf Bifurcation



$\lambda = 15$

Motivating Example: Hopf Bifurcation



$$\lambda = 17.5$$

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Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Can we use persistence to capture this, or a related feature?

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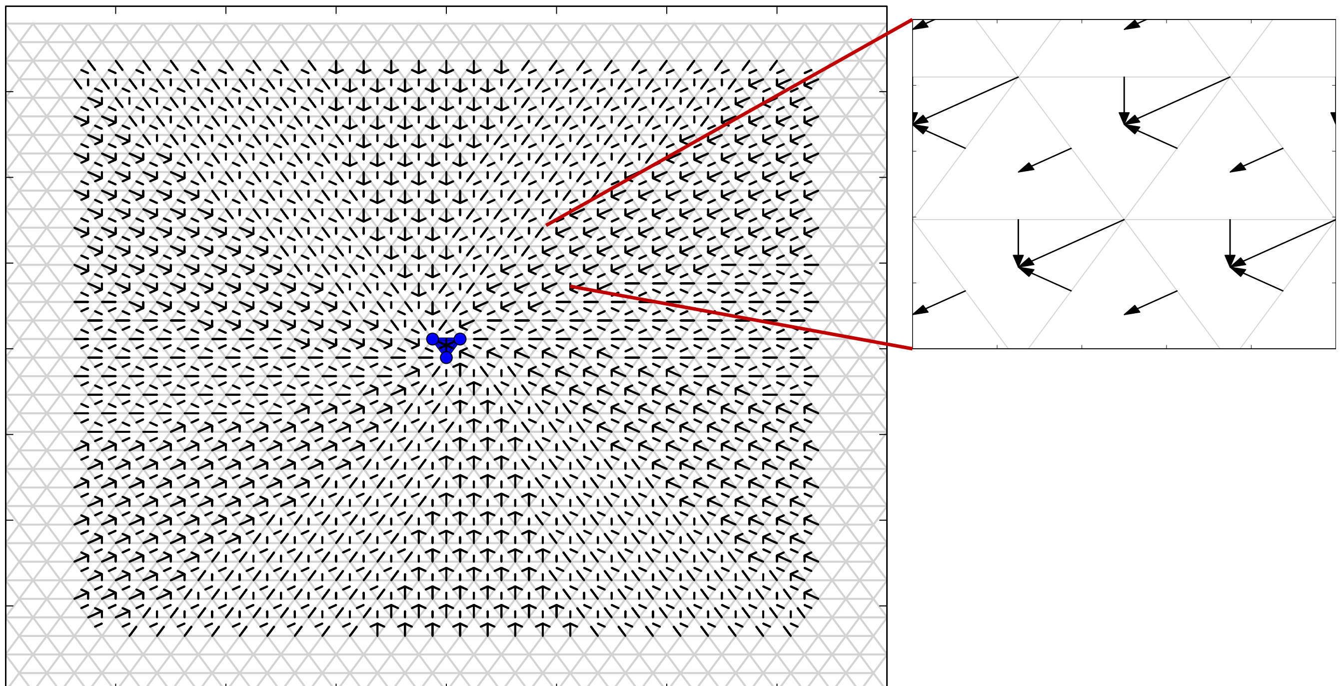
Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Can we use persistence to capture this, or a related feature?

Yes, using the Conley Index

Motivating Example: Hopf Bifurcation



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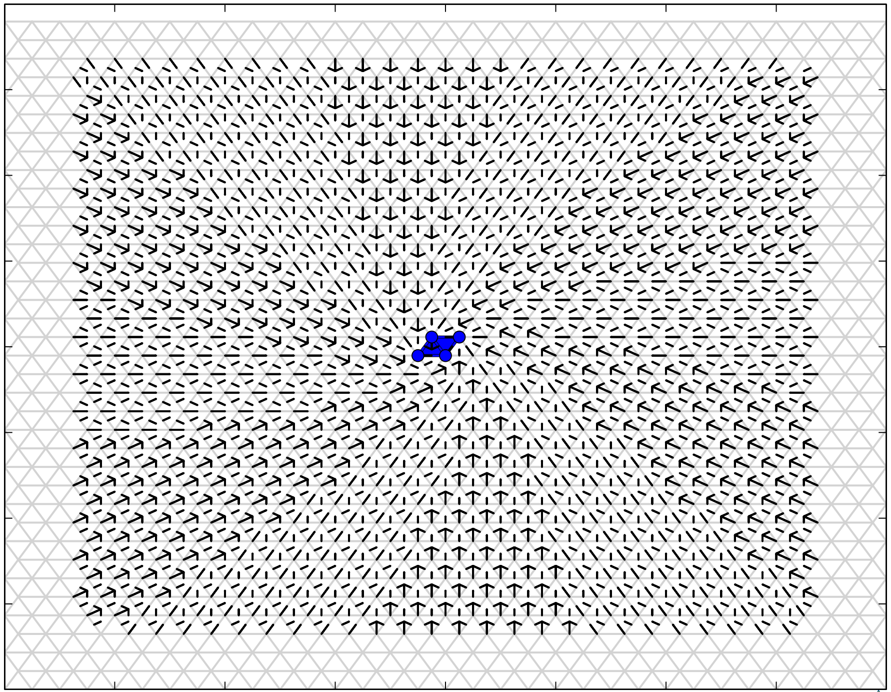

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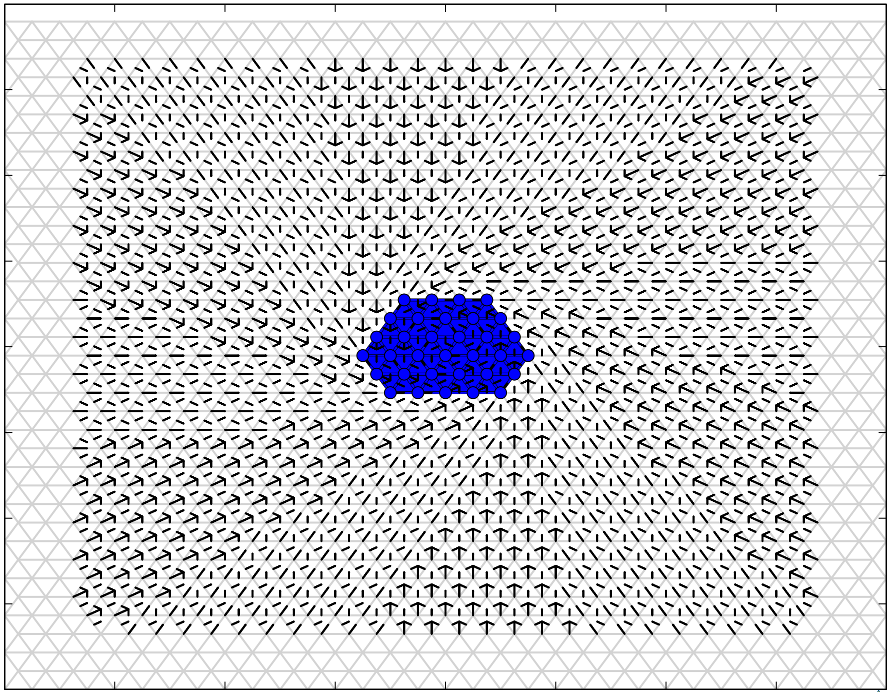

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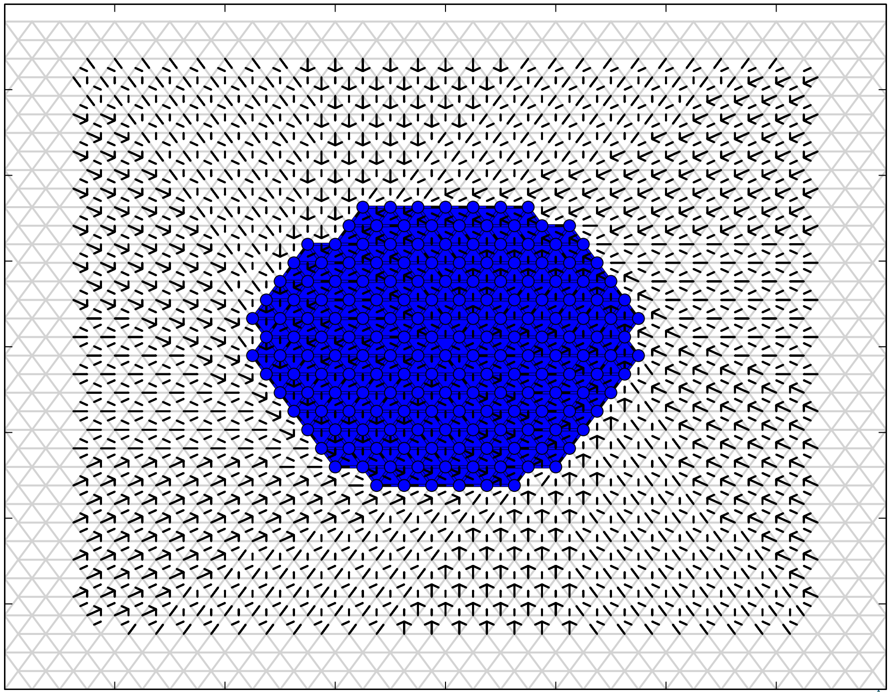

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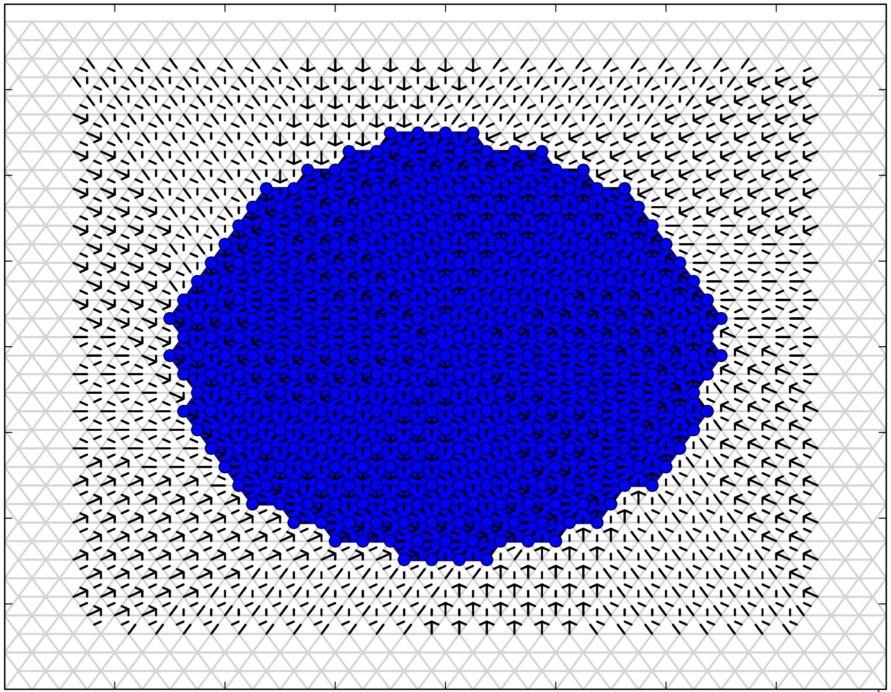

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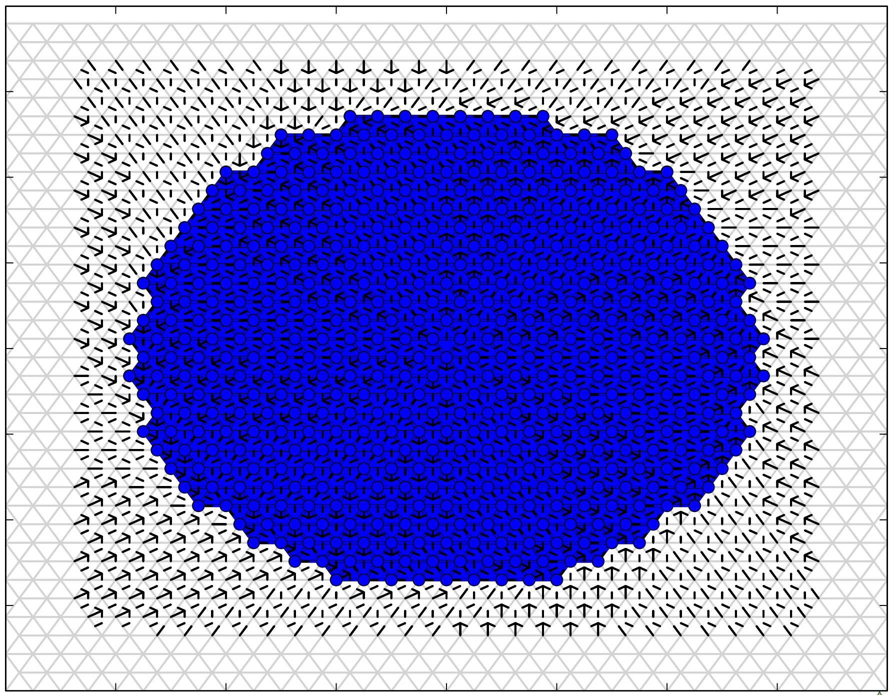

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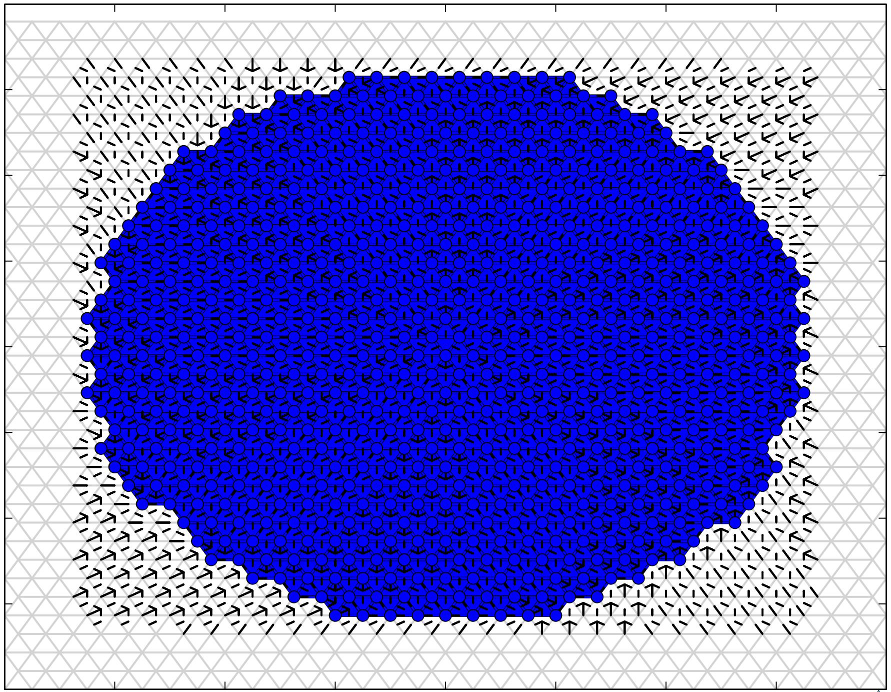

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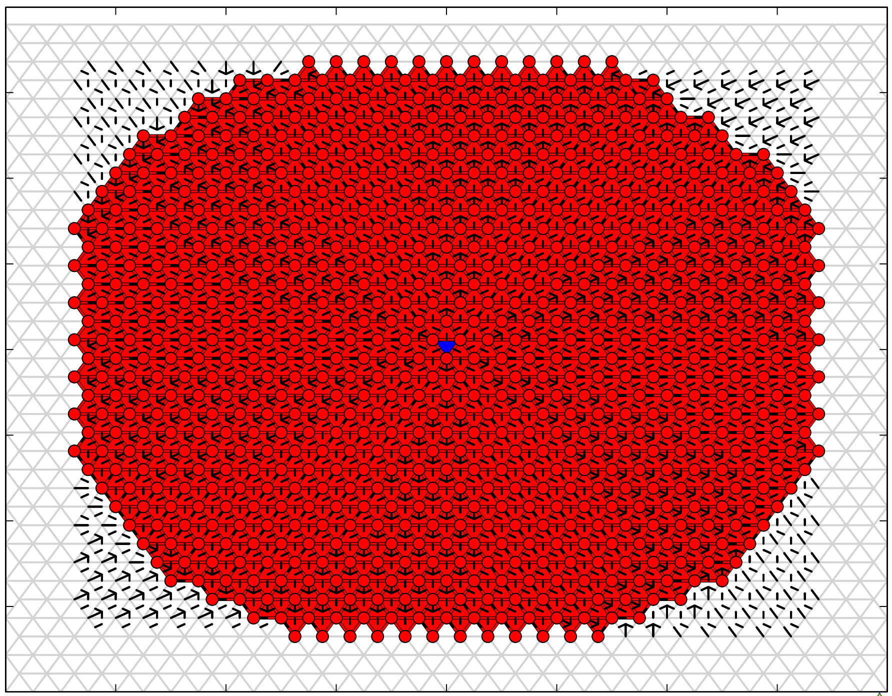

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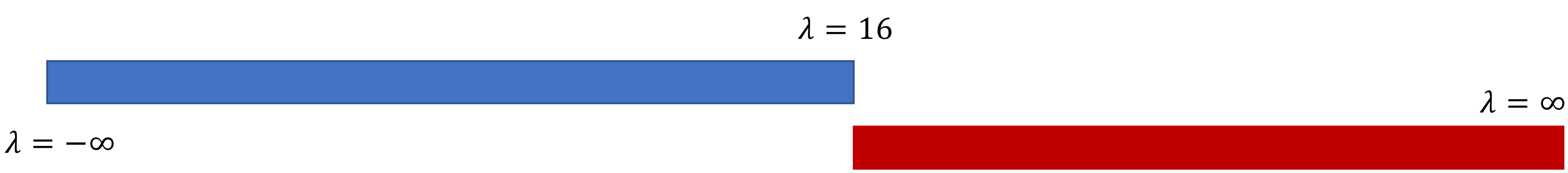
Motivating Example: Hopf Bifurcation

Have sequence of complexes given by blue and red simplices $B_1, B_2, B_3, \ldots, B_n$ and red simplices: $R_1, R_2, R_3, \ldots, R_n$. Use them to obtain relative zigzag filtration:

$$\cdots \subseteq (B_i, R_i) \supseteq (B_i \cap B_{i+1}, R_i \cap R_{i+1}) \subseteq (B_{i+1}, R_{i+1}) \supseteq \cdots$$



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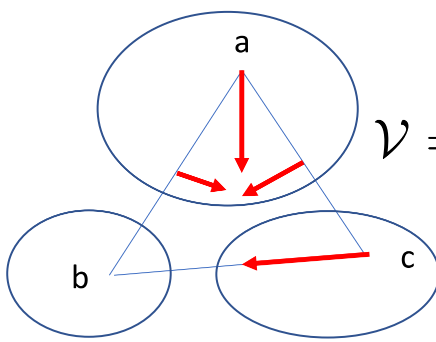
Multivectors

Let K denote a simplicial complex and $\leq$ denote the face relation.

Definition: A <u>multivector</u> V is a convex subset of K with respect to $\leq$.

Definition: A <u>multivector field</u> $\mathcal{V}$ is a partition of K into multivectors.



$$\mathcal{V} = \{\{b\}, \{b, bc\}, \{a, ab, ac, abc\}\}$$

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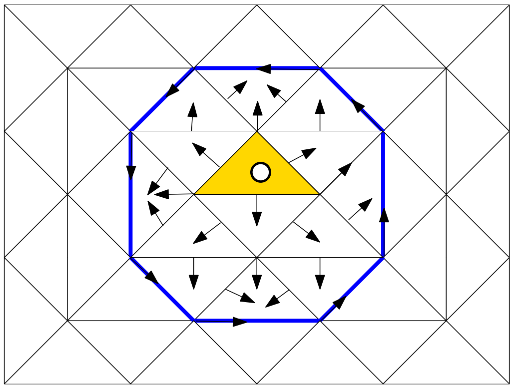
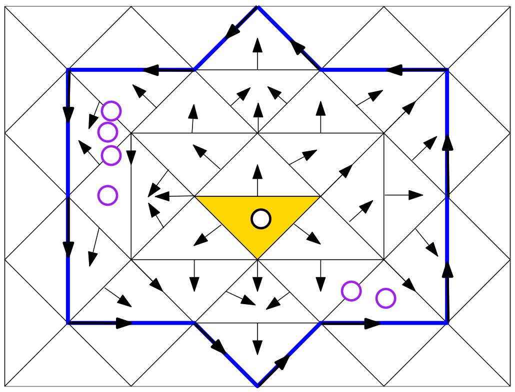
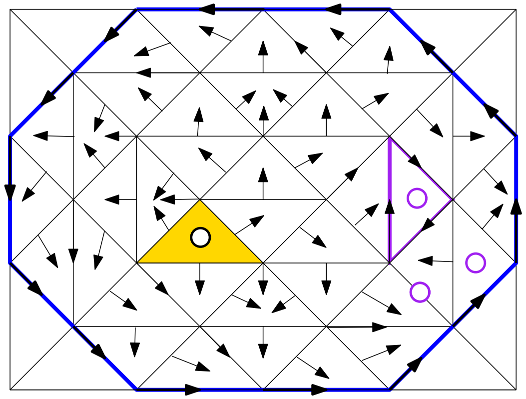
Multivector Fields







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Dynamical Systems from Multivector Fields

Let $\sigma \in K$. Then $\mathrm{cl}(\sigma) = \{\tau \in K \mid \tau \leq \sigma\}$.

$[\sigma]_{\mathcal{V}}$ denotes the vector in $\mathcal{V}$ containing σ

Dynamics generator $F_{\mathcal{V}} : K \multimap K$ defined as:

$$F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \mathrm{cl}(\sigma)$$

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Dynamical Systems from Multivector Fields

Definition: A path is a finite sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$

Definition: A solution is a bi-infinite sequence of simplices $\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$

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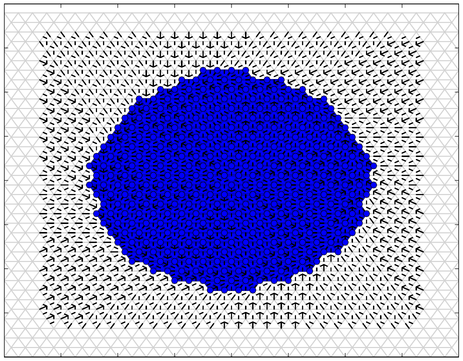
Invariant Sets

Definition: Let $A \subseteq K$. The invariant part of A, denoted $\mathrm{Inv}(A)$, is the set of simplices in A which appear in a solution in A.



If $A = \mathrm{Inv}(A)$, then A is an invariant set.

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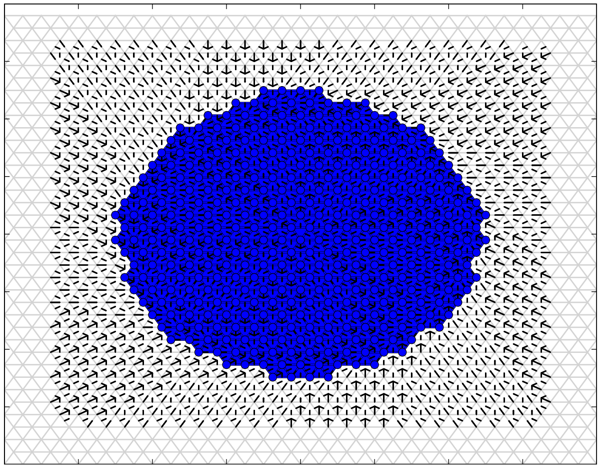
Isolated Invariant Sets

Definition: Let $A \subseteq N \subseteq K$, where A is an invariant set and N is closed (i.e. $N = \mathrm{cl}(N)$). If every path in N with endpoints in A is contained in A, then A is an isolated invariant set, and N is an isolating neighborhood for A.



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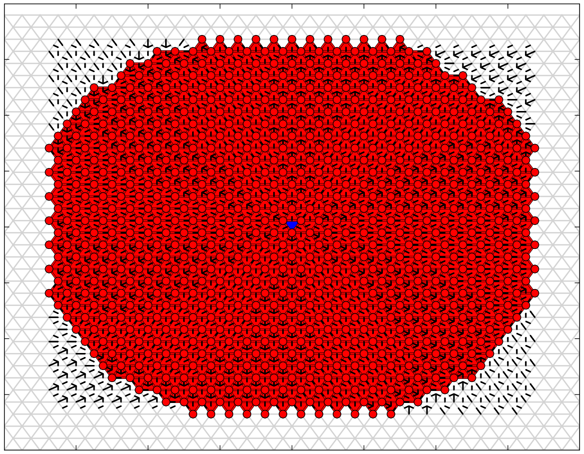
Index Pairs in an Isolating Neighborhood

Let $P, E \subseteq N \subseteq K$ for closed P, E, N, and $A \subseteq N$, If:

1. $F_{\mathcal{V}}(P) \cap N \subseteq P$,
2. $F_{\mathcal{V}}(E) \cap N \subseteq E$,
3. $F_{\mathcal{V}}(P \setminus E) \subseteq N$, and
4. $A = \mathrm{Inv}(P \setminus E)$

then (P, E) is an index pair in N .



<u>Theorem:</u> The push forward in N of $(\mathrm{cl}(A), \mathrm{cl}(A) \setminus A)$ is an index pair in N

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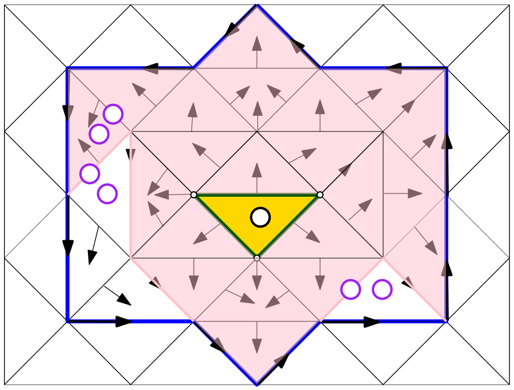
Index Pairs in an Isolating Neighborhood

Let $A \subseteq K$ denote an arbitrary set in some closed N. Then the push forward of A in N is A together with all simplices in N which are reachable from paths originating in A and contained in N.



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Conley Index

Definition: Let (P, E) be an index pair for A. Then the k-dimensional Conley Index is given by $H_k(P, E)$.

Theorem [LKMW 2019]: The k-dimensional Conley Index for A is well defined.

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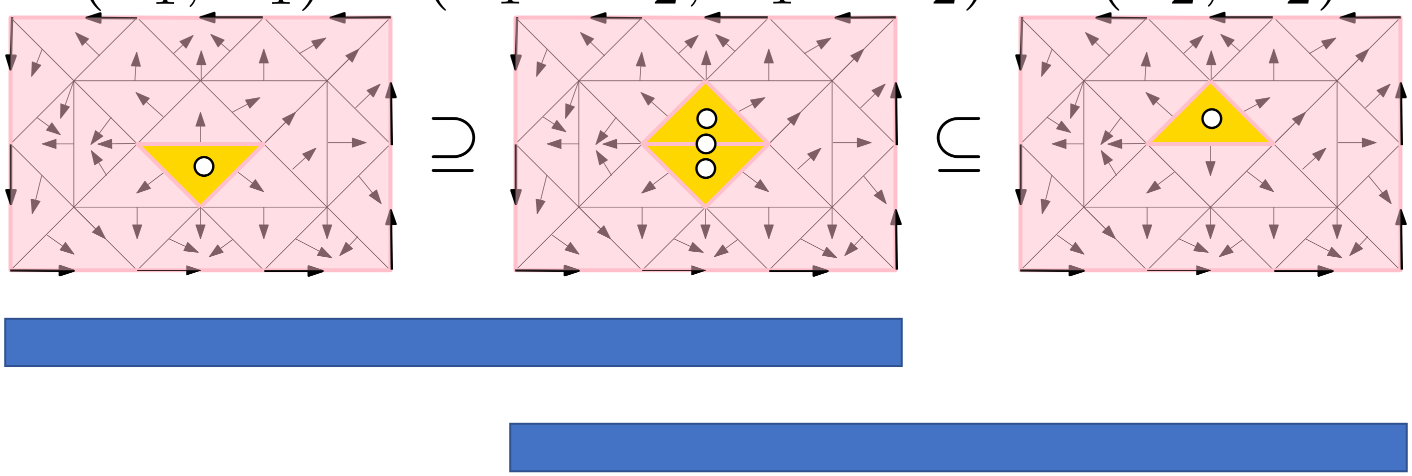
Index Pairs in an Isolating Neighborhood

Theorem: The intersection of two index pairs in N is an index pair in N.

$$\cdots \subseteq (B_i, R_i) \supseteq (B_i \cap B_{i+1}, R_i \cap R_{i+1}) \subseteq (B_{i+1}, R_{i+1}) \supseteq \cdots$$

Problem: Noise Resilience

$$(P_1, E_1) \supseteq (P_1 \cap P_2, E_1 \cap E_2) \subseteq (P_2, E_2)$$



Rectangle: N, P

Pink: P

Gold: P – E

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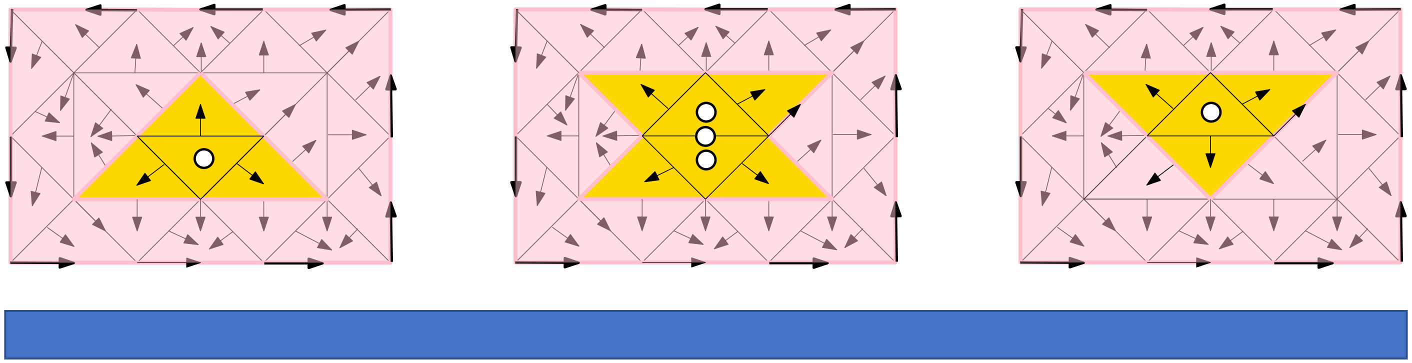
Solution: Make E Smaller



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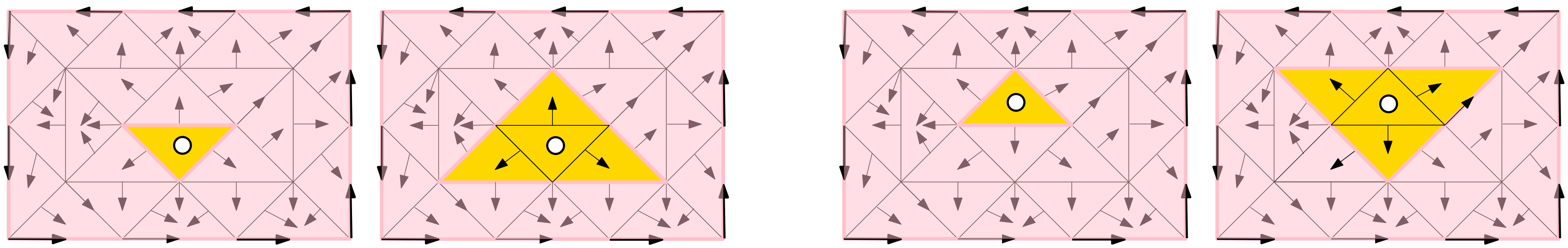
Conley Index Persistence

Theorem: Let (P, E) denote an index pair for A in N. If $V \subseteq E$ is a regular multivector such that $E' := E \setminus V$ is closed, then (P, E') is an index pair in N for A.



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Conclusion & Future Work

- Changing the isolating neighborhood? See paper.

- Stability?

- Inference?

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