Capturing Changes in Combinatorial Dynamical Systems via Persistent Homology

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PURDUE UNIVERSITY

Overview & Outline

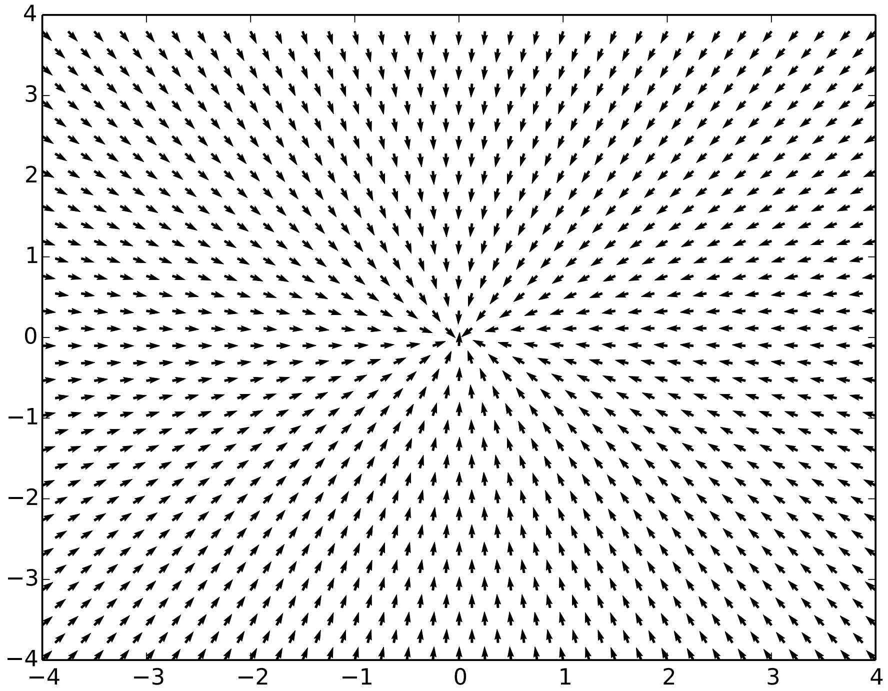
- Motivating Example and Persistence
- Combinatorial Dynamical Systems & Conley Index
- Conley Index Persistence
- Conley-Morse Graph Persistence
- Tracking and Combinatorial Continuation

Motivating Example: Hopf Bifurcation

$$x' = -y + x(\lambda - x^2 - y^2)$$

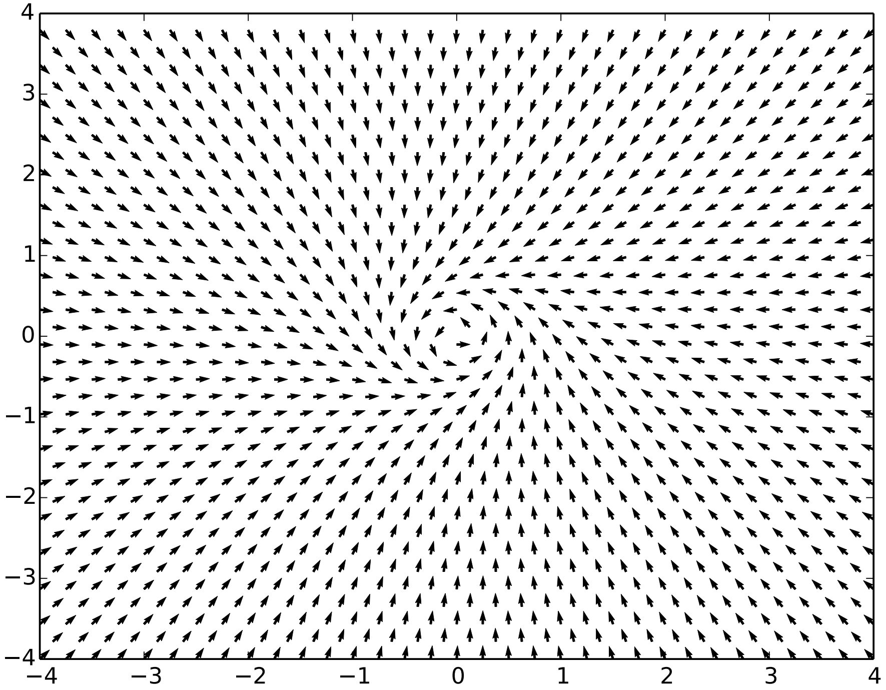
$$y' = x + y(\lambda - x^2 - y^2)$$

Motivating Example: Hopf Bifurcation



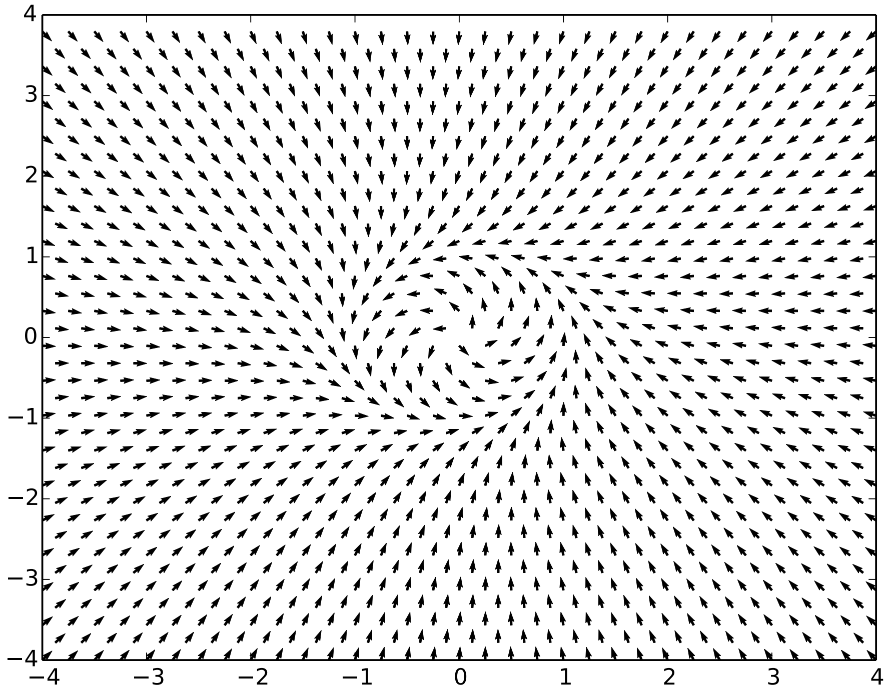
$$\lambda \ll 0$$

Motivating Example: Hopf Bifurcation



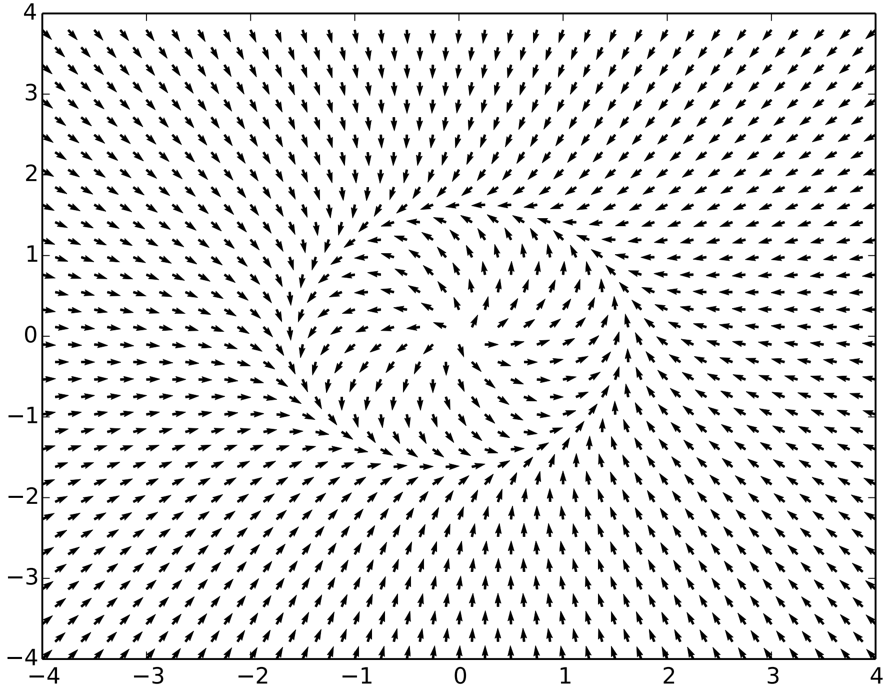
$$\lambda = 0$$

Motivating Example: Hopf Bifurcation



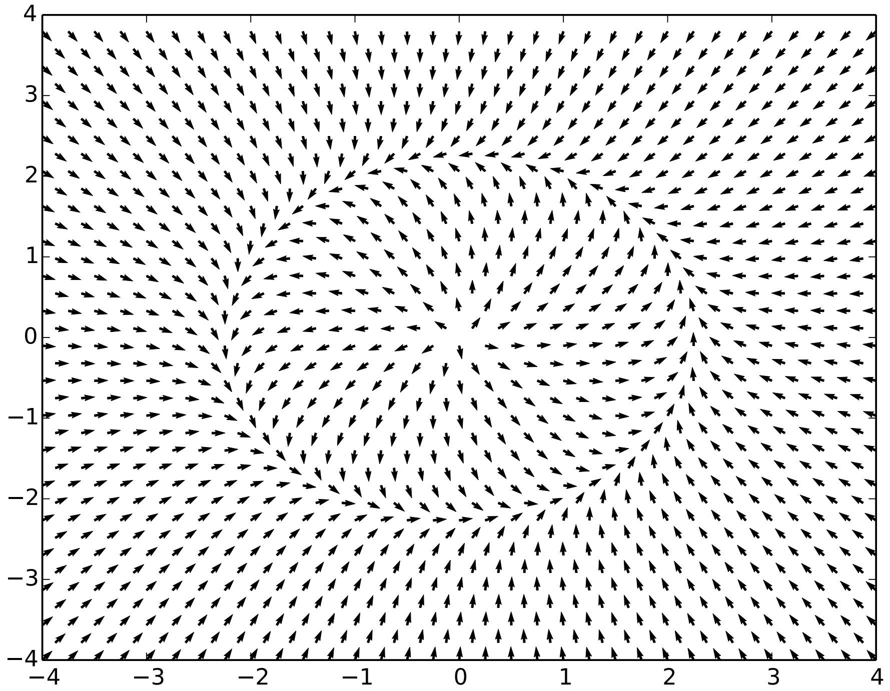
$$\lambda = 1$$

Motivating Example: Hopf Bifurcation



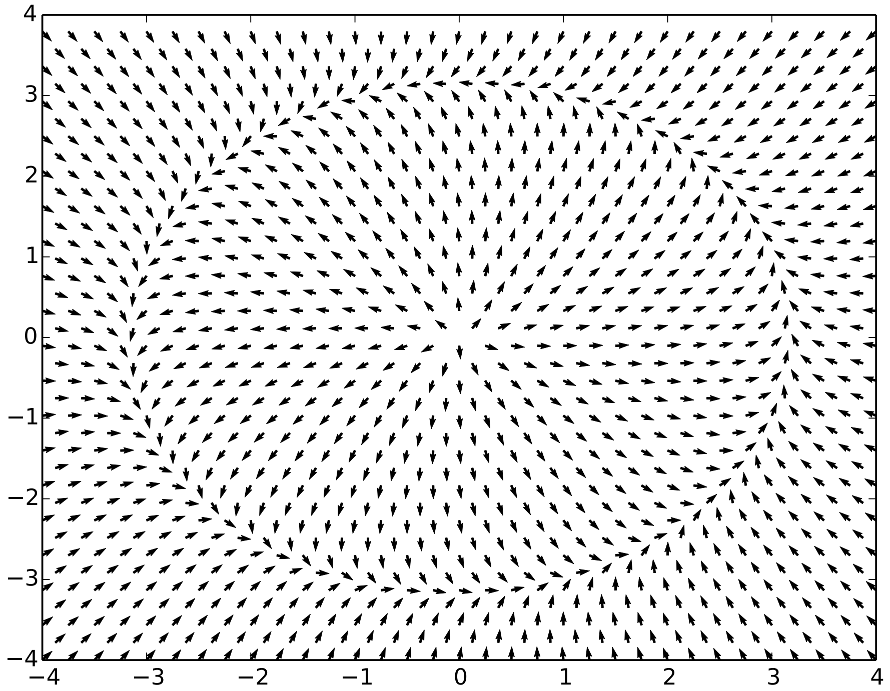
$$\lambda = 2.5$$

Motivating Example: Hopf Bifurcation



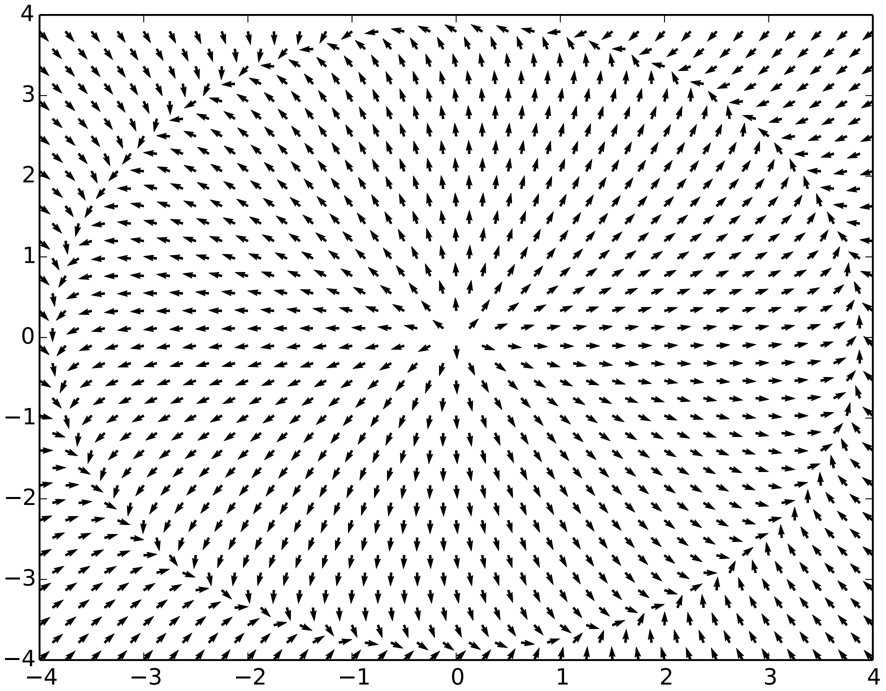
$$\lambda = 5$$

Motivating Example: Hopf Bifurcation



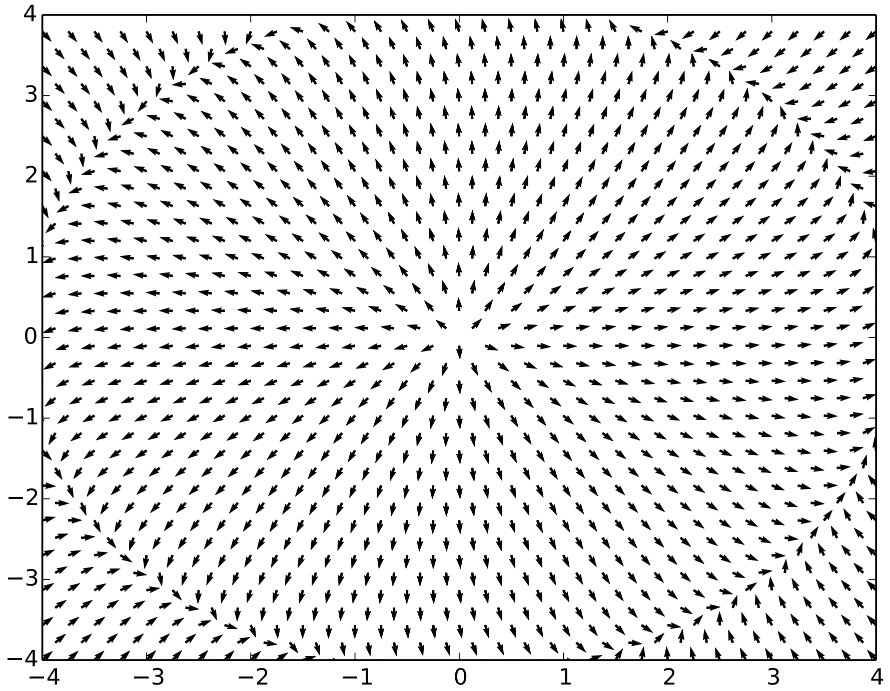
$$\lambda = 10$$

Motivating Example: Hopf Bifurcation



$$\lambda = 15$$

Motivating Example: Hopf Bifurcation



$$\lambda = 17.5$$

Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Repeller from $\lambda = 0$ to $\lambda = \infty$

Can we use computational topology to automatically detect these features?

Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Repeller from $\lambda = 0$ to $\lambda = \infty$

Can we use computational topology to automatically detect these features?

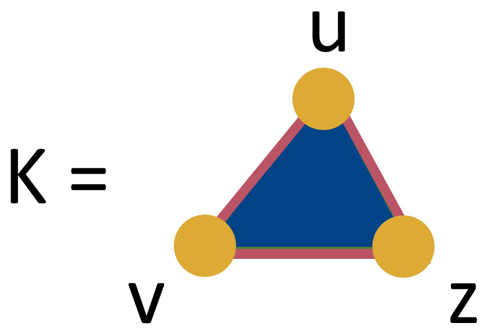
Yes, by using persistence

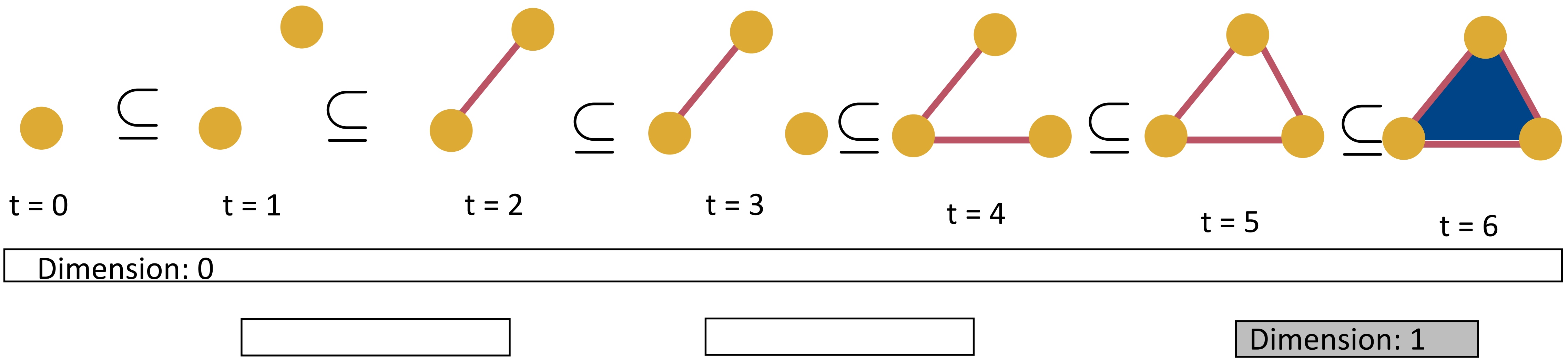
Persistent Homology

Summarizes changing homology of a filtration [ELZ00]

$$K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n = K$$

Persistence Example



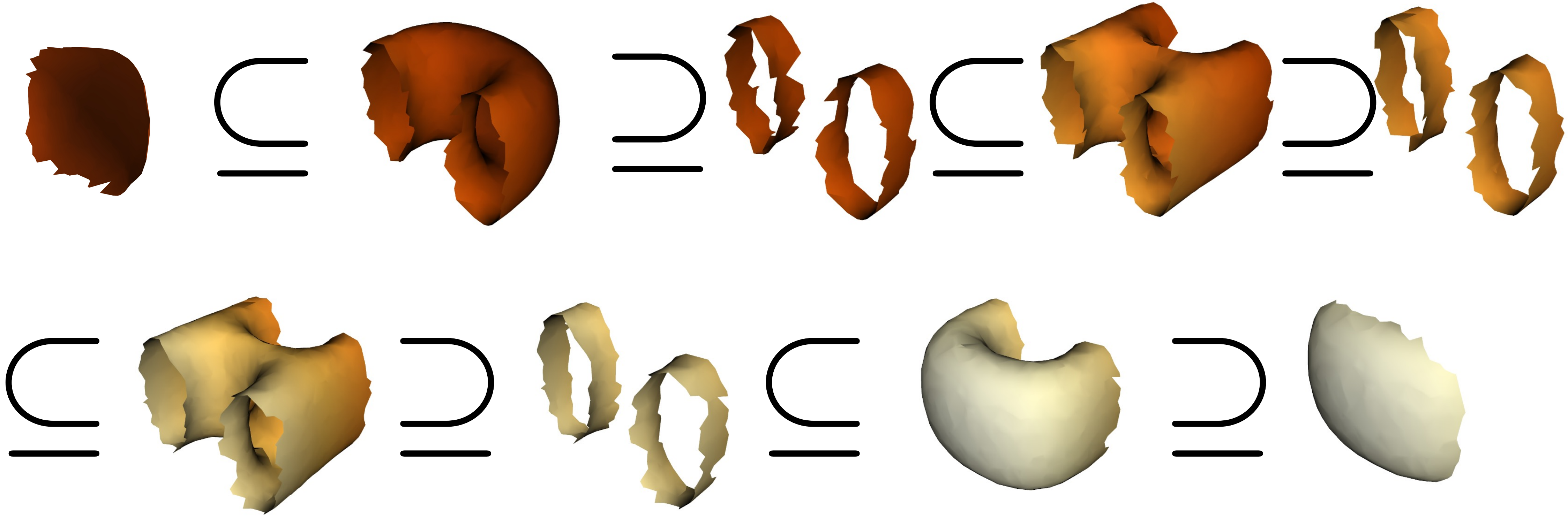


$$K_0 \subset K_1 \subset K_2 \subset K_3 \subset K_4 \subset K_5 \subset K_6$$

Zigzag Persistence

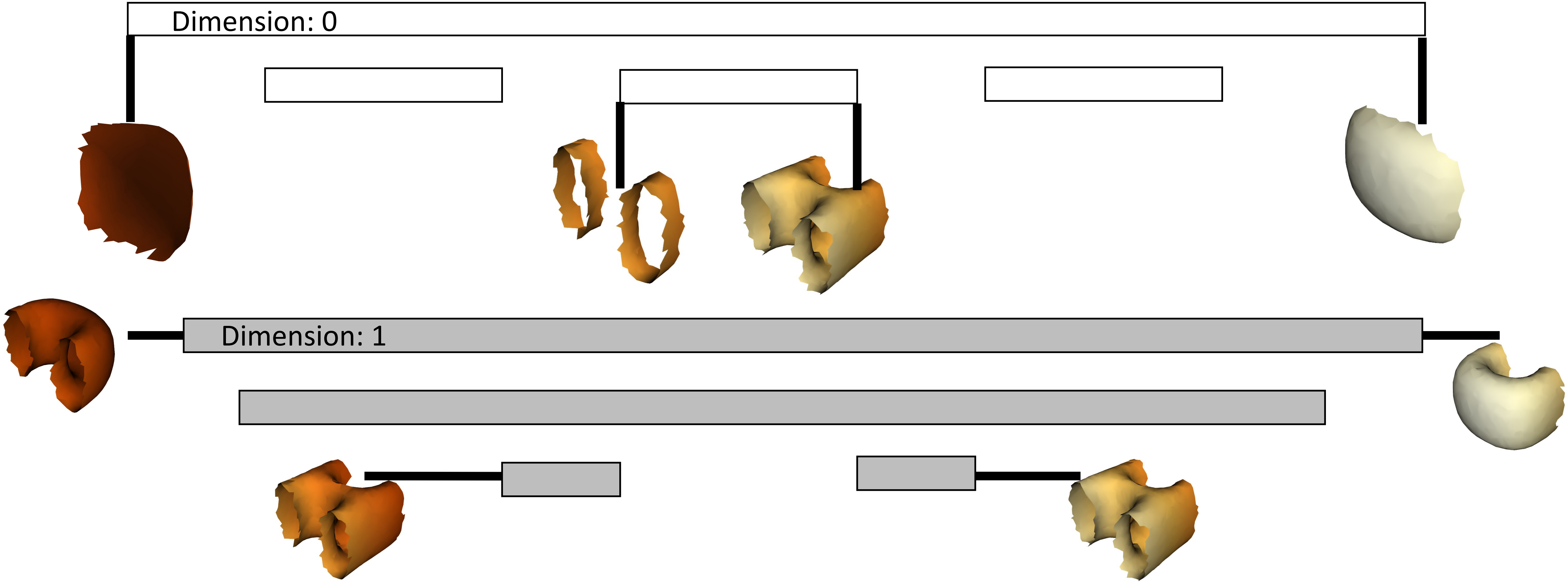
$$K_1 \subseteq K_2 \supseteq K_3 \subseteq \ldots \supseteq K_n$$

“Level Set” Persistence



[CDM09] [DW07]

Level Set Barcode



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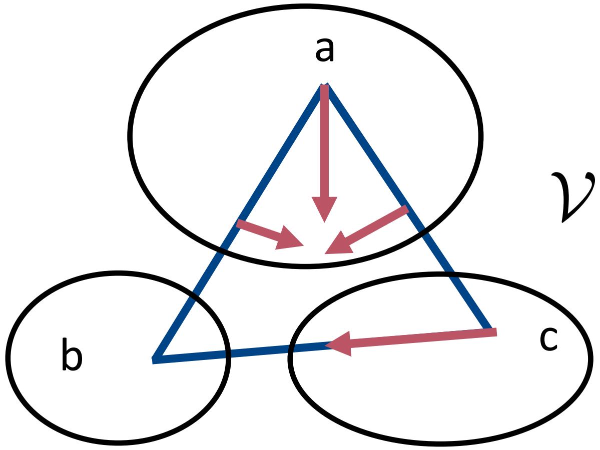
Multivectors

Let K denote a simplicial complex and $\leq$ denote the face relation.

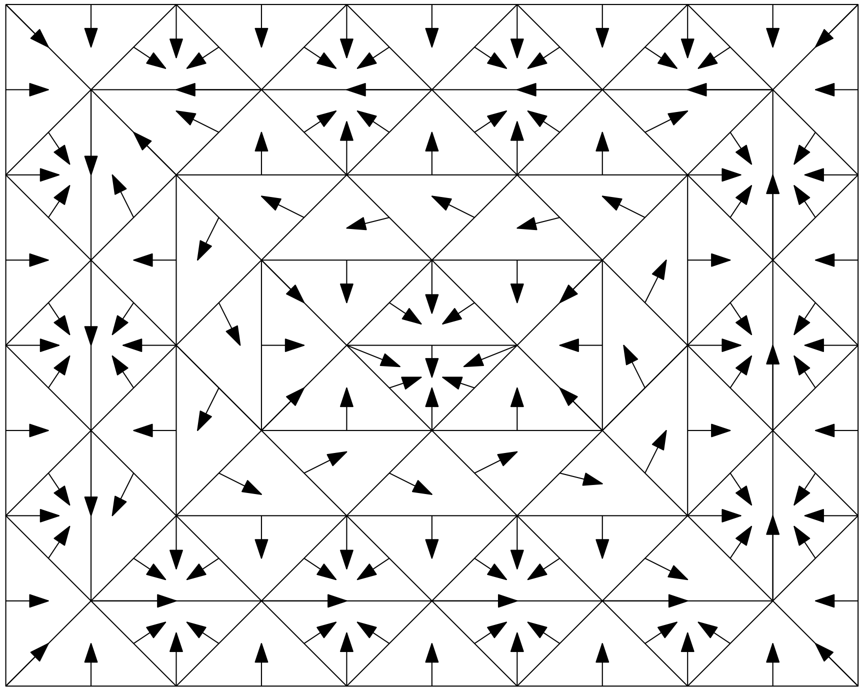
Definition: A <u>multivector</u> V is a convex subset of K with respect to $\leq$.

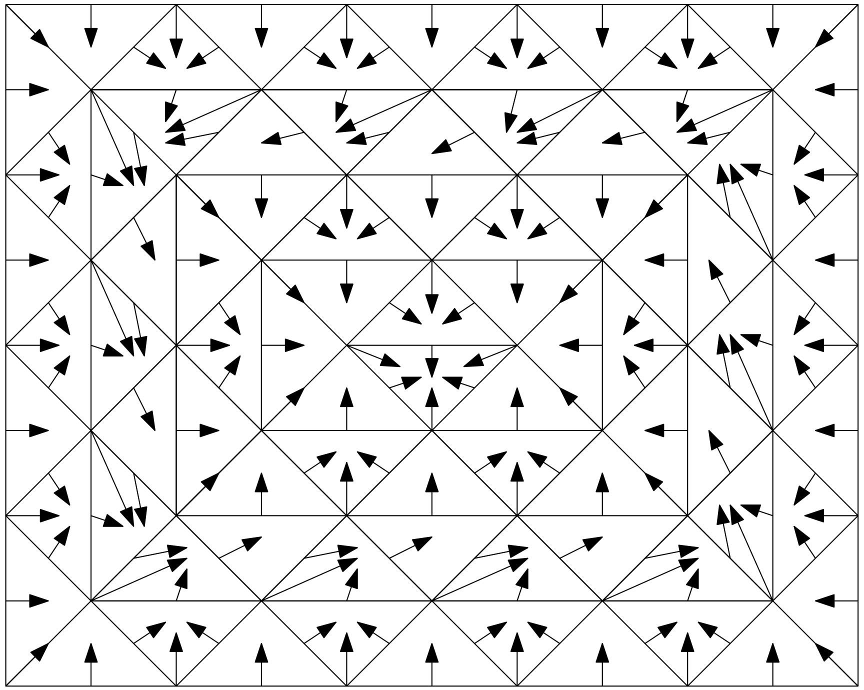
Definition: A <u>multivector field</u> $\mathcal{V}$ is a partition of K into multivectors.



$$\mathcal{V} = \{\{b\}, \{c, bc\}, \{a, ab, ac, abc\}\}$$

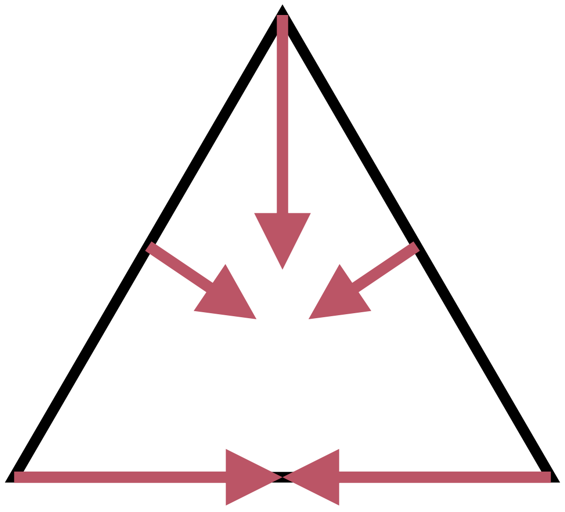
Multivector Fields





Multivector Fields as a Dynamical System

$$F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \mathsf{cl}(\sigma)$$



Multivector field

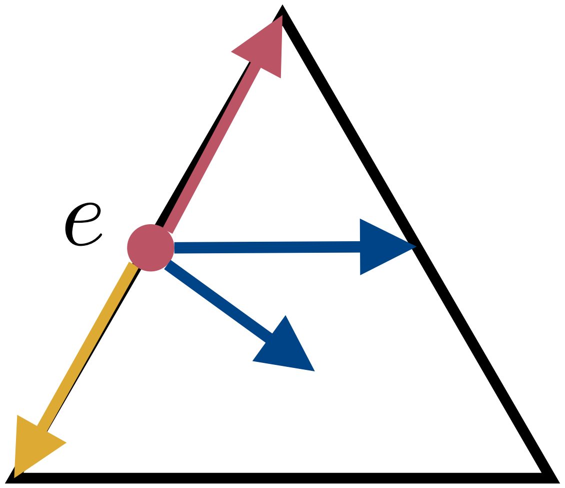


Image of marked edge e

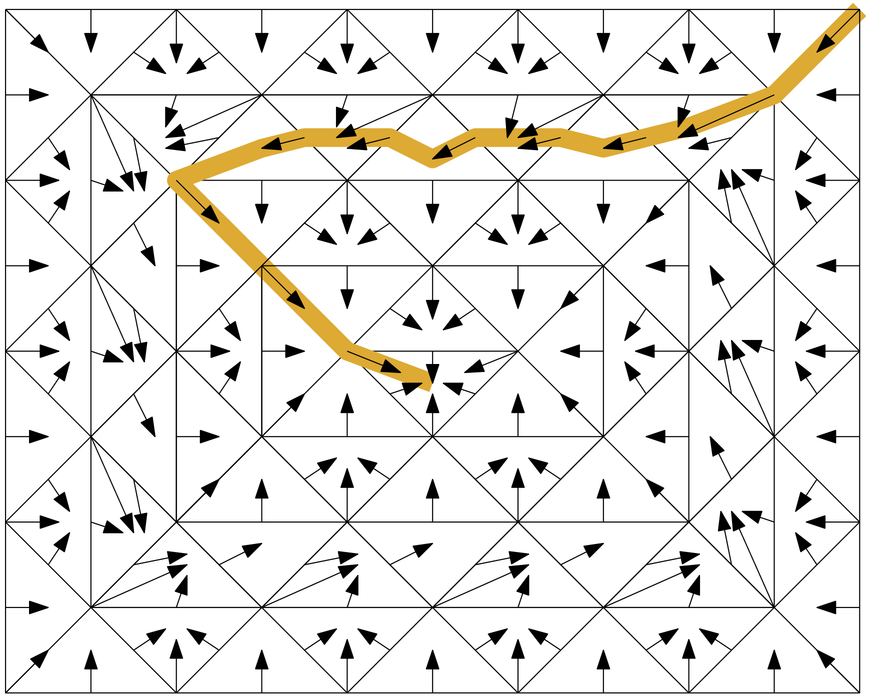
Blue simplices are in $[e]_{\mathcal{V}} \setminus \mathsf{cl}(e)$

Yellow simplices are in $\mathsf{cl}(e) \setminus [e]_{\mathcal{V}}$

Red simplices are in $[e]_{\mathcal{V}} \cap \mathsf{cl}(e)$

Paths

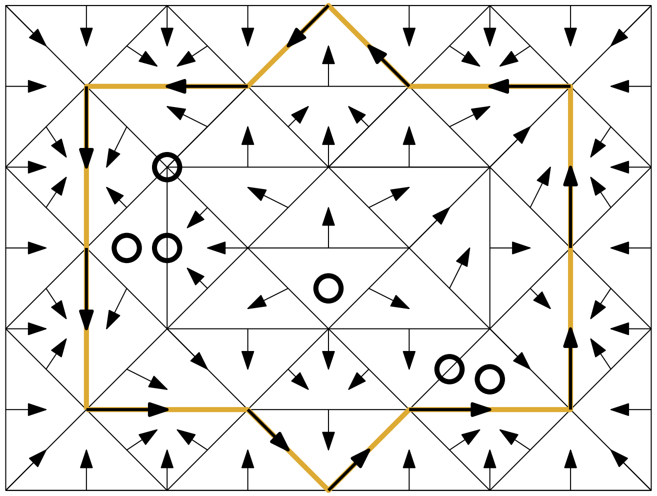
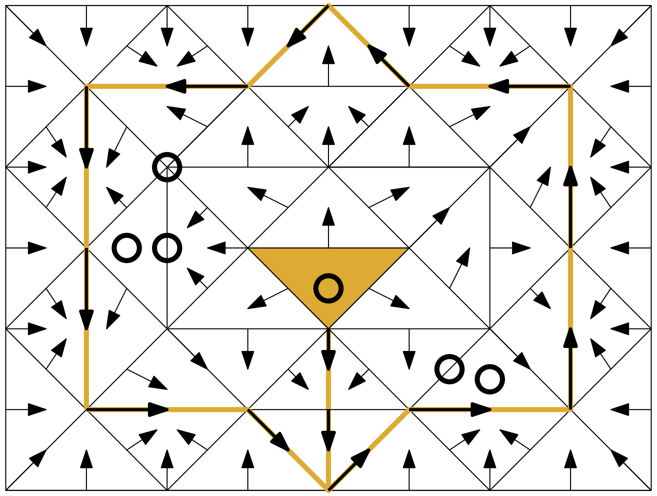
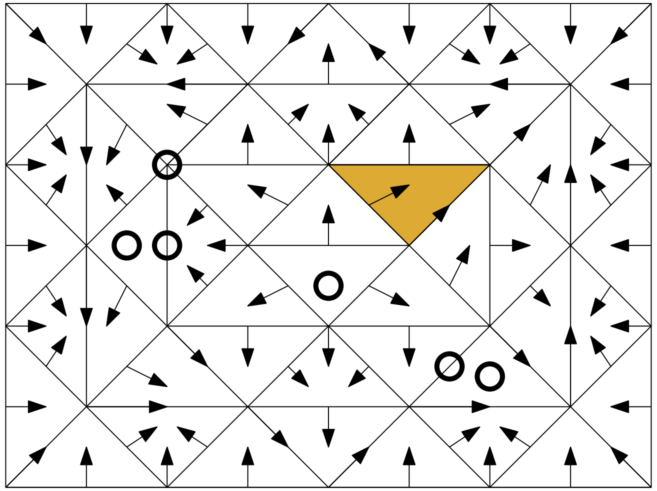
Definition: A path is a finite sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$



Solutions

Definition: A solution is a bi-infinite sequence of simplices

$\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$

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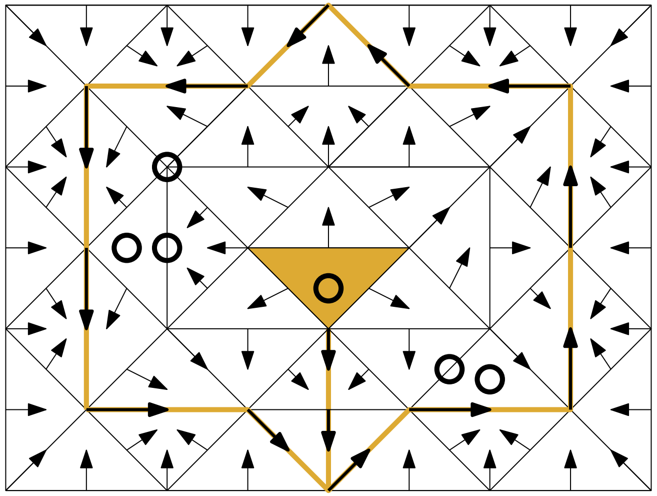
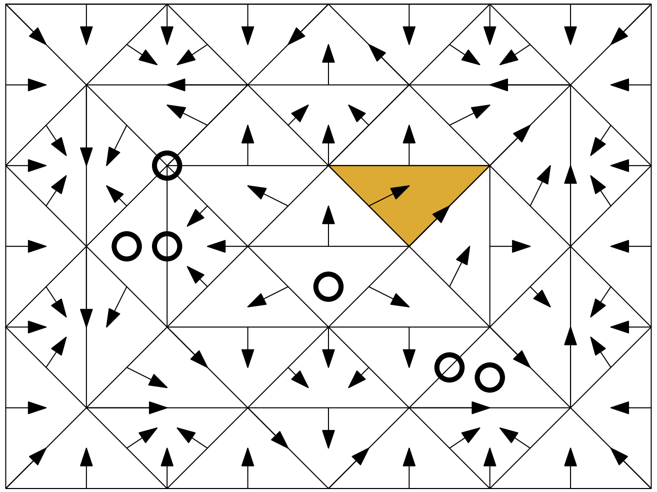




But as $F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \mathrm{cl}(\sigma)$, every simplex gives a solution!

Critical Multivectors

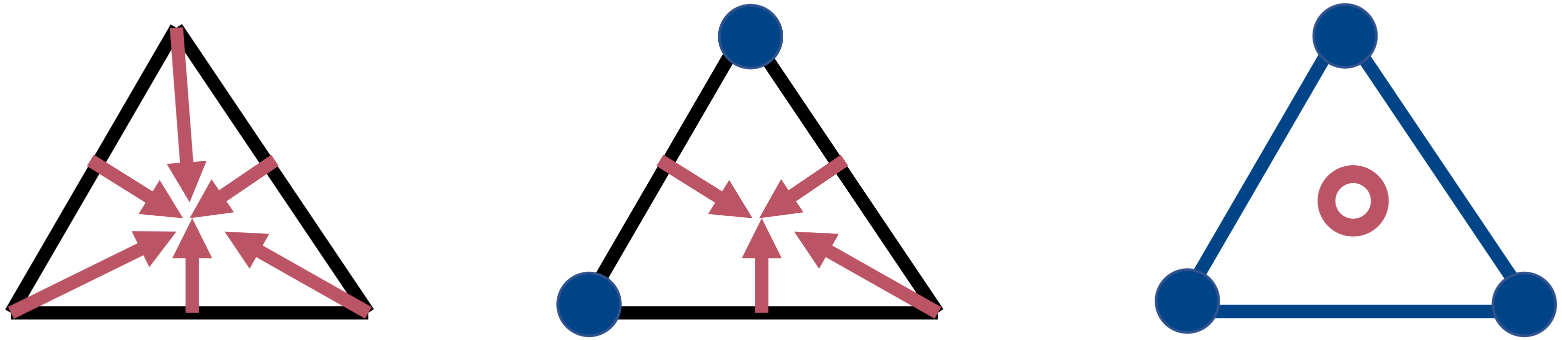
Definition: Let $A \subseteq K$. The mouth of A is defined as

$$\text{mo}\,(A) := \text{cl}\,(A) \setminus A$$

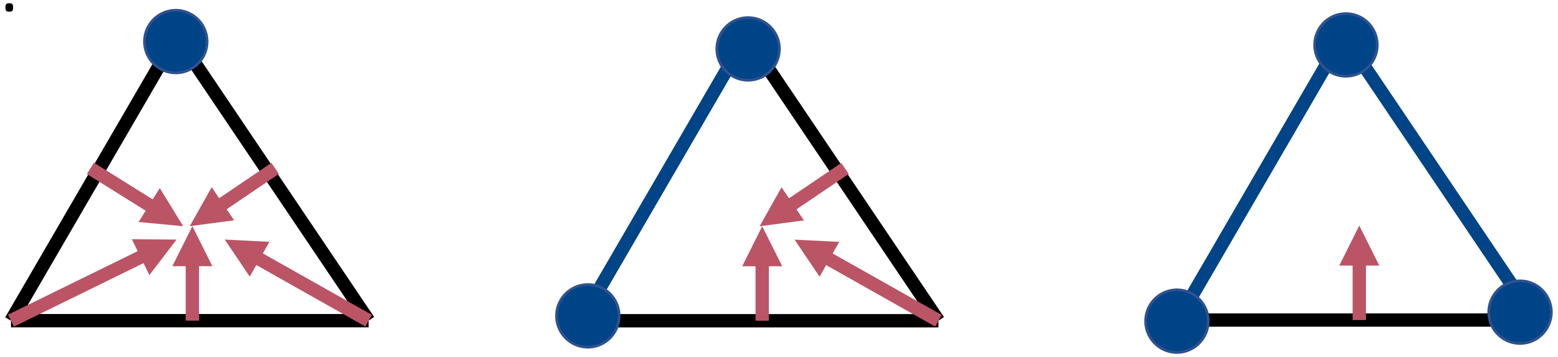
Definition: A multivector $[\sigma]_{\mathcal{V}}$ is critical if there exists a k such that $H_k(\text{cl}\,([\sigma]_{\mathcal{V}})\,, \text{mo}\,([\sigma]_{\mathcal{V}}))$ is nontrivial.

Critical Multivectors

Critical:

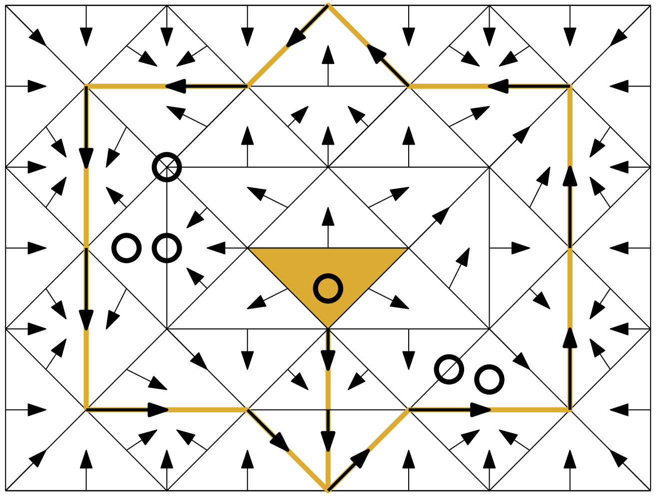


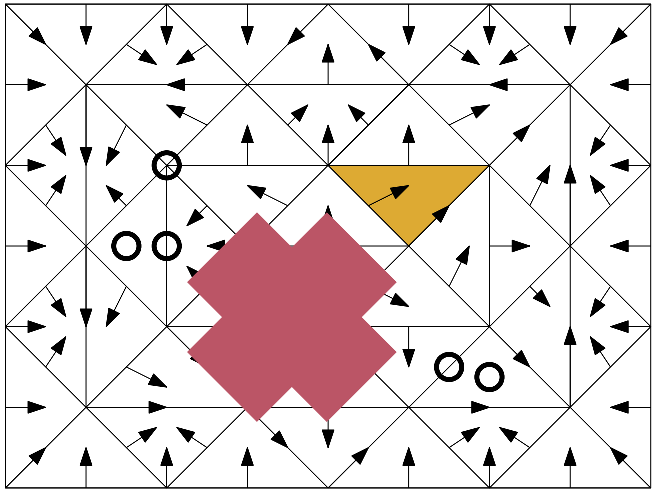
Regular:



Essential Solutions

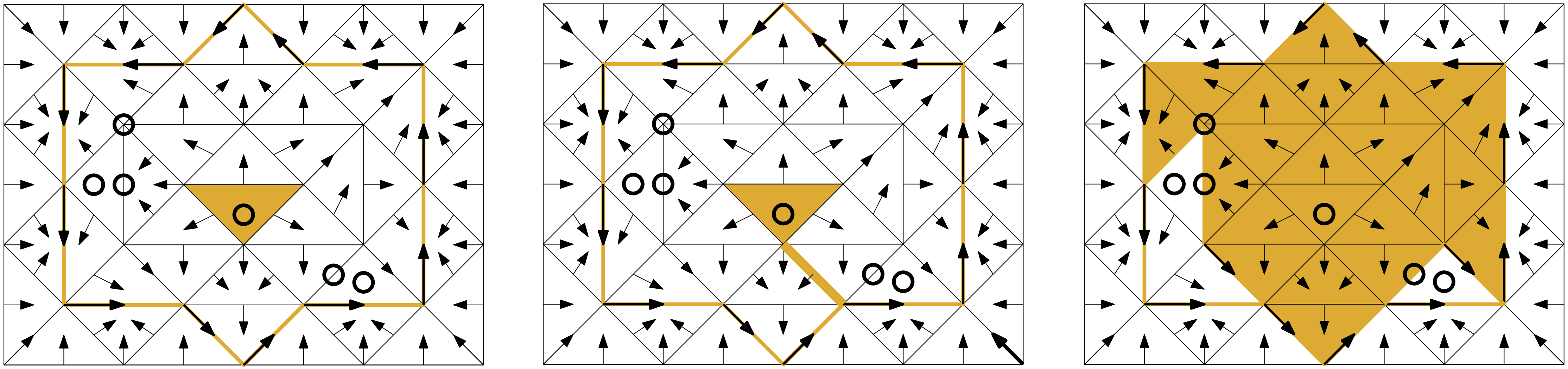
Definition: Let $\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ denote a solution. If for each σ_i where $[\sigma_i]_{\mathcal{V}}$ is noncritical, there exists a $j > i$ and $j' < i$ where $[\sigma_i]_{\mathcal{V}} \neq [\sigma_j]_{\mathcal{V}}$ and $[\sigma_i]_{\mathcal{V}} \neq [\sigma_{j'}]_{\mathcal{V}}$,
then $\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ is an essential solution.





Invariant Sets

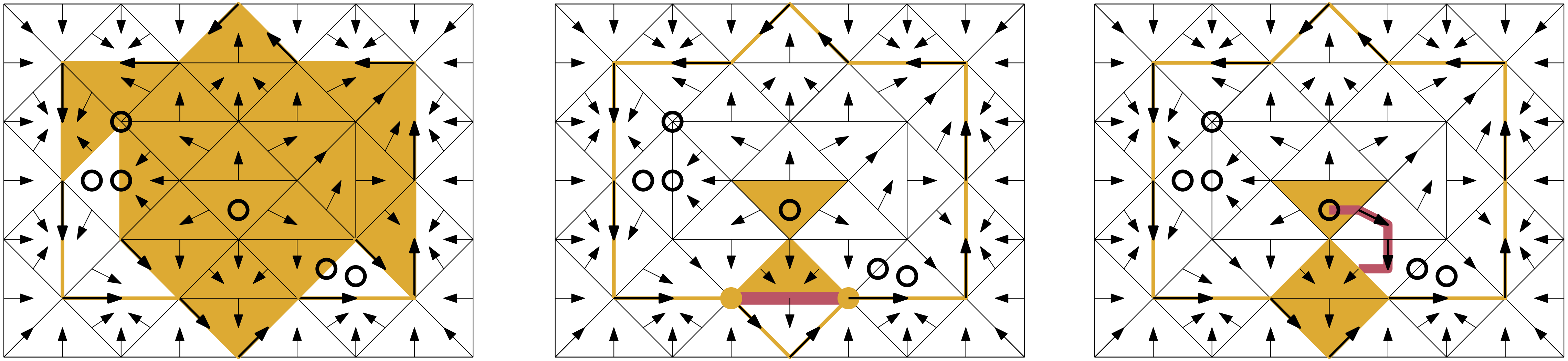
Definition: Let $A \subseteq K$. The invariant part of A, denoted $\text{Inv}(A)$, is the set of simplices in A which appear in an essential solution in A.



If $A = \text{Inv}(A)$, then A is an invariant set.

Isolated Invariant Sets

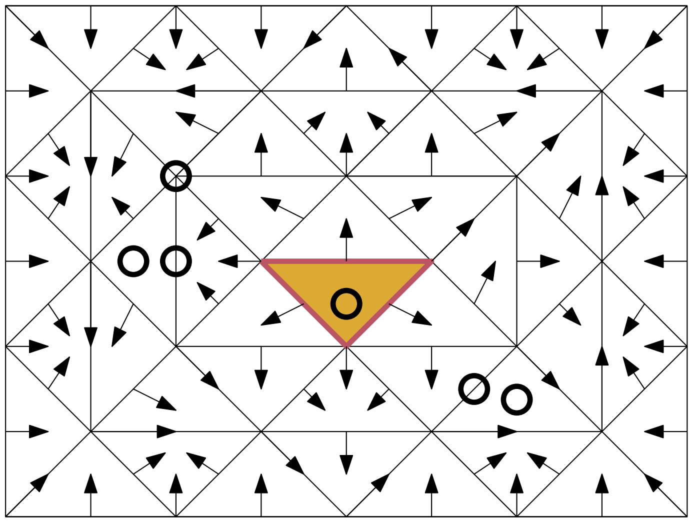
Definition: Let $A \subseteq N \subseteq K$, where A is an invariant set and N is closed (i.e. $N = \mathrm{cl}(N)$). If every path in N with endpoints in A is contained in A, then A is an isolated invariant set, and N is an isolating neighborhood for A.



Index Pairs

Definition: Let A be an isolated invariant set, and E and P closed sets such that $E \subseteq P$. If:

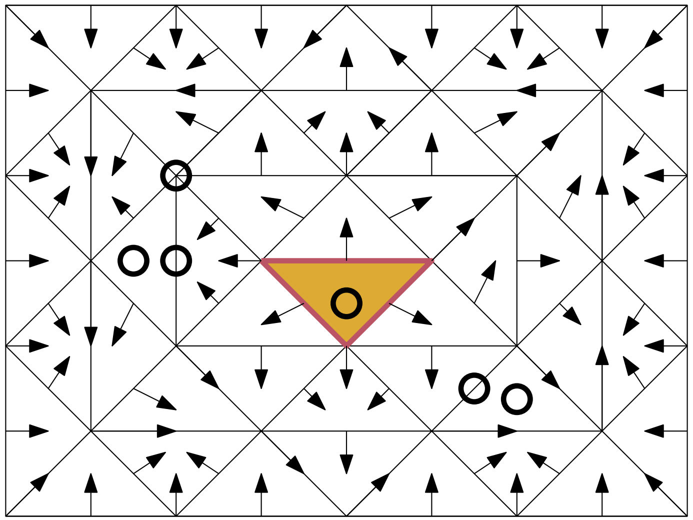
1. $F_{\mathcal{V}}(E) \cap P \subset E$,
2. $F_{\mathcal{V}}(P \setminus E) \subseteq P$, and
3. $A = \mathrm{Inv}(P \setminus E)$



Then (P, E) is an index pair for A.

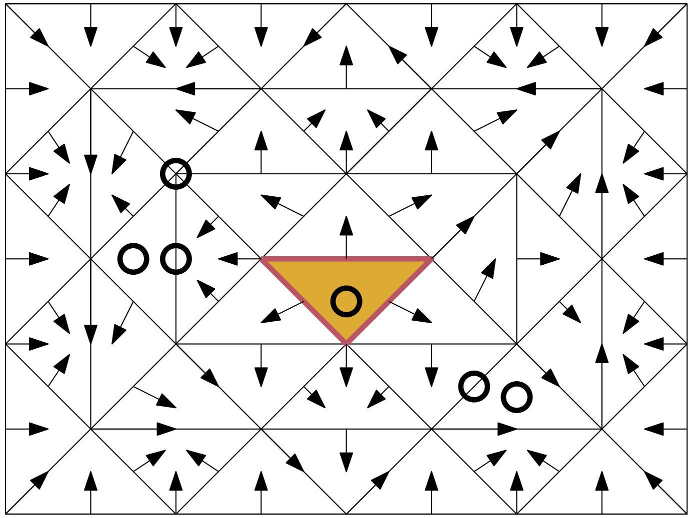
Conley Index

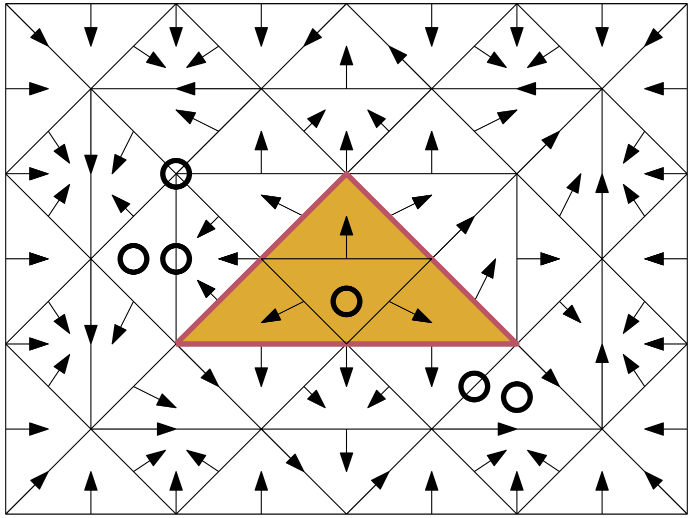
<u>Theorem [LKMW2019]:</u> Let A denote an isolated invariant set. The pair $(\text{cl}(A), \text{mo}(A))$ is an index pair for A.

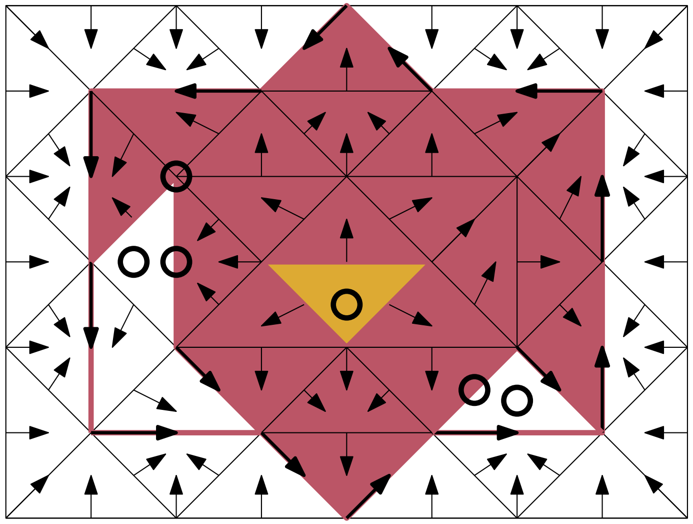


Hide

Index Pairs are Not Unique





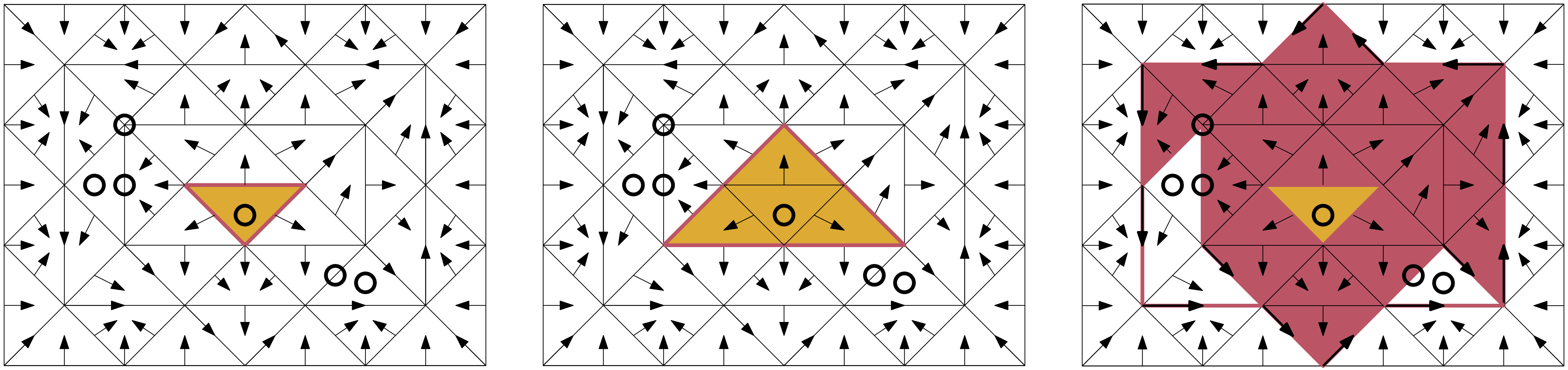


Conley Index

Definition: Let (P, E) be an index pair for A. Then the k-dimensional Conley Index is given by $H_k(P, E)$.

Theorem [LKMW 2019]: The k-dimensional Conley Index for A is well defined.

Conley Indices



$$H_2(R \cup Y, R) = \mathbb{Z}_2$$

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[DMS20] T. K. Dey, M. Mrozek, R. Slechta. "Persistence of the Conley Index in Combinatorial Dynamical Systems." SoCG 2020.

Conley Index Persistence

First attempt: for each $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, compute an isolated invariant set, $A_1, A_2, \ldots, A_n$ and corresponding index pairs.

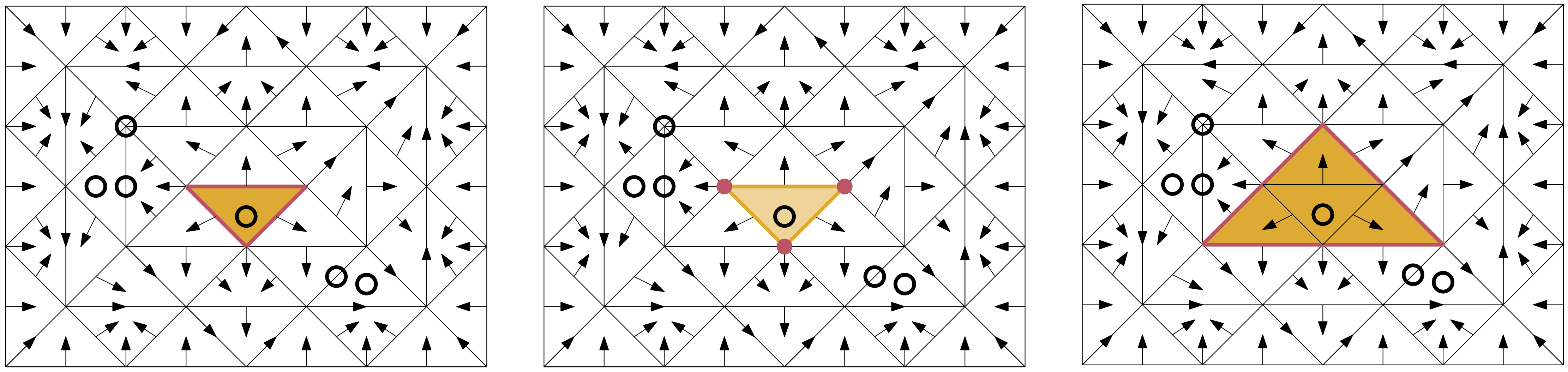
$$(\mathsf{cl}(A_1), \mathsf{mo}(A_1)), (\mathsf{cl}(A_2), \mathsf{mo}(A_2)), \ldots, (\mathsf{cl}(A_n), \mathsf{mo}(A_n))$$

Gives a relative zigzag filtration:

$$\ldots \subseteq (\mathsf{cl}(A_i), \mathsf{mo}(A_i)) \supseteq (\mathsf{cl}(A_i) \cap \mathsf{cl}(A_{i+1}), \mathsf{mo}(A_i) \cap \mathsf{mo}(A_{i+1})) \subseteq (\mathsf{cl}(A_{i+1}), \mathsf{mo}(A_{i+1})) \supseteq \ldots$$

Problem: $(\mathsf{cl}(A_i) \cap \mathsf{cl}(A_{i+1}), \mathsf{mo}(A_i) \cap \mathsf{mo}(A_{i+1}))$ generally not an index pair.

Intersection Example

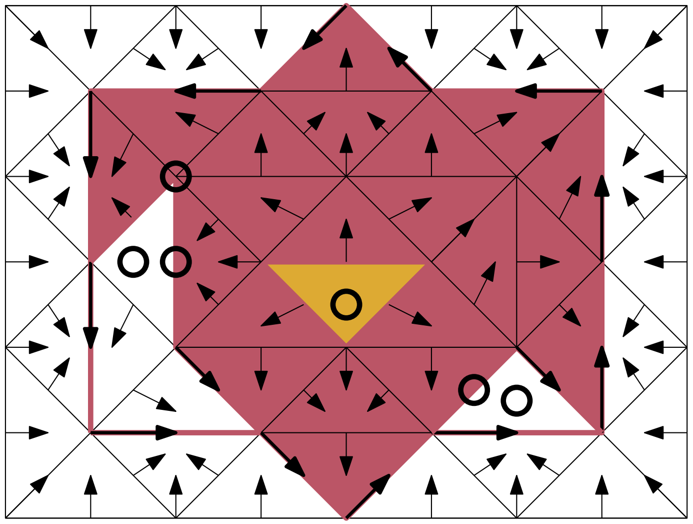


Index Pairs in an Isolating Neighborhood

Let $E \subset P \subseteq N$ for closed P, E, N, and $A \subseteq N$. If:

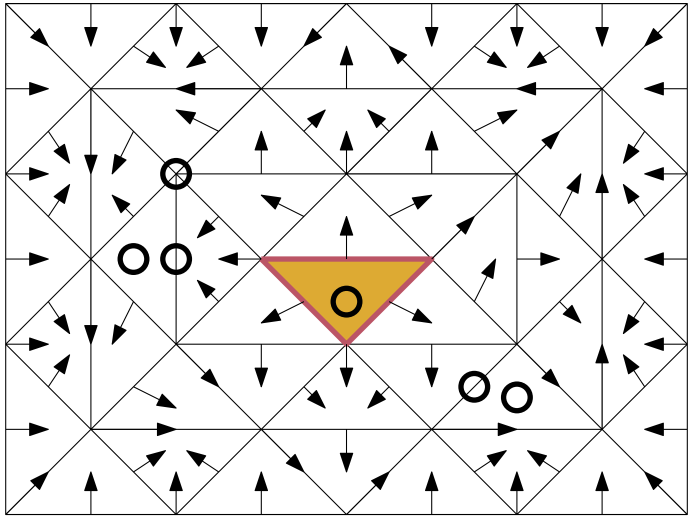
1. $F_{\mathcal{V}}(P) \cap N \subseteq P$,
2. $F_{\mathcal{V}}(E) \cap N \subseteq E$,
3. $F_{\mathcal{V}}(P \setminus E) \subseteq N$, and
4. $A = \mathrm{Inv}(P \setminus E)$

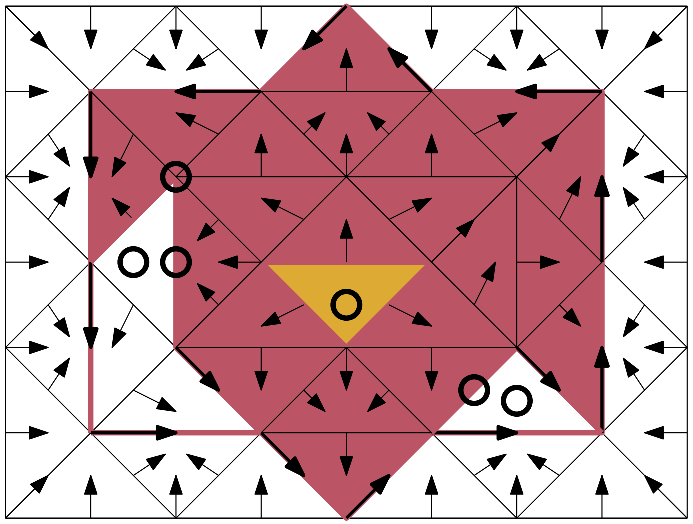
then (P, E) is an index pair in N.



Finding Index Pairs in N

<u>Theorem [DMS20]:</u> Let A denote an isolated invariant set, and let N denote an isolating neighborhood for A. The pair $(\mathrm{pf}(\mathrm{cl}(A)), \mathrm{pf}(\mathrm{mo}(A)))$ is an index pair in N for A.





Index Pairs in an Isolating Neighborhood

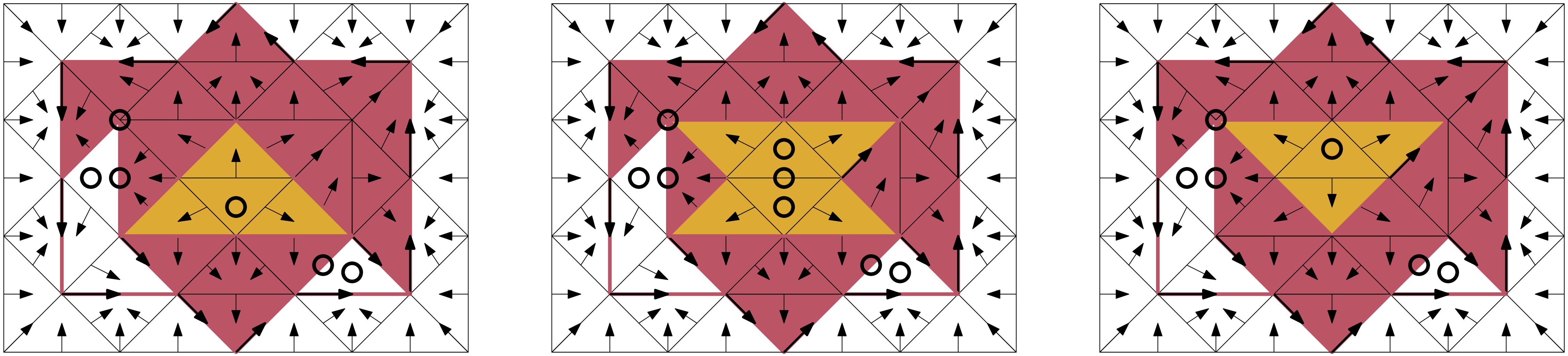
Theorem (DMS20): Index Pairs in N are index pairs.

Definition: Let $\mathcal{V}_1$, $\mathcal{V}_2$ denote multivector fields over K. The intersection of multivector fields is given by

$$\mathcal{V}_1 \overline{\cap} \mathcal{V}_2 = \{V_1 \cap V_2 \mid V_1 \in \mathcal{V}_1,\ V_2 \in \mathcal{V}_2\}$$

Theorem (DMS20): Let (P_1, E_1) , (P_2, E_2) denote index pairs in N under $\mathcal{V}_1, \mathcal{V}_2$. The pair $(P_1 \cap P_2, E_1 \cap E_2)$ is an index pair in N under $\mathcal{V}_1 \overline{\cap} \mathcal{V}_2$ for $\mathrm{Inv}((P_1 \cap P_2) \setminus (E_1 \cap E_2))$

Intersection Example



Dimension: 2

All simplices in N, Yellow union
Red is P, and Red is E

Conley Index Persistence: New Strategy

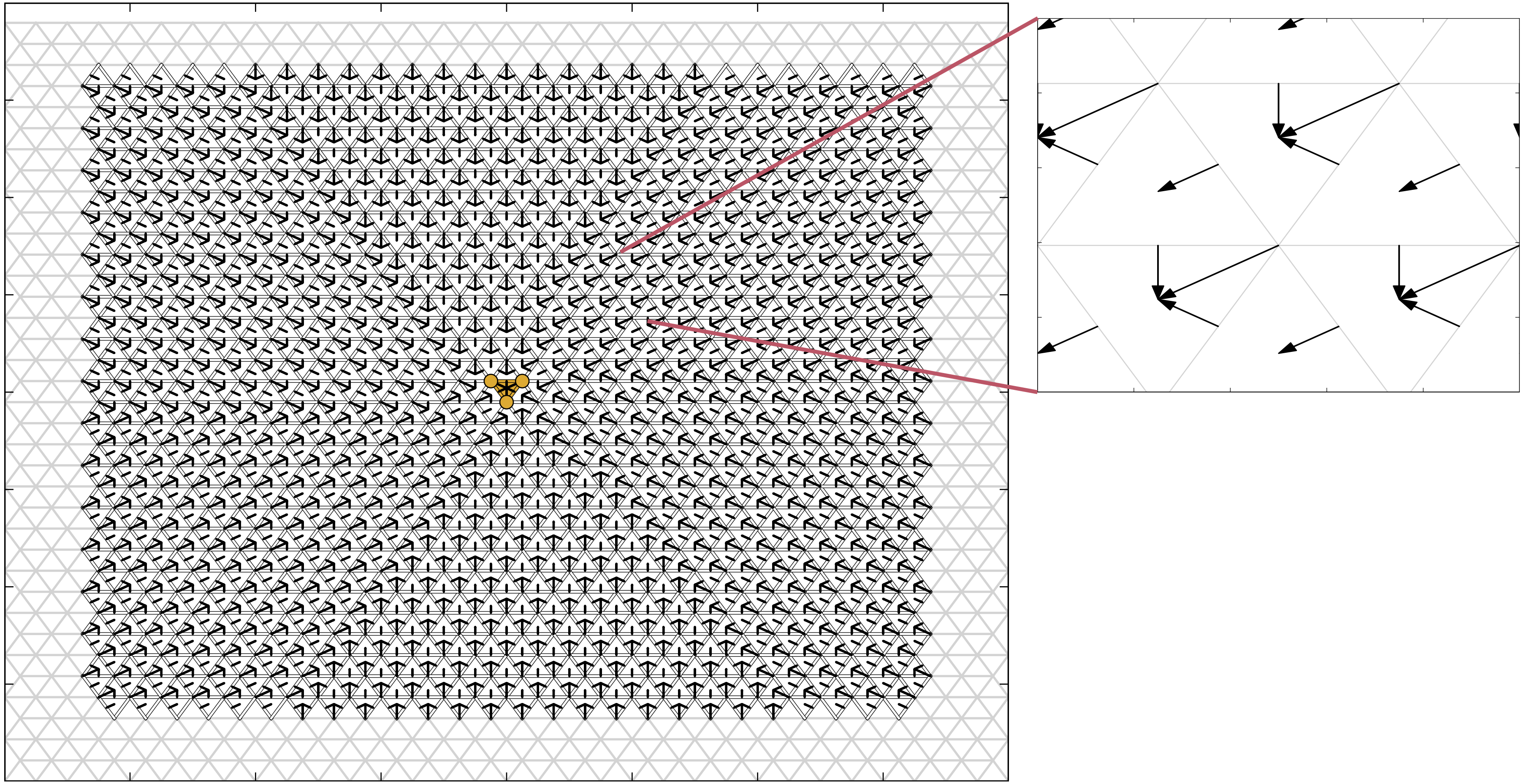
Fix N, and for each $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, compute the maximal invariant set in N, denoted $A_1, A_2, \ldots, A_n$, and corresponding index pairs.

$$(\mathsf{cl}(A_1), \mathsf{mo}(A_1)), (\mathsf{cl}(A_2), \mathsf{mo}(A_2)), \ldots, (\mathsf{cl}(A_n), \mathsf{mo}(A_n))$$

Gives a relative zigzag filtration:

$$(\mathsf{pf}_N(\mathsf{cl}A_i), \mathsf{pf}_N(\mathsf{mo}A_i)) \supseteq (\mathsf{pf}_N(\mathsf{cl}A_i) \cap \mathsf{pf}_N(\mathsf{cl}A_{i+1}), \mathsf{pf}_N(\mathsf{mo}A_i) \cap \mathsf{pf}_N(\mathsf{mo}A_{i+1})) \subseteq (\mathsf{pf}_N(\mathsf{cl}A_{i+1}), \mathsf{pf}_N(\mathsf{mo}A_{i+1}))$$

Motivating Example: Hopf Bifurcation



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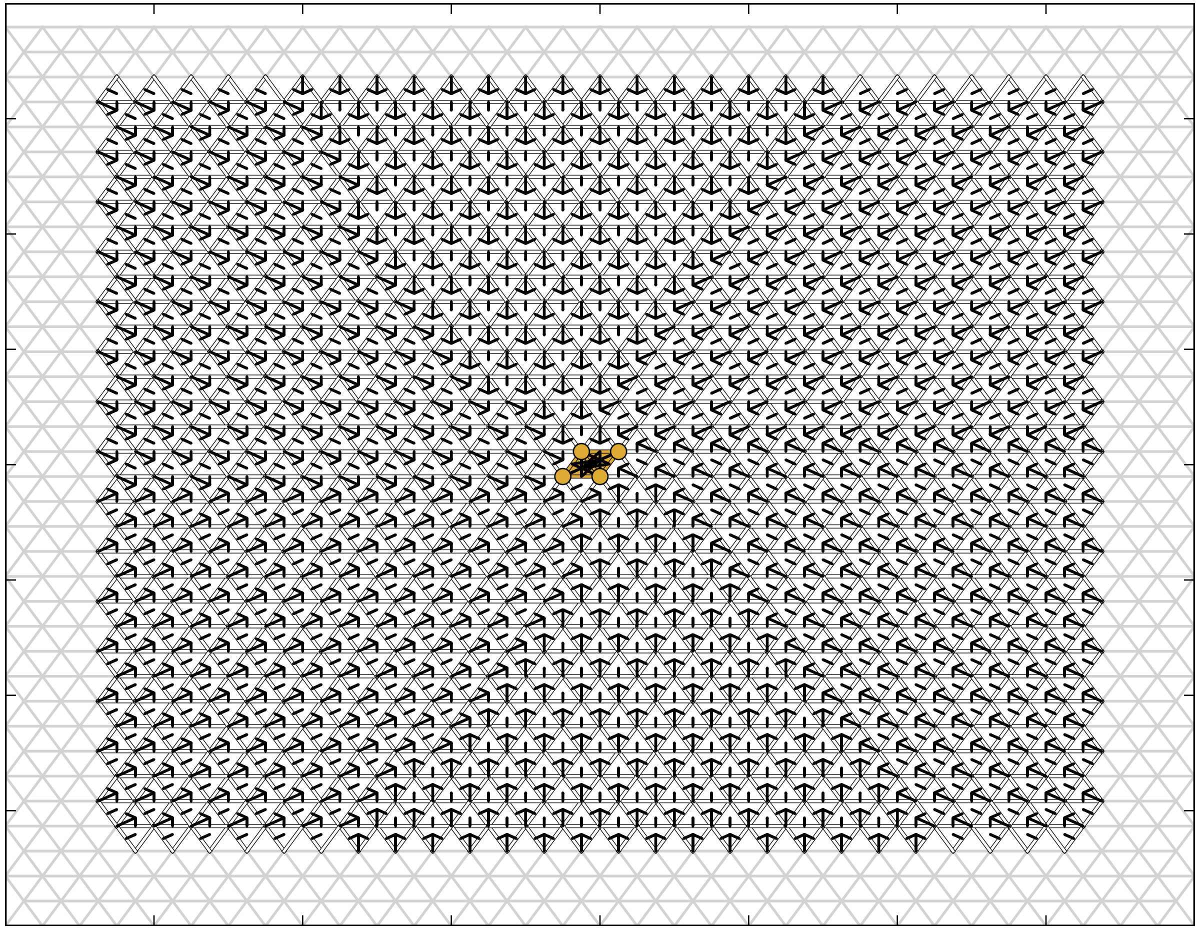

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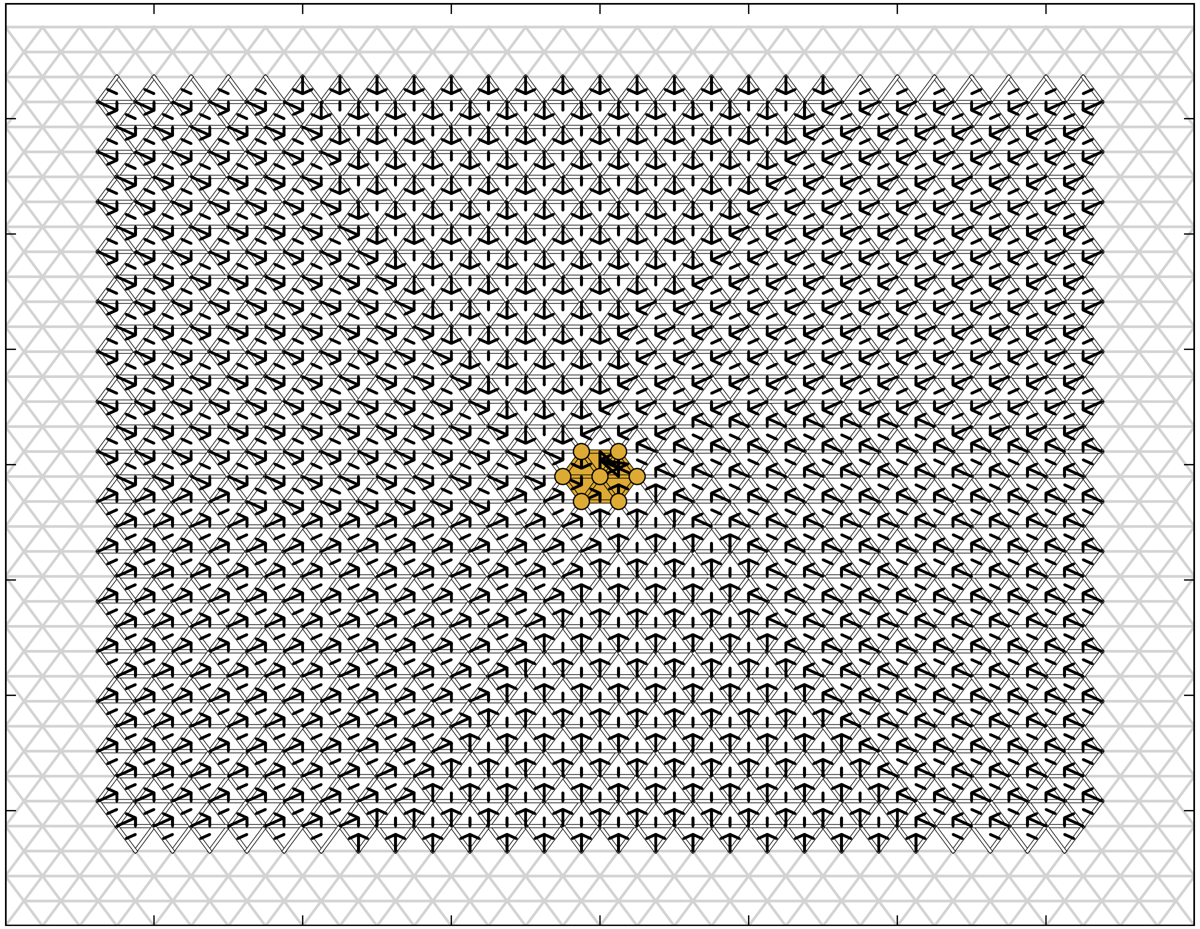

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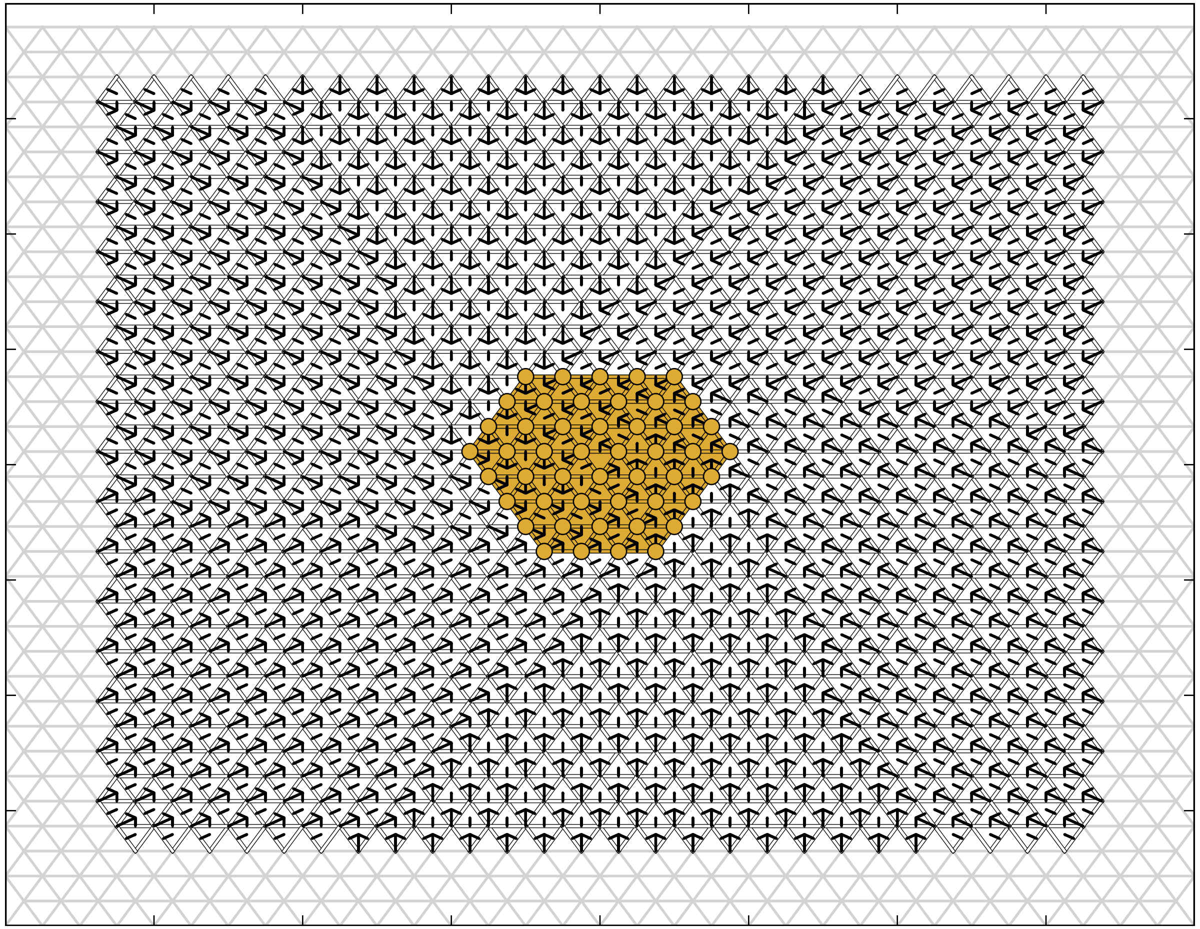

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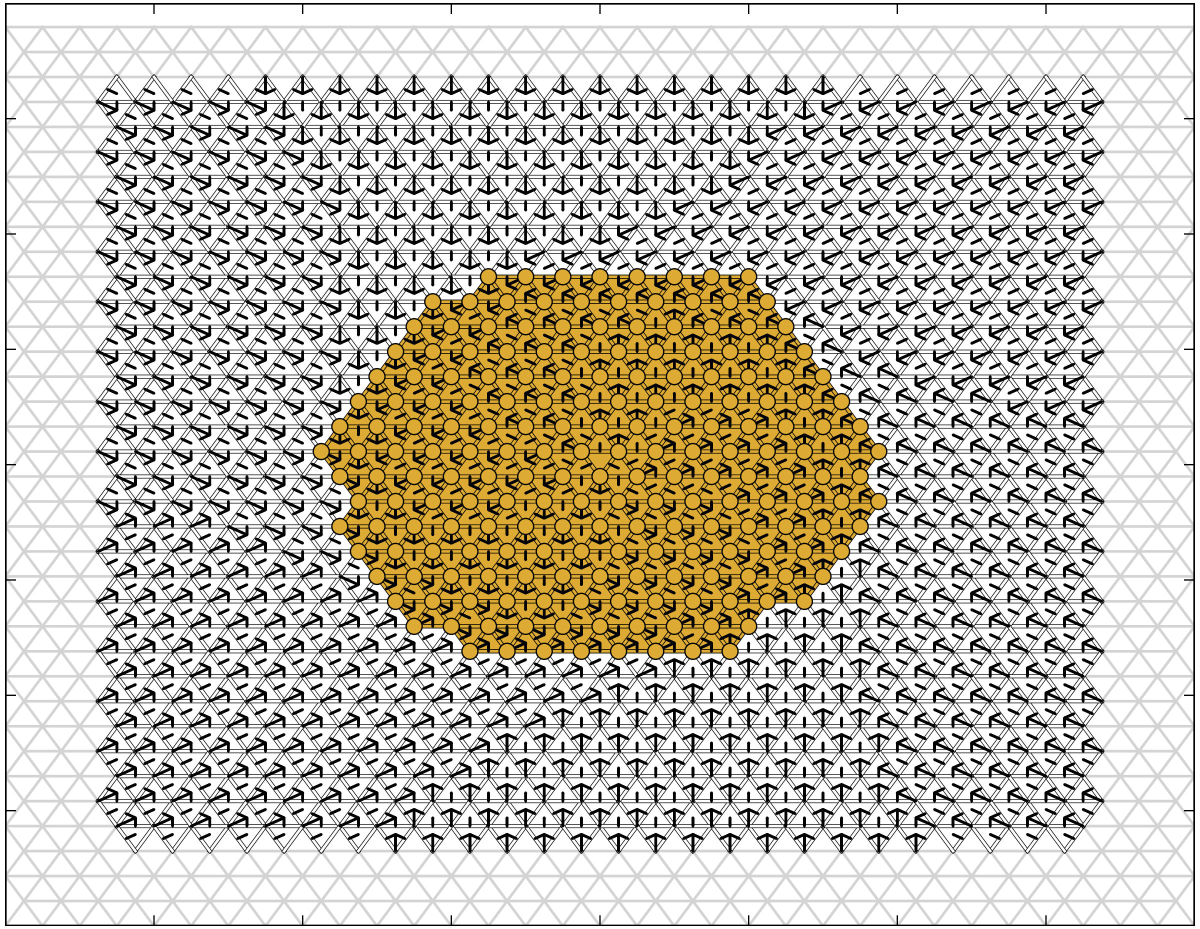

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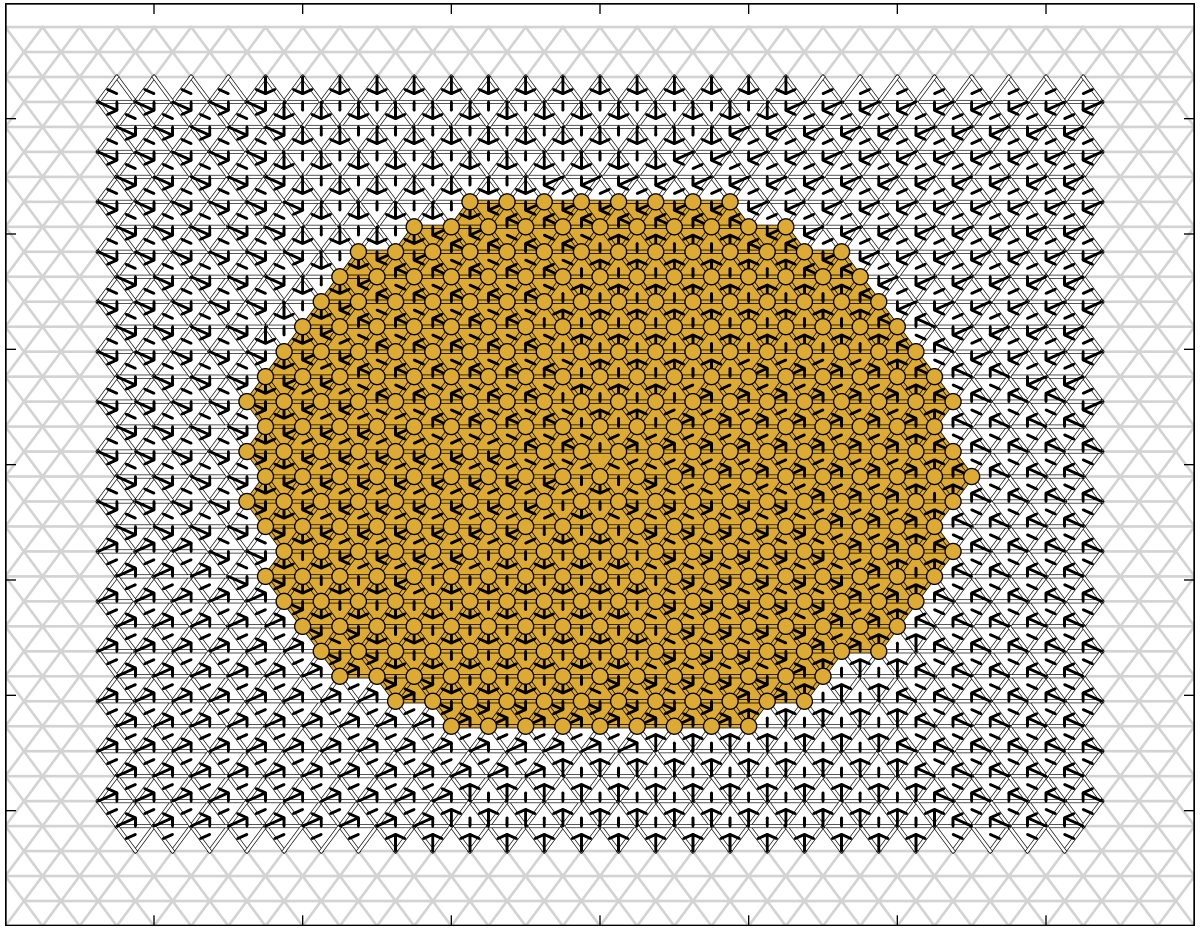

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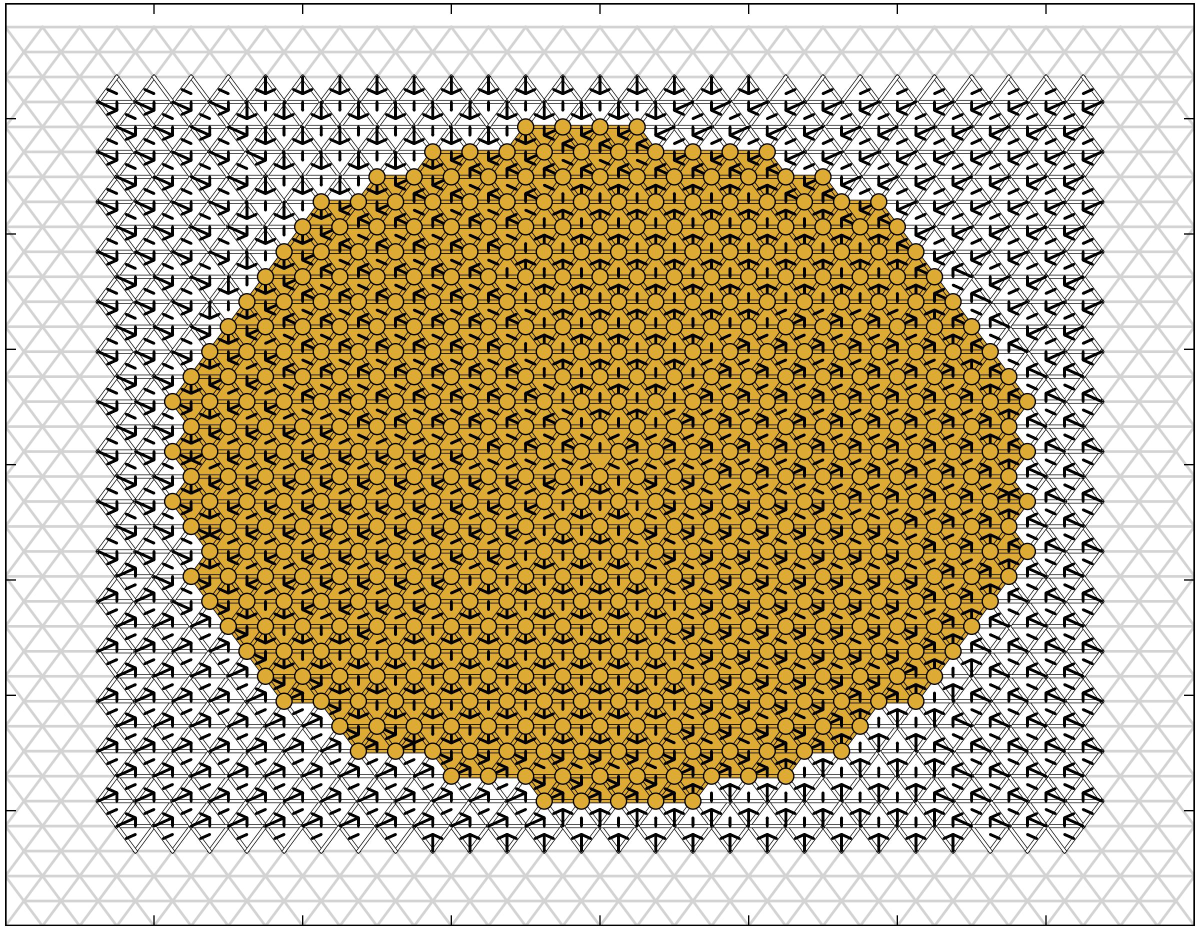

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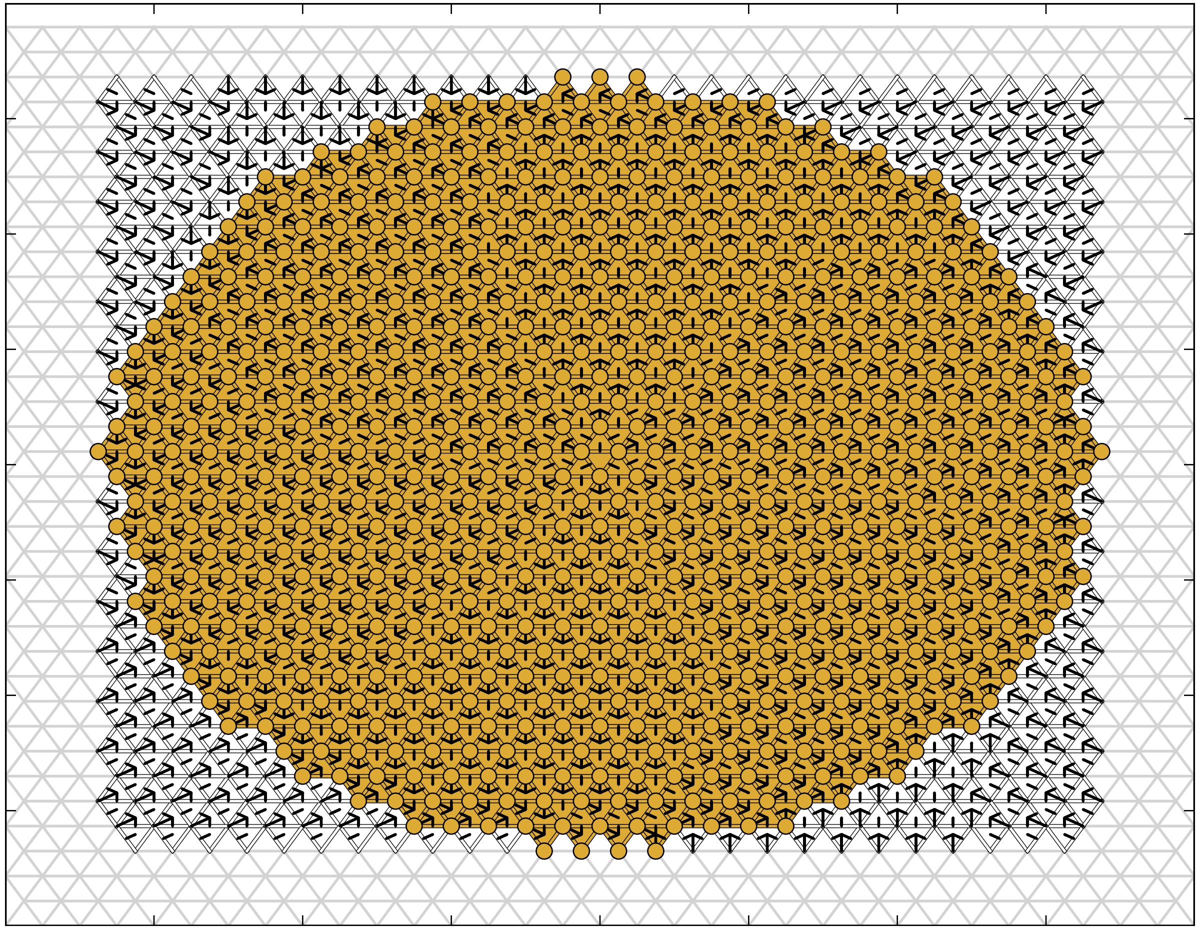
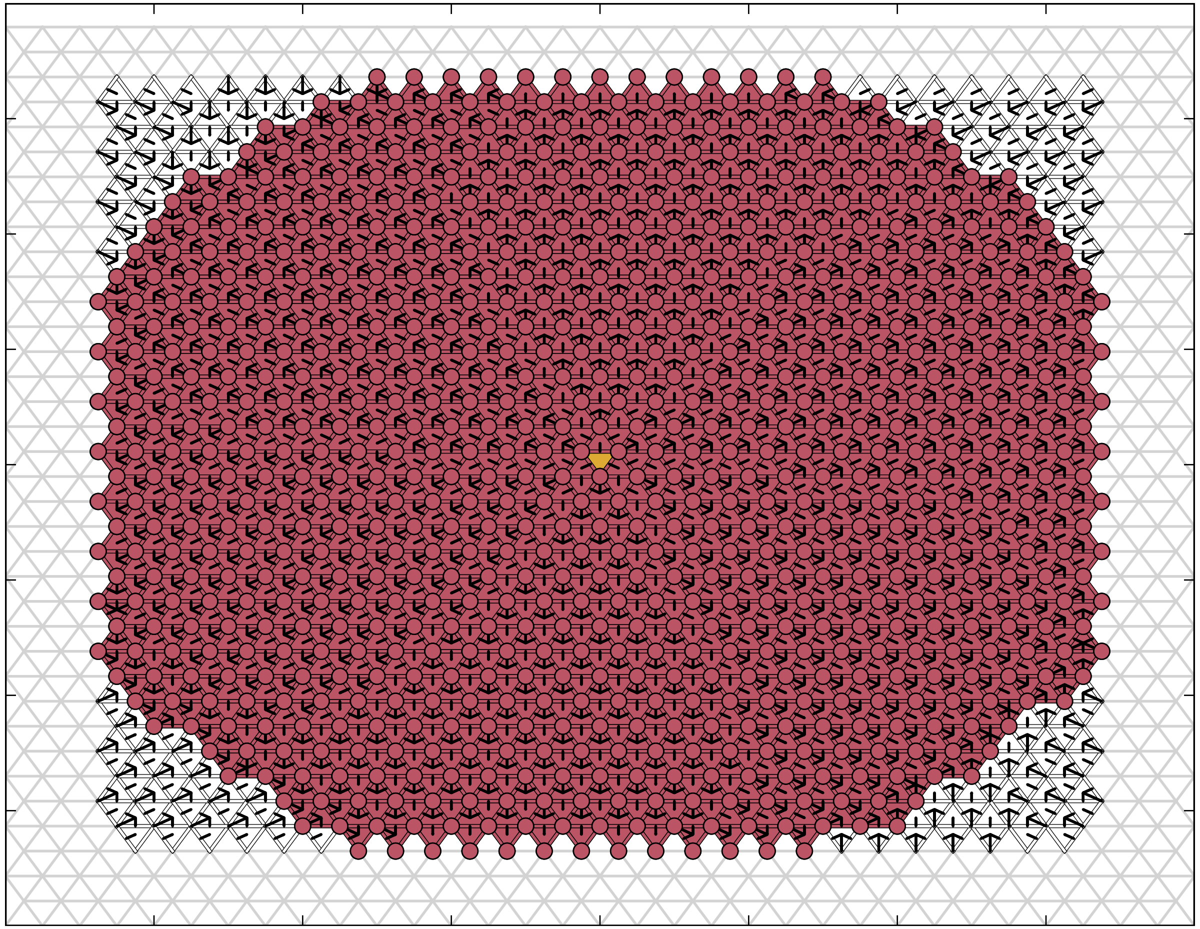
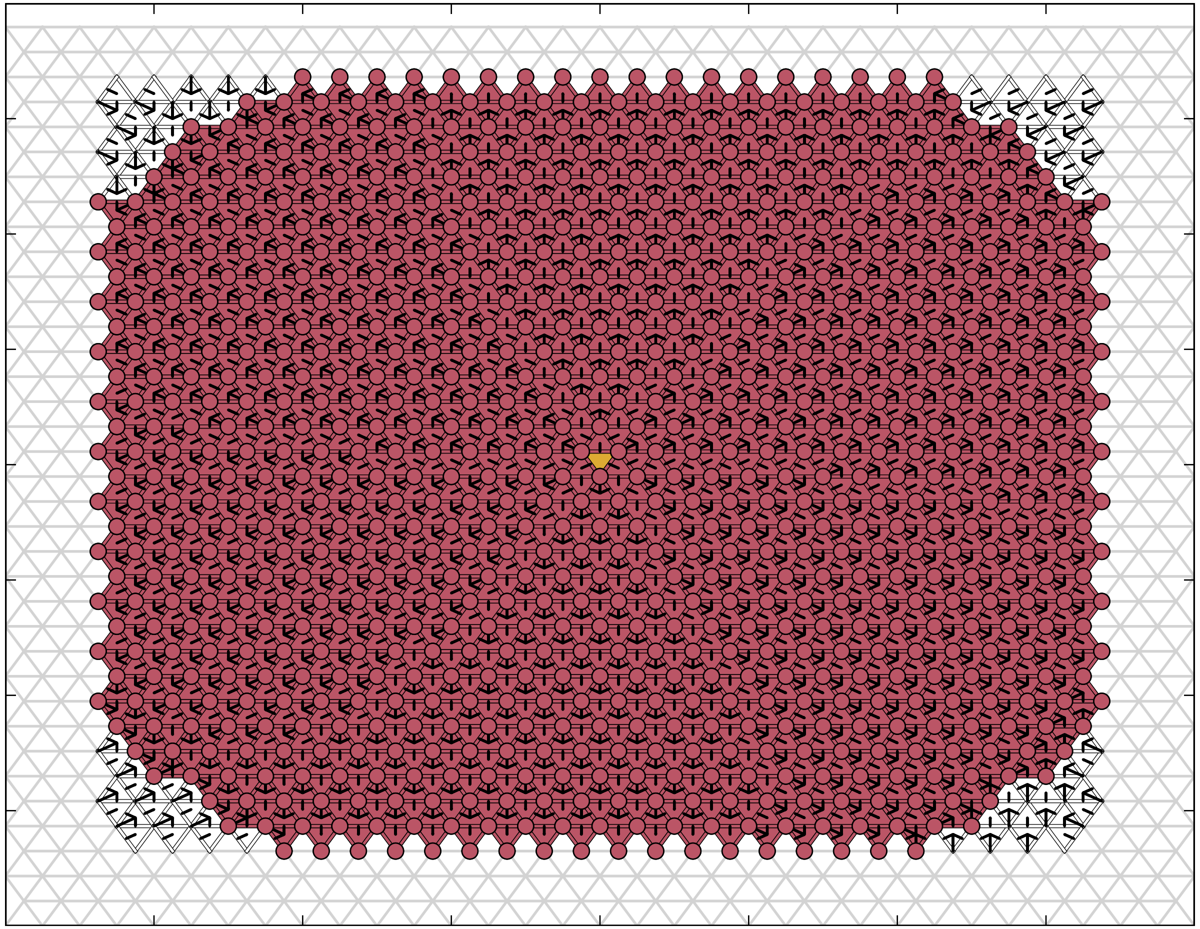


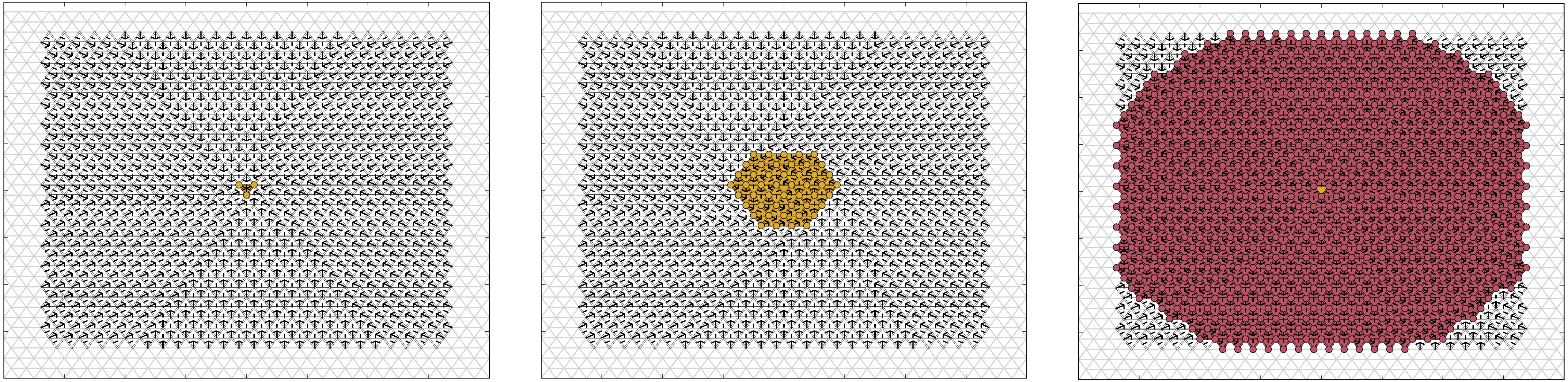
Motivating Example: Hopf Bifurcation



Motivating Example: Hopf Bifurcation



Conley Index Persistence



Dimension: 0

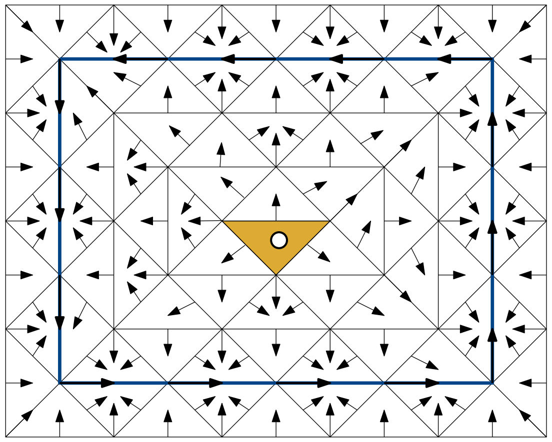
Dimension: 2

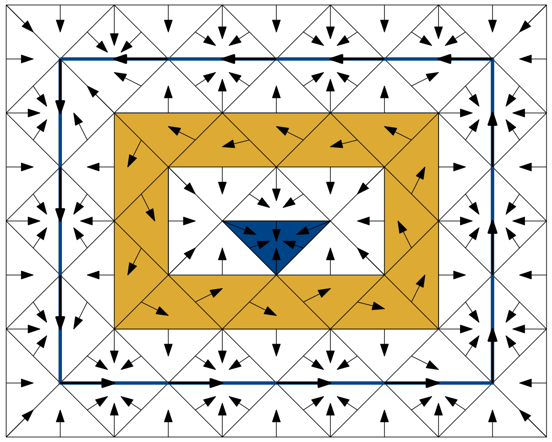
Overview & Outline

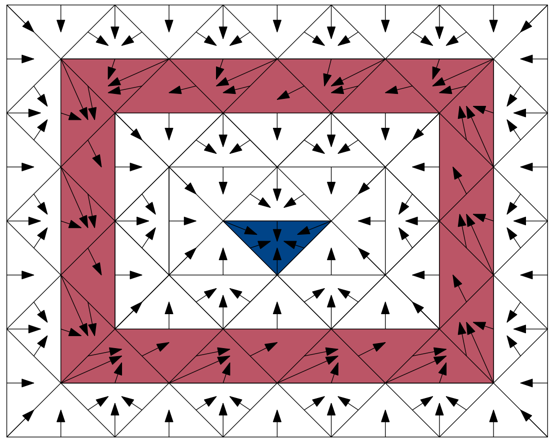
- Motivating Example and Persistence
- Combinatorial Dynamical Systems & Conley Index
- Conley Index Persistence
- Conley-Morse Graph Persistence
- Tracking and Combinatorial Continuation

[DMS22] T. K. Dey, M. Mrozek, R. Slechta. "Persistence of Conley-Morse Graphs in Combinatorial Dynamical Systems." SIADS 2022, to appear.

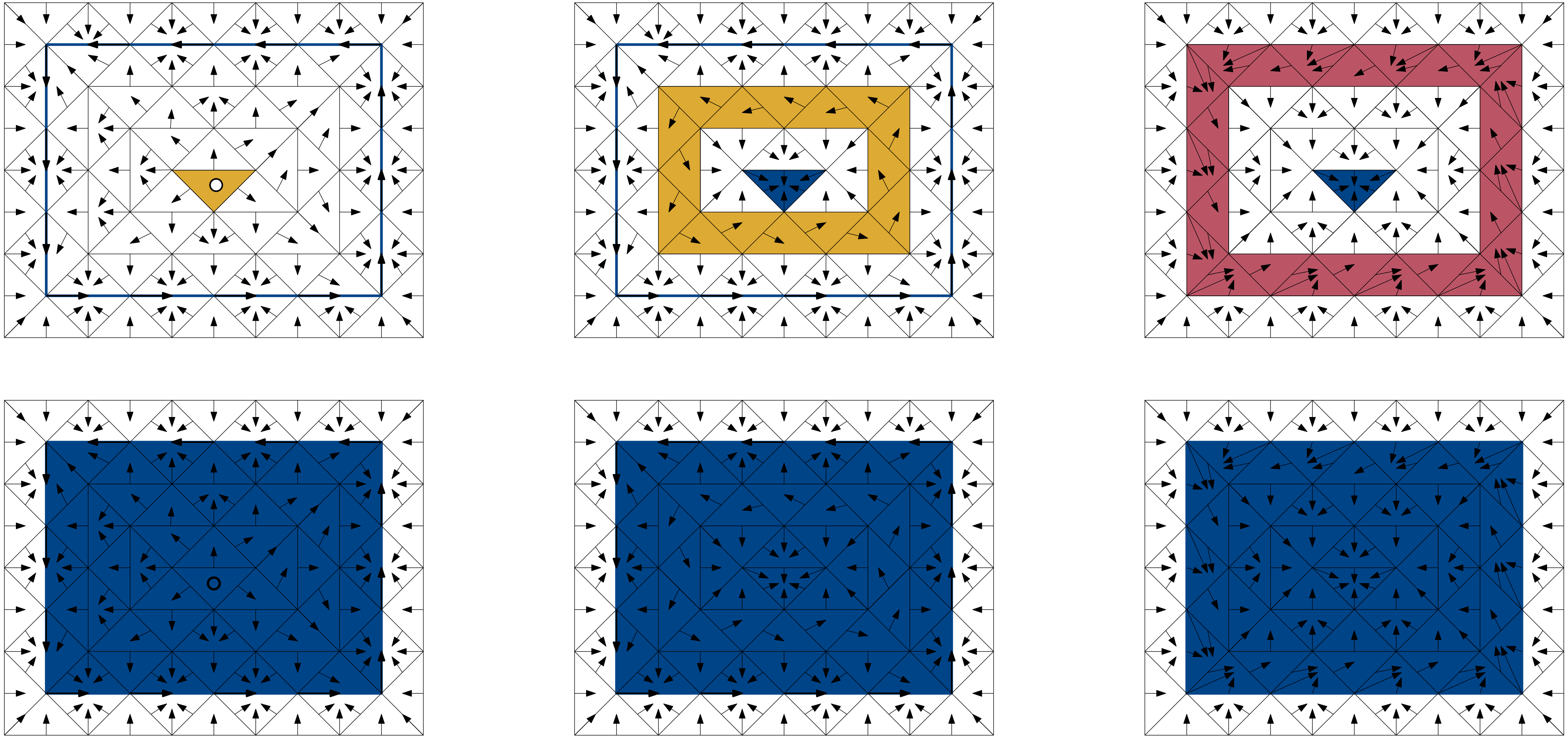
Motivating Example





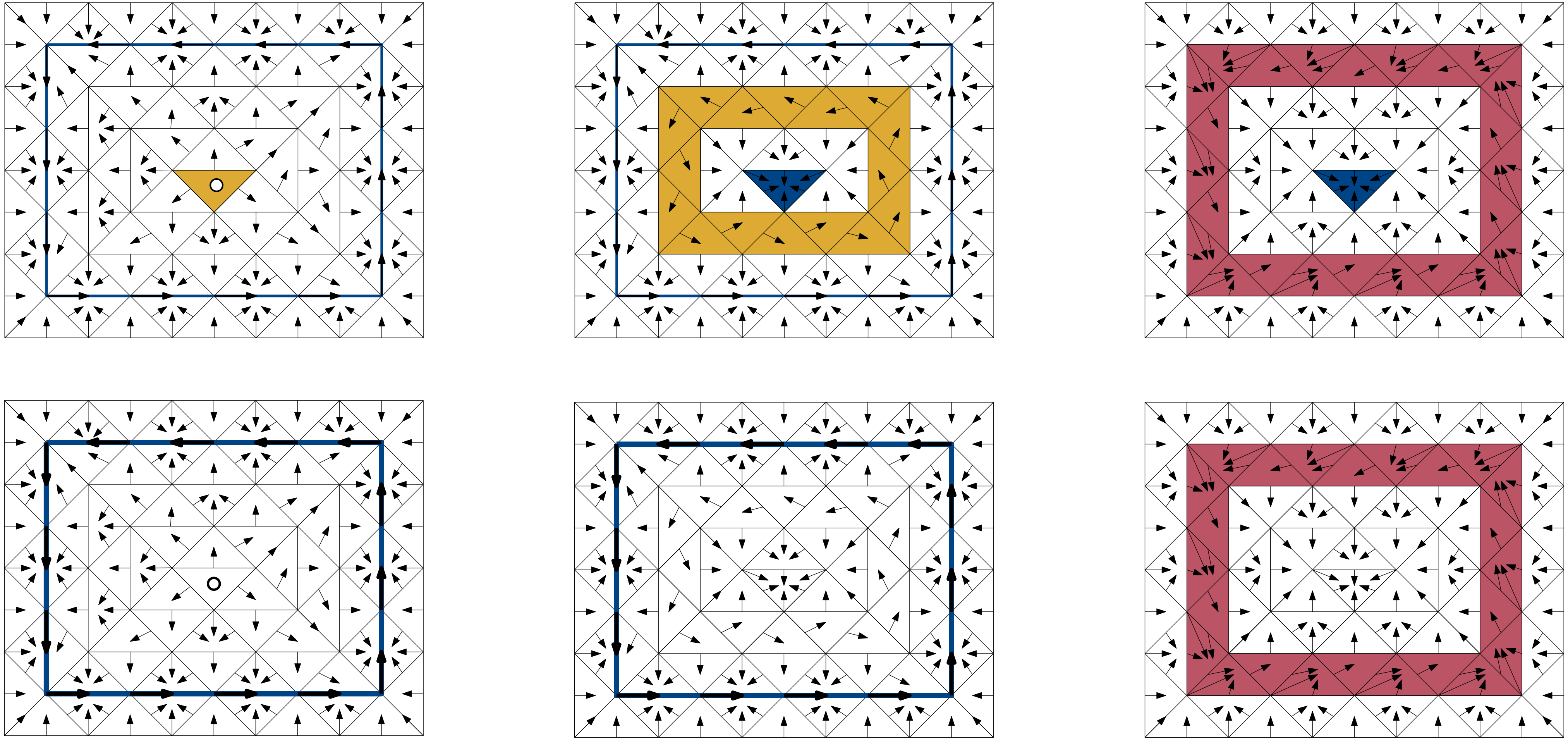


Motivating Example



Dimension: 0

Original Example

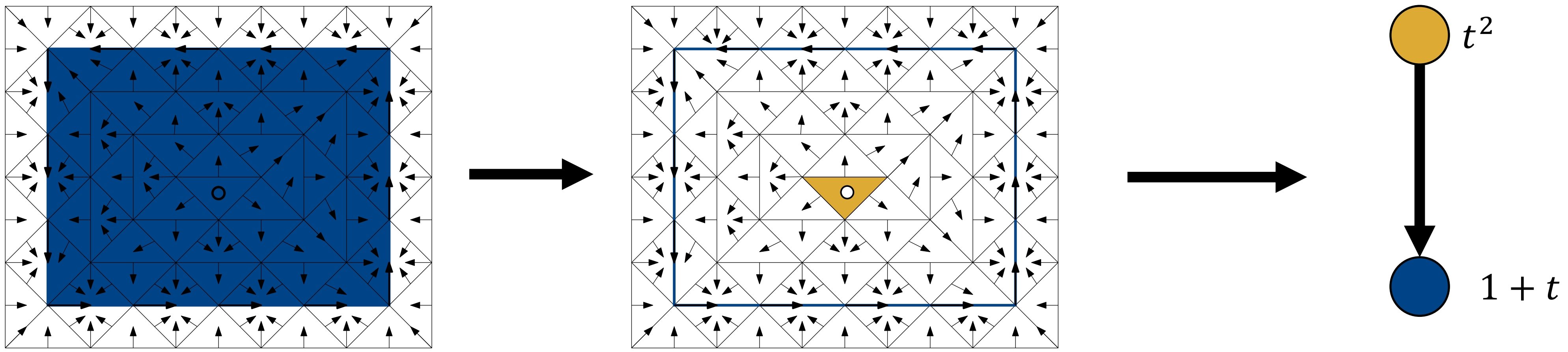


Dimension: 0

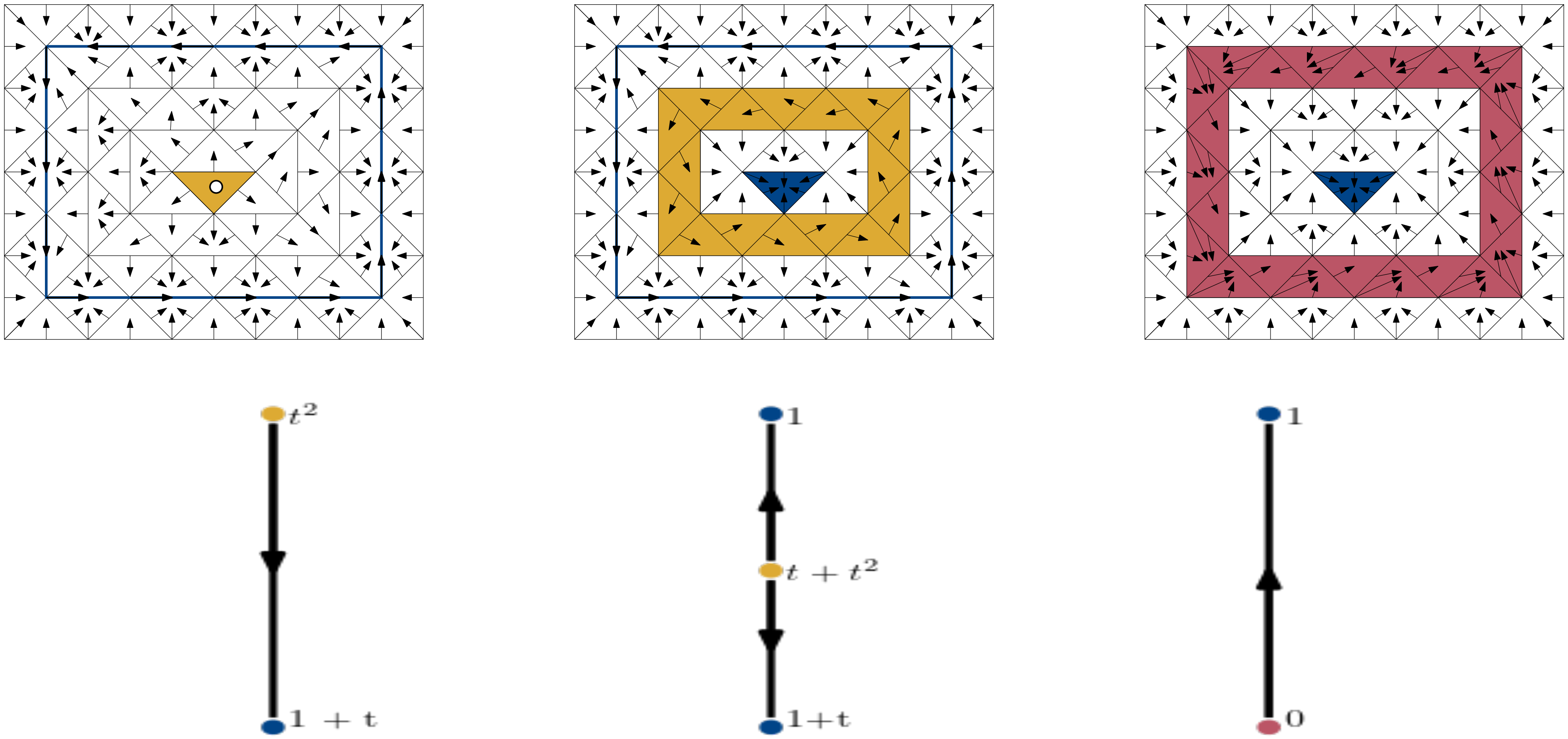
Dimension: 1

Conley-Morse Graph

A Morse decomposition graph equipped with information about the Conley Index



Conley-Morse Graph



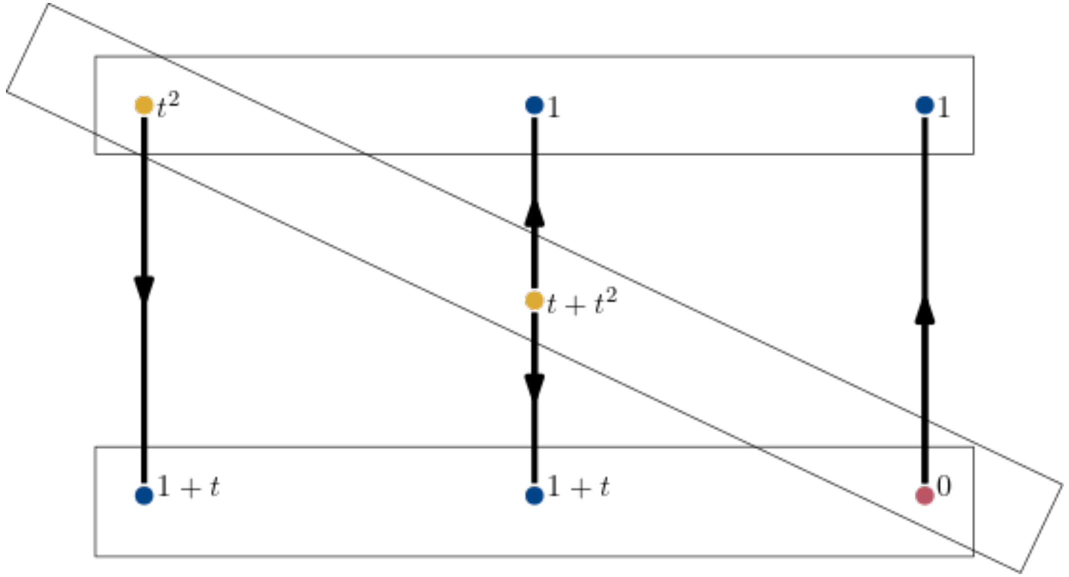
Conley-Morse Graph Persistence

Two types of filtrations:

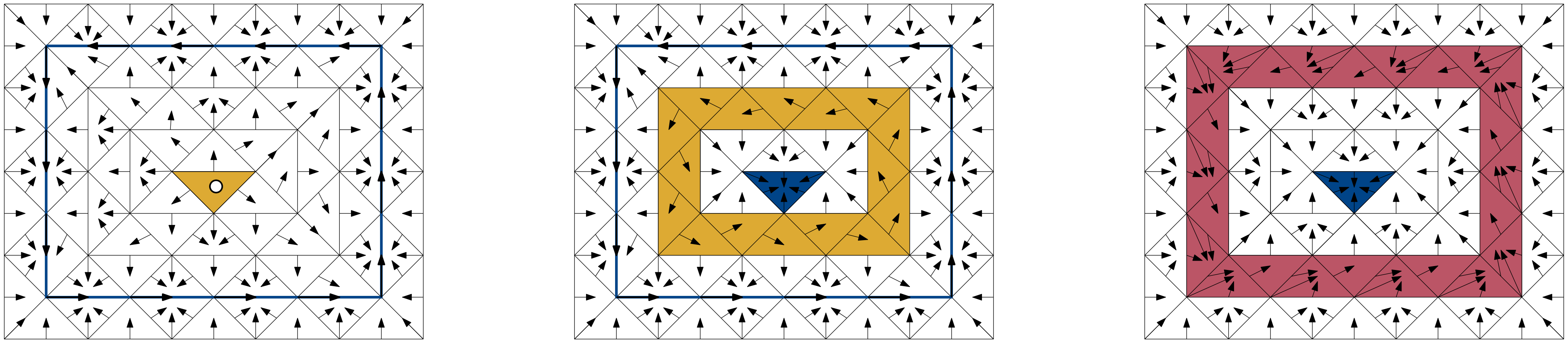
1. Graph Filtrations
2. Conley-Morse Filtrations

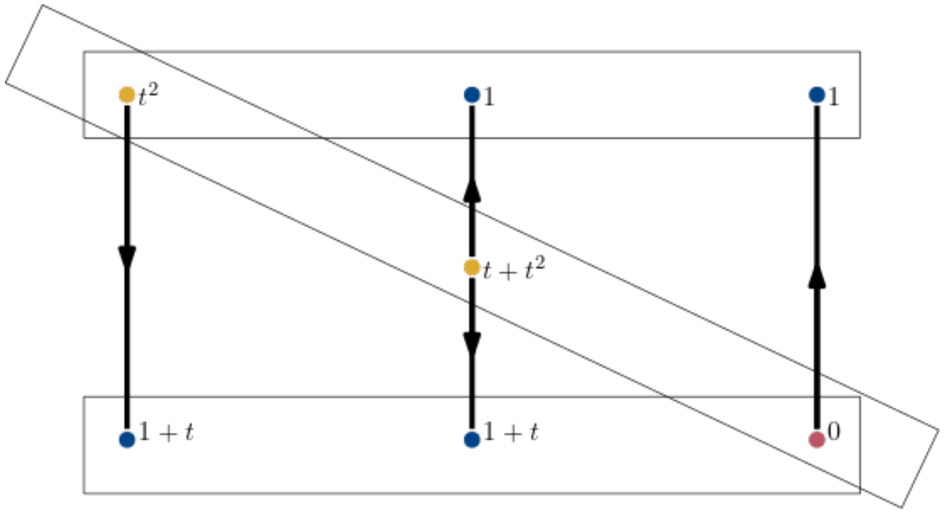
Conley-Morse Filtrations

1. Assume every isolated invariant set is isolated by the same N.
2. Fix index pair for each Morse set.
3. Find all "maximal" sequences of index pairs across Conley-Morse graphs with nontrivial intersection.

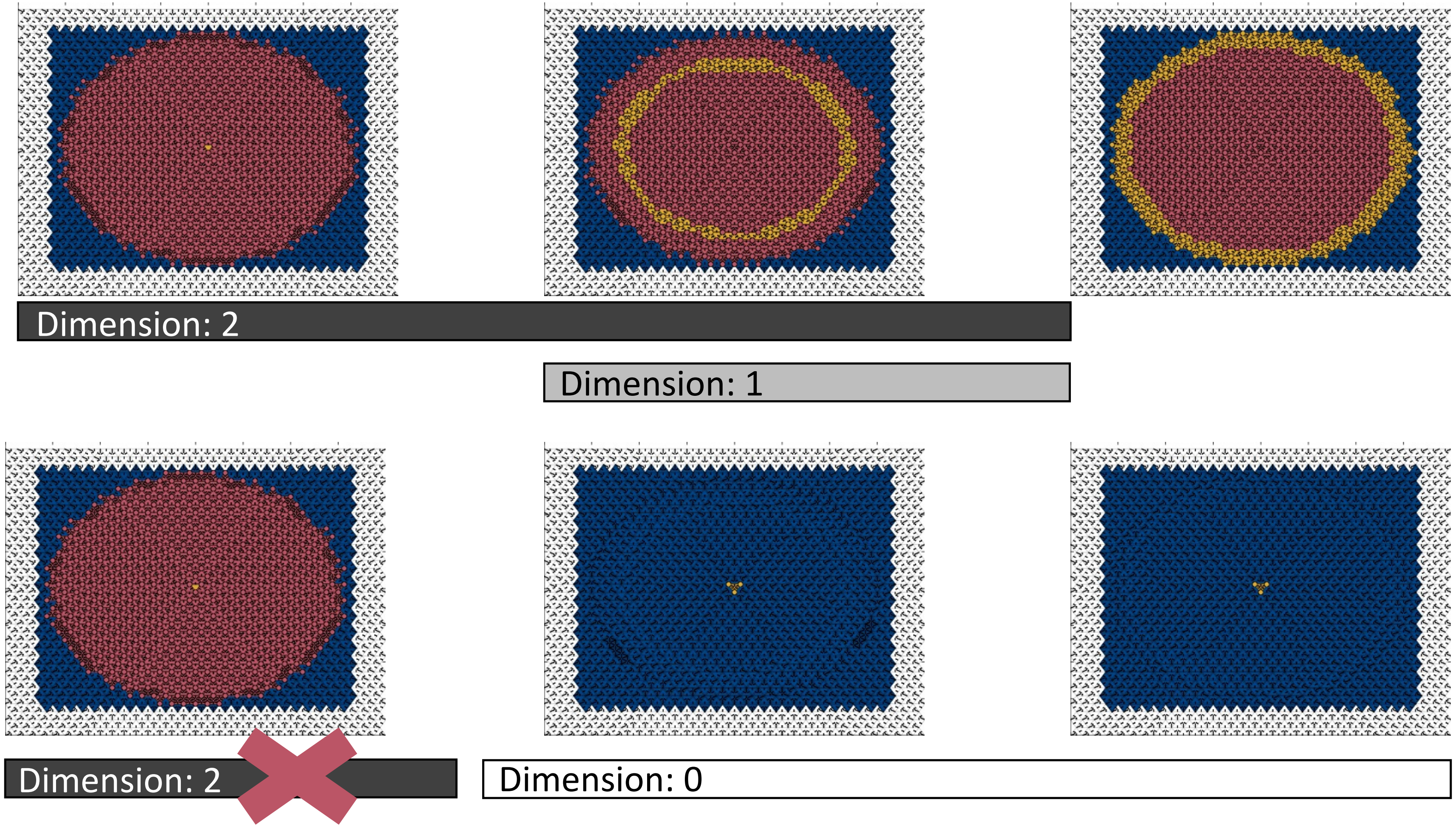


Conley-Morse Filtrations

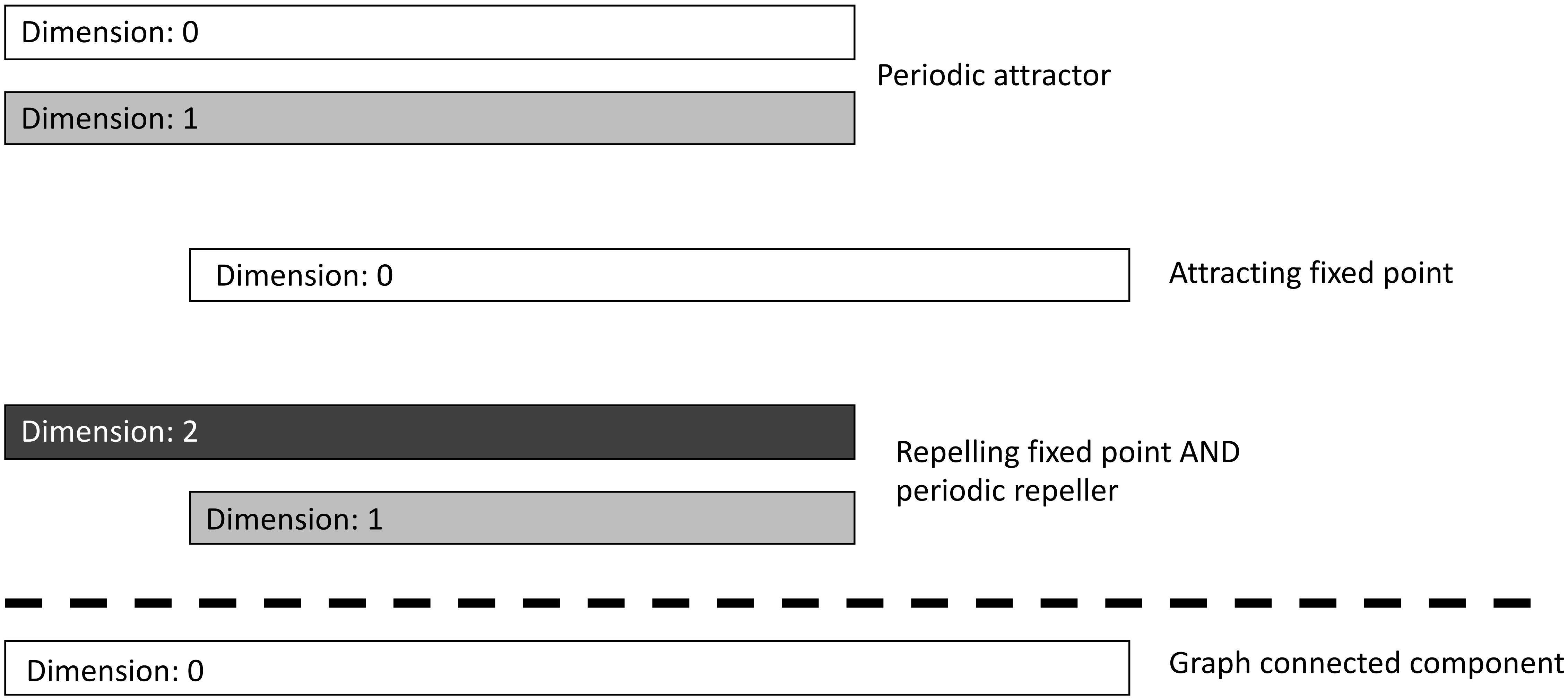




Conley-Morse Barcodes



Conley-Morse Graph Barcodes



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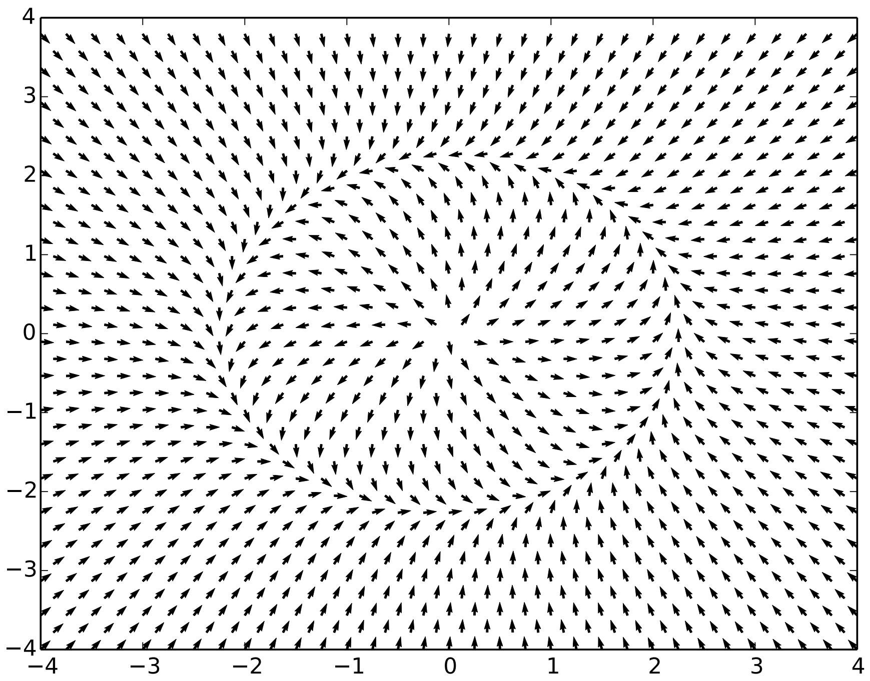
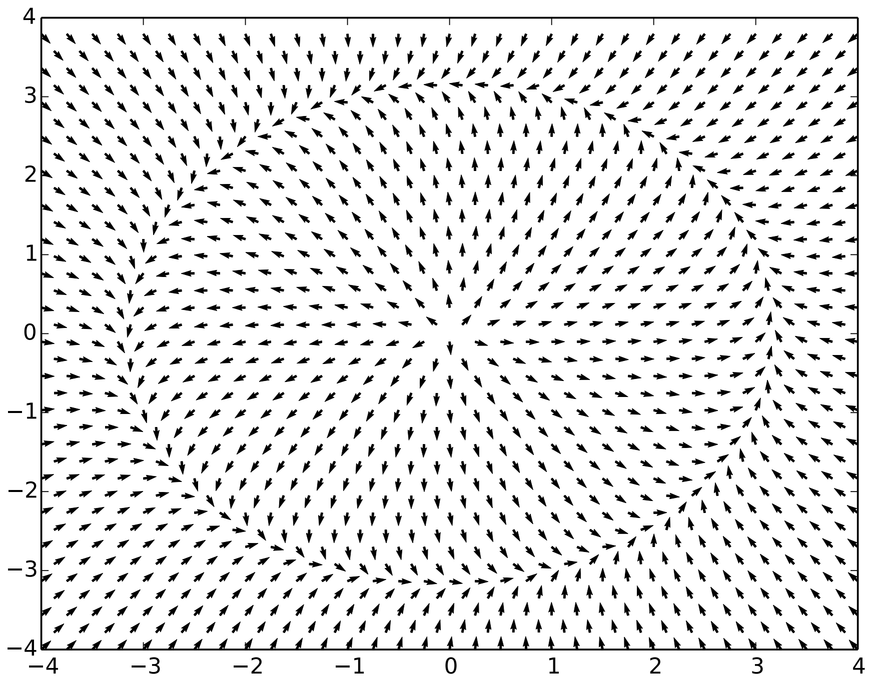
- Tracking and Combinatorial Continuation

[DLMS22] T. K. Dey, M. Lipiński, M. Mrozek, R. Slechta. "Tracking Dynamical Features via Continuation and Persistence." SoCG 2022, to appear.

Classical Continuation



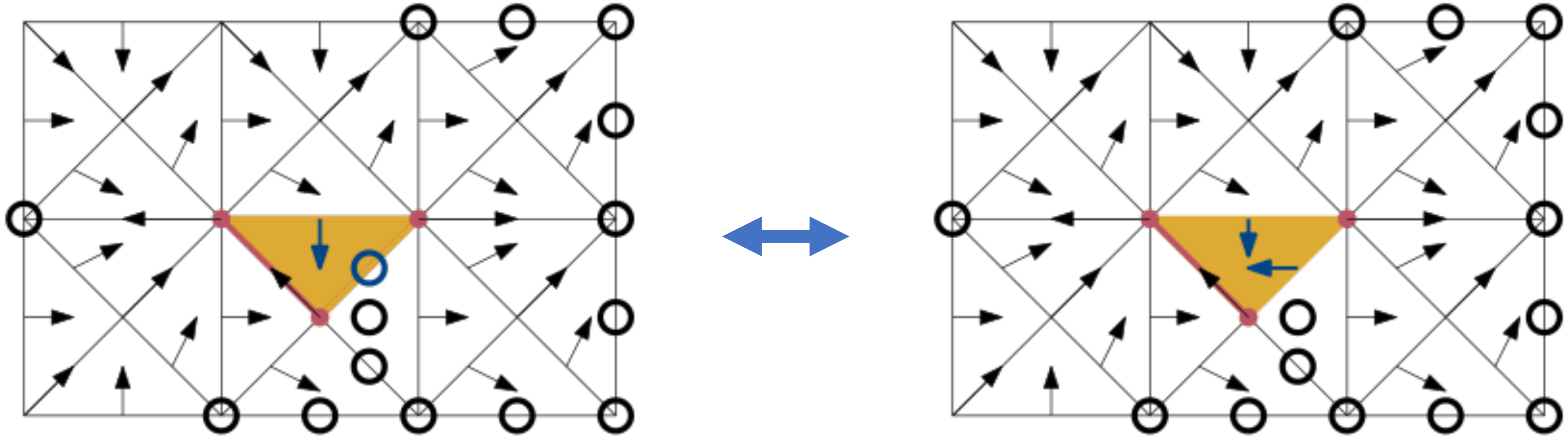
$\lambda = 5$



$\lambda = 10$

Combinatorial Continuation

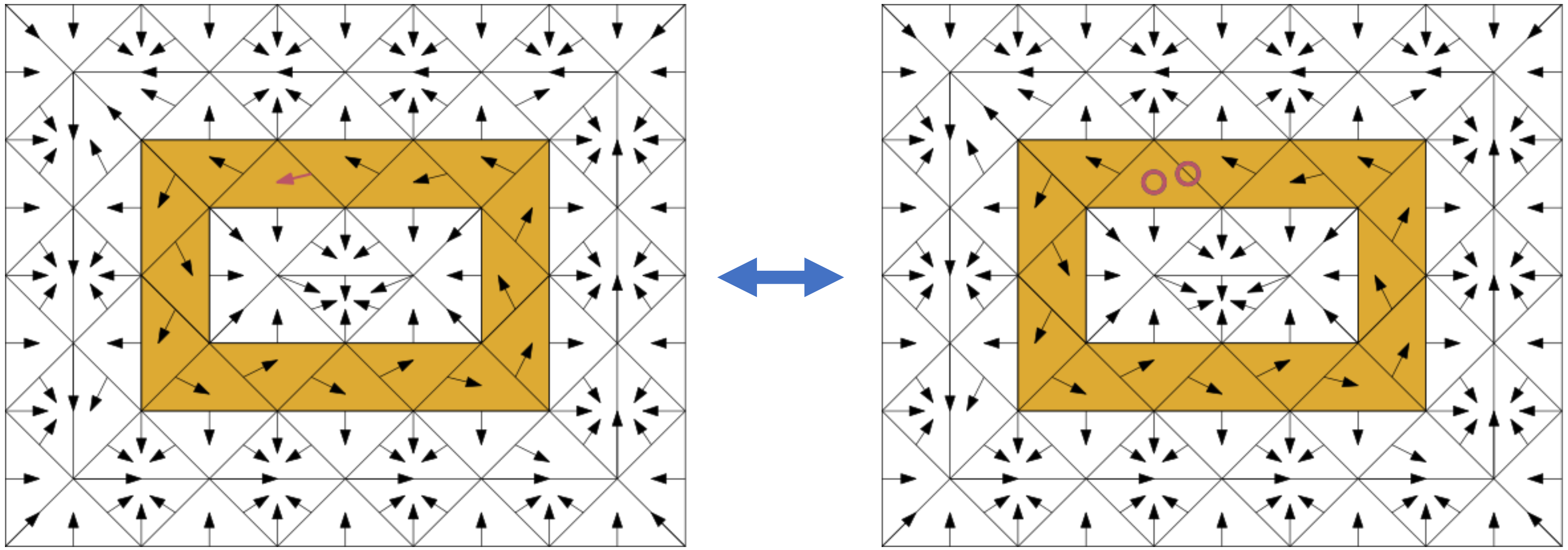
Definition: A sequence of isolated invariant sets is a *continuation* if there is a common index pair for each consecutive pair in the sequence.



Small Perturbations

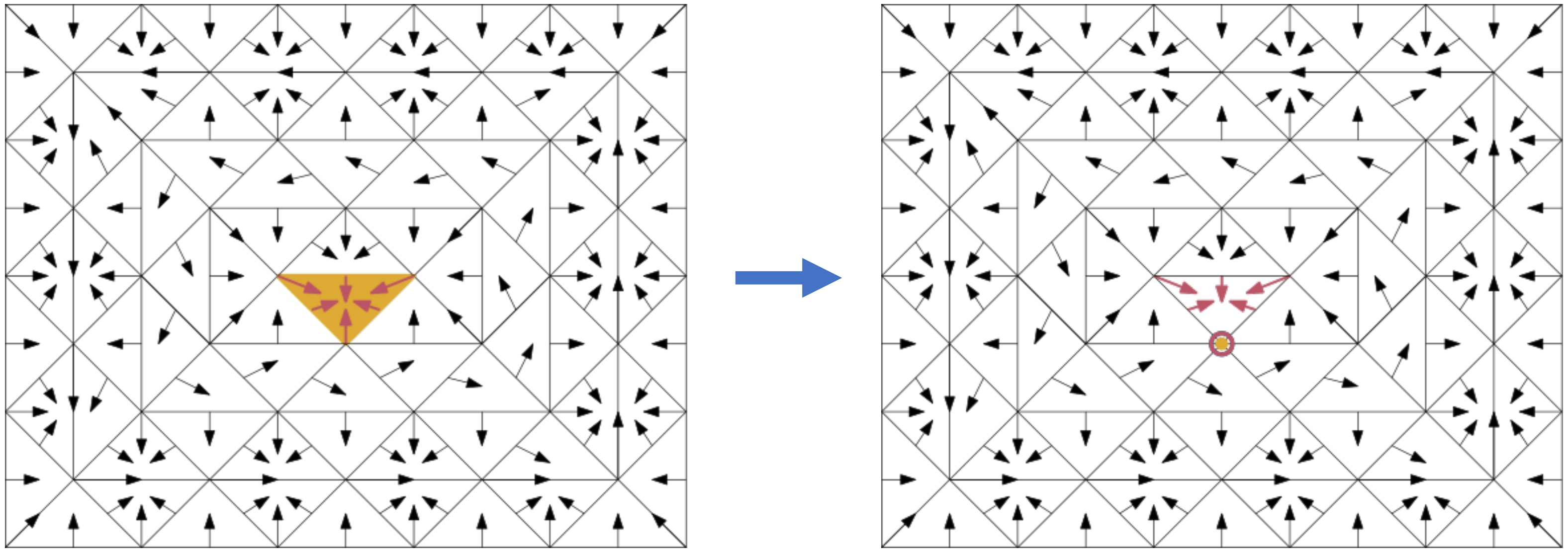
Definition: Let $\mathcal{V}$ denote a multivector field. The multivector field $\mathcal{V}'$ is an *atomic refinement* of $\mathcal{V}$ if $|\mathcal{V} \setminus \mathcal{V}'| = 1$ and $|\mathcal{V}' \setminus \mathcal{V}| = 2$. Symmetrically, $\mathcal{V}$ is an *atomic coarsening* of $\mathcal{V}'$.

Theorem[DLMS22]: Every pair of multivector fields can be transformed into each other by a sequence of atomic rearrangements.



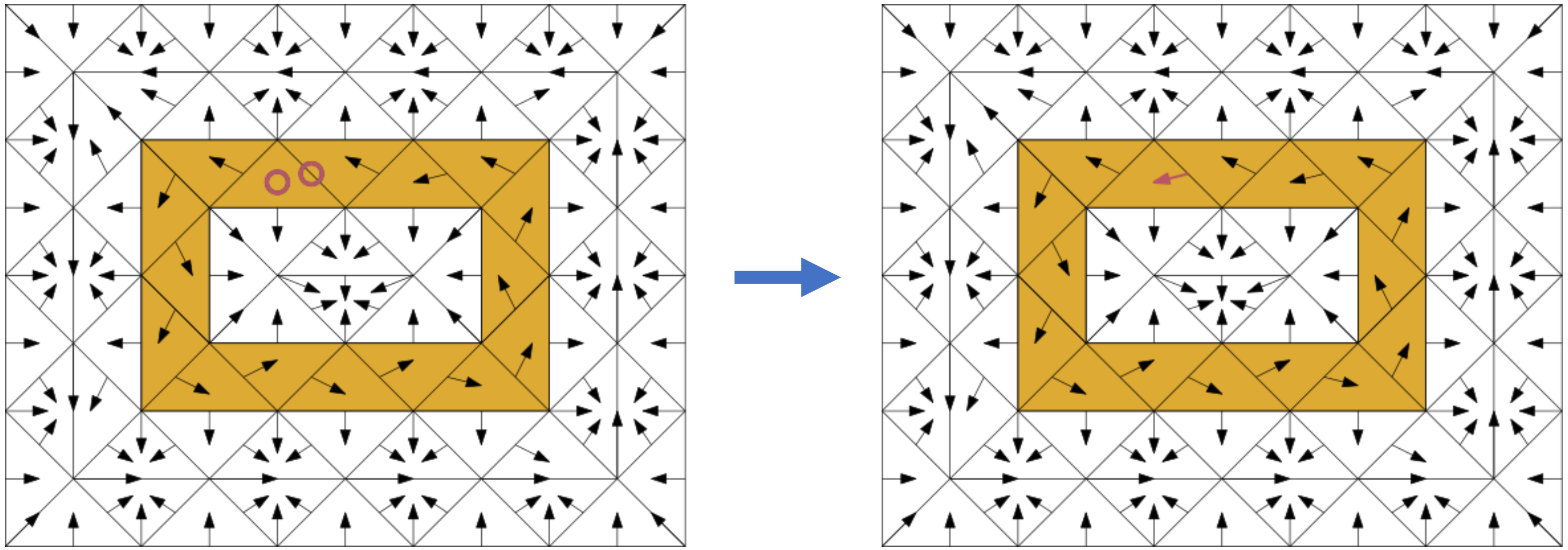
Tracking with Continuation: Case 1

<u>Theorem[DLMS22]:</u> Let S denote an isolated invariant set under $\mathcal{V}$, and let $\mathcal{V}'$ denote an atomic refinement of $\mathcal{V}$. Then $\mathsf{Inv}_{\mathcal{V}'}(\mathsf{S})$ is an invariant set and $(\mathsf{cl}(\mathsf{S}), \mathsf{mo}(\mathsf{S}))$ is an index pair for both.



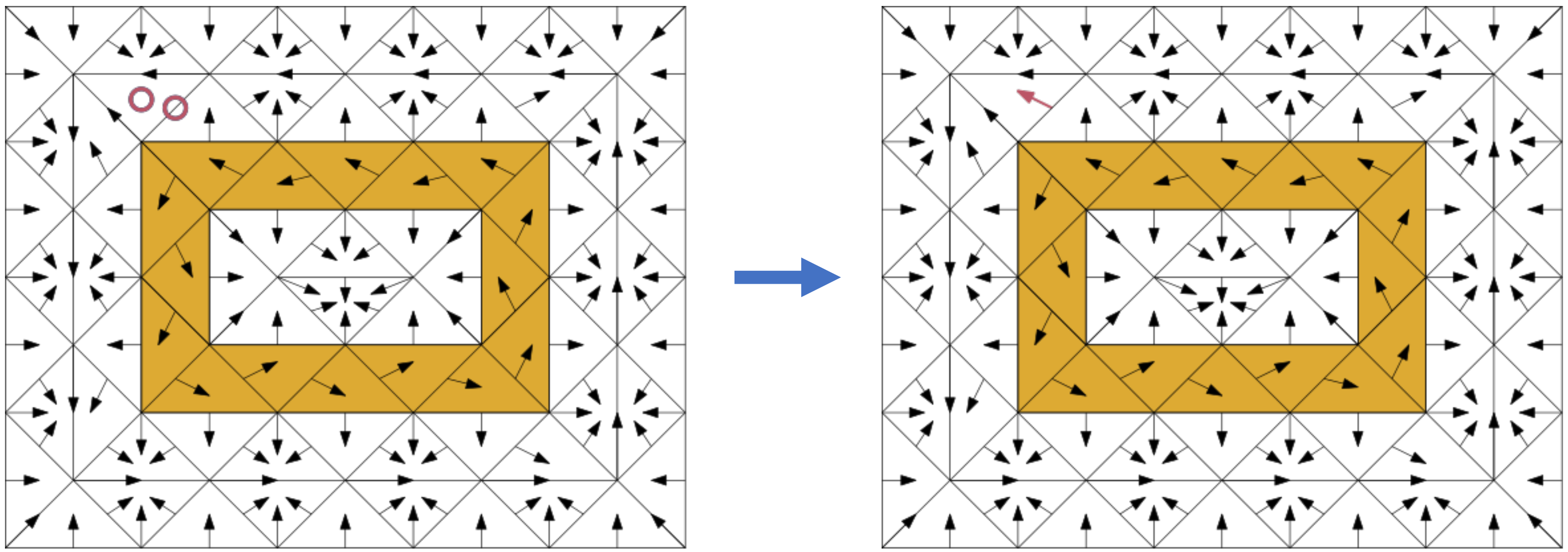
Tracking with Continuation: Case 2

<u>Theorem[DLMS22]:</u> Let S denote an isolated invariant set under $\mathcal{V}$, and let $\mathcal{V}'$ denote an atomic coarsening of $\mathcal{V}$ where the unique bifurcating multivector is contained in S. Then $\mathrm{Inv}_{\mathcal{V}'}(\mathrm{S})$ is an invariant set and $(\mathrm{cl}(\mathrm{S}), \mathrm{mo}(\mathrm{S}))$ is an index pair for both.



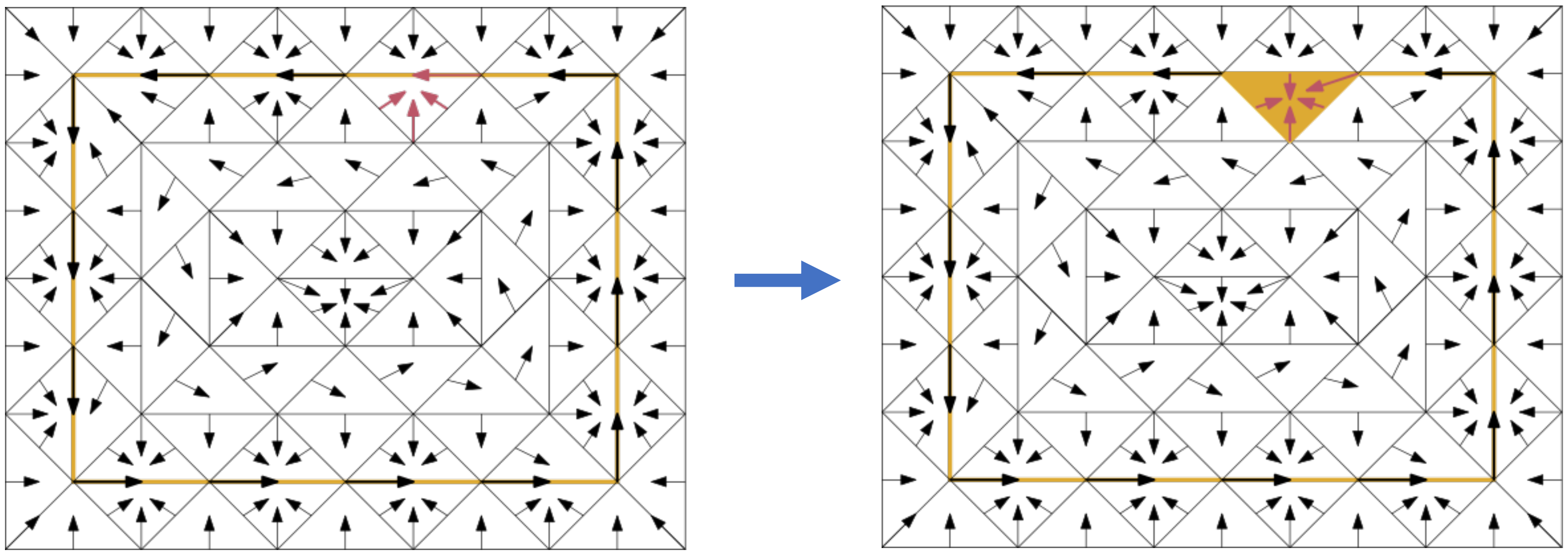
Tracking with Continuation: Case 3

<u>Theorem[DLMS22]:</u> Let S denote an isolated invariant set under $\mathcal{V}$, and let $\mathcal{V}'$ denote an atomic coarsening of $\mathcal{V}$ where the unique bifurcating multivector does not intersect S. Then $\mathrm{Inv}_{\mathcal{V}'}(S)$ is an invariant set and $(\mathrm{cl}(S), \mathrm{mo}(S))$ is an index pair for both.



Tracking with Continuation: Case 4

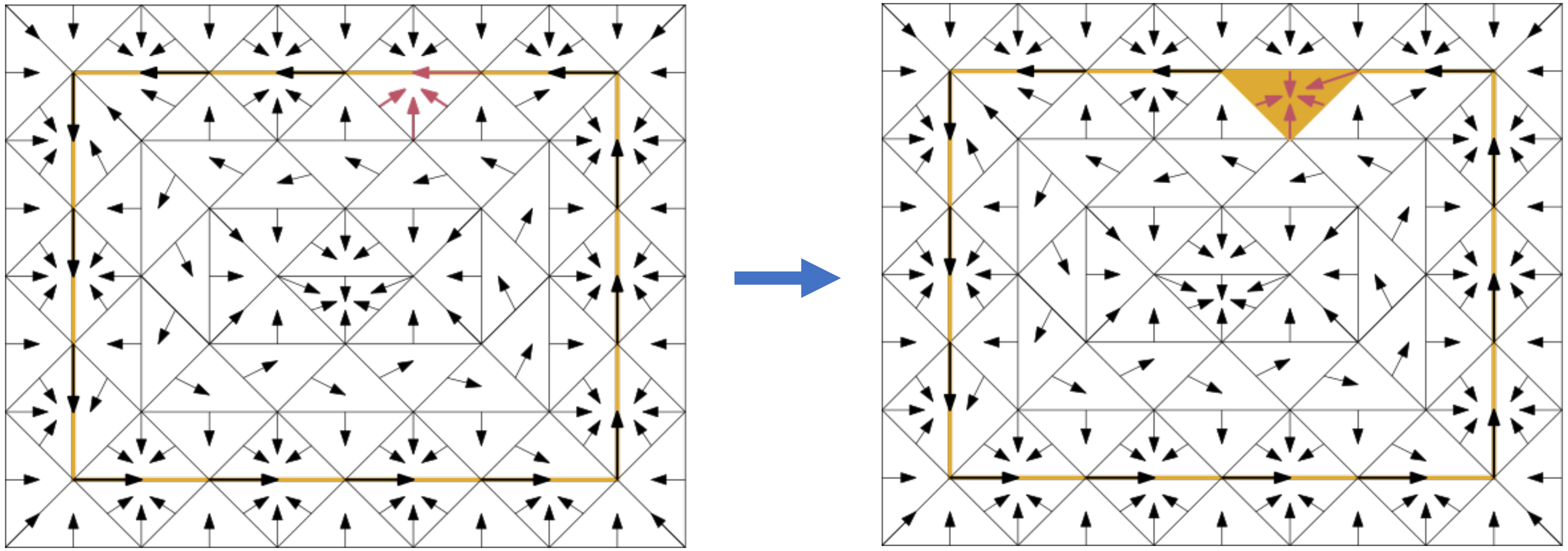
More complicated. Bifurcating vector intersects S but isn't contained in S. Let $A = \langle S \cup V \rangle_{\mathcal{V}}$ denote the minimal $\mathcal{V}$-compatible, convex set containing S.



Tracking with Continuation: Case 4

More complicated. Bifurcating vector intersects S but isn't contained in S. Let $A = \langle S \cup V \rangle_{\mathcal{V}}$ denote the minimal $\mathcal{V}$-compatible, convex set containing S.

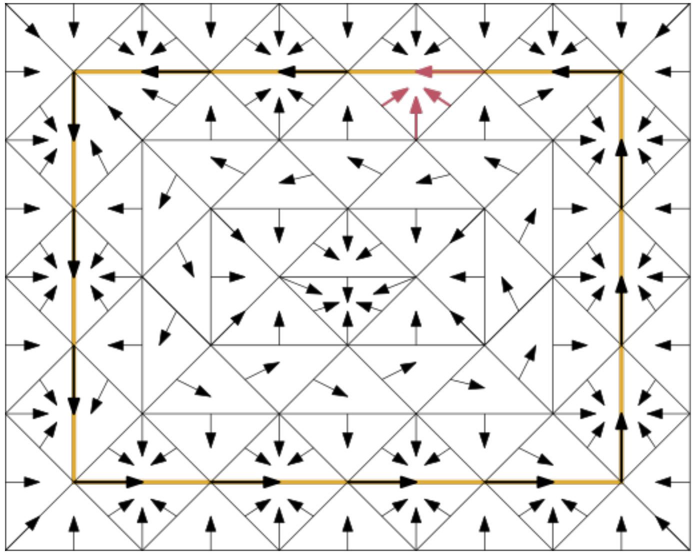
Theorem[DLMS22]: S and $\mathrm{Inv}_{\mathcal{V}'}(\mathrm{A})$ have a common index pair if and only if $(\mathrm{cl}(\mathrm{A}), \mathrm{mo}(\mathrm{A}))$ an index pair for both of them.

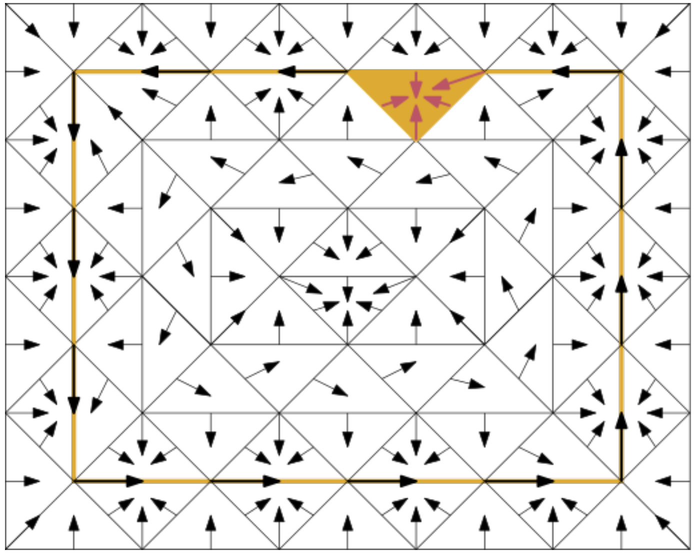


Tracking with Continuation: Case 4

More complicated. Bifurcating vector intersects S but isn't contained in S. Let $A = \langle S \cup V \rangle_{\mathcal{V}}$ denote the minimal $\mathcal{V}$-compatible, convex set containing S.

Theorem[DLMS22]: If S and $\mathrm{Inv}_{\mathcal{V}'}(A)$ do not have a common index pair then S does not have a common index pair with any isolated invariant set under

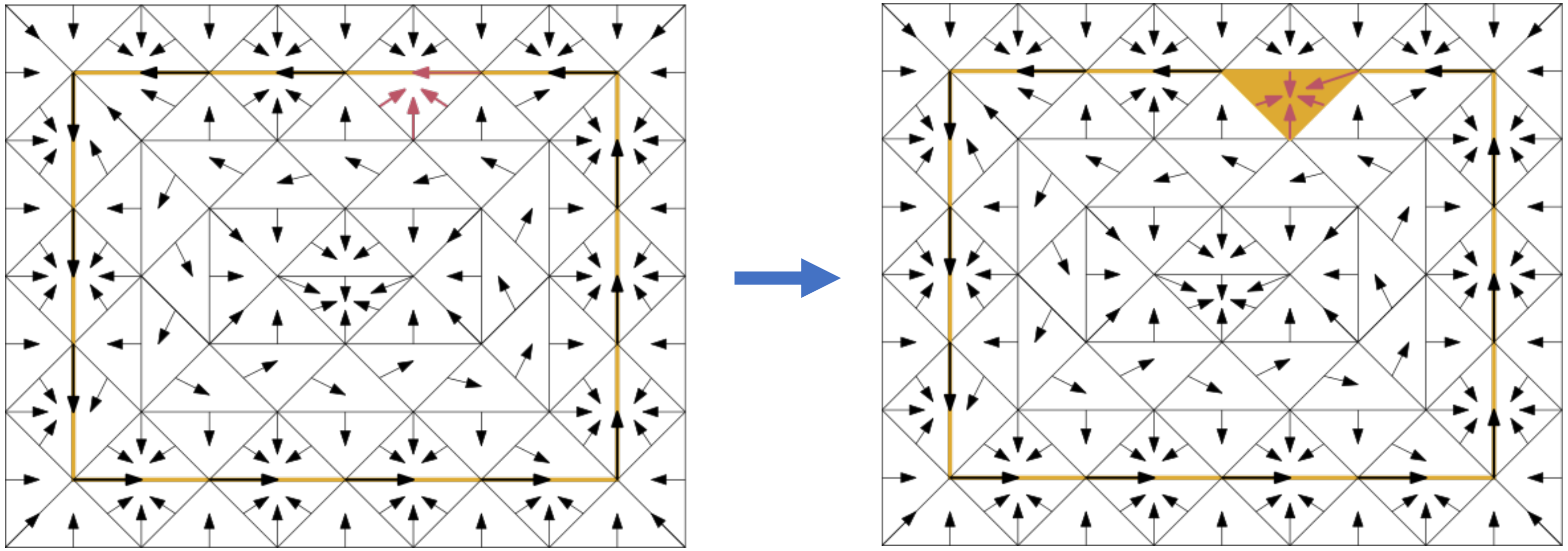




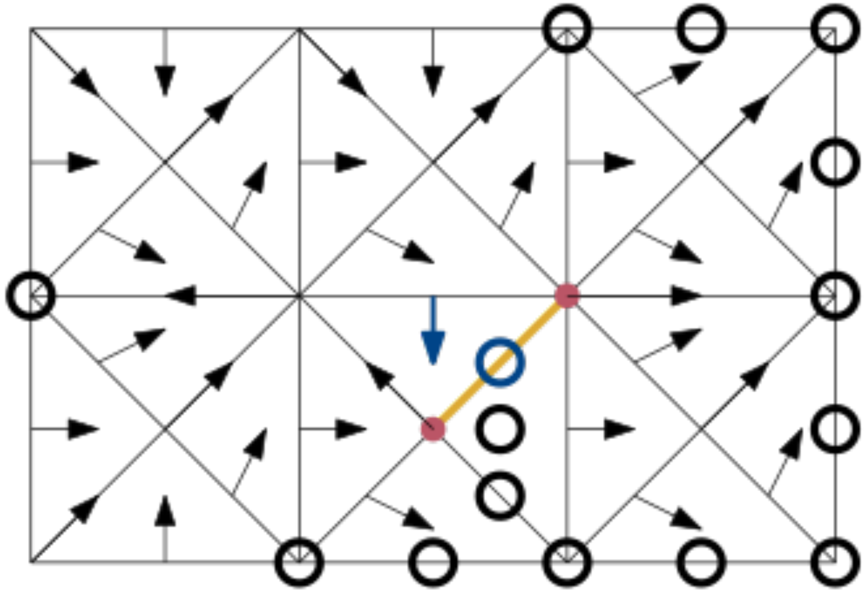
Characterizing Tracked Invariant Sets

Theorem[DLMS22]: If we use the previous rules on a seed S to obtain S’, then S is contained in S’ or S’ is contained in S.

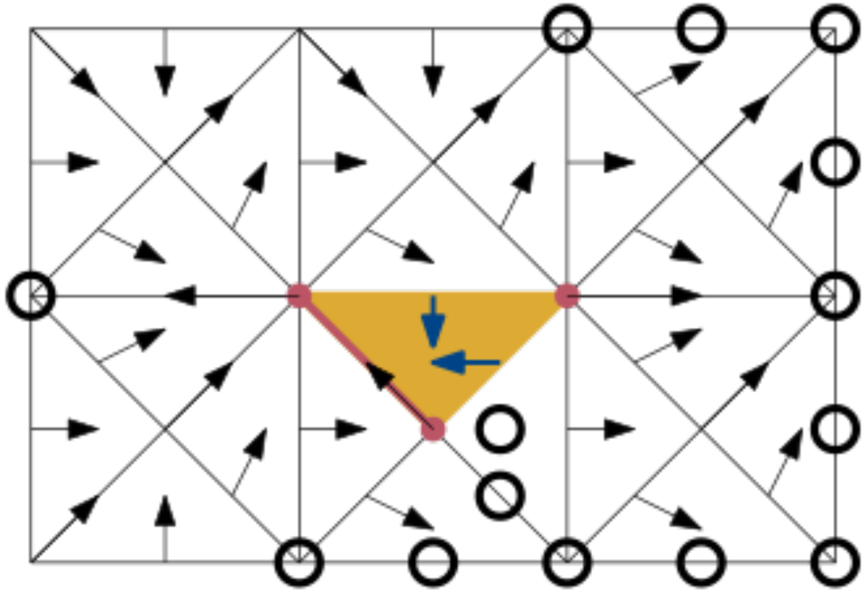
Theorem[DLMS22]: If we use the previous rules on a seed S to obtain S’ where S is contained in S’, then S’ is the smallest isolated invariant set to which S can continue.



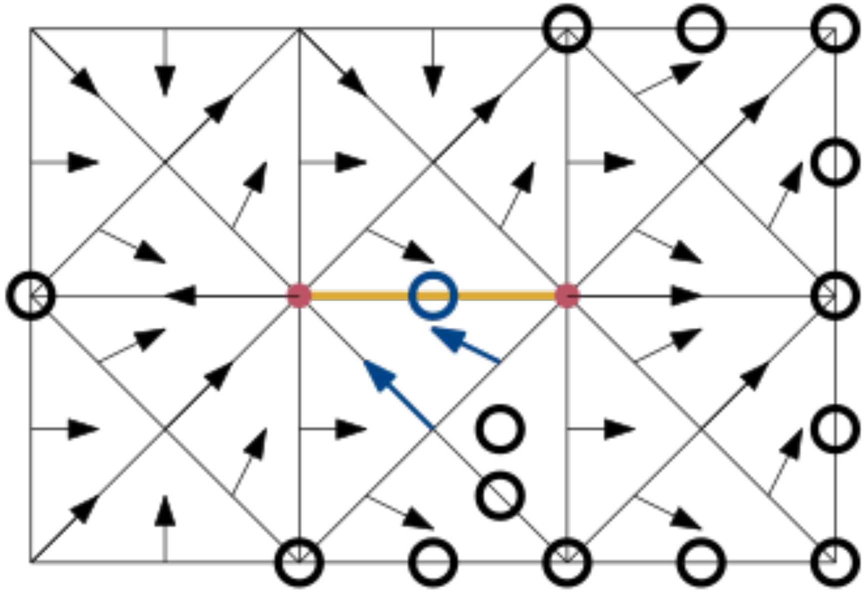
Tracking Example



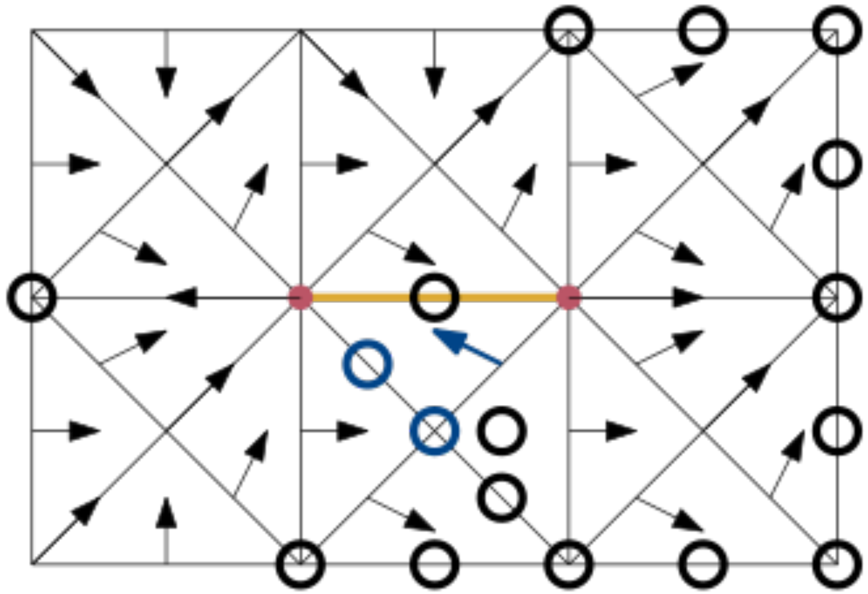
Tracking Example



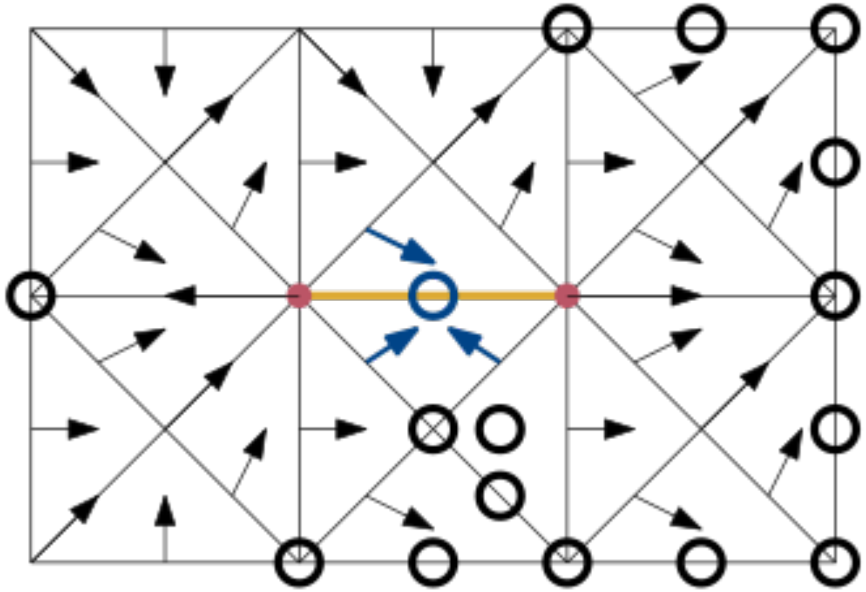
Tracking Example



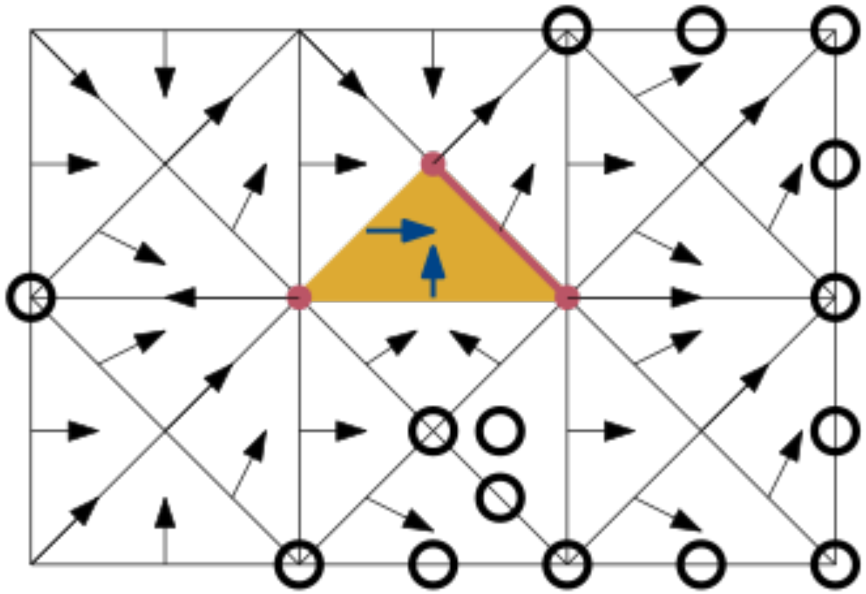
Tracking Example



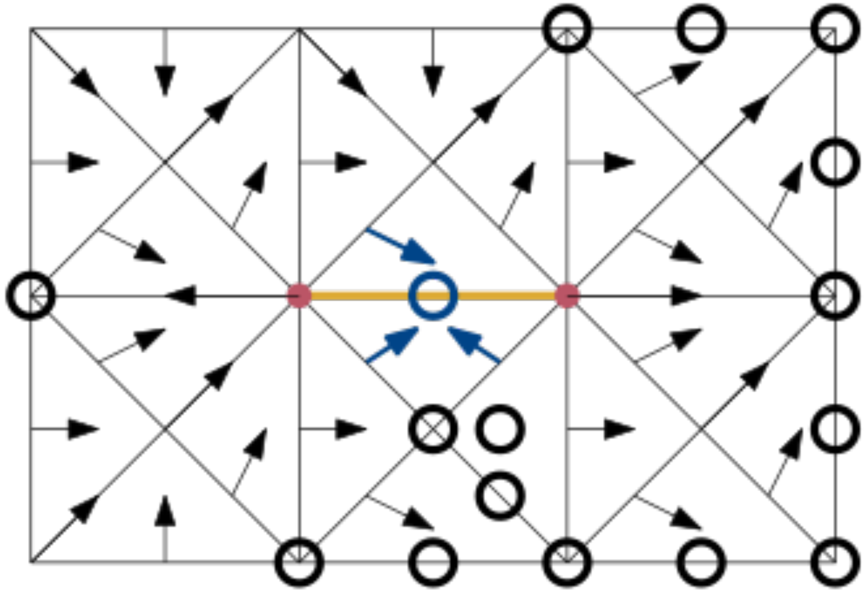
Tracking Example



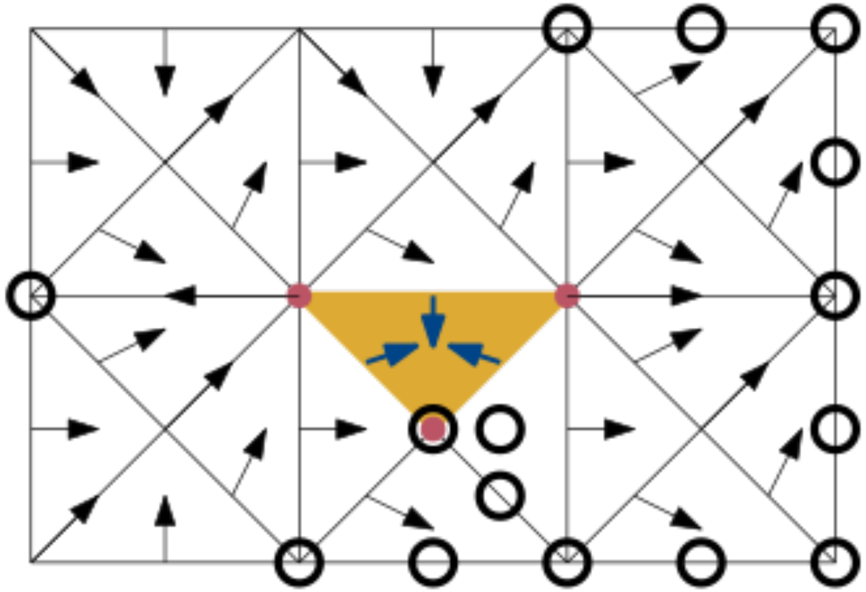
Tracking Example



Tracking Example

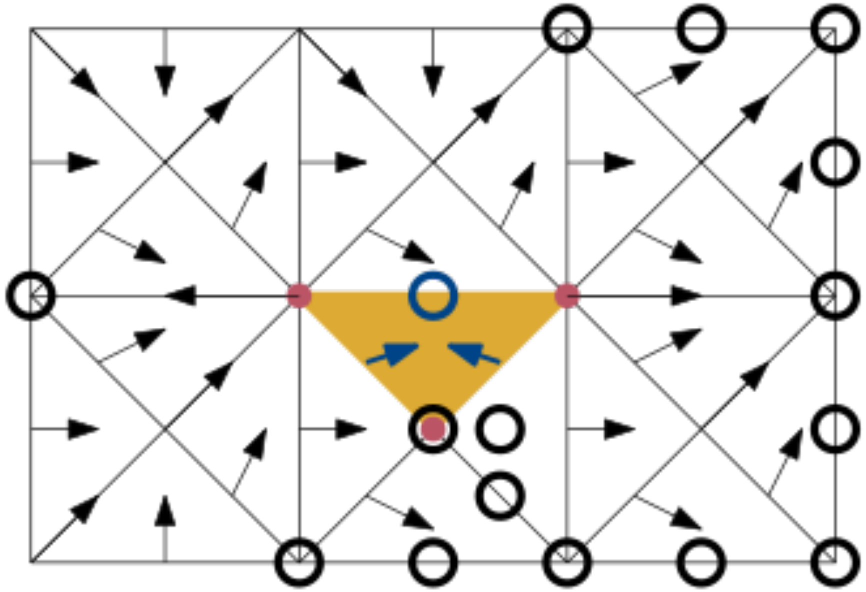


Tracking Example



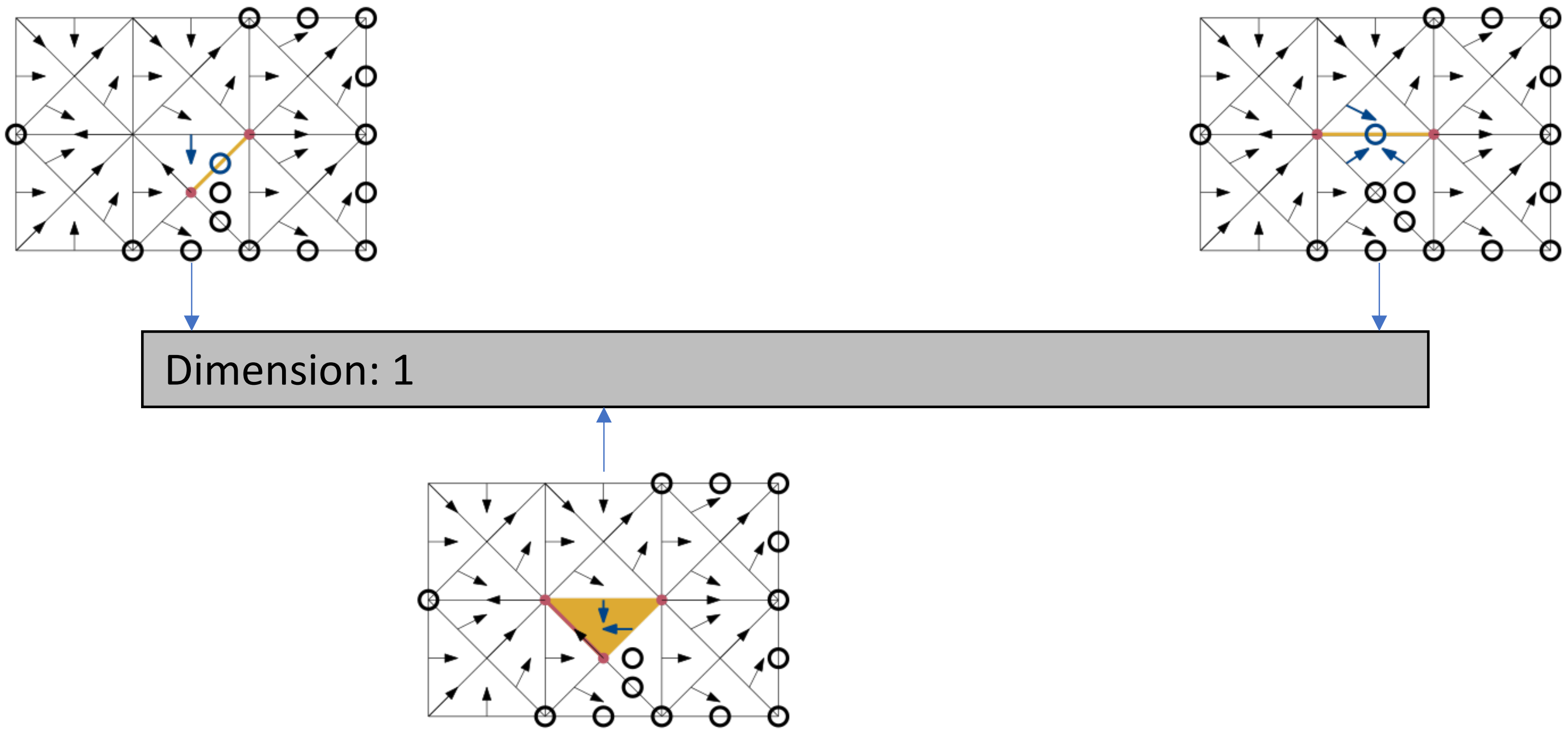
Continuation breaks!!!

Tracking Example



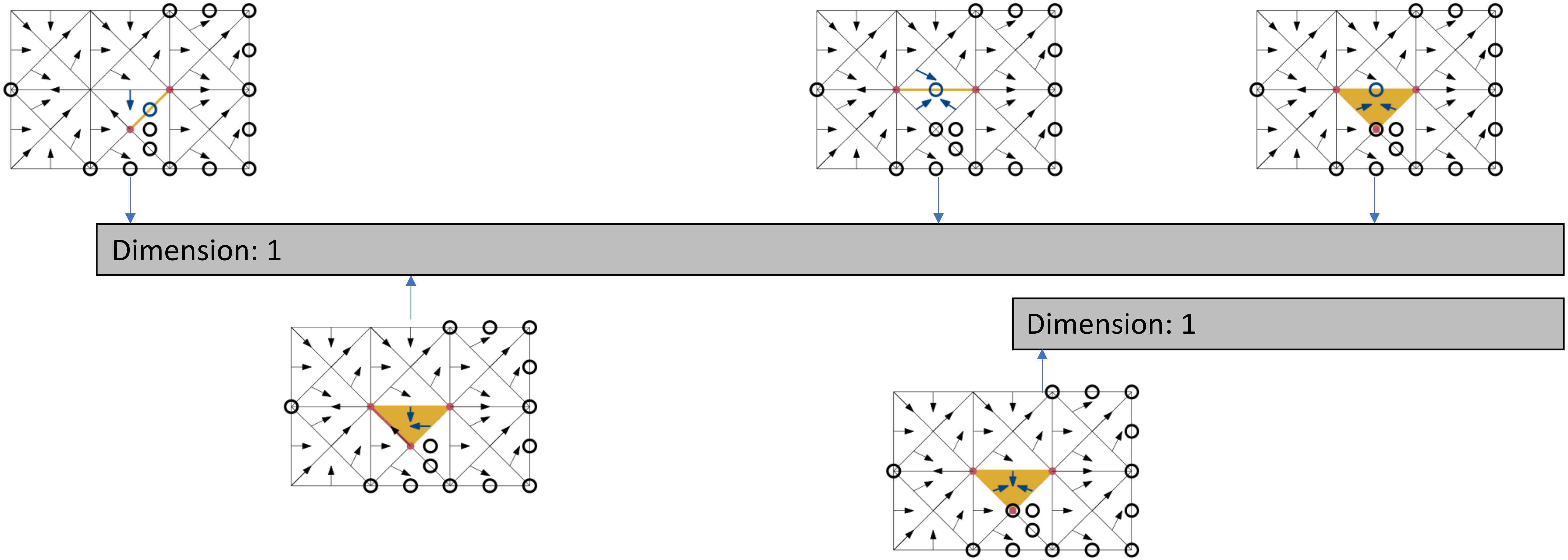
Solution: Persistence

<u>Theorem[DLMS22]:</u> A sequence of index pairs corresponding to a continuation can be converted to a relative zigzag filtration with a “static” barcode.



Solution: Persistence

Theorem[DLMS22]: If S and S' don't share an index pair when using case 4, compute persistence instead!



Conclusion & Future Work

- In this presentation: devised method to capture changes in combinatorial dynamical systems. But…

- Stability?

- Inference?

References

[CDM09] G. Carlsson, V. de Silva, D. Morozov. “Zigzag Persistent Homology and Real Valued Functions.” SoCG ‘09

[DW07] T. Dey, R. Wenger. “Stability of Critical Points with Interval Persistence.” Discret. Comput. Geom. Volume 33, Issue 3.

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[LKMW19] M. Lipinski, J. Kubica, M. Mrozek, T. Wanner. “Conley-Morse-Forman theory for generalized combinatorial multivector fields on finite topological spaces.” Preprint.

[Mr17] M. Mrozek. “Conley-Morse-Forman Theory for Combinatorial Multivector Fields.” FOCM Volume 17, Issue 6.