Controllability Gramian

$$x_{t+1} = Ax_t + Bu_t \qquad x_0$$
$$x_1 = Ax_0 + Bu_0$$
$$x_2 = A(Ax_0 + Bu_0) + Bu_1$$
$$x_3 = A^3 + A^2Bu_0 + ABu_1 + Bu_2$$
$$\vdots$$
$$x_t = A^tx_0 + A^{t-1}Bu_0 + A^{t-2}Bu_1 + \ldots + ABu_{t-2} + Bu_{t-1}$$
$$x_t = A^tx_0 + \begin{bmatrix} A^{t-1}B & \ldots & \ldots & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{t-2} \\ u_{t-1} \end{bmatrix}$$
$$U_{t-1} = (u_0, u_1, \ldots, u_{t-2}, u_{t-1})^T$$
$$\mathcal{W}_c(t) = (A^{t-1}B, A^{t-2}B, \ldots, AB, B)$$
$$x_t = A^tx_0 + \sum_{s=0}^{t-1} A^{t-s-1}Bu_s = A^tx_0 + \mathcal{W}_c(t)U_{t-1}$$

define,

$$\Sigma_t^c = \mathcal{W}_c(t)\mathcal{W}_c^T(t) \qquad n \times n$$

Controllable if Σ^c is non singular. Discrete controllability Gramian:

$$\Sigma_\tau^c = A^{\tau-1}BB^T(A^{\tau-1})^T + \ldots + ABB^TA^T + BB^T$$
$$= \sum_{\tau=0}^{\tau-1} A^iBB^T(A^i)^T \tag{1}$$

upon expansion, satisfies the discrete time Lyapunov equation:

$$\Sigma_{\tau+1}^c = A\Sigma_\tau^cA^T + BB^T \qquad \Sigma_0^c = 0 \tag{2}$$

consider the difference equation:

$$u_{n+1} - a^2u_n - b^2 = 0$$

in continuous time form:

$$\dot{u} - a^2u - b^2 = 0$$

homogeneous sol'n:

$$\dot{u} - a^2u = 0$$
$$\dot{u} = a^2u$$
$$u = Ce^{a^2t}$$

particular sol'n:
let u=d=constant

$$0 - a^2 d - b^2 = 0$$

$$d = -\frac{b^2}{a^2}$$

general sol'n:

$$u = Ce^{a^2 t} - \frac{b^2}{a^2}$$

$$u_0 = c - \frac{b^2}{a^2} = 0 \Rightarrow c = \frac{b^2}{a^2}$$

$$u = \frac{b^2}{a^2} e^{a^2 t} - \frac{b^2}{a^2}$$

$$\dot{u} = b^2 e^{a^2 t} \Rightarrow u = \int b^2 e^{a^2 t} \mathrm{dt}$$

hence, analog to the scalar case (this is admittedly a hand-wavy argument), the continuous controllability Gramian is

$$\Sigma_\tau^c = \int_0^\tau e^{A\tau} B B^T (e^{A\tau})^T \mathrm{d}\tau$$

it turns out that the continuous time version of Lyapunov equation (eq. (2)), is:

$$\dot{\Sigma}_t^c = A\Sigma_t^c + \Sigma_t^c A^T + BB^T$$

Furthermore, suppose we want to drive x_t to a desirable state at time $t = t_1$. Given, $x_t = A^t x_0 + \mathcal{W}_c(t) U_{t-1}$ for $t \geq n$:

$$\text{error :} \quad \tilde{x}_t = x_t - A^t x_0 = (W)_c(t) U_{t-1}$$

define the cost function,

$$J = \frac{1}{2} U_{t-1}^T U_{t-1} \quad \text{s.t.} \quad \mathcal{W}_c(t) U_{t-1} = \tilde{x}_t = x_r - A^t x_0$$

Using the method of Lagrange multipliers,

$$J = \frac{1}{2} U_{t-1}^T U_{t-1} - \lambda^T (\mathcal{W}_c(t) U_{t-1} - (x_r - A^t x_0))$$

set $\frac{\partial J}{\partial U_{t-1}} = 0$ and $\frac{\partial J}{\partial \lambda} = 0$.

$$\frac{\partial J}{\partial U_{t-1}} = U_{t-1} - \mathcal{W}_c(t)^T \lambda = 0 \tag{3}$$

$$\frac{\partial J}{\partial \lambda} = \mathcal{W}_c(t) U_{t-1} - (x_r - A^t x_0) = 0 \tag{4}$$

from (3),

$$U_{t-1} = \mathcal{W}_c(t)^T \lambda \tag{5}$$

substituting (5) to (4),

$$\mathcal{W}_c(t)\mathcal{W}_c(t)^T \lambda - (x_r - A^t x_0) = 0$$

$$\lambda = (\mathcal{W}_c(t)\mathcal{W}_c(t)^T)^{-1}(x_r - A^t x_0)$$

substituting back λ to (5) yields

$$\begin{aligned} U &= \mathcal{W}_c(t)^T(\mathcal{W}_c(t)\mathcal{W}_c(t)^T)^{-1}(x_r - A^t x_0) \\ &= \mathcal{W}_c(t_1)^T(\Sigma^c_{t_1})^{-1}(x_r - A^{t_1} x_0) \end{aligned}$$

note: this is one of the many ways to reach x_1 since the cost function could be formulated differently.

references:

1. https://en.wikipedia.org/wiki/Controllability_Gramian
2. http://www2.imm.dtu.dk/courses/02421/tfoils.pdf
3. solving difference equation, https://www.cl.cam.ac.uk/teaching/2003/Probability/prob07.pdf