Block matrix inversion

The block matrix inversion is often found in texts involving Kalman filter derivation because it eases the computation of covariance matrix inverses. Resulting from the matrix computation analog of the scalar case of *completion of squares* for quadratic polynomials

$$ax^2 + 2bxy + dy^2 = a(x + \frac{b}{a}y)^2 + (d - \frac{b^2}{a})y^2$$

minimizing with respect to x for fixed y. If $A \in R^{n\times n}$ and $D \in R^{m\times m}$ are symmetric matrices and $B \in R^{n\times m}$, then from the quadratic form, we get

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - B^T A^{-1} B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

holds whenever A is invertible. For the general asymmetric case,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1} B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

this form can easily be inverted

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$$

the matrix $S = D - CA^{-1}B$ is called the **Schur complement** of A.

Minimizing with respect to y will give the polynomial,

$$d(y + \frac{b}{d}x)^2 + (a - \frac{b^2}{d})x^2$$

which in turn will result in another form,

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}B^T & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}B^T & I \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -D^{-1}B^T & I \end{bmatrix} \begin{bmatrix} (A - BD^{-1}B^T)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} T^{-1} & -T^{-1}BD^{-1} \\ -D^{-1}CT^{-1} & D^{-1} + D^{-1}CT^{-1}BD^{-1} \end{bmatrix}$$

where $T = A - BD^{-1}C$

Comparing the two different representations, observe

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}$$

rewriting $A = A, B = -B, D = C^{-1}, C = D$, yields the **Woodbury matrix identity**,

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$