proving vector calculus identities using Einstein's index summation

given the rule:

$$\epsilon_{lmi}\epsilon_{jki} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}$$

$$\nabla \cdot (\nabla \times \vec{v}) = div(curl(\vec{v}))$$
$$= \partial_i \epsilon_{ijk} \partial_j v_k = \epsilon_{ijk} \partial_i \partial_j v_k = \epsilon_{ijk} \partial_{ij} v_k = 0$$

Observe that ϵ_{ijk} is antisymmetric but $\partial_i \partial_j$ is symmetric. For example, if $k = 3$,

$$\partial_{ij} v_3 - \partial_{ji} v_3 = 0$$

with the same line of reasoning,

$$\nabla \times (\nabla \phi) = curl(grad(\phi)) = \epsilon_{ijk} \partial_j \partial_k \phi = \epsilon_{ijk} \partial_{jk} \phi = 0$$

$$\nabla \cdot (\phi \vec{v}) = div(\phi \vec{v}) = \partial_i (\phi v_i) = v_i \partial_i (\phi) + \phi \partial_i (v_i)$$
$$= \vec{v} \cdot grad(\phi) + \phi div(\vec{v}) = \vec{v} \cdot \nabla \vec{\phi} + \phi \nabla \cdot \vec{v}$$

$$\nabla \times (\phi \vec{v}) = \epsilon_{ijk} \partial_j (\phi \vec{v})_k$$
$$= \epsilon_{ijk} \partial_k (\phi v_k)$$
$$= \epsilon_{ijk} v_k \partial_j \phi + \epsilon_{ijk} \phi \partial_j v_k$$
$$= \nabla \phi \times \vec{v} + \phi (\nabla \times \vec{v})$$

$$\nabla (\vec{u} \cdot \vec{v}) = grad(u \cdot v) = \partial_i (u_j v_j) = u_j \partial_i v_j + v_j \partial_i u_j$$
$$= u_j \partial_i v_j + v_j \partial_i u_j + u_j \partial_j v_i + v_j \partial_j u_i - u_j \partial_j v_i - v_j \partial_j u$$
$$= u_j (\partial_i v_j - \partial_j v_i) + v_j (\partial_i u_j - \partial_j u_i) + u_j \partial_j v_i + v_j \partial_j v_i$$
$$= u_j \begin{vmatrix} \partial_i & \partial_j \\ v_i & v_j \end{vmatrix}_k + v_j \begin{vmatrix} \partial_i & \partial_j \\ u_i & u_j \end{vmatrix}_k + u_j \partial_j v_i + v_j \partial_j v_i$$
$$= \epsilon_{ijk} u_j \begin{vmatrix} \partial_i & \partial_j \\ v_i & v_j \end{vmatrix}_k + \epsilon_{ijk} v_j \begin{vmatrix} \partial_i & \partial_j \\ u_i & u_j \end{vmatrix}_k + u_j \partial_j v_i + v_j \partial_u v_i$$
$$= \vec{u} \times curl(\vec{v}) + \vec{v} \times curl(\vec{u}) + \vec{u} \cdot grad(\vec{v}) + \vec{v} \cdot grad(\vec{u})$$
$$= \vec{u} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{u}) + \vec{u} \cdot \nabla \vec{v} + \vec{v} \cdot \nabla \vec{u}$$

$$\nabla \cdot (\vec{u} \times \vec{v}) = div(\vec{u} \times \vec{v}) = \partial_i (\epsilon_{ijk} u_j v_k) = v_k \epsilon_{ijk} \partial_i u_j + u_j \epsilon_{ijk} \partial_i v_k = v_k \epsilon_{kij} \partial_i u_j - u_j \epsilon_{jik} \partial_i v_k$$
$$= \vec{v} \cdot curl(\vec{u}) - \vec{u} \cdot curl(\vec{v}) = \vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v})$$

$$\nabla \times (\vec{u} \times \vec{v}) = curl(\vec{u} \times \vec{v}) = \epsilon_{lmi} \partial_m \epsilon_{ijk} u_j v_k = \epsilon_{lmi} \epsilon_{ijk} (v_k \partial_m u_j + u_j \partial_m v_k)$$

$$= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})(v_k\partial_m u_j + u_j\partial_m v_k)$$

$$= \delta_{lj}\delta_{mk}v_k\partial_m u_j + \delta_{lj}\delta_{mk}u_j\partial_m v_k - \delta_{lk}\delta_{mj}v_k\partial_m u_j - \delta_{lk}\delta_{mj}u_j\partial_m v_k$$

$$= v_k\partial_k u_l + u_l\partial_k v_k - v_l\partial_j u_j - u_j\partial_j v_l$$

$$= \vec{v}\cdot grad(\vec{u}) - \vec{u}\cdot grad(\vec{v}) + \vec{u}div(\vec{v}) - \vec{v}div(\vec{u})$$

$$= \vec{u}(\nabla\cdot\vec{v}) - \vec{v}(\nabla\cdot\vec{u}) + \vec{v}\cdot\nabla\vec{u} - \vec{u}\cdot\nabla\vec{v}$$

$$\nabla\times(\nabla\times\vec{v}) = \epsilon_{ijk}\partial_j(\nabla\times v)_k$$

$$= \epsilon_{ijk}\partial_j\epsilon_{kmn}\partial_m v_n = \epsilon_{ijk}\epsilon_{kmn}(\partial_j\partial_m v_m + \partial_m\partial_j v_n)$$

$$= (\delta_{im}\partial_{jn} - \partial_{in}\partial_{jm})(\partial_{jm}v_m + \partial_{mj}v_n)$$

$$= \delta_{im}\delta_{jn}\partial_{jm}v_m + \delta_{im}\delta_{jn}\partial_{mj}v_n - \delta_{in}\delta_{jm}\partial_{jm}v_m - \delta_{in}\delta_{jm}\partial_{mj}v_n$$

The first and third term does not make sense because the index 'm' was repeated 3 times, therefore the only possible value for these terms are zero. Hence,

$$[\nabla\times(\nabla\times\vec{v})]_i = \partial_{in}v_n - \partial_{jj}v_i = \nabla(\nabla\cdot\vec{v}) - \nabla^2\vec{v}$$

$$(\nabla\times\vec{v})\times\vec{v} = \epsilon_{ijk}(\nabla\times v)_j v_k$$

$$= \epsilon_{ijk}jmn\partial_m v_n v_k$$

$$= (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn})\partial_m v_n v_k$$

$$= \partial_k v_i v_k - \partial_i v_k v_k$$

$$= grad(\vec{v})\cdot\vec{v} - grad(\vec{v}\cdot\vec{v}) = (\nabla\vec{v})\cdot\vec{v} - \nabla(\vec{v}\cdot\vec{v})$$

bonus:

$$\vec{u}\times(\nabla\times\vec{u}) = \epsilon_{lmi}u_m\epsilon_{ijk}\partial_j u_k$$

$$= \epsilon_{lmi}\epsilon_{ijk}u_m\partial_j u_k$$

$$= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})u_m\partial_j u_k$$

$$= u_k\partial_l u_k - u_j\partial_j u_l$$

$$= \frac{1}{2}\nabla(u\cdot u) - u\cdot\nabla u$$

given $\nabla(\vec{u}\cdot\vec{u}) = \partial_i(u_j u_j) = 2u_j\partial_i u_j$.

references:

1. William Prager, Introduction to Mechanics of Continua