discrete time algebraic riccati equation

LQR problem
discrete-time system:

$$x_{t+1} = Ax_t + Bu_t, x_0 = x^{init}$$

problem: choose $u_0, u_1, ...$ so that:

- $x_0, x_1, ...$ is 'small', i.e. we get good *regulation* or *control*
- $u_0, u_1, ...$ is 'small', using small*input effort* or *actuator authority*
- there are usually competing objectives, e.g. a large u can drive x to zero fast
- LQR theory addresses this question

we define quadratic cost function:

$$J(U) = \sum_{\tau=0}^{N-1}(x_\tau^T Q x_\tau + u_\tau^T R u_\tau) + x_N^T Q_f x_N$$

where $U = (u_0, ..., u_{N-1})$ and $Q = Q^T \geq 0, Q_f = Q_f^T \geq 0, R = R^T > 0$ are given state cost, final state cost, and input cost matrices. Later, we'll see why we require R¿0.

- N is called time horizon
- first term measures *state deviation*
- second term measures *input size* or *actuator authority*
- last term measures *final state deviation*

Let us consider a simpler scalar example,

$$J = \frac{1}{2}\int_0^\infty (qx^2 + ru^2)\mathrm{dt}$$

- the factor 1/2 is introduced for numerical convenience.
- the weighing factors q and r express the relative importance of keeping x and u near zero.
- if we place more importance on x, then we select q to be large relative to r. In this case, the state x will converge to 0 faster, but the control effort will be bigger, and energy cost higher.
- If we care more about the energy cost rather than the response speed, then we should set higher r.

- Although we are interested in minimizing J, the actual value of J is usually not of interest. This also means that we can set either q or r to unity for convenience because it is their relative weight that is important.

it turns out that feedback-control law that minimizes J is a linear state feedback law,

$$u = -Kx$$

Assume a simple plant, and $r = 1$

$$\dot{x} = x + u$$

$$J = \frac{1}{2}\int_0^\infty (qx^2 + u^2)\mathrm{dt}$$

plug in the control law, we have the closed loop system:

$$\dot{x} = x - Kx = -(K-1)x$$

and for constant K,

$$x = x(0)e^{-(K-1)t}$$

substituting $x(t)$ into J gives,

$$J = \frac{1}{2}(q + K^2)x^2(0)\int_0^\infty e^{-2(K-1)t}\mathrm{dt} = \frac{q + K^2}{4(K-1)}x^2(0)$$

now we can compute $\frac{dJ}{dK} = 0$
we will have $K^2 - 2K - q = 0$. its roots are $K_1 = 1 + \sqrt{1+q}, K_2 = 1 - \sqrt{1+q}$. To ensure the system to be stable we require $K > 1$. K_1 will satisfy this condition.

Lets have more insights by considering other sample problems.

1. Consider a state space plant:

$$\dot{x}_1 = x_1$$

$$\dot{x}_2 = x_2 + u$$

   the performance index:

$$J = \frac{1}{2}\int_0^\infty x_1^2 + u^2\mathrm{dt}$$

   no minimum for J will exist because the state variable x_1 is uncontrollable and of an unstable mode. The response of x_1 will be $x_1(t) = x_1(0)e^t$ regardless of what u and x_2 do, i.e. $x_1(t) \to \infty$ no matter what $u(t)$ does. Hence, we assume that the system, (A, B), is controllable.

2. Consider a state space plant:

$$\dot{x} = x + u$$

$$J = \frac{1}{2}\int_0^\infty (u^2)\mathrm{dt}$$

to minimize J, $u(t) = 0$. but then $x \to \infty$. The system is unstable. We need the performance index to know all the state variables. Therefore, we assume that the system pair (A,H) is completely observabl, where H is any matrix such that $H^T H = Q$.

3. Consider the cost functions:

$$J = \frac{1}{2}\int_0^\infty x^2 - u^2\mathrm{dt}$$

$$J = \frac{1}{2}\int_0^\infty -x^2 + u^2\mathrm{dt}$$

J maybe made as negative as one wishes (when u or $x \to \infty$) therefore we impose the conditions the weighing matrices Q to be semi-positive definite and R to be positive definite respectively.

4. But why do we require $R > 0$
Consider:

$$\dot{x} = -x + u$$

$$J = \frac{1}{2}\int_0^\infty x^2\mathrm{dt}$$

has Q=1 and R=0 (not positive definite). In this case, $K \to \infty$. Which implies the input is infinity. $u(t)$ must be finite due to physical limitations.

this section, considers only **linear** discrete-time time-invariant systems

$$x(k+1) = Ax(k) + Bu(k)$$

and **quadratic** cost functions

$$J(x_0, U) = x_N^T P X_N + \sum_{i=0}^{N-1}(x_i^T Q x_i + u_i^T R U_i)$$

are considered, and we consider only the problem of regulating the state to the origin, **without state or input constraints**.

Recursive approach
using the concept from dynamic programming, the recursive solution relies on Bellman's principle of optimality cf. Bellman's equation in RL,

$$q_*(s,a) = \sum_r \sum_{s'} P(s', r|s,a)[r(s,a,s') + \gamma \max_{a'} q_*(s', a')]$$

. It states, "For any solution for steps j to N with $j \geq 0$, taken from the 0 to N solution, must itself be optimal for the $j - to - N$ problem". Therefore, we have for any $j = 0, ..., N$

$$J_j^*(x_j) = \min_{\mathrm{u_j}} J(x_i, u_i) + J_{j+1}^*(x_{j+1})$$

$$\text{subj. to } x_{j+1} = Ax_j + Bu_j$$

Define the "j-step optimal cost-to-go" as the **optimal** cost attainable for the step j problem:

$$J_j^*(x(j)) = \min_{U_j \to N} x_N^T P x_N + \sum_{i=j}^{N-1} (x_i^T Q x_i + u_i^T R u_i)$$

$$\text{subj. to } x_{i+1} = Ax_i + Bu_i, i = j, ..., N-1$$

$$x_j = x(j)$$

where the final state cost $J(x(j)) = x(j)^T P x(j)$ is an assumed Lyapunov function, $P_{k+1} = P_{k+1}^T \geq 0$. This is the minimum cost attainable for the remainder of the horizon after step j.

- Consider the 1-step problem (solved at time N-1)

$$J_{N-1}^*(x_{N-1}) = \min_{U_{N-1}} x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} + x_N^T P_N x_N \tag{1}$$

$$\text{s.t. } x_N = Ax_{N-1} + Bu_{N-1} \tag{2}$$

$$P_N = P$$

- Substituting (2) into (1)

$$J_{N-1}^*(x_{N-1}) = \min_{u_{N-1}} \{x_{N-1}^T (A^T P_N A + Q) x_{N-1}$$

$$u_{N-1}^T (B^T P_N B + R) u_{N-1}$$

$$+2x_{N-1}^T A^T P_N B u_{N-1}\}$$

- Solving again by setting the gradient to zero leads to the following optimality condition for u_{N-1}

$$2(B^T P_N B + R) u_{N-1} + 2B^T P_N A x_{N-1} = 0$$

Optimal 1-step input:

$$u_{N-1}^* = -(B^T P_N B + R)^{-1} B^T P_N A x_{N-1} = F_{N-1} x_{N-1}$$

1-step cost-to-go:

$$J_{N-1}^*(x_{N-1}) = x_{N-1}^T P_{N-1} x_{N-1} \ ,$$

where

$$P_{N-1} = A^T P_N A + Q - A^T P_N B (B^T P_N B + R)^{-1} B^T P_N A.$$

- We can obtain the solution for any given time step i in the horizon

$$u_t^* = -(B^T P_{t+1} B + R)^{-1} B^T P_{t+1} A x_t$$

$$K_t = -(B^T P_{t+1} B + R)^{-1} B^T P_{t+1} A$$

  where we can find any P_t by recursive evaulation from $P_N = P$, using

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (B^T P_{t+1} B + R)^{-1} B^T P_{t+1} A$$

  which can be initialized with $P_N = P$, the given terminal weight. This is called the **Discrete Time Riccati equation** or **Riccati Difference equation (RDE)**

- The optimal cost-to-go is

$$J_i^*(x_i) = x_I^T P_i x_i$$

  , evaluating down to P_0, we obtain

$$x(0)^T P_0 x(0)$$

references:

1. ee363
2. Model Predictive Control, Melanie Zeilinger