Rigid Body Dynamics

angular velocity
change of basis from basis $\mathcal{I}$ to some basis $\mathcal{B}^*$

$$x = Rx^*$$

$$u = Tv$$
$$Ru^* = TRv^*$$
$$u^* = R^{-1}TRv^*$$
$$T^* = R^{-1}TR$$

let X be a skew-symmetric matrix, then

$$X^* = R^{-1}XR \quad R^{-1} = R^T$$

$$(X^*)^V = (R^T XR)^V \quad (X^*)^V = x^* \quad x = Rx^*$$

where ^{V} denotes the *vect* operation to retrieve the axial vector form the dual matrix.

$$R^T x = (R^T XR)^V \tag{1}$$

let $R \coloneqq R^T$, then we get

$$Rx = (RXR^T)^V \tag{2}$$

. Unless otherwise stated, $({}^A\omega_B)$ means the vector ω rotates with the frame B, and both observed w.r.t *and* resolved in frame A. ${}^C({}^A\omega_B)$, means the vector ω is a property of frame B observed w.r.t frame A resolved in frame C

$${}^A x_B = {}^A_B R\, {}^B x_B$$

$${}^A \dot{x}_B = {}^A_B \dot{R}\, {}^B x_B + {}^A_B R\, {}^B \dot{x}_B$$

we assume that the object does not deform (rigid), in this case ${}^B\dot{x}_B = 0$.

$${}^A x_B = {}^A_B \dot{R} ({}^A_B R^T\, {}^A x_B)$$

Let ${}^A_B R = R$, ${}^A\omega_B = (\dot{R}R^T)^V$ with ${}^A\Omega_B = \dot{R}R^T$ a skew-symmetric tensor. To get the so called body-fixed angular velocity,

$${}^B({}^A\omega_B) = {}^B_A R\, {}^A\omega_B \tag{3}$$

making use of eq. (1)

$${}^B({}^A\omega_B) = {}^A_B R^T\, {}^A\omega_B = (R^T\, {}^A\Omega_B R)^V = (R^T \dot{R} R^T R)^V = (R^T \dot{R})^V$$

In summary, spaced fixed:

$${}^A\omega_B = (\dot{R}R^T)^V$$

body fixed:

$${}^B({}^A\omega_B) = (R^T \dot{R})^V$$

proof that angular velocity are skew-symmetric, i.e. $A = -A^T$:

$$R^T R = I$$

$$\dot{R}^T R + R^T \dot{R} = 0$$
$$R^T \dot{R} = -\dot{R}^T R = -(R^T \dot{R})^T$$
$$R R^T = I$$
$$\dot{R} R^T + R \dot{R}^T = 0$$
$$\dot{R} R^T = -R \dot{R}^T = -(\dot{R} R)^T$$

Point P is in motion w.r.t. the rigid body
Let $\mathcal{A}$ be an inertial frame, B the origin of an intermediate frame $\mathcal{B}^*$. Let p be a point defined in frame $\mathcal{B}^*$. Then,

$$^A r_p = {}^A r_B + {}^A u$$

where u is the position vector of p measure from B. $^B u$ denotes measurement w.r.t and resolved in frame B^*.

$$^A r_p = {}^A r_B + {}^A_B R \, {}^B u$$

velocity

$$\begin{aligned} {}^A v_p &= {}^A v_B + {}^A_B \dot{R} \, {}^B u + {}^A_B R \, {}^B \dot{u} \\ &= {}^A v_B + \dot{R} R^T ({}^A r_p - {}^A r_B) + R \, {}^B \dot{u} \\ &= {}^A v_B + {}^A \widetilde{\omega}_B ({}^A r_p - {}^A r_B) + R \, {}^B \dot{u} \\ &= v_B + {}^A \widetilde{\omega}_B (r_p - r_B) + R \dot{u}^* \end{aligned}$$

$$\begin{aligned} {}^B_A R \, {}^A v_p = R^T \, {}^A v_p &= R^T \, {}^A v_B + R^T \, {}^A \widetilde{\omega}_B (R \, {}^B u) + R^T R \, {}^B \dot{u} \\ &= R^T \, {}^A v_B + {}^B ({}^A \widetilde{\omega}_B) \, {}^B u + {}^B \dot{u} \\ &= R^T v_B + \widetilde{\omega}^* u^* + \dot{u}^* \end{aligned}$$

acceleration

$$^A a_p = {}^A a_B + {}^A \dot{\widetilde{\omega}} ({}^A r_p - {}^A r_B) + {}^A \widetilde{\omega}_B ({}^A v_p - {}^A v_B) + \dot{R} ({}^B \dot{u}) + R \, {}^B \ddot{u}$$

third term:

$$^A \widetilde{\omega}_B ({}^A \widetilde{\omega}_B ({}^A r_p - {}^A r_B) + R \, {}^B \dot{u}) = {}^A \widetilde{\omega}_B \, {}^A \widetilde{\omega}_B ({}^A r_p - {}^A r_B) + {}^A \widetilde{\omega}_B R \, {}^B \dot{u}$$

fourth term:

$$\dot{R} \, {}^B \dot{u} = \dot{R} R^T R \, {}^B \dot{u} = {}^A \widetilde{\omega}_B R \, {}^B \dot{u}$$

summing up:

$$ {}^A a_p = {}^A a_B + {}^A\dot{\widetilde{\omega}}({}^A r_p - {}^A r_B) + {}^A\widetilde{\omega}_B {}^A\widetilde{\omega}_B({}^A r_p - {}^A r_B) + 2{}^A\widetilde{\omega}_B R^B\dot{u} + R^B\ddot{u} $$

the first term is the inertial acceleration of the origin of the body attached frame, the second and third term arises from the effect of angular acceleration and velocity, the fourth term is the coriolis acceleration and the last term is the relative acceleration of point p w.r.t point B.

Here again, the inertial acceleration vector of the point p could be resolved in the body attached basis $\mathcal{B}^*$

$$ {}^B_A R^A a_p = R^T({}^A a_B) + R^T\dot{\widetilde{\omega}}({}^A r_p - {}^A r_B) + R^T\widetilde{\omega}\widetilde{\omega}({}^A r_p - {}^A r_B) + 2R^T\widetilde{\omega}R^B\dot{u} + R^T R({}^B\ddot{u}) $$

second term:

$$ R^T\dot{\widetilde{\omega}}R^B u = (\widetilde{R^T\dot{\omega}})^B u $$

third term:

$$ R^T\widetilde{\omega}\widetilde{\omega}({}^A r_p - {}^A r_B) = (R^T\widetilde{\omega}RR^T\widetilde{\omega}R)^B u = (\widetilde{{}^A_B R^{T A}\omega_B})(\widetilde{{}^A_B R^{T A}\omega_B})^B u = {}^B({}^A\widetilde{\omega}_B)^B({}^A\widetilde{\omega}_B)^B u $$

making use of eqs. (1) and (3). fourth term:

$$ 2R^T\widetilde{\omega}R^B\dot{u} = 2^B({}^A\widetilde{\omega}_B)^B\dot{u} $$

summing up,

$$ \begin{aligned} {}^B_A R^A a_p &= R^T({}^A a_B) + (\widetilde{R^T\dot{\omega}})^B u + {}^B({}^A\widetilde{\omega}_B)^B({}^A\widetilde{\omega}_B)^B u + 2^B({}^A\widetilde{\omega}_B)^B\dot{u} + {}^B\ddot{u} \\ &= R^T a_B + (\widetilde{R^T\dot{\omega}})^B u + \widetilde{\omega}^*\widetilde{\omega}^* u^* + 2\widetilde{\omega}^*\dot{u}^* + \ddot{u}^* \end{aligned} $$

$$ \begin{aligned} R^T\dot{\omega} &= R^T\frac{d}{dt}({}^A_B R^B\omega) \\ &= (R^T\dot{R})^B\omega + R^T R^B\dot{\omega} \\ &= \dot{\widetilde{\omega}}^*\omega^* + \dot{\omega}^* = \dot{\omega}^* \quad \text{since cross product of a vector with itself is zero} \end{aligned} $$

hence,

$$ {}^B_A R^A a_p = R^T a_B + \dot{\widetilde{\omega}}^* u^* + \widetilde{\omega}^*\widetilde{\omega}^* u^* + 2\widetilde{\omega}^*\dot{u}^* + \ddot{u}^* $$

references:

1. O.A. Bauchau, Flexible Multibody Dynamics.