Frenet-Serret and Space Curves

insights from kinematics

Imagine the simple case of a particle tied to the end of a string rotating in a circle. There is a way of computing the radius of curvature ρ that will be independent of the coordinate system used. We make use of the well-known relationship $\rho = v^2/a_n$. We must fine expressions for v^2 and a_n in a general way. For v^2, this can be realized by: $v^2 = \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$. For a_n,

$$a_n \mathbf{e}_n = \mathbf{a} - a_t \mathbf{e}_t$$

$$\mathbf{a} = a_n \mathbf{e}_n + a_t \mathbf{e}_t$$

$$\mathbf{v} \times \mathbf{a} = \mathbf{v} \times (\mathbf{a}_t + \mathbf{a}_n) = \mathbf{v} \times \mathbf{a}_n$$

since $\mathbf{v}$ and $\mathbf{a}_t$ are in the tangential direction. Thus,

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v} \times \mathbf{a}_n| = |\mathbf{v}||\mathbf{a}_n| \sin 90^o = |\mathbf{v}| a_n$$

and,

$$a_n = \frac{|\mathbf{v} \times \mathbf{a}_n|}{|\mathbf{v}|}$$

finally,

$$\rho = \frac{|\mathbf{v}|^3}{|\mathbf{v} \times \mathbf{a}|}$$

tangent vector

In 2-dimensional Euclidean space, an $\mathbb{E}^2$ plane, we parametrize a curve $p(t) = (x(t), y(t))$ by one parameter t with respect to a reference point O with a *fixed* Cartesian coordinate frame. The velocity vector at point $\mathbf{p}$ is given by $\dot{\mathbf{p}}(t) = (\dot{x}(t), \dot{y}(t))$ with the norm

$$|\dot{\mathbf{p}}(t)| = \sqrt{\dot{\mathbf{p}} \cdot \dot{\mathbf{p}}} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

where $\dot{x} = dx/dt$. The arc lenght s in the interval $[a, b]$ can be calculated by

$$s = \int \mathrm{ds} = \int \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\mathrm{dt} = \int_a^b |\dot{\mathbf{p}}(t)|\mathrm{dt}$$

The arc length can be a function of parameter t given by

$$s(t) = \int_a^t |\dot{\mathbf{p}}(t')|\mathrm{dt}'$$

From the fundamental theorem of calculus

$$|\frac{ds}{dt}| \neq 0 \quad \Rightarrow \quad \dot{s}(t) = |\dot{\mathbf{p}}(t)| > 0$$

One can parametrize the curve by arc length s as $\mathbf{p}(s) = (x(s), y(s))$. The corresponding velocity vector should be $\mathbf{p}'(s) = (x'(s), y'(s))$, where we gave $x' = dx/ds$. We can rewrite the derivatives of x and y with respect to s as

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \dot{x}\frac{dt}{ds}$$

$$y' = \dot{y}\frac{dt}{ds}$$

Thus the norm of the velocity vector parametrized by s can be calculated as

$$|\mathbf{p}'(s)| = \sqrt{x'^2 + y'^2} = \sqrt{\dot{x}^2 + \dot{y}^2}\frac{dt}{ds} = \frac{ds}{dt}\frac{dt}{ds} = 1,$$

which implies that the velocity vector $\mathbf{p}'(s)$ is a unit vector, i.e. in the special case when $t = s$ the velocity vector becomes a unit tangent vector. We can define a unit tangent vector as a velocity vector parametrized by s

$$\widehat{\mathbf{T}} = \mathbf{e}_1 = \mathbf{p}'(s)$$

normal vector

Due to $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{p}' \cdot \mathbf{p}'$, we have

$$\mathbf{e}_1' \cdot \mathbf{e}_1 + \mathbf{e}_1 \cdot \mathbf{e}_1' = 0 \quad \Rightarrow \quad \mathbf{e}_1' \cdot \mathbf{e}_1 = 0 \quad \Rightarrow \quad \mathbf{e}_1' \perp \mathbf{e}_1,$$

it indicates that $\mathbf{e}_1'$ is a normal vector. The principal normal vector is defined by

$$\widehat{\mathbf{N}} = \mathbf{e}_2 = \frac{\mathbf{e}_1'}{|\mathbf{e}_1'|}$$

as a unit normal vector at $\mathbf{p}(s)$. The curvature of $\mathbf{p}(s)$ is given by $1/\rho = \kappa(s) = |\mathbf{e}_1'(s)| > 0$, which can be realized as a norm of the acceleration vector $\mathbf{a} = \mathbf{e}_1' = \mathbf{p}''$. Therefore, we have a relation

$$\mathbf{e}_1' = \kappa(s)\mathbf{e}_2$$

The plane spanned by vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ is called osculating plane.

Frenet-Serret formula in 2D

From the orthogonality condition $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (i, j = 1, 2)$ (2 axis), we have

$$\mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' = 0$$

$$\mathbf{e}_1' \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}' = \kappa\mathbf{e}_2 \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}_2' = \kappa + \mathbf{e}_1 \cdot \mathbf{e}_2' = 0$$

$$\mathbf{e}_1 \cdot \mathbf{e}_2' = -\kappa$$

as a result, we have the following relations

$$\begin{cases} \mathbf{p}' = +\mathbf{e}_1 \\ \mathbf{e}_1' = \qquad +\kappa\mathbf{e}_2 \\ \mathbf{e}_2' = -\kappa\mathbf{e}_1 \end{cases}$$

$$\begin{pmatrix} \mathbf{p}' \\ \mathbf{e}_1' \\ \mathbf{e}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}$$

Frenet's frames and equation in 2d,

$$\begin{pmatrix} \frac{d\widehat{\mathbf{T}}(s)}{ds} \\ \frac{d\widehat{\mathbf{N}}(s)}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{T}}(s) \\ \widehat{\mathbf{N}}(s) \end{pmatrix}$$

curve in 3d space

In $\mathbb{E}^3$, a curve is parametrized as $\mathbf{p}(t) = (x(t), y(t), z(t))$ and we have to look for an orthonormal frame at p denoted by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The vector $\mathbf{e}_1 = \mathbf{p}'$ is uniquely defined the same way. Due to $\mathbf{e}_1' \perp \mathbf{e}_1$,vector $\mathbf{e}_1'$ should be proportional to $\mathbf{e}_2$ or $\mathbf{e}_3$. Now we can fix $\mathbf{e}_1' = \kappa\mathbf{e}_2$ as in the 2d case.

binormal vector
Now define a unit vector orthogonal to $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{N}}$ called binormal vector.

$$\widehat{\mathbf{B}} = \widehat{\mathbf{T}} \wedge \widehat{\mathbf{N}}$$

$$= \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$$

where $\wedge$ is the exterior product or wedge product (in 3d space, the exterior product is the same to the usual cross product $\times$ of two vectors.

By orthonormality condition $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (i, j = 1, 2, 3)$ (3 axis), we have

$$\mathbf{e}_1' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' = 0$$

, which implies:

1. If $i = j$, we have $\mathbf{e}_i' \perp \mathbf{e}_i$, for example $\mathbf{e}_2'$ should be the combination of $\mathbf{e}_1$ and $\mathbf{e}_3$.

2. If $i \neq j$, we have

$$\begin{cases} 0 = \mathbf{e}_1' \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}_2' = (\kappa\mathbf{e}_2) \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}_2' = \kappa + \mathbf{e}_1 \cdot \mathbf{e}_2' \quad (i = 1, j = 2) \quad (1a) \\ 0 = \mathbf{e}_1' \cdot \mathbf{e}_3 + \mathbf{e}_1 \cdot \mathbf{e}_3' = (\kappa\mathbf{e}_2) \cdot \mathbf{e}_3 + \mathbf{e}_1 \cdot \mathbf{e}_3' = 0 + \mathbf{e}_1 \cdot \mathbf{e}_3' \quad (i = 1, j = 3) \quad (1b) \end{cases}$$

From 1a, we have to assume that

$$\mathbf{e}_2' = -\kappa(s)\mathbf{e}_1 + \tau(s)\mathbf{e}_3$$

where it now contains an unknown additional term related to $\mathbf{e}_3$. For $(i = 2, j = 3)$, we obtain

$$0 = \mathbf{e}_2' \cdot \mathbf{e}_3 + \mathbf{e}_2 \cdot \mathbf{e}_3' = (-\kappa\mathbf{e}_1 + \tau\mathbf{e}_3) \cdot \mathbf{e}_3 + \mathbf{e}_2 \cdot \mathbf{e}_3' = \tau + \mathbf{e}_2 \cdot \mathbf{e}_3'$$

(1.) tells us that $\mathbf{e}_3'$ is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. (1b) tells us that $\mathbf{e}_3'$ is perpendicular to $\mathbf{e}_1$. Therefore, we have the unique solution that

$$\mathbf{e}_3' = -\tau \mathbf{e}_2$$

, where $\tau(s)$ is called *torsion* of a curve $\mathbf{p}(s)$. The geometric meaning of torsion is that it make the point of the curve leave for the osculating plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ (the torsion of the curve is always related to the binormal vector $\widehat{\mathbf{B}} = \mathbf{e}_3$.

Frenet-Serret formula in 3D
As a result, we have Frenet-Serret formula in 3d

$$\begin{cases} \mathbf{e}_1' = \quad\quad +\kappa \mathbf{e}_2 \\ \mathbf{e}_2' = -\kappa \mathbf{e}_1 \quad\quad + \tau \mathbf{e}_3 \\ \mathbf{e}_3' = \quad\quad -\tau \mathbf{e}_2 \end{cases} \quad \Rightarrow \quad \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{d\widehat{\mathbf{T}}(s)}{ds} \\ \frac{d\widehat{\mathbf{N}}(s)}{ds} \\ \frac{d\widehat{\mathbf{B}}(s)}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{T}}(s) \\ \widehat{\mathbf{N}}(s) \\ \widehat{\mathbf{B}}(s) \end{pmatrix}$$

Note: If one defines $\widehat{\mathbf{B}} = \widehat{\mathbf{N}} \wedge \widehat{\mathbf{T}}$, then one should assume $\mathbf{e}_2' = -\kappa(s)\mathbf{e}_1 - \tau(s)\mathbf{e}_3$ and obtain $\mathbf{e}_3' = +\tau \mathbf{e}_2$.

Space Curves
Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a regular parametrization of $\mathbb{E}^3$. Let s denotes the arc length of the curve. We then have

$$\frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\frac{d\mathbf{r}}{ds} = \frac{ds}{dt}\widehat{\mathbf{T}}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2 s}{dt^2}\widehat{\mathbf{T}} + \left(\frac{ds}{dt}\right)^2 \frac{d\widehat{\mathbf{T}}}{ds} = \frac{d^2 s}{dt^2}\widehat{\mathbf{T}} + \left(\frac{ds}{dt}\right)^2 \kappa \widehat{\mathbf{N}}$$

Hence,

$$\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \left(\frac{ds}{dt}\right)^3 \kappa \widehat{\mathbf{T}} \times \widehat{\mathbf{N}} = \left|\frac{d\mathbf{r}}{dt}\right|^3 \kappa \mathbf{B}$$

the curvature κ is then given by,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

The equation also implies,

$$\widehat{\mathbf{B}} = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}$$

$$\frac{d^3\mathbf{r}}{dt^3} = \frac{d^3 s}{dt^3}\widehat{\mathbf{T}} + \frac{d^2 s}{dt^2}\frac{dT}{dt} + 2\left(\frac{ds}{dt}\right)\left(\frac{d^2 s}{dt^2}\right)\kappa(s) N + \left(\frac{ds}{dt}\right)^2 (\kappa(s)\frac{dN}{dt} + \frac{d\kappa(s)}{dt}N)$$

$$= \frac{d^3s}{dt^3}T + \frac{d^2}{dt^2}\frac{dT}{ds}\frac{ds}{dt} + 2\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\kappa(s)N + \left(\frac{ds}{dt}\right)^3\left(\kappa(s)\frac{dN}{ds} + \frac{d\kappa}{ds}N\right)$$

$$= \frac{d^3s}{dt^3}T + \frac{d^2s}{dt^2}\kappa(s)N\frac{ds}{dt} + 2\frac{ds}{dt}\frac{d^2s}{dt^2}\kappa(s)N + \left(\frac{ds}{dt}\right)^3\kappa(-\kappa T + \tau B) + \left(\frac{ds}{dt}\right)^3\frac{d\kappa}{ds}N$$

$$= \left(\frac{d^3s}{dt^3} - \left(\frac{ds}{dt}\right)^3\kappa^2\right)T + \left(3\frac{d^2s}{dt^2}\frac{ds}{dt}\kappa + \left(\frac{ds}{dt}\right)^3\frac{d\kappa}{ds}\right)N + \left(\frac{ds}{dt}\right)^3\kappa\tau B$$

we also have the scalar triple product,

$$\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right] = \left[\frac{ds}{dt}\widehat{\mathbf{T}}, \frac{d^2s}{dt^2}\widehat{\mathbf{T}} + \left(\frac{ds}{dt}\right)^2\kappa\widehat{\mathbf{N}}, C\widehat{\mathbf{T}} + D\widehat{\mathbf{N}} + \left(\frac{ds}{dt}\right)^3\kappa\tau\widehat{\mathbf{B}}\right]$$

where C and D are *don't care* because they will eventually vanish.

$$\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right] = \left(\frac{ds}{dt}\right)^6\kappa^2\tau[\widehat{\mathbf{T}}, \widehat{\mathbf{N}}, \widehat{\mathbf{B}}] = \left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right|^2\tau$$

the torsion is then given by,

$$\tau(t) = \frac{[\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)]}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

lastly, because $\widehat{\mathbf{T}}, \widehat{\mathbf{N}}, \widehat{\mathbf{B}}$ forms a positively oriented orthonormal basis,

$$\widehat{\mathbf{N}} = \widehat{\mathbf{B}} \times \widehat{\mathbf{T}}$$

1. Ling-Wei Luo, Lecture Note on Elementary Differential Geometry.

2. Jens Gravesen, Differential Geometry and Design of Shape and Motion, http://www2.mat.dtu.dk/people/J.Gravesen/cagd.pdf.

3. Lawrence E. Goodman, Dynamics.