

Frenet-Serret and Space Curves

an insight from kinematics

Imagine the simple case of a particle tied to the end of a string rotating in a circle. There is a way of computing the radius of curvature ρ that will be independent of the coordinate system used. We make use of the well-known relationship $\rho = v^2/a_n$. We must find expressions for v^2 and a_n in a general way. For v^2 , this can be realized by: $v^2 = \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$. For a_n ,

$$a_n \mathbf{e}_n = \mathbf{a} - a_t \mathbf{e}_t$$

$$\mathbf{a} = a_n \mathbf{e}_n + a_t \mathbf{e}_t$$

$$\mathbf{v} \times \mathbf{a} = \mathbf{v} \times (\mathbf{a}_t + \mathbf{a}_n) = \mathbf{v} \times \mathbf{a}_n$$

since \mathbf{v} and \mathbf{a}_t are in the tangential direction. Thus,

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v} \times \mathbf{a}_n| = |\mathbf{v}| |\mathbf{a}_n| \sin 90^\circ = |\mathbf{v}| a_n$$

and,

$$a_n = \frac{|\mathbf{v} \times \mathbf{a}_n|}{|\mathbf{v}|}$$

finally,

$$\rho = \frac{|\mathbf{v}|^3}{|\mathbf{v} \times \mathbf{a}|}$$

tangent vector

In 2-dimensional Euclidean space, an \mathbb{E}^2 plane, we parametrize a curve $p(t) = (x(t), y(t))$ by one parameter t with respect to a reference point O with a *fixed* Cartesian coordinate frame. The velocity vector at point \mathbf{p} is given by $\dot{\mathbf{p}}(t) = (\dot{x}(t), \dot{y}(t))$ with the norm

$$|\dot{\mathbf{p}}(t)| = \sqrt{\dot{\mathbf{p}} \cdot \dot{\mathbf{p}}} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

where $\dot{x} = dx/dt$. The arc length s in the interval $[a, b]$ can be calculated by

$$s = \int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2} dt = \int_a^b |\dot{\mathbf{p}}(t)| dt$$

The arc length can be a function of parameter t given by

$$s(t) = \int_a^t |\dot{\mathbf{p}}(t')| dt'$$

From the fundamental theorem of calculus

$$\left| \frac{ds}{dt} \right| \neq 0 \quad \Rightarrow \quad \dot{s}(t) = |\dot{\mathbf{p}}(t)| > 0$$

One can parametrize the curve by arc length s as $\mathbf{p}(s) = (x(s), y(s))$. The corresponding velocity vector should be $\mathbf{p}'(s) = (x'(s), y'(s))$, where we gave $x' = dx/ds$. We can rewrite the derivatives of x and y with respect to s as

$$x' = \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \dot{x} \frac{dt}{ds}$$

$$y' = \dot{y} \frac{dt}{ds}$$

Thus the norm of the velocity vector parametrized by s can be calculated as

$$|\mathbf{p}'(s)| = \sqrt{x'^2 + y'^2} = \sqrt{\dot{x}^2 + \dot{y}^2} \frac{dt}{ds} = \frac{ds}{dt} \frac{dt}{ds} = 1,$$

which implies that the velocity vector $\mathbf{p}'(s)$ is a unit vector. We can define a unit tangent vector as a velocity vector parametrized by s

$$\hat{\mathbf{T}} = \mathbf{e}_1 = \mathbf{p}'(s)$$

normal vector

Due to $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{p}' \cdot \mathbf{p}'$, we have

$$\mathbf{e}'_1 \cdot \mathbf{e}_1 + \mathbf{e}_1 \cdot \mathbf{e}'_1 = 0 \quad \Rightarrow \quad \mathbf{e}'_1 \cdot \mathbf{e}_1 = 0 \quad \Rightarrow \quad \mathbf{e}'_1 \perp \mathbf{e}_1,$$

it indicates that \mathbf{e}'_1 is a normal vector. The principal normal vector is defined by

$$\hat{\mathbf{N}} = \mathbf{e}_2 = \frac{\mathbf{e}'_1}{|\mathbf{e}'_1|}$$

as a unit normal vector at $\mathbf{p}(s)$. The curvature of $\mathbf{p}(s)$ is given by $1/\rho = \kappa(s) = |\mathbf{e}'_1(s)| > 0$, which can be realized as a norm of the acceleration vector $\mathbf{a} = \mathbf{e}'_1 = \mathbf{p}''$. Therefore, we have a relation

$$\mathbf{e}'_1 = \kappa(s) \mathbf{e}_2$$

The plane spanned by vectors \mathbf{e}_1 and \mathbf{e}_2 is called osculating plane.

Frenet-Serret formula in 2D

From the orthogonality condition $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ ($i, j = 1, 2$) (2 axis), we have

$$\mathbf{e}'_i \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}'_j = 0$$

$$\mathbf{e}'_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}' = \kappa \mathbf{e}_2 \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}'_2 = \kappa + \mathbf{e}_1 \cdot \mathbf{e}'_2 = 0$$

$$\mathbf{e}_1 \cdot \mathbf{e}'_2 = -\kappa$$

as a result, we have the following relations

$$\begin{cases} \mathbf{p}' = +\mathbf{e}_1 \\ \mathbf{e}'_1 = \quad +\kappa \mathbf{e}_2 \\ \mathbf{e}'_2 = -\kappa \mathbf{e}_1 \end{cases}$$

$$\begin{pmatrix} \mathbf{p}' \\ \mathbf{e}'_1 \\ \mathbf{e}'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}$$

Frenet's frames and equation in 2d,

$$\begin{pmatrix} \frac{d\hat{\mathbf{T}}(s)}{ds} \\ \frac{d\hat{\mathbf{N}}(s)}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{T}}(s) \\ \hat{\mathbf{N}}(s) \end{pmatrix}$$

curve in 3d space

In \mathbb{E}^3 , a curve is parametrized as $\mathbf{p}(t) = (x(t), y(t), z(t))$ and we have to look for an orthonormal frame at p denoted by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The vector $\mathbf{e}_1 = \mathbf{p}'$ is uniquely defined the same way. Due to $\mathbf{e}'_1 \perp \mathbf{e}_1$, vector \mathbf{e}'_1 should be proportional to \mathbf{e}_2 or \mathbf{e}_3 . Now we can fix $\mathbf{e}'_1 = \kappa \mathbf{e}_2$ as in the 2d case.

binormal vector

Now define a unit vector orthogonal to $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ called binormal vector.

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \wedge \hat{\mathbf{N}}$$

$$= \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$$

where \wedge is the exterior product or wedge product (in 3d space, the exterior product is the same to the usual cross product \times of two vectors).

By orthonormality condition $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ ($i, j = 1, 2, 3$) (3 axis), we have

$$\mathbf{e}'_1 \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}'_j = 0$$

, which implies:

1. If $i = j$, we have $\mathbf{e}'_i \perp \mathbf{e}_i$, for example \mathbf{e}'_2 should be the combination of \mathbf{e}_1 and \mathbf{e}_3 .
2. If $i \neq j$, we have

$$\begin{cases} 0 = \mathbf{e}'_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}'_2 = (\kappa \mathbf{e}_2) \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \mathbf{e}'_2 = \kappa + \mathbf{e}_1 \cdot \mathbf{e}'_2 & (i=1, j=2) \quad (1a) \\ 0 = \mathbf{e}'_1 \cdot \mathbf{e}_3 + \mathbf{e}_1 \cdot \mathbf{e}'_3 = (\kappa \mathbf{e}_2) \cdot \mathbf{e}_3 + \mathbf{e}_1 \cdot \mathbf{e}'_3 = 0 + \mathbf{e}_1 \cdot \mathbf{e}'_3 & (i=1, j=3) \quad (1b) \end{cases}$$

From 1a, we have to assume that

$$\mathbf{e}'_2 = -\kappa(s)\mathbf{e}_1 + \tau(s)\mathbf{e}_3$$

where it now contains an unknown additional term related to \mathbf{e}_3 . For $(i=2, j=3)$, we obtain

$$0 = \mathbf{e}'_2 \cdot \mathbf{e}_3 + \mathbf{e}_2 \cdot \mathbf{e}'_3 = (-\kappa \mathbf{e}_1 + \tau \mathbf{e}_3) \cdot \mathbf{e}_3 + \mathbf{e}_2 \cdot \mathbf{e}'_3 = \tau + \mathbf{e}_2 \cdot \mathbf{e}'_3$$

(1.) tells us that \mathbf{e}'_3 is a linear combination of \mathbf{e}_1 and \mathbf{e}_2 . (1b) tells us that \mathbf{e}'_3 is perpendicular to \mathbf{e}_1 . Therefore, we have the unique solution that

$$\mathbf{e}'_3 = -\tau \mathbf{e}_2$$

, where $\tau(s)$ is called *torsion* of a curve $\mathbf{p}(s)$. The geometric meaning of torsion is that it make the point of the curve leave for the osculating plane spanned by \mathbf{e}_1 and \mathbf{e}_2 (the torsion of the curve is always related to the binormal vector $\hat{\mathbf{B}} = \mathbf{e}_3$).

Frenet-Serret formula in 3D

As a result, we have Frenet-Serret formula in 3d

$$\begin{cases} \mathbf{e}'_1 = & +\kappa \mathbf{e}_2 \\ \mathbf{e}'_2 = -\kappa \mathbf{e}_1 & + \tau \mathbf{e}_3 \\ \mathbf{e}'_3 = & -\tau \mathbf{e}_2 \end{cases} \Rightarrow \begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{d\hat{\mathbf{T}}(s)}{ds} \\ \frac{d\hat{\mathbf{N}}(s)}{ds} \\ \frac{d\hat{\mathbf{B}}(s)}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{T}}(s) \\ \hat{\mathbf{N}}(s) \\ \hat{\mathbf{B}}(s) \end{pmatrix}$$

Note: If one defines $\hat{\mathbf{B}} = \hat{\mathbf{N}} \wedge \hat{\mathbf{T}}$, then one should assume $\mathbf{e}'_2 = -\kappa(s)\mathbf{e}_1 - \tau(s)\mathbf{e}_3$ and obtain $\mathbf{e}'_3 = +\tau\mathbf{e}_2$.

Space Curves

Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a regular parametrization of \mathbb{E}^3 . Let s denotes the arc length of the curve. We then have

$$\frac{d\mathbf{r}}{dt} = \frac{ds}{dt} \frac{d\mathbf{r}}{ds} = \frac{ds}{dt} \hat{\mathbf{T}}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2s}{dt^2} \hat{\mathbf{T}} + \left(\frac{ds}{dt}\right)^2 \frac{d\hat{\mathbf{T}}}{ds} = \frac{d^2s}{dt^2} \hat{\mathbf{T}} + \left(\frac{ds}{dt}\right)^2 \kappa \hat{\mathbf{N}}$$

Hence,

$$\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \left(\frac{ds}{dt}\right)^3 \kappa \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \left|\frac{d\mathbf{r}}{dt}\right|^3 \kappa \hat{\mathbf{B}}$$

the curvature κ is then given by,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

The equation also implies,

$$\hat{\mathbf{B}} = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}$$

$$\frac{d^3\mathbf{r}}{dt^3} = \frac{d^3s}{dt^3} \hat{\mathbf{T}} + \frac{d^2s}{dt^2} \frac{d\hat{\mathbf{T}}}{dt} + 2 \left(\frac{ds}{dt}\right) \left(\frac{d^2s}{dt^2}\right) \kappa(s) \hat{\mathbf{N}} + \left(\frac{ds}{dt}\right)^2 \left(\kappa(s) \frac{d\hat{\mathbf{N}}}{dt} + \frac{d\kappa(s)}{dt} \hat{\mathbf{N}}\right)$$

$$\begin{aligned}
&= \frac{d^3 s}{dt^3} T + \frac{d^2}{dt^2} \frac{dT}{ds} \frac{ds}{dt} + 2 \left(\frac{ds}{dt} \right) \left(\frac{d^2 s}{dt^2} \right) \kappa(s) N + \left(\frac{ds}{dt} \right)^3 \left(\kappa(s) \frac{dN}{ds} + \frac{d\kappa}{ds} N \right) \\
&= \frac{d^3 s}{dt^3} T + \frac{d^2 s}{dt^2} \kappa(s) N \frac{ds}{dt} + 2 \frac{ds}{dt} \frac{d^2 s}{dt^2} \kappa(s) N + \left(\frac{ds}{dt} \right)^3 \kappa(-\kappa T + \tau B) + \left(\frac{ds}{dt} \right)^3 \frac{d\kappa}{ds} N \\
&= \left(\frac{d^3 s}{dt^3} - \left(\frac{ds}{dt} \right)^3 \kappa^2 \right) T + \left(3 \frac{d^2 s}{dt^2} \frac{ds}{dt} \kappa + \left(\frac{ds}{dt} \right)^3 \frac{d\kappa}{ds} \right) N + \left(\frac{ds}{dt} \right)^3 \kappa \tau B
\end{aligned}$$

we also have the scalar triple product,

$$\left[\frac{d\mathbf{r}}{dt}, \frac{d^2 \mathbf{r}}{dt^2}, \frac{d^3 \mathbf{r}}{dt^3} \right] = \left[\frac{ds}{dt} \hat{\mathbf{T}}, \frac{d^2 s}{dt^2} \hat{\mathbf{T}} + \left(\frac{ds}{dt} \right)^2 \kappa \hat{\mathbf{N}}, C \hat{\mathbf{T}} + D \hat{\mathbf{N}} + \left(\frac{ds}{dt} \right)^3 \kappa \tau \hat{\mathbf{B}} \right]$$

where C and D are *don't care* because they will eventually vanish.

$$\left[\frac{d\mathbf{r}}{dt}, \frac{d^2 \mathbf{r}}{dt^2}, \frac{d^3 \mathbf{r}}{dt^3} \right] = \left(\frac{ds}{dt} \right)^6 \kappa^2 \tau [\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}] = \left| \frac{d\mathbf{r}}{dt} \times \frac{d^2 \mathbf{r}}{dt^2} \right|^2 \tau$$

the torsion is then given by,

$$\tau(t) = \frac{[\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)]}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

lastly, because $\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}$ forms a positively oriented orthonormal basis,

$$\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}$$

1. Ling-Wei Luo, Lecture Note on Elementary Differential Geometry
2. Jens Gravesen, Differential Geometry and Design of Shape and Motion,
<http://www2.mat.dtu.dk/people/J.Gravesen/cagd.pdf>
3. Lawrence E. Goodman, Dynamics