

layer-cake representation

let $F_X(x)$ denotes the cdf of X ,

$$F_X(x) = Pr(X \leq x), x \in \mathbb{R}$$

then,

$$E[X] = \int_0^\infty 1 - F_X(x) dx$$

Proof: what does the cdf look like for a nonneg discrete rv $X \in \{a_0, a_1, a_2, a_3\} \subset \mathbb{R}_+$

Say $a_0 = 0 < a_1 < a_2 < a_3$

Recall that cdf is continuous from the right.

Area under the complementary cdf:

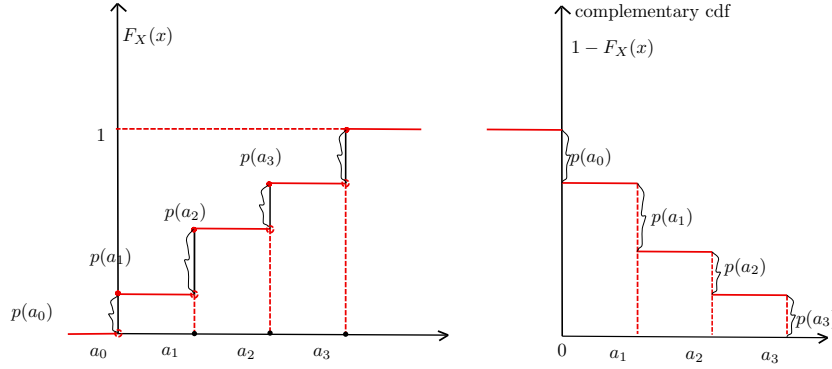


Figure 1: layer cake representation

$$\begin{aligned} \int_0^\infty 1 - F_X(x) dx \\ = a_1 p(a_1) + a_2 p(a_2) + a_3 p(a_3) = \sum_{i=0}^3 a_i p_X(a_i) = E[X] \end{aligned}$$

hence, for non-negative integer-valued rv X ,

$$\begin{aligned} E[X] &= \sum_{n=0}^{\infty} Pr(X > n) \\ &= Pr(X > 0) + Pr(X > 1) + Pr(X > 2) + \dots = [p(1) + p(2) + p(2) + \dots] + [0 + p(2) + p(3) + \dots] + [0 + 0 + p(2) + \dots] \\ &= 1p(2) + 2p(2) + 3p(3) + \dots \\ &= \sum_{n=1}^{\infty} n p_X(n) = E[X] \end{aligned}$$

For the continuous case,

X cts rv & $Pr(X \geq 0) = 1$

$$E[X] = \int_0^\infty x f_X(x) dx = \int_0^\infty F_X^C(x) dx$$

Proof:

$$\begin{aligned} \int_0^\infty Pr(X > x) dx \\ = \int_0^\infty \left(\int_x^\infty f_X(u) du \right) dx \end{aligned}$$

making use of indicator function $\mathbf{1}$

$$= \int_0^\infty \int_0^\infty f_x(u) \mathbf{1}\{u > x\} du dx$$

by Tonelli/Fubini's theorem

$$\begin{aligned} &= \int_0^\infty \int_0^\infty f_x(u) \mathbf{1}\{u > x\} dx du \\ &= \int_0^\infty f_x(u) \left(\int_0^\infty \mathbf{1}\{u > x\} dx \right) du \\ &\quad \int_0^\infty f_x(u) \cdot u du \\ &= \int_0^\infty u f_x(u) du \end{aligned}$$

markov's inequality

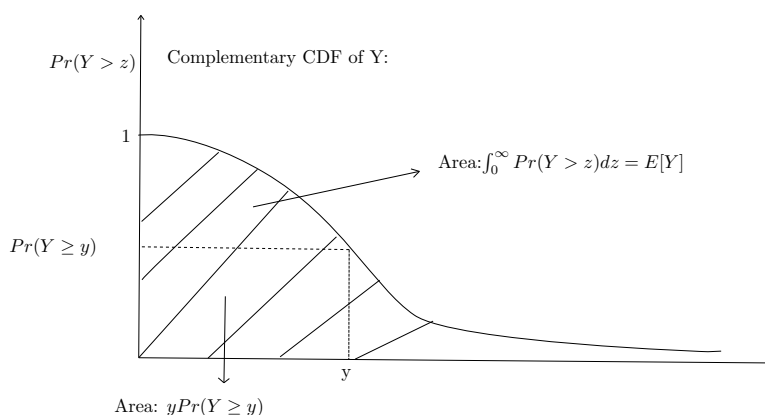


Figure 2: graphical intuition for markov's inequality

Example 1:

If Y represents height of a random student in the class. $E[y] = 1.6m$. What's the chance that Y exceeds $1.5E[Y]$?

$$Pr(Y \geq 1.5E[Y]) \leq \frac{E[Y]}{1.5E[Y]} = 2/3$$

$$Pr(Y \geq 2.4m) \leq \frac{2}{3}$$

comment: using Markov's inequality, we get a loose bound. In general Markov's inequality is quite loose under certain cases.

Example 2:

If $X \in \{0, 5\}$, $Pr(X = 0) = \frac{24}{25}$, $Pr(X = 5) = \frac{1}{25}$

$$Pr(X \geq 5) \leq \frac{E[X]}{5} = \frac{1}{25} = Pr(X = 5)$$

Comment: we now get a tight bound, in fact it is exactl equal to $P(X = 5)$. We can see that in situations where no better bounds can be achieved using only expectation of X , the bound is quite tight.