Tsai-Lenz Hand-eye calibration

we will derive a modified version of Rodrigues formula:

$$R = \left(1 - \frac{1}{2}\|P_r\|^2\right) I_3 + \frac{1}{2}(P_r P_r^T + \sqrt{4 - \|P_r\|^2}[P_r \times])$$

starting with,

$$R = [(q_4^2 - \|\mathbf{q}\|^2)I_3 + 2\mathbf{q}\mathbf{q}^T + 2q_4\mathbf{q}\times]$$

where

$$\mathbf{q}\times = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

$$q_4 = \cos\frac{\theta}{2}$$

$$\mathbf{q} = \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

define,

$$\vec{P}_r = 2\sin\frac{\theta}{2}\hat{n}$$

$$\|\vec{P}_r\|^2 = 4\sin^2\frac{\theta}{2}$$

$$\frac{\|\vec{P}_r\|}{4} = 1 - \cos^2\frac{\theta}{2}$$

$$\cos^2\frac{\theta}{2} = 1 - \frac{\|\vec{P}_r\|^2}{4}$$

$$q_4 = \cos\frac{\theta}{2} = \sqrt{1 - \frac{\|P_r\|^2}{4}}$$

now,

$$\|\vec{q}\|^2 = \sin^2\frac{\theta}{2} = \frac{\|P_r\|^2}{4}$$

For the middle term, $2\mathbf{q}\mathbf{q}^T$

$$2\vec{q}\vec{q}^T = 2\frac{\vec{P}_r}{2}\frac{\vec{P}_r^T}{2} = \frac{1}{2}\vec{P}_r\vec{P}_r^T$$

For the last term $2q_4\mathbf{q}\times$,

$$2q_4\vec{q}\times = 2\sqrt{1 - \frac{\|\vec{P}_r\|^2}{4}}\frac{[\vec{P}_r\times]}{2} = \frac{1}{2}\sqrt{4 - \|\vec{P}_r\|^2}[\vec{P}_r\times]$$

summing up,

$$\begin{aligned} R &= \left(1 - \frac{1}{2}\|P_r\|^2\right) I_3 + \frac{1}{2}P_r P_r^T + \frac{1}{2}\sqrt{4 - \|P_r\|^2}[P_r\times] \\ &= \left(1 - \frac{1}{2}\|P_r\|^2\right) I_3 + \frac{1}{2}(P_r P_r^T + \alpha[P_r\times]) \end{aligned}$$

where $\alpha = \sqrt{4 - \|P_r\|^2}$, and $[P_r\times] = skew(P_r)$

Figure 1 shows the relation:

$$H_{gj} \circ H_{gij} = H_{gi}$$

$$H_{cij} \circ H_{ci} = H_{cj}$$

where $\circ$ denotes "followed by"

$$H_{cij} = H_{cj} H_{ci}^{-1}$$
$$H_{gij} = H_{gj}^{-1} H_{gi}$$

given: H_{gi} defines coordinate transformation from G_i to RW

$$H_{gi} = \begin{bmatrix} & R_{gi} & & T_{gi} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

H_{ci} defines coordinate transformation from CW to C_i

$$H_{ci} = \begin{bmatrix} & R_{ci} & & T_{ci} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

H_{gij} defines coordinate transformation from G_i to G_j

$$H_{gij} = \begin{bmatrix} & R_{gij} & & T_{gij} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

H_{cij} defines coordinate transformation from C_i to C_j

$$H_{cij} = \begin{bmatrix} & R_{cij} & & T_{cij} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

H_{cg} defines coordinate transformation from C_i to G_i

$$H_{cg} = \begin{bmatrix} & R_{cg} & & T_{cg} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the above i,j range from 1 to N, where N is the number of stations.
Lemma 1

$$\begin{gathered} H_{gj} H_{cg} H_{cj} = H_{gi} H_{cg} H_{ci} \\ H_{cg} H_{cj} H_{ci}^{-1} = H_{gj}^{-1} H_{gi} H_{cg} \\ H_{cg} H_{cij} = H_{gij} H_{cg} \\ H_{gij} H_{cg} = H_{cg} H_{cij} \\ AX = XB \end{gathered} \tag{1}$$

from the rotation terms,

$$\text{Lemma 1:} \quad R_{gij} = R_{cg} R_{cij} R_{cg}^T$$

Lemma 2 : $P_{gij} = R_{cg} P_{cij}$
Proof: Suppose we have two coordinate frames, the transformation of a vector measured from one to the other is related by

$$\vec{u} = Q \vec{u}_a$$

where Q is a rotation matrix with the direction cosines of the axes of the rotated frame w.r.t to the original frame. Now, let's say we want to transform the vector $\vec{u}$ to another vector $\vec{v}$,

$$\vec{v} = T \vec{u}$$

then,

$$Q \vec{v}_a = T Q \vec{u}_a$$

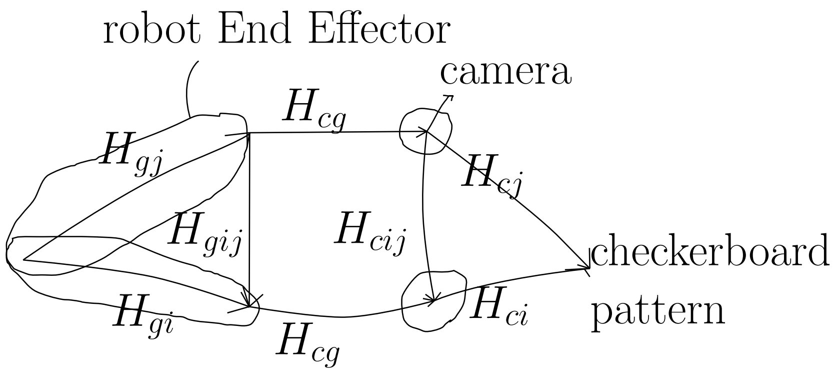


Figure 1: hand-eye calibration

$$\vec{v}_a = Q^{-1}TQ\vec{u}_a$$
$$\vec{v}_a = T_a\vec{u}_a \Rightarrow T_a = Q^{-1}TQ = Q^TTQ$$

now let $B = T_a$ and $A = T$. Then $B = Q^TAQ$

$$B\vec{v} = \lambda\vec{v}$$

with λ and $\vec{v}$ being the eigenvalue and eigenvector pair of B.
Then

$$Q^TAQ\vec{v} = \lambda\vec{v}$$
$$AQ\vec{v} = \lambda Q\vec{v}$$
$$A(Q\vec{v}) = \lambda(Q\vec{v})$$
$$A\vec{w} = \lambda\vec{w}$$

with $\vec{w} = Q\vec{v}$ being the eigenvector of A corresponding to the same eigenvalue λ.
Hence, given the similarity transform $R_{cij} = R_{cg}^TR_{gij}R_{cg}$ from lemma 1 and letting $B = R_{cij}$ and $A = R_{gij}$, by comparison, we can observe

$$P_{gij} = \vec{w} = Q\vec{v} = R_{cg}P_{cij}$$

Lemma 3 : $P_{cg} \perp (P_{gij} - P_{cij})$,
Before we prove that $P_{cg} \perp (P_{gij} - P_{cij})$, we will digress to show that the eigenvalues of a rotation matrix, R are $\lambda_1 = 1$ and $\lambda_{2,3} = e^{\pm i\phi}$ for $\phi \in [0, \pi]$:

$$R\mathbf{x} = \lambda\mathbf{x}$$

Since a rotation matrix consists of only real entries, complex conjugation on both sides yields:

$$\overline{R\mathbf{x}} = R\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$$

$$(R\mathbf{x}) \cdot (R\bar{\mathbf{x}}) = (\lambda\mathbf{x}) \cdot (\bar{\lambda}\bar{\mathbf{x}})$$
$$\mathbf{x}^TR^TR\bar{\mathbf{x}} = \lambda\bar{\lambda}\mathbf{x} \cdot \mathbf{x}$$

Since $R^TR = I$,

$$\mathbf{x} \cdot \mathbf{x} = \lambda\bar{\lambda}\mathbf{x} \cdot \mathbf{x}$$
$$\|\mathbf{x}\|^2 = \lambda\bar{\lambda}\|\mathbf{x}\|^2$$

Since $\|\mathbf{x}\| \neq 0$,

$$\lambda\bar{\lambda} = 1$$

This can be satisfied for the eigenvalues $\lambda_1 = 1$ and $\lambda_{2,3} = e^{\pm i\phi}$ for $\phi \in [0, \pi]$. The eigenvalue $\lambda = 1$ correspond to the eigenvector $\hat{\mathbf{n}}$. This implies $R\hat{\mathbf{n}} = \hat{\mathbf{n}}$. Furthermore, since $det(A^T - \lambda I) = det((A - \lambda I)^T) = det(A - \lambda I)$, implies $R^T\hat{\mathbf{n}} = \hat{\mathbf{n}} \Rightarrow (R^T - I)\hat{\mathbf{n}} = 0$.
Now, to prove $P_{cg} \perp (P_{gij} - P_{cij})$ (we will ommit the subscript ij for clarity).

$$(P_g - P_c)^T P_{cg} = (P_g - P_c)^T R_{cg}^T R_{cg} P_{cg}$$

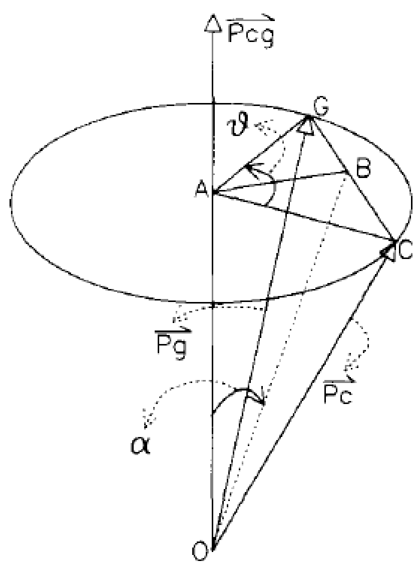
$$= [R_{cg}(P_g - P_c)]^T R_{cg} P_{cg}$$

Obviously $P_{cg} = R_{cg}P_{cg}$,

$$= (R_{cg}P_g - P_g)^T P_{cg}$$
$$= [(R_{cg} - I)P_g]^T P_{cg}$$
$$= P_g^T (R_{cg}^T - I) P_{cg} = 0$$

Since it was shown before that $(R^T - I)\hat{\mathbf{n}} = 0$.

Figure 2: Geometrical relationships, the plane containing the circle is perpendicular to P_{cg}. B is the midpoint of point C and G.



Lemma 4 : $P_{cij} - P_{gij}$ is collinear with $(P_{gij} + P_{cij}) \times P_{cg}$. This means $P_{cij} - P_{gij} = s(P_{gij} + P_{cij}) \times P_{cg}$ for some scale factor s. This follows from the fact that $(P_{gij} - P_{cij})$ is perpendicular to $(P_{gij} + P_{cij})$. Proof:

$$\begin{aligned}
&(P_{gij} - P_{cij})^T (P_{gij} - P_{cij}) \\
&= P_{gij}^T P_{gij} + P_{gij}^T P_{cij} - P_{cij}^T P_{gij} - P_{cij}^T P_{cij} \\
&= \|P_{gij}\|^2 - \|P_{cij}\|^2 \\
&= (R_{cg}P_{cij})^T (R_{cg}P_{cij}) - \|P_{cij}\|^2 \\
&= \|P_{cij}\|^2 - \|P_{cij}\|^2 = 0
\end{aligned}$$

Lemma 5 ; $P_{cij} - P_{gij}$ and $(P_{gij} + P_{cij}) \times P'_{cg}$ have the same length. Define P'_r,

$$P'_r = \frac{P_r}{2\cos\frac{\theta_r}{2}}$$

This definition will be useful in calculating θ later. Since,

$$\frac{2\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \frac{\|P_r\|}{\frac{\|P_r\|}{\|P'_r\|}} \Rightarrow \tan\frac{\theta}{2} = \|P'_r\|$$

$$\theta = 2\tan^{-1}\|P'_r\|$$

Also, with this definition P_r can be calculated using,

$$P_r = \frac{2P'_r}{\sqrt{1+\|P'_r\|^2}}$$

by definition,

$$\begin{aligned}
|(P_g + P_c) \times P'_{cg}| &= |P_g + P_c| 2\sin\frac{\theta}{2} \\
&\cdot \left(4 - 4\sin^2\frac{\theta}{2}\right)^{-1/2} \sin\alpha \\
&= |P_g + P_c| \tan\frac{\theta}{2}\sin\alpha \\
&= 2|\overline{OB}|\sin\alpha\tan\frac{\theta}{2} \\
&= 2|\overline{AB}|\tan\frac{\theta}{2} = 2|\overline{CB}| \\
&= |\overline{CG}| = |P_c - P_g|
\end{aligned}$$

Lemma 6 : $(P_{gij} + P_{cij}) \times P'_{cg} = P_{cij} - P_{gij}$. Proof: direct consequence of Lemmas 4 and 5.
Lemma 7 : Skew$(P_{gij} + P_{cij})$ is singular and has rank 2. Consequence: it is impossible to compute R_{cg} with only two stations, i.e. i=1, j=2 (two stations), we therefore need a minimum of j=3.
Lemma 8: $(R_{gij} - I)T_{cg} = R_{cg}T_{cij} - T_{gij}$. This follows from the translation terms of eq. (1).
Lemma 9 : $R_{gij} - I$ is singular and has rank 2. Consequence: it is impossible to compute R_{cg} with only two stations.
Lemma 10 : If $\theta_{R_{cg}} \neq \pi$,or equivalently, $|P_{cg}| \neq \pm 2$, then

$$\begin{bmatrix} Skew(P_{gi_1j_1} + P_{ci_1j_1}) \\ Skew(P_{gi_2j_2} + P_{ci_2j_2}) \end{bmatrix}$$

has full column rank if and only if $P_{gi_1j_1}$ and $P_{gi_2j_2}$ have different directions (or equivalently $P_{ci_1j_1}$ and $P_{ci_2j_2}$ have different directions.
Lemma 11 :

$$\begin{bmatrix} R_{gi_1j_1} - I \\ R_{gi_2j_2} - I \end{bmatrix}$$

has full column rank if and only if we have different pairs of $P_{gi_1j_1}$ and $P_{gi_2j_2}$.

In summary, procedure for computing R_{cg}:

1. Compute P'_{cg}. For each pair of stations i,j such that the rotation angle R_{gij} or R_{cij} is as large as possible, Figure 2. Set up a system of linear equations with P'_{cg} as the unknown

$$Skew(P_{gij} + P_{cij})P'_{cg} = P_{cij} - P_{gij}$$

Since $Skew(P_{gij} + P_{cij})$ is always singular, it takes at least two pairs of stations to solve for a unique solution for P'_{cg} using least squares technique. **Exception handling:** If $P_{gi_1j_1} + P_{ci_1j_1}$ is collinear with $P_{gi_2j_2} + P_{ci_2j_2}$ while $P_{gi_1j_1}$ is not collinear with $P_{gi_2j_2}$, then the rotation angle of R_{cg} must be 180^o and the rotation axis the same as $P_{gi_1j_1} + P_{ci_1j_1}$.

2. (Optional: since step 3 which computes the rotation axis already captures this information through its magnitude) Compute $\theta_{R_{cg}}$:

$$\theta_{R_{cg}} = 2\tan^{-1}|P'_{cg}|$$

3. Compute P_{cg}

$$P_{cg} = \frac{2P'_{cg}}{\sqrt{1 + |P'_{cg}|^2}}$$

4. Compute T_{cg}. Given at least two distinct pairs of stations (i,j), set up a linear system of three linear equations with T_{cg} as unknowns

$$(R_{gij} - I)T_{cg} = R_{cg}T_{cij} - T_{gij}.$$

For at least two pairs of stations, two sets of the above equation are established and can be solved using least squares solutions.