**layer-cake representation**

let $F_X(X)$ denotes the cdf of X,

$$F_X(x) = Pr(X \leq x), x \in \mathbb{R}$$

then,

$$E[X] = \int_0^\infty 1 - F_X(x)\text{dx}$$

Proof: what does the cdf look like for a nonneg discrete rv $X \in \{a_0, a_1, a_2, a_3\} \subset \mathbb{R}_+$
Say $a_0 = 0 < a_1 < a_2 < a_3$
Recall that cdf is continuous from the right.
Area under the complementary cdf:

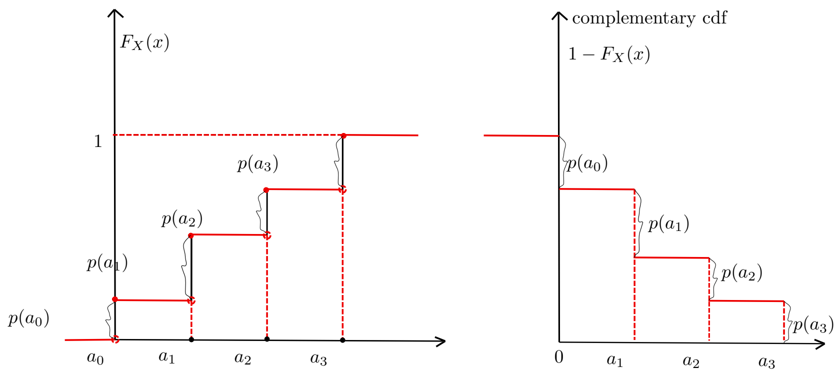


Figure 1: layer cake representation

$$\int_0^\infty 1 - F_X(x)\text{dx}$$

$$= a_1 p(a_1) + a_2 p(a_2) + a_3 p(a_3) = \sum_{i=0}^{3} a_i p_X(a_i) = E[X]$$

hence, for non-negative integer-valued rv X,

$$E[X] = \sum_{n=0}^{\infty} Pr(X > n)$$

$$= Pr(X > 0) + Pr(X > 1) + Pr(X > 2) + ... = [p(1) + p(2) + p(2) + ...] + [0 + p(2) + p(3) + ...] + [0 + 0 + p(2) + ...]$$

$$= 1p(2) + 2p(2) + 3p(3)$$

$$= \sum_{n=1}^{\infty} np_X(N) = E[X]$$

For the continuous case,
X cts rv & $Pr(X \geq 0) = 1$

$$E[X] = \int_0^\infty x f_X(x)dx = \int_0^\infty F_X^C(x)dx$$

Proof:

$$\int_0^\infty Pr(X > x)dx$$

$$= \int_0^\infty \left( \int_x^\infty f_x(u)du \right) dx$$

making use of indicator function $\mathbf{1}$

$$= \int_0^\infty \int_0^\infty f_x(u)\mathbf{1}\{u > x\}dudx$$

by Tonelli/Fubini's theorem

$$= \int_0^\infty \int_0^\infty f_x(u)\mathbf{1}\{u > x\}dxdu$$

$$= \int_0^\infty f_x(u) \left( \int_0^\infty \mathbf{1}\{u > x\}dx \right) du$$

$$\int_0^\infty f_x(u) \cdot udu$$

$$= \int_0^\infty uf_x(u)du$$

**markov's inequality**

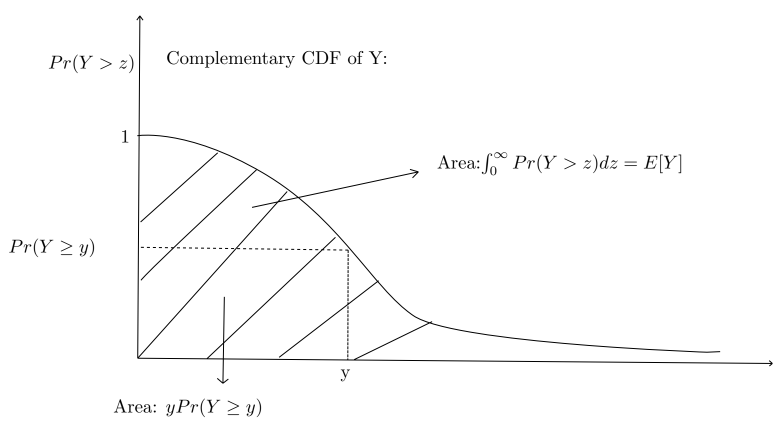


Figure 2: graphical intuition for markov's inequality

Example 1:
If Y represents height of a random student in the class. $E[y] = 1.6m$. What's the chance that Y exceeds $1.5E[Y]$?

$$Pr(Y \geq 1.5E[Y]) \leq \frac{E[Y]}{1.5E[Y]} = 2/3$$

$$Pr(Y \geq 2.4m) \leq \frac{2}{3}$$

comment: using Markov's inequality, we get a loose bound. In general Markov's inequality is quite loose under certain cases.

Example 2:
If $X \in \{0, 5\}, Pr(X = 0) = \frac{24}{25}$, $Pr(X = 5) = \frac{1}{25}$

$$Pr(X \geq 5) \leq \frac{E[X]}{5} = \frac{1}{25} = Pr(X = 5)$$

Comment: we now get a tight bound, in fact it is exactl equal to $P(X = 5)$. We can see that in situations where no better bounds can be achieved using only expectation of X, the bound is quite tight.