Conditional Gaussian Multivariate Distribution

The multivariate Gaussian distribution on R^n is defined as,

$$\rho(\mathbf{x};\boldsymbol{\mu},\Sigma) = \frac{1}{(2\pi)^{n/2}|\det\Sigma|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

calculate the conditional distribution,

$$\rho([\mathbf{x}_1^T,\mathbf{x}_2^T]^T;\boldsymbol{\mu},\Sigma)/\rho(\mathbf{x}_2;\boldsymbol{\mu}_2,\Sigma_2)$$

$$\Sigma = \int_{\mathrm{R}^n}(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T f(\mathbf{x}_1,\mathbf{x}_2)\mathrm{d}\mathbf{x}$$

$$= \begin{pmatrix} \int_{\mathrm{R}^n}(\mathbf{x}_1-\boldsymbol{\mu}_1)(\mathbf{x}_1-\boldsymbol{\mu}_1)^T f(\mathbf{x}_1,\mathbf{x}_2) & \int_{\mathrm{R}^n}(\mathbf{x}_1-\boldsymbol{\mu}_1)(\mathbf{x}_2-\boldsymbol{\mu}_2)^T f(\mathbf{x}_1,\mathbf{x}_2) \\ \int_{\mathrm{R}^n}(\mathbf{x}_2-\boldsymbol{\mu}_2)(\mathbf{x}_1-\boldsymbol{\mu}_1)^T f(\mathbf{x}_1,\mathbf{x}_2) & \int_{\mathrm{R}^n}(\mathbf{x}_2-\boldsymbol{\mu}_2)(\mathbf{x}_2-\boldsymbol{\mu}_2)^T f(\mathbf{x}_1,\mathbf{x}_2) \end{pmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$\Sigma_{12}^T = \Sigma_{21} \qquad \Sigma_{11}^T = \Sigma_{11} \qquad \Sigma_{22}^T = \Sigma_{22}$$

let's ignore the constant term in the front and denote it $A^{'}$. Conditional Gaussian PDF,

$$f(\mathbf{x}_1|\mathbf{x}_2) = A^{'}\exp\left(-\frac{1}{2}[(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) - (\mathbf{x}_2-\boldsymbol{\mu}_2)^T\Sigma_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)]\right)$$

Recall,

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -D^{-1}B^T & I \end{bmatrix}\begin{bmatrix} (A-BD^{-1}B^T)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}\begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix}$$

let $\Sigma_{11} = A$, $\Sigma_{12} = B$, $\Sigma_{21} = B^T$, $\Sigma_{22} = D$.

$$= A^{'}\exp(-\frac{1}{2}\{\begin{pmatrix}(\mathbf{x}_1-\boldsymbol{\mu}_1)^T & (\mathbf{x}_2-\boldsymbol{\mu}_2)^T\end{pmatrix}\begin{bmatrix} I & 0 \\ -D^{-1}B^T & I \end{bmatrix}\begin{bmatrix} (A-BD^{-1}B^T)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}$$

$$\begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix}\begin{pmatrix} \mathbf{x}_1-\boldsymbol{\mu}_1 \\ \mathbf{x}_2-\boldsymbol{\mu}_2 \end{pmatrix} - (\mathbf{x}_2-\boldsymbol{\mu}_2)^T\Sigma_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\})$$

$$= A^{'}\exp(-\frac{1}{2}\{(\mathbf{x}_1-\boldsymbol{\mu}_1-BD^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2))^T(A-BD^{-1}B^T)^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1-BD^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2))+$$

$$(\mathbf{x}_2-\boldsymbol{\mu}_2)^T\Sigma_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2) - (\mathbf{x}_2-\boldsymbol{\mu}_2)^T\Sigma_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\})$$

$$= A^{'}\exp(-\frac{1}{2}\{(\mathbf{x}_1-\boldsymbol{\mu}_1-BD^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2))^T(A-BD^{-1}B^T)^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1-BD^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2))\})$$

hence,

$$\boldsymbol{\mu}_{\mathbf{x}_1|\mathbf{x}_2} = \boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2) \tag{1}$$

$$\Sigma_{\mathbf{x}_1|\mathbf{x}_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \tag{2}$$

Furthermore, suppose $\mathbf{x} \sim \mathcal{N}(\mu_x, \Sigma_x)$ and a variable $\mathbf{y}$ resulting from affine transformation of $\mathbf{x}$, so that $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ where $\mathbf{b} \sim \mathcal{N}(0, \Sigma_b)$.

$$\Sigma_{\mathbf{x}} = E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T]$$

$$\begin{aligned}\Sigma_{\mathbf{xy}} &= E(\tilde{\mathbf{x}}\tilde{\mathbf{y}}^T) \\ &= E[(\mathbf{x} - \bar{\mathbf{x}})(A\mathbf{x} + \mathbf{b} - A\bar{\mathbf{x}})^T] \\ &= E[(\mathbf{x} - \bar{\mathbf{x}})((\mathbf{x} - \bar{\mathbf{x}})^T A^T + \mathbf{b}^T] \\ &= E[(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^T]A^T = \Sigma_x A^T\end{aligned}$$

Similarly,

$$\Sigma_{\mathbf{yx}} = A\Sigma_{\mathbf{x}}$$

and,

$$\begin{aligned}\Sigma_{\mathbf{yy}} &= E[(A(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{b})(A(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{b})^T] \\ &= AE[(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})^T]A^T + E[\mathbf{b}\mathbf{b}^T] \\ &= A\Sigma_{\mathbf{x}}A^T + \Sigma_b\end{aligned}$$

hence, comparing to (1) and (2)

$$\boldsymbol{\mu}_{\mathbf{x}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{x}} + \Sigma_{\mathbf{x}}A^T(A\Sigma_{\mathbf{x}}A^T + \Sigma_b)^{-1}(\mathbf{y} - A\bar{\mathbf{x}})$$

$$\Sigma_{\mathbf{x}|\mathbf{y}} = \Sigma_{\mathbf{x}} - \Sigma_{\mathbf{x}}A^T(A\Sigma_{\mathbf{x}}A^T + \Sigma_b)^{-1}A\Sigma_{\mathbf{x}}$$

references:

1. http://www.cs.columbia.edu/ liulp/pdf/linear_normal_dist.pdf.
2. G.S. Chirikjian. 2009. Stochastic Models, Information Theory, and Lie Groups: Classical Results and Geometric Methods vol 1.
3. Timothy D. Barfoot. 2017. State Estimation for Robotics.