proving vector calculus identities using Einstein's index summation

given the most general identity involving pairs Levi_Civita tensors:

$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}$,

for the special case n=i,

$$\epsilon_{lmi}\epsilon_{jki} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}$$

$$\nabla \cdot (\nabla \times \vec{v}) = div(curl(\vec{v}))$$
$$= \partial_i \epsilon_{ijk}\partial_j v_k = \epsilon_{ijk}\partial_i\partial_j v_k = \epsilon_{ijk}\partial_{ij}v_k = 0$$

Observe that ϵ_{ijk} is antisymmetric but $\partial_i\partial_j$ is symmetric. For example, if $k = 3$,

$$\partial_{ij}v_3 - \partial_{ji}v_3 = 0$$

with the same line of reasoning,$\nabla \times (\nabla\phi) = curl(grad(\phi)) = \epsilon_{ijk}\partial_j\partial_k\phi = \epsilon_{ijk}\partial_{jk}\phi = 0$

$$\nabla \cdot (\phi\vec{v}) = div(\phi\vec{v}) = \partial_i(\phi v_i) = v_i\partial_i(\phi) + \phi\partial_i(v_i)$$
$$= \vec{v} \cdot grad(\phi) + \phi div(\vec{v}) = \vec{v} \cdot \nabla\vec{\phi} + \phi\nabla \cdot \vec{v}$$

$$\nabla \times (\phi\vec{v}) = \epsilon_{ijk}\partial_j(\phi\vec{v})_k$$
$$= \epsilon_{ijk}v_k\partial_j\phi + \epsilon_{ijk}\phi\partial_j v_k$$
$$= \nabla\phi \times \vec{v} + \phi(\nabla \times \vec{v})$$

$$\nabla(\vec{u} \cdot \vec{v}) = grad(u \cdot v) = \partial_i(u_j v_j) = u_j\partial_i v_j + v_j\partial_i u_j$$
$$= u_j\partial_i v_j + v_j\partial_i u_j + u_j\partial_j v_i + v_j\partial_j u_i - u_j\partial_j v_i - v_j\partial_j u$$
$$= u_j(\partial_i v_j - \partial_j v_i) + v_j(\partial_i u_j - \partial_j u_i) + u_j\partial_j v_i + v_j\partial_j v_i$$
$$= u_j \begin{vmatrix} \partial_i & \partial_j \\ v_i & v_j \end{vmatrix}_k + v_j \begin{vmatrix} \partial_i & \partial_j \\ u_i & u_j \end{vmatrix}_k + u_j\partial_j v_i + v_j\partial_j v_i$$
$$= \epsilon_{ijk}u_j \begin{vmatrix} \partial_i & \partial_j \\ v_i & v_j \end{vmatrix}_k + \epsilon_{ijk}v_j \begin{vmatrix} \partial_i & \partial_j \\ u_i & u_j \end{vmatrix}_k + u_j\partial_j v_i + v_j\partial_u v_i$$
$$= \vec{u} \times curl(\vec{v}) + \vec{v} \times curl(\vec{u}) + \vec{u} \cdot grad(\vec{v}) + \vec{v} \cdot grad(\vec{u})$$
$$= \vec{u} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{u}) + \vec{u} \cdot \nabla\vec{v} + \vec{v} \cdot \nabla\vec{u}$$

$$\nabla \cdot (\vec{u} \times \vec{v}) = div(\vec{u} \times \vec{v}) = \partial_i(\epsilon_{ijk}u_j v_k)$$
$$= v_k\epsilon_{ijk}\partial_i u_j + u_j\epsilon_{ijk}\partial_i v_k = v_k\epsilon_{kij}\partial_i u_j - u_j\epsilon_{jik}\partial_i v_k$$

$$= \vec{v} \cdot curl(\vec{u}) - \vec{u} \cdot curl(\vec{v}) = \vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v})$$

$$\nabla \times (\vec{u} \times \vec{v}) = curl(\vec{u} \times \vec{v}) = \epsilon_{lmi}\partial_m \epsilon_{ijk} u_j v_k = \epsilon_{lmi}\epsilon_{ijk}(v_k \partial_m u_j + u_j \partial_m v_k)$$
$$= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})(v_k \partial_m u_j + u_j \partial_m v_k)$$
$$= \delta_{lj}\delta_{mk} v_k \partial_m u_j + \delta_{lj}\delta_{mk} u_j \partial_m v_k - \delta_{lk}\delta_{mj} v_k \partial_m u_j - \delta_{lk}\delta_{mj} u_j \partial_m v_k$$
$$= v_k \partial_k u_l + u_l \partial_k v_k - v_l \partial_j u_j - u_j \partial_j v_l$$
$$= \vec{v} \cdot grad(\vec{u}) - \vec{u} \cdot grad(\vec{v}) + \vec{u} div(\vec{v}) - \vec{v} div(\vec{u})$$
$$= \vec{u}(\nabla \cdot \vec{v}) - \vec{v}(\nabla \cdot \vec{u}) + \vec{v} \cdot \nabla \vec{u} - \vec{u} \cdot \nabla \vec{v}$$

$$\nabla \times (\nabla \times \vec{v}) = \epsilon_{ijk}\partial_j (\nabla \times v)_k$$
$$= \epsilon_{ijk}\partial_j \epsilon_{kmn}\partial_m v_n = \epsilon_{ijk}\epsilon_{kmn}(\partial_j \partial_m v_m + \partial_m \partial_j v_n)$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(\partial_{jm} v_m + \partial_{mj} v_n)$$
$$= \delta_{im}\delta_{jn}\partial_{jm} v_m + \delta_{im}\delta_{jn}\partial_{mj} v_n - \delta_{in}\delta_{jm}\partial_{jm} v_m - \delta_{in}\delta_{jm}\partial_{mj} v_n$$

The first and third term does not make sense because the index 'm' was repeated 3 times, therefore the only possible value for these terms are zero. Hence,

$$[\nabla \times (\nabla \times \vec{v})]_i = \partial_{in} v_n - \partial_{jj} v_i = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$$

$$(\nabla \times \vec{v}) \times \vec{v} = \epsilon_{ijk}(\nabla \times v)_j v_k$$
$$= \epsilon_{ijk}\epsilon_{jmn}\partial_m v_n v_k$$
$$= (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn})\partial_m v_n v_k$$
$$= \partial_k v_i v_k - \partial_i v_k v_k$$
$$= grad(\vec{v}) \cdot \vec{v} - grad(\vec{v} \cdot \vec{v}) = (\nabla \vec{v}) \cdot \vec{v} - \nabla(\vec{v} \cdot \vec{v})$$

bonus:

$$\vec{u} \times (\nabla \times \vec{u}) = \epsilon_{lmi} u_m \epsilon_{ijk}\partial_j u_k$$
$$= \epsilon_{lmi}\epsilon_{ijk} u_m \partial_j u_k$$
$$= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}) u_m \partial_j u_k$$
$$= u_k \partial_l u_k - u_j \partial_j u_l$$
$$= \frac{1}{2}\nabla(u \cdot u) - u \cdot \nabla u$$

Since $\nabla(\vec{u} \cdot \vec{u}) = \partial_i(u_j u_j) = 2u_j \partial_i u_j$.

references:

1. William Prager, Introduction to Mechanics of Continua

2. http://www.physics.usu.edu/Wheeler/ClassicalMechanics/CMnotesLeviCivita.pdf