Curves and Surfaces

insights from kinematics
consider at any point $\mathbf{p}$ of the curve the following basis $\{\mathbf{v}, \mathbf{f}, \mathbf{n}\}$, where

- velocity vector $\mathbf{v}$ is tangent to the curve
- the vector $\mathbf{f}$ is the vector tangent to the surface but orthogonal to the vector $\mathbf{v}$.
- $\mathbf{n}$ is the unit normal to the surface, i.e. it is orthogonal to vectors $\mathbf{v}$ and $\mathbf{f}$.

Decompose acceleration vector over three directions, i.e. over three one-dimensional spaces spanned by vectors $\mathbf{v}, \mathbf{f}$ and $\mathbf{n}$:

$$\mathbf{a} = \mathbf{a}_{\text{orthogonal to surface}} + \mathbf{a}_{\text{tang.to.surf. and orthog. to curve}} + \mathbf{a}_{\text{tangent to curve}}$$

The vector $\mathbf{a}_{\text{orthogonal to surface}}$ which is collinear to normal unit vector $\mathbf{n}$, will be called vector of normal acceleration of the curve on the surface. We denote it by $\mathbf{a}_n$.

The vector $\mathbf{a}_{\text{tang.to surf. and orthog. to curve}}$, collinear to unit vector $\mathbf{f}_C$ will be called *vector of geodesic acceleration*. We denote it by $\mathbf{a}_{geod}$.

The vector $\mathbf{a}_{\text{tangent to curve}}$, collinear to velocity vector $\mathbf{v}$, is just *vector of tangential acceleration*. We denote it by $\mathbf{a}_{tang}$. Thus, the acceleration vector can be rewritten as

$$\mathbf{a} = \mathbf{a}_n + \mathbf{a}_{geod} + \mathbf{a}_{tang}$$

Both vectors $\mathbf{a}_n$ and $\mathbf{a}_{geod}$ are orthogonal to the curve. The vector $\mathbf{a}_{geod}$ is orthogonal to the curve but it is tangent to the surface. The vector $\mathbf{a}_n$ is orthogonal not only to the curve. It is orthogonal to the surface.

The vector $\mathbf{a}_{geod} + \mathbf{a}_n = \mathbf{a}_\perp$ is orthogonal to the curve. It is the vector of normal acceleration of the curve.

Note: When we consider the curves on the surface it could arise the confusion between the vector $\mathbf{a}_n$—normal acceleration of the curve on the surface and the vector $\mathbf{a}_\perp$ of normal acceleration to the curve (see Fig. 2).

When we decompose the acceleration vector $\mathbf{a}$ in the sum of three vectors $\mathbf{a}_n$,$\mathbf{a}_{geod}$, and $\mathbf{a}_{tang}$ then the vector $\mathbf{a}_n$, the normal acceleration of the curve on the surface is orthogonal to the surface not only to the curve. The vector

$$\mathbf{a}_\perp = \mathbf{a}_n + \mathbf{a}_{geod}$$

is orthogonal only to the curve and in general it is not orthogonal to the surface (if $\mathbf{a}_{geod} \neq 0$). It is the normal acceleration of the curve. It depends only on the curve. The normal acceleration $\mathbf{a}_n$ of the curve on the surface which is orthogonal to the surface depends on the surface where the curve lies.

We know that the curvature of the curve is equal to the magnitude of normal acceleration of the curve divided on the square of the speed. We have:

$$\text{curvature of the curve } k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_n + \mathbf{a}_{geod}|}{|\mathbf{v}|^2}$$

parametric representation of surface and curve in $\mathbb{R}^3$
A two-dimensional surface can be described as $\mathbf{P}(\mathbf{q})$ where $\mathbf{P} \in \mathbb{R}^3$ and $\mathbf{q} \in \mathbb{R}^2$. In the ambient space $\mathbb{R}^3$:

$$\mathbf{P}(\mathbf{q}) = \mathbf{P}(u, v) = \begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \end{cases}$$

- All the points of the surface can be obtained by varying u and v.

- Again, the parametrization is not unique
- The interval of definition is limited (limits the surface)

If we were to parametrize the surface described with time t, we will get the equation of a curve

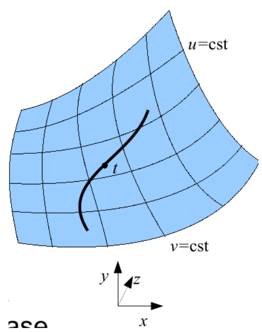
$$\mathbf{q}(t) = \begin{cases} u = u(t) \\ v = v(t) \end{cases}$$

the equation of a curve is given by,

$$\boldsymbol{\Gamma}(t) = \begin{cases} x = f(u(t), v(t)) \\ y = g(u(t), v(t)) \\ z = h(u(t), v(t)) \end{cases}$$

Example a sphere of radius R in $\mathbb{E}^3$:

Figure 1: Parametric Surface



$$\mathbf{r}(\theta, \phi) = \begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}$$

then

$$\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix}, \quad \mathbf{r}_\phi = \begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix}$$

and the unit normal to the surface

$$\mathbf{n} = \frac{\mathbf{r}_\theta \times \mathbf{r}_\phi}{\|\mathbf{r}_\theta \times \mathbf{r}_\phi\|} = \frac{\begin{pmatrix} R^2 s^2\theta c\phi \\ R^2 s^2\theta c\phi \\ R^2 s\theta c\theta \end{pmatrix}}{R^2 s\theta} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

where $\mathbf{r}_\theta$ and $\mathbf{r}_\phi$ are basic tangent vectors.

arc length and the first fundamental form

$$L = \int_a^b |\boldsymbol{\Gamma}'(t)| \mathrm{dt} = \int_a^b \sqrt{\|\boldsymbol{\Gamma}'(t)\|^2} \mathrm{dt}$$

$$\boldsymbol{\Gamma}'(t) = \mathbf{P}_u(u(t), v(t))\frac{du}{dt} + \mathbf{P}_v(u(t), v(t))\frac{dv}{dt}$$

where $\mathbf{P}_u = \frac{\partial \mathbf{P}}{\partial u}$ and $\mathbf{P}_v = \frac{\partial \mathbf{P}}{\partial v}$. The norm squared of $\mathbf{\Gamma}'(t)$ is given by

$$\begin{aligned}|\mathbf{\Gamma}'(t)|^2 &= \mathbf{P}_u^T\mathbf{P}_u u'^2 + \mathbf{P}_u^T\mathbf{P}_v u'v' + \mathbf{P}_v^T\mathbf{P}_u v'u' + \mathbf{P}_v^T\mathbf{P}_v v'^2 \\ &= (eu'^2 + 2fu'v' + gv'^2)\end{aligned}$$

where $e = \mathbf{P}_u^T\mathbf{P}_u, \quad f = \mathbf{P}_u^T\mathbf{P}_v,$ and $\quad g = \mathbf{P}_v^T\mathbf{P}_v$.
In quadratic form,

$$L = \int_a^b \sqrt{\begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}} \mathrm{dt}$$

where we define the *first fundamental form*

$$ds^2 = d\mathbf{P} \cdot d\mathbf{P} = \mathcal{F}^{(1)} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} du \\ dv \end{pmatrix}^T G \begin{pmatrix} du \\ dv \end{pmatrix}$$

with G being the *metric tensor*

$$G = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{P}_u^T\mathbf{P}_u & \mathbf{P}_u^T\mathbf{P}_v \\ \mathbf{P}_v^T\mathbf{P}_u & \mathbf{P}_v^T\mathbf{P}_v \end{pmatrix}$$

surface area

$$dS = |\mathbf{P}_u \mathrm{du} \times \mathbf{P}_v \mathrm{dv}| = |\mathbf{P}_u \times \mathbf{P}_v| dudv$$

$$|\mathbf{P}_u \times \mathbf{P}_v| = \sqrt{(\mathbf{P}_u \times \mathbf{P}_v)^2}$$

by using Lagrange's identity, $\mathbf{|a \times b|^2 = (a \cdot a)(b \cdot b) - (a \cdot b)^2}$ (see appendix).

$$\sqrt{(\mathbf{P}_u \times \mathbf{P}_v)^2} = \sqrt{(\mathbf{P}_u \cdot \mathbf{P}_u)(\mathbf{P}_v \cdot \mathbf{P}_v) - (\mathbf{P}_u \cdot \mathbf{P}_v)^2} = \sqrt{eg - f^2}$$

Thus

$$S = \iint_D \sqrt{eg - f^2} \mathrm{dudv} = \iint_D (\det G)^{\frac{1}{2}} \mathrm{dudv}$$

now the unit normal can be reformulated to become,

$$\mathbf{n} = (\det G)^{-\frac{1}{2}} (\mathbf{P}_u \times \mathbf{P}_v)$$

curvature vectors and the second fundamental form

One could take the representation of a surface $\mathbf{\Gamma}(t)$ and parametrize the curve by arc length s as $\mathbf{p}(s) = (x(s), y(s))$ and form a vector triad $\langle \mathbf{T}, \mathbf{N}, \mathbf{G} \rangle$ moving along the curve (c.f. Frenet-Serret frame in 3d). Where $\widehat{\mathbf{T}}$ is the usual unit tangent or velocity vector, $\widehat{\mathbf{N}}$ is the unit normal vector, $\bar{\mathbf{n}}$ is the unit normal perpendicular to the tangent plane of the surface, $\widehat{\mathbf{G}}$ is a unit vector contained to the tangent plane to the surface at that point and perpendicular to the tangent curve (Figure 2).

$$\begin{aligned}\frac{d\widehat{\mathbf{T}}}{ds} &= \kappa \widehat{\mathbf{N}} \\ &= \kappa_n \bar{\mathbf{n}} + \kappa_g \widehat{\mathbf{G}} \\ \kappa(s)\widehat{\mathbf{N}}(s) &= \kappa(s)_n \bar{\mathbf{n}}(u(s), v(s)) + \kappa(s)_g \widehat{\mathbf{G}}(s) \\ \frac{d\widehat{\mathbf{T}}}{ds} \cdot \bar{\mathbf{n}} &= \kappa_n \bar{\mathbf{n}} \cdot \bar{\mathbf{n}} + 0 = \kappa_n \end{aligned} \tag{1}$$

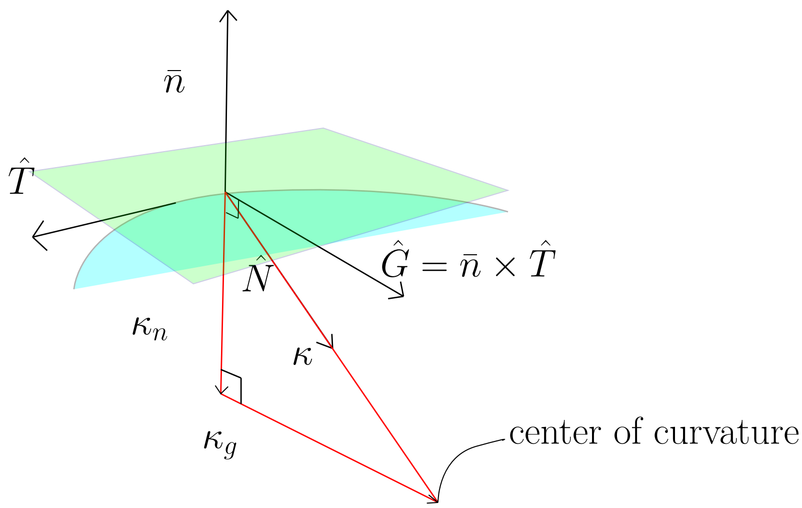


Figure 2: geometry of the surface

$$\frac{d\widehat{\mathbf{T}}}{ds} \cdot \widehat{\mathbf{G}} = 0 + \kappa_g \widehat{\mathbf{G}} \cdot \widehat{\mathbf{G}} = \kappa_g \tag{2}$$

from the perpendicularity of $\widehat{\mathbf{T}}$ and $\bar{\mathbf{n}}$,

$$\frac{d\widehat{\mathbf{T}}}{ds} \cdot \bar{\mathbf{n}} + \widehat{\mathbf{T}} \cdot \frac{d\bar{\mathbf{n}}}{ds} = 0$$

substituting (1),

$$\kappa_n = -\widehat{\mathbf{T}} \cdot \frac{d\bar{\mathbf{n}}}{ds}$$

from (2),

$$\kappa_g = \frac{d\widehat{\mathbf{T}}}{ds} \cdot \widehat{\mathbf{G}}$$

$$\kappa_n = -\frac{d\mathbf{P}(u(s), v(s))}{ds} \cdot \frac{d\bar{\mathbf{n}}}{ds} = -\frac{d\mathbf{P} \cdot d\bar{n}}{ds^2}$$

the total derivative of $P(u, v)$ can be expressed as

$$d\mathbf{P} = \frac{\partial \mathbf{P}}{\partial u} du + \frac{\partial \mathbf{P}}{\partial v} dv$$

similarly for $\bar{\mathbf{n}}(u, v)$,

$$d\bar{\mathbf{n}} = \frac{\partial \bar{\mathbf{n}}}{\partial u} du + \frac{\partial \bar{\mathbf{n}}}{\partial v} dv$$

Thus

$$\kappa_n = \frac{l\, du^2 + 2m\, du\, dv + n\, dv^2}{e\, du^2 + 2f\, du\, dv + g\, dv^2}$$

with

$$l = -\frac{\partial \mathbf{P}}{\partial u} \cdot \frac{\partial \bar{\mathbf{n}}}{\partial u}$$

$$m = -\frac{1}{2} \left(\frac{\partial \mathbf{P}}{\partial u} \cdot \frac{\partial \mathbf{N}}{\partial v} + \frac{\partial \mathbf{P}}{\partial v} \cdot \frac{\partial \mathbf{N}}{\partial u} \right)$$

$$n = -\frac{\partial \mathbf{P}}{\partial v} \cdot \frac{\partial \bar{\mathbf{n}}}{\partial v}$$

The above equations can be reformulated by noting that

$$\frac{\partial \mathbf{P}}{\partial u} \cdot \bar{\mathbf{n}} = 0$$

$$\frac{\partial \mathbf{P}}{\partial v} \cdot \bar{\mathbf{n}} = 0$$

and using integration by parts,

$$\int u\text{dv} = uv - \int v\text{du}$$

for $u = \frac{d\mathbf{P}}{du}$, $dv = \frac{d\mathbf{N}}{du} \cdot du$,

$$l = \frac{\partial^2 \mathbf{P}}{du^2} \cdot \bar{\mathbf{n}},\ m = \frac{\partial^2 \mathbf{P}}{dudv} \cdot \bar{\mathbf{n}},\ n = \frac{\partial^2 \mathbf{P}}{dv^2} \cdot \bar{\mathbf{n}}$$

κ_n can be written as

$$\kappa_n = \frac{\begin{pmatrix} du \\ dv \end{pmatrix}^T L \begin{pmatrix} du \\ dv \end{pmatrix}}{ds^2} = \frac{\mathcal{F}^{(2)}}{\mathcal{F}^{(1)}}$$

where we define the second fundamental form in the numerator as:

$$\mathcal{F}^{(2)} = -d\mathbf{P} \cdot d\bar{\mathbf{n}} = \begin{pmatrix} du \\ dv \end{pmatrix}^T L \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$\text{with} \quad L = \begin{pmatrix} l & m \\ m & n \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \mathbf{P}}{du^2} \cdot \bar{\mathbf{n}} & \frac{\partial^2 \mathbf{P}}{dudv} \cdot \bar{\mathbf{n}} \\ \frac{\partial^2 \mathbf{P}}{dudv} \cdot \bar{\mathbf{n}} & \frac{\partial^2 \mathbf{P}}{dv^2} \cdot \bar{\mathbf{n}} \end{pmatrix}$$

.
The matrix L contains information about how curved the surface is.

mean and Gaussian curvature

Let $\mathcal{P}$ be a plane containing the normal, $\bar{\mathbf{n}}$, to the surface $\mathbb{S}$. Let curve $\mathbb{C}_n$ be at the intersection of plane $\mathcal{P}$ and $\mathbb{S}$. Because curve $\mathbb{C}_n$ is a planar curve, its curvature is in plane $\mathcal{P}$.

Now imagine plane $\mathcal{P}$ rotate about $\bar{\mathbf{n}}$. For each new orientation of the plane, a new curve $\mathbb{C}_n$ is generated with its own normal curvature.

What is the orientation of plane $\mathcal{P}$ that maximized the normal curvature κ_n? In other words, we seek to find the maximum value of $\kappa_n = \mathbf{v}^T L \mathbf{v}$, $\mathbf{v} = d\mathbf{q}/ds$, under normality constraint $\mathbf{v}^T G \mathbf{v} = 1$.

The constrained maximization problem will be solved with the help of Lagrange's multiplier method,

$$\max_{\mathbf{v},\mu} \left[\mathbf{v}^T L \mathbf{v} - \lambda(\mathbf{v}^T G \mathbf{v} - 1)\right]$$

$$\frac{\partial J}{\partial \mathbf{v}} = (L - \lambda G)\mathbf{v} = 0 \tag{3}$$

$$\frac{\partial J}{\partial \mu} = (\mathbf{v}^T G \mathbf{v} - 1) \tag{4}$$

pre-multiplying equation (3) with $\mathbf{v}^T$,

$$\mathbf{v}^T L \mathbf{v} - \lambda \mathbf{v}^T G \mathbf{v} = 0$$

from (4), $\mathbf{v}^T G \mathbf{v} = 1$, substituting

$$\lambda = \mathbf{v}^T L \mathbf{v} = \kappa_n$$

Hence, Lagrange's multiplier λ can be interpreted as the normal curvature itself.
The condition for maximum normal curvature is now given by

$$(L - \kappa_n G)\mathbf{v} = 0$$

the trivial solution $\mathbf{v} = 0$ does not work because it violates the normality constraint. Non-trivial solutions correspond to the eigenvalues and eigenvectors of the eigenproblem $L\mathbf{v} = \kappa_n G\mathbf{v}$. Because L and G are symmetric and G is positive-definite, the eigenvalues are always real, and mutually orthogonal eigenvectors can be constructed. *Note:* $\det(L) \neq 0$ will not always hold (for example, $\det(L) = 0$ for a cylinder or plane).

The eigenvalues are the solution to the characteristic equation $\det(L - \kappa_n G) = 0$, or

$$\kappa_n^2 - 2\kappa_m \kappa_n + \frac{b}{a} = 0$$

where $\kappa_m = (g_{11}l_{22} + g_{22}l_{11} - 2g_{12}l_{12})/2a$. The solutions of this quadratic equation are called principal curvatures

$$\kappa_n^I, \kappa_n^{II} = \kappa_m \pm \sqrt{\kappa_m^2 - b/a}$$

. The mean curvature is defined as

$$\kappa_m = \frac{\kappa_n^I + \kappa_n^{II}}{2} = \frac{g_{11}l_{22} + g_{22}l_{11} - 2g_{12}l_{12}}{2a}$$

and the Gaussian curvature as

$$\kappa_n^I \kappa_n^{II} = \frac{b}{a}$$

when $b/a > 0$, the principal curvatures have the same sign, corresponding to a convex shape. When $/a < 0$, the principal curvatures are of opposite sign, corresponding to a saddle shape. Finally, when $b/a = 0$, one of the principal curvatures is zero, the surface has zero curvature in one of the principal curvature directions.

Note: the Gaussian curvature is an intrinsic or coordinate free surface property i.e. it only depends on the first fundamental form (metric tensor). This can be viewed as Gauss' *Theorema Egregium*, or remarkable theorem. The mean curvature (which has to do with minimal surface theory), on the other hand, is *not* intrinsic.

Christoffel symbols

Now we would like to discuss decomposition formulas of the derivative vector of frame $(\mathbf{P}_u, \mathbf{P}_v, \bar{\mathbf{n}})$. The partial derivative of $\mathbf{P}_u$ and $\mathbf{P}_v$ with respect to u and v can be expressed as the linear combination of themselves and the unit normal vector, $\bar{\mathbf{n}}$, in other words the frame itself. Thus

$$\begin{aligned} \mathbf{P}_{u\mathbf{u}} &= (\Gamma_u)^u_{\mathbf{u}}\mathbf{P}_u + (\Gamma_u)^v_{\mathbf{u}}\mathbf{P}_v + (\Gamma_u)^n_{\mathbf{u}}\bar{\mathbf{n}} \\ &= (\Gamma_u)^u_u\mathbf{P}_u + (\Gamma_u)^v_u\mathbf{P}_v + (\underbrace{\mathbf{P}_{uu} \cdot \bar{\mathbf{n}}}_{l})\mathbf{n} \end{aligned}$$

$$\begin{aligned} \mathbf{P}_{u\mathbf{v}} &= (\Gamma_u)^u_{\mathbf{v}}\mathbf{P}_u + (\Gamma_u)^v_{\mathbf{v}}\mathbf{P}_v + (\Gamma_u)^n_{\mathbf{v}}\bar{\mathbf{n}} \\ &= (\Gamma_u)^u_v\mathbf{P}_u + (\Gamma_u)^v_v\mathbf{P}_v + (\underbrace{\mathbf{P}_{uv} \cdot \bar{\mathbf{n}}}_{m})\mathbf{n} \end{aligned}$$

$$\begin{aligned} \mathbf{P}_{v\mathbf{u}} &= (\Gamma_v)^u_{\mathbf{u}}\mathbf{P}_u + (\Gamma_v)^v_{\mathbf{u}}\mathbf{P}_v + (\Gamma_v)^n_{\mathbf{u}}\bar{\mathbf{n}} \\ &= (\Gamma_v)^u_u\mathbf{P}_u + (\Gamma_v)^v_u\mathbf{P}_v + (\underbrace{\mathbf{P}_{vu} \cdot \bar{\mathbf{n}}}_{m})\mathbf{n} \end{aligned}$$

$$\begin{aligned} \mathbf{P}_{v\mathbf{v}} &= (\Gamma_v)^u_{\mathbf{v}}\mathbf{P}_u + (\Gamma_v)^v_{\mathbf{v}}\mathbf{P}_v + (\Gamma_v)^n_{\mathbf{v}}\bar{\mathbf{n}} \\ &= (\Gamma_v)^u_v\mathbf{P}_u + (\Gamma_v)^v_v\mathbf{P}_v + (\underbrace{\mathbf{P}_{vv} \cdot \bar{\mathbf{n}}}_{n})\mathbf{n} \end{aligned}$$

we identify the coefficients

$$(\Gamma_a)^c_b = \Gamma^c_{ab}$$

by *Einstein*'s index notation

$$\begin{aligned}\mathbf{P}_{ij} &= \Gamma_{ij}^{k}\mathbf{P}_k + \Gamma_{ij}^{n}\bar{\mathbf{n}} \quad (i,j,k=1,2)\\ &= \Gamma_{ij}^{k}\mathbf{P}_k + L_{ij}\bar{\mathbf{n}}\end{aligned}$$

$$\begin{aligned}\langle \mathbf{P}_{ij}, \mathbf{P}_k \rangle &= \Gamma_{ij}^{l}\mathbf{P}_l \cdot \mathbf{P}_k + L_{ij}\bar{\mathbf{n}} \cdot \mathbf{P}_k\\ &= \Gamma_{ij}^{l} g_{lk} + 0\end{aligned}$$

$$\begin{aligned}\langle \mathbf{P}_{ij}, \mathbf{P}_k \rangle\, g^{km} &= \Gamma_{ij}^{l} g_{lk}\, g^{km}\\ &= \Gamma_{ij}^{l}\delta_l^m\\ &= \Gamma_{ij}^{m}\end{aligned}$$

where g^{ij} is used as shorthand for the entries of the inverse of G. That is, $G^{-1} = [g^{ij}]$.

Thus

$$\Gamma_{ij}^{l} = g^{kl}\langle \mathbf{P}_{ij}, \mathbf{P}_k \rangle$$

now, we use Gauss' trick of permuting indices.

$$\begin{aligned}g_{ij,k} = \frac{\partial}{\partial u^k} g_{ij} &= \frac{\partial}{\partial u^k}\langle \mathbf{P}_i, \mathbf{P}_j \rangle\\ &= \langle \frac{\partial \mathbf{P}_i}{\partial u^k}, \mathbf{P}_j \rangle + \langle \mathbf{P}_i, \frac{\partial \mathbf{P}_j}{\partial u^k} \rangle\end{aligned}$$

$$\begin{aligned}&(1) \quad g_{ij,k} = \langle \mathbf{P}_{ik}, \mathbf{P}_j \rangle + \langle \mathbf{P}_i, \mathbf{P}_{jk} \rangle\\ (j \leftrightarrow k) \quad &(2) \quad g_{ik,j} = \langle \mathbf{P}_{ij}, \mathbf{P}_k \rangle + \langle \mathbf{P}_i, \mathbf{P}_{kj} \rangle\\ (i \leftrightarrow j) \quad &((3) \quad g_{ji,k} = \langle \mathbf{P}_{ik}, \mathbf{P}_j \rangle + \langle \mathbf{P}_j, \mathbf{P}_{ki} \rangle\end{aligned}$$

Then (2)+(3)-(1) gives:

$$g_{ik,j} + g_{jk,i} - g_{ij,k} = 2\langle \mathbf{P}_{ij}, \mathbf{P}_k \rangle$$

Finally we arrived at,

$$\Gamma_{ij}^{l} = \frac{1}{2}\sum_k g^{kl}(g_{ik,j} + g_{jk,i} - g_{ij,k})$$

because the metric tensor is symmetric,

$$\Gamma_{ij}^{l} = \frac{1}{2}\sum_k g^{lk}(g_{ki,j} + g_{kj,i} - g_{ij,k})$$

These are known as Christoffel symbols of the second kind.

Appendix: Lagrange's identity

$$\begin{aligned}[(a \times b) \times c]_i &= \epsilon_{ijk}(a \times b)_j c_k\\ &= \epsilon_{ijk}\epsilon_{jmn} a_m b_n c_k\\ &= (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn}) a_m b_n c_k\\ &= \delta_{in}\delta_{km} a_m b_n c_k - \delta_{im}\delta_{kn} a_m b_n c_k\end{aligned}$$

$$= a_k b_i c_k - a_i b_k c_k$$

Hence,

$$\widetilde{\tilde{\mathbf{a}}\mathbf{b}}\mathbf{c} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

similarly,

$$\begin{aligned}
[(a \times (b \times c)]_i &= \epsilon_{ijk} a_j (b \times c)_k \\
&= \epsilon_{ijk} \epsilon_{kmn} a_j b_m c_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) a_j b_m c_n \\
&= \delta_{im}\delta_{jn} a_j b_m c_n - \delta_{in}\delta_{jm} a_j b_m c_n \\
&= a_n b_i c_n - a_m b_m c_i
\end{aligned}$$

Hence,

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}}\mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b})\mathbf{c}$$

Thus,

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} = (\mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b})$$

$$\|(\mathbf{a} \times \mathbf{b})\|^2 = (\tilde{\mathbf{a}}\mathbf{b})^T(\tilde{\mathbf{a}}\mathbf{b}) = \mathbf{b}^T\tilde{\mathbf{a}}^T\tilde{\mathbf{a}}\mathbf{b} = -\mathbf{b}^T\tilde{\mathbf{a}}\tilde{\mathbf{a}}\mathbf{b} = -\mathbf{b}^T(\mathbf{a}\mathbf{a}^T - \mathbf{a}^T\mathbf{a})\mathbf{b} = -(\mathbf{a}^T\mathbf{b})^2 + \|\mathbf{a}\|^2\|\mathbf{b}\|^2$$

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