**proving vector calculus identities using Einstein's index summation**

$$\epsilon_{lmi}\epsilon_{jki} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}$$

$$\nabla(\phi\vec{v}) = div(\phi\vec{v}) = \partial_i(\phi v_i) = v_i\partial_i(\phi) + \phi\partial_i(v_i)$$

$$= v \cdot grad(\phi) + \phi div(\vec{v})$$

$$\nabla\cdot(\vec{u}\times\vec{v}) = div(\vec{u}\times\vec{v}) = \partial_i(\epsilon_{ijk}u_jv_k) = v_k\epsilon_{ijk}\partial_iu_j + u_j\epsilon_{ijk}\partial_iv_k = v_k\epsilon_{kij}\partial_iu_j - u_j\epsilon_{jik}\partial_iv_k$$

$$= \vec{v}\cdot curl(\vec{u}) - \vec{u}\cdot curl(\vec{v})$$

$$\nabla\times(\vec{u}\times\vec{v}) = curl(\vec{u}\times\vec{v}) = \epsilon_{lmi}\partial_m\epsilon_{ijk}u_jv_k = \epsilon_{lmi}\epsilon_{ijk}(v_k\partial_mu_j + u_j\partial_mv_k)$$

$$= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})(v_k\partial_mu_j + u_j\partial_mv_k)$$

$$= \delta_{lj}\delta_{mk}v_k\partial_mu_j + \delta_{lj}\delta_{mk}u_j\partial_mv_k - \delta_{lk}\delta_{mj}v_k\partial_mu_j - \delta_{lk}\delta_{mj}u_j\partial_mv_k$$

$$= v_k\partial_ku_l + u_l\partial_kv_k - v_l\partial_ju_j - u_j\partial_jv_l$$

$$= \vec{v}\cdot grad(\vec{u}) - \vec{u}\cdot grad(\vec{v}) + \vec{u}div(\vec{v}) - \vec{v}div(\vec{u})$$

$$\nabla(\vec{u}\cdot\vec{v}) = grad(u\cdot v) = \partial_i(u_jv_j) = u_j\partial_iv_j + v_j\partial_iu_j$$

$$= u_j\partial_iv_j + v_j\partial_iu_j + u_j\partial_jv_i + v_j\partial_ju_i - u_j\partial_jv_i - v_j\partial_ju$$

$$= u_j(\partial_iv_j - \partial_jv_i) + v_j(\partial_iu_j - \partial_ju_i) + u_j\partial_jv_i + v_j\partial_jv_i$$

$$= u_j\begin{vmatrix}\partial_i & \partial_j\\ v_i & v_j\end{vmatrix}_k + v_j\begin{vmatrix}\partial_i & \partial_j\\ u_i & u_j\end{vmatrix}_k + u_j\partial_jv_i + v_j\partial_jv_i$$

$$= \epsilon_{ijk}u_j\begin{vmatrix}\partial_i & \partial_j\\ v_i & v_j\end{vmatrix}_k + \epsilon_{ijk}v_j\begin{vmatrix}\partial_i & \partial_j\\ u_i & u_j\end{vmatrix}_k + u_j\partial_jv_i + v_j\partial_uv_i$$

$$= \vec{u}\times curl(\vec{v}) + \vec{v}\times curl(\vec{u}) + \vec{u}\cdot grad(\vec{v}) + \vec{v}\cdot grad(\vec{u})$$

$$\nabla\cdot(\nabla\times\vec{v}) = div(curl(\vec{v}))$$

$$= \partial_i\epsilon_{ijk}\partial_jv_k = \epsilon_{ijk}\partial_i\partial_jv_k = \epsilon_{ijk}\partial_{ij}v_k = 0$$

Observe that $\epsilon_{ijk}$ is antisymmetric but $\partial_i\partial_j$ is symmetric. For example, if $k = 3$,

$$\partial_{ij}v_3 - \partial_{ji}v_3 = 0$$

with the same line of reasoning,

$$\nabla\times(\nabla\phi) = curl(grad(\phi)) = \epsilon_{ijk}\partial_j\partial_k\phi = \epsilon_{ijk}\partial_{jk}\phi = 0$$

bonus:

$$\vec{u}\times(\nabla\times\vec{u}) = \epsilon_{lmi}u_m\epsilon_{ijk}\partial_ju_k$$

$$= \epsilon_{lmi}\epsilon_{ijk}u_m\partial_ju_k$$

$$= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})u_m\partial_ju_k$$

$$= u_k\partial_lu_k - u_j\partial_ju_l$$

$$= \frac{1}{2}\nabla(u\cdot u) - u\cdot\nabla u$$

given $\nabla(\vec{u}\cdot\vec{u}) = \partial_i(u_ju_j) = 2u_j\partial_iu_j$.