# Formale Semantik 04. Aussagenlogik

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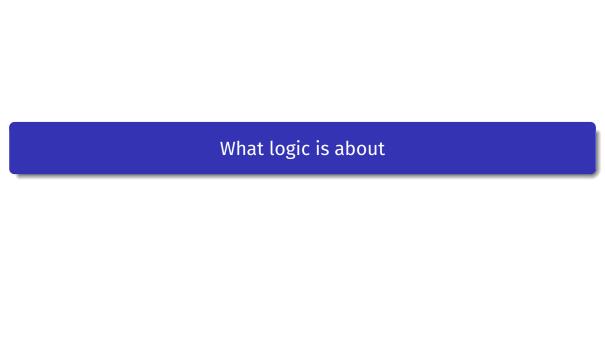
stets aktuelle Fassungen: https://github.com/rsling/VL-Deutsche-Syntax

### Inhalt

- 1 What logic is about
  - On reasoning
  - Where we need logic
- 2 Statement calculus

- Formalization: Recursive Syntax
- Interpretation
- Laws of the PropC
- Rules of Inference
- Proof

The book (PMW:87-246) deals with logic far more in-depth than we do. Only what is mentioned on the slides is relevant for the test. Reading the whole chapter from PMW will do you no harm, though.



#### **Theories**

- a collection of statements (propositions)
- axioms (statements accepted to be true)
- maybe based on observations (induction)
- statements that follow from the axioms (deduction)
- predictions beyond the axioms
- rechecking for usability: e.g., Russell's paradox

#### **Proofs**

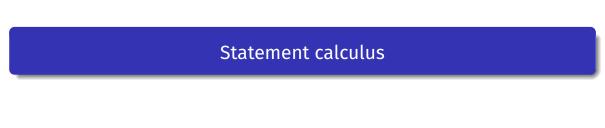
- axioms: atomic truths of your theory
- theorem: a proposition you want to prove
- lemma: subsidiary propositions (used to prove the theorem)
- corollary: propositions proved while proving some axiom

# A method of reasoning

- logic does not generate truths
- formalizing statements, predications etc.
- rules of deduction from axioms to theorems
- empirical (induction) and exact (deduction) science
- aiming at an adequate model of the world (e.g., heliocentric universe)

# Why logic for semantics?

- truth-conditional
- compositional behavior of propositions and connectives
- a logic for entailments
- ullet why, e.g.: It is not the case that someone is happy. o Nobody is happy.



#### Atomic formulas: statements

- statements/propositions = the atoms
- a propositional symbol p: a well-formed formula (wff)
- ex.: Herr Keydana is a passionate cyclist.: k
- [k]=1 or o (depending on corresponding **model**)

# Complex (molecular) formulas

- syntax: restricts the forms of wff's to make them interpretable
- define functors: functions in  $\{0,1\}$
- If p and q are wff's, then
  - → ¬p
  - ▶ p∨q
  - ▶ p∧q
  - p → q
  - $\triangleright p \leftrightarrow q$

is also a wff (a molecular term).

# Complex (molecular) formulas

- syntax: restricts forms of wff's to make them interpretable
- define functors: functions in  $\{\langle 0,1\rangle, \langle 1,0\rangle, 0,1\}$
- If p and q are wff's, then
  - ▶ ¬p (negation)
  - ▶ p∨q (disjunction)
  - ▶ p∧q (conjunction)
  - ▶  $p \rightarrow q$  (conditional)
  - ▶  $p \leftrightarrow q$  (biconditional)

is also a wff.

# Complex (molecular) formulas

- syntax: restricts forms of wff's to make them interpretable
- define functors: functions in  $\{\langle 0,1\rangle, \langle 1,0\rangle, 0,1\}$
- If p and q are wff's, then
  - ▶ ¬p (negation 'not')
  - ▶ p∨q (disjunction 'or')
  - p∧q (conjunction 'and')
  - ▶  $p \rightarrow q$  (conditional 'if')
  - ▶  $p \leftrightarrow q$  (biconditional 'iff')

is also a wff.

#### Functions and truth tables

standard defintion:

$$\llbracket \neg \rrbracket = \left[ \begin{array}{c} 1 \to 0 \\ 0 \to 1 \end{array} \right]$$

• but most widely used: truth tables

# Disjunction

р	$\vee$	q
1	1	1
1	1	0
0	1	1
0	0	0

- Herr Keydana is a passionate cyclist **or** we all love logic.
- *K*∨L

# Conjunction

р	$\land$	q
1	1	1
1	0	0
0	0	1
0	0	0

- Herr Keydana is a passionate cyclist **and** we all love logic.
- K∧L

## Conditional

р	$\rightarrow$	q
1	1	1
1	0	0
0	1	1
0	1	0

- *If* it <u>rains</u>, **then** the <u>s</u>treets get wet.
- $\bullet$   $R \rightarrow S$

# Any problems with that?

#### If it rains, the streets get wet.

- it is raining (1), the streets are wet 1:1
- it is raining (1), the streets are dry 0: 0
- it is not raining (o), the streets are wet 1:1
- it is not raining (o), the streets are dry 0:1
- ex vero non sequitur falsum

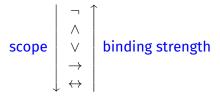
### **Biconditional**

р	$\leftrightarrow$	q
1	1	1
1	0	0
0	0	1
0	1	0

- If and only if your score is above 50, then you pass the semantics exam.
- $S \leftrightarrow P$

# Scope of functors

- brackets are facultative
- or set non-default functor scope
- default scope



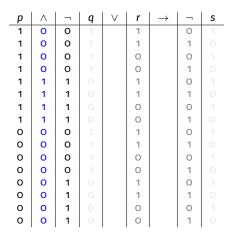
- $p \land \neg q \lor r \rightarrow \neg s$
- $p \wedge (\neg q) \vee r \rightarrow (\neg s)$
- $(p \land (\neg q)) \lor r \rightarrow (\neg s)$
- $((p \land (\neg q)) \lor r) \rightarrow (\neg s)$
- $(((p \land (\neg q)) \lor r) \rightarrow (\neg s))$

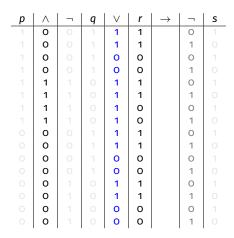
## Large truth tables

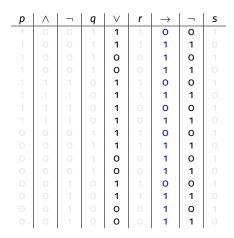
- for n atoms in the term:  $2^n$  lines
- alternating blocks of 1's and 0's under every atom
- $2^{(m-1)}$  times '1' followed by  $2^{(m-1)}$  times '0' for the m-th atom from the right
- until  $2^n$  lines are reached

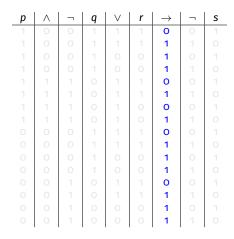
р	$\wedge$	_	q	V	r	$\rightarrow$	_	s
1			1		1			1
1			1		1			0
1			1		0			1
1			1		0			0
1			0		1 1			1
1			0					0
1			0 0		0			1
1			0		0			0
0			1		1 1			1
0			1		1			0
0			1		0			1
0			1		0			0
0			0		1 1			1
0			0					0
1 1 1 1 1 1 0 0 0 0 0			0 0		0			1
0			0		0			0

р	$\wedge$	_	q	V	r	$\rightarrow$	_	s
1		0	1		1		0	1
1		0	1		1		1	0
1		0	1		0		0	1
1		0 0 1 1	1		0		0	0
1		1	0		1		0	1
1			0		1		1	0
1		1	0 0		0 0 1 1 0 0 1 1 0 0		1 0 1	1
1		1	0		0		1	0
0		0	1		1		0	1
0		0 0 0 1 1	1		1		1 0 1	0
0		0	1		0		0	1
0		0	1		0		1	0
0		1	0		1		0 1	1
0		1	0		1			0
1 1 1 1 1 1 1 0 0 0 0		1	0		0		0	1
Ο		1	0		0		1	0









# Assignments: a contingent example

р	$\wedge$	_	q	V	r	$\rightarrow$	_	s
1	0	0	1	1	1	0	0	1
1	0	0	1	1	1	1	1	0
1	0	0	1	0	0	1	0	1
1	0	0	1	0	0 1 1	1	1 0	0
1	1	1		1	1	0	0	1
1	1	1	0	1	1	1	1	0
1	1	1	0	1	0	1 0 1 0 1	1 0 1 0	1
1	1	1	0	1	0	1	1	0
0	0	0	1	1	1			1
0	0	0	1	1		1 1 1	1	0
0	0	0	1	0	0	1	0	1
0	0	0		0		1		0
0	0 0 0	1	0	1	0 1 1	0	1 0 1	1
0	0	1	0	1	1		1	0
1 1 1 1 1 1 0 0 0 0	0	1	0	0	0	1	0	1
0	0	1	0	0	0	1	1	0

# **Tautology**

- take  $p \vee \neg p$
- truth-table:  $\begin{array}{c|cccc} p & \lor & \neg & p \\ \hline 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \end{array}$
- true under every assignment, it is valid
- by law of excluded middle: for every P, P  $\lor \neg$ P is true

#### Contradiction

• take  $p \land \neg p$ 

	р	$\land$	$\neg$	р
• truth-table:	1	0	0	1
	0	0	1	0

• false under every assignment, called contradictory

# Contingency

- take  $p \wedge p$
- truth-table:  $\begin{array}{c|ccc} p & \land & p \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \end{array}$
- the truth value depends on the assignemt

#### What are laws?

- notice: similarities of set theory and logic
- non-trivial exact nature of their equivalence
- laws state equivalences of (types of) wff
- truth-conservative rewriting of wff's
- any subformula which is a tautology (T) or contradiction (F):

ignore by Identity Laws (Id.):

- $P \lor F) \Leftrightarrow P, (P \lor T) \Leftrightarrow T$
- $P \land F) \Leftrightarrow F, (P \land T) \Leftrightarrow P$

# Equivalences: ⇔

- X ⇔ Y: X has the same truth-conditions as Y
- derivability of laws and rules (convenient redundancies)
- Idempotency (Idemp.):
  - $\triangleright (P \lor P) \Leftrightarrow P$
  - $\triangleright$   $(P \land P) \Leftrightarrow P$
  - ▶ Peter walks and Peter walks. ⇔ Peter walks.

# Simple laws

- Associative Laws for ∨ and ∧ (Assoc.):
  - $\qquad \qquad \bullet \ \, ((P \lor Q) \lor R) \Leftrightarrow (P \lor (Q \lor R))$
  - ► ((He walks or she talks) or we walk.) ⇔ (He walks or (she talks or we walk.))
- Commutative Laws for ∨ and ∧ (Comm.):
  - $\blacktriangleright (P \lor Q) \Leftrightarrow (Q \lor P)$
  - ▶ Peter walks or Sue snores. ⇔ Sue snores or Peter walks.
- Distributive Laws for ∨∧ and ∧∨ (Distr.):
  - $\blacktriangleright (P \lor (Q \land R)) \Leftrightarrow ((P \lor Q) \land (P \lor R))$
  - (Sue snores) and (Peter walks or we talk).
    - $\Leftrightarrow$  (Sue snores and Peter walks) or (Sue snores and we talk).

# Laws dealing with tautology and contradiction

#### Complement Laws:

- ▶ Tautology (T):  $(P \lor \neg P) \Leftrightarrow \mathbf{T}$
- ▶ Contradiction (F):  $(P \land \neg P) \Leftrightarrow \mathbf{F}$
- ▶ Double Negation (DN):  $(\neg \neg P) \Leftrightarrow P$
- It is not the case that Sandy is not walking.
  ⇔ Sandy is walking.

### **Conditionals Laws**

• Implication (Impl.):

Ρ	$\rightarrow$	Q	$\Leftrightarrow$		Ρ	$\vee$	_
1	1	1		0	1	1	1
1	0	0		0	1	0	0
0	1	1		1	0	1	1
0	1	0		1	0	1	0

• Contraposition (Contr.):

Ρ	$\rightarrow$	Q	$\Leftrightarrow$	¬	Q	$\rightarrow$	$\neg$	Ρ
1	1	1		0	1	1 0 1 1	0	1
	0			1	0	0	0	1
0	1	1		0	1	1	1	0
0	1	0		1	0	1	1	0

# DeMorgan (DeM)

- DeMorgan's Laws:
  - $\qquad \qquad \neg (P \lor Q) \Leftrightarrow (\neg P \land \neg Q)$
  - ▶ alternatively:  $\overline{P \lor Q} \Leftrightarrow \overline{P} \land \overline{Q}$
  - $\qquad \neg (P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$
  - consequently:  $\overline{\overline{P} \vee \overline{Q}} \Leftrightarrow \overline{\overline{P}} \wedge \overline{\overline{Q}} \Leftrightarrow P \wedge Q$

## The Modus Ponens (MP)

• Definition:

Deminicion:					
P	, –	→ C	Ç	premise 1	
P	•			premise 2	
		C	)	conclusion	

- or:  $(P \rightarrow Q) \land (P) \rightarrow (Q)$
- (1) If It rains, the streets get wet. (2) It is raining.
  - $\rightarrow$  The streets are getting wet.

### MP: a truth table illustration

- Premises are always set to be true!
- the table:
- $P \rightarrow C$ 
  - . . .
  - 1 0 0
  - 011
  - 0 1 C

### MP: a truth table illustration

- The conditional must be true.
- cancel the 'false' row

 $P \rightarrow C$ 

1 1 1

1 0 0

011

0 1 C

#### MP: a truth table illustration

- P must be true.
- cancel the 'false' rows, Q can only be true:

 $P \rightarrow C$ 

1 1

1 0 0

0 1 1

0 1 0

### The Modus Tollens (MT)

Definition:



• the table illustration:

```
P → Q

1 1 1 (by premise 2)

1 0 0 (by premise 1)

0 1 1 (by premise 2)

0 1 0
```

## The Syllogisms

- Hypothetical Syllogism (HS):
  - $\blacktriangleright ((P \to Q) \land (Q \to R)) \to (P \to R)$
  - (1) If it rains, the streets get wet. (2) If the streets get wet, it smells nice. → If it rains, it smells nice.
- Disjunctive Syllogism (DS):
  - $\blacktriangleright ((P \lor Q) \land (\neg P)) \to (Q)$
  - ▶ (1) Either Peter sleeps or Peter is awake. (2) Peter isn't awake.
    - ightarrow Peter sleeps.

### Trivial rules

- Simplification (Simp.):
  - $(P \land Q) \rightarrow P$
  - (1) It is raining and the sun is shining.  $\rightarrow$  It is raining.
- Conjunction (Conj.):
  - $\blacktriangleright (P) \land (Q) \rightarrow (P \land Q)$
  - lacksquare (1) It is raining. (2) The sun is shining. o It is raining and the sun is shining.
- Addition (Add.):
  - $\blacktriangleright \ (P) \to (P \land Q)$
  - lacksquare (1) It is raining. o It is raining or the sun is shining.
  - What if Q is instantiated as true or false by another premise?

# A sample proof

- Prove  $p \lor q$  from  $(p \lor q) \to \neg (r \land \neg s)$  and  $r \land \neg s$
- The proof:

$$\begin{array}{ccc} & & & & & & & \\ 1 & (p \lor q) \to \neg (r \land \neg s) & & & \\ 2 & r \land \neg s & & & & \\ \hline & p \lor q & & & 1,2,MT \end{array}$$

## Literatur I

#### Autor

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