# Formale Semantik o3. Mengen und Funktionen

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stets aktuelle Fassungen: https://github.com/rsling/VL-Deutsche-Syntax

#### Inhalt

- 1 Sets and Functions
  - The naive concept
  - Elements, subsets, power sets
  - Union, intersection, etc.
- 2 Functions and Relations
  - Ordered pairs/sets, n-tuples, Cartesian products
  - Relations

- Functions
- 3 More about relations and sets
  - Relations among themselves
  - Orders
- 4 Cardinalities
  - Denumerability
  - Non-denumerability



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- U: the universal set (contains every discrete object)

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{x:x is human} = {x:x is from the planet earth and x can speak}

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- the inverse: the superset

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- $\bullet \ \{\{a\}\} \not\in \{a,b,c\}$
- $\{\} \subset \{a,b,c\}$  (or any set),  $\{\}$  is sometimes written  $\emptyset$

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- Hence: \*Herr Webelhuth is numerous.

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- ...and why is the empty a set a proper subset of every set?

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- as a general principle (Consitency):  $M \subseteq N$  iff  $M \cup N = N$  and  $M \subseteq N$  iff  $M \cap N = M$

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- the universal complement:  $M' = \{x | x \in U \text{ and } x \notin M\}$  (U the universal set)

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- Identity:  $M \cup \emptyset = X$ ,  $M \cup U = U$  ...what about  $\cap$

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- DeMorgan:  $(M \cup N)' = M' \cap X' \dots$



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- for  $S \times S \times \cdots$ : n-fold products  $S^n = \{\vec{s} | | s_i \in S \text{ for } 1 \le i \le n\}$

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  - ▶  $R^{-1}$  = all pairs  $\langle b, a \rangle$  where a is the teacher of b: Herr Schäfer is the inverse-teacher of Herr Webelhuth.

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- partial function from A to B: for some  $a \in A$  there is no tuple  $\langle a, b \rangle \in A \times B$ , F is not defined for some a

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- one-to-one, onto, and total function: correspondence (bijection)

# Composition

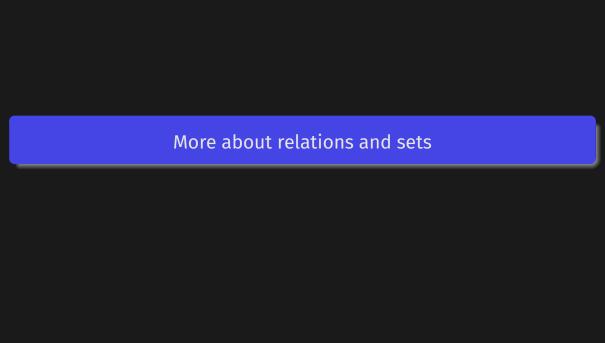
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- the compound function can be empty, it will be total if both A and B are bijections.



# Reflexivity

	if	(ex.)
reflexive	for <b>every</b> $a \in A$ : $\langle a, a \rangle \in R$	is as heavy as
irreflexive non-reflexive	for <b>every</b> $a \in A$ : $\langle a, a \rangle \not\in R$ for <b>some</b> $a \in A$ : $\langle a, a \rangle \not\in R$	A: physical objects is the father of has hurt

## **Symmetry**

	if	(ex.)
symmetric	for every $\langle a,b \rangle \in R$ :	has the same car as
	$\langle b,a  angle \in R$	
asymmetric	for every $\langle a,b angle \in R$ :	has a different car than
	$\langle b,a\rangle  ot\in R$	
non-symmetric	for some $\langle a,b \rangle \in R$ :	is the sister of
	$\langle b,a\rangle \not\in R$	
anti-symmetric	for every $\langle a,b\rangle\in R$ : $a=b$	beats oneself
		not every human does

# Transitivity

	if	(ex.)
transitive	if $\langle a,b \rangle \in R$ and $\langle b,c \rangle \in R$	is to the left of
	then $\langle a,c \rangle \in R$	
intransitive	the above is never the case	is the father of
non-transitive	the above is sometimes not the case	likes

#### Connectedness

	if	(ex.)
connected	for every $a, b \in A$ , $a \neq b$ :	>
	either $\langle a,b\rangle\in R$ or $\langle b,a\rangle\in R$	(A: the natural numbers)
non-connected	for some $a, b \in A$ the above is not the case	likes

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• not {{a}, {b, c}} or {{a, b}, {b, c}, {d}}

# Defining ordering relations

An ordering relation R in A is ...

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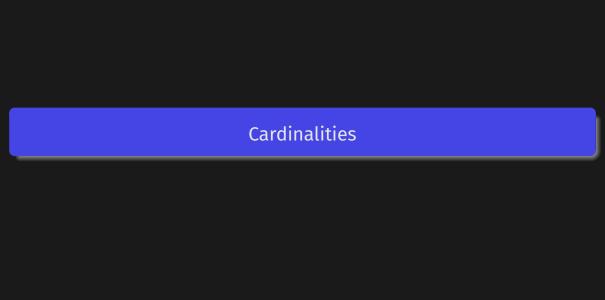
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- well-ordering: total order, every subset has a least element



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- such relations are one-to-one correspondences

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- sets A,B with ||A|| = ||B|| are equivalent
- $\|\mathbb{N}\| = \aleph^0$

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- for some sets there is no such R<sub>care</sub>
- no way of bringing their elements into an exhaustive linear order
- no problem with  $\mathbb{Q}$ :

- $\langle 0, 1 \rangle$
- $\langle 0, 2 \rangle$
- $\langle 0, 3 \rangle$

. . .

 $\langle 1, 0 \rangle$ 

- $\langle 1, 1 \rangle$
- $\langle 1, 2 \rangle$
- $\langle 1, 3 \rangle$
- • •

- $\langle 2, 0 \rangle$
- $\langle 2, 1 \rangle$
- $\langle 2, 2 \rangle$
- $\langle 2, 3 \rangle$

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:

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### The non-denumerable real numbers

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- now:  $\mathbb{R}$
- ullet some elements cannot be represented as an ordered pair of two elements of  ${\mathbb N}$
- in [0,1], every real can be represented as 0.abcdefg...,  $a,b,c,d,e,f,g,... \in \{0,1,2,3,4,5,6,7,8,9\}$

# Trying to enumerate

• an enumeration of [0,1] in  $\mathbb{R}$ ?

```
X_1
                    . a<sub>11</sub>
                                      a<sub>12</sub>
                                                a<sub>13</sub>
                                                           a<sub>14</sub>
        = 0 . a_{21} a_{22} a_{23}
X_2
                                                          a<sub>24</sub>
                                                                     •••
              O . a_{31} a_{32} a_{33}
                                                          a<sub>34</sub>
X_3
                                                                     •••
Xn
               0
                           a<sub>n1</sub>
                                     a_{n2}
                                                a<sub>n3</sub>
                                                           a_{n4}
```

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• What about an  $x_m$  which differs from  $x_n$  at  $a_{nn}$ 

- It won't be in the array...
- R is non-denumerable
- If  $||A|| = \aleph^0$  then  $||\wp(A)|| = 2^{\aleph_0}$  (cf. Partee et al. 62f.)

# Literatur I

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