

# Formale Semantik

## 03. Mengen und Funktionen

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**Achtung: Folien in Überarbeitung. Englische Teile sind noch von 2007!**  
Stets aktuelle Fassungen: <https://github.com/rsling/VL-Semantik>

## 1 Sets and Functions

- The naive concept
- Elements, subsets, power sets
- Union, intersection, etc.

## 2 Functions and Relations

- Ordered pairs/sets, n-tuples, Cartesian products
- Relations

- Functions

## 3 More about relations and sets

- Relations among themselves
- Orders

## 4 Cardinalities

- Denumerability
- Non-denumerability

## Sets and Functions

# What is a set?

- a freely defined unordered collection of discrete objects
  - ▶ numbers,
  - ▶ people,
  - ▶ pairs of shoes,
  - ▶ words, ...
- not necessarily for any purpose
- no object occurs more than once

# Set definition and elements: $\in$

- $M_1 = \{a, b, c\}$
- $N_1 = \{\text{'my book'}\}$   
vs.  $N_2 = \{\text{my book}\}$   
vs.  $N_3 = \{\text{'my'}, \text{'book'}\}$
- ill-formed:  $N_4 = \{\text{'my'}, \text{book}\}$
- defined by a property of its members:  
 $M_2 = \{x : x \text{ is one of the first three letters of the alphabet}\}$
- alternatively:  
 $M_2 = \{x \mid x \text{ is one of the first three letters of the alphabet}\}$
- $U$ : the universal set (contains every discrete object)

# Equality: =

- Two sets with contain exactly the same members are *equal*.
- independent of definition:  
 $\{a,b,c\} =$   
 $\{x:x \text{ is one of the first three letters of the alphabet}\}$
- $\{x:x \text{ is human}\} = \{x:x \text{ is from the planet earth and } x \text{ can speak}\}$

# Subsets: $\subseteq$

- A set  $N$  which holds no member which is not in  $M$  is a *subset* of  $M$ :  $N \subseteq M$
- $\{a\} \subseteq \{a, b, c\}$
- the inverse: the *superset*

## Proper subsets: $\subset$

- A set  $N$  which holds no member which is not in  $M$  and which is not equal to  $M$  is a *proper subset* of  $M$ :  $N \subset M$
- So, actually:  $\{a\} \subset \{a, b, c\}$  and  $\{a, b, c\} \subseteq \{a, b, c\}$ . Note that:
- $M \subseteq M$  but  $M \not\subset M$
- $\{\{a\}\} \not\subseteq \{a, b, c\}$
- $\{\} \subset \{a, b, c\}$  (or any set),  $\{\}$  is sometimes written  $\emptyset$



- *All professors of English Linguistics are human.  
Herr Webelhuth is a professor of English Linguistics.*
- $w$  = Herr Webelhuth  
E = the set of professors of English Linguistics  
H = the set of human beings
- $w \in E \ \& \ E \subset H \Rightarrow w \in H$

- But: *Professors of English Linguistics are numerous.*
- $N$  = the set of sets with numerous members
- $w \in E \ \& \ E \in N \not\Rightarrow w \in P$
- Hence: \*Herr Webelhuth is numerous.

- For any set  $M$ :  $\wp(M) = \{X \mid X \subseteq M\}$
- for  $M = \{a, b, c\}$ :  
 $\wp(M) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}, \{b, c\}\}$
- Why is the empty set in the power set of every set ...
- ...and why is the empty set a proper subset of every set?

# Union $\cup$ and intersection $\cap$

- For any sets  $M$  and  $N$ :  $M \cup N = \{x \mid x \in M \text{ **or** } x \in N\}$
- if  $M = \{a, b, c\}$  and  $N = \{a, b, d\}$  then  $M \cup N = \{a, b, c, d\}$
- For any sets  $M$  and  $N$ :  $M \cap N = \{x \mid x \in M \text{ **and** } x \in N\}$
- if  $M = \{a, b, c\}$  and  $N = \{a, b\}$  then  $M \cap N = \{a, b\}$
- as a general principle (Consistency):  $M \subseteq N$  iff  $M \cup N = N$  and  $M \subseteq N$  iff  $M \cap N = M$

# Generalized union $\bigcup$ and intersection $\bigcap$

- $\bigcup M = \{x \mid x \in Y \text{ for some } Y \in M\}$
- (a) if  $M = \{\{a\}, \{a, b\}, \{a, b, c\}\}$  then  $\bigcup M = \{a, b, c\}$
- (b)  $M_1 = \{a\}, M_2 = \{a, b\}, M_3 = \{a, b, c\}, I = \{1, 2, 3\}; \bigcup_{i \in I} M_i = \{a, b, c\}$
- $\bigcap M = \{x \mid x \in Y \text{ for every } Y \in M\}$
- (a) if  $M = \{\{a\}, \{a, b\}, \{a, b, c\}\}$  then  $\bigcap M = \{a\}$
- (b)  $M_1 = \{a\}, M_2 = \{a, b\}, M_3 = \{a, b, c\}, I = \{1, 2, 3\}; \bigcap_{i \in I} M_i = \{a\}$

# Difference - and complement \ and '

- For any two sets  $M$  and  $N$ :  $M - N = \{x \mid x \in M \text{ and } x \notin N\}$
- $M = \{a, b, c\}$ ,  $N = \{a\}$ ,  $M - N = \{b, c\}$
- For any two sets  $M$  and  $N$ :  $M \setminus N = \{x \mid x \in N \text{ and } x \notin M\}$
- $O = \{a, b, c, k\}$   $M \setminus O = \{k\}$
- the universal complement:  $M' = \{x \mid x \in U \text{ and } x \notin M\}$   
( $U$  the universal set)

# Trivial equalities

- Idempotency:  $M \cup M = M, M \cap M = M$
- Commutativity for  $\cup$  and  $\cap$ :  $M \cup N = N \cup M \dots$
- Associativity for  $\cup$  and  $\cap$ :  $(M \cup N) \cup O = M \cup (N \cup O) \dots$
- Distributivity for  $\cup$  and  $\cap$ :  $M \cup (N \cap O) = (M \cup N) \cap (M \cup O) \dots$
- Identity:  $M \cup \emptyset = X, M \cup U = U \dots$  what about  $\cap$

## More interesting equalities

- Complement laws:  $M \cup \emptyset = M$ ,  $M'' = M$ ,  $M \cap M' = \emptyset$ ,  $X \cap U = U$
- DeMorgan:  $(M \cup N)' = M' \cap N'$  ...



## Functions and Relations

# How to define an ordered pair

- ...without introducing ordered tuples as a new primitive
- take  $S = \{\{a\}, \{a, b\}\}$
- we write:  $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$
- ordered n-tuples defined recursively
- $\langle a, b \rangle \neq \langle b, a \rangle$
- first and second coordinate of the tuple

- sets of ordered pairs
- tupling each member of the first argument with each of the second
- $S_1 \times S_2 = \{\langle x, y \rangle \mid x \in S_1 \text{ and } y \in S_2\}$
- for an arbitrary number of sets:  $S_1 \times \cdots \times S_n = \{\langle x_1, x_2, \dots, x_n \rangle \mid x_i \in S_i\}$
- $\langle x_1, x_2, \dots, x_n \rangle$  abbreviated  $\vec{x}$
- for  $S \times S \times \cdots$ : n-fold products  
 $S^n = \{\vec{s} \mid s_i \in S \text{ for } 1 \leq i \leq n\}$

# Defintion of relations

- hold between (sets of) objects
- *x kicks y, x lives on the same floor as y, ...*
- formalization:  $Rab$ ,  $aRb$
- $a \in A$  and  $b \in B$ :  $R \subseteq A \times B$ ,  
R is from A (**domain**) to B (**range**)
- R from A to A is **in** A

- complement  $R' = \{\langle a, b \rangle \notin R\}$  for  $R \subseteq A \times B$ 
  - ▶  $R$  = the relation of teacherhood between  $a$  and  $b$  (the **arguments**)
  - ▶  $R'$  = all pairs  $\langle b, a \rangle$  s.t. it is false that the first member is the teacher of the second member
- inverse:  $R^{-1} = \{\langle b, a \rangle \mid \langle a, b \rangle \in R\}$  for  $R \subseteq A \times B$ 
  - ▶  $R$  = the relation of teacherhood between  $a$  and  $b$ :  
*Herr Webelhuth is the teacher of Herr Schäfer.*
  - ▶  $R^{-1}$  = all pairs  $\langle b, a \rangle$  where  $a$  is the teacher of  $b$ :  
*Herr Schäfer is the inverse-teacher of Herr Webelhuth.*

- A function  $F$  from  $A$  to  $B$  is a relation s.t. for every  $a \in A$  there is exactly one tuple  $\langle a, b \rangle \in A \times B$  s.t.  $a$  is the first coordinate.
- partial function from  $A$  to  $B$ : for some  $a \in A$  there is no tuple  $\langle a, b \rangle \in A \times B$ ,  $F$  is not *defined* for some  $a$

# Injection, surjection, bijection

- B the range of F, F is **into** B
- F from A to B is **onto (a surjection)** B iff there is no  $b_i \in B$  s.t. there is no  $\langle a, b_i \rangle \in F$
- F from A to B is **one-to-one (an injection)** iff there are no two pairs s.t.  $\langle a_i, b_j \rangle \in F$  and  $\langle a_k, b_j \rangle \in F$
- one-to-one, onto, and total function: correspondence (bijection)

- One can take the range of a function and make it the domain of another function.
- A function  $F_1 : A \rightarrow B$  and a function  $F_2 : B \rightarrow C$  can be composed as  $B(A(a))$ , short  $B \circ A$
- the compound function can be empty, it will be total if both A and B are bijections.



More about relations and sets

A relation  $R$  in  $A = \{a, b, \dots\}$  is...

	if	(ex.)
reflexive	for <b>every</b> $a \in A$ : $\langle a, a \rangle \in R$	is as heavy as
irreflexive	for <b>every</b> $a \in A$ : $\langle a, a \rangle \notin R$	A: physical objects
non-reflexive	for <b>some</b> $a \in A$ : $\langle a, a \rangle \notin R$	is the father of
		has hurt

A relation  $R$  in  $A = \{a, b, \dots\}$  is...

	if	(ex.)
symmetric	for every $\langle a, b \rangle \in R$ : $\langle b, a \rangle \in R$	has the same car as
asymmetric	for every $\langle a, b \rangle \in R$ : $\langle b, a \rangle \notin R$	has a different car than
non-symmetric	for some $\langle a, b \rangle \in R$ : $\langle b, a \rangle \notin R$	is the sister of
anti-symmetric	for every $\langle a, b \rangle \in R$ : $a = b$	beats oneself not every human does

A relation  $R$  in  $A = \{a, b, \dots\}$  is...

	if	(ex.)
transitive	if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$ then $\langle a, c \rangle \in R$	is to the left of
intransitive	the above is never the case	is the father of
non-transitive	the above is sometimes not the case	likes

A relation  $R$  in  $A = \{a, b, \dots\}$  is...

	if	(ex.)
connected	for every $a, b \in A, a \neq b$ : either $\langle a, b \rangle \in R$ or $\langle b, a \rangle \in R$	$>$ ( $A$ : the natural numbers)
non-connected	for some $a, b \in A$ the above is not the case	likes

# Equivalence relations

- reflexive ( $\langle a, a \rangle \in R$  for every  $a$ )
- symmetric ( $\langle b, a \rangle \in R$  for every  $\langle a, b \rangle$ )
- transitive ( $\langle a, b \rangle \in R \ \& \ \langle b, c \rangle \in R \rightarrow \langle a, c \rangle \in R$ )
- *is as stupid as*
- partition the range into equivalence classes:  
 $A = \{a, b, c, d\}$ , for example  $P_{A_1} = \{\{a, b\}, \{c\}, \{d\}\}$
- **not**  $\{\{a\}, \{b, c\}\}$  or  $\{\{a, b\}, \{b, c\}, \{d\}\}$

An ordering relation  $R$  in  $A$  is ...

- transitive ( $\langle a, b \rangle \in R \ \& \ \langle b, c \rangle \in R \rightarrow \langle a, c \rangle \in R$ ) ...plus ...
- irreflexive and asymmetric: **strict order**
- $A = \{a, b, c, d\}, R_1 = \{\langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle\}$
- reflexive and anti-symmetric: **weak order**
- $A = \{a, b, c, d\}, R_1 = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle a, b \rangle, \langle b, c \rangle, \langle a, c \rangle\}$

# Orders: an example

- a strict order: *greater than* ( $>$ ) in  $\mathbb{N}$
- what is the corresponding weak order
- $\geq$



- **minimal:**  $x$  is not preceded
- **least:**  $x$  precedes every other element
- **maximal:**  $x$  is not succeeded
- **greatest:**  $x$  succeeds every other element
- **well-ordering:** total order, every subset has a least element

## Cardinalities

# The number of elements...

- $A = \{a, b, c\}$
- $B = \{a, b, c\}$
- obviously,  $A = B$  (equal)
- there is an  $R$  from  $A$  to  $B$  s.t.  $R = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle\}$
- for every set  $C$  with the same number of elements (e.g.,  $C = \{1, 2, 3\}$ ):  $R = \{\langle a, 1 \rangle, \langle b, 2 \rangle, \langle c, 3 \rangle\}$
- such relations are one-to-one correspondences

- $\mathbb{N}$  is infinite
- for every  $A$  there is some  $R_{\text{card}}$ 
  - ▶ a one-to-one correspondence
  - ▶ from  $A$ 's members to the first  $n$  members of  $\mathbb{N}$
  - ▶ s.t.  $n$  is the **cardinality of  $A$ ,  $\|A\|$**
- sets  $A, B$  with  $\|A\| = \|B\|$  are **equivalent**
- $\|\mathbb{N}\| = \aleph^0$

# A problem

- for some sets there is no such  $R_{\text{card}}$
- no way of bringing their elements into an exhaustive linear order
- no problem with  $\mathbb{Q}$ :

	$\langle 0, 1 \rangle$	$\langle 0, 2 \rangle$	$\langle 0, 3 \rangle$	$\dots$
	$\langle 1, 0 \rangle$	$\langle 1, 1 \rangle$	$\langle 1, 2 \rangle$	$\langle 1, 3 \rangle$ $\dots$
	$\langle 2, 0 \rangle$	$\langle 2, 1 \rangle$	$\langle 2, 2 \rangle$	$\langle 2, 3 \rangle$ $\dots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

# The non-denumerable real numbers

- now:  $\mathbb{R}$
- some elements cannot be represented as an ordered pair of two elements of  $\mathbb{N}$
- in  $[0, 1]$ , every real can be represented as  $0.\textcolor{blue}{abcdefg} \dots$ ,  
 $a, b, c, d, e, f, g, \dots \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- an enumeration of  $[0, 1]$  in  $\mathbb{R}$ ?

$$\begin{array}{rcllclclcl} x_1 & = & 0 & . & a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ x_2 & = & 0 & . & a_{21} & a_{22} & a_{23} & a_{24} & \dots \\ x_3 & = & 0 & . & a_{31} & a_{32} & a_{33} & a_{34} & \dots \\ \vdots & & \vdots & & & & & & \\ x_n & = & 0 & . & a_{n1} & a_{n2} & a_{n3} & a_{n4} & \dots \end{array}$$

- What about an  $x_m$  which differs from  $x_n$  at  $a_{nn}$

$$\begin{array}{rcccccccc} x_1 & = & 0 & \cdot & \mathbf{a_{11}} & a_{12} & a_{13} & a_{14} & \dots \\ x_2 & = & 0 & \cdot & a_{21} & \mathbf{a_{22}} & a_{23} & a_{24} & \dots \\ x_3 & = & 0 & \cdot & a_{31} & a_{32} & \mathbf{a_{33}} & a_{34} & \dots \\ \vdots & & \vdots & & & & & & \\ x_n & = & 0 & \cdot & a_{n1} & a_{n2} & a_{n3} & \mathbf{a_{nn}} & \dots \end{array}$$

- It won't be in the array...
- $\mathbb{R}$  is non-denumerable
- If  $\|A\| = \aleph^0$  then  $\|\wp(A)\| = 2^{\aleph^0}$  (cf. Partee et al. 62f.)





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