Formale Semantik
03. Mengen und Funktionen

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stets aktuelle Fassungen: `https://github.com/rsling/VL-Deutsche-Syntax`

Inhalt

Sets and Functions

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- U: the universal set (contains every discrete object)

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- {x:x is human} = {x:x is from the planet earth and x can speak}

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- $M \subseteq M$ but $M \not\subset M$
- $\{\{a\}\} \not\subseteq \{a, b, c\}$
- $\{\} \subset \{a, b, c\}$ (or any set), $\{\}$ is sometimes written $\emptyset$

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- Hence: *Herr Webelhuth is numerous.

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- ...and why is the empty a set a proper subset of every set?

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- if $M = \{a, b, c\}$ and $N = \{a, b\}$ then $M \cap N = \{a, b\}$
- as a general principle (Consitency): $M \subseteq N$ iff $M \cup N = N$ and $M \subseteq N$ iff $M \cap N = M$

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Difference - and complement \ and ′

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- $O = \{a, b, c, k\}$ $M \backslash O = \{k\}$
- the universal complement: $M' = \{x||x \in U \text{ and } x \notin M\}$
  (U the universal set)

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- Identity: $M \cup \emptyset = X$, $M \cup U = U$ ...what about $\cap$

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- DeMorgan: $(M \cup N)' = M' \cap X'$ ...

Functions and Relations

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- first and second coordinate of the tuple

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- for $S \times S \times \cdots$: n-fold products
  $S^n = \{\vec{s} \| s_i \in S \text{ for } 1 \leq i \leq n\}$

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    Herr Schäfer is the inverse-teacher of Herr Webelhuth.

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- partial function from A to B: for some $a \in A$ there is no tuple $\langle a, b \rangle \in A \times B$, F is not *defined for* some a

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- one-to-one, onto, and total function: correspondence (bijection)

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- the compound function can be empty, it will be total if both A and B are bijections.

More about relations and sets

Reflexivity

A relation R in $A = \{a, b, \ldots\}$ is...

	if	(ex.)
reflexive	for **every** $a \in A$: $\langle a, a \rangle \in R$	is as heavy as
		A: physical objects
irreflexive	for **every** $a \in A$: $\langle a, a \rangle \notin R$	is the father of
non-reflexive	for **some** $a \in A$: $\langle a, a \rangle \notin R$	has hurt

Symmetry

A relation R in $A = \{a, b, \ldots\}$ is...

	if	(ex.)
symmetric	for every $\langle a, b\rangle \in R$: $\langle b, a\rangle \in R$	has the same car as
asymmetric	for every $\langle a, b\rangle \in R$: $\langle b, a\rangle \notin R$	has a different car than
non-symmetric	for some $\langle a, b\rangle \in R$: $\langle b, a\rangle \notin R$	is the sister of
anti-symmetric	for every $\langle a, b\rangle \in R$: $a = b$	beats oneself not every human does

Transitivity

A relation R in $A = \{a, b, \ldots\}$ is...

	if	(ex.)
transitive	if $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ then $\langle a, c\rangle \in R$	is to the left of
intransitive	the above is never the case	is the father of
non-transitive	the above is sometimes not the case	likes

Connectedness

A relation R in $A = \{a, b, \ldots\}$ is...

	if	(ex.)
connected	for every $a, b \in A$, $a \neq b$: either $\langle a, b \rangle \in R$ or $\langle b, a \rangle \in R$	$>$ (A: the natural numbers)
non-connected	for some $a, b \in A$ the above is not the case	likes

Equivalence relations

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- **well-ordering**: total order, every subset has a least element

Cardinalities

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- such relations are one-to-one correspondences

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- for some sets there is no such R_{card}
- no way of bringing their elements into an exhaustive linear order
- no problem with $\mathbb{Q}$:

$$
\begin{array}{ccccc}
 & \langle 0,1\rangle & \langle 0,2\rangle & \langle 0,3\rangle & \cdots \\
\langle 1,0\rangle & \langle 1,1\rangle & \langle 1,2\rangle & \langle 1,3\rangle & \cdots \\
\langle 2,0\rangle & \langle 2,1\rangle & \langle 2,2\rangle & \langle 2,3\rangle & \cdots \\
\vdots & \vdots & \vdots & \vdots &
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The non-denumerable real numbers

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- now: $\mathbb{R}$
- some elements cannot be represented as an ordered pair of two elements of $\mathbb{N}$
- in $[0, 1]$, every real can be represented as $0.abcdefg\ldots$, $a, b, c, d, e, f, g, \ldots \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Trying to enumerate

- an enumeration of $[0, 1]$ in $\mathbb{R}$?

$$
\begin{array}{ccccccccc}
x_1 & = & 0 & . & a_{11} & a_{12} & a_{13} & a_{14} & \ldots \\
x_2 & = & 0 & . & a_{21} & a_{22} & a_{23} & a_{24} & \ldots \\
x_3 & = & 0 & . & a_{31} & a_{32} & a_{33} & a_{34} & \ldots \\
\vdots & & \vdots & & & & & & \\
x_n & = & 0 & . & a_{n1} & a_{n2} & a_{n3} & a_{n4} & \ldots
\end{array}
$$

Failing to enumerate

- What about an x_m which differs from x_n at a_{nn}

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\begin{array}{lllllllll}
x_1 & = & 0 & . & \color{red}{a_{11}} & a_{12} & a_{13} & a_{14} & \ldots \\
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\end{array}
$$

- It won't be in the array...
- $\mathbb{R}$ is non-denumerable
- If $\|A\| = \aleph^0$ then $\|\wp(A)\| = 2^{\aleph_0}$ (cf. Partee et al. 62f.)

Literatur I

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Lizenz