Formale Semantik o3. Mengen und Funktionen

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Achtung: Folien in Überarbeitung. Englische Teile sind noch von 2007! Stets aktuelle Fassungen: https://github.com/rsling/VL-Semantik

Inhalt

- 1 Sets and Functions
 - The naive concept
 - Elements, subsets, power sets
 - Union, intersection, etc.
- 2 Functions and Relations
 - Ordered pairs/sets, n-tuples, Cartesian products
 - Relations

- Functions
- 3 More about relations and sets
 - Relations among themselves
 - Orders
- 4 Cardinalities
 - Denumerability
 - Non-denumerability



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- U: the universal set (contains every discrete object)

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{x:x is human} = {x:x is from the planet earth and x can speak}

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- the inverse: the superset

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- $\bullet \ \{\{a\}\} \not\in \{a,b,c\}$
- $\{\} \subset \{a,b,c\}$ (or any set), $\{\}$ is sometimes written \emptyset

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- Hence: *Herr Webelhuth is numerous.

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- ...and why is the empty a set a proper subset of every set?

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- if $M = \{a, b, c\}$ and $N = \{a, b\}$ then $M \cap N = \{a, b\}$
- as a general principle (Consitency): $M \subseteq N$ iff $M \cup N = N$ and $M \subseteq N$ iff $M \cap N = M$

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- $O = \{a, b, c, k\} M \setminus O = \{k\}$
- the universal complement: $M' = \{x | x \in U \text{ and } x \notin M\}$ (U the universal set)

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- Identity: $M \cup \emptyset = X$, $M \cup U = U$...what about \cap

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- DeMorgan: $(M \cup N)' = M' \cap X' \dots$



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- for $S \times S \times \cdots$: n-fold products $S^n = \{\vec{s} | | s_i \in S \text{ for } 1 \le i \le n\}$

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 - R = the relation of teacherhood between a and b: Herr Webelhuth is the teacher of Herr Schäfer.
 - ▶ R^{-1} = all pairs $\langle b, a \rangle$ where a is the teacher of b: Herr Schäfer is the inverse-teacher of Herr Webelhuth.

Functions

A function F from A to B is a relation s.t. for every a ∈ A there is exactly on tuple
 ⟨a, b⟩ ∈ A × B s.t. a is the first coordinate.

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- A function F from A to B is a relation s.t. for every $a \in A$ there is exactly on tuple $\langle a, b \rangle \in A \times B$ s.t. a is the first coordinate.
- partial function from A to B: for some $a \in A$ there is no tuple $\langle a, b \rangle \in A \times B$, F is not defined for some a

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- one-to-one, onto, and total function: correspondence (bijection)

Composition

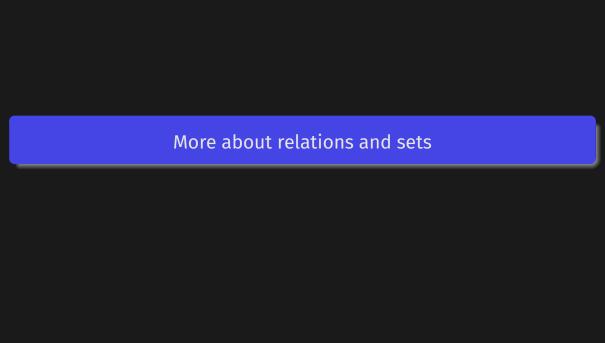
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- One can take the range of a function and make it the domain of another function.
- A function $F_1:A\to B$ and a function $F_2:B\to C$ can be composed as B(A(a)), short $B\circ A$
- the compound function can be empty, it will be total if both A and B are bijections.



Reflexivity

	if	(ex.)
reflexive	for every $a \in A$: $\langle a, a \rangle \in R$	is as heavy as
irreflexive non-reflexive	for every $a \in A$: $\langle a, a \rangle \not\in R$ for some $a \in A$: $\langle a, a \rangle \not\in R$	A: physical objects is the father of has hurt

Symmetry

	if	(ex.)
symmetric	for every $\langle a,b \rangle \in R$:	has the same car as
	$\langle b,a angle \in R$	
asymmetric	for every $\langle a,b angle \in R$:	has a different car than
	$\langle b,a\rangle ot\in R$	
non-symmetric	for some $\langle a,b \rangle \in R$:	is the sister of
	$\langle b,a\rangle \not\in R$	
anti-symmetric	for every $\langle a,b\rangle\in R$: $a=b$	beats oneself
		not every human does

Transitivity

	if	(ex.)
transitive	if $\langle a,b \rangle \in R$ and $\langle b,c \rangle \in R$	is to the left of
	then $\langle a,c \rangle \in R$	
intransitive	the above is never the case	is the father of
non-transitive	the above is sometimes not the case	likes

Connectedness

	if	(ex.)
connected	for every $a, b \in A$, $a \neq b$:	>
	either $\langle a,b\rangle\in R$ or $\langle b,a\rangle\in R$	(A: the natural numbers)
non-connected	for some $a, b \in A$ the above is not the case	likes

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- partition the range into equivalence classes:

```
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```

- reflexive $(\langle a, a \rangle \in R \text{ for every } a)$
- symmetric $(\langle b, a \rangle \in R \text{ for every } \langle a, b \rangle)$
- transitive $(\langle a,b\rangle \in R \& \langle b,c\rangle \in R \to \langle a,c\rangle \in R)$
- is as stupid as
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```

• not {{a}, {b, c}} or {{a, b}, {b, c}, {d}}

Defining ordering relations

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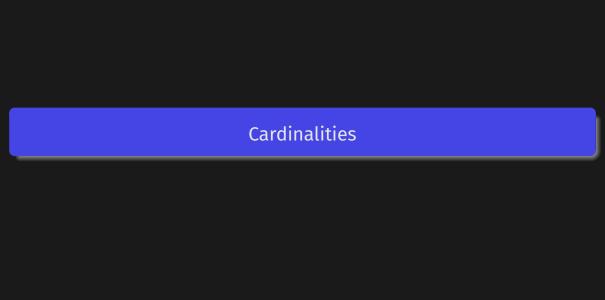
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- well-ordering: total order, every subset has a least element



•
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- such relations are one-to-one correspondences

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- $\|\mathbb{N}\| = \aleph^0$

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- no way of bringing their elements into an exhaustive linear order
- no problem with \mathbb{Q} :

- $\langle 0, 1 \rangle$
- $\langle 0, 2 \rangle$
- $\langle 0, 3 \rangle$

. . .

 $\langle 1, 0 \rangle$

- $\langle 1, 1 \rangle$
- $\langle 1, 2 \rangle$
- $\langle 1, 3 \rangle$
- • •

- $\langle 2, 0 \rangle$
- $\langle 2, 1 \rangle$
- $\langle 2, 2 \rangle$
- $\langle 2, 3 \rangle$

• •

:

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:

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The non-denumerable real numbers

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The non-denumerable real numbers

- now: \mathbb{R}
- ullet some elements cannot be represented as an ordered pair of two elements of ${\mathbb N}$
- in [0,1], every real can be represented as 0.abcdefg..., $a,b,c,d,e,f,g,... \in \{0,1,2,3,4,5,6,7,8,9\}$

Trying to enumerate

• an enumeration of [0,1] in \mathbb{R} ?

```
X_1
                    . a<sub>11</sub>
                                      a<sub>12</sub>
                                                a<sub>13</sub>
                                                           a<sub>14</sub>
        = 0 . a_{21} a_{22} a_{23}
X_2
                                                          a<sub>24</sub>
                                                                     •••
              O . a_{31} a_{32} a_{33}
                                                          a<sub>34</sub>
X_3
                                                                     •••
Xn
               0
                           a<sub>n1</sub>
                                     a_{n2}
                                                a<sub>n3</sub>
                                                           a_{n4}
```

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• What about an x_m which differs from x_n at a_{nn}

- It won't be in the array...
- R is non-denumerable
- If $||A|| = \aleph^0$ then $||\wp(A)|| = 2^{\aleph_0}$ (cf. Partee et al. 62f.)

Literatur I

Autor

Kontakt

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