

# Formale Semantik

## 03. Mengen und Funktionen

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stets aktuelle Fassungen: <https://github.com/rsling/VL-Deutsche-Syntax>

1

## Sets and Functions

- The naive concept
- Elements, subsets, power sets
- Union, intersection, etc.

2

## Functions and Relations

- Ordered pairs/sets, n-tuples, Cartesian products
- Relations

- Functions

3

## More about relations and sets

- Relations among themselves
- Orders

4

## Cardinalities

- Denumerability
- Non-denumerability

## Sets and Functions

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- $U$ : the universal set (contains every discrete object)



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 $\{a,b,c\} = \{x:x \text{ is one of the first three letters of the alphabet}\}$
- $\{x:x \text{ is human}\} = \{x:x \text{ is from the planet earth and } x \text{ can speak}\}$

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- the inverse: the *superset*

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- $\{\{a\}\} \not\subset \{a, b, c\}$
- $\{\} \subset \{a, b, c\}$  (or any set),  $\{\}$  is sometimes written  $\emptyset$

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- ...and why is the empty set a proper subset of every set?

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- if  $M = \{a, b, c\}$  and  $N = \{a, b\}$  then  $M \cap N = \{a, b\}$
- as a general principle (Consistency):  $M \subseteq N$  iff  $M \cup N = N$  and  $M \subseteq N$  iff  $M \cap N = M$

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- the universal complement:  $M' = \{x \mid x \in U \text{ and } x \notin M\}$   
( $U$  the universal set)

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- Identity:  $M \cup \emptyset = X, M \cup U = U \dots$  what about  $\cap$

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- DeMorgan:  $(M \cup N)' = M' \cap N'$  ...

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- first and second coordinate of the tuple

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- for  $S \times S \times \cdots$ : n-fold products  
 $S^n = \{\vec{s} \mid s_i \in S \text{ for } 1 \leq i \leq n\}$

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  - ▶  $R^{-1}$  = all pairs  $\langle b, a \rangle$  where  $a$  is the teacher of  $b$ :  
*Herr Schäfer is the inverse-teacher of Herr Webelhuth.*

- A function  $F$  from  $A$  to  $B$  is a relation s.t. for every  $a \in A$  there is exactly one tuple  $\langle a, b \rangle \in A \times B$  s.t.  $a$  is the first coordinate.

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- partial function from  $A$  to  $B$ : for some  $a \in A$  there is no tuple  $\langle a, b \rangle \in A \times B$ ,  $F$  is not *defined* for some  $a$



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- one-to-one, onto, and total function: correspondence (bijection)

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# Composition

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- the compound function can be empty, it will be total if both A and B are bijections.

More about relations and sets



A relation  $R$  in  $A = \{a, b, \dots\}$  is...

	if	(ex.)
reflexive	for <b>every</b> $a \in A$ : $\langle a, a \rangle \in R$	is as heavy as A: physical objects
irreflexive	for <b>every</b> $a \in A$ : $\langle a, a \rangle \notin R$	is the father of
non-reflexive	for <b>some</b> $a \in A$ : $\langle a, a \rangle \notin R$	has hurt

# Symmetry

A relation  $R$  in  $A = \{a, b, \dots\}$  is...

	if	(ex.)
symmetric	for every $\langle a, b \rangle \in R$ : $\langle b, a \rangle \in R$	has the same car as
asymmetric	for every $\langle a, b \rangle \in R$ : $\langle b, a \rangle \notin R$	has a different car than
non-symmetric	for some $\langle a, b \rangle \in R$ : $\langle b, a \rangle \notin R$	is the sister of
anti-symmetric	for every $\langle a, b \rangle \in R$ : $a = b$	beats oneself not every human does

# Transitivity

A relation  $R$  in  $A = \{a, b, \dots\}$  is...

	if	(ex.)
transitive	if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$ then $\langle a, c \rangle \in R$	is to the left of
intransitive	the above is never the case	is the father of
non-transitive	the above is sometimes not the case	likes

A relation  $R$  in  $A = \{a, b, \dots\}$  is...

	if	(ex.)
connected	for every $a, b \in A, a \neq b$ : either $\langle a, b \rangle \in R$ or $\langle b, a \rangle \in R$	$>$ ( $A$ : the natural numbers)
non-connected	for some $a, b \in A$ the above is not the case	likes

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# Defining ordering relations

An ordering relation  $R$  in  $A$  is ...

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- $\geq$

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- **well-ordering:** total order, every subset has a least element

## Cardinalities



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- such relations are one-to-one correspondences

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- $\|\mathbb{N}\| = \aleph^0$

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- no way of bringing their elements into an exhaustive linear order
- no problem with  $\mathbb{Q}$ :

	$\langle 0, 1 \rangle$	$\langle 0, 2 \rangle$	$\langle 0, 3 \rangle$	$\dots$
$\langle 1, 0 \rangle$	$\langle 1, 1 \rangle$	$\langle 1, 2 \rangle$	$\langle 1, 3 \rangle$	$\dots$
$\langle 2, 0 \rangle$	$\langle 2, 1 \rangle$	$\langle 2, 2 \rangle$	$\langle 2, 3 \rangle$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	



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- now:  $\mathbb{R}$
- some elements cannot be represented as an ordered pair of two elements of  $\mathbb{N}$
- in  $[0, 1]$ , every real can be represented as  $0.\textcolor{blue}{abcdefg} \dots$ ,  
 $a, b, c, d, e, f, g, \dots \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

# Trying to enumerate

- an enumeration of  $[0, 1]$  in  $\mathbb{R}$ ?

$$\begin{array}{rcccccccl} x_1 & = & 0 & . & a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ x_2 & = & 0 & . & a_{21} & a_{22} & a_{23} & a_{24} & \dots \\ x_3 & = & 0 & . & a_{31} & a_{32} & a_{33} & a_{34} & \dots \\ \vdots & & \vdots & & & & & & \\ x_n & = & 0 & . & a_{n1} & a_{n2} & a_{n3} & a_{n4} & \dots \end{array}$$

# Failing to enumerate

- What about an  $x_m$  which differs from  $x_n$  at  $a_{nn}$

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- It won't be in the array...
- $\mathbb{R}$  is non-denumerable
- If  $\|A\| = \aleph^0$  then  $\|\wp(A)\| = 2^{\aleph^0}$  (cf. Partee et al. 62f.)





## Kontakt

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