

# Formale Semantik

## 05. Prädikatenlogik

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stets aktuelle Fassungen: <https://github.com/rsling/VL-Deutsche-Syntax>

## 1 Why predicate calculus?

- ## 2 The construction of PC
- Atoms and syntax
  - Semantics
  - More rules

- ## 3 Laws of PC
- Negation and distribution
  - Movement
  - Some in-class practice

- ## 4 Natural deduction in PC
- Quantifier elimination
  - An example

Why predicate calculus?

- properties/relations vs. individuals
- *Martin is an expert on inversion and Martin is a good climber.*
- ...becomes  $E \wedge C$
- compositionality restricted to level of connected propositional atoms

# Some desirable deductions

- important generalizations about all and some individuals (which have property P)
- '*all P  $\rightarrow$  some P*'
- '*Martin P  $\rightarrow$  some P*'

## The construction of PC

- individual **variables**:  $x, y, z, x_1, x_2 \dots$
- individual **constants**:  $a, b, c, \dots$
- variables and constants: **terms**
- **predicate symbols** (taking individual symbols or tuples of them):  $A, B, C, \dots$
- **quantifiers**: existential  $\exists$  (or  $\vee$ ) and universal  $\forall$  (or  $\wedge$ )
- plus the connectives of SL

- for an  $n$ -ary predicate  $P$  and terms  $t_1 \dots t_n$ ,  
 $P(t_1 \dots t_n)$  or  $Pt_1 \dots t_n$  is a wff.
- possible prefix, function (bracket) and infix notation:  
 $Pxy$ ,  $P(x, y)$ ,  $xPy$
- syntax for connectives from SL
- for any wff  $\phi$  and any variable  $x$ ,  $(\exists x)\phi$  and  $(\forall x)\phi$  are wff's



- denote individuals
- a model  $\mathcal{M}$  contains a set of individuals  $D$
- the valuation function  $V$  (or  $F$ ): from constants to individuals in  $D$
- for some  $\mathcal{M}_1$ :  $D = \{Martin, Kilroy, Scully\}$
- $V_{\mathcal{M}_1}(m) = Martin$
- $V_{\mathcal{M}_1}(k) = Kilroy, V_{\mathcal{M}_1}(s) = Scully$

- denote relations (sets of n-tuples)
- $\llbracket P \rrbracket^{\mathcal{M}_1} = \{\text{Martin}, \text{Kilroy}\}$  or  $V_{\mathcal{M}_1}(P) = \{\text{Martin}, \text{Kilroy}\}$
- $V_{\mathcal{M}_1}(Q) = \{\langle \text{Martin}, \text{Kilroy} \rangle, \langle \text{Martin}, \text{Scully} \rangle, \langle \text{Kilroy}, \text{Kilroy} \rangle, \langle \text{Scully}, \text{Scully} \rangle\}$
- s.t.  $\llbracket P(m) \rrbracket^{\mathcal{M}_1} = \llbracket P \rrbracket^{\mathcal{M}_1}(\llbracket m \rrbracket^{\mathcal{M}_1}) = 1$  iff  $\llbracket m \rrbracket^{\mathcal{M}_1} \in \llbracket P \rrbracket^{\mathcal{M}_1}$

- **connectives**: 'apply to' formulas (semantically truth-valued), semantics as in SL
- $(\forall x)\phi = 1$  iff  $\phi$  is true for every  $d \in D$   
assigned to every occurrence of  $x$  in  $\phi$
- $(\exists x)\phi = 1$  iff  $\phi$  is true for at least one  $d \in D$   
assigned to every occurrence of  $x$  in  $\phi$
- algorithmic instruction to check wff's containing Q's
- check outside-in (unambiguous scoping)

- universal quantifiers can be swapped:  
 $(\forall x)(\forall y)\phi \Leftrightarrow (\forall y)(\forall x)\phi$
- same for existential quantifiers:  
 $(\exists x)(\exists y)\phi \Leftrightarrow (\exists y)(\exists x)\phi$
- whereas:  $(\exists x)(\forall y)\phi \Rightarrow (\forall y)(\exists x)\phi$
- example in  $\mathcal{M}_1$ :
  - ▶  $\llbracket (\forall x)(\exists y)Qxy \rrbracket^{\mathcal{M}_1} = 1$
  - ▶ but:  $\llbracket (\exists y)(\forall x)Qxy \rrbracket^{\mathcal{M}_1} = 0$
  - ▶ direct consequence of algorithmic definition
  - ▶ if  $\exists\forall$  is true,  $\forall\exists$  follows

# Hints on quantifiers

- domain of quantifiers:  $D$  (universe of discourse)
- $\forall x$  checks for truth of some predication for all individuals
- $\exists x(Px \wedge \neg Px)$  is a contradiction
- $\forall x(Wx \wedge \neg Wx)$  is a contradiction,  
 $\forall x$  'checks' for an empty set by def.
- standard form of NL quantification:  
 $\forall x(Wx \rightarrow Bx)$  'All women are beautiful.'
- standard form of NL existential quantification:  
 $\exists x(Wx \wedge Bx)$  'Some woman is beautiful.'

- by def., functors take formulas, not terms:
  - ▶  $\neg Wm$  'Mary doesn't weep.'
  - ▶  $(\exists x)(Gx \wedge Wx)$  'Some girl weeps.'
  - ▶ \*  $W\neg x$
  - ▶ \*  $(\exists\neg x)(Gx)$
- quantifiers take variables, not constants:
  - ▶  $(\forall x)(Ox \rightarrow Wx)$  'All ozelots are wildcats.'
  - ▶ \*  $(\forall o)(Wo)$
- $\neg$  negates the wff, not the q:
  - \*  $(\neg\forall x)Px$  but  $\neg(\forall x)Px$

- quantifiers **bind** variables
- free variables (constants) are unbound
- **no double binding** \*  $(\forall x \exists x)Px$
- **Q scope**: only the first wff to its right:
  - ▶  $(\forall x)Px \vee Qx$
  - ▶  $\frac{(\forall x)(Px \vee Qx)}{(\forall x)Px \vee (\forall x)Qx}$
  - ▶  $\frac{(\exists x)Px \rightarrow (\forall y)(Qy \wedge Ry)}{(\exists x)Px \wedge Qx}$  (second x is a unbound)
- **no double-naming**

## Laws of PC



- $\exists$  and  $\forall$  'or' and 'and' over the universe of discourse (hence:  $\vee$  and  $\wedge$ )
- $(\forall x)Px \Leftrightarrow Px_1 \wedge Px_2 \wedge \dots \wedge Px_n$  for all  $x_n$  assigned to  $d_n \in D$
- $(\exists x)Px \Leftrightarrow Px_1 \vee Px_2 \vee \dots \vee Px_n$  for all  $x_n$  assigned to  $d_n \in D$
- hence:  $\neg(\forall x)Px \Leftrightarrow \neg(Px_1 \wedge Px_2 \wedge \dots \wedge Px_n)$
- with DeM:  $\overline{Px_1 \wedge Px_2 \wedge \dots \wedge Px_n}$
- $\Leftrightarrow \overline{Px_1} \vee \overline{Px_2} \vee \dots \vee \overline{Px_n}$
- $\Leftrightarrow (\exists x)\neg Px$

# Quantifier negation (QN)

- $\neg(\forall x)Px \Leftrightarrow (\exists x)\neg Px$
- $\neg(\exists x)Px \Leftrightarrow (\forall x)\neg Px$
- $\neg(\forall x)\neg Px \Leftrightarrow (\exists x)Px$
- $\neg(\exists x)\neg Px \Leftrightarrow (\forall x)Px$

# The distribution laws

- the conjunction of universally quantified formulas:

$$\underline{(\forall x)(Px \wedge Qx)} \Leftrightarrow \underline{(\forall x)Px} \wedge \underline{(\forall x)Qx}$$

- the disjunction of existentially quantified formulas:

$$\underline{(\exists x)(Px \vee Qx)} \Leftrightarrow \underline{(\exists x)Px} \vee \underline{(\exists x)Qx}$$

- not v.v.:  $(\forall x)Px \vee (\forall x)Qx \Rightarrow (\forall x)(Px \vee Qx)$
- why?

# Quantifier movement (QM)

- desirable format: **prefix + matrix**
- Movement Laws for antecedents of conditionals:  
 $(\exists x)Px \rightarrow \phi \Leftrightarrow (\forall x)(Px \rightarrow \phi)$   
 $(\forall x)Px \rightarrow \phi \Leftrightarrow (\exists x)(Px \rightarrow \phi)$
- Movement Laws for Q's in disjunction, conjunction, and the consequent of conditionals: **Just move them to the prefix!**
- condition: **x must not be free in  $\phi$ .**
- i.e.: Watch your variables!

## Let's formalize:

- Paul Kalkbrenner is a musician and signed on bpitchcontrol.
- Herr S. installed RedHat and not every Linux distribution is easy to install.
- All talkmasters are human and Harald Schmidt is a talkmaster.
- Some talkmasters are not musicians.
- Heiko Laux owns Kanzleramt records and does not like any Gigolo artist.
- Some humans are neither talkmasters nor do they own Kanzleramt records.

## Natural deduction in PC

# Universal instantiation ( $-\forall$ ) and generalization ( $+\forall$ )

- $(\forall x)Px \rightarrow Pa$
- always applies
- can use any variable/constant
- $Pa \rightarrow (\forall x)Px$
- iff  $Pa$  was instantiated by  $-\forall$

# Existential generalization ( $+\exists$ ) and instantiation ( $-\exists$ )

- $Pa \rightarrow (\exists x)Px$  for any individual constant  $a$
- always applies
- $(\exists x)Px \rightarrow Pa$  for some indiv. const.
- always applies (there is a minimal individual for  $\exists x$ )
- for some  $(\exists x)Px$  and  $(\exists x)Qx$  the minimal individual might be different
- hence: **When you apply EI, always use fresh constants!**



# One sample task

- (1) Herr Keydana drives a Golf. (2) Anything that drives a golf is human or a complex program simulating an artificial neural net. (3) There are no programs s.a.a.n.n. which are complex enough to drive a Golf.
- Formalize and prove: **At least one human exists.**
- (1)  $Dk$
- (2)  $(\forall x)(Dx \rightarrow Hx \vee Px)$
- (3)  $\neg(\exists x)(Px \wedge Dx)$
- $(\exists x)Hx$

# The proof

(1)	$Dk$	
(2)	$(\forall x)(Dx \rightarrow Hx \vee Px)$	
(3)	$\neg(\exists x)(Px \wedge Dx)$	
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(4)	$(\forall x)\neg(Px \wedge Dx)$	3,QN
(5)	$(\forall x)(\neg Px \vee \neg Dx)$	4,DeM
(6)	$(\forall x)(Dx \rightarrow \neg Px)$	5,Comm,Impl
(7)	$Dk \rightarrow \neg Pk$	6, $\neg\forall(1)$
(8)	$\neg Pk$	1,7,MP
(9)	$Dk \rightarrow Hk \vee Pk$	2, $\neg\forall(1)$
(10)	$Hk \vee Pk$	1,9,MP
(11)	$Hk$	8,10,DS
$\therefore$	$(\exists x)Hx$	10, $+\exists$



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