Formale Semantik
04. Aussagenlogik

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Achtung: Folien in Überarbeitung. Englische Teile sind noch von 2007!

Stets aktuelle Fassungen: `https://github.com/rsling/VL-Semantik`

Inhalt

1. What logic is about
   - On reasoning
   - Where we need logic
2. Statement calculus
   - Formalization: Recursive Syntax
   - Interpretation
   - Laws of the PropC
   - Rules of Inference
   - Proof

The book (PMW:87-246) deals with logic far more in-depth than we do. Only what is mentioned on the slides is relevant for the test. Reading the whole chapter from PMW will do you no harm, though.

What logic is about

Theories

- a collection of statements (propositions)
- axioms (statements accepted to be true)
- maybe based on observations (induction)
- statements that follow from the axioms (deduction)
- predictions beyond the axioms
- rechecking for usability: e.g., Russell’s paradox

Proofs

- axioms: atomic truths of your theory
- theorem: a proposition you want to prove
- lemma: subsidiary propositions (used to prove the theorem)
- corollary: propositions proved while proving some axiom

A method of reasoning

- logic does not generate truths
- formalizing statements, predications etc.
- rules of deduction from axioms to theorems
- empirical (induction) and exact (deduction) science
- aiming at an adequate model of the world (e.g., heliocentric universe)

Why logic for semantics?

- truth-conditional
- compositional behavior of propositions and connectives
- a logic for entailments
- why, e.g.: *It is not the case that someone is happy.* $\rightarrow$ *Nobody is happy.*

Statement calculus

Atomic formulas: statements

- statements/propositions = the atoms
- a propositional symbol *p*: a well-formed formula (wff)
- ex.: *Herr Keydana is a passionate cyclist.*: *k*
- $[\![k]\!]$=1 or 0 (depending on corresponding **model**)

Complex (molecular) formulas

- syntax: restricts the forms of wff's to make them interpretable
- define functors: functions in $\{0, 1\}$
- If p and q are wff's, then
  - $\neg p$
  - $p \vee q$
  - $p \wedge q$
  - $p \rightarrow q$
  - $p \leftrightarrow q$

  is also a wff (a molecular term).

Complex (molecular) formulas

- syntax: restricts forms of wff's to make them interpretable
- define functors: functions in $\{\langle 0,1\rangle, \langle 1,0\rangle, 0, 1\}$
- If *p* and *q* are wff's, then
  - $\neg p$ (negation)
  - $p \vee q$ (disjunction)
  - $p \wedge q$ (conjunction)
  - $p \rightarrow q$ (conditional)
  - $p \leftrightarrow q$ (biconditional)

  is also a wff.

Complex (molecular) formulas

- syntax: restricts forms of wff's to make them interpretable
- define functors: functions in $\{\langle 0,1\rangle, \langle 1,0\rangle, 0, 1\}$
- If *p* and *q* are wff's, then
  - $\neg p$ (negation - 'not')
  - $p \vee q$ (disjunction - 'or')
  - $p \wedge q$ (conjunction - 'and')
  - $p \rightarrow q$ (conditional - 'if')
  - $p \leftrightarrow q$ (biconditional - 'iff')

  is also a wff.

Functions and truth tables

- standard defintion:

$$[\![\neg]\!] = \begin{bmatrix} 1 \rightarrow 0 \\ 0 \rightarrow 1 \end{bmatrix}$$

- but most widely used: truth tables

$\neg$	p
0	1
1	0

Disjunction

p	$\vee$	q
1	1	1
1	1	0
0	1	1
0	0	0

- *Herr Keydana is a passionate cyclist* ***or*** *we all love logic.*
- $K \vee L$

Conjunction

p	$\wedge$	q
1	1	1
1	0	0
0	0	1
0	0	0

- *Herr Keydana is a passionate cyclist* ***and*** *we all love logic.*
- $K \wedge L$

Conditional

p	$\rightarrow$	q
1	1	1
1	0	0
0	1	1
0	1	0

- ***If*** *it rains,* ***then*** *the streets get wet.*
- $R \rightarrow S$

Any problems with that?

If it rains, the streets get wet.

- it is raining (1) , the streets are wet 1 : **1**
- it is raining (1) , the streets are dry 0 : **0**
- it is not raining (0) , the streets are wet 1 : **1**
- it is not raining (0) , the streets are dry 0 : **1**
- ex vero non sequitur falsum

Biconditional

p	$\leftrightarrow$	q
1	1	1
1	0	0
0	0	1
0	1	0

- ***If and only if*** *your score is above 50,* ***then*** *you pass the semantics exam.*
- $S \leftrightarrow P$

Scope of functors

- brackets are facultative
- or set non-default functor scope
- default scope

$$\text{scope} \downarrow \quad \begin{array}{c} \neg \\ \wedge \\ \vee \\ \rightarrow \\ \leftrightarrow \end{array} \quad \uparrow \text{binding strength}$$

An example

- $p \land \neg q \lor r \rightarrow \neg s$
- $p \land (\neg q) \lor r \rightarrow (\neg s)$
- $(p \land (\neg q)) \lor r \rightarrow (\neg s)$
- $((p \land (\neg q)) \lor r) \rightarrow (\neg s)$
- $(((p \land (\neg q)) \lor r) \rightarrow (\neg s))$

Large truth tables

- for *n* atoms in the term: 2^n lines
- alternating blocks of 1's and 0's under every atom
- $2^{(m-1)}$ times '1' followed by $2^{(m-1)}$ times '0' for the *m*-th atom from the right
- until 2^n lines are reached

An example

p	$\wedge$	$\neg$	q	$\vee$	r	$\rightarrow$	$\neg$	s
1			1		1			1
1			1		1			0
1			1		0			1
1			1		0			0
1			0		1			1
1			0		1			0
1			0		0			1
1			0		0			0
0			1		1			1
0			1		1			0
0			1		0			1
0			1		0			0
0			0		1			1
0			0		1			0
0			0		0			1
0			0		0			0

An example

p	$\wedge$	$\neg$	q	$\vee$	r	$\rightarrow$	$\neg$	s
1		0	1		1		0	1
1		0	1		1		1	0
1		0	1		0		0	1
1		0	1		0		1	0
1		1	0		1		0	1
1		1	0		1		1	0
1		1	0		0		0	1
1		1	0		0		1	0
0		0	1		1		0	1
0		0	1		1		1	0
0		0	1		0		0	1
0		0	1		0		1	0
0		1	0		1		0	1
0		1	0		1		1	0
0		1	0		0		0	1
0		1	0		0		1	0

An example

p	$\wedge$	$\neg$	q	$\vee$	r	$\rightarrow$	$\neg$	s
1	0	0	1		1		0	1
1	0	0	1		1		1	0
1	0	0	1		0		0	1
1	0	0	1		0		1	0
1	1	1	0		1		0	1
1	1	1	0		1		1	0
1	1	1	0		0		0	1
1	1	1	0		0		1	0
0	0	0	1		1		0	1
0	0	0	1		1		1	0
0	0	0	1		0		0	1
0	0	0	1		0		1	0
0	0	1	0		1		0	1
0	0	1	0		1		1	0
0	0	1	0		0		0	1
0	0	1	0		0		1	0

An example

p	$\wedge$	$\neg$	q	$\vee$	r	$\rightarrow$	$\neg$	s
1	0	0	1	1	1		0	1
1	0	0	1	1	1		1	0
1	0	0	1	0	0		0	1
1	0	0	1	0	0		1	0
1	1	1	0	1	1		0	1
1	1	1	0	1	1		1	0
1	1	1	0	1	0		0	1
1	1	1	0	1	0		1	0
0	0	0	1	1	1		0	1
0	0	0	1	1	1		1	0
0	0	0	1	0	0		0	1
0	0	0	1	0	0		1	0
0	0	1	0	1	1		0	1
0	0	1	0	1	1		1	0
0	0	1	0	0	0		0	1
0	0	1	0	0	0		1	0

An example

p	$\wedge$	$\neg$	q	$\vee$	r	$\rightarrow$	$\neg$	s
1	0	0	1	1	1	0	0	1
1	0	0	1	1	1	1	1	0
1	0	0	1	0	0	1	0	1
1	0	0	1	0	0	1	1	0
1	1	1	0	1	1	0	0	1
1	1	1	0	1	1	1	1	0
1	1	1	0	1	0	0	0	1
1	1	1	0	1	0	1	1	0
0	0	0	1	1	1	0	0	1
0	0	0	1	1	1	1	1	0
0	0	0	1	0	0	1	0	1
0	0	0	1	0	0	1	1	0
0	0	1	0	1	1	0	0	1
0	0	1	0	1	1	1	1	0
0	0	1	0	0	0	1	0	1
0	0	1	0	0	0	1	1	0

An example

p	$\wedge$	$\neg$	q	$\vee$	r	$\rightarrow$	$\neg$	s
1	0	0	1	1	1	0	0	1
1	0	0	1	1	1	1	1	0
1	0	0	1	0	0	1	0	1
1	0	0	1	0	0	1	1	0
1	1	1	0	1	1	0	0	1
1	1	1	0	1	1	1	1	0
1	1	1	0	1	0	0	0	1
1	1	1	0	1	0	1	1	0
0	0	0	1	1	1	0	0	1
0	0	0	1	1	1	1	1	0
0	0	0	1	0	0	1	0	1
0	0	0	1	0	0	1	1	0
0	0	1	0	1	1	0	0	1
0	0	1	0	1	1	1	1	0
0	0	1	0	0	0	1	0	1
0	0	1	0	0	0	1	1	0

Assignments: a contingent example

p	$\wedge$	$\neg$	q	$\vee$	r	$\rightarrow$	$\neg$	s
1	0	0	1	1	1	0	0	1
1	0	0	1	1	1	1	1	0
1	0	0	1	0	0	1	0	1
1	0	0	1	0	0	1	1	0
1	1	1	0	1	1	0	0	1
1	1	1	0	1	1	1	1	0
1	1	1	0	1	0	0	0	1
1	1	1	0	1	0	1	1	0
0	0	0	1	1	1	0	0	1
0	0	0	1	1	1	1	1	0
0	0	0	1	0	0	1	0	1
0	0	0	1	0	0	1	1	0
0	0	1	0	1	1	0	0	1
0	0	1	0	1	1	1	1	0
0	0	1	0	0	0	1	0	1
0	0	1	0	0	0	1	1	0

Tautology

- take $p \vee \neg p$
- truth-table:

p	$\vee$	$\neg$	p
1	1	0	1
0	1	1	0

- true under every assignment, it is **valid**
- by *law of excluded middle*: for every P, $P \vee \neg P$ is true

Contradiction

- take $p \wedge \neg p$
- truth-table:

p	$\wedge$	$\neg$	p
1	0	0	1
0	0	1	0

- false under every assignment, called **contradictory**

Contingency

- take $p \wedge p$
- truth-table:

p	$\wedge$	p
1	1	1
0	0	0

- the truth value depends on the assignemt

What are laws?

- notice: similarities of set theory and logic
- non-trivial exact nature of their equivalence
- laws state equivalences of (types of) wff
- truth-conservative rewriting of wff's
- any subformula which is a tautology (T) or contradiction (F):

  ignore by Identity Laws (Id.):
  - $(P \vee F) \Leftrightarrow P$, $(P \vee T) \Leftrightarrow T$
  - $(P \wedge F) \Leftrightarrow F$, $(P \wedge T) \Leftrightarrow P$

Equivalences: $\Leftrightarrow$

- X $\Leftrightarrow$ Y: X has the same truth-conditions as Y
- derivability of laws and rules (convenient redundancies)
- Idempotency (Idemp.):
  - $(P \lor P) \Leftrightarrow P$
  - $(P \land P) \Leftrightarrow P$
  - *Peter walks and Peter walks.* $\Leftrightarrow$ *Peter walks.*

Simple laws

- Associative Laws for $\vee$ and $\wedge$ (Assoc.):
  - $((P \vee Q) \vee R) \Leftrightarrow (P \vee (Q \vee R))$
  - *((He walks or she talks) or we walk.)* $\Leftrightarrow$ *(He walks or (she talks or we walk.))*
- Commutative Laws for $\vee$ and $\wedge$ (Comm.):
  - $(P \vee Q) \Leftrightarrow (Q \vee P)$
  - *Peter walks or Sue snores.* $\Leftrightarrow$ *Sue snores or Peter walks.*
- Distributive Laws for $\vee\wedge$ and $\wedge\vee$ (Distr.):
  - $(P \vee (Q \wedge R)) \Leftrightarrow ((P \vee Q) \wedge (P \vee R))$
  - *(Sue snores) and (Peter walks or we talk).* $\Leftrightarrow$ *(Sue snores and Peter walks) or (Sue snores and we talk).*

Laws dealing with tautology and contradiction

- Complement Laws:
  - Tautology (T): $(P \vee \neg P) \Leftrightarrow \mathbf{T}$
  - Contradiction (F): $(P \wedge \neg P) \Leftrightarrow \mathbf{F}$
  - Double Negation (DN): $(\neg\neg P) \Leftrightarrow P$
  - *It is not the case that Sandy is not walking.*
    $\Leftrightarrow$ *Sandy is walking.*

Conditionals Laws

- **Implication** (Impl.):

P	$\rightarrow$	Q	$\Leftrightarrow$	$\neg$	P	$\vee$	Q
1	1	1		0	1	1	1
1	0	0		0	1	0	0
0	1	1		1	0	1	1
0	1	0		1	0	1	0

- **Contraposition** (Contr.):

P	$\rightarrow$	Q	$\Leftrightarrow$	$\neg$	Q	$\rightarrow$	$\neg$	P
1	1	1		0	1	1	0	1
1	0	0		1	0	0	0	1
0	1	1		0	1	1	1	0
0	1	0		1	0	1	1	0

DeMorgan (DeM)

- DeMorgan's Laws:
  - $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$
  - alternatively: $\overline{P \vee Q} \Leftrightarrow \overline{P} \wedge \overline{Q}$
  - $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$
  - consequently: $\overline{\overline{P} \vee \overline{Q}} \Leftrightarrow \overline{\overline{P}} \wedge \overline{\overline{Q}} \Leftrightarrow P \wedge Q$

The Modus Ponens (MP)

- Definition:

P	$\rightarrow$	Q	premise 1
P			premise 2
		Q	conclusion

- or: $(P \rightarrow Q) \wedge (P) \rightarrow (Q)$
- *(1) If It rains, the streets get wet. (2) It is raining.*
  $\rightarrow$ *The streets are getting wet.*

MP: a truth table illustration

- Premises are always set to be true!
- the table:

P	$\rightarrow$	Q
1	1	1
1	0	0
0	1	1
0	1	0

MP: a truth table illustration

- The conditional must be true.
- cancel the 'false' row

P	$\rightarrow$	Q
1	1	1
1	0	0
0	1	1
0	1	0

MP: a truth table illustration

- P must be true.
- cancel the 'false' rows, Q can only be true:

P	$\rightarrow$	Q
1	1	1
1	0	0
0	1	1
0	1	0

The Modus Tollens (MT)

- Definition:

P	$\rightarrow$	Q
		$\neg Q$
$\neg P$		

- the table illustration:

P	$\rightarrow$	Q	
1	1	1	(by premise 2)
1	0	0	(by premise 1)
0	1	1	(by premise 2)
0	1	0	

The Syllogisms

- Hypothetical Syllogism (HS):
  - $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$
  - *(1) If it rains, the streets get wet. (2) If the streets get wet, it smells nice.* $\rightarrow$ *If it rains, it smells nice.*
- Disjunctive Syllogism (DS):
  - $((P \vee Q) \wedge (\neg P)) \rightarrow (Q)$
  - *(1) Either Peter sleeps or Peter is awake. (2) Peter isn't awake.* $\rightarrow$ *Peter sleeps.*

Trivial rules

- Simplification (Simp.):
  - $(P \wedge Q) \rightarrow P$
  - *(1) It is raining and the sun is shining.* $\rightarrow$ *It is raining.*
- Conjunction (Conj.):
  - $(P) \wedge (Q) \rightarrow (P \wedge Q)$
  - *(1) It is raining. (2) The sun is shining.* $\rightarrow$ *It is raining and the sun is shining.*
- Addition (Add.):
  - $(P) \rightarrow (P \wedge Q)$
  - *(1) It is raining.* $\rightarrow$ *It is raining or the sun is shining.*
  - What if Q is instantiated as true or false by another premise?

A sample proof

- Prove $p \vee q$ from $(p \vee q) \to \neg(r \wedge \neg s)$ and $r \wedge \neg s$
- The proof:

$$p \vee q$$

1	$(p \vee q) \to \neg(r \wedge \neg s)$	
2	$r \wedge \neg s$	
	$p \vee q$	1,2,MT

Literatur I

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