Formale Semantik
03. Mengen und Funktionen

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Achtung: Folien in Überarbeitung. Englische Teile sind noch von 2007!

Stets aktuelle Fassungen: `https://github.com/rsling/VL-Semantik`

Inhalt

Mengen und Funktionen

Was ist eine Menge?

Eine frei definierbare ungeordnete Sammlung von diskreten Objekten

- Zahlen
- Menschen
- Schuhe
- Wörter
- ...

- nicht unbedingt zweckgebunden
- jedes Objekt maximal einmal in jeder Menge

Das Wesentliche von heute in Partee u. a. (1990: Kapitel 1–4)

Notation und Beispiele für Mengen

Mengendefinition {}, Elementstatus $\in$

- $M_1 = \{a, b, c\}$ (Menge von Buchstaben)
- N_1 = {'my book'} (einelementige Menge, enthält eine NP)
  - vs. N_2 = {my book} (einelementige Menge, enthält mein Buch)
  - vs. N_3 = {'my', 'book'} (Menge von Wörtern)
- möglich, aber ungewöhnlich: N_4 = {'my', book}
- definiert über eine Eigenschaft der Elemente (zwei Notationen):
  M_2 = {x: x is one of the first three letters of the alphabet}
  M_2 = {x|| x is one of the first three letters of the alphabet}
- U: die universelle Menge (alle Objekte)

Identität von Mengen

Zwei Mengen mit exakt den gleichen Elementen sind identisch.

- {a, b, c} = {x: x is one of the first tree letter of the alphabet}
- {x: x is human} = {x: x is from the Earth, a primate but not an ape}

Teilmengen und Obermengen

Teilmenge | Eine Menge N, die kein Element enthält,
das nicht auch in Menge M enthalten ist (umg. Obermenge).

Teilmenge oder Identität $\subseteq$
Obermenge oder Identität $\supseteq$

- $\{a\} \subseteq \{a,b,c\}$ und $\{a,b,c\} \supseteq \{a\}$
- $\{a\} \subseteq \{a,b,c\}$ und $\{a,b,c\} \supseteq \{a\}$
- $\{a,b,c\} \subseteq \{a,b,c\}$
- $\{a,b,c,d\} \nsubseteq \{a,b,c\}$ und $\{a,b,c\} \nsupseteq \{a,b,c,d\}$
- {x: x is human} $\subseteq$ {x: is an ape}

Echte Teilmengen und Obermengen

Echte Teilmenge | Eine Menge N, die kein Element enthält,
das nicht auch in Menge M enthalten ist, und die nicht mit M identisch ist.

Echte Teilmenge $\subset$
Echte Obermenge $\supset$

- $\{a\}\subset\{a,b,c\}$ und $\{a\} \subset \{a,b,c\}$
- $\{a,b,c\}\not\subset\{a,b,c\}$ aber $\{a,b,c\}\subseteq\{a,b,c\}$

Elemente vs. Teilmengen

- Achtung bei Mengen von Mengen
  - $\{\{a\}\} \not\subset \{a,b,c\}$
  - $\{\{a\}\} \not\subseteq \{a,b,c\}$
  - $\{\{a\}\} \notin \{a,b,c\}$
- für leere Menge $\{\}$ oder $\emptyset$
  - $\{\} \subset$ jede anderen Menge
  - $\{\} \notin \{\}$

Logik mit Mengen, Teilmengen und Elementen

- Logik mit Mengen
  - *Alle Anglistikprofessoren sind menschlich.*
    Herr Webelhuth ist Anglistikprofessor.
  - w = Herr Webelhuth
    E = {x: x is professors of English Linguistics}
    H = {x: x is human}
  - Aus $w \in E$ und $E \subset H$ folgt $w \in H$
- Aber
  - *Die Anglistikprofessoren sind zahlreich.*
  - N = {x: x is a set with many members}
  - Aus $\mathrm{w} \in \mathrm{E}$ und $\mathrm{E} \in \mathrm{N}$ folgt nicht $\mathrm{w} \in \mathrm{N}$
  - Vergleiche: *Herr Webelhuth ist zahlreich.

Potenzmenge $\wp(\cdot)$ | Für jede Menge M: $\wp(M) = \{X : X \subseteq M\}$

- Beispiel
  - M={a,b,c}
  - $\wp(M) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{\}\}$
- Warum ist die leere Menge in der Potenzmenge jeder Menge?
- Warum ist die leere Menge eine echte Teilmenge jeder Menge?

Union ∪ and intersection ∩

- For any sets M and N: $M \cup N = \{x \| x \in M \textbf{ or } x \in N\}$
- if $M = \{a, b, c\}$ and $N = \{a, b, d\}$ then $M \cup N = \{a, b, c, d\}$
- For any sets M and N: $M \cap N = \{x \| x \in M \textbf{ and } x \in N\}$
- if $M = \{a, b, c\}$ and $N = \{a, b\}$ then $M \cap N = \{a, b\}$
- as a general principle (Consitency): $M \subseteq N$ iff $M \cup N = N$ and $M \subseteq N$ iff $M \cap N = M$

- $\bigcup M = \{x \| x \in Y$ **for some** $Y \in M\}$
- (a) if $M = \{\{a\}, \{a, b\}, \{a, b, c\}\}$ then $\bigcup M = \{a, b, c\}$
- (b) $M_1 = \{a\}$, $M_2 = \{a, b\}$, $M_3 = \{a, b, c\}$, $I = \{1, 2, 3\}$; $\bigcup_{i \in I} M = \{a, b, c\}$
- $\bigcap M = \{x \| x \in Y$ **for every** $Y \in M\}$
- (a) if $M = \{\{a\}, \{a, b\}, \{a, b, c\}\}$ then $\bigcap M = \{a\}$
- (b) $M_1 = \{a\}$, $M_2 = \{a, b\}$, $M_3 = \{a, b, c\}$, $I = \{1, 2, 3\}$; $\bigcap_{i \in I} M = \{a\}$

Difference - and complement $\setminus$ and $'$

- For any two sets M and N: $M - N = \{x \| x \in M \text{ and } x \notin N\}$
- $M = \{a, b, c\}$, $N = \{a\}$, $M - N = \{b, c\}$
- For any two sets M and N: $M \setminus N = \{x \| x \in N \text{ and } x \notin M\}$
- $O = \{a, b, c, k\}$ $M \setminus O = \{k\}$
- the universal complement: $M' = \{x \| x \in U \text{ and } x \notin M\}$
  (U the universal set)

Trivial equalities

- Idempotency: $M \cup M = M$, $M \cap M = M$
- Commutativity for $\cup$ and $\cap$: $M \cup N = N \cup M$ ...
- Associativiy for $\cup$ and $\cap$: $(M \cup N) \cup O = M \cup (N \cup O)$ ...
- Distributivity for $\cup$ and $\cap$: $M \cup (N \cap O) = (M \cup N) \cap (M \cup O)$ ...
- Identity: $M \cup \emptyset = X$, $M \cup U = U$ ...what about $\cap$

More interesting equalities

- Complement laws: $M \cup \emptyset = M$, $M'' = M$, $M \cap M' = \emptyset$, $X \cap U = U$
- DeMorgan: $(M \cup N)' = M' \cap X'$ ...

Funktionen und Relationen

How to define an ordered pair

- ...without introducing ordered tuples as a new primitive
- take S=$\{\{a\},\{a,b\}\}$
- we write: $\langle a,b\rangle = \{\{a\},\{a,b\}\}$
- orderend n-tuples defined recursively
- $\langle a,b\rangle \neq \langle b,a\rangle$
- first and second coordinate of the tuple

Cartesian products

- sets of ordered pairs
- tupling each member of the first argument with each of the second
- $S_1 \times S_2 = \{\langle x, y \rangle \| x \in S_1 \text{ and } y \in S_2\}$
- for an arbitrary number of sets: $S_1 \times \cdots \times S_n = \{\langle x_1, x_2, \ldots, x_n \rangle \| x_i \in S_i\}$
- $\langle x_1, x_2, \ldots, x_n \rangle$ abbreviated $\vec{x}$
- for $S \times S \times \cdots$: n-fold products
  $S^n = \{\vec{s} \| s_i \in S \text{ for } 1 \leq i \leq n\}$

Defintion of relations

- hold between (sets of) objects
- *x kicks y*, *x lives on the same floor as y*, ...
- formalization: Rab, aRb
- $a \in A$ and $b \in B$: $R \subseteq A \times B$,
  R is from A (**domain**) to B (**range**)
- R from A to A is in A

Complement, inverse

- complement $R' = \{\langle a, b \rangle \notin R\}$ for $R \subseteq A \times B$
  - R = the relation of teacherhood between a and b (the **arguments**)
  - R′ = all pairs $\langle b, a \rangle$ s.t. it is false that the first member is the teacher of the second member
- inverse: $R^{-1} = \{\langle b, a \rangle \| \langle a, b \rangle \in R\}$ for $R \subseteq A \times B$
  - R = the relation of teacherhood between a and b:
    Herr Webelhuth is the teacher of Herr Schäfer.
  - R^{-1} = all pairs $\langle b, a \rangle$ where *a* is the teacher of *b*:
    Herr Schäfer is the inverse-teacher of Herr Webelhuth.

Functions

- A function F from A to B is a relation s.t. for every $a \in A$ there is exactly on tuple $\langle a, b \rangle \in A \times B$ s.t. a is the first coordinate.
- partial function from A to B: for some $a \in A$ there is no tuple $\langle a, b \rangle \in A \times B$, F is not *defined for* some a

Injection, surjection, bijection

- B the range of F, F is **into** B
- F from A to B is **onto (a surjection)** B iff there is no $b_i \in B$ s.t. there is no $\langle a, b_i \rangle \in F$
- F from A to B is **one-to-one (an injection)** iff there are no two pairs s.t. $\langle a_i, b_j \rangle \in F$ and $\langle a_k, b_j \rangle \in F$
- one-to-one, onto, and total function: correspondence (bijection)

Composition

- One can take the range of a function and make it the domain of another function.
- A function $F_1 : A \rightarrow B$ and a function $F_2 : B \rightarrow C$ can be composed as $B(A(a))$, short $B \circ A$
- the compound function can be empty, it will be total if both A and B are bijections.

Mehr über Relationen und Mengen

Reflexivity

A relation R in $A = \{a, b, \ldots\}$ is...

	if	(ex.)
reflexive	for **every** $a \in A$: $\langle a, a \rangle \in R$	is as heavy as A: physical objects
irreflexive	for **every** $a \in A$: $\langle a, a \rangle \notin R$	is the father of
non-reflexive	for **some** $a \in A$: $\langle a, a \rangle \notin R$	has hurt

Symmetry

A relation R in $A = \{a, b, \ldots\}$ is...

	if	(ex.)
symmetric	for every $\langle a, b \rangle \in R$: $\langle b, a \rangle \in R$	has the same car as
asymmetric	for every $\langle a, b \rangle \in R$: $\langle b, a \rangle \notin R$	has a different car than
non-symmetric	for some $\langle a, b \rangle \in R$: $\langle b, a \rangle \notin R$	is the sister of
anti-symmetric	for every $\langle a, b \rangle \in R$: $a = b$	beats oneself not every human does

Transitivity

A relation R in $A = \{a, b, \ldots\}$ is...

	if	(ex.)
transitive	if $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ then $\langle a, c\rangle \in R$	is to the left of
intransitive	the above is never the case	is the father of
non-transitive	the above is sometimes not the case	likes

Connectedness

A relation R in $A = \{a, b, \ldots\}$ is...

	if	(ex.)
connected	for every $a, b \in A$, $a \neq b$: either $\langle a, b\rangle \in R$ or $\langle b, a\rangle \in R$	$>$ (A: the natural numbers)
non-connected	for some $a, b \in A$ the above is not the case	likes

Equivalence relations

- reflexive ($\langle a, a \rangle \in R$ for every a)
- symmetric ($\langle b, a \rangle \in R$ for every $\langle a, b \rangle$)
- transitive ($\langle a, b \rangle \in R$ & $\langle b, c \rangle \in R \rightarrow \langle a, c \rangle \in R$)
- *is as stupid as*
- partition the range into equivalence classes:
  $A = \{a, b, c, d\}$, for example $P_{A_1} = \{\{a, b\}, \{c\}, \{d\}\}$
- not $\{\{a\}, \{b, c\}\}$ or $\{\{a, b\}, \{b, c\}, \{d\}\}$

Defining ordering relations

An ordering relation R in A is ...

- transitive ($\langle a, b\rangle \in R$ & $\langle b, c\rangle \in R \rightarrow \langle a, c\rangle \in R$) ...plus ...
- irreflexive and asymmetric: **strict order**
- $A = \{a, b, c, d\}$, $R_1 = \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$
- reflexive and anti-symmetric: **weak order**
- $A = \{a, b, c, d\}$, $R_1 = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$

Orders: an example

- a strict order: *greater than* ($>$) in $\mathbb{N}$
- what is the corresponding weak order
- $\geq$

- **minimal**: x is not preceded
- **least**: x precedes every other lement
- **maximal**: x is not succeeded
- **greatest**: x succeeds every other element
- **well-ordering**: total order, every subset has a least element

Literatur I

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