# Formale Semantik
# 03. Mengen und Funktionen

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stets aktuelle Fassungen: `https://github.com/rsling/VL-Deutsche-Syntax`

# Inhalt

# Sets and Functions

# What is a set?

- a freely defined unordered collection of discrete objects
  - numbers,
  - people,
  - pairs of shoes,
  - words, ...
- not necessarily for any purpose
- no object occurs more than once

# Set definition and elements: $\in$

- $M_1 = \{a, b, c\}$
- $N_1$ = {‘my book’}
  vs. $N_2$ = {my book}
  vs. $N_3$ = {‘my’, ‘book’}
- ill-formed: $N_4$ = {‘my’, book}
- defined by a property of its members:
  $M_2$ = {x:x is one of the first three letters of the alphabet}
- alternatively:
  $M_2$ = {x||x is one of the first three letters of the alphabet}
- U: the universal set (contains every discrete object)

# Equality: =

- Two sets with contain exactly the same members are *equal*.
- independent of definition:
  {a,b,c} =
  {x:x is one of the first three letters of the alphabet}
- {x:x is human} = {x:x is from the planet earth and x can speak}

# Subsets: $\subseteq$

- A set N which holds no member which is not in M is a *subset* of M: N $\subseteq$ M
- $\{a\} \subseteq \{a, b, c\}$
- the inverse: the superset

# Proper subsets: $\subset$

- A set N which holds no member which is not in M and which is not equal to M is a *proper subset* of M: N $\subset$ M
- So, actually: $\{a\} \subset \{a, b, c\}$ and $\{a, b, c\} \subseteq \{a, b, c\}$. Note that:
- $M \subseteq M$ but $M \not\subset M$
- $\{\{a\}\} \not\in \{a, b, c\}$
- $\{\} \subset \{a, b, c\}$ (or any set), $\{\}$ is sometimes written $\emptyset$

# Elements vs. subsets

- *All professors of English Linguistics are human.*
  *Herr Webelhuth is a professor of English Linguistics.*
- w = Herr Webelhuth
  E = the set of professors of English Linguistics
  H = the set of human beings
- $w \in E \;\&\; E \subset H \Rightarrow w \in H$

# Elements vs. subsets

- But: *Professors of English Linguistics are numerous.*
- N = the set of sets with numerous members
- $w \in E$ & $E \in N \nRightarrow w \in P$
- Hence: *Herr Webelhuth is numerous.

# Power sets: $\wp$

- For any set M: $\wp(M) = \{X \| X \subseteq M\}$
- for M=$\{a, b, c\}$:
  $\wp(M) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}, \{b, c\}\}$
- Why is the empty set in the power set of every set ...
- ...and why is the empty a set a proper subset of every set?

# Union ∪ and intersection ∩

- For any sets M and N: $M \cup N = \{x \| x \in M \textbf{ or } x \in N\}$
- if $M = \{a, b, c\}$ and $N = \{a, b, d\}$ then $M \cup N = \{a, b, c, d\}$
- For any sets M and N: $M \cap N = \{x \| x \in M \textbf{ and } x \in N\}$
- if $M = \{a, b, c\}$ and $N = \{a, b\}$ then $M \cap N = \{a, b\}$
- as a general principle (Consitency): $M \subseteq N$ iff $M \cup N = N$ and $M \subseteq N$ iff $M \cap N = M$

# Generalized union $\bigcup$ and intersection $\bigcap$

- $\bigcup M = \{x || x \in Y$ **for some** $Y \in M\}$
- (a) if $M = \{\{a\}, \{a, b\}, \{a, b, c\}\}$ then $\bigcup M = \{a, b, c\}$
- (b) $M_1 = \{a\}$, $M_2 = \{a, b\}$, $M_3 = \{a, b, c\}$, $I = \{1, 2, 3\}$; $\bigcup_{i \in I} M = \{a, b, c\}$
- $\bigcap M = \{x || x \in Y$ **for every** $Y \in M\}$
- (a) if $M = \{\{a\}, \{a, b\}, \{a, b, c\}\}$ then $\bigcap M = \{a\}$
- (b) $M_1 = \{a\}$, $M_2 = \{a, b\}$, $M_3 = \{a, b, c\}$, $I = \{1, 2, 3\}$; $\bigcap_{i \in I} M = \{a\}$

# Difference - and complement $\setminus$ and $'$

- For any two sets M and N: $M - N = \{x \| x \in M \text{ and } x \notin N\}$
- $M = \{a, b, c\}$, $N = \{a\}$, $M - N = \{b, c\}$
- For any two sets M and N: $M \setminus N = \{x \| x \in N \text{ and } x \notin M\}$
- $O = \{a, b, c, k\}$ $M \setminus O = \{k\}$
- the universal complement: $M' = \{x \| x \in U \text{ and } x \notin M\}$
  (U the universal set)

# Trivial equalities

- Idempotency: $M \cup M = M$, $M \cap M = M$
- Commutativity for $\cup$ and $\cap$: $M \cup N = N \cup M$ ...
- Associativiy for $\cup$ and $\cap$: $(M \cup N) \cup O = M \cup (N \cup O)$ ...
- Distributivity for $\cup$ and $\cap$: $M \cup (N \cap O) = (M \cup N) \cap (M \cup O)$ ...
- Identity: $M \cup \emptyset = X$, $M \cup U = U$ ...what about $\cap$

# More interesting equalities

- Complement laws: $M \cup \emptyset = M$, $M'' = M$, $M \cap M' = \emptyset$, $X \cap U = U$
- DeMorgan: $(M \cup N)' = M' \cap X'$ ...

# Functions and Relations

# How to define an ordered pair

- ...without introducing ordered tuples as a new primitive
- take S=$\{\{a\},\{a,b\}\}$
- we write: $\langle a,b\rangle = \{\{a\},\{a,b\}\}$
- orderend n-tuples defined recursively
- $\langle a,b\rangle \neq \langle b,a\rangle$
- first and second coordinate of the tuple

# Cartesian products

- sets of ordered pairs
- tupling each member of the first argument with each of the second
- $S_1 \times S_2 = \{\langle x, y\rangle \| x \in S_1 \text{ and } y \in S_2\}$
- for an arbitrary number of sets: $S_1 \times \cdots \times S_n = \{\langle x_1, x_2, \ldots, x_n\rangle \| x_i \in S_i\}$
- $\langle x_1, x_2, \ldots, x_n\rangle$ abbreviated $\vec{x}$
- for $S \times S \times \cdots$: n-fold products
  $S^n = \{\vec{s} \| s_i \in S \text{ for } 1 \leq i \leq n\}$

# Defintion of relations

- hold between (sets of) objects
- *x kicks y*, *x lives on the same floor as y*, ...
- formalization: Rab, aRb
- $a \in A$ and $b \in B$: $R \subseteq A \times B$,
  R is from A (**domain**) to B (**range**)
- R from A to A is in A

# Complement, inverse

- complement $R' = \{\langle a, b \rangle \ \notin R\}$ for $R \subseteq A \times B$
  - R = the relation of teacherhood between a and b (the **arguments**)
  - R′ = all pairs $\langle b, a \rangle$ s.t. it is false that the first member is the teacher of the second member
- inverse: $R^{-1} = \{\langle b, a \rangle \| \langle a, b \rangle \in R\}$ for $R \subseteq A \times B$
  - R = the relation of teacherhood between a and b:
    *Herr Webelhuth is the teacher of Herr Schäfer.*
  - $\mathrm{R}^{-1}$ = all pairs $\langle b, a \rangle$ where *a* is the teacher of *b*:
    *Herr Schäfer is the inverse-teacher of Herr Webelhuth.*

# Functions

- A function F from A to B is a relation s.t. for every $a \in A$ there is exactly on tuple $\langle a, b \rangle \in A \times B$ s.t. $a$ is the first coordinate.
- partial function from A to B: for some $a \in A$ there is no tuple $\langle a, b \rangle \in A \times B$, F is not *defined for* some $a$

# Injection, surjection, bijection

- B the range of F, F is **into** B
- F from A to B is **onto (a surjection)** B iff there is no $b_i \in B$ s.t. there is no $\langle a, b_i \rangle \in F$
- F from A to B is **one-to-one (an injection)** iff there are no two pairs s.t. $\langle a_i, b_j \rangle \in F$ and $\langle a_k, b_j \rangle \in F$
- one-to-one, onto, and total function: correspondence (bijection)

# Composition

- One can take the range of a function and make it the domain of another function.
- A function $F_1 : A \rightarrow B$ and a function $F_2 : B \rightarrow C$ can be composed as $B(A(a))$, short $B \circ A$
- the compound function can be empty, it will be total if both A and B are bijections.

# More about relations and sets

# Reflexivity

A relation R in $A = \{a, b, \ldots\}$ is...

| | if | (ex.) |
|---|---|---|
| reflexive | for **every** $a \in A$: $\langle a, a \rangle \in R$ | is as heavy as |
| | | A: physical objects |
| irreflexive | for **every** $a \in A$: $\langle a, a \rangle \notin R$ | is the father of |
| non-reflexive | for **some** $a \in A$: $\langle a, a \rangle \notin R$ | has hurt |

# Symmetry

A relation R in $A = \{a, b, \ldots\}$ is...

| | if | (ex.) |
|---|---|---|
| symmetric | for every $\langle a, b \rangle \in R$: $\langle b, a \rangle \in R$ | has the same car as |
| asymmetric | for every $\langle a, b \rangle \in R$: $\langle b, a \rangle \notin R$ | has a different car than |
| non-symmetric | for some $\langle a, b \rangle \in R$: $\langle b, a \rangle \notin R$ | is the sister of |
| anti-symmetric | for every $\langle a, b \rangle \in R$: $a = b$ | beats oneself<br>not every human does |

# Transitivity

A relation R in $A = \{a, b, \ldots\}$ is...

| | if | (ex.) |
|---|---|---|
| transitive | if $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ then $\langle a, c\rangle \in R$ | is to the left of |
| intransitive | the above is never the case | is the father of |
| non-transitive | the above is sometimes not the case | likes |

# Connectedness

A relation R in $A = \{a, b, \ldots\}$ is...

| | if | (ex.) |
|---|---|---|
| connected | for every $a, b \in A$, $a \neq b$: either $\langle a, b\rangle \in R$ or $\langle b, a\rangle \in R$ | $>$ (A: the natural numbers) |
| non-connected | for some $a, b \in A$ the above is not the case | likes |

# Equivalence relations

- reflexive ($\langle a, a \rangle \in R$ for every $a$)
- symmetric ($\langle b, a \rangle \in R$ for every $\langle a, b \rangle$)
- transitive ($\langle a, b \rangle \in R$ & $\langle b, c \rangle \in R \rightarrow \langle a, c \rangle \in R$)
- *is as stupid as*
- partition the range into equivalence classes:
  $A = \{a, b, c, d\}$, for example $P_{A_1} = \{\{a, b\}, \{c\}, \{d\}\}$
- not $\{\{a\}, \{b, c\}\}$ or $\{\{a, b\}, \{b, c\}, \{d\}\}$

# Defining ordering relations

An ordering relation R in A is ...

- transitive ($\langle a, b\rangle \in R$ & $\langle b, c\rangle \in R \rightarrow \langle a, c\rangle \in R$) ...plus ...
- irreflexive and asymmetric: **strict order**
- $A = \{a, b, c, d\}$, $R_1 = \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$
- reflexive and anti-symmetric: **weak order**
- $A = \{a, b, c, d\}$, $R_1 = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$

# Orders: an example

- a strict order: *greater than* ($>$) in $\mathbb{N}$
- what is the corresponding weak order
- $\geq$

- **minimal**: x is not preceded
- **least**: x precedes every other lement
- **maximal**: x is not succeeded
- **greatest**: x succeeds every other element
- **well-ordering**: total order, every subset has a least element

# Cardinalities

# The number of elements...

- $A = \{a, b, c\}$
- $B = \{a, b, c\}$
- obviously, $A = B$ (equal)
- there is an $R$ from A to B s.t. $R = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle\}$
- for every set C with the same number of elements (e.g., $C = \{1, 2, 3\}$): $R = \{\langle a, 1\rangle, \langle b, 2\rangle, \langle c, 3\rangle\}$
- such relations are one-to-one correspondences

# Denumerable sets

- $\mathbb{N}$ is infinite
- for every A there is some $R_{card}$
  - a one-to-one correspondence
  - from A's members to the first *n* members of $\mathbb{N}$
  - s.t. *n* is the cardinality of A, $\|A\|$
- sets A,B with $\|A\| = \|B\|$ are equivalent
- $\|\mathbb{N}\| = \aleph^0$

# A problem

- for some sets there is no such $R_{card}$
- no way of bringing their elements into an exhaustive linear order
- no problem with $\mathbb{Q}$:

$$
\begin{array}{ccccc}
 & \langle 0,1\rangle & \langle 0,2\rangle & \langle 0,3\rangle & \cdots \\
\langle 1,0\rangle & \langle 1,1\rangle & \langle 1,2\rangle & \langle 1,3\rangle & \cdots \\
\langle 2,0\rangle & \langle 2,1\rangle & \langle 2,2\rangle & \langle 2,3\rangle & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

# The non-denumerable real numbers

- now: $\mathbb{R}$
- some elements cannot be represented as an ordered pair of two elements of $\mathbb{N}$
- in $[0, 1]$, every real can be represented as $0.abcdefg\ldots$,
  $a, b, c, d, e, f, g, \ldots \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

# Trying to enumerate

- an enumeration of $[0, 1]$ in $\mathbb{R}$?

$$
\begin{array}{llllllllll}
x_1 & = & 0 & . & a_{11} & a_{12} & a_{13} & a_{14} & \dots \\
x_2 & = & 0 & . & a_{21} & a_{22} & a_{23} & a_{24} & \dots \\
x_3 & = & 0 & . & a_{31} & a_{32} & a_{33} & a_{34} & \dots \\
\vdots & & \vdots & & & & & & \\
x_n & = & 0 & . & a_{n1} & a_{n2} & a_{n3} & a_{n4} & \dots
\end{array}
$$

# Failing to enumerate

- What about an $x_m$ which differs from $x_n$ at $a_{nn}$

$$
\begin{array}{cccccccccc}
x_1 & = & 0 & . & \color{red}{a_{11}} & a_{12} & a_{13} & a_{14} & \ldots \\
x_2 & = & 0 & . & a_{21} & \color{red}{a_{22}} & a_{23} & a_{24} & \ldots \\
x_3 & = & 0 & . & a_{31} & a_{32} & \color{red}{a_{33}} & a_{34} & \ldots \\
\vdots & & \vdots & & & & & & \\
x_n & = & 0 & . & a_{n1} & a_{n2} & a_{n3} & \color{red}{a_{nn}} & \ldots
\end{array}
$$

- It won't be in the array...
- $\mathbb{R}$ is non-denumerable
- If $\|A\| = \aleph^0$ then $\|\wp(A)\| = 2^{\aleph_0}$ (cf. Partee et al. 62f.)

# Literatur I

# Autor

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# Lizenz

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