# Formale Semantik 05. Prädikatenlogik

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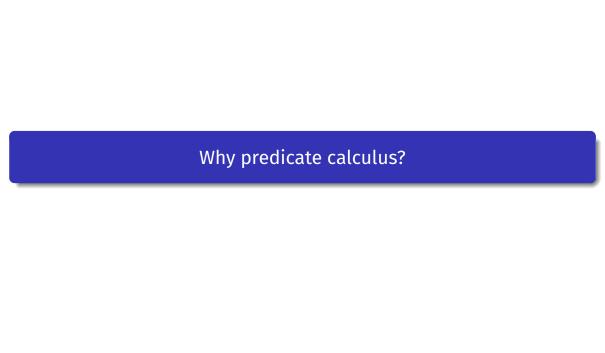
stets aktuelle Fassungen: https://github.com/rsling/VL-Deutsche-Syntax

### Inhalt

- 1 Why predicate calculus?
- 2 The construction of PC
  - Atoms and syntax Semantics
  - More rules

- 3 Laws of PC
  - Negation and distributionMovement

  - Some in-class practice
  - Natural deduction in PC
    - Quantifier elimination
    - An example



## Weak compositionality in SL

- properties/relations vs. individuals
- Martin is an expert on inversion and Martin is a good climber.
- ...becomes  $E \wedge C$
- compositionality resticted to level of connected propositional atoms

### Some desirable deductions

- important generalizations about all and some individuals (which have property P)
- 'all P  $\rightarrow$  some P'
- 'Martin P  $\rightarrow$  some P'



#### Atoms of PC

- individual variables:  $x, y, z, x_1, x_2 \dots$
- individual constants: a, b, c, . . .
- variables and constants: terms
- predicate symbols (taking individual symbols or tuples of them):  $A, B, C, \ldots$
- quantifiers: existential  $\exists$  (or  $\lor$ ) and universal  $\forall$  (or  $\land$ )
- plus the connectives of SL

### Some syntax

- for an *n*-ary predicate P and terms  $t_1 ldots t_n$ ,  $P(t_1 ldots t_n)$  or  $Pt_1 ldots t_n$  is a wff.
- possible prefix, function (bracket) and infix notation:
   Pxy, P(x, y), xPy
- syntax for connectives from SL
- for any wff  $\phi$  and any variable x,  $(\exists x)\phi$  and  $(\forall x)\phi$  are wff's

### Semantic for individual constants

- denote individuals
- a model  $\mathcal{M}$  contains a set of individuals D
- the valuation function V (or F): from constants to individuals in D
- for some  $\mathcal{M}_1$ :  $D = \{Martin, Kilroy, Scully\}$
- $V_{\mathcal{M}_1}(m) = Martin$
- $V_{\mathcal{M}_1}(k) = Kilroy$ ,  $V_{\mathcal{M}_1}(s) = Scully$

## Semantics for predicate symbols

- denote relations (sets of n-tuples)
- $\llbracket P \rrbracket^{\mathcal{M}_1} = \{ Martin, Kilroy \}$  or  $V_{\mathcal{M}_1}(P) = \{ Martin, Kilroy \}$
- $V_{\mathcal{M}_1}(Q) = \{\langle Martin, Kilroy \rangle, \langle Martin, Scully \rangle, \langle Kilroy, Kilroy \rangle, \langle Scully, Scully \rangle \}$
- s.t.  $\llbracket P(m) \rrbracket^{\mathcal{M}_1} = \llbracket P \rrbracket^{\mathcal{M}_1}(\llbracket m) \rrbracket^{\mathcal{M}_1}) = 1$  iff  $\llbracket m \rrbracket^{\mathcal{M}_1} \in \llbracket P \rrbracket^{\mathcal{M}_1}$

## Semantics for connectives and quantifiers

- connectives: 'apply to' formulas (semantically truth-valued), semantics as in SL
- $(\forall x)\phi$  = 1 iff  $\phi$  is true for every  $d \in D$  assigned to every occurrence of x in  $\phi$
- $(\exists x)\phi$  = 1 iff  $\phi$  is true for at least one  $d \in D$  assigned to every occurrence of x in  $\phi$
- algorithmic instruction to check wff's containing Q's
- check outside-in (unambiguous scoping)

## Dependencies

• universal quantifiers can be swapped:

$$(\forall x)(\forall y)\phi \Leftrightarrow (\forall y)(\forall x)\phi$$

• same for existential quantifiers:

$$(\exists x)(\exists y)\phi \Leftrightarrow (\exists y)(\exists x)\phi$$

- whereas:  $(\exists x)(\forall y)\phi \Rightarrow (\forall y)(\exists x)\phi$
- example in  $\mathcal{M}_1$ :
  - $\blacktriangleright \ \llbracket (\forall \mathbf{x})(\exists \mathbf{y})\mathbf{Q}\mathbf{x}\mathbf{y} \rrbracket^{\mathcal{M}_1} = 1$
  - $but: \llbracket (\exists y)(\forall x)Qxy \rrbracket^{\mathcal{M}_1} = 0$
  - direct consequence of algorithmic definition
  - if ∃∀ is true, ∀∃ follows

## Hints on quantifiers

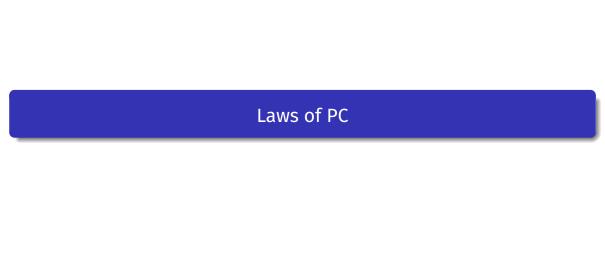
- domain of quantifiers: D (universe of discourse)
- $\forall x$  checks for truth of some predication for all individuals
- $\exists x (Px \land \neg Px)$  is a contradiction
- $\forall x (Wx \land \neg Wx)$  is a contradiciton,  $\forall x$  'checks' for an empty set by def.
- standard form of NL quantification:  $\forall x (Wx \rightarrow Bx)$  'All women are beautiful.'
- standard form of NL existential quantification:  $\exists x (Wx \land Bx)$  'Some woman is beautiful.'

## Functor/quantifier practice

- by def., functors take formulas, not terms:
  - ► ¬Wm 'Mary doesn't weep.'
  - ▶  $(\exists x)(Gx \land Wx)$  'Some girl weeps.'
  - ▶ \*W¬x
  - $*(\exists \neg x)(Gx)$
- quantifiers take variables, not constants:
  - ▶  $(\forall x)(Ox \rightarrow Wx)$  'All ozelots are wildcats.'
  - ▶ \*(∀o)(Wo)
- negates the wff, not the q:
  - \* $(\neg \forall x)Px$  but  $\neg(\forall x)Px$

### Scope

- quantifiers bind variables
- free variables (constants) are unbound
- no double binding  $*(\forall x \exists x) Px$
- Q scope: only the first wff to its right:
  - ▶  $(\forall x)Px \lor Qx$
  - $(\forall x)(Px \lor Qx) = (\forall x)Px \lor (\forall x)Qx$
  - $\overline{(\exists x)Px \to (\forall y)}(Q\overline{y \wedge Ry})$
  - ▶  $(\exists x)Px \land Qx$  (second x is a unbound)
- no double-naming



### Universal $\lor$ and $\land$

- $\exists$  and  $\forall$  'or' and 'and' over the universe of discourse (hence:  $\bigvee$  and  $\bigwedge$ )
- $(\forall x)$ Px  $\Leftrightarrow$  Px<sub>1</sub>  $\land$  Px<sub>2</sub>  $\land \dots \land$  Px<sub>n</sub> for all x<sub>n</sub> assigned to  $d_n \in D$
- $(\exists x)Px \Leftrightarrow Px_1 \lor Px_2 \lor \ldots \lor Px_n$  for all  $x_n$  assigned to  $d_n \in D$
- hence:  $\neg(\forall x) Px \Leftrightarrow \neg(Px_1 \land Px_2 \land \ldots \land Px_n)$
- with DeM:  $\overline{Px_1 \wedge Px_2 \wedge \ldots \wedge Px_n}$
- $\Leftrightarrow \overline{Px_1} \vee \overline{Px_2} \vee \ldots \vee \overline{Px_n}$
- $\Leftrightarrow (\exists x) \neg Px$

## Quantifier negation (QN)

- $\neg(\forall x)Px \Leftrightarrow (\exists x)\neg Px$
- $\neg(\exists x)Px \Leftrightarrow (\forall x)\neg Px$
- $\neg(\forall x)\neg Px \Leftrightarrow (\exists x)Px$
- $\bullet \ \neg(\exists x)\neg Px \Leftrightarrow (\forall x)Px$

### The distribution laws

• the conjunction of universally quantified formulas:

$$(\forall x)(Px \land Qx) \Leftrightarrow (\forall x)Px \land (\forall x)Qx$$

• the disjunction of existentially quantified formulas:

$$(\exists x)(Px \lor Qx) \Leftrightarrow (\exists x)Px \lor (\exists x)Qx$$

- not v.v.:  $(\forall x)Px \lor (\forall x)Qx \Rightarrow (\forall x)(Px \lor Qx)$
- why?

## Quantifier movement (QM)

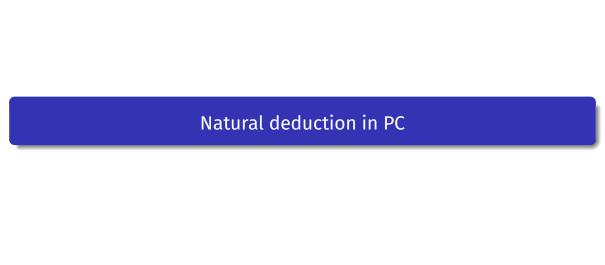
- desirable format: prefix + matrix
- Movement Laws for antecedents of conditionals:

$$(\exists x) Px \to \phi \Leftrightarrow (\forall x) (Px \to \phi)$$
$$(\forall x) Px \to \phi \Leftrightarrow (\exists x) (Px \to \phi)$$

- Movement Laws for Q's in disjunction, conjunction, and the consequent of conditionals: Just move them to the prefix!
- condition: x must not be free in  $\phi$ .
- i.e.: Watch your variables!

### Let's formalize:

- Paul Kalkbrenner is a musician and signed on bpitchcontrol.
- Herr <u>S</u>. installed <u>RedHat</u> and not every <u>Linux</u> distribution is <u>e</u>asy to install.
- All talkmasters are human and Harald Schmidt is a talkmaster.
- Some talkmasters are not musicians.
- Heiko Laux owns Kanzleramt records and does not like any Gigolo artist.
- Some <u>h</u>umans are neither <u>t</u>alkmasters nor do they <u>o</u>wn <u>K</u>anzleramt records.



# Universal instantiation ( $-\forall$ ) and generalization ( $+\forall$ )

- $(\forall x)Px \rightarrow Pa$
- always applies
- can use any variable/constant
- $Pa \rightarrow (\forall x)Px$
- iff Pa was instantiated by  $-\forall$

# Existential generalization $(+\exists)$ and instantiation $(-\exists)$

- $Pa \rightarrow (\exists x)Px$  for any individual constant a
- always applies
- $(\exists x)Px \rightarrow Pa$  for some indiv. const.
- always applies (there is a minimal individual for  $\exists x$ )
- for some  $(\exists x)Px$  and  $(\exists x)Qx$  the minimal individual might be different
- hence: When you apply EI, always use fresh constants!

## One sample task

- (1) Herr Keydana drives a Golf. (2) Anything that drives a golf is human or a complex program simulating an artificial neural net. (3) There are no programs s.a.a.n.n. which are complex enough to drive a Golf.
- Formalize and prove: At least one human exists.
- (1) Dk
- (2)  $(\forall x)(Dx \rightarrow Hx \lor Px)$
- (3)  $\neg(\exists x)(Px \wedge Dx)$
- (∃x)Hx

## The proof

```
(1)
          Dk
(2)
         (\forall x)(Dx \rightarrow Hx \lor Px)
(3)
         \neg(\exists x)(Px \wedge Dx)
(4)
          (\forall x) \neg (Px \wedge Dx)
                                            3,QN
(5)
         (\forall x)(\neg Px \vee \neg Dx)
                                         4.DeM
(6)
      (\forall x)(Dx \rightarrow \neg Px)
                                            5,Comm,Impl
(7)
       \mathsf{Dk} 	o \neg \mathsf{Pk}
                                           6.−∀(1)
(8)
         \neg Pk
                                            1.7.MP
(9)
                                            2,−∀(1)
       \mathsf{Dk} \to \mathsf{Hk} \lor \mathsf{Pk}
(10) Hk \lor Pk
                                            1,9,MP
(11)
          Hk
                                            8,10,DS
          (\exists x)Hx
                                            10.+∃
```

# Literatur I

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