A tricky inequality

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1 Problem

Let $a+b+c=1$. Prove the inequality

$$\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}\leq\sqrt{21}. \tag{1}$$

2 Proof

I assume $a,b,c\in\mathbb{R}_{\geq 0}$. The assumption to stay on the positive real is justified below. We perform some preliminary algebra,

$$\begin{aligned}
\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}&\leq\sqrt{21}, \qquad (2)\\
4a+4b+4c+3+2(\sqrt{(4a+1)(4b+1)}+\sqrt{(4a+1)(4c+1)}+\sqrt{(4c+1)(4b+1)})&\leq 21,\\
4\underbrace{(a+b+c)}_{1}+3+2(\sqrt{(4a+1)(4b+1)}+\sqrt{(4a+1)(4c+1)}+\sqrt{(4c+1)(4b+1)})&\leq 21,\\
\sqrt{(4a+1)(4b+1)}+\sqrt{(4a+1)(4c+1)}+\sqrt{(4c+1)(4b+1)}&\leq 9.
\end{aligned}$$

Simplifying the left side we get

$$2(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})+2(\sqrt{a}+\sqrt{b}+\sqrt{c})\leq 3.$$

Using the general formula $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ac)$, we modify the left side and simplify

$$\begin{aligned}
(\sqrt{a}+\sqrt{b}+\sqrt{c})^2\underbrace{-\sqrt{a}^2-\sqrt{b}^2-\sqrt{c}^2}_{-1}+2(\sqrt{a}+\sqrt{b}+\sqrt{c})&\leq 3, \qquad (3)\\
(\sqrt{a}+\sqrt{b}+\sqrt{c})^2+2(\sqrt{a}+\sqrt{b}+\sqrt{c})&\leq 4.
\end{aligned}$$

At this point, we use Jensen's inequality for concave functions that says $f(ax+by+cz)\geq af(x)+bf(y)+cf(z)$ as long as $a+b+c=1$. Applied to the square root function here, $\sqrt{a}+\sqrt{b}+\sqrt{c}\leq\sqrt{a+b+c}$. So,

$$\begin{aligned}
(\sqrt{a+b+c})^2+2(\sqrt{a+b+c})&\leq 4, \qquad (4)\\
(1)^2+2(1)&\leq 4,\\
3&\leq 4.
\end{aligned}$$

Hence, proven.