A tricky inequality

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1 Problem

Let $a+b+c=1$. Prove the inequality

$$\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}\leq\sqrt{21}. \tag{1}$$

2 Proof

I assume $a,b,c\in\mathbb{R}_{\geq 0}$. The assumption to stay on the positive real is justified below. We perform some preliminary algebra,

$$\begin{aligned}
\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}&\leq\sqrt{21}, \qquad (2)\\
4a+4b+4c+3+2(\sqrt{(4a+1)(4b+1)}+\sqrt{(4a+1)(4c+1)}+\sqrt{(4c+1)(4b+1)})&\leq 21,\\
4\underbrace{(a+b+c)}_{1}+3+2(\sqrt{(4a+1)(4b+1)}+\sqrt{(4a+1)(4c+1)}+\sqrt{(4c+1)(4b+1)})&\leq 21,\\
\sqrt{(4a+1)(4b+1)}+\sqrt{(4a+1)(4c+1)}+\sqrt{(4c+1)(4b+1)}&\leq 9.
\end{aligned}$$