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Homework Assignment 5
Linear Algebra

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1 Question 1

1.1 Part 1

Let $\mathbf{v} \in V$ then we have from the second property of Linear Transformations that $\mathbf{T}(-\mathbf{v}) = \mathbf{T}(-\mathbf{1}.\mathbf{v}) = -\mathbf{1}.\mathbf{T}(\mathbf{v}) = -\mathbf{T}(\mathbf{v})$ which proves the second part of the question.

Now , we have from the additivity of Linear Transformations that $T(\mathbf{0_v}) = \mathbf{T}(\mathbf{v} + (-\mathbf{v})) = \mathbf{T}(\mathbf{v}) + \mathbf{T}(-\mathbf{v}) = \mathbf{T}(\mathbf{v}) - \mathbf{T}(\mathbf{v}) = \mathbf{0_w}$ which proves the first part. ■

1.2 Part 2

Given that T is a bijection this means that $\forall\, w \in W\ \exists!\, v \in V\ such\ that\ T(v) = w$

Now as T is a bijection , T^{-1} is a valid function from $W \mapsto V$.

Now , consider any $v_1, v_2 \in V$ such that $T(v_1) = w_1, T(v_2) = w_2$. By definition we have that, $T^{-1}(w_1) = v_1$ and $T^{-1}(w_2) = v_2$

$w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$ (from additivity of linear transformations).

Thus , $T^{-1}(w_1 + w_2) = v_1 + v_2 = T^{-1}(w_1) + T^{-1}(w_2)$, therefore T^{-1} satisfies additivity.

Next , for any $c \in \mathbb{F}$ we have , $cw_1 = cT(v_1) = T(cv_1)$ from scaling property of Linear Transformations but then ,

$T^1(cw_1) = cv_1 = cT^{-1}(w_1)$ and thus T^{-1} satisfies scaling property as well , concurrently therefore T^{-1} is also a linear transformation ■

2 Question 2

Let S be the set of all $\mathbb{F}$ vector spaces . Let $V, W, U \in S$ then we have that,

$V \cong V\ \forall\, V \in S$ this is because we can choose our isomorphism to be the identity isomorphism from $V \mapsto V$ i.e $\phi(x) = x\ \forall x \in V$

Therefore $\cong$ is reflexive.

Next , if $V \cong W$ under an isomorphism $\phi : V \mapsto W$ then $\phi^{-1} : W \mapsto V$ is also an isomorphism and therefore $W \cong V$ and the relation is symmetric.

Finally if $V \cong W$ under the isomorphism ϕ and $W \cong U$ under the isomorphism Γ then consider η such that $\eta : V \mapsto U$ and $\eta(x) = \Gamma(\phi(x))\ \forall x \in V$. Clearly , this is an isomorphism and therefore $V \cong U$ and thus the relation is transitive.

Since the relation is reflexive , symmetric and transitive it is equivalent.

3 Question 3

To show that $Lin(V, W)$ is a vector space we will do the following :

Define $\bigoplus : Lin(V, W) \times L(V, W) \mapsto Lin(V, W)$ in the following way :
given $T_1, T_2 \in Lin(V, W)$ we have that $(T_1 \bigoplus T_2)(x) = T_1(x) + T_2(x)\ \forall\, x \in V$

Next define , $\bigotimes : \mathbb{F} \times Lin(V, W) \mapsto Lin(V, W)$ in the following way :
given $T_1 \in Lin(V, W)$ and $c \in \mathbb{F}$ we have that $(c \bigotimes T_1)(x) = cT_1(x) \, \forall \, x \in V$

Clearly , both of our operations are valid from the properties of Linear Transformations. (i.e) $(T_1 \bigoplus T_2)$ and $(c \bigotimes T_1)$ are both in $Lin(V, W)$

If we define 0_{Lin} as the **zero linear transformation** (i.e the transformations that maps every element in V to 0_W) then using the fact that V, W are vector spaces it is trivial to verify that $\bigoplus$ with 0_{Lin} as the identity satisfies all properties of vector addition with $(-1 \bigotimes T)$ being the additive inverse of T . Similarly , $\bigotimes$ with $1_{\mathbb{F}}$ as the identity satisfies all properties of scalar multiplication. Distributivity follows trivially from the fact that V, W are vector spaces. $\blacksquare$

4 Question 4

Suppose that $A = B$ then we have $A - B = 0_{m \times n}$ and thus for any $v \in \mathbb{F}^{n \times 1}$ we have , $(A - B)v = 0_{n \times 1}$ which implies that $Av - Bv = 0_{n \times 1}$ and thus $Av = Bv$

To proceed in the backward direction , suppose that we are given that $Av = Bv$ Since this is true $\forall \, v \in \mathbb{F}^{n \times 1}$ in particular, it is true for the basis vectors $e_1, ...e_n$. Clearly , multiplying any matrix say $M \in \mathbb{F}^{m \times n}$ with e_j gives the j^{th} column of M.Thus , we have that j^{th} column of A is the same as that of the j^{th} column of B $\forall \, j \in \{1, ...n\}$ but these are all the columns of both matrices! Therefore all elements of the two matrices are common and consequently they are equal $\blacksquare$.

5 Question 5

5.1 Part 1

Let $[T]_{B,B} = [c_1|c_2]$ for some $c_1, c_2 \in \mathbb{F}^2$.

Given that $\{e_1, e_2\}$ is the basis
Then we have that , $[T(e_1)]_{B,B} = [T]_{B,B} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and therefore $c_1 = T(e_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Similarly $c_2 = T(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Therefore , $[T]_{B,B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

5.2 Part 2

We have that $T_{B'} = T_B \circ Id$ where Id is the Identity transformation which transforms vectors from basis B' to B which means we have that $[T]_{B,B'} = [T]_{B,B}[Id]_{B,B'}$
(Logically this can be thought of as first shifting the basis and then doing the transformation in the standard basis)

Again , let $[Id]_{B,B'} = [c_1|c_2]$ for some $c_1, c_2 \in \mathbb{F}^2$. Let e'_1, e'_2 be the new coordinates of the basis vectors of B in B' (Firstly note that if the basis vectors of B' are f_1, f_2 then $e'_1 = -f_2$ and $e_2 = f_1 + f_2$) then we have from the definition of the Identity transformation that $Id(e'_1) = e_1$ and $Id(e'_2) = e_2$ which means $[c_1|c_2] \times \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -c_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $[c_1|c_2] \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 + c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Therefore , $c_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \, and \, c_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ which means that ,

$Id = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

Finally $[T]_{B,B'} = [T]_{B,B}[Id]_{B,B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

5.3 Part 3

The matrix P is clearly the matrix Id which has been discussed above since $Id[v]_{B'} = [v]_B$ (this is true for the basis vectors in B and therefore true for all vectors by properties of matrix multiplication and vector spaces)

5.4 Part 4

We want to transform B to B' so we need the transformation A such that if $A = [c_1|c_2]$ for some $c_1, c_2 \in \mathbb{F}^2$ then $A(e_1) = e'_1$ and $A(e_2) = e'_2$ which implies that :

$[c_1|c_2] \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $[c_1|c_2] \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which means that $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

Therefore if v is the requisite vector we have $v = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

A quick check reveals that $-1 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + -3 \times \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ which means that we have the right coordinates. ■

6 Question 6

Switching to polar coordinates is a natural choice in this case. Given a vector v we may represent it as (r, β) where r refers to it's **norm** (usually Euclidean) which is a measure of its 'length' and β is the counterclockwise angle that it makes with the $'x'$ axis which is a measure of its 'direction'

Thus , the x coordinate of the vector in terms of its polar coordinates is $x = rcos(\beta)$ and similarly the y coordinate $y = rsin(\beta)$

Rotation by an angle θ changes the direction while preserving the length thus moving the vector to $(r, \theta + \beta)$.

To get the transformation matrix , let's understand how the basis vectors are transformed in this process.

The length or norm of each basis vector is unit with their angles being $\theta = 0$ and $\theta = \frac{\pi}{2}$
therefore the new points are $(1, \theta)$ and $(1, (\theta + \frac{\pi}{2}))$ respectively which correspond to the following two vectors in standard basis : $\begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$ and $\begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$ respectively.

Thus if $T = [c_1|c_2]$ for some $c_1, c_2 \in \mathbb{R}$ is the required matrix , we have that for the standard basis vectors e_1, e_2

$T(e_1) = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$ which gives :

$[c_1|c_2] \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$ and $[c_1|c_2] \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_2 = \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$

Thus we have that , $T = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

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