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Homework Assignment 4
Linear Algebra

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1 Question 1

Given an upper triangular matrix with non-zero diagonals proving that it is Row equivalent to Id is equivalent to stating that it is invertible. Since given a vector $\mathbf{x} \in \mathbb{F}^{\mathbf{n}\times\mathbf{1}}$ the solutions to the equation $\mathbf{Ax} = \mathbf{0}$ would only be $\mathbf{x} = \mathbf{0}$.
We state an algorithm to transform any such upper triangular matrix into an Identity matrix :

Algorithm 1: Transforming an Upper triangular matrix into an Identity matrix

i=1;
while $i \leq n$ **do**
  Set the pivot to 1 ;
  $R_i \mapsto R_i/A_{ii}$
  Set all other column entries to 0 ;
  j = 1 ;
  while $j < i$ **do**
    $R_j \mapsto R_j - A_{ji}R_i$
    $j \mapsto j+1$
  end
  $i \mapsto i+1$
end

After applying this algorithm all diagonal entries are $1_{\mathbb{F}}$ and all non diagonal entries are $0_{\mathbb{F}}$ and thus the conclusion follows.

For the converse, given an invertible upper triangular matrix we wish to show that all its diagonal elements are non-zero. Suppose to the contrary this was not the case i.e $\exists\ m : 1_{\mathbb{F}} \leq m \leq n$ number of rows in the matrix such that each has the diagonal element 0.

Applying Algorithm 1 on this matrix by skipping these m rows guarantees that we would eventually get a matrix with $n-m$ pivot and m free rows where the free rows are the ones with the diagonal elements $0_{\mathbb{F}}$

Since the RREF form of this matrix has free rows there exists a non-trivial solution to the equation $\mathbf{Ax} = \mathbf{0}$ which means it is not invertible , a contradiction given our assumption. Thus we are done

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2 Question 2

Consider the sequence of Sets indexed by $n \in \mathbb{N}$ such that $S_n = \{x^i : 0 \leq i \leq n\}$

Observe that each S_i is linearly independent and that its span forms a subspace of $\mathbb{F}[x]$ (this subspace happens to be the space of all polynomials of degree less than or equal to n)

Consider any finite set of Polynomials say P Then, let $k = max\{deg(u) : u \in P\}$ (k exists as P is finite !)

Then $Span(P) \subseteq Span(S_k)$.

Notice however that $Span(S_i) \neq \mathbb{F}[x]\ \forall i$ as x^j for $j > i$ is not in $Span(S_i)$
The conclusion then is straightforward. ■

3 Question 3

Clearly $Fun(D, \mathbb{F}) \subset \{f : D \mapsto \mathbb{F}\}$ where $\{f : D \mapsto \mathbb{F}\}$ which is the set of all functions mapping D to $\mathbb{F}$ is a vector space over $\mathbb{F}$ with the needed properties.

Now , let $f_1, f_2 \in Fun(D, \mathbb{F})$ then :

Let $f_3 = af_1 + bf_2$ for some $a, b \in \mathbb{F}$

Let $A_1 = \{x \in D : f_1(x) \neq 0\}$ Similarly define A_2 for f_2. Note that both the sets defined above are finite sets.

Thus , Let $|A_1| = k$ and $|A_2| = l$. Consider the set A_3 defined analogously for f_3.

It is clear that $|A_3| \leq k + l$ where the equality is attained when the sets A_1 and A_2 are disjoint

But this means that A_3 is finite !

Or $f_3 \in Fun(D, \mathbb{F})$

$Fun(D, \mathbb{F})$ therefore is a subspace of $\{f : D \mapsto \mathbb{F}\}$ and thus it is a vector-space over $\mathbb{F}$

Now , to construct a basis we will use the Kronecker Delta Function as an inspiration.
Define $\delta_i(x) : D \mapsto \mathbb{F}$ as follows :
$\delta_i(x) = \begin{Bmatrix} 1_{\mathbb{F}} & if\ x = i \\ 0_{\mathbb{F}} & otherwise \end{Bmatrix}$
Note that $\delta_i(x) \in Fun(D, \mathbb{F})\ \forall\ i \in D$

For any arbitrary $f \in Fun(D, \mathbb{F})$ define $\Phi_f = \sum_{j \in A_f} f(j)\delta_j$ where A_f is once again the finite set of all elements not mapped to $0_{\mathbb{F}}$ by f.

Clearly , $\Phi_f = f$ as both functions have the same values at all points.

Since f was arbitrary such a decomposition exists for all functions in our vector space and therefore the set $\{\delta_i : i \in D\}$ spans $Fun(D, \mathbb{F})$.

Finally , observe that the set $\{\delta_i : i \in D\}$ is linearly independent since in any linear combination if one term maps to a non-zero point in $\mathbb{F}$ the others have to map to $0_{\mathbb{F}}$ by our definition and thus there is no non-trivial combination that maps to the zero function.

Thus , $\{\delta_i : i \in D\}$ is a basis for our vector subspace ■

4 Question 4

Suppose to the contrary that $\mathbb{R}$ has a finite dimension over $\mathbb{Q}$ let this be $n \in \mathbb{N}$. We can thus define a basis of $\mathbb{R}$ over $\mathbb{Q}$ let this be $B = \{q_1, ...q_n\}$, where each $q_i \in \mathbb{Q}$ and $q_i \neq q_j\ \forall\ i \neq j$

By the definition of the basis each $x \in \mathbb{R}$ can be represented as a linear combination of the elements of this basis.

Thus, let $f : \mathbb{Q}^n \mapsto \mathbb{R}$ be a function defined in the following manner : Let $x = (x_1, ..x_n) \in \mathbb{Q}^n$ then :

$f(x) = \sum_{i=1}^{n} q_i x_i$

But as $\mathbb{Q}$ is a field $Range(f) = \mathbb{Q}$ and thus no linear combination of the basis can ever map to an irrational number! What this means that B does not span $\mathbb{R}$ which contradicts our assumption that B is a basis.

Now , given that V has a countably infinite Basis. First , we will use the following lemma :

Lemma : Every subset of a linearly independent set is a linearly independent set

Proof : Let the Linearly Independent Set be $S = \{a_1, ...a_n\}$. Fix any j of these elements to be our subset if this subset is not linearly independent $\exists\ c_1, ...c_j$ constants such that the linear combination sums to 0. Now chose these constants and add $n - j$ $0's$ for all the other $n - j$ elements in S so that we have a new non-trivial linear combination for S. This sums to 0 which contradicts the fact that S is linearly independent ! □

Now if B is the Basis of V such that $B = \{v_1, v_2...\}$ consider the sequence of sets B_i constructed in the following manner :

$B_i = \{v_1, ...v_i\}$

Let V_i be the subspace spanned by B_i. Clearly as each set in our sequence is linearly independent then we have that ,
$V_i \subset V_j \, \forall \, i < j$

Implying that $\lim_{n \to +\infty} \cup_{i=1}^{n} V_i = \lim_{n \to +\infty} \, V_n = V$

Or V is a union of its finite dimensional subspaces spanned by the subsets of its basis.

Finally , we show that $\mathbb{R}$ does not have a countably infinite basis over $\mathbb{Q}$ for otherwise, from the previous part it could be represented as a union of its finite dimensional subspaces. But as we saw earlier each such subspace can never have an irrational number thus the union cannot have an irrational number and is not equal to $\mathbb{R}$ giving the required contradiction. ■

5 Question 5

We know that $|\mathbb{F}^2| = q^2$. Consider the $q + 1$ subspaces defined as follows : $L_0 = Span((0_{\mathbb{F}}, 1_{\mathbb{F}}))$

$L_i = Span((1_{\mathbb{F}}, (i-1)_{\mathbb{F}}))$ where the elements of $\mathbb{F}$ are precisely $\{0_{\mathbb{F}}, ...(q-1)_{\mathbb{F}}\}$

Now any two subspaces have a singleton intersection which is precisely $(0_{\mathbb{F}}, 0_{\mathbb{F}})$

Thus $| \cup_{i=0}^{q} \, L_i| = (q-1)(q+1) + 1 = q^2 - 1 + 1 = q^2$

But $\cup_{i=0}^{q} \, L_i \subseteq \mathbb{F}^2$ which means that ;

$\mathbb{F}^2 = \cup_{i=0}^{q} \, L_i$ or a union of q+1 proper subspaces as claimed.

To prove that this works for an arbitrary $\mathbb{F}$ vector space V of dimension atleast 2 , we will do the following :

Let a basis of V be represented as $\{v_1, v_2\} \cup B$ for some set B and vectors v_1, v_2 if the dimension is 2 then $B = \phi$.

Let $L(\alpha, \beta)$ be the proper space generated by $\alpha v_1 + \beta v_2$ and B for $\alpha, \beta \in \mathbb{F}$. Note that $L(\alpha, \beta) = L(\gamma, \delta)$ for some $\gamma, \delta \in \mathbb{F}$ if and only if

$\frac{\gamma}{\alpha} = \frac{\delta}{\beta} \neq 0_{\mathbb{F}}$

The number of distinct choices for α, β is $(q-1)(q+1) = q^2 - 1$ but since there would be common subspaces generated the actual number of distinct subspaces is precisely $\frac{q^2-1}{q-1} = q + 1$

We claim that V is precisely the union of these distinct subspaces.Indeed for any $v \in V$ we may write $v = \alpha_1 v_1 + \beta_1 v_2 + u$ where $\alpha_1, \beta_1 \in \mathbb{F}$ and $u \in Span(B)$.

Now if $\alpha_1 = \beta_1 = 0_{\mathbb{F}}$ then $v \in L(\alpha, \beta) \, \forall \, \alpha, \beta \in \mathbb{F}$ otherwise $v \in L(k\alpha_1, k\beta_1)$ for $k \neq 0_{\mathbb{F}}$ and the conclusion follows. ■

6 Question 6

Suppose $v \in V_1 + V_2$ decompose v into a sum $t_1 + t_2$ where $t_1 \in V_1$ and $t_2 \in V_2$

If either $t_1 \in V_1 \cap V_2$ or $t_2 \in V_1 \cap V_2$ or both are in this set in each case atleast one can be represented as a linear combination of vectors $w_1, ..w_k$ and thus is in the Span of all of the other vectors similarly if $t_1 \in V_1 \backslash V_1 \cap V_2$ then it can be represented as a linear combination of vectors $u_1, ...u_m$ as the whole set $w_1, ...w_k, u_1, ...u_m$ is a Basis for V_1, finally if $t_2 \in V_2 \setminus V_1 \cap V_2$ we can represent it as a linear combination of vectors $v_1, ...v_n$ analogously. This covers all possible choices of v and therefore the Span of the vectors $w_1, ...w_k, u_1...u_m, v_1, ..v_n$ is indeed $V_1 + V_2$.

Now to check for Linear Independence as $w_1, ...w_k, u_1, ...u_m$ and $w_1, ...w_k, v_1, ...v_n$ are bases and therefore linearly independent sets any possible subsets of these are linearly independent by the lemma proved in Question 4.

Finally as $V_1 \setminus V_1 \cap V_2$ and $V_2 \setminus V_1 \cap V_2$ are disjoint, this means that $u_1, ...u_m, v_1, ...v_n$ are linearly independent as well. Thus the entire set $\{w_1, ...w_k, u_1...u_m, v_1, ..v_n\}$ is linearly independent and therefore forms a basis for $V_1 + V_2$ ■

7 Question 7

We are given a sequence of increasing linearly independent sets. To show that $\bigcup_{i>0} S_i$ is linearly independent, we need to consider linear combinations of finitely many elements from this set.

But clearly as $S_0 \subset S_1...$ if we chose any finite set of elements it must lie in S_k for some k. To see this, we can understand each S_j to be $S_{j-1} \cup L$ where L is some Linearly independent set. Clearly all elements of S_{j-1} are contained in S_j apart from those that come for L. Since j was arbitrary our claim follows.

Now, if all of our elements are in some S_k by the lemma proved in Question 4 we are done! ■