MA219 – Linear Algebra
2020 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 2 (*due by Thursday, October 22 by 5pm, on Microsoft Teams*) (Updated.)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Suppose $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$ for integers $m, n, p \geq 1$.

(1) Show that $(AB)^T = B^T A^T$.
(2) Suppose C is also a matrix over $\mathbb{F}$ such that BC is defined. Prove that $A(BC) = (AB)C$.
(3) Suppose $C \in \mathbb{F}^{n\times p}$ instead. Show that $A(B + C) = AB + AC$.

Question 2. Suppose $A \in \mathbb{F}^{m\times n}$. Show the following statements *without* computing individual entries. You can use that $\mathbb{F}^{r\times s}$ forms a vector space for all integers $r, s \geq 1$.

(1) $A \cdot \mathbf{0}_{n\times p} = \mathbf{0}_{m\times p}$.
(2) $(P + Q)A = PA + QA$ for $P, Q \in \mathbb{F}^{p\times m}$.
(3) $\mathbf{0}_{p\times m} \cdot A = \mathbf{0}_{p\times n}$.

Question 3. Suppose $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$ for integers $m, n, p \geq 1$. Also suppose A, B are *invertible*. In other words, (by results from class) there exist unique matrices

$$A^{-1} \in \mathbb{F}^{n\times m}, \quad B^{-1} \in \mathbb{F}^{p\times n}$$

such that

$$AA^{-1}, \quad A^{-1}A, \quad BB^{-1}, \quad B^{-1}B$$

are identity matrices of suitable orders. Prove the following.

(1) AB is invertible, with inverse $B^{-1}A^{-1}$.
(2) cA is invertible for a scalar $0 \neq c \in \mathbb{F}$, with inverse $c^{-1}A^{-1}$.
(3) If A is square, and $k \geq 1$ is an integer, then A^k is invertible, with inverse $(A^{-1})^k$.
(4) What is the inverse of A^{-1}? Give reasons.

Question 4. Suppose $A \in \mathbb{F}^{m\times n}$ as above. Show that A is invertible if and only if A^T is invertible, and if this happens then $(A^T)^{-1} = (A^{-1})^T$.

Question 5. (Elementary matrices.)

(1) Let $E_1^{(i,c)}$ denote the elementary row operation where the ith row is rescaled by $0 \neq c \in \mathbb{F}$. Show that

$$E_1^{(i,c)}(\mathrm{Id})^{-1} = E_1^{(i,c^{-1})}(\mathrm{Id}).$$

(2) Let $E_2^{(i,j)}$ denote the elementary row operation where the ith and jth rows are interchanged. Show that

$$E_2^{(i,j)}(\mathrm{Id})^{-1} = E_2^{(i,j)}(\mathrm{Id}).$$

(3) Let $E_3^{(i,c,j)}$ denote the elementary row operation where c times the ith row is added to the jth row. Show that

$$E_3^{(i,c,j)}(\mathrm{Id})^{-1} = E_3^{(i,-c,j)}(\mathrm{Id}).$$

Question 6. Solve the following systems of linear equations (over real numbers, or over any field of characteristic zero). Use row operations to obtain the RREF in all cases.

(1) $x + 2y + 3z = 0, \quad x + y + z = 1, \quad -x + z = 1.$
(2) $x + 2y + 3z = 3, \quad x + y + z = 1, \quad -x + z = 1.$
(3) $x + 6y + 9z = 1, \quad y - z = 3, \quad z = 7.$