MA219 – Linear Algebra
2020 Autumn Semester

[You are expected to write proofs / arguments with reasoning provided, in solving these questions.]

Homework Set 4 (*due by Wednesday, November 4 by 11:59 pm, on Microsoft Teams*)
(Updated.)

Throughout this homework (and this course), $\mathbb{F}$ denotes an arbitrary field.

Question 1. Let $n \geq 1$ be an integer, and let $A \in \mathbb{F}^{n \times n}$ be an upper triangular matrix. Thus, $a_{ij} = 0$ for all $i > j$. Prove that A is invertible if and only if the diagonal entries of A are all non-zero.

Question 2. Prove that the space of polynomials $\mathbb{F}[x]$ does not have a finite basis.

Question 3. Suppose D is a domain set, and $Fun(D, \mathbb{F})$ is the set of functions $f : D \to \mathbb{F}$ such that $f(x) = 0$ for all but finitely many $x \in D$.

(1) Show that $Fun(D, \mathbb{F})$ is a vector space under pointwise addition and scalar multiplication.
(2) Find a basis of this vector space (over the ground field $\mathbb{F}$).

Question 4. Let $\mathbb{F} = \mathbb{Q}$ and $V = \mathbb{R}$, a $\mathbb{Q}$-vector space.

(1) Show that $\mathbb{R}$ is not a finite-dimensional $\mathbb{Q}$-vector space.
(2) Suppose V is a *countable*-dimensional $\mathbb{Q}$-vector space, i.e. a vector space with a countably infinite basis $v_1, v_2, \ldots, v_n, \ldots$ Show that V is the union of its finite-dimensional subspaces V_n spanned by $v_1, \ldots, v_n$.
(3) Show that $\mathbb{R}$ is not a countable-dimensional $\mathbb{Q}$-vector space.

Question 5. Continuing from HW3 Q6: Suppose $\mathbb{F}$ is a finite field of size $q \geq 2$, and V is an $\mathbb{F}$-vector space. You showed in HW3 Q6 that V is not a union of $n \leq q$ proper subspaces. The goal is now to show that if we instead had $n \geq q + 1$ (in fact $n = q + 1$), then this is not so.

(1) First show that $\mathbb{F}^2$ is a union of $q + 1$ proper subspaces.

(2) Now suppose $V \neq 0$ is an arbitrary $\mathbb{F}$-vector space of dimension at least 2 (and possibly infinite), and B is a basis of V. (Assume B exists.) Show that V is a union of $q+1$ proper subspaces.

Question 6. Suppose W_1, W_2 are finite-dimensional subspaces of an $\mathbb{F}$-vector space V. Suppose $W_1 \cap W_2$ has a basis $w_1, \dots, w_k$. Extend this separately to bases of W_1 and W_2 by adding vectors $u_1, \dots, u_m$; and vectors $v_1, \dots, v_n$, respectively.

Prove that all of the above vectors *together* form a basis of $W_1 + W_2$.

Question 7. Suppose S is a linearly independent subset of a vector space W (over a field $\mathbb{F}$). Consider a chain of linearly independent subsets in W:

$$S = S_0 \subset S_1 \subset S_2 \subset \cdots$$

Prove that $\bigcup_{i \geq 0} S_i$ is also a linearly independent subset. (In a special case, this is the 'upper bound' of a 'chain' that was used in proving that every vector space has a basis, via Zorn's Lemma.)