Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 23 (3—4), (1972), pp. 341—342.

ON THE REPRESENTATION OF VECTOR SPACES AS A FINITE UNION OF SUBSPACES

By

J. LUH (Raleigh)

It has been proved in [1] that if a vector space V over a field F is a union of n (finite) proper subspaces then the order of F is necessarily less than n.

The purpose of this note is to sharpen this result. We shall prove the following:

THEOREM. *Let V be a vector space over a finite field F and q be the order of F. Then V can be expressed as a union of $q+1$ proper subspaces, and such expression is unique up to an automorphism of V.*

PROOF. Let $\{v_1, v_2\} \cup B$ be a basis of V. For $\alpha, \beta \in F$, not all zero, let $W(\alpha, \beta) = \langle \alpha v_1 + \beta v_2, B\rangle$, the proper subspace generated by B and $\alpha v_1 + \beta v_2$. We note that $W(\alpha, \beta) = W(\gamma, \delta)$ if and only if $\alpha v_1 + \beta v_2 \equiv \sigma(\gamma v_1 + \delta v_2) \pmod{\langle B\rangle}$ for some $\sigma \neq 0$ in F, if and only if $\alpha = \sigma\gamma$ and $\beta = \sigma\delta$ for some $\sigma \neq 0$ in F. Thus there are exactly $(q^2-1)/(q-1) = q+1$ distinct subspaces $W(\alpha, \beta)$ with $\alpha, \beta \in F$, not all zero.

We claim that $V = \bigcup W(\alpha, \beta)$, the union of these $q+1$ subspaces. Indeed, let v be an arbitrary vector in V. Then $v = \alpha_1 v_1 + \beta_1 v_2 + u$, where $\alpha_1, \beta_1 \in F$ and $u \in \langle B\rangle$. If $v \in \langle B\rangle$ then $\alpha_1 = \beta_1 = 0$ and $v = u \in W(\alpha, \beta)$ for all $\alpha, \beta \in F$. If $v \notin \langle B\rangle$ then α_1, β_1 are not all zero, and $v \in W(\alpha_1, \beta_1)$. Hence $V = \bigcup W(\alpha, \beta)$ as we desired.

To show the uniqueness of such expression up to automorphism, it suffices to show that if $V = V_1 \cup V_2 \cup \ldots \cup V_{q+1}$ then there exist $v_1, v_2 \in V$ and $B \subset V$ such that $\{v_1, v_2\} \cup B$ forms a basis of V and $V_i = W(\alpha, \beta) = \langle \alpha v_1 + \beta v_2, B\rangle$ for some $\alpha, \beta \in F$, not all zero.

In view of [1], $V_i \subsetneqq \bigcup_{j \neq i} V_j$, $i = 1, 2, \ldots, q+1$. Let us first show that the codimension of V_1 is 1. Suppose codim $V_1 \geqq 2$. Let $V = V_1 \oplus U_1$, where $\dim U_1 = \operatorname{codim} V_1 \geqq 2$, let $v_1 \in V_1 \setminus (V_2 \cup \ldots \cup V_{q+1})$, and let $u_1, u_2 \in U_1$ be linearly independent. For any $\alpha, \beta \in F$, let $U(\alpha, \beta) = \langle \alpha v_1 + u_1, \beta v_1 + u_2\rangle$, the subspace generated by the two vectors $\alpha v_1 + u_1$ and $\beta v_1 + u_2$. It is easy to see that $U(\alpha, \beta) \cap V_1 = \{0\}$. Since $V = \bigcup_{i=1}^{q+1} V_i$, $U(\alpha, \beta) = U(\alpha, \beta) \cap V = \bigcup_{i=1}^{q+1} (U(\alpha, \beta) \cap V_i) = \bigcup_{i=2}^{q+1} (U(\alpha,\beta) \cap V_i)$, a union of q subspaces. It would be a contradiction unless $U(\alpha, \beta) \cap V_j = U(\alpha, \beta)$ for some $j \geqq 2$, i.e., $U(\alpha, \beta) \subseteqq V_j$. This shows that each $U(\alpha, \beta)$ is contained in some V_j with $j \geqq 2$. Now we have q^2 distinct $U(\alpha, \beta)$'s and have q V_j's with $j \geqq 2$. Hence there exist two distinct pairs (α_1, β_1) and (α_2, β_2) of elements in F such that $U(\alpha_1, \beta_1)$ and $U(\alpha_2, \beta_2)$ both are contained in the same V_j with $j \geqq 2$. Without loss of generality, we may assume that $\alpha_1 \neq \alpha_2$. Then, since $\alpha_1 v_1 + u_1$, $\alpha_2 v_1 + u_1 \in V_j$, $(\alpha_1 - \alpha_2)v_1 \in V_j$ and hence $v_1 \in V_j$, where $j \geq 2$, a contradiction. Therefore, codim $V_1 = 1$.

Next, let $R = V_1 \cap V_2$. We assert that $R = \bigcap_{i=1}^{q+1} V_i$. Assume that $R = \bigcap_{i=1}^{t} V_i$, where $2 \leqq t < q+1$. Suppose $R \neq \bigcap_{i=1}^{t+1} V_i$. Let $u \in R \setminus \bigcap_{i=1}^{t+1} V_i$, $v_{t+1} \in V_{t+1} \setminus \bigcup_{i \neq t+1} V_i$. Then, for any $\alpha \neq 0$ in F, clearly $u + \alpha v_{t+1} \notin V_j$ for $1 \leqq j \leqq t$. Furthermore, $u + \alpha v_{t+1} \notin V_{t+1}$. For otherwise, $u \in V_{t+1}$ would imply that $u \in \bigcap_{i=1}^{t+1} V_i$. Hence $u + \alpha v_{t+1} \in V_j$ for some $j > t+1$. Consider the set $\{u + \alpha v_{t+1} \mid \alpha \in F, \alpha \neq 0\}$. It has exactly $q-1$ vectors and is contained in $\bigcup_{i=t+2}^{q+1} V_i$, a union of $q-t$ V_j's. Since $t \geqq 2$, $q - t < q - 1$ and hence there exist two non-zero elements $\alpha, \beta \in F$, $\alpha \neq \beta$, such that $u + \alpha v_{t+1}, u + \beta v_{t+1} \in V_j$ for some $j > t+1$. Consequently $(\alpha - \beta)v_{t+1} \in V_j$ and hence $v_{t+1} \in V_j$, a contradiction. Thus $R = \bigcap_{i=1}^{t+1} V_i$. By induction, therefore, we have $R = \bigcap_{i=1}^{q+1} V_i$.

Similarly, we obtain $V_k \cap V_j = \bigcap_{i=1}^{q+1} V_i = R$, $V_k + V_j = V$ for all $k \neq j$, and codim $V_i = 1$ for all i.

Now since $V_1 + V_2 = V$ and $V_1 \cap V_2 = R$, $V/V_1 \cong V_2/R$, $V/V_2 \cong (V/R)/(V_2/R)$ and codim V_2 = codim $R - \dim(V_2/R)$, so codim R = codim $V_2 + \dim(V_2/R)$ = = codim V_2 + codim $V_1 = 2$. Therefore $V_1 = R \oplus \langle v_1 \rangle$, $V_2 = R \oplus \langle v_2 \rangle$ for some $v_1, v_2 \in V$, and $V = V_1 + V_2 = R \oplus \langle v_1 \rangle \oplus \langle v_2 \rangle = \langle B \rangle \oplus \langle v_1 \rangle \oplus \langle v_2 \rangle$, where B is a basis of R. For each V_i, since $\langle B \rangle$ is contained in V_i properly, there is $\alpha v_1 + \beta v_2 + u \in V_i$ for some $\alpha, \beta \in F$, not all zero, and $u \in R$. Let $W(\alpha, \beta) = \langle \alpha v_1 + \beta v_2, B \rangle$. Then, since $\alpha v_1 + \beta v_2 \in V_i$ and $B \subseteqq V_i$, $W(\alpha, \beta) \subseteqq V_i$. But codim $W(\alpha, \beta)$ = = codim $V_i = 1$, so $V_i = W(\alpha, \beta)$. This completes the proof.

(Received 12 January 1971)

DEPARTMENT OF MATHEMATICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, N. C. 27607
U.S.A.

Reference

[1] A. Bialynicki-Birula, J. Browkin and A. Schinzel, On the representation of fields as finite union of subfields, *Colloquium Mathematicum*, 7 (1959), pp. 31—32.