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Homework Assignment 2
Linear Algebra

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1 Question 1

We are given that A $\in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$.

1.1

To prove that $(AB)^T = B^T A^T$.
$(AB)_{ij} = (AB)^T_{ji}$ hence, $(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki}$ on the other hand,
$(B^T A^T)_{ij} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = \sum_{k=1}^{n} B_{ki}A_{jk} = \sum_{k=1}^{n} A_{jk}B_{ki}$
Since $(AB)^T_{ij} = (B^T A^T)_{ij}$ for all $i \in \{1, 2....p\}$ and $j \in \{1, 2...m\}$ we have that $(AB)^T = B^T A^T$
■

1.2

Given that $C \in \mathbb{F}^{p\times u}$ such that BC is also defined , we wish to show that $A(BC) = (AB)C$
We have , $(BC)_{ij} = \sum_{k=1}^{p} B_{ik}C_{kj}$ and , $(A(BC))_{ij} = \sum_{k=1}^{n} A_{ik}(BC)_{kj}$ Similarly $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$ and , $((AB)C)_{ij} = \sum_{k=1}^{p}(AB)_{ik}C_{kj}$
Combining , we have $(A(BC))_{ij} = \sum_{k=1}^{n} A_{ik}(\sum_{l=1}^{p} B_{kl}C_{lj})$
Since a field distributes multiplication over addition, we can rearrange the above terms to get,
$\sum_{l=1}^{p}(\sum_{k=1}^{n} A_{ik}B_{kl})C_{lj} = \sum_{l=1}^{p}(AB)_{il}C_{lj} = ((AB)C)_{ij}$
Since our choice of i, j was arbitrary we are done ■

1.3

Given that $C \in \mathbb{F}^{n\times p}$ We have that , $(A(B + C))_{ij} = \sum_{k=1}^{n} A_{ik}(B + C)_{kj} = \sum_{k=1}^{n} A_{ik}(B_{kj} + C_{kj}) = \sum_{k=1}^{n} A_{ik}B_{kj} + \sum_{k=1}^{n} A_{ik}C_{kj} = AB_{ij} + AC_{ij}$
Since this is true for arbitrary i, j we are done ■

2 Question 2

2.1

Consider any matrix $C \in \mathbb{F}^{n\times p}$, we have from the previous question and the fact that $\mathbb{F}^{n\times p}$ forms a vector space with additive identity $\mathbf{0_{n\times p}}$,

$AC = A(C + 0_{n\times p}) = AC + A0_{n\times p}$
It is quite clear since this is true for arbitrary C we have the fact $A0_{n\times p}$ is the additive identity of the vector space $\mathbb{F}^{m\times p}$ and we are done. ■

2.2

In this case, since A is an element of the vector space $\mathbb{F}^{m\times n}$, We can represent it as a *linear combination* of basis vectors for this vector space. We will choose the basis represented by the matrices of size $m \times n$ such that one element is $1_{\mathbb{F}}$ and all others are $0_{\mathbb{F}}$. i.e we have ,$A = \sum_{i=1}^{mn} c_iE_i$ where E_i is the matrix with the i^{th} element $1_{\mathbb{F}}$ and all others as $0_{\mathbb{F}}$ where $1 \leq i \leq mn$ and is part of the basis we chose and $c_i \in \mathbb{F}$ are arbitrary constants which in fact are precisely equal to A_i or the i^{th} element of our original matrix.
Now from Question 1 , $(P + Q)(A) = (P + Q)(\sum_{i=1}^{mn} A_iE_i) = \sum_{i=1}^{mn}(P + Q)(A_iE_i) = \sum_{i=1}^{mn}(A_i)(P + Q)(E_i) = \sum_{i=1}^{mn}(A_i)(PE_i + QE_i)$
Where the last equality came from distributivity over the vector space $\mathbb{F}^{p\times m}$ to which both P, Q belong and the fact that our choice of E_i allows for a procedure analogous to *scalar multiplication* , What follows now is quite trivial,
$\sum_{i=1}^{mn}(A_i)(PE_i + QE_i) = \sum_{i=1}^{mn}(P(A_i)E_i + Q(A_i)E_i) = P\sum_{i=1}^{mn} A_iE_i + Q\sum_{i=1}^{mn} A_iE_i = PA + QA$
■

2.3

Using the previous part we will adopt a strategy similar to the first part. Let C be any arbitrary matrix in $\mathbb{F}^{p\times m}$,The fact that $\mathbb{F}^{p\times m}$ forms a vector space with additive identity $\mathbf{0_{p\times m}}$ gives,
We have , $CA = (C + 0_{p\times m})A = CA + 0_{p\times m}A$
Since our choice of C was arbitrary,we have the fact $0_{p\times m}A$ is the additive identity of the vector space $\mathbb{F}^{p\times n}$ and we are done. ■

3 Question 3

3.1

Clearly as inverses exist neither of the matrices is a zero matrix and thus AB is also not a zero matrix , let us now confirm that $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(Id)A^{-1} = AA^{-1} = Id$
$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}(Id)B = B^{-1}B = Id$
Thus $B^{-1}A^{-1}$ is indeed the inverse of AB ■

3.2

As long as $c \neq 0_{\mathbb{F}}$ we have that $c^{-1}\exists! \in \mathbb{F}$

Therefore , let us check by a similar strategy as the previous part
$cA(c^{-1}A^{-1}) = cAc^{-1}A^{-1} = cc^{-1}AA^{-1} = cc^{-1}(Id) = 1_{\mathbb{F}}(Id) = Id$
$(c^{-1}A^{-1})cA = c^{-1}A^{-1}cA = c^{-1}A^{-1}Ac = c^{-1}(Id)c = c^{-1}c(Id) = 1_{\mathbb{F}}(Id) = Id$
Thus we are done ■

3.3

As A is a square matrix we have the fact that A^k is defined $\forall\ k \geq 1$
We will prove the required claim by strong induction for $\forall k \geq 2$, Now for the base case of $k = 2$

Clearly setting $B = A$ in part 1 of this question is enough to prove for this case.
Let us assume now that our claim is valid up to all $k \leq n$ for some $n \in \mathbb{N}$ Then ,

$(A^{n+1})^{-1} = (A^nA)^{-1} = A^{-1}(A^n)^{-1} = A^{-1}A^{-n}$ from the inductive hypothesis , which is precisely $A^{-(n+1)}$
We are thus done by strong induction , since the statement is trivially true for $k = 1$ This is true $\forall n \in \mathbb{N}$ ■

3.4

Notice that by definition , $AA^{-1}, A^{-1}A$ are both identity matrices. This is in fact the definition of the inverse of any matrix i.e if X is a matrix then X^{-1} is a matrix then both XX^{-1} and $X^{-1}X$ are identity matrices. From the definition thus , the inverse of A^{-1} is precisely A ■

4 Question 4

If A is invertible then $\exists$ a matrix B such that AB and BA are identity matrices.
We have then that $(AB)^T = B^TA^T = (Id)^T = Id$ and similarly $(BA)^T = A^TB^T = (Id)^T = Id$ and thus by definition $B^T = (A^T)^{-1}$ but $B = A^{-1}$ therefore A^T is invertible with the inverse $(A^{-1})^T$ since A was arbitrary we can replace A with A^T to obtain the reverse implication.
■

5 Question 5

5.1

We wish to show that ,$(E_1^{i,c}(Id))^{-1} = E_1^{i,c^{-1}}(Id)$ (Note that c^{-1} exists as long as $c \neq 0_{\mathbb{F}}$) The matrix on the right hand side is simply the Identity matrix with one row re-scaled by c^{-1}.
Now , $E_1^{i,c^{-1}}(Id) * (E_1^{i,c}(Id))$ is simply the diagonal matrix with the i^{th} diagonal element $c^{-1}c = 1_{\mathbb{F}}$ as the two matrices are diagonal matrices themselves. But this product is precisely the Identity matrix , similarly we can show that $(E_1^{i,c}(Id)) * E_1^{i,c^{-1}}(Id)$ is also the Identity matrix and the result follows ■

5.2

In this case we have switched rows i, j of the Identity matrix. Let this matrix be A Let the size of this matrix be $n \times n$ then we have that $(AA)_{ii} = \sum_{k=1}^{n} A_{ik}A_{kn} = A_{ij}A_{ji} = 1_{\mathbb{F}}$
Similarly $(AA)_{jj} = \sum_{k=1}^{n} A_{jk}A_{kn} = A_{ji}A_{ij} = 1_{\mathbb{F}}$
It is easy to check that all other diagonal elements are $1_{\mathbb{F}}$ and all non diagonal elements are precisely $0_{\mathbb{F}}$ which means that $AA = Id$ and the conclusion follows $\blacksquare$

5.3

In this case let the new transformed matrix with c be A and the one with $-c$ be B again with sizes $n \times n$ now $(AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = A_{ji}B_{ii} + A_{jj}B_{ji} = c * 1_{\mathbb{F}} + 1_{\mathbb{F}} * (-c) = c - c = 0_{\mathbb{F}}$ Similarly $(AB)_{jj} = \sum_{k=1}^{n} A_{jk}B_{kj} = A_{ji}B_{ij} + A_{jj}B_{jj} = c * 0_{\mathbb{F}} + 1_{\mathbb{F}} * 1_{\mathbb{F}} = 1_{\mathbb{F}}$ We can prove easily that all other diagonal elements are $1_{\mathbb{F}}$ and non diagonal elements are $0_{\mathbb{F}}$. That is $(AB) = (Id)$ Switching A, B it is easy to prove that $BA = (Id)$ $\blacksquare$

6 Question 6

6.1

We have the system of equations
$x + 2y + 3z = 0$
$x + y + z = 1$
$-1 \times x + 0 \times y + z = 1$
The augmented matrix for the following system is
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{array}\right)$$
Our first row operation would be to interchange R_2 and R_1 to obtain the new matrix :
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ -1 & 0 & 1 & 1 \end{array}\right)$$
Next transform R_2 to $R_2 - R_1$ and R_3 to $R_3 + R_1$ to give ,
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$
Next transform R_3 to $R_3 - R_2$ to give the matrix ,
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{array}\right)$$
Finally , transform R_1 to $R_1 - R_2$ to obtain the final RREF matrix,
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{array}\right)$$
Clearly , reading off the solutions from the RREF we have the absurd inference from the last row that $0 \times x + 0 \times y + 0 \times z = 0 + 0 + 0 = 0 = 3$! This is absurd since the underlying field has characteristic 0
Thus , **no solutions** exist for this system in this field $\blacksquare$

6.2

$x + 2y + 3z = 3$
$x + y + z = 1$
$-1 \times x + 0 \times y + z = 1$
The augmented matrix for the following system is
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{array}\right)$$
Our first row operation would be to interchange R_2 and R_1 to obtain the new matrix :
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 \\ -1 & 0 & 1 & 1 \end{array}\right)$$
Next transform R_2 to $R_2 - R_1$ and R_3 to $R_3 + R_1$ to give ,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

Next transform R_3 to $R_3 - R_2$ to give matrix ,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Finally , transform R_1 to $R_1 - R_2$ to obtain the final RREF matrix,

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now , reading off from the matrix we have the following constraints $x - z = -1$
$y + 2z = 2$ The last row since it does not provide a unique condition simply says that there are infinite number of solutions for this system. Choose some $u \in \mathbb{F}$ where $\mathbb{F}$ is the underlying field. We have that $(u - 1, 2 - 2u, u)$ represents a valid solution. ■

6.3

We have the system ,
$x + 6y + 9z = 1$
$y - z = 3$
$z = 7$
The augmented matrix is therefore,

$$\left(\begin{array}{ccc|c} 1 & 6 & 9 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

We will perform the following row operations, R_2 to $R_2 + R_3$ and R_1 to $R_1 - 9R_3$
This gives the following augmented matrix ,

$$\left(\begin{array}{ccc|c} 1 & 6 & 0 & -62 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

Now we will perform the following Row operation R_1 to $R_1 - 6R_2$ which gives the following augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -122 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

This is precisely the RREF of the matrix that we started with.Now reading off the solutions from this matrix we have,
$x = -122$
$y = 10$
$z = 7$
Which is the unique solution of the system that we started with ■