An Analysis of Stochastic Approximations through the method of Ordinary Differential Equations

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I. Introduction

A stochastic approximation is a discrete time recursive process with an inherent stochastic nature. This formulation serves as a basis for a wide class of problems, of particular interest in real time applications are the so called *learning problems* forming the basis of modern **Reinforcement Learning** theory , **Game Theory** and **Control Theory** thus having wide reaching influences ranging from economics to efficient engineering design. Here we provide a foundation for the theory of such approximation algorithms using the familiar theory of Ordinary Differential Equations (ODE).

II. Formulation and Preliminaries

A. The Standard Formulation

Our goal shall to be solve an equation of the form :

$$h(\theta) = 0 \tag{1}$$

where h is some unknown function on a parameter which we wish to optimize in some manner. Considering that this is a recursive process at each time step t we can *query* an *"oracle"* to obtain the value of $h(\theta_t)$ as this process is stochastic the value obtained has some noise added to it. In general, we may represent our problem as follows :

$$\theta_{t+1} = \theta_t + \alpha_t(h(\theta_t) + M_t) \tag{2}$$

where t represents the current time step , $\{\alpha_t\}$ represents the step sizes taken by our algorithm which are positive reals, the sequence $\{M_t\}$ is the sequence of noises or fluctuations that are encountered in our process. The equation above is termed as the **Robbins-Munro update** after those who studied it for the first time.

B. Some Preliminaries

We cover some preliminary frameworks here which will be of use to us soon. The results are covered in a succinct manner whilst providing detail only to the level that is needed for the further analysis.

1) Some basics from Probability Theory: Consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is our sample space , $\mathcal{F}$ is our σ algebra and $\mathcal{P}$ is an appropriately defined probability measure.Any real valued function $X : \Omega \mapsto R$ which is measurable is a **random variable**. A d dimensional **random vector** is a function $X : \Omega \mapsto R^d$ which is similarly measurable i.e all sets of the form $\{w \in \Omega : X(w) \leq \alpha\}$ are events in $\mathcal{F}$.

Consider two random variables X, Y. Suppose that $\mathcal{F}_X = \sigma\{X\}$ i.e, the σ algebra generated by X , we define the conditional expectation , $E[Y|X]$ to be $E[Y|\mathcal{F}_X]$. Naturally this definition can be extended to multiple random variables $X_1, ...X_n$ let $\mathcal{F}_n = \sigma\{X_1, ...X_n\}$ then we define $E[Y|X_1, ...X_n]$ to be $E[Y|\mathcal{F}_n]$. Observe that the sequence $\mathcal{F}_1, ..., \mathcal{F}_{n-1}, \mathcal{F}_n$ is *necessarily* a **filtration** i.e, $\mathcal{F}_1 \subseteq ..., \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$ as each new σ algebra has to necessarily contain the previous one (the inequality is strict for distinct random variables). We define Y to be $\mathcal{F}_n$ measurable if $Y = f(X_1, ...X_n)$ where f is some **Borel measurable function**.

We now define a **Martingale Difference Sequence**, as follows :

A sequence $\{(X_n, \mathcal{F}_n)\}$ is a martingale difference sequence on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ if the follwoing conditions hold :

1) The sequence $\{\mathcal{F}_n\}$ is a filtration , i.e $\mathcal{F}_{n-1} \subset \mathcal{F}_n \subset \mathcal{F}$
2) X_n is $\mathcal{F}_n$ measurable $\forall\ n$
3) $E[X_n] < \infty\ \forall\ n$
4) $E[X_{n+1}|\mathcal{F}_n] = 0\ \forall\ n$

The basic idea is to construct a sequence which is **tractable** and where new information is independent to the previous ones.

2) Lyapunov Functions: Lyapunov functions are an extremely convenient device for proving that a dynamical system converges. We assume that $f : R^n \mapsto R^n$ is a continuous function and $\exists\ x(t)$ such that :

$$\frac{dx}{dt} = f(x(t)) \tag{3}$$

where $t \in R^+$. A **Lyapunov function** is a continuously differentiable function $L : R^n \mapsto R$ with an unique minimum at x^* such that we have :

$$\nabla L \cdot f(x) < 0 \, \forall x \neq x^* \tag{4}$$

We further assume that the set $\{x : L(x) < l\}$ is **compact** $\forall \, l \in R$. We now prove the following result :

Theorem 1. *If a Lyapunov function L exists for a differential equation as defined above then $L(x(t)) \to L(x^*)$ as $t \to \infty$ further we have that, $x(t) \to x^*$ as $t \to \infty$*

Proof. We first note that $\frac{dL(x(t))}{dt} = \nabla L(x(t)) \cdot f(x(t))$ which is less than 0 by definition so $L(x(t))$ is decreasing suppose that it decreases to a value l then we have from the **Fundamental Theorem of Calculus** that ,

$l - L(x(t)) = \int_t^\infty \frac{dL(x(u))}{du} du$.. Let the integral on the right hand side be I_t , we have that $I_t \to 0$ as $t \to \infty$, so $\exists$ a sequence $\{s_k\}$ such that $\frac{dL(x(s_k))}{dt} \to 0$ as $s_k \to \infty$.

From our earlier assumptions we know that the set , $\{x : L(x) < L(x_0)\}$ is compact , thus we have that $\exists$ a subsequence of $\{s_k\}$ say $\{t_k\}$ where k is indexed from 0 onwards which also converges. Suppose this sequence converges to $x^{'}$.From continuity we obtain ,

$$0 = \lim_{k\to\infty} \frac{dL(x(t_k))}{dt} = \lim_{k\to\infty} f(x(t_k)) \cdot \nabla L(x^{'}) = f(x^{'})) \cdot \nabla L(x^{'}) \tag{5}$$

This forces $x^{'} = x^*$ and by continuity of L we obtain that $x(t) \to x^*$ as needed $\square$

We can extend the above result to generic point sets of the form $\{x : \nabla L(x) \cdot f(x) = 0\}$. Let this set be termed as $\mathcal{X}^*$ or the set of invariant points. Now assume that $\exists$ a compact set $\mathcal{X}$ such that if $x(0) \in \mathcal{X}$ then $x(t) \in \mathcal{X} \; \forall t$. Define the Lyapunov function L within the domain $\mathcal{X}$ the following result termed as **Le Salle's Invariant Principle** states that:

Theorem 2. *Le Salle's Invariant Principle :*
$x(t)$ converges to $\mathcal{X}^*$ uniformly over all initial conditions in $\mathcal{X}$ i.e, $\forall \, x(0) \in \mathcal{X}, \epsilon > 0 \; \exists \; T > 0$ such that :

$min_{x^* \in \mathcal{X}^*} |x(T) - x^*| < \epsilon.$

Proof. Suppose that $x(0)$ belongs to a compact set , say $\mathcal{X}$. Then by definition we have that, $L(x(0)) \leq l$ for some $l \geq 0$. For any $\delta > 0$ define $\mathcal{X}^*(\delta) \doteq \{x \in \mathcal{X} : f(x) \cdot \nabla L(x) < \delta\}$.Since $\frac{dL}{dt} = f(x) \cdot \nabla L(x)$ then as $L(x(0)) < l$ we have that for $T > \frac{l}{\delta}$, $x(t) \in X^*(\delta) \; \forall \, t \geq T$.
Now we show that $\forall \, \epsilon > 0 \; \exists \delta > 0$ such that if $x(t) \in \mathcal{X}^*$ $|x(t) - x^*| < \epsilon$ for some $x^* \in \mathcal{X}^*$.The claim follows by a simple contradiction argument based on the compactness of $\mathcal{X}$ and the continuity of $f(x) \cdot \nabla L(x)$ analogous to the previous theorem. $\square$

Now , we continue onto the main analysis.

III. Analysis

To make the analysis easier and tractable we take some further assumptions which are valid most of the time in practical scenarios. Consider once again the formulation of a Stochastic Approximation problem :

$$\theta_{t+1} = \theta_t + \alpha_t(h(\theta_t) + M_t) \tag{6}$$

We take the following assumptions for any such system :

1) We assume that $\sum_{t=0}^{\infty} \alpha_n = \infty$ and that $\sum_{t=0}^{\infty} \alpha_n^2 < \infty$. The first condition informally ensures that the process keeps on moving (even in vanishingly small increments) whilst the second ensures that the noise from the process tends to zero. (More formally it is related to the convergence of martingale sequences and has been explored in much detail in **Borkar , [1]** for example, the analysis being rather involved has been omitted here.)
2) We assume that $\{M_n\}$ forms a **Martingale Difference Sequence** i.e $E[M_n|\mathcal{F}_{n-1}] = 0$ where $\mathcal{F}_{n-1} = \sigma\{\theta_0, M_0, ...\theta_{n-1}, M_{n-1}\}$ further we assume the existence of constants $c_1, c_2 \in R^+$ and a norm $\|.\|$ in R^d such that we have

   $$E[\|M_n^2\||\mathcal{F}_{n-1}] \leq c_1 + c_2\|\theta_n\| \; \forall n \; a.s.$$

   The role of this condition is to control the fluctuations of $\{\theta_n\}$ around the solution of the ODE. Of frequent interest is the case when $\{M_n\}$ is an independent sequence with bounded variance.
3) We now assume that $sup_n\|\theta_n\| < \infty$ a.s.
   This assumption is crucial since it often doesn't follow from prior information. Combining this and the last condition of the previous we can easily obtain that $\exists$ a finite constant $C \in R$ such that

   $$E[\|M_n^2\||\mathcal{F}_{n-1}] \leq C \; \forall n \; a.s.$$

   As the iterates are now bounded we have much more room to work with.
4) We next assume that the function h is **Lipschitz Continuous** with respect to some suitable metric in R^d.This assumption ensures that for a given initial condition the ODE has a unique solution and that this solution depends continuously on the initial condition after a given amount of time. These existence, uniqueness and continuity features of the ODE solution are due to the **Picard-Lindelöf** theorem.
5) Finally we assume that $\exists$ a positive, continuously differentiable and radially unbounded function $V : R^d \mapsto R$ such that $\nabla V(x) \cdot h(x) \leq 0 \; \forall x$ and further $\nabla V(x) \cdot h(x) \neq 0$ if $V(x) \neq 0$. Therefore V is a **Lyapunov function** for h.

With these assumptions in hand we shall continue onto the analysis.

Let t_n be the partial sum $\sum_{k=0}^{n} \alpha_k$ representing in some sense the time spent by the process. Suppose that $\mathcal{T} = \{t_n | n \in$

$N \cup \{0\}\}$. Let $x(t) = \theta_n$ for $t = t_n$. Now supose that x_m is the solution of the ODE started at θ_m and time t_m,i.e,

$$\frac{dz_m}{dt} = h(z_m(t))\,, z_m(t_m) = \theta_m \tag{7}$$

We are then interested in the error accumulated since time t_m i.e., in the following quantity

$$\sup_{t\in[t_m,t_m+T]\cap\mathcal{T}} \|x(t) - z_m(t)\| \tag{8}$$

where T is a time interval of interest. We will now prove the following theorem :

Theorem 3. $\lim_{m\to\infty} \sup_{t\in[t_m,t_m+T]\cap\mathcal{T}} \|x(t) - z_m(t)\| = 0$ *where convergence holds with probability 1 and the metric involved is the standard euclidean norm.*

Proof. Notice that :
$z_m(t_n) = \theta_n + \int_{t_m}^{t_n} z_m(u)\,du$ where we simply manipulated our **Stochastic Approximation** update. Now we have that ,

$$x_{t_n} = \theta_m + \sum_{k=m}^{n-1} \alpha_k h(\theta_k) + \sum_{k=m}^{n-1} \alpha_k M_k = \theta_m + \int_{t_m}^{t_n} h(x([u]_\alpha)\,du + \sum_{k=m}^{n-1} \alpha_k M_k$$

where $[u]_\alpha \doteq max\{t_k : t_k \leq u\}$

So we have that ,

$$\|x(t) - z_m(t)\| \leq \|\sum_{k=m}^{n-1} \alpha_k M_k\| + \int_{t_m}^{t_n} \|h(x([u]_\alpha) - h(z_m(u))\|\,du$$

from the triangle inequality. Now from the definition of the ODE we get ,

$$\|\sum_{k=m}^{n-1} \alpha_k M_k\| + \int_{t_m}^{t_n} \|h(x([u]_\alpha) - h(z_m(u))\|\,du \leq \|\sum_{k=m}^{n-1} \alpha_k M_k\| + \int_{t_m}^{t_n} \|x([u]_\alpha - z_m(u)\|\,du$$

so finally we obtain ,

$$\|x(t) - z_m(t)\| \leq \|\sum_{k=m}^{n-1} \alpha_k M_k\| + \int_{t_m}^{t_n} \|x([u]_\alpha - z_m(u)\|\,du$$

we may now use the familiar **Gronwall's Inequality** to get :

$$\|x(t) - z_m(t)\| \leq \|\sum_{k=m}^{n-1} \alpha_k M_k\| e^{I(t_n - t_m)}$$

we used the fact that the summation function is non decreasing **(Assumption 1.)** and I is the integral in the earlier inequality, but then we get

$$\|\sum_{k=m}^{n-1} \alpha_k M_k\| e^{I(t_n - t_m)} \leq \|M_n - M_m\| e^{IT} \quad when\ T \geq t_n - t_m.$$

Now, from **assumption 3** we get that $E[\|M_n^2\||\mathcal{F}_{n-1}] \leq C$ so $\{M_n\}$ being bounded in the space equipped with **Euclidean norm** converges there with probability 1 thus, $\{M_n\}$ forms a **cauchy sequence** in the euclidean normed space, consequently as $m \to \infty$ the RHS tends to 0 and the LHS as to as well being a non-negative quantity.We are done ☐

Finally, we are ready to prove the main result of this work. Assume that we have a lyapunov function as in **Assumption 5** say , V. Let $\mathcal{V}^*$ represent the set of its invariant points i.e $\mathcal{V}^* = \{x : \nabla V(x) \cdot h(x) = 0\}$. Our aim is to prove the following result :

Theorem 4. *Convergence of Stochastic Approximation schema:*
For the standard stochastic approximation update defined as $\theta_{t+1} = \theta_t + \alpha_t(h(\theta_t) + M_t)$ when the five assumptions discussed earlier hold we have that, with probability 1 $\lim_{n\to\infty} \theta_n \to \mathcal{V}^*$

Proof. Our aim is to use **La Salle's Invariant Principle** combined with **Theorem 3** to progress towards a solution. Now from **Assumption 3** we have that the iterates are bounded to be exact let us define $sup_n\|\theta_n\| = S$. Now , define $c = \max\{V(x) : \|x\| \leq S\}$, further take $\mathcal{V} = \{x : V(x) \leq c\}$ as $V(z(t))$ decreases for all solutions to the ODE considered in **Theorem 3** we get that $z_m(t) \in \mathcal{V}$ $\forall m$ and for $t \geq t_m$.

Now, from **La Salle's Invariant Principle** we obtain that $\forall \epsilon \geq 0\ \exists T \geq 0$ such that $\forall t \geq T$ and $\forall m$ we have that :

$$min_{x^*}\|z_m(t_m + T) - x^*\| < \epsilon$$

Taking this T we know from **Theorem 3** that there exists m^* such that $\forall m > m^*$ we have :

$$\sup_{t\in[t_m,t_m+2T]\cap\mathcal{T}} \|x(t) - z_m(t)\| < \epsilon$$

Now, we can take m^* suitably large so that $t_m - t_{m-1} \leq T$ whenever $m \geq m^*$ this implies that :

$$\bigcup_{m:m\geq m^*} [t_m + T, t_m + 2T] = [t_{m^*} + T, \infty)$$

Now we note that if n is such that $t_n \in [t_m + T, t_m + 2T]$ for some $m \geq m^*$ we obtain the following :

$$\min_{x^*} \|x_n - x^*\| \leq \|x_n - z_m(t_n)\| + \min_{x^*} \|z_m(t_n) - x^*\| < \epsilon + \epsilon = 2\epsilon$$

Using the union above we see that for all $t_n \geq t_m + T$ we have that $\min_{x^*} \|x_n - x^*\| < 2\epsilon$ but then this means that $\lim_{n\infty} x_n \to \mathcal{V}^*$ as needed. $\square$

With this we end the preliminary analysis on convergence of Stochastic Approximation Schemes.

IV. Conclusion

In practice many methods used in **Control theory and Game theory** can be reduced to Stochastic Approximation algorithms. One famous class is the class of **Temporal Difference Algorithms** or **TD Algorithms** from the domain of **Reinforcement Learning**, the situation in this case is to learn an optimal **policy** that an agent needs to follow in order to perform some tasks that are demanded of it. The underlying environment being stochastic, the learning algorithm is reduced to a Stochastic Approximation system. An excellent analysis of such a situation is the work by **Thoppe,et.al , [2]** While convergence results such as the one presented above provide the existence of a solution they do not provide a direct formula for the same. One often has to resort to "guessing" or optimizing suitably chosen Lyapunov functions. Convergence results can however suggest suitable step sizes and lyapunov functions as starting points and rate of convergence analysis is always useful to understand the worst , best and average case dynamics of a system. Such results however are beyond the scope of this preliminary work. They can be found in much detail in the excellent work by **Borkar ,[1]** for example, which has been the guide for this exposition. We have presented a simplified treatment here omitting most of the probability theory, instead choosing to concentrate on the ODE based analysis of the underlying system so as to provide an intutive understanding into the underpinnings of such a system.

References

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